A field guide for Hilbert transforms
with new estimates on an associated maximal
directional operator

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Abstract

We give an overview of Hilbert transforms, followed by new results concerning maximal directional Hilbert transforms. Historically, the Hilbert transform motivated the development of many tools in harmonic analysis, such as interpolation theorems and more general singular integrals. Over time, variants of the Hilbert transform were studied as prototypical examples of singular integrals and maximal directional operators. In our research, we are especially concerned with maximal directional Hilbert transforms. After rigorously constructing the Hilbert transform and directional Hilbert transforms, we proceed to define the maximal directional Hilbert transforms. We then prove general $L^2$ mapping estimates for maximal directional Hilbert transforms, followed by specific examples which sharpen these estimates. Finally, we prove sharp $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ estimates for a large class of maximal directional Hilbert transforms.
Lay Summary

We examine an object from harmonic analysis called the Hilbert transform. Harmonic analysis is a branch of mathematics which studies how to break apart a complicated signal into more understandable pieces. The Hilbert transform is a key object in harmonic analysis, enabling us to represent signals with certain frequencies in a very simple way. Besides cataloging the development of Hilbert transform and some simple variations, we also examine modern research on Hilbert transforms. Using this overview as motivation, we then provide new results concerning maximal directional Hilbert transforms, which are a powerful and somewhat-mysterious variation on the classical Hilbert transform.
Preface

This thesis contains new and unpublished research. This is the content of Chapter 6. This research was conducted through ongoing discussion and meetings with Dr. Izabella Laba and Dr. Malabika Pramanik. Each of us contributed to the proof of the main results in this text. However, I was responsible for writing this document in its entirety.
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I love you to the moon. And back.
Chapter 1

Introduction

In this thesis, we provide an exposition for the Hilbert transform and related functional operators on \( L^2(\mathbb{R}^n) \). This first chapter provides a broad overview of the maximal directional Hilbert transform, including the statement of new results concerning maximal directional Hilbert transforms with certain extremal behavior. This chapter also includes a short introduction to the Hilbert transform on \( L^2(\mathbb{R}) \), its extension to the directional Hilbert transforms on \( L^2(\mathbb{R}^n) \), and the resulting modifications which lead to the maximal directional Hilbert transforms on \( L^2(\mathbb{R}^n) \).

Together, this overview provides a superficial understanding of the contents of this thesis, without providing detailed cumbersome definitions or rigorous proofs. This more-relaxed overview prepares the reader for the remaining chapters. These include the construction of the Hilbert transform as an operator on \( L^2(\mathbb{R}) \), which is the subject of Chapter 2, the discussion of its higher-dimensional variants, which are the subjects of Chapters 2-5, and proofs of novel results, which are given in full detail in Chapter 6.

1.1 Major Pieces

1.1.1 Fourier transforms

We first discuss the Fourier transform. For this introduction, we see results pertaining to the spaces \( L^1(\mathbb{R}^n) \) and \( L^2(\mathbb{R}^n) \), sparing most details. For information on \( L^p \) spaces, see [4]. For more detailed information on the Fourier transform, one can reference [5] or [16].

We begin with the \( L^1 \) Fourier transform.

**Definition 1.1.1.** The Fourier transform of \( f \in L^1(\mathbb{R}^n) \) is given by,

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx
\]  

(1.1)

For \( f \in L^1(\mathbb{R}^n) \), it is easy to see the integral in (1.1) converges absolutely.
for all $\xi \in \mathbb{R}^n$. In fact, we have,

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)e^{-2\pi i x \cdot \xi}| dx = \int_{\mathbb{R}^n} |f(x)| dx = ||f||_{L^1(\mathbb{R}^n)} \tag{1.2}$$

Hence, whenever $f \in L^1$, we have that $\hat{f} \in L^\infty$ with $||\hat{f}||_\infty \leq ||f||_1$. So long as $f \in L^1(\mathbb{R}^n)$, $\hat{f}$ is also a continuous function. This is a direct consequence of inequality (1.2) and the dominated convergence theorem.

A key result in the study of the Fourier analysis is the Fourier inversion formula. Although a relatively simple statement, this result underpins much of what follows in this thesis.

**Proposition 1.1.2.** Suppose that $f, g \in L^1(\mathbb{R}^n)$ and also that $\hat{f} \in L^1(\mathbb{R}^n)$. Then, for almost-every $x \in \mathbb{R}^n$, we have:

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi \quad \text{or, alternatively,} \quad f(-x) = \hat{\hat{f}}(x)$$

This leads us to the inverse Fourier transform,

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi i x \cdot \xi} d\xi. \tag{1.3}$$

where again, we assume $f \in L^1(\mathbb{R}^n)$. As with the Fourier transform, the integral in (1.3) converges absolutely for $f \in L^1(\mathbb{R}^n)$. Beyond this, the transformation $f \mapsto \hat{f}$ on $L^1$ shares all relevant properties as the Fourier transform $f \mapsto \hat{f}$. With this piece of notation, we can write the inversion formula as,

$$f(x) = (\hat{\hat{f}})(x) \quad \text{for almost-every} \quad x \in \mathbb{R}^n.$$  

Continuing with our enumeration of the properties of Fourier transforms, we come to the duality relationship. We state this again as a proposition.

**Proposition 1.1.3.** Suppose that $f, g \in L^1(\mathbb{R}^n)$. Then, we have the following relation:

$$\int_{\mathbb{R}^n} \hat{f}(x)g(x) dx = \int_{\mathbb{R}^n} f(x)\hat{g}(x) dx.$$

Combining Proposition 1.1.2 and Proposition 1.1.3 leads to our final basic formula for the Fourier transform on $L^1$. This is the following proposition.

**Proposition 1.1.4.** Let $f, g \in L^1(\mathbb{R}^n)$. The following relationship then holds,

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)} dx = \int_{\mathbb{R}^n} \hat{f}(x)\overline{\hat{g}(x)} dx.$$
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We choose to refer to this formula as Plancherel’s theorem.

Using 1.1.4, we can extend the Fourier transform to some ‘nice’ operator on $L^2$. Unfortunately, representation (1.1) does not hold for every $f \in L^2(\mathbb{R}^n)$. However, if we choose $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, Plancherel’s theorem gives,

$$||f||_2^2 = \int_{\mathbb{R}^n} f(x) \overline{f(x)} dx = \int_{\mathbb{R}^n} \hat{f}(x) \overline{f(x)} dx = ||\hat{f}||_2^2.$$

So, the $L^1$ Fourier transform $f \mapsto \hat{f}$ maps $L^1 \cap L^2$ into $L^2$ isometrically. Also, $L^1 \cap L^2$ is a dense in $L^2$ with respect to the norm $||\cdot||_2$. Since $L^2$ is a Banach space, the mapping $f \mapsto \hat{f}$ on $L^1 \cap L^2$ admits a unique linear extension to $L^2$.

The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is thus the unique linear operator such that $\mathcal{F} f = \hat{f}$ for every $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Moreover, the operator $\mathcal{F}$ is remarkably well-behaved, as we had hoped. We summarize in the following Proposition, which is taken from [13, pg. 185]

**Proposition 1.1.5.** There exists a unique, bounded linear operator $\mathcal{F} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ with the following properties:

1. If $f \in L^1 \cap L^2$, then $\mathcal{F} f = \hat{f}$, where $\hat{f}$ is defined via (1.1).

2. $||\mathcal{F} f||_2 = ||f||_2$ for every $f \in L^2(\mathbb{R}^n)$.

3. $\mathcal{F}$ is an isomorphism of the (Hilbert) space $L^2(\mathbb{R}^n)$ onto itself.

Because of this first property, we continue to call the mapping $\mathcal{F} : L^2 \rightarrow L^2$ the Fourier transform and use the notation $\hat{f}$ and $\mathcal{F} f$ interchangeably. For those unfamiliar with this convention, I suggest reading the first four chapters in [16].

We come to our final word on the Fourier transform on $L^2$. For functions in $L^2(\mathbb{R}^n)$, the following kind of "inversion formula" continues to hold.

**Proposition 1.1.6.** Suppose that $f \in L^2(\mathbb{R}^n)$. Let,

$$\phi_A(\xi) = \int_{[-A,A]^n} f(x)e^{-2\pi ix\cdot \xi} d\xi \quad \text{and} \quad \psi_A(x) = \int_{[-A,A]^n} \hat{f}(\xi)e^{2\pi i x\cdot \xi} d\xi.$$

Then,

$$\lim_{A \rightarrow \infty} ||\phi_A - \hat{f}||_2 = 0 \quad \text{and} \quad \lim_{A \rightarrow \infty} ||\psi_A - f||_2 = 0.$$

This concludes our brief exposition on the Fourier transform, and prepares us for the introduction of the main object of interest in this thesis.
1.1. Major Pieces

1.1.2 Hilbert transforms

Key objects of interest in this thesis are Hilbert transforms acting on $L^p(\mathbb{R})$ for $1 < p < \infty$. In this exposition, we present the Hilbert transform as an operator on $L^2$. The following definition is taken from [15, pg. 62].

**Definition 1.1.7.** The Hilbert transform of $f \in L^2(\mathbb{R})$ is defined via the formula,

$$Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - y) \frac{dy}{y} = \lim_{\epsilon \to 0} \int_{|y| > \epsilon} f(x - y) \frac{dy}{y}.$$

Justifying the existence of the Hilbert transform as an operator on $L^2(\mathbb{R})$, as given by equation 1.1.2, will occupy the majority of Chapter 2. We will also demonstrate that the Hilbert transform on $L^2$ has the following amenable properties:

1. The Hilbert transform is an isometric, bijective linear operator on $L^2$, with $H \circ H = -\text{Id}_{L^2}$.

2. For each $f \in L^2(\mathbb{R})$, we have the following alternative formulation of the Hilbert transform:

$$Hf(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{\text{sgn}(\xi)}{i} e^{2\pi i x \xi} d\xi \quad (1.4)$$

The first of these statements asserts that the Hilbert transform is a unitary operator on $L^2(\mathbb{R})$. The second statement introduces the Hilbert transform as a Fourier multiplier operator, a property which will be further explored in Chapter 2.

To acquaint ourselves with Hilbert transforms, let us consider the following example. Suppose we take the $f$ to be the characteristic function of the interval $[-1, 1] \subset \mathbb{R}$. Then, for any $\epsilon > 0$,

$$H_\epsilon f(x) = \int_{|y| > \epsilon} 1_{[-1,1]}(x - y) \frac{dy}{y}.$$

Using the change of variables $y \to x - y$ and after a little algebra, one obtains:

$$H_\epsilon f(x) = \log \left( \frac{x - 1}{x + 1} \right) + \log \left( \frac{x + \epsilon}{x - \epsilon} \right).$$
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So, for almost-every \( x \in \mathbb{R} \), \( H_\epsilon 1_{[-1,1]}(x) \to \log(\frac{x+1}{x-1}) \) as \( \epsilon \to 0 \). This function has mild singularities at \( x = -1, 1 \), preventing \( H_\epsilon f \) from being an \( L^\infty \) function. Moreover, for \( x > 1 \), we have:

\[
\log \left( \frac{x+1}{x-1} \right) \geq \log \left( 1 + \frac{1}{x} \right) = \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \frac{x^{-(n+1)}}{n+1}.
\]

Since \( \log(1 + \frac{1}{x}) \) behaves like \( \frac{1}{x} \) for large \( x \), \( Hf \) cannot be in \( L^1(\mathbb{R}) \). So, even for simple functions, the Hilbert transform can behave quite poorly. Since \( 1_{[-1,1]} \in L^1 \cap L^\infty \), this example justifies our restriction to the range \( 1 < p < \infty \) when examining the mapping properties of \( H \).

We now consider simple variations of the Hilbert transform for functions in \( L^p(\mathbb{R}^n) \), again for \( 1 < p < \infty \). This is discussed in more detail in Chapter 3. We are especially concerned with the directional Hilbert transforms. We take the definition from [5, pg.272], but restrict our attention to \( L^2(\mathbb{R}^n) \) for the moment.

**Definition 1.1.8.** Given some \( f \in L^2(\mathbb{R}^n) \) and some \( v \in S^{n-1} \), we define the directional Hilbert transform along \( v \) at \( x \in \mathbb{R}^n \) as:

\[
H_v f(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - tv) \frac{dt}{t}
\]

The directional Hilbert transforms can be pointwise-controlled by the univariate Hilbert transform. In fact, the directional Hilbert transforms can be written in terms of the univariate Hilbert transform composed with orthogonal matrices on \( \mathbb{R}^n \). This greatly simplifies the study of the directional Hilbert transforms, as many useful properties of the Hilbert transform over \( \mathbb{R} \) have direct analogs for directional Hilbert transform.

### 1.1.3 The maximal directional Hilbert transforms

Having defined the Hilbert transform as an operator on \( L^2 \), and introduced the directional Hilbert transforms, we can now define the maximal directional Hilbert transforms of multivariate functions. This is the main subject of Chapter 4. First, we introduce a simple, yet central, piece of nomenclature. In \( \mathbb{R}^n \), a non-empty subset \( \Theta \subset S^{n-1} \) is called a set of directions or direction set. At the moment, we place no restrictions on the cardinality of such an \( \Theta \). This will be a central consideration in Chapters 4 and 5.

We now define some central objects of interest for this thesis: the maximal directional Hilbert transforms. We take the formal definition from [9], again avoiding questions of convergence:
Definition 1.1.9. Given a fixed direction set $\Theta \subset \mathbb{S}^{n-1}$, and some measurable $f : \mathbb{R}^n \to \mathbb{C}$, we define the maximal directional Hilbert transform of $f$ over $\Theta$ at the point $x \in \mathbb{R}^n$ as:

$$H_\Theta f(x) := \sup_{v \in \Theta} |H_v f(x)| = \sup_{v \in \Theta} \left| \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} f(x - tv) \frac{dt}{t} \right|. \quad (1.5)$$

Thus, for each point $x \in \mathbb{R}^n$, $H_\Theta f(x)$ gives an optimal pointwise upper bound for each directional Hilbert transform defined over $\Theta$.

Equation (1.5) suggests that maximal directional Hilbert transforms are an entirely different class of operators from the directional Hilbert transforms. One obvious difference is that, whereas the directional Hilbert transforms are linear operators, maximal directional Hilbert transforms are usually sub-linear. Moreover, while $H_v$ was associated to a single vector $v \in \mathbb{S}^{n-1}$, $H_\Theta$ now depends on $\Theta \subset \mathbb{S}^{n-1}$, which itself can have a high degree of combinatorial complexity.

For any fixed $\Theta \subset \mathbb{S}^{n-1}$, one can ask whether the maximal directional Hilbert $H_\Theta$ defines a bounded operator on $L^p$. Since, for each $v \in \Theta$, $H_v$ is a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, it seems reasonable to guess that $H_\Theta$ would itself be a bounded operator on $L^p(\mathbb{R}^n)$ for the same range of exponent. When $\Theta$ is a finite set in the plane, this is indeed the case. We give a simple proof of this fact in Chapter 4. On more restricted classes of functions than general $L^p(\mathbb{R}^n)$, one can obtain bounded maximal directional Hilbert transforms. [2] Through the late 1900’s to the early 2000’s, the $L^p(\mathbb{R}^n)$ boundedness of maximal directional Hilbert transforms over infinite direction sets remained unproven. This was likely because no such result could be proven.

In 2008, it was discovered that all maximal directional Hilbert transforms associated to infinite direction sets must be unbounded operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and $n = 2$. [6] This is a consequence of the following asymptotic lower-bound. For every direction set $\Theta$ of cardinality $N$, there always exists an absolute constant $C$ independent of $N$ such that:

$$||H_\Theta f||_{L^p(\mathbb{R}^2)} \geq C\sqrt{\log N}||f||_{L^p(\mathbb{R}^2)}. \quad (1.6)$$

This result is in sharp contrast to the $L^p(\mathbb{R}^2)$ boundedness of all maximal directional Hilbert transforms over finite direction sets in the plane.

The asymptotic lower bound (1.6) has generated much study into directional maximal Hilbert transforms over finite direction sets. While (1.6) provides a general lower bound on all directional maximal Hilbert transforms over a set of $N$ directions in the plane, there are known examples of
direction sets which exhibit much larger operator norm. [8] We will discuss one such example in Chapter 4, and also prove a new lower bound which far exceeds (1.6), and is applicable to a large class of direction sets in the plane. While we delay the proof of these results until the final chapter, we briefly present the result below.

**Theorem 6.1.1.** Let \( \Theta = \{(1, v_j) : 0 \leq v_1 < v_2 < \cdots < v_N \leq 1\} \) be a set of directions in the plane. For each \( 1 \leq j \leq N \), call \( \delta_j = v_{j+1} - v_j > 0 \). Further, let:

\[
\Lambda_+ := \{j \in \mathbb{N} : 1 \leq j \leq N, \text{ and } \delta_j N > 1\}
\]
\[
\Lambda_- := \{j \in \mathbb{N} : 1 \leq j \leq N, \text{ and } \delta_j N \leq 1\}
\]

Then, for the associated directional maximal Hilbert transform \( H_\Theta \), we have the following operator norm estimate:

\[
||H_\Theta||_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \geq C \sum_{j \in \Lambda_+} \delta_j [\log^2 \frac{1}{\delta_j} + \log \frac{1}{\delta_j}] + \frac{C}{\log N} \sum_{j \in \Lambda_-} \delta_j \log^3 \frac{1}{\delta_j},
\]

where \( 0 < C < 1 \) is some absolute constant independent of \( N \).

Thus, by studying the angular distance between adjacent directions \( v_j, v_{j+1} \), we prove new lower bounds on certain direction sets. As an example, fix some \( \frac{1}{1000} < \alpha < 1 \), and consider the set of directions \( \Theta_\alpha := \{(1, v_j) \in \mathbb{R}^2 : 0 = v_1 < \cdots < v_N = 1\} \), where \( v_j \in \mathbb{R} \) is given by equation:

\[
v_{j+1} - v_j = \frac{1}{(1 + j)^\alpha} \implies v_j = \sum_{k=1}^{j} \frac{1}{(1 + k)^\alpha}.
\]

Utilizing Theorem 6.1.1, we obtain the following lower bound,

\[
||H_\Theta||_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \geq C(\alpha) \log N,
\]

where \( 0 < C(\alpha) < 1 \) depends on \( \alpha \), but is absolute with respect to \( N \). This estimate is proven in detail in Chapter 6.
Chapter 2

The Hilbert transform on \(\mathbb{R}\)

Maximal directional Hilbert transforms emerged from the rich theory of the Hilbert transform on \(L^p(\mathbb{R})\). In this chapter, we rigorously construct the Hilbert transform on \(L^2(\mathbb{R})\). To accomplish this, we take the complex analytic approach developed by M. Riesz, which also uses a fair amount of Fourier analysis. This approach is similar to E. Stein’s approach in [15]. But first, let us consider an overview of this chapter, which is more detailed than some of the others.

We first examine the relationship between \(L^2\) functions of the torus \(T = [0, 2\pi]\) and analytic functions of the complex unit disc \(D = \{z \in \mathbb{C} : |z| < 1\}\). As we shall see, Fourier series and the Hilbert transform on \(T\) enable us to consider these functions as ‘essentially’ the same.

For any \(f \in L^2([0, 2\pi])\), we can interpret \(f\) as a periodic function of the unit circle \(\partial D = \{z \in \mathbb{C} : |z| = 1\}\). From this more-geometric vantage point, utilize the Poisson integral to identify a certain harmonic function \(u_f : \mathbb{D} \to \mathbb{R}\) related to \(f\). In fact, \(u_f\) is the unique harmonic in the disc such that, for almost every \(\theta \in S^1\), we have \(u_f(re^{i\theta}) \to f(\theta)\) as \(r \to 1\). In this sense, we view \(f \in L^2([0, 2\pi])\) as the boundary value of \(u_f\).

We continue to examine \(u_f\). From the theory of harmonic functions, there exists a unique harmonic \(Q_f : \mathbb{D} \to \mathbb{R}\), such that \(u_f + iQ_f\) is analytic in \(\mathbb{D}\). This conjugate function \(Q_f\) is identifiable using the conjugate Poisson kernel. We then are left to determine the boundary value of \(Q_f\). This is precisely the role played by the Hilbert transform: uniquely specifying the boundary values of \(Q_f\), and guaranteeing their square-integrability on \(T\).

Leaving behind the disc \(\mathbb{D}\), we then consider the half-plane,

\[
\mathbb{H}^+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}
\]

A Fourier analytic construction of the Hilbert transform of \(L^2(\mathbb{R})\) functions then follows. This is the natural extension to \(L^2(\mathbb{R})\) of the general scheme presented for \(L^2([0, 2\pi])\). Following this construction, we prove that the Fourier analytic construction of the Hilbert transform coincides with its spatial representation as a singular integral operator. This is the main content of this chapter and is given in full detail.
2.1 \( L^2 \) functions of the torus

In this thesis, we are mostly interested in the Hilbert transform as an operator on \( L^2(\mathbb{R}) \). However, one of the earliest applications of the Hilbert transform was in specifying the boundary values of analytic functions of the disc \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \). This is primarily a complex-analytic problem.

We begin by letting \( T = T^1 \) denote the one-dimensional unit torus. There are two homeomorphically-equivalent definitions of \( T \),

\[ T = \{ \theta \in \mathbb{R} : 0 \leq \theta \leq 2\pi \} \simeq \{ e^{i\theta} \in \mathbb{C} : 0 \leq \theta \leq 2\pi, \text{ and } 0 \equiv 2\pi \} \]

with homeomorphism given by the mapping \( \phi : e^{i\theta} \mapsto \theta \). This allows us to think of functions \( f \in L^2([0,2\pi]) \) as periodic functions \( f_{\text{circ}} : \partial \mathbb{D} \to \mathbb{C} \) via the identification:

\[ f_{\text{circ}}(e^{i\theta}) = (f \circ \phi)(e^{i\theta}), \text{ for every } \theta \in [0,2\pi]. \]

Because of this simple identification, we associate \( T \) with the interval \([0,2\pi]\) in our calculations. However, most of our intuition will come from considering \( T \) as the boundary of the unit disc \( \mathbb{D} \).

