Infrared Quantum Information

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Abstract

Scattering amplitudes in massless gauge field theories have long been known to give rise to infrared divergent effects from the emission of very low energy gauge bosons. The traditional way of dealing with those divergences has been to abandon the idea of measuring amplitudes by only focusing on inclusive cross-sections constructed out of physically equivalent states. An alternative option, found to be consistent with the S-matrix framework, suggested to *dress* asymptotic states of charged particles by shockwaves of low energy bosons. In this formalism, the clouds of soft bosons, when tuned appropriately, cancel the usual infrared divergences occurring in the standard approach. Recently, the dressing approach has received renewed attention for its connection with newly discovered asymptotic symmetries of massless gauge theories and its potential role in the black hole information paradox.

We start by investigating quantum information properties of scattering theory while having only access to a subset of the outgoing state. We give an exact formula for the von Neuman entanglement entropy of an apparatus particle scattered off a set of system particles and show how to obtain late-time expectation values of apparatus observables.

We then specify to the case of quantum electrodynamics (QED) and gravity where the unobserved system particles are low energy photons and gravitons. Using the standard inclusive cross-section formalism, we demonstrate that those soft bosons decohere nearly all momentum superpositions of hard particles. Repeating a similar computation using the dressing formalism, we obtain an analogous result: In either framework, outgoing hard momentum states at late times are fully decohered from not having access to the soft bosons.

Finally, we make the connection between our results and the framework of asymptotic symmetries of QED and gravity. We give new evidence for the use of the dressed formalism by exhibiting an inconsistency in the scattering of wavepackets in the original inclusive cross-section framework.

Lay Summary

Field theories like quantum electrodynamics and perturbative gravity have long been known to have issues arising from the emission of long wavelength photons and gravitons. The standard approach to curing those problems has been to accept that such particles can not be observed by finite sized detectors and trace them out of any computation. However, a more recent proposal suggests that using states of charged matter dressed by incoming radiation in a very specific way can also cure the infrared problems of the theory.

In this thesis, we investigate quantum information properties of the long wavelength radiation after scattering. We evaluate relevant quantities such as the entanglement entropy of the radiation and demonstrate that both approaches predict complete decoherence of the charged particles at late times. We then demonstrate that the dressed formalism is the correct framework to perform scattering.

Preface

A version of chapter 2 has been uploaded to arxiv.org. Dan Carney, Laurent Chaurette & Gordon Semenoff, *Scattering with partial information, arXiv:1606.0310*. My main contributions were related to establishing the setup in terms of density matrices and calculations of entanglement entropy.

A version of chapter 3 has been published. Dan Carney, Laurent Chaurette, Dominik Neuenfeld & Gordon Semenoff, *Infrared quantum information*, Phys.Rev.Lett. 119 (2017) no.18, 180502. My role was primarily related to calculations of the decoherence exponent and the proof of its positivity.

A version of chapter 4 has been published. Dan Carney, Laurent Chaurette, Dominik Neuenfeld & Gordon Semenoff, *Dressed infrared quantum information*, Phys.Rev. D97 (2018) no.2, 025007. My contribution was mostly related to calculations on the damping factor D and its connection to decoherence

A version of chapter 5 was submitted for publication and is currently undergoing peer-review. Dan Carney, Laurent Chaurette, Dominik Neuenfeld & Gordon, Semenoff, *On the need for soft dressing*, arXiv:1803.02370. My contributions were mostly related to decoherence calculations for entangled superpositions and wavepackets. I worked on the evaluation of the decoherence exponents and the proof of their positivity as well as the relation of the decoherence in terms of conserved charges.

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Chapter 1

Introduction

It has long been known that field theories with massless gauge bosons are plagued with infrared divergences which effectively force the transition amplitudes between any two states to be exactly zero. Historically, the method of choice for dealing with such divergences was introduced by Bloch and Nordsieck for quantum electrodynamics (QED) [1] and Weinberg for gravity [2] and has been to evaluate inclusive cross-sections between every physically indistinguishable state. While this approach agrees with experiments on the transition probabilities between various states, it has the shortcoming of abandoning the S-matrix description of the theory as amplitudes are all zero. A second framework consistent with an S-matrix picture was proposed by Chung and Faddeev-Kulish [3, 4], suggesting that asymptotic states of charged particles could be *dressed* by a cloud of soft radiation. When the cloud is chosen to be a specific coherent state of soft photons and gravitons, the S-matrix elements between such states becomes non-singular. The two approaches were then mostly considered to be equivalent: choosing to evaluate probabilities could be done in either framework and would simply come to a matter of choice.

Seemingly unrelated recent findings [5–7] have demonstrated the existence of an infinite number of broken symmetries in QED and gravity leading to an infinitely degenerate vacuum for the theory. While scattering would allow transitioning from one vacuum to an other, the vanishing of the S-matrix can be seen as a statement about the conservation of the charges of those broken symmetries. A recent paper [8] showed there is in fact a strong connection between the conservation of these charges and Faddeev-Kulish states, leading to believe that dressed states could be the actual states found in nature.

Additionally, a new proposition [9] suggested dressed states could potentially hold the key to the resolution of the Black hole information paradox: If a black hole was formed from the collision of high energy dressed states, the long wavelength radiation contained in the dressing would not fall in the black hole and could perhaps hold enough information to help distinguish between different states after the black hole evaporated.

In this dissertation, we investigate quantum information properties of the long wavelength radiation emitted from scattering using both the inclusive cross section and dressed approaches. We investigate the late time decoherence effects found in each formalism when radiation is left unobserved and argue that scattering requires the use of Faddeev-Kulish states. First, we wish to review some of the previous literature on the key concepts we will be using in the thesis. In section 1.1, we review how the infrared catastrophe comes about for QED and PG. Sections 1.2 and 1.3 give an overview of the inclusive cross-section and dressing approaches respectively. Finally, section 1.4 reviews the program of broken asymptotic symmetries of QED and gravity and how charge conservation is linked to the vanishing of the S-matrix.

1.1 Infrared Catastrophe

When dealing with massless gauge theories like QED, one finds that the probability of charged particles to emit photons diverges as the energy of the gauge bosons go to zero. Every scattering event is then dominated by outgoing states which contain an infinite amount of soft photons rendering the probability to emit only a finite number of photons to be zero. This is the infrared catastrophe. In this section, we review how soft gauge boson emission implies divergent amplitudes between any two Fock space states.

We will start by reviewing the amplitudes of Feynman diagrams containing the emission of bremsstrahlung. Let us consider a scattering process $\alpha \to \beta$ where one of the outgoing legs emits a photon of momentum k. This adds a propagator to the diagram which has the effect of multiplying the amplitude by a factor of

$$\left[ie\frac{-i\left(2p^{\mu}+k^{\mu}\right)}{(p+k)^2+m^2-i\epsilon}\right] \tag{1.1}$$
$$\left[ie\gamma^{\mu}\frac{-i(\not{p}+\not{k})+m}{(p+k)^2+m^2-i\epsilon}\right],$$

for spin 0 and spin 1/2 particles respectively. Here, e, p and m are the charge, momentum and mass of the particle emitting the photon. In the case when the photon momentum k is nearly zero, both expressions have the same behaviour

$$\frac{ep^{\mu}}{p\cdot k-i\epsilon}. \tag{1.2}$$

To obtain this result we used properties of gamma matrices, the fact that p is on-shell ($p^2=-m^2$) while keeping only the terms at lowest order in

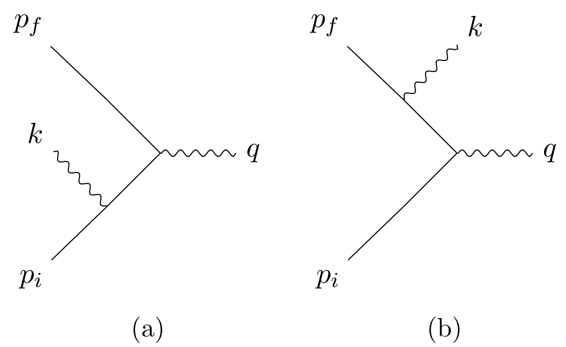

Figure 1.1: The two diagrams contributing to first order bremsstrahlung emission for $1 \to 1$ potential scattering. The soft photon can either be emitted a) before or b) after scattering off the potential

the photon momentum $\mathbf{k}$. This expression is indeed universal in the sense that it does not depend on the spin of the emitting particle. If instead of an outgoing line, it was an incoming line that emitted the soft photon, the denominator in equation (1.1) would instead behave as $(p+k)^2 \to (p-k)^2$. After taking the $k \to 0$ limit, this effectively changes the sign of the pole to

$$\frac{ep^\mu}{-p \cdot k - i\epsilon}. \tag{1.3}$$

Accounting for both possible first order emissions (fig 1.1), the amplitude gets modified by

$$M^{(1)}_{\beta\alpha}(k,l) = M^{(0)}_{\beta\alpha} \frac{1}{\sqrt{(2\pi)^3 2|\mathbf{k}|}} \sum_{n \in \alpha,\beta} \frac{\eta_n e_n p_n \cdot \epsilon^*_l(\mathbf{k})}{p_n \cdot k - i\eta_n \epsilon}, \tag{1.4}$$

where the index n runs over all particles in α and β and η_n is defined as 1(-1) if particle n is incoming(outgoing). The factor of $\frac{1}{\sqrt{(2\pi)^3 2|\mathbf{k}|}}$ came from the normalization of the emitted photon wavefunction while $\epsilon_l(\mathbf{k})$ denotes its polarization. For gravity, the situation is completely analogous: the poles take the form

$$\frac{M_p^{-1} \eta_n \, p_n^\mu p_n^\nu}{p_n \cdot k - i\eta_n \epsilon}, \tag{1.5}$$

where M_p is the Plank mass. The amplitudes are then modified accordingly

$$M^{(1)}_{\beta\alpha}(k,l) = M^{(0)}_{\beta\alpha}\frac{1}{\sqrt{(2\pi)^3 2|\mathbf{k}|}}\sum_{n\in\alpha,\beta}\frac{M_p^{-1}\eta_n p_n^\mu p_n^\nu \epsilon^*_{\mu\nu,l}(\mathbf{k})}{p_n\cdot k - i\eta_n\epsilon}, \tag{1.6}$$

following the same infrared behavior as for QED. We will therefore solely focus on photon emission from now on and simply give the results for gravity at the end.

The probability to emit any low energy photon is given by the square of the amplitude summed over all possible outgoing photons

$$\begin{aligned}
P^{(1)}_{\beta\alpha} &= \int d^3k \sum_{l=1}^{2}|M^\mu_{\beta\alpha}(k,l)|^2 \\
&= P^{(0)}\int\frac{d^3k}{(2\pi)^3 2k}\sum_{l=1}^{2}\sum_{n,m\in\alpha,\beta} e_n e_m\eta_n\eta_m\frac{p_n^\mu p_m^\nu\epsilon^*_{\mu,l}(\mathbf{k})\epsilon_{\nu,l}(\mathbf{k})}{[p_n\cdot k - i\eta_n\epsilon]\,[p_m\cdot k - i\eta_m\epsilon]} \\
&= P^{(0)}I(\alpha,\beta) \\
&\propto P^{(0)}\int_\lambda^\Lambda dk\frac{k^2}{k^3},
\end{aligned} \tag{1.7}$$

which has a logarithmic divergence for small k. The probability of emitting a soft photon therefore seems to be infinitely larger than having no emission. We could then add a second soft photon and notice that the amplitude gets multiplied by the square of the soft factor found in equation (1.4), that is every emission is independent from each other and the amplitude of emitting N soft photons and M soft gravitons is simply

$$M^{(N)}_{\beta\alpha}(k_1,...,k_N,k'_1,...,k'_M) \to M^{(0)}_{\beta\alpha}F_{\alpha\beta}(k_1,...,k_N)G_{\alpha\beta}(k'_1,...,k'_M) \tag{1.8}$$

with the functions F and G being the contributions from soft photons and soft gravitons respectively

$$F_{\beta\alpha}(k_1,...,k_N) = \prod_{j=1}^{N}\sum_{l_j=1}^{2}\sum_{n\in\alpha,\beta}\frac{\eta_n e_n}{\sqrt{(2\pi)^3|\mathbf{k}_j|}}\frac{p_n^\mu\epsilon^*_{\mu,l_j}(\mathbf{k}_j)}{p_n\cdot k_j - i\eta_n\epsilon} \tag{1.9}$$

$$G_{\beta\alpha}(k'_1,...,k'_M) = \prod_{j=1}^{M}\sum_{l_j=1}^{2}\sum_{n\in\alpha,\beta}\frac{M_p^{-1}\eta_n}{\sqrt{(2\pi)^3|\mathbf{k}'_j|}}\frac{p_n^\mu p_n^\nu\epsilon^*_{\mu\nu,l_j}(\mathbf{k}'_j)}{p_n\cdot k'_j - i\eta_n\epsilon}. \tag{1.10}$$

The probabilities are then found to follow a Poisson distribution

$$P^{(N)}_{\beta\alpha} = P^{(0)}_{\beta\alpha} \frac{I(\alpha,\beta)^N}{N!}, \tag{1.11}$$

where the factor of $N!$ in the denominator comes from all possible permutations of emission of N bosons. The expectation value of the number of emitted soft bosons is infinite

$$\bar{N} = \sum_{N=0}^{\infty} N P^{(N)}_{\beta\alpha} = I(\alpha,\beta) \to \infty. \tag{1.12}$$

This shows that any scattering between charged particles in QED and gravity generate an infinite amount of soft radiation in average. It gets even worst when we look at the normalization arising from

$$\begin{aligned} 1 &= \sum_N P^{(N)}_{\beta\alpha} \qquad (1.13) \\ &= P^{(0)}_{\beta\alpha} \sum_N \frac{I(\alpha,\beta)^N}{N!} \\ &= P^{(0)}_{\beta\alpha} e^{I(\alpha,\beta)}, \end{aligned}$$

implying the probability that scattering process $\alpha \to \beta$ will emit no photon is $P^{(0)}_{\beta\alpha} = e^{-I(\alpha,\beta)} = 0$. For any finite number of emitted photons N, the probability remains exactly zero because of the negative divergence in the exponential.

$$P^{(N)}_{\beta\alpha} = e^{-I(\alpha,\beta)} \frac{I(\alpha,\beta)^N}{N!} = 0. \tag{1.14}$$

Therefore, every scattering between charged particles always emits an infinite number of low energy bosons and the transition probabilities between any two states in Fock space is zero. These dramatic results have been known as the infrared catastrophe and can actually be resolved. Two methods of getting rid of those divergences have been found through the years and we will review each of them in the next sections.