2.1.1 Harmonic functions on \( \mathbb{D} \) and their conjugates

We are, for now, primarily interested in the connection between real-valued \( f \in L^2([0,2\pi]) \) and analytic functions of the disc \( \mathbb{D} \). We begin by introducing the **Poisson integral** of real-valued \( f \in L^2([0,2\pi]) \). This is the function \( u = u_f : \mathbb{D} \to \mathbb{C} \) where, for \( 0 \leq r < 1 \) and \( 0 \leq \theta \leq 2\pi \), we have:

\[ u(re^{i\theta}) = u(r,\theta) = \int_0^{2\pi} P_r(\theta - t)f(t)dt. \]

Here, \( P_r : [0,2\pi] \to \mathbb{C} \) is the **Poisson kernel on the disc**, given by expression:

\[ P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{i n \theta} = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}. \]

The Poisson integral of a real-valued function \( f \in L^2([0,2\pi]) \) has two distinct properties. First, we have that \( u : \mathbb{D} \to \mathbb{R} \) is a harmonic function of the disc. This is quite an improvement. Recall, we only assume that \( f \) is square-integrable on the circle. This improvement follows from the smoothness of the Poisson kernel, which is itself a harmonic function on the disc.
2.1. \( L^2 \) functions of the torus

For the Poisson integral, a simple application of the Lebesgue dominated convergence theorem gives,

\[
\Delta u(r, \theta) = \frac{1}{2\pi} \Delta \int_0^{2\pi} P_r(\theta - t) f(t) dt = \frac{1}{2\pi} \int_0^{2\pi} [\Delta P_r(\theta - t)] f(t) dt = 0.
\]

This guarantees that the Poisson integral of every \( f \in L^2([0, 2\pi]) \) is indeed a harmonic function.

We now come to the second important property of the Poisson integral. If \( u : \mathbb{D} \rightarrow \mathbb{R} \) denotes the Poisson integral of some \( f \in L^2([0, 2\pi]) \), one has:

\[
\lim_{r \to 1} u(r, \theta) = f(\theta), \quad \text{for almost-every } 0 \leq \theta \leq 2\pi. \tag{2.1}
\]

This is a consequence of some straightforward calculations which we do not discuss here.

Utilizing the Poisson integral, for each \( f \in L^2([0, 2\pi]) \), we have identified a harmonic function \( u : \mathbb{D} \rightarrow \mathbb{R} \) such that \( u \rightarrow f \) along \( \partial \mathbb{D} \). The following theorem motivates a crucial question regarding the Poisson integral of a function \( f \in L^2([0, 2\pi]) \).

**Proposition 2.1.1.** Let \( D \subset \mathbb{C} \) be a simply-connected domain. Then, if \( u : D \rightarrow \mathbb{R} \) is harmonic in \( D \), then there exists a unique harmonic function \( v : D \rightarrow \mathbb{R} \) such that:

\[ F(r, \theta) = u(r, \theta) + iv(r, \theta), \quad \text{is analytic throughout } D. \]

Such a \( v \) is called the harmonic conjugate function of \( u \).

Proposition 2.1.1 ensures the existence of a harmonic conjugate \( v : \mathbb{D} \rightarrow \mathbb{R} \) so that \( u + iv \) is analytic in the disc. At this point, we do not have an explicit description of the functions \( F : \mathbb{D} \rightarrow \mathbb{C} \) or \( v : \mathbb{D} \rightarrow \mathbb{R} \). As we shall see in the following section, they are given by the formulas:

\[
F(r, \theta) = \sum_{n=0}^{\infty} \hat{f}(n) r^n e^{in\theta}
\]

\[
v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} Q_r(\theta - t) f(t) dt,
\]

where \( \hat{f}(n) \) denotes the \( n \)-th Fourier coefficient of \( f \), and \( Q_r \) is the conjugate Poisson kernel, given by:

\[
Q_r(\theta) = \sum_{n=-\infty}^{\infty} \frac{\text{sgn}(n)}{i} r^{|k|} e^{ik\theta}, \quad \text{for } 0 \leq r < 1 \text{ and } 0 \leq \theta \leq 2\pi. \tag{2.4}
\]
Although far from obvious, equation (2.2) provides the first glimpse at the Hilbert transform on the torus. In order to adequately motivate this definition, we must first discuss the Fourier series of functions $f \in L^2([0, 2\pi])$.

### 2.1.2 Fourier series and analytic functions on $\mathbb{D}$

Historically, the Hilbert transform first arose in the study of Fourier series of $L^p$ functions on the torus. Given some $f \in L^2([0, 2\pi])$, one associates $f$ with its Fourier series. We use the notation,

$$ f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}, \text{ where } \hat{f}(n) = \frac{1}{2\pi} \int_{0}^{2\pi} f(s)e^{-ins}ds. $$

For each $f$, we let

$$ S_N f(\theta) = \sum_{|n| \leq N} \hat{f}(n)e^{in\theta}. $$

These are the $N$-th partial sums of the Fourier series of $f$. It is a well-known that, for $f \in L^2([0, 2\pi])$, we have:

$$ \lim_{N \to \infty} ||S_N f - f||_{L^2([0,2\pi])} = 0 $$

Since $L^2([0, 2\pi])$ is a Banach space, limits in the $L^2$-norm must be unique. As a consequence, each $f \in L^2([0, 2\pi])$ is uniquely identifiable via its Fourier series. Finally, we also have Parseval’s relationship,

$$ \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |f(s)|^2 ds. \quad (2.5) $$

At this point, we want to connect our previous discussion of analytic functions on the disc with the Fourier series of $f \in L^2([0, 2\pi])$. We begin by letting $F: \mathbb{D} \to \mathbb{C}$ be the function given by the formal power series:

$$ F(r, \theta) = \sum_{n=0}^{\infty} \hat{f}(n)r^ne^{in\theta}, \text{ where } 0 \leq r < 1 \text{ and } 0 \leq \theta \leq 2\pi. \quad (2.6) $$

Notice that, by the Cauchy-Schwarz inequality and (2.5), we have,

$$ |\sum_{n=0}^{\infty} \hat{f}(n)r^ne^{in\theta}| \leq \left( \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} r^{2n} \right)^{1/2} < \infty, $$
2.1. $L^2$ functions of the torus

justifying the convergence of the power series in (2.6). Under the assumption that $f \in L^2([0, 2\pi])$ is real-valued, there is a simple formulation for the real and imaginary parts of the analytic function (2.6). We outline the necessary calculations below.

First, utilizing the Fourier coefficients of $f$, we have,

$$F(r, \theta) = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi} \int_{0}^{2\pi} f(s) e^{-ins} ds \right) r^n e^{in\theta}$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} f(s) r^n e^{in(\theta-s)} ds.$$

Now, utilizing Euler’s formula, we can write the above as:

$$F(r, \theta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{0}^{2\pi} f(s) r^n \cos(n(\theta-s)) ds \quad (2.7)$$

$$+ \frac{i}{2\pi} \sum_{n=0}^{\infty} \int_{0}^{2\pi} f(s) r^n \sin(n(\theta-s)) ds.$$

This expresses $F : \mathbb{D} \to \mathbb{C}$ in terms of its real and imaginary parts. However, we can say more than this.

First, we examine the real part of $F$. Since $\cos(\omega) = \frac{e^{i\omega} + e^{-i\omega}}{2}$, we have:

$$\sum_{n=0}^{\infty} r^n \cos(n(\theta-s)) = \frac{1}{2} \sum_{n=0}^{\infty} r^n e^{in(\theta-s)} + r^n e^{-in(\theta-s)}$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-s)}.$$

This final equality is exactly the Poisson kernel $\frac{1}{2} P_r(\theta-s)$. Thus, applying the Lebesgue dominated convergence theorem in (2.7), we have:

$$Re[F(r, \theta)] = \frac{1}{4\pi} \int_{0}^{2\pi} f(s) P_r(\theta-s) ds \quad (2.8)$$

For the imaginary part of $F$, we use the identity $\sin(\omega) = \frac{e^{i\omega} - e^{-i\omega}}{2i}$. 
2.1. \(L^2\) functions of the torus

Substituting into the second term in (2.7) gives:

\[
\sum_{n=0}^{\infty} r^n \sin(n(\theta - s)) = \frac{1}{2} \sum_{n=0}^{\infty} r^n \frac{e^{in(\theta - s)} - e^{-in(\theta - s)}}{i} \\
= \frac{1}{2} \sum_{n=1}^{\infty} r^{|n|} \frac{e^{in(\theta - s)}}{i} - \frac{1}{2} \sum_{n=-\infty}^{\infty} r^{|n|} \frac{e^{in(\theta - s)}}{i} \\
= \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{\text{sgn}(n)}{i} r^{|n|} e^{in(\theta - s)}
\]

Recalling equation (2.4), this final quantity is the conjugate Poisson kernel \(Q_r(\theta - s)\). Using dominated convergence, we again obtain,

\[
\text{Im}[F(r, \theta)] = \frac{1}{4\pi} \int_0^{2\pi} f(s) Q_r(\theta - s) ds \quad (2.9)
\]

Combining equations (2.7), (2.8) and (2.9), we see that:

\[
F(r, \theta) = \frac{1}{4\pi} \int_0^{2\pi} f(s) P_r(\theta - s) ds + \frac{i}{4\pi} \int_0^{2\pi} f(s) Q_r(\theta - s) ds. \quad (2.10)
\]

If we prefer, both (2.8) and (2.9) have natural representations in terms of the Fourier coefficients of \(f\). As given by Plancherel, these are given by:

\[
\frac{1}{4\pi} \int_0^{2\pi} f(s) P_r(\theta - s) ds = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta} \\
\frac{1}{4\pi} \int_0^{2\pi} f(s) Q_r(\theta - s) ds = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{f}(n) \frac{\text{sgn}(n)}{i} r^{|n|} e^{in\theta}
\]

Moreover, through a series of simple but tedious calculations, one can show that:

\[
F(r, \theta) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(s)}{e^{is} - re^{i\theta}} e^{is} ds \quad (2.11)
\]

The right-hand side of equation (2.11) is the Cauchy integral projection of \(f\) into \(\mathbb{D}\). This representation arises from expression (2.10) and re-appears when we consider the Hilbert transform of \(f \in L^2(\mathbb{R})\) later in this chapter.

We summarize what we have discussed so far. We began with the Poisson integral \(u : \mathbb{D} \to \mathbb{R}\) of a real-valued function \(f \in L^2([0, 2\pi])\). Proposition 2.1.1 guaranteed the existence of a unique conjugate harmonic function.
2.1. \( L^2 \) functions of the torus

\( v : \mathbb{D} \to \mathbb{R} \) such that \( F = u + iv \) was analytic in \( \mathbb{D} \). Having derived equation (2.10), we see that this harmonic conjugate \( v \) is uniquely given by the expression:

\[
v(r, \theta) = \frac{1}{4\pi} \int_0^{2\pi} f(s)Q_r(\theta - s)ds.
\]

Finally, having determined the harmonic functions \( u, v : \mathbb{D} \to \mathbb{R} \), we can explicitly write the unique analytic function \( F : \mathbb{D} \to \mathbb{C} \) whose real boundary-values converge to \( f \). This \( F \) is given by the equivalent expressions:

\[
F(r, \theta) = \sum_{n=-\infty}^{\infty} \tilde{f}(n)r^n e^{in\theta}
\]

\[
F(r, \theta) = \frac{1}{4\pi} \int_0^{2\pi} f(s)P_r(\theta - s)ds + \frac{i}{4\pi} \int_0^{2\pi} f(s)Q_r(\theta - s)ds \quad (2.12)
\]

In concluding this section, we arrive at the heart-of-the-matter. Presently, we have only considered the harmonic conjugate function:

\[
v(r, \theta) = \frac{1}{4\pi} \int_0^{2\pi} Q_r(\theta - s)f(s)ds = \sum_{n=-\infty}^{\infty} \tilde{f}(n)\frac{\text{sgn}(n)}{n}r^n e^{in\theta}, \quad (2.13)
\]

in terms of its complex-analytic properties. However, there is a powerful functional operator hiding in expression (2.13).

Namely, if we let \( a_n = \tilde{f}(n) \) and \( \tilde{a}_n = \frac{\text{sgn}(n)}{i} \tilde{f}(n) \) for each non-zero \( n \in \mathbb{Z} \), we have:

\[
|a_n| = |\tilde{a}_n| \implies \sum_{n=-\infty}^{\infty} |a_n|^2 = \sum_{n=-\infty}^{\infty} |\tilde{a}_n|^2 + |a_0|^2.
\]

So, the sequence \( \tilde{a}_n \) must be square-summable. Proposition 2.5 then furnishes a unique \( \tilde{f} \in L^2([0, 2\pi]) \) such that \( \tilde{f}(n) = a_n \). As a consequence, we must have that:

\[
\tilde{f}(\theta) \sim \sum_{n=-\infty}^{\infty} \tilde{f}(n)\frac{\text{sgn}(n)}{i} e^{i n \theta}. \quad (2.14)
\]

This \( \tilde{f} \) in equation (2.14) is the conjugate function associated to \( f \in L^2([0, 2\pi]) \). As shown above, the mapping \( f \mapsto \tilde{f} \) is bounded on \( L^2([0, 2\pi]) \), guaranteeing that \( \tilde{f} \in L^2([0, 2\pi]) \).

This is especially useful when considering our previous discussion of the boundary values of analytic functions. For, if \( F : \mathbb{D} \to \mathbb{C} \) denotes the Cauchy integral projection of some \( f \in L^2([0, 2\pi]) \), we then know that:

\[
\lim_{r \to 1} F(r, \theta) = \frac{1}{2}[f(\theta) + i\tilde{f}(\theta) + a_0], \text{ for almost-every } 0 \leq \theta \leq 2\pi.
\]
2.2 Constructing the Hilbert transform on $L^2(\mathbb{R})$

Even more, we are always guaranteed that the real and imaginary boundary-values of the analytic function $F$ are square-integrable on the circle.

We end this section with a brief discussion of the general case, where $L^2([0,2\pi])$ is replaced by $L^p([0,2\pi])$ for some $1 < p < \infty$. For $f \in L^p$, we still have the Fourier and Allied series,

$$
\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{i n \theta} = \sum_{n=-\infty}^{\infty} \hat{f}(n) \frac{\text{sgn}(n)}{i} e^{i n \theta}.
$$

It is a celebrated fact that $S_N f \to f$ in $L^p$ norm for each $f \in L^p([0,2\pi])$. Even more satisfying is the realization that the mapping $f \mapsto \tilde{f}$ is bounded on $L^p([0,2\pi])$. This result was first attributed to M. Riesz, and was a strong motivation for the development of interpolation methods between $L^p([0,2\pi])$ spaces. We do not discuss this result any further, but refer the interested reader to Chapter 2 in [15].

2.2 Constructing the Hilbert transform on $L^2(\mathbb{R})$

To construct the Hilbert transform on $L^2(\mathbb{R})$, we will take the complex analytic approach of M. Riesz. This has two advantages. First, it exploits the unique Fourier analytic properties of $L^2(\mathbb{R})$. Secondly, it mirrors our discovery of the conjugate function, with the upper half-space $\mathbb{H}^+ = \{ z \in \mathbb{C} : \text{Im}[z] > 0 \}$ now replacing the disc $\mathbb{D}$.

2.2.1 The Hilbert transform as a conjugate operator

For $f \in L^2(\mathbb{R})$, we let:

$$
C_f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt, \text{ where Im}(z) > 0. \tag{2.15}
$$

This is the Cauchy integral projection of $f$ into $\mathbb{H}^+$. Compare equation (2.15) with (2.11) to see how we pass from the disc to the upper half space. For each fixed $z = x + iy$ in $\mathbb{H}^+$, we have:

$$
\int_{-\infty}^{\infty} \frac{dt}{|t - z|^2} = \int_{-\infty}^{\infty} \frac{dt}{(t - x)^2 + y^2} < \infty \implies \frac{1}{t - z} \in L^2(\mathbb{R}, dt).
$$

Letting $\epsilon(z) = -\frac{1}{\pi iz}$ for $z \in \mathbb{H}^+$, we then obtain,

$$
|C_f(z)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{f(t)}{t - z} \right| dt \leq \| f \|_{L^2(\mathbb{R}, dt)} \| \epsilon(\cdot) \|_{L^2(\mathbb{R}^+, dt)} < \infty.
$$
2.2. Constructing the Hilbert transform on $L^2(\mathbb{R})$

As a consequence of (2.15), $C_f$ converges absolutely for $f \in L^2(\mathbb{R})$. This is analogous to the absolute convergence of the series,

$$
\sum_{n=0}^{\infty} \hat{f}(n)r^n e^{i\theta}, \text{ when we had } f \in L^2([0, 2\pi]).
$$

In fact, $C_f$ has a simple representation in terms of $\hat{f}$, the Fourier transform of $f \in L^2(\mathbb{R})$. This is,

$$
C_f(z) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i z \xi} d\xi = \int_{-\infty}^{\infty} 1_{[0,\infty)}(\xi)\hat{f}(\xi)e^{2\pi i z \xi} d\xi, \quad (2.16)
$$

To see this, notice for $w \in \mathbb{H}^+$, we have:

$$
\int_{0}^{\infty} e^{2\pi i w \xi} d\xi = \frac{1}{2\pi i w} \lim_{\xi \to \infty} [e^{2\pi i w \xi} - 1] = -\frac{1}{2\pi i w}.
$$

So, for any $z \in \mathbb{H}^+$, $t \in \mathbb{R}$, we obtain:

$$
\frac{1}{2\pi i(t - z)} = -\frac{1}{2\pi i(z - t)} = \int_{0}^{\infty} e^{2\pi i(z-t) \xi} d\xi, \quad (2.17)
$$

Substituting (2.17) into (2.15) and utilizing Fubini’s theorem, we obtain:

$$
C_f(z) = \int_{-\infty}^{\infty} f(t)\left(\int_{0}^{\infty} e^{2\pi i(z-t) \xi} d\xi\right) dt
= \int_{0}^{\infty} e^{2\pi i z \xi} \left(\int_{-\infty}^{\infty} f(t)e^{-2\pi i t \xi} dt\right) d\xi.
$$

Here, the application of Fubini’s theorem is justified, since $C_f(z)$ converges absolutely for $z \in \mathbb{H}^+$. This last equality is (2.16). Notice that, for $x + iy = z \in \mathbb{H}^+$, we have:

$$
|e^{2\pi i z \xi}| = |e^{2\pi i x \xi} e^{-2\pi y \xi}| = e^{-2\pi y \xi} \in L^2(\mathbb{R}, d\xi).
$$

Since $\hat{f} \in L^2(\mathbb{R})$, the integral in (2.16) converges absolutely.

Representation (2.16) motivates the following. For $f \in L^2(\mathbb{R})$, define:

$$
u_f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)1_{[0,\infty)}(\xi)e^{2\pi i x \xi} d\xi, \text{ for } x \in \mathbb{R}.$$
Notice that, for any \( x \in \mathbb{R} \) and \( \epsilon > 0 \), we always have:

\[
C_f(x + i \epsilon) - u_f(x) = \int_0^\infty \hat{f}(\xi)e^{2\pi i (x + \epsilon)\xi}d\xi - \int_0^\infty \hat{f}(\xi)e^{2\pi i x\xi}d\xi = \int_{\mathbb{R}} \hat{f}(\xi)1_{[0,\infty)}(\xi)[e^{2\pi i (x + \epsilon)\xi} - e^{2\pi i x\xi}]d\xi
\]

As a consequence of Plancharel’s theorem (Proposition 1.1.5), we thus obtain:

\[
\lim_{\epsilon \to 0} \|C_f(\cdot + i \epsilon) - u_f(\cdot)\|_{L^2(\mathbb{R},dx)} = 0.
\]

This allows us to interpret \( u_f(\cdot) \) as the \( L^2 \) boundary value of \( C_f(\cdot + i \epsilon) \), since we can always pass to some subsequence \( \epsilon_j \to 0 \) so that \( C_f(x + i \epsilon) \to u_f(x) \) for each \( x \in \mathbb{R} \). Compare this expression with equation (2.1).

Letting \( p(\xi) = 1_{[0,\infty)} \), we then have:

\[
u_f(x) = \int_{-\infty}^\infty \hat{f}(\xi)1_{[0,\infty)}(\xi)e^{2\pi i x\xi}d\xi = (\mathcal{P}\hat{f})(x).
\]

where equality is considered in the \( L^2(\mathbb{R}) \) norm. Notice that, if \( f \in L^2(\mathbb{R}) \) and \( \hat{f}(\xi) = 0 \) for almost every \( \xi < 0 \), Proposition 1.1.5 guarantees that,

\[
u_f(x) = (\mathcal{P}\hat{f})(x) = (\mathcal{F}\check{f})(x) = f(x).
\]

(2.18)

where again equality is considered with respect to the \( L^2 \) norm. So, utilizing Parseval’s formula and basic properties of the Fourier transform (see [5]), for any \( f, g \in L^2(\mathbb{R}) \) we have,

\[
\int_{-\infty}^\infty u_f(x)\overline{g}(x)dx = \int_{-\infty}^\infty f(x)\overline{P_{f_0}(x)}dx
\]

(2.19)

The \( L^2 \) norm equalities (2.18) and (2.19) imply that the mapping \( f \mapsto \nu_f \) is an orthonormal projection of \( L^2 \) onto the subspace,

\[
\{\phi \in L^2 : \hat{\phi}(\xi) = 0, \text{ for almost-every } \xi < 0\}.
\]

We denote this mapping by \( P : L^2 \to L^2 \), so that \( Pf = \nu_f \) for each \( f \in L^2(\mathbb{R}) \). Keep in mind that, as an operator, \( P : L^2 \to L^2 \) is a ‘tangential limit’ of the Cauchy integral projection operator \( f \mapsto C_f \) for \( f \in L^2 \).

We can finally introduce the Hilbert transform on \( L^2(\mathbb{R}) \), as given by equation (1.4). First notice, for \( \xi \in \mathbb{R} \), we have:

\[
1 + \text{sgn}(\xi) = 0, \text{ whenever } \xi < 0, \text{ and } 1 + \text{sgn}(\xi) = 2, \text{ whenever } \xi > 0.
\]
Utilizing Proposition 1.1.5, we obtain the $L^2$-norm inequality:

\[ u_f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) 1_{[0,\infty)}(\xi) e^{2\pi i x \xi} d\xi \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi + \frac{i}{2} \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{\text{sgn}(\xi)}{i} e^{2\pi i x \xi} d\xi \]

\[ = \frac{1}{2} [f(x) + iHf(x)] \]

We thus arrive at the beautiful equality relating the projection operator $P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and the Hilbert transform $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$:

\[ P = \frac{1}{2} [I + iH] \implies H = \frac{1}{i} [2P - I]. \quad (2.21) \]

Recall that $P$ is the real-variable analog of the Cauchy integral (2.15). Equation (2.21) demonstrates that the Hilbert transform $H$ is the unique operator on $L^2$ which conjugates the projection operator $P$. This is the analogous conclusion to our consideration of the Cauchy integral projection into the disc, and its relationship to the conjugate function.

2.2.2 The Hilbert transform as a singular integral operator

Until now, we have treated the Hilbert transform on $L^2(\mathbb{R})$ as a uniquely Fourier analytic object. In every sense, the action of the Hilbert transform on $L^2(\mathbb{R})$ is is analogous to the conjugate function on $L^2([0,2\pi])$. Most importantly, both operators map $L^2$ to itself unitarily, and conjugate the respective Cauchy integral projections into $\mathbb{H}^+$ and $\mathbb{D}$, respectively.

In this subsection, we develop the theory unique to the Hilbert transform on $L^2(\mathbb{R})$. This is accomplished by representing the Hilbert transform as a singular integral operator. This theory is unique to the real-line, and inspired a wide range ideas in modern Harmonic Analysis, such as the Calderón-Zygmund theory of singular integral operators and Carleson-type theorems.

The heart-of-the-matter is presented in the following proposition, whose statement and proof are taken from [15].

**Proposition 2.2.1.** For $f \in L^2(\mathbb{R})$ and $\epsilon > 0$, let:

\[ H_\epsilon f(x) = \frac{1}{\pi} \int_{|y| \geq \epsilon} f(x - y) \frac{dy}{y}, \quad Hf(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{\text{sgn}(\xi)}{i} e^{2\pi i x \xi} d\xi. \]

Then, we have the following convergence in $L^2$ norm,

\[ \lim_{\epsilon \to 0^+} ||Hf - H_\epsilon f||_{L^2(\mathbb{R})} = 0. \]
2.2. Constructing the Hilbert transform on $L^2(\mathbb{R})$

Because of Proposition 2.2.1, we always have,

$$Hf(x) = \lim_{\epsilon_j \to 0} \int \frac{f(x - y) dy}{y}$$  \hspace{1cm} (2.22)

for almost-every $x \in \mathbb{R}$ along some subsequence $\epsilon_j \to 0$. Proposition 2.2.1 allows us to represent the Hilbert transform $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ without explicit reference to the Fourier transform.

In preparation of proving Proposition 2.2.1, we introduce two important integral kernels. These are:

(Poisson kernel) $P_y(x) = \frac{y}{\pi(x^2 + y^2)}$

(conjugate Poisson kernel) $Q_y(x) = \frac{x}{\pi(x^2 + y^2)}$.

These two kernels arise from the Cauchy integral kernel $c$, via the identity:

$$c(z) = -\frac{1}{\pi i z} = \frac{i\bar{z}}{\pi|z|^2} = \frac{y + ix}{\pi(x^2 + y^2)} = P_y(x) + iQ_y(x).$$  \hspace{1cm} (2.25)

As such, (2.25) demonstrates that the real and imaginary parts of $c$ on $\mathbb{H}^+$ are given by $Q_y$ and $P_y$, respectively.

Moreover, for $z = x + iy$ in $\mathbb{H}^+$, we have,

$$C_f(z) = \frac{1}{2} \int_{-\infty}^{\infty} f(t) \left( \frac{-1}{i\pi(z - t)} \right) dt = \frac{1}{2} (f \ast c)(z).$$  \hspace{1cm} (2.26)

where $C_f$ is the Cauchy integral projection of $f$ into $\mathbb{H}^+$. Using (2.25) and (2.26), we have:

$$C_f(z) = (f \ast c)(z) = \frac{1}{2} \left[ (f \ast P_y)(x) + i(f \ast Q_y)(x) \right]$$  \hspace{1cm} (2.27)

It is worth comparing (2.27) with (2.27). As before, equation (2.27) shows $f \ast Q_y$ is the unique harmonic conjugate to $f \ast P_y$ in $\mathbb{H}^+$.

We are able to prove Proposition 2.2.1 by calculating $\widehat{P_y}$ and $\widehat{Q_y}$. To do this, we appeal to (2.16). So long as $y > 0$, by equating (2.16) and (2.27), we obtain:

$$C_f(x, y) = \int_{-\infty}^{\infty} \hat{f}(\xi) 1_{[0, \infty)}(\xi) e^{2\pi i x \xi} e^{-2\pi y \xi} d\xi = \frac{1}{2} \left[ (f \ast P_y)(x) + i(f \ast Q_y)(x) \right]$$

Viewing $y > 0$ as fixed, Plancherel gives the $L^2$-norm equalities:

$$\widehat{P_y}(\xi) = e^{-2\pi y |\xi|}$$

$$\widehat{Q_y}(\xi) = e^{-2\pi y |\xi|} \frac{\text{sgn}(\xi)}{i}.$$
2.2. Constructing the Hilbert transform on $L^2(\mathbb{R})$

The requisite calculations are given in [15]. This establishes another analog between $f \ast Q_y$ and the conjugate function $\hat{f}$: the appearance of the multiplier $-i \text{sgn}(\xi)$. We are now equipped to prove Proposition 2.2.1.

**Proof of Proposition 2.2.1.** We begin by noticing that (2.28) guarantees the norm equality:

$$\lim_{y \to 0} \hat{Q}_y(\xi) = \lim_{y \to 0} e^{-2\pi y|\xi|} \frac{\text{sgn}(\xi)}{i} = \frac{\text{sgn}(\xi)}{i}.$$

Let $f \in L^2(\mathbb{R})$. Utilizing the convolution formula for the Fourier transform (see [16]):

$$\hat{Hf}(\xi) - (Q_y \ast f)(\xi) = \frac{\text{sgn}(\xi)}{i} \hat{f}(\xi) - e^{-2\pi y|\xi|} \frac{\text{sgn}(\xi)}{i} \hat{f}(\xi). \quad (2.30)$$

Consequently, $(Q_y \ast f) \to \hat{Hf}$ in $L^2$ norm. Equation (2.30), alongside Plancherel and the continuity of the $L^2(\mathbb{R})$ norm, then gives:

$$\lim_{y \to 0} \|Hf - (Q_y \ast f)\|_{L^2(\mathbb{R})} = \lim_{y \to 0} \|\hat{Hf} - (Q_y \ast f)\|_{L^2(\mathbb{R})} = 0. \quad (2.31)$$

Recall the operator $H_\epsilon : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined:

$$H_\epsilon f(x) = \frac{1}{\pi} \int_{|y| \geq \epsilon} f(x - t) \frac{dt}{t}, \text{ for each } \epsilon > 0.$$

For some $\epsilon > 0$ small, we consider the difference,

$$H_\epsilon f(x) - (f \ast Q_\epsilon)(x) = (f \ast \Delta_\epsilon)(x), \quad (2.32)$$

where the kernel $\Delta_\epsilon$ is given by:

$$\Delta_\epsilon(x) = \begin{cases} \frac{1}{\pi x} - Q_\epsilon(x), & \text{for } |x| \geq \epsilon \\ -Q_\epsilon(x), & \text{for } |x| < \epsilon \end{cases}.$$

The reader is encouraged to prove the following basic facts concerning $\Delta_\epsilon$:

1. For each $x \in \mathbb{R}$ and every $\epsilon > 0$, $\Delta_\epsilon(x) = \epsilon^{-1} \Delta_1(\epsilon^{-1}x)$

2. There exists an $A > 0$ such that for every $x \in \mathbb{R}$, $|\Delta_1(x)| \leq \frac{A}{1 + x^2}$.

3. We have the integral estimate:

$$\int_{-\infty}^{\infty} \Delta_\epsilon(x) dx = 0.$$
2.2. Constructing the Hilbert transform on $L^2(\mathbb{R})$

Facts (1.) and (2.) are dilation and decay estimates usually associated to approximations to the identity. Fact (3.) is a significant departure, since approximate identities must have integral 1. This is exactly what we need. For any $\epsilon > 0$, we have by (3.) that,

$$(f * \Delta_\epsilon)(x) = \int_{-\infty}^{\infty} f(x-t)\Delta_\epsilon(t)dt = \int_{-\infty}^{\infty} [f(x-t) - f(x)]\Delta_\epsilon(t)dt.$$\hspace{1cm}\text{(2.32)}$$

Using (1.), we dilate the kernel and perform a change-of-variables,

$$(f * \Delta_\epsilon)(x) = \int_{-\infty}^{\infty} \epsilon^{-1}[f(x-t) - f(x)]\Delta_1(\epsilon^{-1}t)dt$$

$$= \int_{-\infty}^{\infty} [f(x-\epsilon t) - f(x)]\Delta_1(t)dt$$

This puts us in an excellent position to evaluate the $L^2$ norm of $f * \Delta_\epsilon$.

In fact, simply writing out this norm gives:

$$||f * \Delta_\epsilon||_{L^2(\mathbb{R})}^2 \leq \int_{-\infty}^{\infty} ||(f * \Delta_\epsilon)(t)||_{L^2(\mathbb{R},dx)}^2 dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f(x-\epsilon t) - f(x)]\Delta_1(t)dt||_{L^2(\mathbb{R},dx)}^2 dx.$$

The Minkowski integral inequality then gives:

$$||f * \Delta_\epsilon||_{L^2(\mathbb{R})} \leq \int_{-\infty}^{\infty} ||f(x-\epsilon t) - f(x)||_{L^2(\mathbb{R},dx)}||\Delta_1(t)||_{L^2(\mathbb{R})}dt.$$ \hspace{1cm}(2.33)$$

For any $t \in \mathbb{R}$, the triangle inequality gives,

$$||f(x-\epsilon t) - f(x)||_{L^2(\mathbb{R},dx)} \leq ||f(x-\epsilon t)||_{L^2(\mathbb{R},dx)} + ||f||_{L^2(\mathbb{R})} = 2||f||_{L^2(\mathbb{R})}.$$

So, for each $t \in \mathbb{R}$, we have the estimate,

$$||f(x-\epsilon t) - f(x)||_{L^2(\mathbb{R},dx)}||\Delta_1(t)||_{L^2(\mathbb{R})} \leq \frac{C_0}{1 + x^2} \in L^1(\mathbb{R}),$$

which follows from fact (2.). Here, $C_0 > 0$ depends on $\Delta_\epsilon$ and $f$.

Applying dominated convergence in equation (2.33) gives:

$$\lim_{\epsilon \to 0} ||f * \Delta_\epsilon||_{L^2(\mathbb{R})} = 0$$

Appealing to (2.31) and (2.32), this gives:

$$\lim_{\epsilon \to 0} H_\epsilon f = \lim_{\epsilon \to 0} (f * Q_\epsilon) = Hf, \text{ in } L^2 \text{ norm.}$$
2.3. Conclusion

Having proven Proposition 2.2.1, we take a short diversion to discuss the general case $p \neq 2$.

2.2.3 A brief mention of $L^p(\mathbb{R})$

In this thesis, we mostly work in $L^2(\mathbb{R}^n)$, for $n = 1, 2, 3$. This motivated our focus on the $L^2(\mathbb{R})$ Hilbert transform. In fact, $L^2$ is the ideal testing-space for constructing the Hilbert transform, and sleuthing-out its more amenable properties.

However, it was the general $L^p(\mathbb{R})$ theorem which captivated M. Riesz’s early attention on the Hilbert transform on the torus. Through a combination of lovely complex analysis, interpolation and duality arguments, one can indeed show that the Hilbert transform extends as a bounded operator on $L^p(\mathbb{R})$, for all $1 < p < \infty$. However, we do not focus on this now, and instead refer the interested reader to Chapter 2 in [15].

2.3 Conclusion

In this chapter, we have encountered three distinct incarnations of the Hilbert transform. First, its role in determining an explicit identity for the conjugate function $\tilde{f}$. Secondly, its role in conjugating the projection operator $P : L^2 \rightarrow L^2$ arising from the Cauchy integral. Lastly, its role as a prototypical singular integral operator.

On a personal note, I find the Hilbert transform to be a captivating piece of mathematics. From one perspective, the Hilbert transform is a dyed-in-the-wool Fourier analytic object, as evidenced by equations like (2.14) or (2.21). This was my first encounter of the Hilbert transform, as an undergraduate reading through [15].

And yet, expressions like (2.12) or (2.22) make no mention of the Fourier transform at all! These equations could come from a Complex Analysis or Partial Differential Equations textbook. In fact, L. Evans’s classic textbook, *Partial Differential Equations*, introduces analogues of the Hilbert transform and associated boundary-value problems within the first thirty pages. [3]

What position, then, does the Hilbert transform occupy in the story of mathematics? As with any beautiful thing, it belongs exactly where it naturally appears; which, in this case, is the intersection of many seemingly-distinct disciplines. Moving forward, we restrict ourselves to a Fourier-analytic view of Hilbert transforms. However, this limitation is for practicality’s sake, and should not discourage the reader from viewing the Hilbert transform as a wonderfully diverse piece of mathematics.
Chapter 3

Higher dimensional analogs

As we saw in the last chapter, the Hilbert transform provides a key link between harmonic functions of the disc or upper half-space and their boundary values. The Hilbert transform also uniquely conjugates the projection operator \( P : L^2 \to L^2 \) arising from the Cauchy integral operator. Finally, the Hilbert transform gives a glimpse into the broader theory of general singular integrals.

Underpinning most of Chapter 2 was the notion of conjugation. Our discussion of the Hilbert transform centered on its role as a conjugating operator for the Poisson integrals of the disc and half-space. As we generalize the Hilbert transform to higher dimensions, this theme of conjugation and symmetrization for certain ‘nice’ projection operators will continue to hold in higher dimensions.

We begin by considering simple generalizations of the Hilbert transform: the directional Hilbert transforms \( H_\theta \). These operators resemble the univariate Hilbert transform embedded in \( n \)-dimensional space and oriented in some direction \( \theta \in S^{n-1} \). To motivate the introduction of directional Hilbert transforms, we re-examine the Hilbert transform and its relationship to the aforementioned projection operator on \( L^2(\mathbb{R}) \). We then show that \( H_\theta \) has \( L^p(\mathbb{R}^n) \) operator norm identical to the \( L^p(\mathbb{R}) \) operator norm of \( H \).

We then introduce the maximal directional Hilbert transforms \( H_\Theta \). Here, the underlying object is a collection of directions \( \Theta \subset S^{n-1} \). Maximal directional Hilbert transforms are the subject of the new results in the final chapter of this thesis. With this in mind, we provide a naive introduction to the object at the end of this chapter, before examining it in detail in Chapter 4.
3.1 Hilbert transforms in higher dimensions

3.1.1 Motivation for directional Hilbert transforms

To motivate the directional Hilbert transforms in \( \mathbb{R}^n \), let us reconsider the projection operator \( P : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \), with formula:

\[
P f(x) = \int_0^\infty \hat{f}(\xi) e^{2\pi i \xi x} d\xi = (\hat{f}(\cdot) p(\cdot))(x)
\]  (3.1)

where \( p = 1_{[0, \infty)}(\xi) \) is the indicator function of the half-line \([0, \infty) \subset \mathbb{R}\).

We now attempt to ‘directionalize’ our discussion of the projection operator arising from the Cauchy integral. Recall the following representation formula for real-valued \( f \in L^2(\mathbb{R}) \):

\[
P f(x) = \frac{1}{2} [f(x) + iH f(x)]
\]  (3.2)

where,

\[
H f(x) = \int_{-\infty}^\infty \frac{\text{sgn}(\xi)}{i} \hat{f}(\xi) e^{2\pi i \xi x} d\xi,
\]

is the Hilbert transform on \( L^2(\mathbb{R}) \).

We also consider the projection operator \( \tilde{P} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) given by:

\[
\tilde{P} f(x) = \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \xi x} d\xi = (\hat{f}(\cdot) \tilde{p}(\cdot)) (x)
\]

where \( \tilde{p}(\xi) = 1_{(-\infty, 0]}(\xi) \) is the indicator function of the half-line \((-\infty, 0]\).

As with the operator \( P \), we have the following expression:

\[
\tilde{P} = \frac{1}{2} [I - iH], \text{ as operators on } L^2(\mathbb{R}).
\]

As such, the operator \(-H : L^2(\mathbb{R}) \to L^2(\mathbb{R})\) uniquely conjugates the orthogonal projection of \( L^2(\mathbb{R}) \) onto the subspace,

\[
\{ f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for almost-every } \xi > 0 \}.
\]

This suggests a path forward for developing the directional Hilbert transforms \( H_\theta \). When \( n = 1 \), the unit circle \( S^0 \) is simply \( \theta_+ = 1 \) and \( \theta_- = -1 \).

We consider the half-spaces:

\[
\Gamma_{\theta_+} = \{ \xi \in \mathbb{R} : \theta_+ \cdot \xi = \xi \geq 0 \}, \Gamma_{\theta_-} = \{ \xi \in \mathbb{R} : \theta_- \cdot \xi = -\xi \geq 0 \}.
\]
3.1. Hilbert transforms in higher dimensions

With this notation, we see that our multipliers $p$ and $\tilde{p}$ are simply the indicator functions of $\Gamma_{\theta^+}$ and $\Gamma_{\theta^-}$ respectively. Our work above demonstrates these sets induce specific projection operators $P$ and $\tilde{P}$ on $L^2(\mathbb{R})$. In turn, these operators are conjugated by the Hilbert transforms $H$ and $\tilde{H}$. This exhausts all potential Fourier projections onto half-spaces in $\mathbb{R}$. We now want to generalize this heuristic to projection operators for half-spaces in general dimension.

To begin, for each $\theta \in S^{n-1}$, let:

$$
\Gamma_\theta := \{ \xi \in \mathbb{R}^n : \xi \cdot \theta \geq 0 \}.
$$

So, $\Gamma_\theta$ is the half-plane with unit normal vector $\theta$ and boundary line passing through the origin in $\mathbb{R}^n$. We construct a new operator via the formula,

$$
P_\theta f(x) = \int_{\Gamma_\theta} \hat{f}(\xi)e^{2\pi i \xi \cdot x} d\xi = (\hat{f}(\cdot) p_\theta(\cdot))(x). \tag{3.3}
$$

This is the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the subspace,

$$\{ f \in L^2(\mathbb{R}^n) : \text{supp } f \subset \Gamma_\theta \} = \{ f \in L^2 : \hat{f}(\xi) = 0, \text{ for } \xi \text{ with } \xi \cdot \theta = 0 \}$$

Observe that, for any $\xi \in \mathbb{R}^n$ and $\theta \in S^{n-1}$, we have:

$$1 + \text{sgn}(\xi \cdot \theta) = \begin{cases} 
2, & \text{if } \xi \cdot \theta > 0 \\
1, & \text{if } \xi \cdot \theta = 0 \\
0, & \text{if } \xi \cdot \theta < 0
\end{cases}$$

As a consequence, we know that,

$$\frac{1}{2} \left[ 1 + \text{sgn}(\xi \cdot \theta) \right] = 1_{\Gamma_\theta}(\xi),$$

for almost-every $\xi \in \mathbb{R}^n$. Recall, the set $\{ \xi \cdot \theta = 0 \}$ is a Lebesgue null set. Substituting this into equation (3.3) gives,

$$P_\theta f(x) = \int_{\mathbb{R}^n} 1_{\Gamma_\theta}(\xi) \hat{f}(\xi)e^{2\pi i \xi \cdot x} d\xi \tag{3.4}$$

Motivated by this calculation, we define the directional Hilbert transform of $f$ along $\theta$ as,

$$H_\theta f(x) = \int_{\mathbb{R}^n} \frac{\text{sgn}(\xi \cdot \theta)}{i} \hat{f}(\xi)e^{2\pi i \xi \cdot x} d\xi. \tag{3.5}$$
Hilbert transforms in higher dimensions

It is worth comparing equations (3.4) and (3.5) with the comparable equations (2.20) and (2.21) for the Hilbert transform.

Appealing to Proposition 1.1.5, we have the almost-everywhere equality for $L^2$ functions,

$$P_\theta f = \frac{1}{2} [f + H_\theta f].$$

This, in turn, gives the following operator-valued equality,

$$P_\theta = \frac{1}{2} [I + H_\theta].$$

Hence, $H_\theta$ is a bounded operator on $L^2(\mathbb{R}^n)$. If we further assume that $f \in L^2(\mathbb{R}^n)$ is real-valued, then $H_\theta$ is the unique operator conjugating the projection operator $P_\theta$.

In summary, for each $\theta \in S^{n-1}$, we identified the associated half-plane,

$$\Gamma_\theta = \{ \xi \in \mathbb{R}^n : \xi \cdot \theta \geq 0 \},$$

and its associated orthonormal projection operator,

$$P_\theta f(x) = \int_{\Gamma_\theta} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi. \quad (3.6)$$

We then demonstrated that the operator $H_\theta$, given by (3.5), uniquely conjugates the operator $P_\theta$, leading to the operator-valued identity:

$$P_\theta = \frac{1}{2} [I + iH_\theta]. \quad (3.7)$$

Comparing equations (3.1) and (3.2) with equations (3.6) and (3.7) shows that $H$ and $H_\theta$ possess analogous properties as conjugation operators for half-space projection operators on $L^2$.

3.1.2 General properties of $H_\theta$

We now show that each $H_\theta$ is bounded on $L^p(\mathbb{R}^n)$. Following this, we realize each $H_\theta$ as a singular integral operator on $L^p(\mathbb{R}^n)$. Each of these results follows from the following proposition.

**Proposition 3.1.1.** Let $\theta \in S^{n-1}$. Choose the orthogonal matrix $A \in O(n)$ such that $A(e_1) = \theta$. Then, if $H_\theta$ denotes the directional Hilbert transform acting on $S(\mathbb{R}^n)$, we have the identity:

$$H_\theta f(x) = H_{e_1} (f \circ A)(A^{-1}x), \text{ for all } f \in S(\mathbb{R}^n).$$
3.1. Hilbert transforms in higher dimensions

As such, the operator $H_\theta$ can be viewed as a composition of the Hilbert transform $H_{e_1}$ in the variable $x_1$ composed with an appropriate rotation.

Proof. Let $\theta \in S^{n-1}$ and choose $A \in O(n)$ so that $A(e_1) = \theta$. For $f \in \mathcal{S}(\mathbb{R}^n)$, we then have:

$$H_\theta f(x) = H_{A(e_1)} f(x) = \int_{\mathbb{R}^n} \frac{\text{sgn}(\xi \cdot A(e_1))}{i} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$ 

Recall, for any vectors $u, v \in \mathbb{R}^n$,

$$u \cdot Av = A^* u \cdot v,$$

where $A^*$ is the conjugate transpose of $A$.

Since $A \in O(n)$, we have $A^* = A^t = A^{-1}$. Applying this, we obtain:

$$H_\theta f(x) = \int_{\mathbb{R}^n} \frac{\text{sgn}(A^{-1}(\xi) \cdot e_1)}{i} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$ 

We perform the changes of variables $\xi \mapsto \xi'$, with $\xi' = A^{-1} \xi$, which gives:

$$H_\theta = \int_{\mathbb{R}^n} \frac{\text{sgn}(\xi' \cdot e_1)}{i} \hat{f}(A\xi') e^{2\pi i \xi' \cdot x} d\xi'.$$

Again, since $A \in O(n)$, we have the formula $\hat{f}(A\xi') = \hat{f} \circ A(\xi')$. Combining these facts with the previous equality gives:

$$H_\theta = \int_{\mathbb{R}^n} \frac{\text{sgn}(\xi' \cdot e_1)}{i} \hat{f} \circ A(\xi') e^{2\pi i \xi' \cdot A^{-1}(x)} d\xi'$$

$$= H_{e_1} (f \circ A)(A^{-1}x).$$

We are done. 