1.2 Inclusive Formalism

Introduced by Bloch-Nordsiek [1] for QED and extended to gravity by Weinberg [2], the most widely used method of dealing with the divergences arising from soft photon emission is to calculate inclusive cross-sections. That is, to account for the emission of soft photons but to trace them out at the end of the computation as these states are physically indistinguishable for any finite sized detector. In this context, the divergences coming from soft emissions are exactly canceled by divergences coming from loop diagrams. However, the cancellation of divergences does not work at the level of the amplitudes but only for probabilities. The inclusive cross-section paradigm therefore abandons the idea of a well-defined S-matrix for solely calculating transition rates between processes. Losing the S-matrix description of the theory may sound disturbing, but the inclusive formalism has enjoyed wide success as it is in strong agreement with every experiment ever performed.

1.2.1 Virtual Divergences

There is a second type of infrared divergence occurring in the computation of Feynman diagrams for QED and gravity. Indeed, adding loop corrections to diagrams also creates logarithmic divergences when the momentum in the loop approaches zero.

Considering some arbitrary process $\alpha \to \beta$, let us add photon loops between external legs and calculate the correction to the amplitude. For each loop added, the photon propagator provides a factor of

$$\int_{\lambda}^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{k^2 - i\epsilon}. \tag{1.15}$$

Here, the upper and lower bounds of the integral need to be carefully defined. We must cut the integral up to momentum Λ which is taken to be the upper bound on what we define to be *soft* photons. However, the lower cutoff λ is akin to a photon mass and needs to be taken to zero.

Adding a loop also adds a propagator to the diagram for each particle connected by that loop. The total contribution to the diagram at lowest order in $|\mathbf{k}|$ is

$$\frac{-i\, p_n \cdot p_m}{(2\pi)^4} e_n e_m \eta_n \eta_m \int_{\lambda}^{\Lambda} \frac{d^4k}{[k^2 - i\epsilon]\,[p_n \cdot k - i\eta_n \epsilon]\,[-p_m \cdot k - i\eta_m \epsilon]}, \tag{1.16}$$

if the loop connects particles n and m. Let us define this quantity to be $e_n e_m \eta_n \eta_m J_{nm}$. Considering the fact that the loop can connect any incoming and outgoing lines, all first order loop diagrams can be counted by summing the result for n, m in α, β. Note that we did not need to include the contribution from loops that are connected to internal lines as the propagator $\left[(p \pm k)^2 + m^2 - i\epsilon\right]^{-1}$ arising there would not be on-shell, leaving only finite

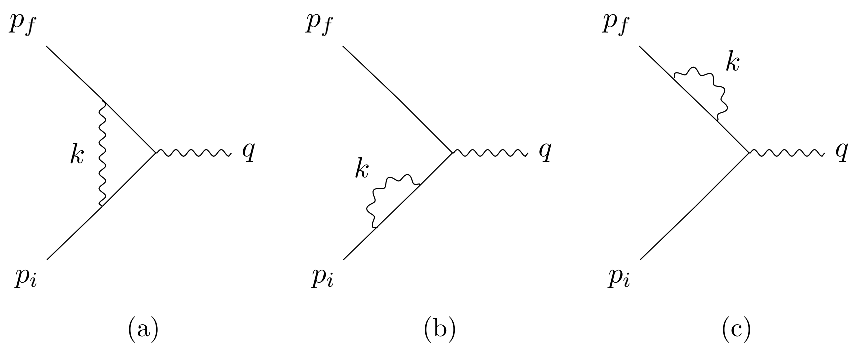

Figure 1.2: The three diverging loop diagrams at order e^2 for $1 \to 1$ potential scattering. a) is the correction to the vertex diagram while b) and c) correspond to mass renormalization for the incoming and outgoing legs.

terms in $|\mathbf{k}|$. The integral over d^4k would not be singular, allowing us to drop those diagrams as we only interest ourselves in the divergent parts of the scattering.

When adding multiple loops, each loop is independent from one another and the contributions simply multiply. For any number of internal loops N, we find

$$M^{N,\lambda}_{\beta\alpha} = M^{\Lambda}_{\beta\alpha} \frac{1}{2^N N!} \left[\sum_{n,m\in\alpha,\beta} e_n e_m \eta_n \eta_m J_{nm} \right]^N . \tag{1.17}$$

The upper indices $\lambda(\Lambda)$ meaning that the amplitudes are computed solely with loops of momentum above $\lambda(\Lambda)$. Summing the contribution of virtual boson loops for any number of such loops exponentiates

$$M^{\lambda}_{\beta\alpha} = M^{\Lambda}_{\beta\alpha} \exp\left(\frac{1}{2} \sum_{n,m\in\alpha,\beta} e_n e_m \eta_n \eta_m J_{nm} \right) . \tag{1.18}$$

The quantity J_{nm} can be evaluated by first performing the k^0 integral by the method of residues. The resulting integral is simple and can be found in [2]

$$J_{nm} = \frac{2\pi^2}{\beta_{nm}} \ln\left[\frac{1+\beta_{nm}}{1-\beta_{nm}}\right] \ln\left(\frac{\Lambda}{\lambda}\right) + \; phase, \tag{1.19}$$

where $\beta_{nm} = \left(1 - \frac{m_n^2 m_m^2}{(p_n \cdot p_m)^2}\right)^{1/2}$ is the relative velocity between particles n and m, satisfying $0 \leq \beta_{nm} \leq 1$. J_{nm} also has a divergent imaginary part but we will not bother writing it explicitly as it will drop out of the calculations once we square the amplitudes to get probabilities. Having an expression for J_{nm} we can finally compute the contribution of soft virtual loops to the amplitudes

$$M^{\lambda}_{\beta\alpha} = M^{\Lambda}_{\beta\alpha}\left(\frac{\lambda}{\Lambda}\right)^{A_{\beta\alpha}/2}, \tag{1.20}$$

where the exponent $A_{\beta\alpha}$ is defined as

$$A_{\beta\alpha} = -\frac{1}{8\pi^2}\sum_{n,m\in\alpha,\beta}\frac{e_n e_m \eta_n \eta_m}{\beta_{nm}}\ln\left[\frac{1+\beta_{nm}}{1-\beta_{nm}}\right]. \tag{1.21}$$

Once again, the story is the same for gravity where a similar integral over d^4k needs to be performed. We then find that the amplitudes get a factor of $\left(\frac{\lambda}{\Lambda}\right)^{B_{\beta\alpha}/2}$ with the gravity exponent B defined as

$$B_{\beta\alpha} = \frac{1}{16\pi^2 M_p^2}\sum_{n,m\in\alpha,\beta} m_n m_m \eta_n \eta_m \frac{1+\beta_{nm}^2}{\beta_{nm}\sqrt{1-\beta_{nm}^2}}\ln\left[\frac{1+\beta_{nm}}{1-\beta_{nm}}\right]. \tag{1.22}$$

It is important to note that the exponents A and B are positive numbers for any scattering states α,β. Therefore, virtual boson loops always contribute to make the amplitudes vanish as we take the limit $\lambda \to 0$.

Probabilities are computed by squaring the amplitudes yielding

$$\Gamma^{\lambda}_{\beta\alpha} = \Gamma^{\Lambda}_{\beta\alpha}\left(\frac{\lambda}{\Lambda}\right)^{A_{\beta\alpha}+B_{\beta\alpha}}. \tag{1.23}$$

1.2.2 Cancellation of Divergences

We have now encountered two types of divergences occurring during scattering for massless gauge field theories: Bremsstrahlung of soft photons and virtual loop diagrams. Let us now review how the two cancel each other at the level of probabilities.

Coming back to our expression for the soft factor $F_{\beta\alpha}$, this time we will be more careful and investigate the differential rate for the emission of N soft photons. That is the amplitude squared taken only over an element of volume where all N photons are soft

$$d\Gamma^{\lambda}_{\beta\alpha}(\mathbf{k}_1, ..., \mathbf{k}_N) = \Gamma^{\lambda}_{\beta\alpha} \prod_{j=1}^{N} \frac{d^3k_j}{(2\pi)^3 2|\mathbf{k}_j|} \tag{1.24}$$

$$\sum_{l=1}^{2} \sum_{n,m\in\alpha,\beta} e_n e_m \eta_n \eta_m \frac{p_n^{\mu} p_m^{\nu} \epsilon^*_{\mu,l}(\mathbf{k}_j)\epsilon_{\nu,l}(\mathbf{k}_j)}{[p_n \cdot k_j - i\eta_n\epsilon]\,[p_m \cdot k_j - i\eta_m\epsilon]}.$$

The sum over polarizations can be performed and simply gives a factor of the metric $\eta_{\mu\nu}$. The integral over angles then happens to be exactly the same as we encountered in equation (1.16) which allows us to write the differential rate only in terms of frequencies emitted

$$d\Gamma^{\lambda}_{\beta\alpha}(\omega_1, ..., \omega_N) = \Gamma^{\lambda}_{\beta\alpha} A^N_{\beta\alpha} \frac{d\omega_1}{\omega_1} ... \frac{d\omega_N}{\omega_N}, \tag{1.25}$$

where we recognize $A_{\beta\alpha}$ as defined in section (1.2.1) arising from the angular integral. A nuance now comes from the integral over frequencies. Naively, we would expect to integrate each frequency from the lower bound λ, which will eventually need to be taken to zero, all the way to Λ which was our definition of soft. However, the problem with this is that as the number of emitted photons goes to infinity, so would the energy carried by those photons. Instead, we need to make sure there is only a finite amount of energy carried away by soft photons. Let us denote that energy E_T and the maximum energy of each individual photon as E. These energies may seem somewhat arbitrary but would in fact correspond to the energy resolution of the experimenter's detector.

With this in mind, the total probability of emitting any number of soft photons can be written as

$$\Gamma^{\lambda}_{\beta\alpha}(E, E_T) = \Gamma^{\lambda}_{\beta\alpha} \sum_{N=1}^{\infty} \frac{A^N_{\beta\alpha}}{N!} \int_{\substack{\omega_j < E \\ \sum_j \omega_j < E_T}} \prod_{j=1}^{N} \frac{d\omega_j}{\omega_j}. \tag{1.26}$$

Now we will spare the details of the calculation of this integral which can be found in [2] to simply give the result

$$\Gamma^{\lambda}_{\beta\alpha}(E, E_T) = \Gamma^{\lambda}_{\beta\alpha} \left(\frac{E}{\lambda}\right)^{A_{\beta\alpha}} \mathcal{F}(A_{\beta\alpha}, E, E_T), \tag{1.27}$$

with $\mathcal{F}$ being a complicated but smooth function that depends on the scattering data and the detection cutoff. We finally have everything in hand to see the cancellation of divergences. Virtual photons provide a factor $\left(\frac{\lambda}{\Lambda}\right)^A$ while emission of real soft photons yield $\left(\frac{E}{\lambda}\right)^A$, the final probability being

$$\Gamma_{\beta\alpha}(E, E_T) = \Gamma^{\Lambda}_{\beta\alpha} \left(\frac{E}{\Lambda}\right)^{A_{\beta\alpha}} \mathcal{F}(A_{\beta\alpha}, E, E_T). \tag{1.28}$$

As before, the calculation for gravity goes in the same way where the exponent $A_{\beta\alpha}$ gets replaced by $B_{\beta\alpha}$ defined above.

In the inclusive cross-section formalism, we had to accept that the scattering amplitudes are ill defined with divergences coming from both real soft photon emission and virtual loops. However, the dependence on the photon regulator λ drops out of the calculation when evaluating probabilities. The method claims that while the QED S-matrix is zero everywhere, we can still recover the correct values for scattering probabilities which correspond to what is observed in experiments.

1.3 Dressed Formalism

While the inclusive cross-section approach to dealing with infrared divergences has enjoyed wide success for its accurate agreement with experiments, the idea of abandoning the S-matrix description of QED and gravity is unsettling. A second program for curing the infrared behavior of QED was initiated by the works of Chung [3] and Faddeev-Kulish [4] where it was shown that one can obtain a finite S-matrix between states dressed by long-wavelength photons. In this section, we will review what it means for charged particle states to be dressed and how the cancellation of divergences comes about at the level of the S-matrix.

Following the work of Chung, we will focus on scattering of a single electron off a potential. The generalization of the following procedure is explained in details in [3] but the $1 \to 1$ case is sufficient to get the proper understanding of this approach. Instead of working with standard free-field Fock space electron states $|\mathbf{p}\rangle$, each electron state gets promoted to a dressed state, denoted $\|\mathbf{p}\rangle\!\rangle$. To define the dressing that accompanies the electron, let us define the soft factor

$$F_l(\mathbf{k}, \mathbf{p}) = \frac{p \cdot e_l(\mathbf{k})}{p \cdot k} \phi(\mathbf{k}, \mathbf{p}), \tag{1.29}$$

which is the bremsstrahlung emission pole found above multiplied by an arbitrary function ϕ. This function captures the non-divergent behavior of the emission and we do not care about what ϕ is except that it must smoothly approach 1 as $|\mathbf{k}|$ goes to zero. Then for every soft momentum $\lambda < |\mathbf{k}| < E$, we construct a coherent state of transversely polarized photons from the soft factor

$$R_{\mathbf{p}} = e \sum_{l=1}^{2} \int_{\lambda<|\mathbf{k}|<E} \frac{d^3\mathbf{k}}{\sqrt{2k}} \left[F_l(\mathbf{k},\mathbf{p}) a_l^\dagger(\mathbf{k}) - F_l^*(\mathbf{k},\mathbf{p}) a_l(\mathbf{k}) \right], \tag{1.30}$$

with the dressing operator being defined as

$$W_{\mathbf{p}} = \exp(R_{\mathbf{p}}). \tag{1.31}$$

The electron states we consider are simply given by this dressing operator acting on free-field electron states

$$\|\mathbf{p}\rangle\rangle = W_{\mathbf{p}} \, |\mathbf{p}\rangle \,. \tag{1.32}$$

These states correspond to standard free field electrons accompanied by a shockwave of low energy photons. λ is once again the low energy cutoff for each photon and we will be interested in the $\lambda \to 0$ limit.

We can simplify this expression of dressed states by using the Baker-Campbell-Hausdorff formula

$$\begin{aligned} \|\mathbf{p}\rangle\rangle &= \exp\left\{ e \sum_{l=1}^{2} \int_{\lambda<|\mathbf{k}|<E} \frac{d^3\mathbf{k}}{\sqrt{2k}} \left[F_l(\mathbf{k},\mathbf{p}) a_l^\dagger(\mathbf{k}) - F_l^*(\mathbf{k},\mathbf{p}) a_l(\mathbf{k}) \right] \right\} |\mathbf{p}\rangle \qquad (1.33) \\ &= N_{\mathbf{p}} \exp\left\{ e \sum_{l=1}^{2} \int_{\lambda<|\mathbf{k}|<E} \frac{d^3\mathbf{k}}{\sqrt{2k}} F_l(\mathbf{k},\mathbf{p}) a_l^\dagger(\mathbf{k}) \right\} \\ &\qquad \times \exp\left\{ -e \sum_{l=1}^{2} \int_{\lambda<|\mathbf{k}|<E} \frac{d^3\mathbf{k}}{\sqrt{2k}} F_l^*(\mathbf{k},\mathbf{p}) a_l(\mathbf{k}) \right\} |\mathbf{p}\rangle \\ &= N_{\mathbf{p}} \exp\left\{ e \sum_{l=1}^{2} \int_{\lambda<|\mathbf{k}|<E} \frac{d^3\mathbf{k}}{\sqrt{2k}} F_l(\mathbf{k},\mathbf{p}) a_l^\dagger(\mathbf{k}) \right\} |\mathbf{p}\rangle \,, \end{aligned}$$

where $N_{\mathbf{p}}$ is the normalization of the coherent state given by

$$N_{\mathbf{p}} = \exp\left\{ -\frac{e^2}{2} \sum_{l=1}^{2} \int_{\lambda<|\mathbf{k}|<E} \frac{d^3\mathbf{k}}{2k} |F_l(\mathbf{k},\mathbf{p})|^2 \right\}. \tag{1.34}$$

1.3.1 Second Order Cancellation of Divergences

The dressing defined above was chosen precisely such that the standard S-matrix defined between such states shows no sign of infrared divergence. We will now review the argument leading to the cancellation to second order in the electron charge e. For small values of the electric charge, we expand the dressed state into a first part containing only an electron and a second one containing an electron plus a single soft photon

$$\|\mathbf{p}\rangle\!\rangle \simeq \left(1-\frac{e^2}{2}\sum_{l=1}^{2}\int\frac{d^3\mathbf{k}}{\sqrt{2k}}|F_l(\mathbf{k},\mathbf{p})|^2\right)\left(1+e\sum_{l=1}^{2}\int\frac{d^3\mathbf{k}}{2k}F_l(\mathbf{k},\mathbf{p})a_l^\dagger(\mathbf{k})\right)|\mathbf{p}\rangle\,. \tag{1.35}$$

Applying the S-matrix on such a state provides multiple types of interactions. First of all, we need to account for the vertex diagrams described in figure (1.2). As discussed before, these diagrams provide a divergence of

$$M_{p,p'}\sum_{n,m\in p,p'}e^2\eta_n\eta_m J_{nm}, \tag{1.36}$$

where $M_{p,p'}$ is the tree level vertex diagram between incoming electron $\mathbf{p}$ and outgoing electron $\mathbf{p}'$. Note that up to order e^2, the loop diagrams can only be acting on the part of the dressed states that contain no photon. Their contribution therefore ends up being

$$\begin{aligned}\langle\!\langle\mathbf{p}'\|S^{loop}\|\mathbf{p}\rangle\!\rangle &= M_{p,p'}\left(1-\frac{e^2}{2}\sum_{l=1}^{2}\int\frac{d^3\mathbf{k}}{\sqrt{2k}}|F_l(\mathbf{k},\mathbf{p}')|^2\right)\left(1+\sum_{n,m\in p,p'}e^2\eta_n\eta_m J_{nm}\right)\\ &\quad\times\left(1-\frac{e^2}{2}\sum_{l=1}^{2}\int\frac{d^3\mathbf{k}}{\sqrt{2k}}|F_l(\mathbf{k},\mathbf{p})|^2\right) \qquad (1.37)\\ &= M_{p,p'}\left(1+e^2\sum_{n,m\in p,p'}\eta_n\eta_m J_{nm}\right.\\ &\qquad\left.-\frac{e^2}{2}\sum_{l=1}^{2}\int\frac{d^3\mathbf{k}}{\sqrt{2k}}|F_l(\mathbf{k},\mathbf{p})|^2+|F_l(\mathbf{k},\mathbf{p}')|^2\right).\end{aligned}$$

Turning to the bremsstrahlung diagrams, we notice an important difference from the previous computation: This time the incoming state can also contain a photon. At zeroth order, the incoming photon does not interact and simply goes through unscattered (fig 1.3 a). At first order, the incoming

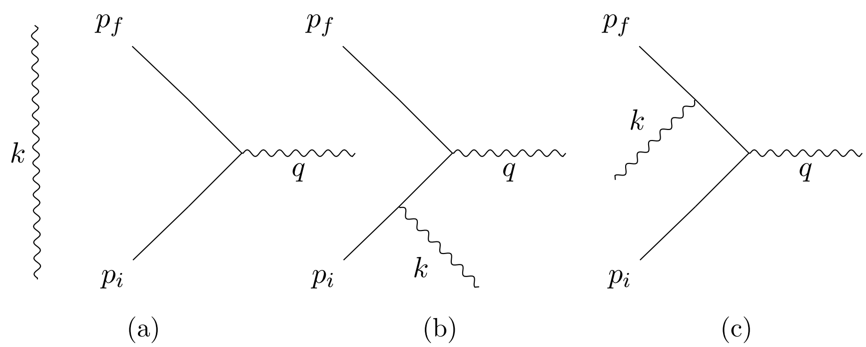

Figure 1.3: Three new diagrams for Bremsstrahlung up to first order in e. The incoming photon can either a) not interact or b), c) by absorbed by either the incoming or outgoing electron leg

electron can be absorbed by either the incoming or the outgoing electron leg (fig 1.3 b,c). We then need to account for three new interactions as well as the two old ones from figure (1.1). Taking into account the normalization of states containing one photon, diagram 1.3 a provides a factor of

$$\begin{aligned}\langle\!\langle \mathbf{p'} \| S^{1.3a} \| \mathbf{p} \rangle\!\rangle &= \langle \mathbf{p'}| \left(e \sum_{l'=1}^{2} \int \frac{d^3\mathbf{k'}}{2k} F_l^*(\mathbf{k'}, \mathbf{p'}) a_l(\mathbf{k'}) \right) \\ &\quad \times \left(e \sum_{l=1}^{2} \int \frac{d^3\mathbf{k}}{2k} F_l(\mathbf{k}, \mathbf{p}) a_l^\dagger(\mathbf{k}) \right) |\mathbf{p}\rangle \\ &= e^2 M_{p,p'} \sum_{l=1}^{2} \int \frac{d^3\mathbf{k}}{2k} F_l^*(\mathbf{k}, \mathbf{p'}) F_l(\mathbf{k}, \mathbf{p}).\end{aligned} \tag{1.38}$$

The bremsstrahlung diagrams (fig 1.1) have already been computed but this time we account for the normalization of the dressed states

$$\begin{aligned}\langle\!\langle \mathbf{p'} \| S^{1.1} \| \mathbf{p} \rangle\!\rangle &= \langle \mathbf{p'}| \left(e \sum_{l'=1}^{2} \int \frac{d^3\mathbf{k'}}{2k} F_l^*(\mathbf{k'}, \mathbf{p'}) a_l(\mathbf{k'}) \right) \\ &\quad \times \left(e \sum_{l=1}^{2} \int \frac{d^3\mathbf{k}}{2k} \left[F_l(\mathbf{k}, \mathbf{p'}) - F_l(\mathbf{k}, \mathbf{p}) \right] a_l^\dagger(\mathbf{k}) \right) |\mathbf{p}\rangle \\ &= e^2 M_{p,p'} \sum_{l=1}^{2} \int \frac{d^3\mathbf{k}}{2k} F_l^*(\mathbf{k}, \mathbf{p'}) \left[F_l(\mathbf{k}, \mathbf{p'}) - F_l(\mathbf{k}, \mathbf{p}) \right],\end{aligned} \tag{1.39}$$

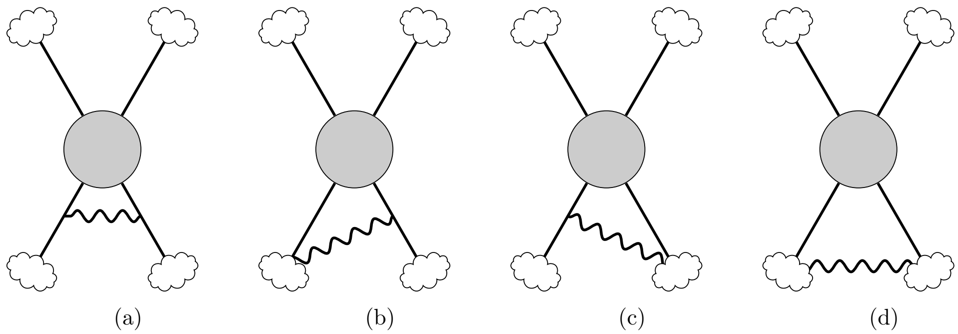

Figure 1.4: In the dressing formalism, divergences from standard loop diagrams are canceled by similar diagrams involving exchanges between the soft clouds

while diagrams (fig 1.3 b,c) give similar results

$$\langle\!\langle \mathbf{p}' \| S^{1.3b,c} \| \mathbf{p} \rangle\!\rangle = e^2 M_{p,p'} \sum_{l=1}^{2} \int \frac{d^3\mathbf{k}}{2k} \left[-F_l^*(\mathbf{k}, \mathbf{p}') + F_l^*(\mathbf{k}, \mathbf{p}) \right] F_l(\mathbf{k}, \mathbf{p}). \tag{1.40}$$

Adding the contribution of all these terms, we find at order e^2 that the S-matrix is

$$\langle\!\langle \mathbf{p}' \| S \| \mathbf{p} \rangle\!\rangle = e^2 M_{p,p'} \left(\sum_{l=1}^{2} \int \frac{d^3\mathbf{k}}{2k} \left[\frac{1}{2} |F_l(\mathbf{k}, \mathbf{p})|^2 + \frac{1}{2} |F_l(\mathbf{k}, \mathbf{p}')|^2 - F_l^*(\mathbf{k}, \mathbf{p}') F_l(\mathbf{k}, \mathbf{p}) \right] \right.$$
$$\left. \sum_{n,m \in p,p'} \eta_n \eta_m J_{nm} \right). \tag{1.41}$$

Finally, we recall the form of the real part of J_{nm} after performing the k^0 integral to be

$$Re \left(\sum_{n,m \in p,p'} \eta_n \eta_m J_{nm} \right) = \frac{2}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2k} \left[\frac{p'_\mu}{2p' \cdot k} - \frac{p_\mu}{2p \cdot k} \right]^2. \tag{1.42}$$

Evaluating the sum over polarizations on the top line of equation (1.41) provides a factor of $-\eta^{\mu\nu}$ and it now becomes clear that all infrared divergent terms cancel out of the S-matrix at second order.

The full cancellation of IR divergences in the S-matrix between dressed states can be shown order by order in a similar way, leaving the S-matrix only with the non-divergent pieces.