We now derive the $L^p \to L^p$ boundedness of $H_\theta$. The proof is built on the following observation. For any $g \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, one can write:

$$H_{e_1} g(x) = \int_{\mathbb{R}^n} \frac{\text{sgn}(\xi \cdot e_1)}{i} \hat{g}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

$$= \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} \frac{\text{sgn}(\xi_1)}{i} \hat{g}(\xi_1, \xi') e^{2\pi i \xi_1 x_1} d\xi_1 \right) e^{2\pi i \xi' \cdot x} d\xi'$$

where $\xi' = (\xi_2, \ldots, \xi_n)$ and $x' = (x_2, \ldots, x_n)$. Notice that the integral,

$$\int_{\mathbb{R}} \frac{\text{sgn}(\xi_1)}{i} \hat{g}(\xi_1, \xi') e^{2\pi i \xi_1 x_1} d\xi_1$$
3.1. Hilbert transforms in higher dimensions

is precisely the Hilbert transform of the function \( \tilde{g}(:, \xi') : \mathbb{R} \to \mathbb{C} \), where we view \( \xi' \) as fixed. Notice that, for almost-every \( \xi' \in \mathbb{R}^{n-1} \), we have the equality,

\[
\tilde{g}(x, \xi') = \int_{\mathbb{R}^{n-1}} g(x_1, x') e^{-2\pi i x' \cdot \xi'} \, dx', \quad \text{for almost every } (x, \xi') \in \mathbb{R}^n.
\]

So, almost-every \( \xi' \in \mathbb{R}^{n-1} \), \( \tilde{g} \) coincides with the Fourier transform of \( g \) in the variables \( (x_2, \ldots, x_n) \).

As discussed in [5], the Hilbert transform maps \( S(\mathbb{R}) \) into the Lebesgue space,

\[
L_0(\mathbb{R}) = \bigcap_{1 < p \leq \infty} L^p(\mathbb{R}).
\]

Hence, \( H\tilde{g} \in L_0(\mathbb{R}) \) for each \( \xi' \in \mathbb{R}^{n-1} \). In particular, for fixed \( x_1 \in \mathbb{R} \), we can apply Fourier inversion in \( H g(x, :) : \mathbb{R}^{n-1} \to \mathbb{C} \) to obtain the almost-everywhere equality:

\[
H_{e_1} g(x_1, x') = \int_{\mathbb{R}^{n-1}} H g(x, \xi') e^{2\pi i \xi' \cdot x'} \, d\xi' = H_{1} g(x_1, x').
\]

Here, we let \( H_{1} g(x_1, x') \) denote an application of the Hilbert transform to the function \( g : \mathbb{R}^n \to \mathbb{C} \) in the variable \( x_1 \), while viewing the coordinates \( x' = (x_2, \ldots, x_n) \) as fixed.

This notation is somewhat pedantic. However, we are simply making rigorous the intuition that the directional Hilbert transform \( H_{e_1} \) on \( \mathbb{R}^n \) can be viewed as a pointwise application of the univariate Hilbert transform in the variable \( x_1 \). This observation leads to the following proposition.

**Proposition 3.1.2.** Let \( \theta \in S^{n-1} \) and \( 1 < p < \infty \). Then, if \( f \in S(\mathbb{R}^n) \), there exists an absolute constant \( C_p > 0 \) such that,

\[
||H_{\theta} f||_{L^p(\mathbb{R}^n)} \leq C_p ||f||_{L^p(\mathbb{R}^n)}.
\]

Moreover, we have the estimate,

\[
C_p \leq ||H||_{L^p(\mathbb{R}) \to L^p(\mathbb{R})},
\]

where \( H \) denotes the univariate Hilbert transform.

Utilizing density arguments, the directional Hilbert transforms \( H_{\theta} \) extend to bounded operators on \( L^p(\mathbb{R}^n) \) for each \( 1 < p < \infty \).
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Proof. Let \( \theta \in S^{n-1} \) and \( 1 < p < \infty \). Choose \( A \in O(n) \) such that \( A(e_1) = \theta \). Letting \( f \in S(\mathbb{R}^n) \), set \( g = f \circ A \). Then \( g \in S(\mathbb{R}^n) \). Utilizing Proposition 3.1.1, we have,

\[
\|H_\theta f\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |H_\theta f(x)|^p \, dx = \int_{\mathbb{R}^n} |H_{e_1}g(A^{-1}x)|^p \, dx.
\]

The rotational-invariance of Lebesgue measure then gives,

\[
\|H_\theta f\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |H_{e_1}g(x)|^p \, dx = \int_{\mathbb{R}^n-1} \left( \int_{\mathbb{R}} |H_1 g(x_1, x')|^p \, dx_1 \right) \, dx'.
\]

Here, \( H_1 \) again denotes the Hilbert transform applied in the first variable. Choosing \( C_p = \|H_1\|_{p \rightarrow p} = \|H\|_{p \rightarrow p} \), we have,

\[
\|H_\theta f\|_{L^p(\mathbb{R}^n)}^p \leq \int_{\mathbb{R}^n-1} \left( C_p \int_{\mathbb{R}} |g(x_1, x')|^p \, dx_1 \right) \, dx' = C_p \int_{\mathbb{R}^n} |g(x)|^p \, dx.
\]

However, recalling that \( g = f \circ A \), this gives:

\[
\|H_\theta f\|_{L^p(\mathbb{R}^n)}^p \leq C_p \|g\|_{L^p(\mathbb{R}^n)}^p \leq C_p \|f\|_{L^p(\mathbb{R}^n)}^p.
\]

Taking \( p \)-th roots gives the result. \( \square \)

Proposition 3.1.2 further reinforces our understanding of \( H_\theta \) as an embedding of the univariate Hilbert transform in \( n \)-dimensional space. To conclude this comparison, we realize each \( H_\theta \) as an appropriate singular integral operator. This is the content of the following proposition.

**Proposition 3.1.3.** Let \( \theta \in S^{n-1} \). Let \( H_\theta \) be the directional Hilbert transform, defined by equation (3.5). Then, for each \( f \in S(\mathbb{R}^n) \), we have,

\[
H_\theta f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - t\theta) \frac{dt}{t}, \text{ for almost-every } x \in \mathbb{R}^n.
\]

Proof. Let \( \theta \in S^{n-1} \) and \( f \in S(\mathbb{R}^n) \). For almost-every \( x \in \mathbb{R}^n \), we have,

\[
H_{e_1} f(x) = H_1 f(x_1, x') = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} f(x_1 - t, x') \frac{dt}{t}.
\]

This can be rewritten as,

\[
H_{e_1} f(x) = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - e_1t) \frac{dt}{t}.
\]
3.1. Hilbert transforms in higher dimensions

Applying Proposition 3.1.1 then gives,

$$H_\theta f(x) = H_{e_1}[(f \circ A)(A^{-1}x)] = \frac{1}{\pi} \int_{-\infty}^{\infty} (f \circ A)(A^{-1}x - e_1t) \frac{dt}{t}. $$

Simplifying this expression, we have, for almost-every $x \in \mathbb{R}^n$,

$$H_\theta f(x) = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - A(e_1)t) \frac{dt}{t} = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - \theta t) \frac{dt}{t}. $$

This completes our construction of the directional Hilbert transforms on $\mathbb{R}^n$ from the one-dimensional Hilbert transform. Proposition 3.1.1 gave a formula for expressing $H_\theta$ in terms of a Hilbert transform in one variable. Propositions 3.1.2 and 3.1.3 then utilized this formula to show that $H_\theta$ shares the $L^p$-mapping properties and singular integral representation of the Hilbert transform.

3.1.3 Beyond directional Hilbert transforms

Having concluded our discussion of Hilbert transforms in higher dimensions, we can now define the key objects of study in this thesis: the maximal directional Hilbert transforms. The purpose of this short subsection is to introduce these objects on an intuitive level, before proving more $L^2$-mapping properties in Chapters 4 and 5.

To begin, we start with a collection of unit vectors $\Theta \subset S^{n-1}$. For each $\theta \in \Theta$, there is an associated directional Hilbert transform $H_\theta$. From the previous section, we know that $H_\theta$ is a bounded multiplier on $L^p$ for each $\theta \in \Theta$. Motivated by these results, we now define a maximal operator associated to the set $\Theta \subset S^{n-1}$ and acting on $\mathcal{S}(\mathbb{R}^n)$:

$$\mathcal{H}_\Theta f(x) = \sup_{\theta \in \Theta} |H_\theta f(x)|, \text{ for each } x \in \mathbb{R}^n.$$  

This $\mathcal{H}_\Theta$ is the maximal Hilbert transform of $f$ associated with $\Theta$.

Suppose, for a moment, that we require $\Theta = \{\theta_1, \ldots, \theta_N\} \subset S^{n-1}$ to be a finite set. Then, it is not difficult to see that $\mathcal{H}_\Theta f \in L_0(\mathbb{R}^n)$ when $f \in \mathcal{S}(\mathbb{R}^n)$. Indeed, for $x \in \mathbb{R}^n$, we have,

$$\mathcal{H}_\Theta f(x) = \max_{1 \leq j \leq N} |H_{\theta_j} f(x)| \leq \sum_{j=1}^{N} |H_{\theta_j} f(x)|.$$
3.1. Hilbert transforms in higher dimensions

We thus obtain the norm inequality,

$$
\|\mathcal{H}_\Theta f\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \left( \max_{1 \leq j \leq N} |H_{\theta_j} f(x)| \right)^p dx \right)^{\frac{1}{p}}
\lesssim \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{N} |H_{\theta_j} f(x)|^p dx \right)^{\frac{1}{p}}. 
$$

Minkowski’s integral inequality then gives,

$$
\|\mathcal{H}_\Theta f\|_{L^p(\mathbb{R}^n)} \leq \sum_{j=1}^{N} \left( \int_{\mathbb{R}^n} |H_{\theta_j} f(x)|^p dx \right)^{\frac{1}{p}} = \sum_{j=1}^{N} \|H_{\theta_j} f\|_{L^p(\mathbb{R}^n)}. 
$$

Proposition 3.1.2 then guarantees each $H_{\theta_j}$ is bounded in $L^p(\mathbb{R}^n)$ norm of the Hilbert transform. Letting $C_p = \|H\|_{p \rightarrow p}$, we obtain:

$$
\|\mathcal{H}_\Theta f\|_{L^p(\mathbb{R}^n)} \leq C_p \sum_{j=1}^{N} \|f\|_{L^p(\mathbb{R}^n)} = C_p N \|f\|_{L^p(\mathbb{R}^n)} \quad (3.8)
$$

Equality (3.8) guarantees that $\mathcal{H}_\Theta$ admits a bounded extension to $L^p(\mathbb{R}^n)$, subordinate to the assumption that $\Theta$ is a finite set.

The latter portion of this thesis examines precise estimates for the asymptotic growth of $\|\mathcal{H}_\Theta\|_{p \rightarrow p}$ as $\#\Theta \to \infty$. For direction sets of large cardinality, inequality (3.8) is quite inefficient. This reflects the generality of techniques which lead to inequality (3.8). Indeed, if we replace the family $\{H_{\theta_j}\}_{1 \leq j \leq N}$ with an arbitrary finite family of $L^p(\mathbb{R}^n)$ bounded operators $T_j$, we obtain an identical bound for the associated maximal operator $T = \max_j T_j$ (of course, with an adjusted constant).

Historically, the maximal directional Hilbert transforms were inspired by the work of E. Stein on directional maximal operators. Stein studied the more sophisticated planar Hilbert transforms, defined for $f \in \mathcal{S}(\mathbb{R}^2)$ as:

$$
H_{v,\epsilon} f(x) = \text{p.v.} \int_{-\epsilon}^{\epsilon} f(x - tv(x)) \frac{dt}{t}.
$$

Here, $v : \mathbb{R}^2 \to S^1$ is a measurable vector field, which is assumed to satisfy some regularity assumptions. For example, under the assumption that $v$ is a Lipschitz vector field, Stein conjectured the following.

**Conjecture 3.1.4** (E. Stein, 1986, [14]). There is an absolute constant $K > 0$ so that, if $\epsilon = (K\|v\|_{L_\infty})^{-1}$, we have the weak-type estimate:

$$
\sup_{\lambda > 0} \lambda \left( \|H_{v,\epsilon} f > \lambda\right)^{1/2} \lesssim \|f\|_{L^2(\mathbb{R}^2)}.
$$
3.1. Hilbert transforms in higher dimensions

At this time, Stein’s conjecture is unproven. In fact, it is currently believed to be exceedingly difficult. As shown by the work of M. Lacey and X. Li ([10], [11]), the $L^2 \to L^{2,\infty}$ mapping properties of $H_{v,\epsilon}$ are intimately related to many other themes in harmonic analysis, such as the almost-everywhere convergence of Fourier series and the existence of Besicovitch sets with high Hausdorff dimension. These connections are well-documented in [11]. We do not consider $H_{v,\epsilon}$ any more in this monogram.

In the next chapter, we will significantly improve inequality 3.8 for arbitrary finite $\Theta \subset S^1$. We will also discuss a complementary lower bounds on $\|H_{\Theta}\|_{2\to2}$ for general $\Theta \subset S^{n-1}$. In Chapter 5, we will discuss specific direction sets $\Theta \subset S^{n-1}$ which obtain vastly different operator norms with respect to $\#\Theta$. This motivates the results in Chapter 6, which provide an asymptotic lower bound for a large family of direction sets $\Theta \subset S^{n-1}$.
Chapter 4

The maximal directional Hilbert transform

We now discuss the main focus of this thesis: the boundedness of $\mathcal{H}_\Theta$ as an operator on $L^p(\mathbb{R}^n)$. For most of what follows, we specifically consider the case $p = 2$ and $n = 2$. This combination of Lebesgue exponent and ambient dimension were the first examined in the literature. As we saw in Chapters 2 and 3, the Hilbert transform and directional Hilbert transforms have simple expressions in terms of projection operators on $L^2(\mathbb{R}^n)$. We will utilize this special property of Hilbert transforms extensively to establish $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ bounds for many types of maximal directional Hilbert transforms.

The proof of norm bounds for generic direction sets will be accomplished through certain key results. I present these in rough chronological order of publication. For general direction set $\Theta$ of cardinality $N$, and some Schwartz function $f$, these are:

1. An improvement of the upper bound (3.8) from Chapter 3. We prove that,
   \[ ||\mathcal{H}_\Theta f||_{L^2(\mathbb{R}^2)} \leq C \log N ||f||_{L^2(\mathbb{R}^2)}. \] 
   (4.1)
   We also discuss a more-recent extension of this result to $p > 2$. [1]

2. A lower bound complementing the upper bound (3.8) in Chapter 3. This is the estimate,
   \[ ||\mathcal{H}_\Theta f||_{L^p(\mathbb{R}^2)} \geq C \sqrt[\log N]{} ||f||_{L^p(\mathbb{R}^2)}. \] 
   (4.2)
   We also discuss [9], a very recent extension of this result for $n \geq 3$.

3. A glimpse into upper bounds for $n \geq 3$. Specifically, we introduce and discuss recent estimates from J. Kim and M. Pramanik in [7].

By comparing equations (4.1) and (4.2), we obtain the following “qualitative” theorem concerning maximal directional Hilbert transforms.
4.1. An upper bound for $||\mathcal{H}_\Theta||_{2\to 2}$

**Theorem 4.0.1.** Suppose that $\Theta \subset \mathbb{S}^{n-1}$. Then, for each $n \geq 2$ and $1 < p < \infty$, the maximal directional Hilbert transform $\mathcal{H}_\Theta$ admits a continuous extension to $L^p(\mathbb{R}^n)$ if-and-only-if $\Theta$ has finite cardinality.

Because of the $\sqrt{\log N}$ term in (4.2) we usually only examine $\mathcal{H}_\Theta$ associated to finite direction sets. We are especially interested in the asymptotic growth of $||\mathcal{H}_\Theta||_{p\to p}$ when $\{\Theta_j\}$ is an increasing family of direction sets, with $\#\Theta_j \to \infty$. We discuss this in the following chapter. For now, we turn our attention to the estimates (4.1) and (4.2).

### 4.1 An upper bound for $||\mathcal{H}_\Theta||_{2\to 2}$

We first want to prove the upper bound (4.1), which we now give as a proposition.

**Theorem 4.1.1.** Let $f \in \mathcal{S}(\mathbb{R}^2)$ and $\Theta \subset \mathbb{S}^1$ with $\#\Theta = N$. Then, there is an absolute constant $C > 0$ such that,

$$||\mathcal{H}_\Theta f||_{L^2(\mathbb{R}^2)} \leq C \log N ||f||_{L^2(\mathbb{R}^2)}.$$

As a consequence, $\mathcal{H}_\Theta$ admits a continuous extension to $L^2(\mathbb{R}^2)$.

To prove Proposition 4.1.1, we utilize the following nifty result concerning orthonormal functions in $L^2(\mathbb{R}^2)$. An orthonormal list or orthonormal system in $L^2$ is a list of $L^2$ functions $\{\phi_j\}$ such that,

$$\int_{\mathbb{R}^2} \phi_j \overline{\phi_k} = 0, \text{ for } j \neq k \text{ and } \int_{\mathbb{R}^2} |\phi_k|^2 = 1, \text{ for all } k.$$

We have the following theorem concerning orthonormal lists. This theorem is known in the literature as the Menshov-Rademacher theorem. (see [6])

**Theorem 4.1.2.** Let $\{\phi_j\}$ be an orthonormal list in $L^2(\mathbb{R}^2)$ and $a_1, \ldots, a_N \in \mathbb{C}$ be certain coefficients. Then, there is a function $F \in L^2(\mathbb{R})$ such that,

$$\max_{1 \leq \nu \leq N} \left| \sum_{j=1}^{\nu} a_j \phi_j(x) \right| \leq F(x), \text{ for every } x \in \mathbb{R}^2. \quad (4.3)$$

Moreover, we have the $L^2$ to $\ell^2$ norm inequality:

$$||F||_{L^2(\mathbb{R}^2)} \leq C \cdot \log N \left( \sum_{j=1}^{N} |a_j|^2 \right)^{1/2}.$$
4.1. An upper bound for $||\mathcal{H}_\Theta||_{2\to 2}$

As a consequence of this theorem, the ‘maximal summation functions’,

$$\max_{1 \leq \nu \leq N} \left| \sum_{j=1}^\nu a_j \phi_j(x) \right|$$

satisfies the following $L^2(\mathbb{R}^2)$ norm estimate,

$$||F||_{L^2(\mathbb{R}^2)} \leq C \cdot \log N \left( \sum_{j=1}^N |a_j|^2 \right)^{1/2}.$$  

To prove Theorem 4.1.1, we will show that the maximal directional Hilbert transform $H_\Theta$ is pointwise controlled by certain maximal summation functions. Written explicitly, that is:

$$||H_\Theta f(x)||_{L^2(\mathbb{R}^2)} \leq ||F_{a_j^+,\phi_j^+}(x)|| + ||F_{a_j^-\phi_j^-}(x)||.$$  \hspace{1cm} (4.4)

Here, the orthonormal lists $\phi_j^\pm$ and coefficients $a_j^\pm$ are chosen so as to guarantee that $\sum_j |a_j^\pm|^2 \leq ||f||_2^2$. From here, the upper bound of Theorem 4.1.1 follows immediately from Theorem 4.1.2.

Of course, the main difficulty is proving the pointwise upper bound (4.4). As such, we prove the following Lemma in detail.

**Lemma 4.1.3.** Let $\Theta \subset S^1$ with $\#\Theta = N$. Then, for every $f \in \mathcal{S}(\mathbb{R}^2)$, there are two lists of coefficients $a_1^\pm, ..., a_N^\pm > 0$ and two orthonormal systems $\phi_1^\pm, ..., \phi_N^\pm$ in $L^2(\mathbb{R}^2)$ such that:

$$||\mathcal{H}_\Theta f||_{L^2(\mathbb{R}^2)} \leq ||F_{a_j^+,\phi_j^+}||_{L^2(\mathbb{R}^2)} + ||F_{a_j^-\phi_j^-}||_{L^2(\mathbb{R}^2)}.$$  \hspace{1cm} (4.5)

Moreover, the coefficients $a_j^\pm$ satisfy the $\ell^2$ to $L^2$ estimate:

$$\sum_{j=1}^N |a_j^\pm| \leq ||f||_{L^2(\mathbb{R}^2)}.$$  \hspace{1cm} (4.6)

Theorem 4.1.1 then follows directly from Theorem 4.1.2 and Lemma 4.1.3.

**Proof of Lemma 4.1.3.** We introduce a piece of notation. For a measurable region $S \subset \mathbb{R}^2$, let $T_S : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the Fourier projection operator,

$$T_S f(x) = \int_S \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi = (1_S \cdot \hat{f} \cdot)$$
4.1. An upper bound for $||H_{\theta}||_{2 \rightarrow 2}$

From equation (3.7) in Chapter 3, we have the following representation formula for the directional Hilbert transform along $\theta \in S^1$:

$$\hat{H}_{\theta} f(\xi) = i \text{sgn}(\xi \cdot \theta) \hat{f}(\xi) = 2i \cdot 1_{\Gamma_{\theta}}(\xi) \hat{f}(\xi) - i \hat{f}(\xi).$$  \hspace{1cm} (4.7)

Here, $\Gamma_{\theta} = \{ \xi \in \mathbb{R}^2 : \xi \cdot \theta \geq 0 \}$ is the half-plane with unit normal vector $\theta \in S^1$. So, for any $\Theta \subset S^1$, equation (4.7) together with Plancherel then gives:

$$||H_{\Theta} f||_{L^2(\mathbb{R}^2)} \leq 2||T_{\Theta} f||_{L^2(\mathbb{R}^2)} + ||f||_{L^2(\mathbb{R}^2)}.$$  \hspace{1cm} (4.8)

Luckily, $T_{\Theta}$ is the much simpler ‘maximal projection operator’,

$$T_{\Theta} f(x) = \sup_{\theta \in \Theta} |T_{\Gamma_{\theta}} f(x)|.$$

We now assume that $\# \Theta = N < \infty$. By appealing to the rotational- and dilation-invariance of the operator norm of $H_{\Theta}$, we can assume that $\Theta$ is contained in the region bounded by the $x$-axis and the line $y = x$. We enumerate $\Theta = \{ \theta_1, ..., \theta_N \}$ so that the $\theta_j$ are arranged in anti-clockwise order.