1.3.2 Dressed States as Eigenstates of the Asymptotic Hamiltonian

Having found one set of states between which the S-matrix is non-singular, it is reasonable to ask why infrared divergences cancel for this precise choice of dressing and if there exists other dressings that give non-zero amplitudes. The answer to this question was partially answered by investigating the asymptotic dynamics of charged particles [4]. The S-matrix is defined as the scattering amplitudes between asymptotic states and, in Quantum Field Theory, the LSZ procedure considers these asymptotic states to be free field Fock space states. However, this is not correct for QED and gravity that are long range forces with a $1/r$ potential. We can see this with from a simple non-relativistic quantum mechanical argument. We start by writing the energy of the system as

$$H = H_0 + V = \frac{\mathbf{p}^2}{2m} + \frac{k}{r}, \tag{1.43}$$

and notice that the potential dies off at infinity. A particle flying off to timelike infinity will, after a very long time, follow a trajectory of the form

$$\mathbf{r}(t) = \frac{\mathbf{p}}{m}t + \mathbf{r}_0, \tag{1.44}$$

with $\mathbf{r}_0$ being an arbitrary vector representing the position of the particle at a given time. When time increases to infinity, $\mathbf{r}(t) \to \mathbf{p}t/m$ and the potential becomes

$$\lim_{t\to\infty} V(t) = V_{as} = \frac{mk}{pt}. \tag{1.45}$$

Asymptotic states should therefore respect the Schrodinger equation with the asymptotic Hamiltonian

$$i\frac{d}{dt}\Psi(r,t) = \left[H_0(t) + V_{as}(t)\right]\Psi(r,t). \tag{1.46}$$

The family of solutions to this differential equation takes the form

$$\Psi(r,t) = i\int \frac{d^3\mathbf{p}}{(2\pi)^3}\, c(\mathbf{p}) \exp\left\{-it\frac{\mathbf{p}^2}{2m} - i\frac{mk}{p}\ln\frac{t}{t_0}\right\} e^{ipt}, \tag{1.47}$$

with the function c being an arbitrary function of the momentum and t_0

the time at which initial conditions are given. This argument of course is a quantum mechanical treatment of the asymptotic dynamics exhibiting the problem of expanding incoming and outgoing fields into free fields. The full field theory derivation needs to take into account that the interaction potential is made out of free Dirac and gauge fields

$$V = -e \int d^3\mathbf{x} : \bar{\psi}(\mathbf{x})\gamma^\mu\psi(\mathbf{x}) : A_\mu(\mathbf{x}), \tag{1.48}$$

where $\psi, \bar{\psi}, A_\mu$ are expanded in momentum creation and annihilation operators in the usual way

$$\begin{aligned}
\psi(\mathbf{x},t) &= \sum_{\pm s}\int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\sqrt{\frac{m}{p_0}}\left[b(\mathbf{p},s)u(\mathbf{p},s)e^{-ip\cdot x} + d^\dagger(\mathbf{p},s)v(\mathbf{p},s)e^{ip\cdot x}\right] \\
\bar{\psi}(\mathbf{x},t) &= \sum_{\pm s}\int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\sqrt{\frac{m}{p_0}}\left[b^\dagger(\mathbf{p},s)\bar{u}(\mathbf{p},s)\gamma_0 e^{ip\cdot x} + d(\mathbf{p},s)\bar{v}(\mathbf{p},s)\gamma_0 e^{-ip\cdot x}\right] \\
A_\mu(\mathbf{x},t) &= \sum_{l=1}^{2}\int \frac{d^3\mathbf{k}}{\sqrt{2\omega}\,(2\pi)^{3/2}}\left[\epsilon_{l,\mu}(\mathbf{k})a_l(\mathbf{k})e^{-ik\cdot x} + \epsilon^*_{l,\mu}(\mathbf{k})a^\dagger_l(\mathbf{k})e^{ik\cdot x}\right].
\end{aligned} \tag{1.49}$$

At large time, terms in the potential which maintain a non-suppressed exponential in time will oscillate rapidly and cancel out, which allows us to write the asymptotic potential as

$$\begin{aligned}
V_{as}(t) = -e\sum_{l=1}^{2}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{\sqrt{2\omega}\,(2\pi)^{3/2}}&\left[b^\dagger(\mathbf{p})b(\mathbf{p}) - d^\dagger(\mathbf{p})d(\mathbf{p})\right] \\
&\frac{p^\mu}{p_0}\left[\epsilon_{l,\mu}(\mathbf{k})a^\dagger_l(-\mathbf{k}) + \epsilon^*_{l,\mu}(\mathbf{k})a_l(\mathbf{k})\right]e^{i\frac{\mathbf{p}\cdot\mathbf{k}}{p_0}t}.
\end{aligned} \tag{1.50}$$

The asymptotic time evolution operator is given by the free time evolution operator times the time ordered exponential of the asymptotic potential

$$U_{as}(t) = e^{-iH_0t}T\exp\Big\{-i\int dt' e^{iH_0t'}V_{as}(t')e^{-iH_0t'}\Big\}. \tag{1.51}$$

The result was evaluated in [4] and gives

$$U_{as}(t) = \exp\{-iH_0t\}\exp\{R(t)\}\exp\{i\Phi(t)\}, \tag{1.52}$$

with

$$R(t) = e \sum_{l=1}^{2} \int \frac{d^3\mathbf{k} d^3\mathbf{p}}{\sqrt{2\omega}\,(2\pi)^{3/2}} \left[b^\dagger(\mathbf{p}) b(\mathbf{p}) - d^\dagger(\mathbf{p}) d(\mathbf{p}) \right] \left[F_l(\mathbf{k},\mathbf{p}) a^\dagger(\mathbf{k}) e^{i\frac{p\cdot k}{p_0}t} - F_l^*(\mathbf{k},\mathbf{p}) a(\mathbf{k}) e^{-i\frac{p\cdot k}{p_0}t} \right] \tag{1.53}$$

and $\Phi(t)$ is a phase which we will not care about. The S-matrix of a given process then becomes

$$S(t_1, t_2) = \lim_{\substack{t_1 \to \infty \\ t_2 \to -\infty}} U_{as}^\dagger(t_1) \exp\{-iH(t_1 - t_2)\} U_{as}(t_2), \tag{1.54}$$

which differs from the standard QED S-matrix from the fact that $U_{as}(t)$ now has an extra factor of $\exp\{R(t)\}\exp\{i\Phi(t)\}$. However one can keep the Dyson definition of the S-matrix if it is defined between states dressed by $W = \lim_{t\to\pm\infty} \exp\{R(t)\}$. Applied on a free electron state of momentum $\mathbf{p}$, we recover the dressed states defined in (1.32).

When taking into account the long-ranged properties of the interaction potential in QED, we notice that the S-matrix does not act between free electron states as the Dyson S-matrix but as this standard S-matrix dressed by W operators. The alternative which we adopt is to dress the in/out states and keep the conventional definition of the S-matrix. Doing so, we recover the dressed states of Chung for which the S-matrix elements are free of infrared divergences to all orders.

1.4 Asymptotic Symmetries

It has recently been discovered that abelian gauge theories possess an infinite number of symmetries. These symmetries are generated by large gauge transformations and take a simple form at null infinity. Asymptotic symmetries are spontaneously broken which results in an infinite degeneracy of the vacuum. At first glance, this topic seems unrelated to the study of soft theorems which are the primary focus of our attention in this thesis. However, it was found that the conservation of charges associated to these symmetries is directly related to the vanishing of the S-matrix in standard QED. By assuming unicity of the vacuum, conventional QED violates charge conservation and forces the S-matrix to be null. However, when one takes into account vacuum transitions, the S-matrix is restored. In those terms, the large gauge transformation program is a reformulation of Fadeev-Kulish dressed states. We will now review the content of these large gauge transformations for QED and gravity and how charge conservation is linked to the vanishing of the S-matrix. Finally, we will review the relation between FK states and conserved charges for QED.

1.4.1 Matching Conditions

As large gauge transformations take a simple form at null infinity, it is simpler to work in a coordinate system which exhibits this property. A judicious choice is to use advanced ($v = t + r$) and retarded ($u = t - r$) coordinates with metric

$$\begin{aligned} ds^2 &= -dv^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \end{aligned} \tag{1.55}$$

to parametrize points at negative and positive time t respectively. The coordinate r represents the radial distance while the location on the two-sphere is depicted via a stereographic projection unto the complex plane via coordinates $z, \bar{z}$. The metric on the complex plane is given by

$$\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}. \tag{1.56}$$

As we interest ourselves in the infrared behavior of QED, it is useful to look at the expansion of fields at past/future null infinity denoted $\mathcal{I}^-/\mathcal{I}^+$ (figure 1.5). In particular, we can examine the Liénard-Wiechert gauge field strength generated by charged particles moving at constant velocity

$$F_{rt}(\vec{x}, t) = \frac{e^2}{4\pi}\sum_{k=1}^{N}\frac{Q_k\gamma_k\left(r - t\hat{x}\cdot\vec{\beta}_k\right)}{|\gamma_k^2\left(t - r\hat{x}\cdot\vec{\beta}_k\right) - t^2 + r^2|^{3/2}}, \tag{1.57}$$

where Q_k is the charge of particle k in units of e, $\vec{\beta}_k$ its velocity and γ_k is the relativistic factor $\gamma_k = \left(1 - \beta_k^2\right)^{-1/2}$. In advanced (retarded) coordinates, $F_{rt}(\vec{x}, t) = F_{rv(u)}(\vec{x}, t)$. $\mathcal{I}^\pm$ is located at null infinity and we can take the

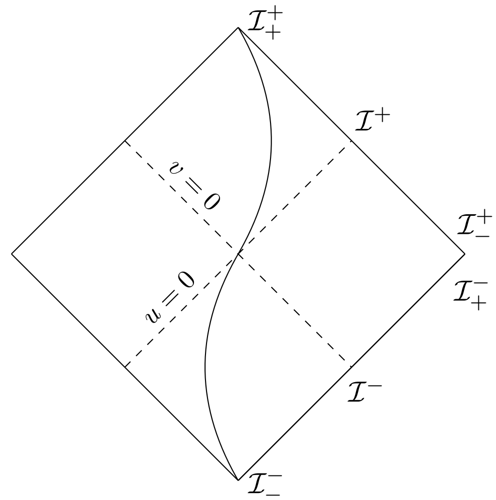

Figure 1.5: Minkowski space Penrose diagram. $\mathcal{I}^{\pm}$ represent past and future null infinity with their S^2 boundaries identified by $\mathcal{I}^{\pm}_{\pm}$. Each point (r,t) identifies two points on the diagram, one on the right and one on the left, which are related by the antipodal mapping. The curved line represents the motion of massive particles while lightrays move along straight lines of constant u or v.

large r limit to find the expansion of the gauge field strength there

$$F_{rv}\big|_{\mathcal{I}^-} = \frac{e^2}{4\pi r^2}\sum_{k=1}^{N}\frac{Q_k}{\gamma_k^2\left(1+\hat{x}\cdot\vec{\beta}_k\right)^2} \tag{1.58}$$
$$F_{ru}\big|_{\mathcal{I}^+} = \frac{e^2}{4\pi r^2}\sum_{k=1}^{N}\frac{Q_k}{\gamma_k^2\left(1-\hat{x}\cdot\vec{\beta}_k\right)^2}.$$

However, Lorentz invariance dictates that at the junction of $\mathcal{I}^-$ and $\mathcal{I}^+$ the two fields must agree up to antipodal matching condition

$$F_{rv}(\hat{x})\big|_{\mathcal{I}^-_+} = F_{ru}(-\hat{x})\big|_{\mathcal{I}^+_-}, \tag{1.59}$$

which is related to the fact that a boost towards the north pole in the past corresponds to a boost towards the south pole in the future. Instead of working with this antipodal matching, it is convenient to define coordinates such that $\hat{x} \to -\hat{x}$ in retarded coordinates so that we get the equality

$$F_{rv}\big|_{\mathcal{I}^-_+} = F_{ru}\big|_{\mathcal{I}^+_-}. \tag{1.60}$$

1.4.2 Conserved Charges

This matching condition for the gauge field strength is at the core of the argument leading to an infinity of conserved charges. Consider any function $\epsilon(t, r, z, \bar{z})$ respecting the same matching conditions

$$\epsilon(z, \bar{z})\big|_{\mathcal{I}^-_+} = \epsilon(z, \bar{z})\big|_{\mathcal{I}^+_-}, \tag{1.61}$$

then we can define charges

$$Q^+_\epsilon = \frac{1}{e^2}\int_{\mathcal{I}^+_-} \epsilon * F = \frac{1}{e^2}\int_{\mathcal{I}^-_+} \epsilon * F = Q^-_\epsilon. \tag{1.62}$$

These charges being defined as integrals on a boundary, we can write them as full derivatives on $\mathcal{I}^\pm$ and use Maxwell's equations in form notation $e^2 j = *d*F$ to rewrite them as

$$\begin{aligned} Q^+_\epsilon &= \frac{1}{e^2}\int_{\mathcal{I}^+} d\epsilon \wedge *F + \int_{\mathcal{I}^+} \epsilon * j + \frac{1}{e^2}\int_{\mathcal{I}^+_+} \epsilon * F \\ Q^-_\epsilon &= \frac{1}{e^2}\int_{\mathcal{I}^-} d\epsilon \wedge *F + \int_{\mathcal{I}^-} \epsilon * j + \frac{1}{e^2}\int_{\mathcal{I}^-_-} \epsilon * F. \end{aligned} \tag{1.63}$$

Equation (1.62) and its rewritten form (1.63) exhibits the conservation of an incoming charge Q^-_ϵ defined on past data $\mathcal{I}^-$ into an outgoing charge Q^+_ϵ defined on future data $\mathcal{I}^+$. There exists an infinite number of functions ϵ which satisfy the matching conditions and therefore QED possesses an infinite number of such conserved charges. While in theory any function ϵ obeying the matching conditions will provide a conserved quantity, it is more instructive to stick to a set of functions $\epsilon = \epsilon(z, \bar{z})$ which only depend on the two sphere coordinates. One simple choice of basis for these functions could be spherical harmonics. Then, we can direcly notice using Gauss' law that the conserved charge associated to Y^0_0 is simply the total charge of the state. $Q^+_{Y^0_0} = Q^-_{Y^0_0}$ is the statement that total charge is conserved during scattering. In the Y^m_l basis, the infinite number of conservation laws is an extension to electric charge conservation stating that every incoming multipole moments of the electric field are also antipodally conserved.

Massless charged particles

A simple situation arises when all charged particles are massless. Then charged particles fly out to $\mathcal{I}^+$ and never reach $\mathcal{I}^+_+$ leaving the electric field to vanish at that point. This has the effect of canceling the last term on the rhs of (1.63). We can thus express the charges as a sum of two terms

$$Q^+_\epsilon = \frac{1}{e^2}\int_{\mathcal{I}^+} dud^2z\left(\partial_z\epsilon F_{u\bar{z}} + \partial_{\bar{z}}\epsilon F_{uz}\right) + \int_{\mathcal{I}^+} dud^2z\epsilon\gamma_{z\bar{z}}j_u \tag{1.64}$$

The point of dividing the charges in such a way can now become apparent. The second term on the rhs is constructed from the charged current and corresponds to hard outgoing data. We will therefore call this term the hard charge denoted $Q^+_{H,\epsilon}$. On the other hand, the first term on the rhs is generated by zero energy photons and thus called the soft charge $Q^+_{S,\epsilon}$. While this statement may not seem obvious at first sight, one can expand the integral of the gauge field strength on $\mathcal{I}^+$ in its Fourier components

$$N^+_z = \int_{-\infty}^{\infty} duF_{uz} = \lim_{\omega\to 0}\int_{-\infty}^{\infty} duF_{uz}e^{i\omega u}, \tag{1.65}$$

where only the zero mode contributes. $Q^+_{S,\epsilon}$ is therefore the contribution to the charge coming from creating and annihilating zero energy photons with polarization $\partial_z\epsilon$ and $\partial_{\bar{z}}\epsilon$. On the other hand, the hard charge being made of the conserved current of the global U(1) symmetry j_u, its action on a state of N incoming particles of charge Q_k is simply

$$Q^-_{H,\epsilon}\left|in\right\rangle = \int_{\mathcal{I}^-} dud^2z\epsilon\gamma_{z\bar{z}}j_u\left|in\right\rangle \tag{1.66}$$

$$= \sum_{k=1}^{N} Q^{in}_k\epsilon(z_k,\bar{z}_k)\left|in\right\rangle. \tag{1.67}$$

This stems from the fact that j_u acts on a momentum eigenstate of charge Q as

$$j_u(u',\omega,\bar{\omega})\left|Q(u,z,\bar{z})\right\rangle = Q\gamma^{z\bar{z}}\delta^2(\omega - z)\delta(u'-u)\left|Q(u,z,\bar{z})\right\rangle. \tag{1.68}$$

Massive charged particles

The case for massive particles is slightly more complicated. As massive particles follow trajectories that start at $\mathcal{I}^-_-$ and end at $\mathcal{I}^+_+$, they can never reach the regions $\mathcal{I}^-$ and $\mathcal{I}^+$. Tt is therefore only the third term in (1.63) which will contribute. While advanced and retarded coordinate v, u are convenient to describe the geometry of Minkowski space near null infinity, these are poor choices of coordinates at past/future timelike infinity. There, we will instead consider the hyperbolic slicing consisting of surfaces of constant

$$\tau^2 = t^2 - r^2 > 0, \tag{1.69}$$

denoted $\mathbb{H}_3$. We define a new coordinate ρ such that

$$\rho = \frac{r}{\sqrt{t^2 - r^2}} = \frac{r}{\tau}, \tag{1.70}$$

with the metric taking the form

$$ds^2 = -d\tau^2 + \tau^2 \left(\frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega_2^2 \right). \tag{1.71}$$

This set of coordinates is well justified when analyzing the motion of a particle moving with constant velocity at late times. Then the motion of the particle follows a trajectory of the form

$$\mathbf{r} = \frac{\mathbf{p}}{p_0} t + \mathbf{r}_0 \tag{1.72}$$

and at late times $t \to \infty$, we find

$$\rho = \frac{|\mathbf{p}|}{m}, \qquad \tau = \frac{m}{p_0} t.$$

Massive particles therefore approach $\mathcal{I}_+^+$ on trajectories of constant ρ. To evaluate the action of the hard charge on massive particles, we need to extend the definition of $\epsilon(z, \bar{z})$ beyond $\mathcal{I}^\pm$ and into the bulk of Minkowski space. To do so, it is simpler to work in Lorenz gauge $\nabla^\mu A_\mu = 0$. Then, the gauge parameter ϵ follows the wave equation

$$\Box \epsilon = 0. \tag{1.73}$$

We can solve this equation in the bulk of Minkowski space by using a Green's function integration kernel

$$\epsilon_{\mathbb{H}}(\rho, \hat{x}) = \int d^2\hat{q}\; G(\rho, \hat{x}; \hat{q}) \epsilon(\hat{q}), \tag{1.74}$$

where the kernel needs to satisfy

$$\Box G(\rho, \hat{x}; \hat{q}) = 0 \tag{1.75}$$
$$\lim_{r\to\infty} G(\rho, \hat{x}; \hat{q}) = \delta^2(\hat{x} - \hat{q}).$$

This Green's function has solution

$$\frac{\gamma_{\omega\bar{\omega}}^{1/2}}{4\pi\left(\sqrt{1+\rho^2} - \rho\hat{q}\cdot\hat{x}\right)^2}, \tag{1.76}$$

where $\hat{q}$ points in the direction of $\omega, \bar{\omega}$. We can then simply evaluate the action of the massive hard charge using Gauss' law. When applied on an outgoing state of momentum $\mathbf{p}$ and charge Q

$$\int_{\mathcal{I}_+^+} \epsilon\,(\rho, \hat{x}) * F\,|out\rangle = Q\epsilon(\frac{|\mathbf{p}|}{m}, \hat{p})\,|out\rangle\,, \tag{1.77}$$

the massive hard charge singles out the electric charge of the outgoing particle times the function ϵ evaluated along lines of constant $\rho = \frac{|\mathbf{p}|}{m}$.

1.4.3 Vanishing of the S-matrix and Vacuum Transitions

Having derived an expression for the hard and soft charges, we are now ready to explain via the large gauge transformation formalism why conventional QED finds a vanishing S-matrix. Assuming some incoming state $|in\rangle$ evolves into an out state $|out\rangle$, the amplitude for that process is given by the S-matrix element

$$\langle out|S|in\rangle\,. \tag{1.78}$$

In those terms, the statement that charge is conserved can be written

$$\langle out|Q_\epsilon^+ S|in\rangle = \langle out|SQ_\epsilon^-|in\rangle\,. \tag{1.79}$$

The equality (1.79) is valid for any choice of the function ϵ and in particular it holds for $\epsilon(\omega, \bar{\omega}) = \frac{1}{z-\omega}$ for which the soft charge integral simplifies substantially. In this special case, we can use the identity

$$\partial_{\bar{z}}\frac{1}{z-\omega} = 2\pi\delta^2(z-\omega), \tag{1.80}$$

which follows from Stoke's theorem and Cauchy's integral theorem. Equation (1.79) then becomes

$$4\pi \langle out|N_z^+ S - S N_z^- |in\rangle = \left(\left[\sum_{k\in massless} \frac{Q_k^{in}}{z-z_k} - \sum_{k\in massless} \frac{Q_k^{out}}{z-z_k}\right]\right. \tag{1.81}$$

$$\left. + \left[\sum_{k\in massive} Q_k^{in} \epsilon\left(\frac{|\mathbf{p}_k|}{m_k}, \hat{p}_k\right) - \sum_{k\in massive} Q_k^{out} \epsilon\left(\frac{|\mathbf{p}_k|}{m_k}, \hat{p}_k\right)\right]\right) \langle out|S|in\rangle,$$

where we used the definition of N_z from (1.65). The lhs then represents the difference of soft charges between the outgoing and incoming states. On the rhs, the index k lists all incoming (outgoing) particles with charge Q_k exiting (entering) $\mathcal{I}^-$ ($\mathcal{I}^+$) and $\mathcal{I}^-_-$ ($\mathcal{I}^+_+$) at position z_k on the two-sphere at infinity . $N_z^\pm$ being comprised of zero-mode photon creation and annihilation operators does not affect the hard content of the *in* and *out* states. We can then consider incoming and outgoing states of the form $|in\rangle = |in; N_z^{in}\rangle$ such that

$$N_z^- \,|in; N_z^{in}\rangle = N_z^{in}\,|in; N_z^{in}\rangle\,. \tag{1.82}$$

N_z^{in} is the zero-mode photon content of the incoming vacuum and corresponds to the infinite degeneracy of the vacuum state due to the broken large gauge symmetries. The eigenvalue on the rhs of (1.81) is usually referred to as $4\pi\Omega_z$ and the conservation of charges becomes

$$\left(N_z^{out} - N_z^{in}\right)\langle out; N_z^{out}|S|in; N_z^{in}\rangle = \Omega_z \,\langle out; N_z^{out}|S|in; N_z^{in}\rangle\,. \tag{1.83}$$

For this equality to hold, two options are possible: Either the difference of soft charges equals the difference of hard charges $\left(N_z^{out} - N_z^{in}\right) = \Omega_z$, or every S-matrix element has to be zero

$$\langle out; N_z^{out}|S|in; N_z^{in}\rangle = 0. \tag{1.84}$$

In standard QED, the vacuum is taken to be non-degenerate which results in the vanishing of the S-matrix generally attributed to infrared divergences as explained in section 1.2. However, this problem can be avoided if one works with degenerate vacua containing soft photons. Then the outgoing vacuum is highly correlated with the hard data as the sum of soft and hard charges need to remain conserved. This is reminiscent of the Faddeev-Kulish formalism where a pure state is formed out of hard data entangled with a coherent state of soft radiation. In fact, this connection was made more rigorous recently [8, 10] where it was argued that large gauge symmetries are a reformulation of the FK dressing.

1.5 Soft Hair and the Black Hole Information Paradox

An analogous situation arises in gravity where the asymptotic symmetries of asymptotically flat Minkowski space were found by Bondi, van der Burg, Metzner and Sachs (BMS) [11, 12]. In this case, the relevant symmetries form a subgroup of the BMS group and are not large gauge transformations, but angle-dependent gauge-preserving diffeomorphisms called supertranslations. Just as in QED, the associated conserved charges can be divided into a hard and a soft component, where the soft charge identifies the zero-mode graviton content of the vacuum. It was recently argued [9] that this degenerate vacuum may play a crucial role in the black hole information paradox. In this section, we will review the long-standing argument leading to the information paradox and how asymptotic symmetries of spacetime could help resolve it.

1.5.1 Black Hole Information Paradox

In 1975, Hawking's seminal paper [13] demonstrated the process from which black holes evaporate. While classical black holes may only absorb particles, quantum mechanics allow for the vacuum to create particle-antiparticle pairs in the vicinity of the black hole horizon. When a positive energy particle escapes the black hole region and leaves the negative energy one to fall in, the black hole loses a small amount of mass. Over long periods of time, this process leads to the evaporation of any black hole in Minkowski space. Furthermore, Hawking argued using the no-hair theorem that the outgoing radiation does not contain any information about the black hole formation process but follows the spectrum of a blackbody with temperature

$$T_{Hawking} = \frac{\hbar}{8\pi GM}. \tag{1.85}$$

The final state is therefore a completely mixed thermal state where all the information about the arrangement of particles that created the black hole in the first place has been lost during the formation/evaporation process. This however contradicts unitarity. If one sends in a pure state made of well localized incoming particles which will form into a black hole, unitary time evolution states that the global state of the system remains pure at all times.

Hawking's proposal of a pure state time evolving into a mixed thermal state therefore violates unitarity. This is the longstanding black hole information paradox. If information is indeed lost during the formation/evaporation process, then we must accept the disturbing fact that unitarity is not a fundamental restriction of nature. Alternatively, we need to come up with an explanation as to where all that information is stored.

1.5.2 Black Hole Soft Hair

Even though the paradox is still unresolved, the asymptotic symmetries of QED and gravity program may provide some insight towards a resolution. Recent work initiated by [9] claimed that the soft graviton vacuum degeneracy plays an important role. While the hard matter content of the state is lost inside the black hole, the soft gravitons of the degenerate vacuum, having wavelengths far larger than the black hole size, are very unlikely to be lost inside the black hole. As we have previously seen, charge conservation implies a strong correlation between the soft vacuum and the hard data. Contrary to classical belief, black hole states would be characterized not solely by their mass, charge and angular momentum, but also by an infinite number of supertranslation charges. Supertranslations would then play the role of infinitely many soft hairs distinguishing the state of any differently formed black holes. An even more recent proposal [14] speculated that the graviton vacuum, being strongly entangled with the hard data content, could be sufficient to purify the outgoing Hawking radiation. If this were true, such an explanation could be sufficient to resolve the paradox as the incoming pure state would then unitarily evolve into an other pure state made of thermal radiation entangled with soft gravitons. The global outgoing state would take the form of a tensor product between thermal and soft graviton states

$$|\psi\rangle = \sum_a |a\rangle_{Thermal} |a\rangle_{soft} , \tag{1.86}$$

but an observer with a finite sized detector would not observe the graviton vacuum and only see the mixed thermal state predicted by Hawking

$$\mathrm{Tr}_{soft} |\psi\rangle \langle\psi| = \rho_{Thermal}. \tag{1.87}$$

Chapter 2

Scattering with Partial Information

We study relativistic scattering when one only has access to a subset of the particles, using the language of quantum measurement theory. We give an exact, non-perturbative formula for the von Neumann entanglement entropy of an apparatus particle scattered off an arbitrary set of system particles, in either the elastic or inelastic regime, and show how to evaluate it perturbatively. We give general formulas for the late-time expectation values of apparatus observables. Some simple example applications are included: in particular, a protocol to verify preparation of coherent superpositions of spatially localized system states using position-space information in the outgoing apparatus state, at lowest order in perturbation theory in a weak apparatus-system coupling.