With these assumptions, the operator $T_{\Theta}$ becomes,

$$T_{\Theta} f(x) = \sup_{1 \leq j \leq N} |T_{\Gamma_{\theta_j}} f(x)|.$$

We now re-express each $\Gamma_{\theta}$ and its associated projection operator in terms of disjoint frequency cones rooted at the origin. For $1 \leq j \leq N - 1$, define:

$$S_j^+ := \{ x \in \mathbb{R}^2 : x_1 \geq 0; x \cdot \theta_{N-j} \geq 0, x \cdot \theta_{N-j+1} \leq 0 \};$$

$$S_j^- := \{ x \in \mathbb{R}^2 : x_1 \leq 0; x \cdot \theta_j \leq 0, x \cdot \theta_{j+1} \geq 0 \}.$$

Also, let,

$$S_0^+ := \{ x \in \mathbb{R}^2 : x \geq 0; x \cdot \theta_N \geq 0 \};$$

$$S_0^- := \{ x \in \mathbb{R}^2 : x \leq 0; x \cdot \theta_1 \geq 0 \}.$$

By construction, the sets $S_j^+, S_j^-$ are interior-disjoint sets. Moreover, for every $0 \leq k \leq N - 1$,

$$\Gamma_{\theta_k} = \left( \bigcup_{j=0}^{N-k} S_j^+ \right) \cup \left( \bigcup_{j=0}^{k} S_j^- \right).$$
Because of the essential disjointness of the \( S_k \), we can rewrite each \( T_{\gamma_k} \) as:

\[
T_{\gamma_k} f(x) = \sum_{j=0}^{N-k} T_{S_j^+} f(x) + \sum_{j=0}^{k} T_{S_j^-} f(x).
\]

This gives the pointwise upper bound,

\[
|T_{\gamma_k} f(x)| \leq \left| \sum_{j=0}^{N-k} T_{S_j^+} f(x) \right| + \left| \sum_{j=0}^{k} T_{S_j^-} f(x) \right|.
\]

Re-indexing and taking supremums in \( k \) gives,

\[
|T_{\Theta} f(x)| \leq \sup_{1 \leq k \leq N} \left| \sum_{j=1}^{k} T_{S_j^+} f(x) \right| + \sup_{1 \leq k \leq N} \left| \sum_{j=1}^{k} T_{S_j^-} f(x) \right| \quad (4.9)
\]

Let us denote the first ‘supremum summand’ as \( F^+ \) and the second as \( F^- \).

From Plancherel, we know that \( ||T_{S_k^+} f||_2 \leq ||f||_2 \) for each \( 1 \leq k \leq N \).

So \( T_{S_k^+} f \in L^2(\mathbb{R^2}) \). Letting \( a_j^+ = ||T_{S_j^+} f||_2 \) we rewrite \( F^+ \) and \( F^- \) as:

\[
F^+(x) = \sup_{1 \leq k \leq N} \left| \sum_{j=1}^{k} a_j^+ \phi_j^+(x) \right|, \quad F^-(x) = \sup_{1 \leq k \leq N} \left| \sum_{j=1}^{k} a_j^- \phi_j^-(x) \right|,
\]

where,

\[
\phi_j^+(x) = (a_j^+)^{-1} \cdot T_{S_j^+} f(x), \text{ for every } 1 \leq j \leq N.
\]

Since the \( S_j^\pm \) are interior disjoint, the functions \( \phi_j^+ \) and \( \phi_j^- \) form orthonormal systems in \( L^2(\mathbb{R^2}) \). Combined with (4.8) and (4.9), this proves (4.5).

A final application of Plancherel then gives:

\[
\sum_{j=1}^{N} |a_j^\pm|^2 = \sum_{j=1}^{N} ||T_{S_j^\pm} f||_2^2 = \sum_{j=1}^{N} ||T_{S_j^\pm} f||_2^2 = \sum_{j=1}^{N} \int_{S_j^\pm} |\hat{f}(\xi)|^2 d\xi.
\]

Since the \( S_k \) are interior-disjoint, this gives the desired estimate (4.6). \( \square \)

The linchpin of this argument is the \( L^2 \)-orthogonality of the functions \( T_{S_k^\pm} f \). This is expressed in the representation,

\[
T_{\gamma_k} f(x) = \sum_{j=0}^{N-k} T_{S_j^+} f(x) + \sum_{j=0}^{k} T_{S_j^-} f(x).
\]
4.1. An upper bound for $||\mathcal{H}_\Theta||_{2 \to 2}$

This reflects a simple geometric property concerning the overlap of half-spaces in the plane. Namely, $N$ unique half spaces $\Gamma_{\theta_j}$ in $\mathbb{R}^2$ determine precisely $2N$ disjoint cones $S_j^\pm$. To prove estimate (4.1), we needed only realize that $\mathcal{H}_\Theta$ is pointwise controlled by the half-space projection operators $\{T_{\Gamma_{\theta_j}}\}_{\theta \in \Theta}$.

Although we do not consider general $L^p \to L^p$ estimates in this monograph, it is worth noting that $L^p$ orthogonality is key in proving upper bounds for maximal directional Hilbert transforms. For example, in [1], the authors prove the upper bound:

$$||\mathcal{H}_\Theta||_{p \to p} \leq C \log (\# \Theta), \text{ for } 2 < p < \infty.$$  

Again, C. Demeter and F. Di Plinio’s proof relies on the fact that $\mathcal{H}_\Theta$ is controlled by half-space operators. The complexity here is in deriving an $L^p$ estimate similar to Theorem 4.1.2 for a certain modulation operator. Besides this piece of $L^p$ machinery, the spirit of the proof remains unchanged.

4.1.1 Upper bounds in higher dimensions

While the new results in this thesis specifically focus on the plane, we do mention recent progress on upper bounds for $L^2(\mathbb{R}^n)$ when $n \geq 3$. Specifically, J. Kim and M. Pramanik [7] have proven the following.

**Theorem 4.1.4.** Let $n \geq 3$ and suppose that $\Theta \subset S^{n-1}$ has cardinality $N < \infty$. Then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ depending only on $n$ and $\epsilon$ such that,

$$||\mathcal{H}_\Theta f||_{L^2(\mathbb{R}^n)} \leq C_\epsilon N \frac{n-2}{2n-2} \epsilon ||f||_{L^2(\mathbb{R}^n)}$$

Notice that, as $n \to \infty$, we have $N \frac{n-2}{2n-2} \epsilon$ strictly increases to $N^{1/2} + \epsilon$.

As such, this upper bound is a significant improvement over the first suboptimal bound we considered last chapter. Moreover, the upper bound of Theorem 4.1.4 is sharp (up to the factor of $N^\epsilon$). This is due to an important counterexample, provided by Joonil Kim in [8]. We will return to both [8] and the work of J. Kim and M. Pramanik next chapter.

In Chapter 5, we consider one specific direction set $\Theta \subset S^1$ which attain this asymptotic upper bound. This proves that (4.1) is the sharpest general upper bound which holds for all maximal directional Hilbert transforms. In Chapter 6, we prove that many direction sets attain this upper bound, proving that maximal $L^2$-norm is not a ‘unique’ property of one specific direction set. For now, we turn to the question of general lower bounds for maximal directional Hilbert transforms.
4.2 Lower bounds for $\|\mathcal{H}_\Theta\|_{p\rightarrow p}$

We now turn our attention to the lower bound (4.2). We formalize this result as a proposition.

**Theorem 4.2.1.** Suppose that $\Theta \subset S^1$ with $\# \Theta = N \leq \infty$. Then, for every $1 < p < \infty$, there exists a function $f \in L^2(\mathbb{R}^2)$ such that,

$$\|\mathcal{H}_\Theta f\|_{L^p(\mathbb{R}^2)} \geq C \sqrt{\log N} \|f\|_{L^2(\mathbb{R}^2)}.$$

As a consequence, $\mathcal{H}_\Theta$ is unbounded on $L^2(\mathbb{R}^2)$ whenever $\# \Theta = \infty$.

This is a stunning result, both in its proof and consequences. We cannot flesh-out the proof of Theorem 4.2.1 in its entirety. However, it is enlightening to consider the key pieces in Karagulyan’s proof of the lower bound (4.2), especially as it pertains to orthogonal systems in $L^2(\mathbb{R}^2)$ and Menshov-Rademacher type arguments.

A key component of Karagulyan’s proof is the discovery of a single permutation which furnishes an $f \in L^2(\mathbb{R}^2)$ tailored to the direction set such that the upper bound of Theorem 4.1.1 can be ‘almost reversed’. This permutation has something in common with the Haar system on $L^2$, and relies on the special structure of *tree systems*. We examine these objects in superficial detail here.

Suppose $f_n$ is a list of functions supported on the square $Q = [-\pi, \pi] \times [-\pi, \pi]$. For our purposes, we suppose that $n = 1, 2, ..., 2^m - 1$ for some $m \in \mathbb{N}$. Under this assumption, we can write:

$$n = 2^k + j - 1,$$

for some $1 \leq j \leq 2^k$, and $k = 1, 2, ..., m - 1$.

This gives the double-numbering $f_n = f_j^k$. We have the following definition.

**Definition 4.2.2.** Suppose that $f_n = f_j^k$ is a system of functions. Then, the $f_j^k$ form a tree system provided,

$$\text{supp } f_{2j-1}^{k+1} \subset \{x \in Q : f_j^k > 0\} \quad \text{and} \quad \text{supp } f_{2j}^{k+1} \subset \{x \in Q : f_j^k < 0\}.$$

Functions forming a tree systems possess a rigid nesting property in their supports. In particular, the supports of each child $f_j^{k+1}$ are entirely determined by the supports of their parent $f_j^k$. This structuring leads to the following powerful result concerning maximal summations of the $f_n$. 

---
4.2. Lower bounds for $||H_\Theta||_{p \to p}$

**Lemma 4.2.3.** Let $n = 1, 2, ..., 2^m - 1 = \nu$ and suppose that $f_n$ is a tree system supported in $Q = [-\pi, \pi] \times [-\pi, \pi]$. Then, there exists a permutation $\sigma$ of the numbers $\{1, 2, ..., \nu\}$ such that,

$$\sup_{1 \leq l \leq \nu} \left| \sum_{n=1}^{l} f_{\sigma(n)}(x) \right| \geq \frac{1}{3} \sum_{n=1}^{2^m-1} |f_n(x)|, \text{ for } x \in Q. \quad (4.10)$$

Lemma 4.2.3 is at the heart of Karagulyan’s proof. One can see the parallels between equations (4.3) and (4.10). What is remarkable about inequality (4.10) is the fact that the permuted and signed maximal operator,

$$F_{\sigma}(x) = \sup_{1 \leq l \leq \nu} \left| \sum_{n=1}^{l} f_{\sigma(n)}(x) \right|$$

dominates (up to some trivial constant) the absolute sum $\sum_n |f_n(x)|$. Armed with this incredible inequality, Karagulyan then constructs the following system of functions. We follow Karagulyan’s lead and state this construction as a lemma.

**Lemma 4.2.4.** For $n = 1, 2, ..., 2^m - 1 = \nu$, let $S_n$ be sectors in the plane. Then, there exist functions $f_1, ..., f_\nu$ and constants $c_1, c_2, c_3$ such that,

$$\text{supp } \widehat{f_n} \subset S_n \quad \text{and} \quad \sum_{j=1}^{\nu} ||f_j||_{L^2}^2 \leq c_1. \quad (4.11)$$

Moreover, if $\sigma$ is the permutation identified in Lemma 4.2.3, then the following estimate holds,

$$|\{x \in Q : \sup_{1 \leq n \leq \nu} \left| \sum_{j=1}^{n} f_{\sigma(j)}(x) \right| > c_3 \sqrt{\log \nu} \}| > c_2 \quad (4.12)$$

Roughly, here is how one utilizes Lemma 4.2.4 to prove Theorem 4.2.1. Again let $\Theta = \{\theta_1, ..., \theta_N\} \subset S^1$ be finite with anti-clockwise ordering. Recall, from Lemma 4.1.3, the ‘maximal projection operator’,

$$T_\Theta f(x) = \sup_{1 \leq j \leq N} |T_{\Gamma_{\theta_j}}(x)|, \text{ where } T_{\Gamma_{\theta_j}} f(x) = \int_{\Gamma_{\theta_j}} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

and $\Gamma_{\theta_j} = \{x \in \mathbb{R}^2 : x \cdot \theta_j \geq 0\}$. We have the representation formula (4.7), which gives for $1 < p < \infty$ and $f \in S(\mathbb{R}^2)$,

$$||H_\Theta||_{L^p(\mathbb{R}^2)} \geq C_p ||T_\Theta f||_{L^p(\mathbb{R}^2)}. \quad (4.13)$$
4.2. Lower bounds for $||\mathcal{H}_\Theta||_p - p$

To prove Theorem 4.2, we will furnish an $f \in L^2(\mathbb{R}^2)$ with,

$$||T_\Theta f||_{L^1,\infty(\mathbb{R}^2)} \geq \sqrt{\log N} ||f||_{L^2(\mathbb{R}^2)};$$

Here, $L^1,\infty$ denotes the weak $L^1$ norm, which is defined in Chapter 6 of [4].

Using Lemma 4.2.4 as a black-box, we outline the construction of the desired $f \in L^2(\mathbb{R}^2)$. To begin, for \(k = 1, 2, ..., N\), let,

$$S_k := \{x \in \mathbb{R}^2 : x \geq 0, x \cdot \theta_k \geq 0, x \cdot \theta_{k+1} \leq 0\}.$$

These sectors should appear familiar, as they originally appeared in the proof of Lemma 4.1.3. Now, let $S'_k = S_{\sigma^{-1}(k)}$. So, $S'_1, ..., S'_N$ is a re-ordering of the sectors $S_k$ subordinate to the permutation $\sigma$ provided by Lemma 4.2.3. Lemma 4.2.4 now guarantees the existence of functions $f_k$ associated to the sectors $S'_k$, which satisfy (4.11) and (4.12).

We now define our test function, via the identity:

$$f(x) = \sum_{n=1}^{N} f_n(x).$$

As a consequence of the orthogonality of the $f_n$, we know that:

$$|| \sum_{n=1}^{N} \hat{f}_n ||_2 = \sum_{n=1}^{N} || \hat{f}_n ||_2.$$

Alongside Plancherel, this gives the estimate,

$$||f||_2 = || \sum_{n=1}^{N} f_n ||_2 = \sum_{n=1}^{N} ||f_n||_2 = \sum_{n=1}^{N} ||\hat{f}_n||_2 \leq c_1$$

So, $f \in L^2(\mathbb{R}^2)$. Recall that $\text{supp } \hat{f}_n \subset S'_n$. But $S'_1, ..., S'_N$ is simply a re-ordering of the sectors $S_k$. So, we have:

$$\text{supp } \hat{f} \subset \bigcup_{n=1}^{N} S_n,$$

This has interesting implications for the projection operators $T_{S_k}$. Recall, in proving Lemma 4.1.3, we had the following decomposition,

$$T_{\Gamma_{\theta_k}} f(x) = \sum_{j=0}^{N-k} T_{S'_j} f(x) + \sum_{j=0}^{k} T_{S'_j} f(x).$$
4.2. Lower bounds for $||\mathcal{H}_\Theta||_{p\to p}$

where the sets $S^+_j$ and $S^-_j$ were obtained from the intersection of the half-planes $\Gamma_{\theta_j}$ and $\Gamma_{\theta_{j+1}}$. Assumption (4.13) simplifies this, allowing us to write,

$$T_{\Theta_{\theta_k}} f(x) = \sum_{n=1}^k T_{S_k} f(x), \text{ for each } 1 \leq k \leq N.$$  

Hence, we can re-express $T_\Theta f$ as,

$$T_\Theta f(x) = \sup_{1 \leq k \leq N} \left| \sum_{n=1}^k T_{S_k} f(x) \right|. \quad (4.14)$$

The permutation $\sigma$ now flexes its muscles.  
Recall that,  

$$\text{supp } \hat{f}_n \subset S_{\sigma^{-1}(n)} \implies \text{supp } \hat{f}_{\sigma(n)} \subset S_n.$$ 

A simple consequence of the disjointness of the $S_k$ is then,

$$\hat{T}_{S_k} f(\xi) = \sum_{n=1}^N 1_{S_k}(\xi) \hat{f}_n(\xi) = \hat{f}_{\sigma(k)}(\xi) \Rightarrow T_{S_k} f(x) = f_{\sigma(k)}(x), \quad (4.15)$$

for almost-every $x \in \mathbb{R}^2$. Together, equations (4.14) and (4.15) imply that,

$$T_\Theta f(x) = \sup_{1 \leq k \leq N} \left| \sum_{n=1}^k f_{\sigma(n)}(x) \right|.$$ 

We now apply Lemma 4.10 in earnest. Equation 4.12 guarantees that,

$$|\{x \in \mathbb{R}^2 : T_\Theta(x) > c_3 \sqrt{\log N}\}| > c_2.$$ 

Since $||f||_2 \leq c_1$, we obtain, for any $1 < p < \infty$, the weak-type inequality,

$$\lambda_N |\{x \in \mathbb{R}^2 : T_\Theta(x) > c_3 \lambda_N\}|^{1/p} \geq \frac{c_2}{c_1} \sqrt{\log N} ||f||_{L^2(\mathbb{R}^2)}.$$ 

with $\lambda_N = \sqrt{\log N}$. This, in turn, implies the strong-type inequality:

$$||T_\Theta f||_{L^p(\mathbb{R}^2)} \geq C_p \sqrt{\log N} ||f||_{L^2(\mathbb{R}^2)}.$$ 

This quick overview highlights the ingenuity of Lemma 4.2.4. That $\mathcal{H}_\Theta$ satisfies the estimate,

$$||\mathcal{H}_\Theta||_{2\to 2} \geq C \cdot \sqrt{\log N}$$
4.2. Lower bounds for $||\mathcal{H}_\Theta||_{p\to p}$

for any $\Theta \subset S^1$ reflects the existence of a system of functions in $L^2(\mathbb{R}^2)$ simultaneously satisfying (4.11) and (4.12). This begs the question: do similar systems of functions exist in higher dimensions? This was confirmed quite recently by I. Laba, A. Marinelli and M. Pramanik in [9]. They proved the following result:

**Theorem 4.2.5.** Let $\Theta \subset S^{n-1}$ with $\#\Theta = N < \infty$. Then, for $1 < p < \infty$ there exists a positive constant $C_{n,p}$ such that,

$$||\mathcal{H}_\Theta||_{L^p(\mathbb{R}^n)} \geq C_{n,p}\sqrt{\log N}.$$  

As a consequence, $\mathcal{H}_\Theta$ is necessarily unbounded on $L^p(\mathbb{R}^n)$ for infinite direction sets $\Theta \subset S^{n-1}$.

We will not explore the proof this result. However, there are many similarities between Karagulyan’s proof of Theorem 4.2.1 and I. Laba’s, A. Marinelli’s, and M. Pramanik’s proof of the above proposition. In [9], one constructs sectors $S_k \subset \mathbb{R}^n$ associated to the chosen direction set $\Theta \subset S^{n-1}$. A test function of the form $f = \sum_k f_k$ is then identified, where the $f_k$ have Fourier support in $S_k$. Two key estimates are then proven,

$$||f||_p \leq C_p\sqrt{\#\Theta} \quad |\{x \in \mathbb{R}^n : T_\Theta f(x) \geq c_1\sqrt{\#\Theta}\}| \geq c_2$$

where $T_\Theta$ is again the ‘maximal projection operator’.

Although these inequalities are obtained through more complicated combinatorial arguments, they rely on the structure of tree system. Here, the nesting property of tree systems again guarantee an inequality of the form,

$$\max_{1 \leq k \leq N} \sum_{n=1}^k |f_{\sigma(n)}(x)| \geq C_n \sum_{n=1}^N |f_n(x)|$$

Moreover, this permutation is identical to the permutation utilized by G. Karagulyan in the plane! That a single permutation guarantees the above estimate in all dimensions is a stunning piece of combinatorics in its own right. This results also likely has other applications, and further examination is warranted to understand its exact role in problems of this nature.

4.2.1 Considering the directional maximal function

I have so far insisted that the lower bound of Theorem 4.2 should come as a surprise to the reader, without much justification as to why. I attempt to
remedy this with the following discussion of the maximal directional functions.

The situation is as follows: take some $f \in L^1_{loc}(\mathbb{R}^n)$. For any $\theta \in S^1$ and $h > 0$, we can calculate the directional averages of $f$ at $x$ along $\theta$, which are:

$$A_{h,\theta}f(x) = \frac{1}{2h} \int_{-h}^{h} f(x - t\theta) \, dt.$$ 

We then induce a family of operators $\{M_\theta\}_{\theta \in S^{n-1}}$ on $L^1_{loc}$ by considering the following supremum,

$$M_\theta f(x) = \sup_{h>0} A_{h,\theta} |f|(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x - t\theta)| \, dt.$$ 

Similar to the directional Hilbert transforms, the operators $M_\theta$ are strongly-bounded operators from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and $n \geq 2$. Whereas the directional Hilbert transforms are controlled in norm by the Hilbert transform on $L^p(\mathbb{R})$, the maximal directional functions $M_\theta$ are themselves controlled by the Hardy-Littlewood function on $L^p(\mathbb{R})$. More information can be found in the monogram [2].

Similar to our construction of maximal directional Hilbert transforms, for some $\Theta \subset S^{n-1}$ we let:

$$M_\Theta f(x) = \sup_{\theta \in \Theta} M_\theta f(x) = \sup_{\theta \in \Theta} \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x - t\theta)| \, dt,$$

where again $f \in L^1_{loc}(\mathbb{R}^n)$. This is the directional maximal function associated to $\Theta$. Not surprisingly, the directional maximal function $M_\Theta$ is to $M_\theta$ what the maximal directional Hilbert transform is to $H_\theta$. Since $H_\theta$ and $M_\theta$ share analogous $L^p \to L^p$ mapping properties, it is good guess that $M_\Theta$ would always have asymptotic growth as $\#\Theta \to \infty$. However, this is patently false.

**Proposition 4.2.6.** For any $n \geq 2$, there exists an infinite direction set $\Theta \subset S^{n-1}$ such that, for all $1 < p < \infty$,

$$||M_\Theta||_{p\to p} \leq C_p,$$

where $C_p > 0$ is some absolute constant.

As we shall see in the following chapter, the direction set in Proposition 4.2.6 possesses a kind of ‘extreme’ combinatorial structure. Moreover, this same direction set has a unique role in determining the sharpness of the lower bound in Theorem 4.2.1. We explore this in the following chapter.
Chapter 5

Some specific examples

Having encountered general bounds for maximal directional Hilbert transforms, we discuss refinements of these bounds for specific direction sets. At present, this area-of-study is largely untapped. Beyond introducing some well-understood direction sets, we also discuss apparent barriers in moving from general bounds for all direction sets to precise bounds for a given direction sets. This also motivates the results of the following chapter, which expand upon the single example in [8]. As our new results only focus on direction sets in the plane, we restrict most of our attention to direction sets $\Theta \subset S^1$ in detail, saving higher dimensional variants for the end of this chapter.

5.1 Asymptotics and sharp estimates

What is gained by improving a known operator norm bound? We will use two specific direction sets to examine this question. Begin by letting $N$ be some large positive integer. Define,

\[ \Theta_L = \Theta_{L,N} = \{ \theta_j \in S^1 : \frac{\theta_2}{\theta_1} = 2^{-(N-j+1)}, \text{ for } 1 \leq j \leq N \} \quad (5.1) \]

\[ \Theta_U = \Theta_{U,N} = \{ \theta_j \in S^1 : \frac{\theta_j}{\theta_1} = (N-j+1)^{-1} \text{ for } 1 \leq j \leq N \} \quad (5.2) \]

so that $\#\Theta_L, \#\Theta_U = N$. Even more, there exists a sequence of positive integers $N = M_1 < M_2 < \cdots < M_K < \cdots$, with:

\[ \Theta_{L,N} = \Theta_{L,M_1} \subsetneq \Theta_{L,M_2} \subsetneq \cdots \subsetneq \Theta_{L,M_K} \subsetneq \cdots \]

\[ \Theta_{U,N} = \Theta_{U,M_1} \subsetneq \Theta_{U,M_2} \subsetneq \cdots \subsetneq \Theta_{U,M_K} \subsetneq \cdots , \]

and the inclusion holds simultaneously across both families of direction sets. For example, a crude choice is $M_k = N^k$ for each $k \in \mathbb{N}$.

We now apply the upper bound of Proposition 4.1.1 and lower bound of Theorem 4.2.1 from Chapter 4 to these direction sets. These theorems
5.1. Asymptotics and sharp estimates

furnish constants $c, C > 0$, independent of $N$, with:

$$ c \sqrt{\log M_k} = \sqrt{k} \sqrt{\log N} \leq \|\mathcal{H}_{\Theta_{L, M_k}}\|_{2 \to 2} \leq C \cdot k \cdot \log N $$

$$ c \sqrt{\log M_k} = \sqrt{k} \sqrt{\log N} \leq \|\mathcal{H}_{\Theta_{U, M_k}}\|_{2 \to 2} \leq C \cdot k \cdot \log N. $$

As we increase $k$, our direction sets $\Theta_{*, M_k}$ grow in size. Consequently, the operators $\mathcal{H}_{\Theta_{*, M_k}}$ grow in $L^2 \to L^2$ norm. While we know that this growth cannot be bounded, we are unable to discriminate between the growth of $\|\mathcal{H}_{\Theta_{L, M_k}}\|_{2 \to 2}$ and $\|\mathcal{H}_{\Theta_{U, M_k}}\|_{2 \to 2}$ as $\#\Theta_{*, M_k} = N^k \to \infty$.

Our goal now is to disentangle the asymptotic growth of $\mathcal{H}_{\Theta_{L}}$ and $\mathcal{H}_{\Theta_{U}}$.

We first record what we mean by a 'precise' asymptotic estimate. I give this definition in the language of sequences, since we are mostly concerned with the growth of $\|\mathcal{H}_{\Theta}\|_{2 \to 2}$ as $\#\Theta = M_k \to \infty$ along the positive integers.

**Definition 5.1.1.** Suppose that $x_k, y_k$ are sequences of non-negative real numbers. We say that $x_k$ is asymptotically equivalent to $y_k$ if there exists some $N \in \mathbb{N}$ and constants $c, C \geq 0$ such that, for $n \geq N$, we have:

$$ c \cdot y_n \leq x_n \leq C \cdot y_n. \tag{5.3} $$

In this case, we write $x_n \approx y_n$. If $y_n \to 0$ as $n \to \infty$, we say that $x_n$ ‘decays’ like $y_n$; similarly, if $y_n \to \infty$, we say that $x_n$ ‘grows’ like $y_n$.

We refer to an inequality of type (5.3) as a sharp estimate.

A key component of this definition is the identity of the constants $c, C > 0$. Often, the constants depend on ambient factors, such as the Lebesgue space or dimension under consideration. For an inequality of type (5.3) to be useful, we must assume that $c, C$ are independent of the index $n \in \mathbb{N}$. Finally, as a piece of notation, if we have $x_n \leq C \cdot y_n$, we will write $x_n \preceq y_n$; similarly, if $x_n \geq c \cdot y_n$, we will write $x_n \succeq y_n$. Therefore, we have:

$$ x_n \approx y_n \iff x_n \preceq y_n \text{ and } x_n \succeq y_n. $$

We can now explain what a ‘precise’ estimate for $\|\mathcal{H}_{\Theta}\|_{2 \to 2}$ entails. This is a sharp estimate of the form,

$$ c \cdot \Phi(\#\Theta) \leq \|\mathcal{H}_{\Theta}\|_{2 \to 2} \leq C \cdot \Phi(\#\Theta), \tag{5.4} $$

As a consequence of Theorems 4.1.1 and 4.2.1, $\Phi : \mathbb{N} \to \mathbb{R}^+$ will always be some logarithmic function of $\#\Theta$. Comparing these general norm bounds with (5.4) demonstrates that, regardless of the identity of $\Phi$, we have:

$$ c \cdot \Phi(\#\Theta) \leq C \cdot \log \#\Theta \implies \Phi(\#\Theta) \ll \log \#\Theta $$

$$ C \cdot \Phi(\#\Theta) \geq c \cdot \sqrt{\log \#\Theta} \implies \Phi(\#\Theta) \gg \sqrt{\log \#\Theta} $$

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This is simply a rephrasing of the main theorems from Chapter 4 in terms of asymptotic growth, which is the main subject of the remainder of this thesis.

As a final piece of housekeeping, when dealing with a specific direction set \( \Theta \subset S^1 \), we will primarily be concerned with the ‘slope-set’ of \( \Theta \). This is the set,

\[
sl(\Theta) = \{v_j \in \mathbb{R} : v_j = \frac{\theta_j^2}{\theta_1^1} \text{ for some } \theta_j \in \Theta\}.
\]

Fortunately, we have the following reduction which allows us to efficiently work with the slope-set \( sl(\Theta) \).

**Lemma 5.1.2.** Let \( \Theta \subset S^1 \) be a set of \( N \) directions, and let \( sl(\Theta) \) be its slope-set. Let,

\[
\tilde{\Theta} = \{(1, v_k) : v_k \in sl(\Theta), v_1 < \cdots < v_N\}.
\]

Then, for every \( f \in S(\mathbb{R}^2) \) and \( x \in \mathbb{R}^2 \), we have:

\[
\mathcal{H}_{\tilde{\Theta}} f(x) = \mathcal{H}_{\Theta} f(x),
\]

where,

\[
\mathcal{H}_{\tilde{\Theta}} f(x) = \sup_{1 \leq k \leq N} |H_{(1,v_k)} f(x)| = \sup_{1 \leq k \leq N} \left| \int_{-\infty}^{\infty} f(x_1 - t, x_2 - v_k) \frac{dt}{t} \right|.
\]

As a consequence of equation (5.5), we always have:

\[
||\mathcal{H}_{\tilde{\Theta}}||_{2\rightarrow 2} = ||\mathcal{H}_{\Theta}||_{2\rightarrow 2}.
\]

Usually, we prove estimates for \( ||\mathcal{H}_{\tilde{\Theta}}||_{2\rightarrow 2} \). Often, this is because it is easier to work with the slope-set of a direction set. From equation (5.7), estimates on slope-sets automatically translate to any direction set of interest.

**Proof of Lemma 5.1.2.** Suppose that \( v_k \in sl(\Theta) \). Then, \( v_k = \theta_k^2/\theta_k^1 \) for some \( \theta_k \in \Theta \). The proof is essentially showing the following pointwise equality,

\[
H_{(1,v_k)} f(x) = H_{\theta_k} f(x), \text{ for every } f \in S(\mathbb{R}^2).
\]

An appropriate change-of-variables in equation (5.6) gives the result. Details are left to the reader. \( \square \)
5.2 Some examples

In this section, we discuss sharp estimates for the direction sets $\Theta_L$ and $\Theta_U$, previously defined in equations (5.1) and (5.2). These are key examples in the literature for their relationship to known upper and lower bounds for $H_{\Theta}$ across general $\Theta \subset S^1$.

To demonstrate the sharpness of Theorems 4.1.1 and 4.2.1, it suffices to show, for each large $N \in \mathbb{N}$, there exist direction sets $\Theta_1, \Theta_2 \subset S^1$ of cardinality $N$, so that:

$$||H_{\Theta_1}||_{2 \rightarrow 2} \geq \log N \quad \text{and} \quad ||H_{\Theta_2}||_{2 \rightarrow 2} \leq \sqrt{\log N}.$$  

This is exactly the roles which $\Theta_L$ and $\Theta_U$ play in the literature. Moreover, in the next chapter, we present new results demonstrating that sharpening examples are far from unique, with many direction sets attaining maximal asymptotic growth.

5.2.1 The uniform direction set

Let us begin with $\Theta_U$, the uniform direction set, defined for large cardinality $N \gg 1$ as:

$$\Theta_U = \{\theta \in S^1 : \frac{\theta_2}{\theta_1} = \frac{j}{N}, \text{ for some integer } 1 \leq j \leq N\}$$

$$= \{(\theta_1, \theta_2) = (\frac{1}{\sqrt{1+(j/N)^2}}, \frac{j/N}{\sqrt{1+(j/N)^2}}), 1 \leq j \leq N\}.$$  

$\Theta_U$ was first studied by Joonil Kim [8], who proved the following lower bound.

**Theorem 5.2.1.** Suppose that $\Theta_U$ is the uniform direction set of large cardinality $N \gg 1$. Then, there exists an $f \in L^2(\mathbb{R}^2)$ and an absolute constant $C > 0$ independent of $N$ such that,

$$||H_{\Theta_U}f||_{L^2(\mathbb{R}^2)} \geq C \log N ||f||_{L^2(\mathbb{R}^2)}.$$  

By comparison with Theorem 4.1.1 from the previous chapter, we obtain constants $c, C > 0$ such that, for $\Theta_U$ of large enough cardinality $N \gg 1$, we always have:

$$c \log N \leq ||H_{\Theta_U}||_{2 \rightarrow 2} \leq C \log N.$$  

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Of course, the constants $c, C > 0$ are independent of the cardinality parameter $N \in \mathbb{N}$. This then implies that $||\mathcal{H}_{\Theta_U}||_{2 \rightarrow 2}$ is asymptotically equivalent to $\log \#\Theta_U$ as $\#\Theta_U \to \infty$.

The results in the next chapter are a direct extension of Theorem 5.2.1. As such, we present Joonil Kim’s proof in detail in preparation for what follows.

Proof of Theorem 5.2.1. The goal is to construct an $f \in L^2(\mathbb{R}^2)$ that satisfies the norm-inequality in Theorem 5.2.1. As such, let $\Theta_U$ be a uniform direction set of large cardinality $N \gg 1$. The argument relies upon the geometric interplay of the following sets:

$$A_N := \{y \in \mathbb{R}^2 : 1 < |y| < N, |y_2| < |y_1|, y_1 > 5\},$$

$$B_N := \{y \in \mathbb{R}^2 : 100 < |x| < \frac{N}{100}, |x_2| < \frac{|x_1|}{100}, x_1 < -100, x_2 < 0\}.$$

Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ associated to the set $A_N$:

$$f(y_1, y_2) = \frac{1_{A_N}(y_1, y_2)}{|(y_1, y_2)|}. \quad (5.8)$$

Notice that $A_N$ is a truncated cone, rooted at the origin. This ‘cone’ has aperture $\frac{\pi}{2}$ and is centered along the line $x_2 = 0$. By construction, $A_N$ excludes the origin and is a bounded subset of the plane. This guarantees that the test function defined by (5.8) is in $L^2(\mathbb{R}^2)$.

By Lemma 5.1.2, it suffices to consider the set,

$$\tilde{\Theta}_U := \{(1, v_k) \in \mathbb{R}^2 : v_k = \frac{k}{N}, \text{ for } 1 \leq k \leq N\},$$

and its maximal directional Hilbert transform,

$$\mathcal{H}_{\tilde{\Theta}_U} f(x_1, x_2) = \max_{1 \leq k \leq N} \left| \int_{-\infty}^{\infty} f(x_1 - t, x_2 - v_k t) \frac{dt}{t} \right|.$$

Let $\tilde{\Theta}_U = \Theta$. For the remainder of the proof, we work with $\mathcal{H}_\Theta f$.

We come to the geometric component of the argument. Notice the family of lines,

$$l_v := \{(x_1 - t, x_2 - tv) : t \in \mathbb{R}, v \in sl(\Theta_U)\},$$

evenly partition the truncated cones $A_N$ and $B_N$. This is a consequence of the fact that $A_N$ and $B_N$ are both bounded by the lines $x_2 = x_1$ and $x_2 = -x_1$. This partitioning is illustrated in the following figures.
Figure 5.1: The set $A_N$ when $N = 20,000$.

Figure 5.2: The set $B_N$ when $N = 20,000$. 
5.2. Some examples

Now, for each \( x \in B_N \) there is a \( v = v(x) \in SL(\Theta_U) \) such that the line \( l_{v(x)} \) intersects the unit disc in \( \mathbb{R}^2 \). This is a consequence of the uniform partition of \( S^1 \) by the unit vectors in \( \Theta_U \). Hence, we have the following inclusion,

\[
\tilde{l}_{v(x)} = \{(x_1 - t, x_2 - tv(x)) : 5 < x_1 - t < \frac{N}{2}\} \subset A_N.
\]

In essence, \( A_N \) is just large enough to trap the portion of the line \( l_{v(x)} \) whose distance is proportional to the distance from \( (x_1, x_2) \) to \( A_N \).

For \( x \in B_N \), consider the Hilbert transform of \( f \) in the direction \((1, v)\):

\[
H_v f(x_1, x_2) = H_{(1,v)} f(x_1, x_2) = \int_{-\infty}^{\infty} \frac{1_{A_N}(x_1 - t, x_2 - vt)}{|(x_1 - t, x_2 - vt)|} \frac{dt}{t}, \quad (5.9)
\]

where we have suppressed the dependence \( v = v(x) \). The integrand in (5.9) is non-zero so long as,

\[
x_1 - t \in A_N \iff x_1 - t > 5 \iff t < x_1 - 5 < -105.
\]

Consequently, the integrand in (5.9) is always negative. We also restrict \( t \) so that \( x_1 - t < |x_1| \). Combining these two facts gives,

\[
|H_v f(x_1, x_2)| \geq \left| \int_{-\infty}^{\infty} \frac{1_{A_N}(x_1 - t, x_2 - vt)}{|(x_1 - t, x_2 - vt)|} \frac{dt}{t} \right| \geq \left| \int_{\{t : 5 < x_1 - t < |x_1|\}} \frac{1_{A_N}(x_1 - t, x_2 - vt)}{|(x_1 - t, x_2 - vt)|} \frac{dt}{t} \right|
\]

Because \( x \in B_N \), we must have \( |x_1| < \frac{N}{100} \). This gives the set inclusion,

\[
\{t : 5 < x_1 - t < |x_1|\} \subset \{t : 5 < x_1 - t < N/2\}. \quad (5.10)
\]

This implies that \( 1_{A_N}(x_1 - t, x_2 - vt) = 1 \) when \( 5 < x_1 - t < |x_1| \). Since \( x_1 < -100 \), we know that:

\[
-2|x_1| < t < -|x_1|. \quad (5.11)
\]

Also, having \( y \in A_N \) guarantees that \( |y_2| < |y_1| \). As such:

\[
\frac{1_{A_N}(x_1 - t, x_2 - vt)}{|(x_1 - t, x_2 - vt)|} \geq \frac{1_{A_N}(x_1 - t, x_2 - vt)}{2|x_1 - t|} \quad (5.12)
\]
5.2. Some examples

Recalling that the integrand was always negative, we combine (5.10), (5.11), and (5.12) to obtain,

\[
|H_v f(x_1, x_2)| \geq \int_{5|x_1-t| < |x_1|} \frac{1}{2|x_1-t|/2|x_1|} dt = \log \left( \frac{|x_1|}{5} \right) / 4|x_1| \geq \frac{\log(|x_1|)}{5|x_1|}.
\]

Using this pointwise estimate, we obtain an \(L^2(\mathbb{R}^2)\) estimate for \(\mathcal{H}_\Theta\):

\[
\int \int_{\mathbb{R}^2} |\mathcal{H}_\Theta f(x_1, x_2)|^2 dx_1 dx_2 \geq \int \int_{B_N} |H_v f(x_1, x_2)|^2 dx_1 dx_2 \geq \arctan \left( \frac{1}{100} \right) \int_{100}^{N/100} \left( \frac{1}{5} \log(r/5) \right)^2 r dr \geq \frac{1}{10,000} (\log N)^3.
\]

However, we also have the \(L^2\) estimate,

\[
\int \int_{\mathbb{R}^2} |f(x)|^2 dx = \int \int_{\mathcal{A}_N} dx / |x|^2 = \arctan \left( \frac{\pi}{4} \right) \int_5^N \frac{1}{r^2} dr \leq \log N.
\]

This gives,

\[
\frac{||\mathcal{H}_\Theta f||^2_{L^2(\mathbb{R}^2)}}{||f||^2_{L^2(\mathbb{R}^2)}} \geq \frac{1}{10,000} (\log N)^3 / \log N = \frac{1}{10,000} (\log N)^2.
\]

Taking square-roots gives the estimate \(||\mathcal{H}_\Theta||_{2-2} \geq \log N\). Appealing to Lemma 5.1.2 gives an identical asymptotic estimate for \(||\mathcal{H}_{\Theta_U}||_{2-2}\). This completes the proof of Theorem 5.2.1.

Joonil Kim’s proof demonstrates a clear connection between the geometric structure of \(\Theta_U\) and the growth of \(\mathcal{H}_{\Theta_U}\) over \(L^2(\mathbb{R}^2)\) as \(#\Theta_U \to \infty\). As we shall see in the next chapter, with a relatively simple partitioning of the set \(B_N\), we can prove lower bounds for many direction sets in the plane. Even more, many direction sets give rise to maximal directional Hilbert transforms whose norm is close to the upper bound of Theorem 4.1.1, a fact which was previously unknown in the literature.
5.2. Some examples

5.2.2 Lacunary direction sets

We now consider the direction set \( \Theta_L \). Recall that,

\[
\Theta_L = \Theta_{L,N} = \{ \theta_j \in S^1 : \frac{\theta_2}{\theta_1} = 2^{-(N-j+1)}, \text{ for } 1 \leq j \leq N \}
= \{ (\theta_1, \theta_2) = \left( \frac{1}{\sqrt{1+4^{-j}}}, \frac{2^{-j}}{\sqrt{1+4^{-j}}} \right), \text{ for } 1 \leq j \leq N \}.
\]

I have intentionally reversed the index in moving from the first line to the second line. This is to guarantee the ordering of the set,

\[
sl(\Theta_L) = \{ v_j : v_j = 2^{-(N-j+1)}, \text{ for some integer } 1 \leq j \leq N \}
\]

so that \( v_1 < v_2 < \cdots < v_N \). The set \( \Theta_L \subset S^1 \) is one specific example of a larger class of direction sets. These are the lacunary set of directions of finite cardinality.

**Definition 5.2.2.** Suppose that \( \Theta = \{ \theta_j = (\theta^j_1, \theta^j_2) \in S^1 \} \) is a (finite or countable) set of directions, ordered so as to be increasing in slopes. Then, \( \Theta \) is said to be lacunary with node \( v_\infty \in \mathbb{R} \) if, for every \( v_j \in sl(\Theta) \), we have:

\[
|v_k - v_\infty| \leq \frac{1}{2} |v_{k+1} - v_\infty|, \text{ for } 1 \leq k \leq N - 1.
\]

This definition of lacunarity is slightly modified from the definition which appears in [1]. Here, I utilize a definition of lacunarity which appeals to the slope-set, as it aligns with the notation used in the next chapter. With this definition, one easily checks that \( \Theta_L \) is lacunary with node \( v_\infty = 0 \), since:

\[
|v_k - v_\infty| = 2^{-(N-k+1)} = \frac{1}{2} 2^{-(N-k)} = |v_{k+1} - v_\infty|, \text{ for } 1 \leq k \leq N - 1.
\]

In their monograph on planar singular integrals, C. Demeter and F. Di Plinio proved a very general upper bound for a wide variety of multiplier-type operators on \( L^p(\mathbb{R}^2) \). I limit the scope of this result (which is Theorem 2 in [1]), stating it as it applies to maximal directional Hilbert transforms.

**Theorem 5.2.3.** Suppose that \( \Theta \subset S^1 \) is a lacunary set of directions of finite but large cardinality \( N \gg 1 \). Then, there exists an absolute constant \( C_p > 0 \) independent of \( N \), such that:

\[
||H_\Theta||_{L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)} \leq C_p \cdot \sqrt{\log N}, \text{ where } 1 < p < \infty.
\]
5.2. Some examples

The proof of this result comprises roughly two pages in [1]. However, it builds on several substantial earlier results. As such, we refer the reader to [1] and its predecessors for more details on the subject. In particular, [1] is an excellent paper for obtaining an overview of areas of research parallel to the ones discussed here.

Comparing Theorem 5.2.3 with Theorem 4.2.1 shows that $\|H_{\Theta_L}\|_{2\rightarrow 2}$ is asymptotically equivalent to $\sqrt{\log \#\Theta_L}$. This demonstrates that Theorem 5.2.3 cannot be improved.

5.2.3 A short note on $M_{\Theta_L}$

Before departing from lacunarity, we again examine the directional maximal function,

$$M_{\Theta_L}f(x_1, x_2) := \sup_{\theta \in \Theta_L} \sup_{h > 0} \frac{1}{2h} \int_{-h}^{h} |f(x_1 - \theta_1 t, x_2 - \theta_2 t)| dt.$$

From the work of A. Nagel, E. Stein and S. Wainger [12], we have the following result concerning the lacunary directional maximal function:

**Theorem 5.2.4.** Suppose that $\Theta_L \subset S^1$ is a countably-infinite lacunary set of directions. Then we have the following estimate,

$$\|M_{\Theta_L}\|_{L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)} \leq C_p, \text{ for every } 1 < p \leq \infty.$$  

In particular, since every finite lacunary set of directions is contained in a lacunary set of directions which is countable, $M_{\Theta}$ is bounded on $L^p(\mathbb{R}^2)$ whenever $\Theta$ is a lacunary direction set.

This celebrated result generated much of the research into maximal directional operators, including research on maximal directional Hilbert transforms. Theorem 5.2.4 preceded the work of [9] and its predecessor [6] by almost forty-years. It is no exaggeration, then, to say that [12] has inspired four decades of research into maximal directional operators in harmonic analysis, including this very thesis.