2.1 Introduction

The purpose of this paper is to make contact between concepts from quantum information and relativistic scattering theory. In particular, we study how to use interacting fields as measurement devices.

In standard formulations of measurement theory, one imagines performing a measurement of a system S by coupling it to an apparatus A. We start the apparatus in some register state $|0\rangle_A$ while the system is in an arbitrary superposition, and then entangle these in such a way that measurements on A can determine the initial state of S. Schematically, one writes things like

$$|0\rangle_A \otimes \sum_i c_i \, |i\rangle_S \to \sum_i c_i \, |i\rangle_A \otimes |i\rangle_S \,, \tag{2.1}$$

with the arrow referring to time evolution under some total Hamiltonian (see eg. [15, 16]). This process necessarily generates entanglement between S and A. The goals of this paper are to study to what extent we can understand the scattering of system particles S by another particle A in this language and to quantify how much entanglement is generated in such scattering events.

To this purpose, we consider an arbitrary system of fields and append an apparatus field ϕ_A which we can scatter off the system, so we consider Hilbert spaces formed by tensor products of apparatus and system fields. The S-matrix generates entanglement between the factors. This approach differs from and complements other ways of dividing field-theoretic systems; one can also consider, for example, divisions by spatial area [17, 18], momentum scale [19], or multiple non-interacting CFTs [20].

We begin by reviewing and slightly extending the textbook treatment [21] of scattering theory to incorporate density matrices as initial conditions in section 2.2. We explain how to calculate expectation values of operators probing only the apparatus. In section 2.3, we present an exact, non-perturbative formula for the von Neumann entropy of the apparatus A after the scattering event, assuming only that the state at early and late times contains exactly one particle of ϕ_A.

We then apply these results to the simplest possible example, in which the apparatus and system both consist of a single particle of some scalar fields $\phi_{A,S}$, with A and S weakly coupled. In section 2.4.1 we give an explicit formula for the entropy generated when we scatter a product momentum state $|\mathbf{p}\rangle_A |\mathbf{q}\rangle_S$, recovering and slightly correcting a result of [22, 23].

In section 2.4.2, we consider a somewhat different problem. Suppose that we think we are preparing the system S in a superposition of two well-localized position states. We show how to do a measurement with A to verify that the superposition is really coherent, as opposed to (say) having decohered into a classical ensemble. We find that a good observable to use to determine the coherence of S is position-space interference fringes in the outgoing distribution for the apparatus particle A. These show up at lowest order in perturbation theory in the S-A coupling λ, whereas the momentum-space distribution of A is only sensitive at second order.

2.2 Scattering with Density Matrices

2.2.1 General Considerations

Let's consider the general problem of scattering where we know the state of the total system at very early times $t \to -\infty$, and we want to know how this evolves at very late times due to a scattering event. We want to consider any density matrix for the full system as an initial condition. The treatment here is a straightforward generalization of Weinberg's textbook [21], and our conventions throughout follow his. In particular, the metric signature is $-+++$ and $\hbar = c = 1$.

Assume the total Hamiltonian can be written

$$H = H_0 + V, \tag{2.2}$$

and denote the energy eigenstates of the free Hamiltonian H_0 as

$$H_0 \left|\alpha\right\rangle = E_\alpha \left|\alpha\right\rangle . \tag{2.3}$$

Here the label $\alpha = \mathbf{p}_1\sigma_1 n_1, \mathbf{p}_2\sigma_2 n_2, \ldots$ covers the momentum, spin, and particle species of the free-particle states. We define in- and out-states as Heisenberg-picture states which have the energies E_α but are eigenstates of the <u>full</u> Hamiltonian,[1]

$$H \left|\alpha^\pm\right\rangle = E_\alpha \left|\alpha^\pm\right\rangle , \tag{2.4}$$

satisfying the condition that as $t \to \mp\infty$, for any reasonably smooth functions $g^\pm(\alpha)$ of the particle labels,

$$\left|\psi\right\rangle = \int d\alpha \; e^{-iE_\alpha t} g^\pm(\alpha) \left|\alpha^\pm\right\rangle \to \int d\alpha \; e^{-iE_\alpha t} g^\pm(\alpha) \left|\alpha\right\rangle . \tag{2.5}$$

This condition says that at very early or late times, the in/out states behave like the free-particle states of the corresponding particle labels α. The notation is that $+$ indicates an in-state while $-$ denotes an out-state. Both the free and scattering states are taken to be Dirac delta-normalizable $\langle\alpha|\alpha'\rangle = \langle\alpha^\pm|\alpha'^\pm\rangle = \delta(\alpha - \alpha')$.

If the system is in a wavepacket like (2.5), and we know the matrix elements $\langle\alpha|\mathcal{O}|\alpha'\rangle$ of some observable in terms of free-particle states, we can compute the expectation value of $\mathcal{O}$ at early or late times in the state $|\psi\rangle$ as follows. In the Heisenberg picture we have $\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt}$, so using (2.4) and (2.5), we have that as $t \to \mp\infty$,

$$\langle\psi|\mathcal{O}(t)|\psi\rangle \to \int d\alpha d\alpha' \; e^{i(E_\alpha - E_{\alpha'})t} g^*(\alpha) g(\alpha') \langle\alpha|\mathcal{O}|\alpha'\rangle . \tag{2.6}$$

More generally, the system may be in a density matrix. This can be decomposed into any complete basis, including the scattering states:

$$\rho = \int d\alpha d\alpha' \rho^\pm(\alpha, \alpha') \left|\alpha^\pm\right\rangle \left\langle\alpha^\pm\right| . \tag{2.7}$$

[1] Notice that the conditions (2.3) and (2.4) mean that the "free" states and scattering states have the same energy spectrum. This means in particular that the masses appearing in the Hamiltonian are the physical ("renormalized" or "dressed") masses of the particles.

Then the expectation value of $\mathcal{O}$ is given asymptotically by

$$\langle \mathcal{O}(t) \rangle = \operatorname{tr} \rho \mathcal{O}(t) \to \int d\alpha d\alpha' \ \rho^{\pm}(\alpha, \alpha') e^{i(E_\alpha - E_{\alpha'})t} \langle \alpha | \mathcal{O} | \alpha' \rangle \tag{2.8}$$

as $t \to \mp\infty$.

Since the states $|\alpha^+\rangle$ and $|\alpha^-\rangle$ separately form complete bases for positive-energy states of the system, we can express one base in terms of the other. The S-matrix is the unitary operator with elements given by the inner product

$$S_{\beta\alpha} = \langle \beta^- | \alpha^+ \rangle \, . \tag{2.9}$$

The in- and out-coefficients of the density matrix are thus related by

$$\rho^-(\beta, \beta') = \int d\alpha d\alpha' \ S_{\beta\alpha} S^*_{\beta'\alpha'} \rho^+(\alpha, \alpha'). \tag{2.10}$$

We will always consider Poincaré-invariant systems. We can therefore write the S-matrix as an identity term plus a term with the total four-momentum invariance factored out,

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2\pi i M_{\beta\alpha} \delta^4(p_\beta - p_\alpha). \tag{2.11}$$

In appendix A, we use the unitarity of the S-matrix,

$$\int d\beta S_{\beta\alpha} S^*_{\beta\alpha} = \delta(\alpha - \alpha') \tag{2.12}$$

to derive the optical theorem, (A.4), which will play a role repeatedly in the calculations that follow.

Box Normalizations

In computing various quantities it will be useful to work with discrete states. We can do this by putting the entire process into a large spacetime volume of duration T and spatial volume $V = L^3$. Periodic boundary conditions on V allow us to retain exact translation invariance. We define dimensionless, box-normalized states

$$|\alpha^{\pm}\rangle^{box} = \tilde{N}^{n_\alpha/2} \, |\alpha^{\pm}\rangle \, , \quad \tilde{N} = \frac{(2\pi)^3}{V}, \tag{2.13}$$

where n_α is the number of particles in the state α. When working directly with box-normed states, delta functions and S-matrix elements are all dimensionless, integrals over states are replaced by sums, and the delta-functions are Kroneckers. We have

$$S^{box}_{\beta\alpha} = \tilde{N}^{(n_\alpha+n_\beta)/2} S_{\beta\alpha}, \tag{2.14}$$

by definition of the S-matrix. Delta functions are then regulated as

$$\delta^3_V(\mathbf{p}-\mathbf{p}') = \tilde{N}^{-1}\delta_{\mathbf{p},\mathbf{p}'}, \quad \delta_T(E-E') = \frac{1}{2\pi}\int_{-T/2}^{T/2} dt\ e^{i(E-E')t}. \tag{2.15}$$

Note in particular that this implies $\delta_T(0) = T/2\pi$. We then define a box-normalized transition amplitude:

$$S^{box}_{\beta\alpha} = \delta_{\beta\alpha} - 2\pi i M^{box}_{\beta\alpha}\delta_{\mathbf{p}_\beta\mathbf{p}_\alpha}\delta_{E_\beta E_\alpha} \iff M^{box}_{\beta\alpha} = \tilde{N}^{(n_\alpha+n_\beta-2)/2} M_{\beta\alpha}. \tag{2.16}$$

Note that M^{box} has mass dimension one, since $\delta_T(E)$ has dimensions of inverse mass.

2.2.2 Measuring the Apparatus State

Suppose now that we divide the total system into an apparatus A and system S and only have direct access to A. Here we work out a formula for computing observables only of A, and for the von Neumann entropy of A.

In what follows, we assume that A and S are distinguishable; a simple way to achieve this is to just have A and S described by different fields. We will make this assumption in everything that follows. We will hereafter make a slight abuse of the previous notation and label states with two indices (a,α) where a labels apparatus eigenstates and α labels system eigenstates. We can decompose the total Hilbert space as a product over free, in, or out states:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_S = \mathcal{H}^\pm_A \otimes \mathcal{H}^\pm_S. \tag{2.17}$$

The total S-matrix provides a unitary map between the in- and out-state decompositions. In particular, a product in-state is a generally non-separable mixture of out-states:

$$|a\alpha\rangle^+ = \int dbd\beta\ S_{b\beta,a\alpha}\,|b\beta\rangle^- . \tag{2.18}$$

At early or late times, we want to compute the expectation value of any observable $\mathcal{O}_A : \mathcal{H}_A \to \mathcal{H}_A$. Note that here $\mathcal{O}_A$ is an operator on the <u>free</u> apparatus Hilbert space factor in (2.17). Take $\mathcal{O} = \mathcal{O}_A \otimes \mathbf{1}_S$ and apply (2.8).

By the asymptotic conditions on the scattering states, a simple calculation shows that at early or late times

$$\langle \mathcal{O}_A(t) \rangle := \langle \mathcal{O}(t) \rangle \to \int dada'd\alpha \; \rho^{\pm}(a,\alpha;a',\alpha) e^{i(E_a - E_{a'})t} \langle a|\mathcal{O}_A|a'\rangle \,. \tag{2.19}$$

To derive this formula, we assumed that the free Hamiltonian has an additive spectrum $H_0 |a\alpha\rangle = (E_a + E_\alpha) |a\alpha\rangle$. The result (2.19) holds for any density matrices; in particular, we do not need to assume that the total state factors into a product of a density matrix for A and a density matrix for S at either early or late times.

We would also like to define the entanglement entropy between apparatus and system. To do this, we again use the decomposition (2.17) to perform partial traces over the system. We can do this using either in- or out-states,

$$\rho_A^{\pm} := \mathrm{tr}_{\mathcal{H}_S^{\pm}} \rho \tag{2.20}$$

from which we can in turn define the entanglement entropy

$$S_A^{\pm} = -\,\mathrm{tr}_{\mathcal{H}_A^{\pm}} \rho_A^{\pm} \ln \rho_A^{\pm}. \tag{2.21}$$

2.3 A-S Entanglement Entropy

Our goal in this section is to calculate the entanglement entropy between the system and apparatus at late times. Consider the system and apparatus both prepared in definite momentum eigenstates at early times,

$$|\psi\rangle = |\mathbf{p}^{+}\rangle_A \, |\alpha^{+}\rangle_S \,. \tag{2.22}$$

Here as before $\alpha = \mathbf{q}_1 n_1 \sigma_1, \mathbf{q}_2 n_2 \sigma_2, \ldots$ labels all the momenta, species, and spin of the system particles, while $\mathbf{p}$ is simply the initial momentum of the apparatus, which we take to be a scalar for notational simplicity. For the entirety of this section until the end, we will work in a spacetime box as described above, but will refrain from writing "box" superscripts. At the end of the computation we will discuss the continuum limit.

We assume that one and only one apparatus particle exists in both the initial and final state. This can be arranged for example by assigning ϕ_A some global charge, or by taking ϕ_A to have high mass and studying scattering events below its production threshold.