5.2.4 Between uniformity and lacunarity

At this stage, we have discussed the following asymptotic estimates,

$$\|H_{\Theta_U}\|_{2\rightarrow 2} \approx \log \#\Theta_U \quad \text{and} \quad \|H_{\Theta_L}\|_{2\rightarrow 2} \approx \sqrt{\log \#\Theta_L},$$

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where $\Theta_U$ is the uniform set of directions and $\Theta_L$ is any lacunary set of directions in the plane. As discussed earlier, for some large integer $N = M_1$, we can find integers $M_1 < M_2 < \cdots < M_k < \cdots$ such that, for all $j \geq 1$,

$$\Theta_{U,M_j} \subsetneq \Theta_{U,M_{j+1}}, \quad \|\mathcal{H}_{\Theta_{U,M_j}}\|_{2 \to 2} = C_U \cdot \log M_j, \quad (5.13)$$

where the constant $C_U$ is independent of the cardinality parameters $M_j$. A similar statement holds for $\Theta_L$,

$$\Theta_{L,M_j} \subsetneq \Theta_{L,M_{j+1}}, \quad \|\mathcal{H}_{\Theta_{L,M_j}}\|_{2 \to 2} = C_L \cdot \sqrt{\log M_j}, \quad (5.14)$$

with $C_L$ is again some absolute constant. Because of equations (5.13) and (5.14), we know that $\|\mathcal{H}_{\Theta_{s,N}}\|_{2 \to 2}$ ‘blows-up’ exactly like the sequences $\log N$ and $\sqrt{\log N}$ along scaled families of uniform and lacunary direction sets, respectively.

Very recently, Jongchon Kim and M. Pramanik considered the following question. Suppose that $\frac{1}{2} < \beta < 1$ is some fixed parameter. Is there some nested sequence of direction sets $\Theta_{M_j} \subset S^1$, with $\#\Theta_{M_j} = M_j \not\to \infty$ such that, for all $j \geq 1$, we have:

$$\Theta_{M_j} \subsetneq \Theta_{M_{j+1}}, \quad \|\mathcal{H}_{\Theta_{M_j}}\|_{2 \to 2} = C \cdot (\log M_j)^\beta,$$

and the constant $C > 0$ is independent of the cardinality parameters $M_j$? This is answered in the affirmative by Jongchon Kim and M. Pramanik in [7]. We present their findings below (which is Theorem 3.1 in [7]).

**Theorem 5.2.5.** For any exponent $\beta \in \left[\frac{1}{2}, 1\right]$, there exists an infinite subset $\Theta_{\infty} = \Theta_{\infty}(\beta) \subset S^1$ and subsets $\Theta_{M_j}$ of cardinality $M_j$ such that,

$$\Theta_{M_1} \subsetneq \Theta_{M_2} \subsetneq \cdots \subsetneq \Theta_{M_k} \subsetneq \cdots, \quad \text{with } M_j \not\to \infty,$$

such that,

$$c (\log M_j)^\beta \leq \|\mathcal{H}_{\Theta_{M_j}}\|_{2 \to 2} \leq C (\log M_j)^\beta,$$

and the constants $c, C > 0$ are independent of $\beta$ and $M_j$.

This is a lovely book-end to our discussion of maximal directional Hilbert transforms as operators on $L^2(\mathbb{R}^2)$. Not only have we derived sharp upper and lower bounds for the operator norm of $\mathcal{H}_\Theta$, we now know that all allowable types of asymptotic growth for maximal directional Hilbert transforms are in fact attained by some ascending chain of direction sets $\Theta_{M_j} \subset S^1$.

It is worth noting, in proving Theorem 5.2.5, Kim and Pramanik again rely on lacunary and uniform structure within direction sets. Essentially,
5.2. Some examples

each $\Theta_{M_j}$ contains the lacunary direction set $\Theta_L$ of some prescribed cardinality. At each ‘arm’ of this lacunary subset are placed rescaled copies of the uniform direction set $\Theta_U$, each of which lives at scale $2^{-j}$. By combining the known estimates for uniform and lacunary direction sets, this guarantees some form of middling asymptotic growth. As one might expect, the larger the lacunary subset, the more slowly the quantities $\|H_{\Theta_{M_j}}\|_{2\rightarrow 2}$ grow as $\#\Theta_{M_j} \rightarrow \infty$.

This result of J. Kim and M. Pramanik furnishes a spectrum of maximal directional Hilbert transforms whose operator norms attain the prescribed asymptotic growth in $\#\Theta$. Moreover, their constructions suggest a qualitative relationship between lacunary structure and slow growth versus uniform structure and fast growth. We will discuss this suggestion more in the following chapter.
Chapter 6

New results

In this chapter, we present original research concerning maximal directional Hilbert transforms. We first prove a general lower bound for all maximal directional Hilbert transforms. This is the content of Theorem 6.1.1 below. A corollary of this result is a ‘simplified’ lower bound, which is the content of Corollary 6.2.1. We prove these results in the first section of this chapter. In the second section, we construct several direction sets $\Theta \subset S^1$ for which Theorem 6.1.1 provides new lower bounds on $||H_\Theta||_{2\to 2}$.

6.1 Proof of main theorem

The main goal of this section is to prove the following result.

**Theorem 6.1.1.** For some large integer $N \gg 1$, let $\Theta = \{(1, v_j) : 0 \leq v_1 < v_2 < \cdots < v_N \leq 1\}$ be a set of directions in the plane. For each $1 \leq j \leq N$, we set $\delta_j = v_{j+1} - v_j$ and define complementary indexing sets:

$$
\Lambda_+ := \{j \in \mathbb{N} : 1 \leq j \leq N, \text{ and } \delta_jN > 1\}
$$

$$
\Lambda_- := \{j \in \mathbb{N} : 1 \leq j \leq N, \text{ and } \delta_jN \leq 1\}
$$

Then we have the following operator norm estimate:

$$
||H_\Theta||_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \geq C \sum_{j \in \Lambda_+} \delta_j [\log^2 \delta_j^{-1} + \log \delta_j^{-1}] + \frac{C}{\log N} \sum_{j \in \Lambda_-} \delta_j \log^3 \delta_j^{-1},
$$

where $0 < C < 1$ is some absolute constant independent of $N$.

The proof of Theorem 6.1.1 is as follows:

1. The construction of a test function. This test function is supported in an almost-square rectangle in the first-quadrant, with side length roughly $N = \# \Theta$. 

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6.1. Proof of main theorem

2. A geometric construction and associated lemma. Using pairs of directions $v_j, v_{j+1} \in \Theta$, we design $N - 1$ disjoint sectors in the third-quadrant. This allows us to encode the geometric information of $\Theta$ in more useable way relative to our test function.

3. A pointwise lower-bound on directional Hilbert transforms. For our test function $f \in L^2$, we prove a pointwise lower bound on the directional Hilbert transforms $H_{v_j} f$ along the conic sector associated to $v_{j+1}, v_j$.

4. A norm estimate on the maximal directional Hilbert transform. We then use our pointwise estimates over certain regions in the plane to obtain a global $L^2$ estimate on $\mathcal{H}_\Theta f$. This requires a division of the conic sectors subordinate to the indexing sets $\Gamma_+ \text{ and } \Gamma_-.$

6.1.1 The Test Function

We begin by fixing some direction set $\Theta \subset \mathbb{R}^2$ of large cardinality $N \gg 1$. Throughout this chapter, we will write:

$$\Theta := \{(1, v_k) \in \mathbb{R}^2 : v_1 < v_2 < \cdots < v_N \},$$

so that each direction $\vec{v} \in \Theta$ is uniquely determined by its second coordinate.

We then consider the set $A_N = [1, 2N] \times [0, 2N]$. For large $N$, this set approximates a square of side-length $2N$, with bottom-left corner at the origin. We define a test function adapted to $A$ via,

$$f(y_1, y_2) = \frac{\chi_{A_N(y_1, y_2)}}{y_1 + y_2} = \chi_{A_N(y_1, y_2)} \phi(y_1, y_2)$$

where,

$$\phi(y_1, y_2) = \frac{1}{y_1 + y_2}.$$  

Since $(y_1, y_2) \in A_N$ implies that $y_1 + y_2 \geq 1$, our test function is bounded and of compact support.

To evaluate the $L^2$ norm of $f$, we split $A_N$ into three pieces: two con-
6.1. Proof of main theorem

gruent right-triangles, and a remainder rectangle. This gives:

$$\|f\|_{L^2}^2 = \int_{A_N} dy_1 dy_2 \frac{\phi^2(y_1, y_2)}{(y_1 + y_2)^2}$$

$$= \int_1^{2N} y_1 \phi^2(y_1, y_2) dy_2 dy_1 + \int_1^{2N} y_2 \phi^2(y_1, y_2) dy_1 dy_2$$

$$+ \int_1^{2N} \int_0^1 \phi^2(y_1, y_2) dy_1 dy_2$$

Examining $\phi$, we see the first two integrals are symmetric in $y_1$ and $y_2$. So, we merely estimate the first integral:

$$\int_1^{2N} y_1 \phi^2(y_1, y_2) dy_2 dy_1 = \int_1^{2N} \int_1^{2N} \frac{\phi^2(y_1, y_2)}{(y_1 + y_2)^2} < \int_1^{2N} \left( \int_0^{y_1} \frac{dy_2}{y_1^2} \right) dy_1 = \log 2N.$$ 

For the remainder integral, we have:

$$\int_1^{2N} \int_0^1 \phi^2(y_1, y_2) dy_1 dy_2 = \int_1^{2N} \int_0^1 \frac{dy_1 dy_2}{(y_1 + y_2)^2} < \int_1^{2N} \frac{dy_1}{y_1^2} < 1$$

Therefore, for large $N >> 1$, we have that:

$$\|f\|_{L^2(\mathbb{R}^2)}^2 < 2 \log 2N + 1 < 4 \log N$$

6.1.2 Geometric Lemma

We now prove a geometric lemma, which enables our later pointwise and norm estimates. For each integer $1 \leq j \leq N$, let us define the truncated conic sectors:

$$B_j = \{(x_1, x_2) \in \mathbb{R}^2 : -N < x_1 < -1; v_{j+1} x_1 < x_2 < v_j x_1\},$$

and also, for each $(x_1, x_2) \in B_j$, the line segment:

$$l_j(x_1, x_2) = \{(x_1 - t, x_2 - v_{j+1}t) ; 1 \leq x_1 - t \leq N\}.$$

We claim for each $(x_1, x_2) \in B_j$, we know that $l(x_1, x_2) \subset A_N$. We illustrate the connection between the sets $A_N$ and $B_N$ in Figure 6.1.2, which follows after the proof of our geometric lemma.
6.1. Proof of main theorem

Proof. We will have the results if we can demonstrate that: $1 \leq x_1 - t \leq N$ implies $0 \leq x_2 - v_{j+1}t \leq 2N$. We provide the necessary calculations below.

First, we know that:

$$x_2 - v_{j+1}t = x_2 - v_{j+1}x_1 + v_{j+1}(x_1 - t) > v_{j+1} > 0.$$  

This follows since $x_2 - v_{j+1}x_1 > 0$ and $x_1 - t \geq 1$ when $(x_1, x_2) \in B_j$.

For the second calculation, we have:

$$x_2 - v_{j+1}t < v_jx_1 - v_{j+1}t = v_{j+1}(x_1 - t) + (v_j - v_{j+1})x_1$$

$$\leq v_{j+1}N + (v_{j+1} - v_j)N = (2v_{j+1} - v_j)N$$

$$\leq 2N,$$

and the final inequality follows since $v_{j+1} \leq 1$ uniformly in $j$. \qed
Figure 6.1: The sets $A_N$ and $B_N$ for a direction set of cardinality 10.
6.1. Proof of main theorem

6.1.3 Bounding the Directional Hilbert Transform

We now obtain a pointwise bound on the function $H_{v_1}f$ for $0 \leq j \leq N - 1$. This aspect of our proof mirrors the work of Joonil Kim in [8]. For $x \in B_j$, we have:

$$|H_{v_1}f(x)| = \left| \int_{-\infty}^{\infty} \frac{1}{x_1 + x_2 - (1 + v_j) t} \frac{1}{t} \right| \leq \left| \int_{1 \leq x_1 - t \leq N} \frac{1}{x_1 + x_2 - (1 + v_j) t} \right|.$$

Here, the inequality follows because the support condition of $A_N$ forces the integral to be single-signed. In fact, the integrand is always non-positive. Now, utilizing our geometric lemma, we have that:

$$|H_{v_1}f(x)| \leq \left| \int_{1 \leq x_1 - t \leq N} \frac{1}{x_1 + x_2 - (1 + v_j) t} \right|.$$

A partial fractions decomposition then gives:

$$|H_{v_1}f(x)| \geq \left| \frac{1}{x_1 + x_2} \left| \int_{1 \leq x_1 - t \leq N} \frac{1 + v_j}{x_1 + x_2 - (1 + v_j) t} + \frac{1}{t} dt \right| \right|.$$

We examine both logarithmic terms in the above to obtain our final pointwise bound. In particular, we will find that the first logarithmic term is negligible when compared to the first. Let us begin with this first term.

$$\log \left( \frac{1 + v_j}{1 - x_1} \right)$$
6.1. Proof of main theorem

First, we know that:

\[
\frac{|x_1 + x_2|}{N + |x_1|} \leq \frac{2|x_1|}{|x_1|} = 2
\]

Similarly, since \(x_1 + x_2 < 0\), and because we know that \(v_{j+1}|x_1| > |x_2|\) and \(N > |x_1|\), we have:

\[
1 + v_{j+1} + \frac{x_1 + x_2}{N - x_1} = 1 + v_{j+1} - \frac{|x_1| + |x_2|}{N + |x_1|}
\geq 1 + v_{j+1} - \frac{(1 + v_{j+1})|x_1|}{2|x_1|}
= \frac{1}{2}(1 + v_{j+1})
\geq \frac{1}{2}.
\]

Thus, for all \(1 \leq j \leq N - 1\), we have:

\[
\left| \log(1 + v_{j+1} + \frac{x_1 + x_2}{N - x_1}) \right| \leq \log 2
\]

Secondly, we can rewrite the second logarithm as:

\[
- \log \left( \frac{x_1 + x_2 + (1 + v_{j+1})(1 - x_1)}{1 - x_1} \right) = \log \left( \frac{1 + |x_1|}{x_2 - v_{j+1}x_1 + (1 + v_{j+1})} \right)
\geq \log \left( \frac{|x_1|}{x_2 - v_{j+1}x_1 + (1 + v_{j+1})} \right)
\]

Finally, since \(\frac{1}{|x_1 + x_2|} \geq \frac{1}{2|x_1|}\), we obtain a clean pointwise lower bound.

For every \(x \in B_j\), \(|H_{\Theta} f(x)| \geq \frac{C}{2|x_1|} \log \left( \frac{|x_1|}{x_2 - v_{j+1}x_1 + (1 + v_{j+1})} \right)\).

Here, \(0 < C < 100\) is some absolute constant, utilized to control the logarithmic factor bounded by \(\log 2\).

6.1.4 Norm Estimates for \(H_\Theta\)

Having derived estimate (6.2), we turn to norm estimates for \(H_\Theta\) on \(L^2\). Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be the test function defined in equation (6.1). By construction, whenever \(1 \leq k \neq l \leq N\), we have \(B_k \cap B_l = \emptyset\). This gives:

\[
||H_\Theta f||_2^2 = \iint_{\mathbb{R}^2} |H_\Theta f(x)|^2 \, dx \geq \sum_{j=1}^{N} \int_{B_j} |H_\Theta f(x)|^2 \, dx
\]
6.1. Proof of main theorem

We also have the following pointwise inequality for \( (x_1, x_2) \in B_j \),

\[
|H_\Theta f(x_1, x_2)|^2 \geq |H_{v_{j+1}} f(x_1, x_2)|^2 \\
\geq \left( \frac{C}{2|x_1|} \log\left( \frac{|x_1|}{x_2 - v_{j+1} x_1 + (1 + v_{j+1})} \right) \right)^2
\]

Combining (6.3) and (6.4) gives:

\[
||H_\Theta f||_2^2 \geq \frac{1}{4} \sum_{j=1}^{N} \int_{B_j} \left( \frac{C}{x_1} \log\left( \frac{|x_1|}{x_2 - v_{j+1} x_1 + (1 + v_{j+1})} \right) \right)^2 dx_1 dx_2
\]

We make the change-of-variables \( (x_1, x_2) \mapsto (y_1, y_2) \), with \( y_1 = x_1 \) and \( y_2 = \frac{x_2 - v_{j+1} x_1}{|x_1|} \), leading to:

\[
||H_\Theta f||_2^2 \geq C \sum_{j=1}^{N-1} \int_{\tilde{B}_j} |y_1|^{-1} \log\left( \frac{1}{y_2 + \frac{1}{|y_1|}} \right)^2 dy_1 dy_2. \quad (6.5)
\]

Here \( \tilde{B}_j \) is the transformed region of integration, defined by:

\[
\tilde{B}_j = \{(y_1, y_2) \in \mathbb{R}^2 : -N < y_1 < -1; 0 < y_2 < v_{j+1} - v_j\},
\]

and the constant \( 0 < C < 1 \) in (6.5) is absolute.

We further divide the regions of integration into disjoint sets:

\[
\tilde{B}_j^+ = \{(y_1, y_2) \in \tilde{B}_j : y_2 > \frac{1 + v_{j+1}}{|y_1|}\}
\]

and,

\[
\tilde{B}_j^- = \{(y_1, y_2) \in \tilde{B}_j : y_2 < \frac{1 + v_{j+1}}{|y_1|}\}.
\]

Then, for the same constant \( C > 0 \) as before, we have:

\[
||H_\Theta f||_2^2 \geq C \sum_{j=1}^{N-1} \int_{\tilde{B}_j^+} |y_1|^{-1} \log\left( \frac{1}{y_2 + \frac{1}{|y_1|}} \right)^2 dy_1 dy_2 \\
+ \sum_{j=1}^{N-1} \int_{\tilde{B}_j^-} |y_1|^{-1} \log\left( \frac{1}{y_2 + \frac{1}{|y_1|}} \right)^2 dy_1 dy_2.
\]

Let \( I^+ \) and \( I^- \) denote the first and second summands, respectively. We bound each of these quantities separately, which is the content of the following two lemmas. Recall that \( \delta_j = v_{j+1} - v_j > 0 \).
6.1. Proof of main theorem

**Lemma 6.1.2.** We have following estimate,

\[
I^+ \geq C_+ \log N \sum_{j \in \Lambda_+} \left[ \delta_j \log^2 \delta_j^{-1} + \delta_j \log \delta_j^{-1} \right] - C_+ \sum_{j \in \Lambda_+} \delta_j \log^3 \delta_j^{-1},
\]

where,

\[
\Lambda_+ = \{ j \in \mathbb{N} : 1 \leq j \leq N, \text{ and } \delta_j N > 1 \},
\]

and \(0 < C_+ < 1\) is independent of \(N\).

**Lemma 6.1.3.** We have the following estimate:

\[
I^- \geq C_- \sum_{j=1}^{N-1} \delta_j^{-1} \log^3 (\delta_j^{-1})
\]

where \(0 < C_- < 1\) is independent of \(N\).

Assuming the previous Lemmata, we now prove the main theorem.

**Proof.** We have already shown that:

\[
||\mathcal{H} \Theta f||^2 \geq I^+ + I^-.
\]

So, applying the estimate for \(I_+\) from Lemma 6.1.2 and the estimate for \(I_-\) from Lemma 6.1.3, we obtain:

\[
||\mathcal{H} \Theta f||^2 \geq C_+ \sum_{j \in \Lambda_+} \left[ \log N \left[ \delta_j \log^2 \delta_j^{-1} + \delta_j \log \delta_j^{-1} \right] - \delta_j \log^3 \delta_j^{-1} \right] + C_- \sum_{j=1}^{N} \delta_j \log^3 (\delta_j^{-1})
\]

Choosing \(C_0 = \min\{C_+, C_-\}\), the above gives:

\[
||\mathcal{H} \Theta f||^2 \geq C_0 \log N \sum_{j \in \Lambda_+} \delta_j \left[ \log^2 \delta_j^{-1} + \log \delta_j^{-1} \right] + C_0 \sum_{j \in \Lambda_-} \delta_j \log^3 \delta_j^{-1}
\]

where,

\[
\Lambda_- = \{ j \in \mathbb{N} : 1 \leq j \leq N, \text{ and } \delta_j N \leq 1 \}.
\]

Recalling that \(||f||^2 \leq C \log N\), we obtain:

\[
\frac{||\mathcal{H} \Theta f||^2}{||f||^2} \geq C \log N \left[ \log N \sum_{j \in \Lambda_+} \delta_j \left[ \log^2 \delta_j^{-1} + \log \delta_j^{-1} \right] + \sum_{j \in \Lambda_-} \delta_j \log^3 \delta_j^{-1} \right]
\]

\[
\geq C \sum_{j \in \Lambda_+} \delta_j \left[ \log^2 \delta_j^{-1} + \log \delta_j^{-1} \right] + \frac{C}{\log N} \sum_{j \in \Lambda_-} \delta_j \log^3 \delta_j^{-1}.
\]

This proves the main theorem, modulo proofs of Lemma 6.1.2 and Lemma 6.1.3. \(\square\)
6.1.5 Proof of Lemma for $I^+$

Let us first prove Lemma 6.1.2. First, for any $(y_1, y_2) \in B^+_j$, we know that:

$$y_2 > \frac{1 + v_{j+1}}{|y_1|} \Rightarrow y_2 + \frac{1 + v_{j+1}}{|y_1|} < 2y_2$$

$$\Rightarrow (y_2 + \frac{1 + v_{j+1}}{|y_1|})^{-1} > \frac{1}{2|y_2|}.$$  

This gives the inequality:

$$I^+ \geq \sum_{j=1}^{N-1} \int_{\tilde{B}^+_j} \frac{1}{|y_1|} \log^2 \left( \frac{1}{2y_2} \right) dy_2 dy_1$$

Now, we have some flexibility in how we determine the bounds of integration in the above double integral. We choose to think of $\tilde{B}^+_j$ as:

$$\tilde{B}^+_j = \{-N < y_1 < -1, |y_1| > \frac{1 + v_{j+1}}{y_2}, 0 < y_2 < v_{j+1} - v_j\},$$

so our bounds on $y_2$ are "independent" of $y_1$. To determine the bounds for $y_1$, we make an observation regarding the quantities $\delta_j = v_{j+1} - v_j$ and $N$. Namely, if $(y_1, y_2) \in B^+_j$, then we have:

$$y_2 > \frac{1 + v_{j+1}}{|y_1|} \Rightarrow v_{j+1} - v_j > \frac{1 + v_{j+1}}{|y_1|} > \frac{1}{N} \Rightarrow (v_{j+1} - v_j)N > 1$$

$$\Rightarrow \delta_j^{-1} = \frac{1}{v_{j+1} - v_j} < N$$

In other words, in order for $\tilde{B}^+_j$ to be non-empty, we must have that $\delta_j N > 1$. With this in mind, we let:

$$\Lambda_+ := \{1 \leq j \leq N-1 : \delta_j N > 1\}.$$

We then take for $y_1$ the following bounds:

$$-N < y_1 < -\frac{1 + v_{j+1}}{y_2},$$

which holds, since $y_2 < \delta_j$, and $\delta_j < 1$ uniformly in $j$.  