Using the formalism from section 2.2, we can express the density matrix

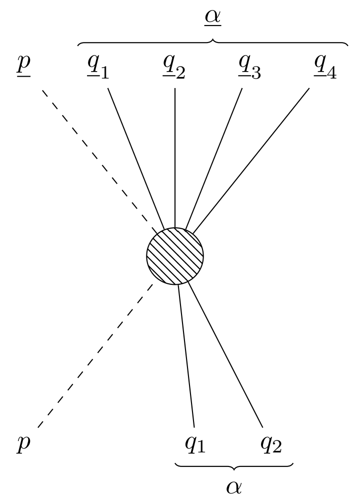

Figure 2.1: A typical apparatus-system scattering process. Dotted lines denote the apparatus, solid lines the system. Time runs from bottom to top.

in terms of out-states,

$$\rho = \sum_{\underline{\mathbf{p}}\underline{\mathbf{p}}'\underline{\alpha}\underline{\alpha}'} S_{\underline{\mathbf{p}}\underline{\alpha}\mathbf{p}\alpha} S^*_{\underline{\mathbf{p}}'\underline{\alpha}'\mathbf{p}\alpha} \left|\underline{\mathbf{p}}\underline{\alpha}^-\right\rangle \left\langle \underline{\mathbf{p}}'\underline{\alpha}'^-\right| . \tag{2.23}$$

From here out we use underlines to denote outgoing variables. Expanding the S-matrix with (2.16), one can see from this expression that ρ will have the correct norm $\operatorname{tr}\rho = 1$ if and only if the optical theorem (A.4) is satisfied (see appendix A). In particular, if one is working in perturbation theory, the optical theorem mixes orders, so one needs to be careful about including the correct set of loop and tree diagrams at a given order.

Now trace over the system, using out-states:

$$\rho_A^- = \sum_{\underline{\mathbf{p}}\underline{\mathbf{p}}'\underline{\alpha}} S_{\underline{\mathbf{p}}\underline{\alpha}\mathbf{p}\alpha} S^*_{\underline{\mathbf{p}}'\underline{\alpha}\mathbf{p}\alpha} \left|\underline{\mathbf{p}}^-\right\rangle \left\langle \underline{\mathbf{p}}'^-\right| . \tag{2.24}$$

Decompose the S-matrix with (2.16). We get three types of terms: from the delta-squared we get a term on the diagonal with momentum given by the initial momentum $\mathbf{p}$:

$$\rho_{A,1}^- = \left|\mathbf{p}^-\right\rangle \left\langle \mathbf{p}^-\right| . \tag{2.25}$$

The cross-terms $-iM\rho + i\rho M^\dagger$ give a contribution

$$\rho^-_{A,2} = -2T\ \mathrm{Im}\left[M_{\mathbf{p}\alpha\mathbf{p}\alpha}\right] |\mathbf{p}^-\rangle \langle \mathbf{p}^-| \,, \tag{2.26}$$

again to the density matrix element for the initial momentum $\mathbf{p}$. This is the forward scattering term that appears in the optical theorem. Finally, we need the terms from $M\rho M^\dagger$. One obtains

$$\rho^-_{A,3} = 2\pi T \sum_{\underline{\mathbf{p}}\underline{\alpha}} \left| M_{\underline{\mathbf{p}}\underline{\alpha}\mathbf{p}\alpha} \right|^2 \delta_{\underline{\mathbf{p}}+\mathbf{p}_{\underline{\alpha}},\mathbf{p}+\mathbf{p}_\alpha} \delta(E^A_{\underline{\mathbf{p}}} + E^S_{\underline{\alpha}} - E^A_{\mathbf{p}} - E^S_\alpha) \, |\underline{\mathbf{p}}^-\rangle \langle \underline{\mathbf{p}}^-| \,. \tag{2.27}$$

We see that the reduced density matrix for A is diagonal in an arbitrary reference frame. This is due entirely to translation invariance and our assumption that we always have precisely one apparatus particle. Writing the apparatus state in matrix form, we have

$$\rho^-_A = \begin{pmatrix} 1 + I_0 + F(\mathbf{p}) & & & \\ & F(\underline{\mathbf{p}}_1) & & \\ & & F(\underline{\mathbf{p}}_2) & \\ & & & \ddots \end{pmatrix} , \tag{2.28}$$

where the $\underline{\mathbf{p}}_i$ are all the outgoing apparatus momenta $\underline{\mathbf{p}} \neq \mathbf{p}$. The coefficients are

$$\begin{aligned} I_0 &= -2T\ \mathrm{Im} M_{\mathbf{p}\alpha\mathbf{p}\alpha} \\ F(\underline{\mathbf{p}}) &= 2\pi T \sum_{\underline{\alpha}} \left| M_{\underline{\mathbf{p}}\underline{\alpha}\mathbf{p}\alpha} \right|^2 \delta_{\underline{\mathbf{p}}+\mathbf{p}_{\underline{\alpha}},\mathbf{p}+\mathbf{p}_\alpha} \delta(E^A_{\underline{\mathbf{p}}} + E^S_{\underline{\alpha}} - E^A_{\mathbf{p}} - E^S_\alpha). \end{aligned} \tag{2.29}$$

The coefficients $F(\underline{\mathbf{p}})$ could be called "conditional transition probabilities". They are given by fixing an apparatus out-momentum $\underline{\mathbf{p}}$ and then summing over the transition probabilities to all the possible system states consisent with total momentum conservation. Note that $F(\underline{\mathbf{p}}) = 0$ for momenta violating energy conservation, that is when $E^A_{\underline{\mathbf{p}}} > E^A_{\mathbf{p}} + E^S_\alpha - E^S_0$.[2]

[2] In $2 \to 2$ scattering, we can write the return-amplitude term $I_0 + F(\mathbf{p})$ in a way that treats the two particles more symmetrically: by the optical theorem (A.4), we have

$$I_0 = -(2\pi)^2 \sum_{\underline{\mathbf{p}}\underline{\mathbf{q}}} \left| M_{\underline{\mathbf{p}}\underline{\mathbf{q}}\mathbf{p}\mathbf{q}} \right|^2 \delta_{\underline{\mathbf{p}}+\underline{\mathbf{q}},\mathbf{p}+\mathbf{q}} \delta_{E^A_{\underline{\mathbf{p}}}+E^S_{\underline{\mathbf{q}}},E^A_{\mathbf{p}}+E^S_{\mathbf{q}}} \tag{2.30}$$

while by definition, $F(\mathbf{p}) = (2\pi)^2 \left| M_{\mathbf{p}\mathbf{q}\mathbf{p}\mathbf{q}} \right|^2$. So the shift in the initial-momentum density

The von Neumann entanglement entropy of the apparatus is given by

$$S_A = -(1 + I_0 + F(\mathbf{p})) \ln(1 + I_0 + F(\mathbf{p})) - \sum_{\underline{\mathbf{p}} \neq \mathbf{p}} F(\underline{\mathbf{p}}) \ln F(\underline{\mathbf{p}}). \tag{2.32}$$

The result (2.32) is exact and non-perturbative. It follows completely from Lorentz invariance and our assumption that precisely one A particle is in both the initial and final state. It can be simplified by invoking perturbation theory: we assume that the scattering amplitudes are significantly less than unity. Then $|I_0 + F(\mathbf{p})| \ll 1$, so we can Taylor expand the first term in (2.32) and get a term linear in this expression. But the other terms still have logarithms, so we have an expression like small $+ \sum$ small ln(small), and the log terms will dominate. So we are left with

$$S_A = -\sum_{\underline{\mathbf{p}}} F(\underline{\mathbf{p}}) \ln F(\underline{\mathbf{p}}). \tag{2.33}$$

In a large box, it is immaterial if the sum on outgoing apparatus momenta $\underline{\mathbf{p}}$ includes $\underline{\mathbf{p}} = \mathbf{p}$ or not, since this term is individually of measure zero.

2.4 Examples with Two Scalar Fields

We will now consider some simple applications of the above theory, with both system and apparatus described by scalar fields $\phi_{A,S}$ with a weak coupling λ. Throughout, we will assume that the initial energies are below the threshold for on-shell pair-production, so that we can work entirely with $2 \to 2$ matrix elements.

In the first subsection, we study entropy generated during a $2 \to 2$ scattering event. In the second subsection, we show how to verify that the system S has been prepared in a spatial superposition by scattering with A. More precisely, we show how to read out the coherence of such a superposition using position-space information in A, at lowest order in λ.

Let us fix our conventions. We take the apparatus and system to be

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matrix eigenvalue is

$$\Delta_0 = -(I_0 + F(\mathbf{p})) = (2\pi)^2 \sum_{(\underline{\mathbf{p}},\underline{\mathbf{q}}) \neq (\mathbf{p},\mathbf{q})} \left| M_{\underline{\mathbf{p}}\underline{\mathbf{q}}\mathbf{p}\mathbf{q}} \right|^2 \delta_{\underline{\mathbf{p}}+\underline{\mathbf{q}},\mathbf{p}+\mathbf{q}} \delta_{E^A_{\underline{\mathbf{p}}}+E^S_{\underline{\mathbf{q}}},E^A_{\mathbf{p}}+E^S_{\mathbf{q}}}. \tag{2.31}$$

described by the action

$$S = -\int d^4x \; \frac{1}{2}(\partial_\mu \phi_S)^2 + \frac{1}{2}(\partial_\mu \phi_A)^2 + \frac{1}{2}m_S^2\phi_S^2 + \frac{1}{2}m_A^2\phi_A^2 + \frac{\lambda}{4}\phi_S^2\phi_A^2 + \frac{\lambda_A}{4!}\phi_A^4 + \frac{\lambda_S}{4!}\phi_S^4 + \mathcal{L}_{ct}. \tag{2.34}$$

In particular, the fields $\phi_{S,A}$ are considered to be distinguishable and renormalized. The term $\mathcal{L}_{ct}$ contains the counterterms; here we use the standard on-shell renormalization conditions that the on-shell propagators have unit residue at the physical masses and the interactions are given exactly by their physical couplings at threshold. This way we can work with amputated diagrams only, and the lowest order in perturbation theory is just tree level. We will take up loop corrections in a future publication. We assume that the self-couplings $\lambda_{A,S} \ll 1$ and ignore them hereafter.

The free single-particle states and operators are normalized as

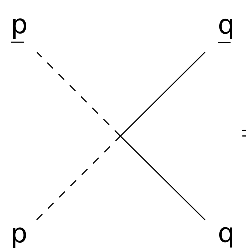
$$\langle \mathbf{k}' | \mathbf{k} \rangle = \left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger \right] = \delta^3(\mathbf{k} - \mathbf{k}'). \tag{2.35}$$

More generally, a free n-particle state of a given species is $|\mathbf{k}_1 \cdots \mathbf{k}_n\rangle = a_{\mathbf{k}_n}^\dagger \cdots a_{\mathbf{k}_1}^\dagger |0\rangle$, where $|0\rangle$ is the free vacuum. In what follows we use $\mathbf{p}$ to denote the 3-momentum of the apparatus and $\mathbf{q}$ that of the system. The relevant S-matrix elements are then

$$S_{\underline{\mathbf{p}}\underline{\mathbf{q}}\mathbf{p}\mathbf{q}} = \delta^3(\underline{\mathbf{p}} - \mathbf{p})\delta^3(\underline{\mathbf{q}} - \mathbf{q}) - 2\pi i M_{\underline{\mathbf{p}}\underline{\mathbf{q}}\mathbf{p}\mathbf{q}} \delta^4(\underline{p} + \underline{q} - p - q) \tag{2.36}$$

with the amplitude given by, to lowest order in perturbation theory,

$$iM_{\underline{\mathbf{p}}\underline{\mathbf{q}}\mathbf{p}\mathbf{q}} =$$



$$= \frac{i\lambda}{(2\pi)^3\sqrt{16 E_{\underline{\mathbf{p}}}^A E_{\underline{\mathbf{q}}}^S E_{\mathbf{p}}^A E_{\mathbf{q}}^S}}. \tag{2.37}$$

Here the single-particle energies are

$$E_{\mathbf{k}}^{S,A} = \sqrt{m_{S,A}^2 + \mathbf{k}^2}. \tag{2.38}$$

2.4.1 Entropy from $2 \to 2$ Scattering

To begin, we study the simplest possible process: scattering with the system and apparatus both prepared in definite momentum eigenstates at early times,

$$|\psi\rangle = |\mathbf{p}^+\rangle_A \, |\mathbf{q}^+\rangle_S \, . \tag{2.39}$$

This is precisely what we studied in section 2.3 and, as we did there, we will work with box-normalized states until the end of the calculation.