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Now, we arrive at the following somewhat-messy inequality:

\[
I^+ \geq \sum_{j=1}^{N-1} \int_{\mathcal{B}^+_j} \int \frac{1}{|y_1|} \log^2 \left( \frac{1}{2y_2} \right) dy_2 dy_1 \\
\geq C_1 \sum_{j \in \Lambda^+} \int_{0}^{\delta_j} \left( \int_{-N}^{-y_2} \frac{1}{|y_1|} \right) \log^2 \left( \frac{1}{2y_2} \right) dy_2.
\]

Here, the constant \(0 < C_1 < 1\) is included to account for the constants we removed in the bounds of integration for the inner integral. Integrating this inner integral gives:

\[
I^+ \geq C_1 \sum_{j \in \Lambda^+} \int_{0}^{\delta_j} \left[ \log N \log^2 \left( \frac{1}{2y_2} \right) - \log^3 \left( \frac{1}{2y_2} \right) \right] dy_2 
\]

(6.6)

Here, \(0 < C_2 < 1\) is some absolute constant which allows us to remove the \(\frac{1}{2}\) from the logarithms. Now, recall a simple recursion for integrating logarithms divided by polynomials. For \(0 < A < B \leq \infty\), we have:

\[
\int_{A}^{B} \frac{\log^m x}{x^n} dx = - \frac{\log^m x}{(n-1)x^{n-1}} \bigg|_{x=A}^{x=B} + \frac{m}{n-1} \int_{A}^{B} \frac{\log^{m-1} x}{x^n} dx.
\]

This follows via a simple integration-by-parts. Applying this formula recursively to the first logarithm in (6.6) gives:

\[
\log N \int_{\delta_j^{-1}}^{\delta_j} \frac{\log^2 s}{s^2} ds = C_3 \log N \left[ \delta_j \log^2 \delta_j^{-1} + \delta_j \log \delta_j^{-1} + \delta_j \right],
\]

(6.7)

and to the second

\[
\int_{\delta_j^{-1}}^{\delta_j} \frac{\log^3 s}{s^2} ds = C_4 \left[ \delta_j \log^3 \delta_j^{-1} + \delta_j \log^2 \delta_j^{-1} + \delta_j \log \delta_j^{-1} + \delta_j \right].
\]

(6.8)

Here, \(0 < C_3, C_4 < 2\) are chosen to take care of the constants of integration in the recursion formula for the logarithm. Combining (6.6) with (6.7) and (6.8) gives:

\[
I^+ \geq C_+ \log N \sum_{j \in \Lambda^+} \left[ \delta_j \log^2 \delta_j^{-1} + \delta_j \log \delta_j^{-1} \right] - C_+ \sum_{j \in \Lambda^+} \delta_j \log^3 \delta_j^{-1}.
\]

Where \(0 < C_+ < 1\) is again some absolute constant.
6.1. Proof of main theorem

6.1.6 Proof of Lemma for $I^-$

Now, let us prove Lemma 6.1.3, which concerns $I^-$. We make some initial observations. First, we have the estimate:

$$y_2 < \frac{1 + v_{j+1}}{|y_1|} \Rightarrow y_2 + \frac{1 + v_{j+1}}{|y_1|} \leq \frac{4}{|y_1|}.$$

Hence, we know that:

$$I^- = \sum_{j=1}^{N-1} \int_{B_j^-} \frac{1}{|y_1|} \log \left( \frac{1}{y_2 + \frac{1 + v_{j+1}}{|y_1|}} \right)^2 dy_1 dy_2$$

$$\geq \sum_{j=1}^{N-1} \int_{B_j^-} \frac{1}{|y_1|} \log^2 \left( \frac{|y_1|}{4} \right) dy_1 dy_2$$

We also have the two estimates:

$$y_2 < \frac{1 + v_{j+1}}{|y_1|} \Rightarrow |y_1| < \frac{1 + v_{j+1}}{y_2}$$

and also,

$$y_2 < v_{j+1} - v_j \Rightarrow \delta_j^{-1} < \frac{1}{y_2} < \frac{1 + v_{j+1}}{y_2}.$$

Therefore, we have the following estimate for $I^-$. Namely:

$$I^- \geq \sum_{j=1}^{N-1} \int_{B_j^-} \frac{1}{|y_1|} \log^2 \left( \frac{|y_1|}{4} \right) dy_1 dy_2$$

$$= \sum_{j=1}^{N-1} \int_{\delta_j^0}^{\delta_j^y} dy_2 \left( \int_{1}^{y_2} \frac{1}{|y_1|} \log^2 \left( \frac{|y_1|}{4} \right) dy_1 \right)$$

$$\geq C_1 \sum_{j=1}^{N-1} \delta_j^{-1} \int_{1}^{\delta_j^y} \frac{1}{|y_1|} \log^2 (|y_1|) dy_1$$

$$\geq C_- \sum_{j=1}^{N-1} \delta_j^{-1} \log^2 (\delta_j^{-1}).$$

Here, the constants $0 < C_1, C_- < 1$ are again absolute constants, simply chosen to clean-up the integration. Notice that our final estimate for $I^-$ places no restriction on our indexing set.
6.2 Lower bounds for specific direction sets

Having proven Theorem 6.1.1, we now apply it to several direction sets of interest. We begin by considering two well-known examples: the uniformly spaced direction set and a lacunary direction set.

6.2.1 Uniform and Lacunary Direction Sets

Let us first consider the uniformly-spaced direction set, given by:

$$\Theta_U := \{(1, v_j) \in \mathbb{R}^2 : v_j = \frac{j}{N}, \text{ for each } 1 \leq j \leq N\},$$

and was first considered in the work of Joonil Kim [8]. For any $1 \leq j \leq N-1$, we have $\delta_j = \frac{j+1}{N} - \frac{j}{N} = \frac{1}{N}$, which shows that $(v_{j+1} - v_j)N = 1$. Applying the operator norm estimate of Theorem 6.1.1, we obtain:

$$\|\mathcal{H}_U\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \geq \frac{C}{\log N} \sum_{j=1}^{N-1} \delta_j \log^3 \delta_j^{-1} \geq \frac{1}{\log N} \sum_{j=1}^{N-1} \frac{1}{N} \log^3 (N) \geq \frac{1}{4} \log^2 N.$$

As such, our main result is able to recover the sharp $L^2 \to L^2$ norm estimate for uniform direction sets,

$$\|\mathcal{H}_{\Theta_U}\|_{2 \to 2} \geq C \cdot \log \left( \#\Theta_U \right),$$

a result which was first proven in [8], and whose proof we discussed in the previous chapter. We can also test a lacunary set of directions. For our purposes, let:

$$\Theta_L = \{(1, v_j) \in \mathbb{R}^2 : v_j = 2^{j-N-1} \text{ for } 1 \leq j \leq N\}$$

Then, for any $1 \leq j \leq N-1$, we have that:

$$\delta_j = 2^{-(N+1-j-1)} - 2^{-(N+1-j)} = 2^{-(N+1-j)}.$$

As a consequence, we must have:

$$\delta_j N > 1 \iff \log \left( 2^{-(N+1-j)}N \right) > 0 \iff j - N - 1 + \log N > 0 \iff j > N + 1 - \log N$$
6.2. Lower bounds for specific direction sets

where, in the middle implication, we lose one direction since we drop a factor of log 2 from the calculation. Applying Theorem 6.1.1, we thus obtain:

$$|| \mathcal{H}_{\Theta} ||_{2\rightarrow 2} \geq C \sum_{j=N+1-\log N}^{N-1} \delta_j \left[ \log^2 \delta_j + \log \delta_j \right] + \frac{C}{\log N} \sum_{j=1}^{N+1-\log N} \delta_j \log^3 \delta_j^{-1}$$

Let us evaluate each of these sums separately. For the first sum, we ignore the second logarithm in the sum, since it is dominated by the log^2 \delta_j term. This gives:

$$C \sum_{j=N+1-\log N}^{N-1} \delta_j \log^2 \delta_j = C \sum_{j=N+1-\log N}^{N-1} 2^{j-N-1} (j - N - 1)^2.$$ 

This final quantity is summable. Hence, we obtain:

$$C \sum_{j=N+1-\log N}^{N-1} \delta_j \log^2 \delta_j = O(1).$$

Similarly, for the second sum, we have:

$$\frac{C}{\log N} \sum_{j=1}^{N+1-\log N} \delta_j \log^3 \delta_j^{-1} = \frac{C}{\log N} \sum_{j=1}^{N+1-\log N} 2^{j-N-1} (N + 1 - j)^3$$

which is again a convergent series. Even worse, this second term approaches zero as N → ∞ because of the (log N)^{-1} term. As a consequence, we see that Theorem 6.1.1 provides the trivial lower bound:

$$|| \mathcal{H}_L ||_{2\rightarrow 2} \geq C,$$ for some absolute constant 0 < C < 1.

This illustrates a key weakness of Theorem 6.1.1. Namely, let \( \Theta \subset \mathbb{S}^1 \) be some direction set and let \( \delta_j \) denote the difference of slopes for adjacent directions in \( \Theta \). Then, if \( \delta_j \log^\alpha \delta_j \) is summable in \( j \in \mathbb{N} \), Theorem 6.1.1 will not provide any useful information. Fortunately, there are plenty of direction sets which avoid this summability.
6.2. Lower bounds for specific direction sets

6.2.2 Direction sets $\Theta$ and $\mathcal{H}_\Theta$ with maximal norm

In this subsection, we prove a sharp lower bound estimate for a general class of direction sets, which are related to the uniform direction sets of [8]. Before attacking this more general case, we consider an illustrative example.

Fix some $\frac{1}{1000} < \alpha < 1$. We then define a direction set:

$$\Theta_\alpha = \{(1, v_j) \in \mathbb{R}^2 : 0 = v_1 < v_2 < \cdots < v_N = 1\}$$

via the formula,

$$v_{j+1} - v_j = C_N(1 + j)^{-\alpha}, \text{ for a constant } C_N \text{ determined below} >$$

By construction, we choose $C_N$ so that:

$$C_N \sum_{j=1}^{N-1} (1 + j)^{-\alpha} = \sum_{j=1}^{N-1} (v_{j+1} - v_j) = v_N - v_1 = 1. \quad (6.9)$$

From the integral test for series, there is an absolute constant $C_0 > 0$ such that:

$$\sum_{j=1}^{N-1} (v_{j+1} - v_j) = C_0 \int_1^{N-1} \frac{C_N ds}{(1 + s)^\alpha} = C_N \times \frac{C_0 N^{1-\alpha}}{2}. \quad (6.10)$$

where $C_0$ is perhaps adjusted to account for the lower limit of integration. Combining (6.9) and (6.10) gives:

$$1 = C_N \times \frac{C_0 N^{1-\alpha}}{2} \Rightarrow C_N = \frac{2N^{\alpha-1}}{C_0}$$

We therefore let $C_N = c_0 N^{\alpha-1}$ for some $0 < c_0 < 1$ which is absolute.

Let us now apply Theorem 6.1.1 to $\mathcal{H}_\Theta$. From our previous examination of $C_N$, we now know that:

$$\delta_j = v_{j+1} - v_j = c_0 N^{\alpha-1}(1 + j)^{-\alpha}.$$  

As a consequence, we obtain the following key inequality:

$$\delta_j N = \frac{c_0 N^{\alpha-1}}{(1 + j)^\alpha} \times N = c_0 \left(\frac{N}{1 + j}\right)^\alpha \geq 1, \text{ for every } 1 \leq j \leq N - 1.$$
6.2. Lower bounds for specific direction sets

Our main theorem then furnishes the following \( L^2 \to L^2 \) estimate,

\[
\| \mathcal{H}_{\Theta_\alpha} \|_{2 \to 2}^2 \geq C \sum_{j=1}^{N-1} \delta_j \log^2 \delta_j^{-1}
\]

\[
\geq c_0 C N^{\alpha-1} \sum_{j=1}^{N-1} (1 + j)^{-\alpha} \left( \log \left[ N^{1-\alpha} (1 + j)^\alpha \right] \right)^2
\]

\[
\geq C'_\alpha N^{\alpha-1} \log^2 N \sum_{j=1}^{N-1} (1 + j)^{-\alpha}.
\]

Here, the constant \( C_\alpha \) depends upon \( \frac{1}{1000} < \alpha < 1 \), but is independent of \( N = \# \Theta_\alpha \). Again, because:

\[
\sum_{j=1}^{N-1} (1 + j)^{-\alpha} \approx N^{1-\alpha}
\]

we take square-roots and arrive at the lower bound:

\[
\| \mathcal{H}_{\Theta_\alpha} \|_{2 \to 2} \geq C'(\alpha) \log N.
\]

Therefore, so long as we restrict \( \frac{1}{1000} < \alpha < 1 \), the class of direction sets \( \Theta_\alpha \) generate maximal directional Hilbert transforms of maximal operator norm.
6.2. Lower bounds for specific direction sets

6.2.3 Proof of Corollary 6.2.1

Using the previous example as motivation, we now prove Corollary 6.2.1. For convenience, we restate the corollary below.

**Corollary 6.2.1.** Let \( \Theta = \{(1, v_j) : 0 = v_1 < v_2 < \cdots < v_N\} \) be a set of directions in the plane, with \( N \gg 1 \). Suppose there were numbers \( a_1, \ldots, a_N > 0 \) so that,

\[
\frac{v_j}{\sum_{k=1}^{j-1} a_k} \quad \text{for each } 1 \leq j \leq N - 1,
\]

and satisfying the estimates,

\[
\sup_k a_k \leq C_1 N^{\epsilon_2} \quad \text{and} \quad \sum_{k=1}^{N-1} a_k \leq C_0 N^{\epsilon_1},
\]

where \( 0 \leq \epsilon_2 \leq \epsilon_1 < \infty \) and the \( C_0, C_1 \) are constants independent of \( N \). Then, there exists some constant \( C \), which is independent of \( N \) and the \( a_n \), such that

\[
\|H_\Theta\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \geq C(\epsilon_1 - \epsilon_2)^{3 \log N}
\]

The significance of this corollary is in the \( \epsilon_1 \) and \( \epsilon_2 \) terms, which will depend upon the direction set \( \Theta \).

**Proof.** Suppose we have a direction set \( \Theta \) satisfying (6.11) for some \( a_1, \ldots, a_N \) which satisfy estimates (6.12) for some \( 0 \leq \epsilon_2 \leq \epsilon_1 < \infty \) and absolute constants \( C_0, C_1 \). Notice first, for any \( 1 \leq j \leq N - 1 \), we have:

\[
\delta_j = v_{j+1} - v_j = \frac{\sum_{k=1}^{j} a_k}{\sum_{k=1}^{N-1} a_k} - \frac{\sum_{k=1}^{j-1} a_k}{\sum_{k=1}^{N-1} a_k} = \frac{a_j}{A_N}, \quad \text{where } A_N = \sum_{k=1}^{N-1} a_k
\]

Let us now apply Theorem 6.1.1 to \( H_\Theta \). First, for the summand in the \( A_+ \) term, we have:

\[
\delta_j \log^2 \delta_j^{-1} = \frac{a_j}{A_N} \log^2 (a_j^{-1} A_N) = \frac{a_j}{A_N} \left[ \log A_N - \log a_j \right]^2.
\]

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Estimates (6.12) then imply:
\[
\delta_j \log^2 \delta_j^{-1} \geq \frac{a_j}{A_N} \left[ \log(C_0^{-1} N^{\epsilon_1}) - \log(C_1 N^{\epsilon_2}) \right]^2
\]
\[
= \frac{a_j}{A_N} \left[ \epsilon_1 \log N - \epsilon_2 \log N - \log(C_0 C_1) \right]^2
\]
\[
= (\epsilon_1 - \epsilon_2)^2 \frac{a_j}{A_N} \log^2 \left( \frac{N}{C_0 C_1} \right)
\]

(6.13)

A similar application of estimate (6.12) for the \( \Lambda_- \) summand gives:
\[
\delta_j \log^3 \delta_j^{-1} \geq \frac{a_j}{A_N} (\epsilon_1 - \epsilon_2)^3 \log^3 \left( \frac{N}{C_0 C_1} \right)
\]

(6.14)

Now, choose some \( C_2 < 1 \) so that:
\[
\log^3 \left( \frac{N}{C_0 C_1} \right) \geq C_2 \log^3 N \text{ and } \log^2 \left( \frac{N}{C_0 C_1} \right) \geq C_2 \log^2 N.
\]

Combining estimates (6.13) and (6.14) with our lower bound gives:
\[
\|H_\Theta\|_{2 \rightarrow 2}^2 \geq C \sum_{j \in \Lambda^+} \delta_j \log^2 \delta_j^{-1} + \frac{C}{\log N} \sum_{j \in \Lambda^+} \delta_j \log^3 \delta_j^{-1}
\]
\[
\geq C \cdot C_2 (\epsilon_1 - \epsilon_2)^2 \log^2 N \sum_{j \in \Lambda^+} \frac{a_j}{A_N}
\]
\[
+ C \cdot C_2 (\epsilon_1 - \epsilon_2)^3 \log^2 N \sum_{j \in \Lambda^-} \frac{a_j}{A_N}.
\]

(6.15)

Now, suppose that \( \epsilon_1 - \epsilon_2 \geq 1 \). Then,
\[
\delta_j N = \frac{a_j}{N^{\epsilon_1}} \times N \leq N^{\epsilon_2 - \epsilon_1} \times N = N^{1 - (\epsilon_1 - \epsilon_2)} \leq 1.
\]

So, when \( \epsilon_1 - \epsilon_2 \geq 1 \), we have \( \Lambda_\neq = \{1, 2, ..., N - 1\} \) and \( \Lambda_+ = \emptyset \). Appealing to (6.15) then gives,
\[
\|H_\Theta\|_{2 \rightarrow 2}^2 \geq C_2 \cdot C(\epsilon_1 - \epsilon_2)^2 \log^2 N \sum_{j=1}^{N-1} \frac{a_j}{A_N} = C_2 \cdot C(\epsilon_1 - \epsilon_2)^3 \log^2 N,
\]
where the final equality follows since \( A_N = \sum_j a_j \). This gives the result when \( \epsilon_1 - \epsilon_2 \geq 1 \).
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Now suppose that $\epsilon_1 - \epsilon_2 < 1$. Then, we know that $(\epsilon_1 - \epsilon_2)^3 < (\epsilon_1 - \epsilon_2)^2$. Hence, (suppressing constants for the moment) equation (6.15) becomes:

$$||H_\Theta||^2_{l^2,N} \geq (\epsilon_1 - \epsilon_2)^2 \log^2 N \sum_{j \in \Lambda_+} \frac{a_j}{A_N} + (\epsilon_1 - \epsilon_2)^3 \log^2 N \sum_{j \in \Lambda_-} \frac{a_j}{A_N}$$

$$\geq (\epsilon_1 - \epsilon_2)^3 \log^2 N \sum_{j \in \Lambda_+} \frac{a_j}{A_N} + (\epsilon_1 - \epsilon_2)^3 \log^2 N \sum_{j \in \Lambda_-} \frac{a_j}{A_N}$$

$$= (\epsilon_1 - \epsilon_2)^3 \log^2 N \sum_{j=1}^{N-1} \frac{a_j}{A_N}.$$

Here, the final equality follows since $\Lambda_+ \cup \Lambda_- = \{1, 2, ..., N - 1\}$. Again, since $\sum_j a_j = A_N$, we must have:

$$||H_\Theta||^2_{l^2,N} \geq C \cdot C_2 (\epsilon_1 - \epsilon_2)^3 \log^2 N$$

Taking square-roots again gives the result when $\epsilon_1 - \epsilon_2 < 1$. \hfill \Box

Corollary 6.2.1 is included to demonstrate the flexibility of Theorem 6.1.1. Heuristically, given any $\frac{1}{2} < \beta < 1$, Corollary 6.2.1 suggests that, for every large $N$, there is a direction set $\Theta_\beta$ such that Theorem 6.1.1 provides the lower bound:

$$||H_{\Theta_\beta}||^2_{l^2,N} \geq \sqrt{C \cdot C_2} (\log N)^\beta.$$  \hspace{1cm} (6.16)

Indeed, for our suspected $\Theta_\beta$, to prove (6.16), it suffices to find $a_1, ..., a_N$ satisfying (6.11) and (6.12) with,

$$\epsilon_1 - \epsilon_2 = \left(\log N\right)^{\frac{2\beta - 2}{3}}.$$

Then, Corollary 6.2.1 furnishes the lower bound:

$$||H_{\Theta_\beta}||^2_{l^2,N} \geq \sqrt{C \cdot C_2} (\epsilon_1 - \epsilon_2)^{\frac{3}{2}} \log N$$

$$= \sqrt{C \cdot C_2} (\log N)^{\frac{2\beta - 2}{3}} \log N$$

$$= \sqrt{C \cdot C_2} (\log N)^{\beta - 1} \log N$$

$$= \sqrt{C \cdot C_2} (\log N)^\beta$$

As a final example, we construct such a direction set associated to each $\frac{1}{2} < \beta < 1$.

Let $N >> 1$ be some large, fixed integer. Similarly, fix an exponent $\frac{1}{2} < \beta < 1$. Define a new quantity,

$$\gamma = \left(\log N\right)^{\frac{2\beta - 2}{3}}.$$
which depends upon $N$. Notice that, since $\frac{1}{2} < \beta < 1$, we know that $\frac{2\beta - 2}{3} < 0$. Hence, we must have $\gamma << 1$. Now, let $a_1, ..., a_N > 0$ be defined by $a_j = j^{\gamma - 1}$ for each $1 \leq j \leq N - 1$. Using the integral test, we know that

$$\sum_{j=1}^{N-1} a_j = \sum_{j=1}^{N-1} \frac{1}{j^{1-\gamma}} = C_0 \int_1^{N-1} \frac{dx}{x^{1-\gamma}} = CN^\gamma,$$

where $0 < C_0, C < 1$ are independent of $N$ and $\beta$. Again, since $\gamma - 1 < 0$, we know that:

$$\max_{1 \leq j \leq N-1} a_j = \max_{1 \leq j \leq N-1} j^{\gamma - 1} \leq 1 = N^0.$$

Hence, in the language of Corollary 6.2.1, we see that $a_1, ..., a_N$ satisfy (6.12) with $\epsilon_1 = \gamma$ and $\epsilon_2 = 0$. Now, define a direction set $\Theta_\beta$ via the recursive formula:

$$v_1 = 0 \quad \text{and then} \quad v_{j+1} - v_j = \delta_j = \frac{a_j}{CN^\gamma}, \quad \text{for } 1 \leq j \leq N - 1,$$

Since $\epsilon_1 - \epsilon_2 = \gamma$, Corollary 6.2.1 gives:

$$\|\mathcal{H}_{\Theta_\beta}\|_2 \geq C(\epsilon_1 - \epsilon_2)^{\frac{3}{2}} \log N = C\gamma^{\frac{3}{2}} \log N = C' \left( \log N \right)^{\frac{3(\beta - 2)}{3}} \log N = C' \left( \log N \right)^{\beta - 1} \log N = C' \left( \log N \right)^{\beta}.$$

This is precisely the lower bound we desired. We are done.
Bibliography


Bibliography