After the scattering event, the von Neumann entropy of the apparatus is given directly by our formula (2.33), viz.

$$S_A = -\sum_{\underline{\mathbf{p}}} F(\underline{\mathbf{p}}) \ln F(\underline{\mathbf{p}}). \tag{2.40}$$

Again the sum runs over all outgoing apparatus momenta $\underline{\mathbf{p}}$, and the coefficients $F(\underline{\mathbf{p}})$ are defined in (2.29). Because scattering in this theory is isotropic, it is straightforward to compute the apparatus density matrix eigenvalues explicitly. Move to the center-of-momentum frame $\mathbf{p} = -\mathbf{q}$. Then

$$\begin{aligned} F(\underline{\mathbf{p}}) &= 2\pi T \sum_{\underline{\mathbf{q}}} \left| M_{\underline{\mathbf{p}}\underline{\mathbf{q}}\mathbf{p}\mathbf{q}} \right|^2 \delta_{\underline{\mathbf{p}}+\underline{\mathbf{q}},\mathbf{p}+\mathbf{q}} \delta(E^A_{\underline{\mathbf{p}}} + E^S_{\underline{\mathbf{q}}} - E^A_{\mathbf{p}} - E^S_{\mathbf{q}}) \\ &= 2\pi T \left| M(p_{cm}) \right|^2 \delta(f(|\underline{\mathbf{p}}|)), \end{aligned} \tag{2.41}$$

where we used isotropy of the interaction to write this as

$$M(p_{cm}) = M_{\underline{\mathbf{p}},-\underline{\mathbf{p}};\mathbf{p},-\mathbf{p}}, \quad p_{cm} = |\mathbf{p}| = |\underline{\mathbf{p}}|, \quad f(|\underline{\mathbf{p}}|) = E^A_{|\underline{\mathbf{p}}|} + E^S_{|\underline{\mathbf{p}}|} - E^A_{|\mathbf{p}|} - E^S_{|\mathbf{p}|}. \tag{2.42}$$

The entropy of A at late times is thus given by

$$S_A = -2\pi T \sum_{\underline{\mathbf{p}}} \left| M(p_{cm}) \right|^2 \delta(f(|\underline{\mathbf{p}}|)) \ln \left[2\pi T \left| M(p_{cm}) \right|^2 \delta(f(|\underline{\mathbf{p}}|)) \right]. \tag{2.43}$$

At this stage, we can take the continuum limit. We replace the sum $\sum_{\underline{\mathbf{p}}} \to V/(2\pi)^3 \int d^3\underline{\mathbf{p}}$, and do the integral in spherical coordinates. The delta-function outside the log enforces energy conservation, and so the delta inside the log is replaced by $\delta_T(0) = T/2\pi$. We also have to insert the appropriate factors of $\tilde{N} = (2\pi)^3/V$ to convert from the box-normalized amplitude to the continuum-normalized one, see eq. (2.16). Finally, we obtain

$$S_A = -2(2\pi)^5 \frac{T}{V} p_{cm}^2 (E^A + E^S) \left|M(p_{cm})\right|^2 \ln\left[(2\pi)^6 \frac{T^2}{V^2} \left|M(p_{cm})\right|^2\right], \quad (2.44)$$

where the energies are understood to be evaluated at p_{cm}. This holds at any order of perturbation theory. If we wanted to work to lowest order in perturbation theory, we can use our matrix element (2.37) given above, in which case we have explicitly[22]

$$S_A = -\frac{T}{V} \frac{\lambda^2}{16\pi} \frac{p_{cm}(E^A + E^S)}{(E^A E^S)^2} \ln\left[\frac{T^2}{V^2} \frac{\lambda^2}{16(E^A E^S)^2}\right]. \quad (2.45)$$

This formula bears some remarking. For one thing, recall that the total cross-section for this theory at this order of perturbation theory is given by $\sigma = \lambda^2/16\pi E^A E^S$ in the center-of-momentum frame. So we have that the entropy is proportional to this quantity, integrated over time and against the flux of incoming particles.[3] We always have a large spatial volume V in mind, so $S_A \geq 0$. The argument of the logarithm likewise cannot be too small: if $T\lambda/16VE^A E^S \leq 1$ then the entropy will be negative. This is essentially the statement that the Compton wavelengths of the particles need to be within the spacetime box. As we take the spatial volume $V \to \infty$ with T fixed, S_A goes to zero from above; this follows from the fact that the probability of the waves to interact at all goes to zero. Finally, one might worry about V fixed and $T \to \infty$, in which case the entropy goes to $-\infty$, but this corresponds to an infinite number of repeated interactions, which would also violate the basic assumption of the S-matrix setup that we are describing an isolated event.

2.4.2 Verifying Spatial Superpositions

Let's consider now a rather different problem. Suppose we prepare the system and apparatus in a separable state, but the system state may or may not be pure. We would like to know how this system information would show up in the outgoing apparatus state.

For definiteness, we consider the following problem: suppose that some black box machine in our lab prepares the system as either a classical ensemble or coherent superposition of two system states, each localized to a different point in real space. The question is: how do we verify the coherence

[3] In this frame, the flux is $\Phi = u/V$ with the relative velocity $u = p_{cm}(E^A + E^S)/E^A E^S$.

of the superposition from a scattering experiment?

We will see that it is sufficient to look at the position-space wavefunction of the outgoing apparatus at order λ. The signature of the system superposition is interference fringes in the apparatus state. They show up at order λ because the position-space projector $|\mathbf{x}\rangle\langle\mathbf{x}|$ is sensitive to off-diagonal momentum-space apparatus density matrix elements, which are generated at first order in the perturbation, as we now demonstrate explicitly.

We begin by defining a pair of states $|L\rangle, |R\rangle$ that describe the apparatus prepared in an incoming state of momentum $\mathbf{p}$ and the system centered at different positions $\mathbf{x}_{L,R}$ in real space.[4] Define the usual Gaussian wavefunction

$$g(\mathbf{q}) = N_S \exp\left\{-\mathbf{q}^2/4\sigma_S\right\}, \quad N_S = \frac{1}{(2\pi\sigma_S)^{3/4}} \tag{2.47}$$

and take the system to be initialized at rest in a lab frame, so we define the state as follows: let $i \in \{L, R\}$ and put

$$\begin{aligned} |i\rangle &= N_A \left|\mathbf{p}^+\right\rangle_A \int d^3\mathbf{q}\; f_i(\mathbf{q}) \left|\mathbf{q}^+\right\rangle_S, \\ f_i(\mathbf{q}) &= g(\mathbf{q}) \exp\left\{i\mathbf{q}\cdot\mathbf{x}_i\right\}, \quad N_A = \sqrt{\frac{(2\pi)^3}{V}}. \end{aligned} \tag{2.48}$$

See figure 2.2. These states are not orthogonal; their overlap is

$$\epsilon = \langle L|R\rangle = \exp\left\{-\sigma_S \left|\Delta\mathbf{x}_0\right|^2/2\right\}, \quad \Delta\mathbf{x}_0 = \mathbf{x}_L - \mathbf{x}_R. \tag{2.49}$$

We have in mind that the system states are localized in real space, so that the momentum spread σ_S is large. The two states are well-separated if $\epsilon \ll 1$; we assume this below for mathematical ease, but the results do not depend qualitatively on this condition.[5] We are assume that the scattering is done in a sufficiently short time so that we can ignore the spreading of these wavepackets.

Now consider an arbitrary density matrix in the space spanned by the

---

[4] In this section we will use continuum-normalized states, regulating squares of Dirac deltas as

$$\left[\delta^3(\mathbf{p}-\mathbf{p}')\right]^2 = \frac{V}{(2\pi)^3}\delta^3(\mathbf{p}-\mathbf{p}'), \quad \left[\delta(E-E')\right]^2 = \frac{T}{2\pi}\delta(E-E'). \tag{2.46}$$

[5] When working with the following formulas, the non-orthogonality of $|L\rangle, |R\rangle$ should be kept in mind; in particular traces should be done with momentum eigenstates. A useful relation is $\operatorname{tr}|i\rangle\langle j| = \langle i|j\rangle = \epsilon$ for $i \neq j$ and 1 for $i = j$.

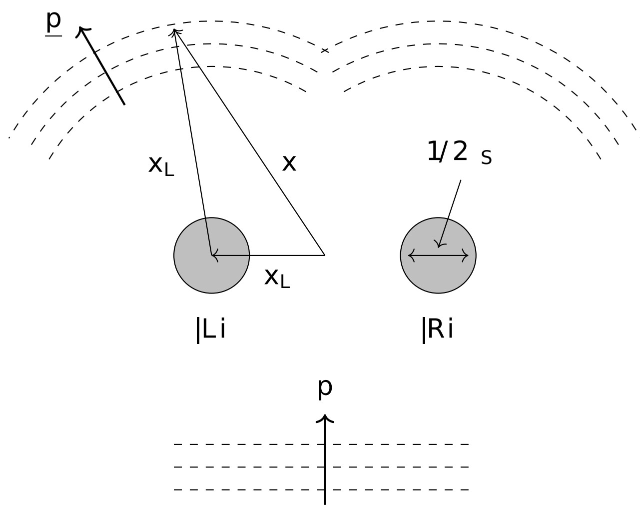


Figure 2.2: Verifying spatial superpositions of the system states $|L\rangle$, $|R\rangle$.

$|L\rangle\,,|R\rangle$ states:

$$\rho = \Gamma^{ij}\,|i\rangle\,\langle j|\,,\quad i,j\in\{L,R\}\,. \tag{2.50}$$

For example, we can form a convex family of density matrices, with coefficients

$$\Gamma^{ij}(\alpha) = \frac{1}{2(1+\epsilon)}\begin{pmatrix} 1+\epsilon-\alpha\epsilon & \alpha \\ \alpha & 1+\epsilon-\alpha\epsilon \end{pmatrix},\quad 0\le\alpha\le 1. \tag{2.51}$$

These linearly interpolate between the classical ensemble proportional to $|L\rangle\,\langle L|+|R\rangle\,\langle R|$ at $\alpha=0$ and the perfect coherent superposition proportional to $(|L\rangle+|R\rangle)(\langle L|+\langle R|)$ at $\alpha=1$. These all have unit trace, while the purity $\mathrm{tr}\,\rho^2(\alpha)=[1+(\alpha+\epsilon-\alpha\epsilon)^2]/2$ vanishes when $\epsilon=\alpha=0$ and goes up to unity if either $\epsilon=1$ or $\alpha=1$. We will refer to α as the coherence parameter. Note in particular that the off-diagonal element Γ^{LR} is linear in α. The reduced density matrix for the apparatus expressed with out-states is

$$\rho_A^- = N_A^2\sum_{ij}\int d^3\underline{\mathbf{p}}d^3\underline{\mathbf{p}}'d^3\underline{\mathbf{q}}d^3\mathbf{q}d^3\mathbf{q}'\Gamma^{ij}f_i(\mathbf{q})f_j^*(\mathbf{q}')S_{\underline{\mathbf{p}}\underline{\mathbf{q}}\mathbf{p}\mathbf{q}}S^*_{\underline{\mathbf{p}}'\underline{\mathbf{q}}\mathbf{p}\mathbf{q}'}\,|\underline{\mathbf{p}}^-\rangle\,\langle\underline{\mathbf{p}}'^-|\,. \tag{2.52}$$

Let's study some outgoing apparatus observables. Consider first the outgoing momentum distribution $P(\underline{\mathbf{p}})$ of the apparatus, so that we take $\mathcal{O}_A=|\underline{\mathbf{p}}\rangle\,\langle\underline{\mathbf{p}}|$ and use (2.19); the expectation value can be read off from the diagonal elements of (2.52). We can work these out a bit more explic-

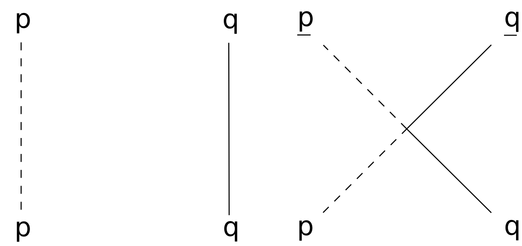


Figure 2.3: Diagrams contributing to the lowest-order position-space distribution of the apparatus.

itly. The identity-squared term from decomposing the S-matrix with (2.11) contributes to $P(\underline{\mathbf{p}})$ as $P_0(\underline{\mathbf{p}}) = \delta^3(\underline{\mathbf{p}} - \mathbf{p})$. The interaction terms give

$$P_{int}(\underline{\mathbf{p}}) = (2\pi)^3 \frac{T}{V} \int d^3\mathbf{q} \, |g(\mathbf{q})|^2 \left(1 + \alpha \cos 2\mathbf{q} \cdot \Delta\mathbf{x}_0\right) \Big\{ -2\mathrm{Im} M_{pqpq} \delta^3(\underline{\mathbf{p}} - \mathbf{p}) + \left| M_{\underline{\mathbf{p}},\mathbf{q}-\mathbf{k};\mathbf{p},\mathbf{q}} \right|^2 \delta\left(E^A_{\underline{\mathbf{p}}} + E^S_{\mathbf{q}-\mathbf{k}} - E^A_{\mathbf{p}} - E^S_{\mathbf{q}}\right) \Big\} \tag{2.53}$$

where here $\mathbf{k} = \underline{\mathbf{p}} - \mathbf{p}$ is the momentum transfer, and we took $\epsilon \ll 1$ to write the result in a simple way. We see that the overall probability is proportional to T/V, as expected. Both terms receive a contribution from the coherence α of the initial superposition. In our specific theory (2.37), both of these contributions are of order λ^2, with the forward-scattering term in (2.53) coming in only at one-loop order. So to measure α by doing such an observation, we would have to be sensitive at order λ^2.

However, it is possible to see signatures of the coherence α at first order in λ if we instead look at position-space observables. Consider the position-space probability distribution for the apparatus at late times after the scattering, $P(\mathbf{x})$. This can be obtained by again applying (2.19) but now using the observable $\mathcal{O}_A = |\mathbf{x}\rangle \langle\mathbf{x}|$, the single-particle position projector. The delta-squared terms from the S-matrix result in $P_0(\mathbf{x}, t) = V^{-1}$ by direct computation. Next we need both the cross terms $M\rho - \rho M^\dagger$ and the amplitude-square $M\rho M^\dagger$ term; the latter will start at $\mathcal{O}(\lambda^2)$, so let us consider the former. A straightforward calculation using hermiticity of Γ^{ij}

gives

$$P_1(\mathbf{x}) = \frac{4\pi}{V}\int d^3\mathbf{q}d^3\underline{\mathbf{q}}\delta(E^A_{\mathbf{p}+\mathbf{q}-\underline{\mathbf{q}}} + E^S_{\underline{\mathbf{q}}} - E^A_{\mathbf{p}} - E^S_{\mathbf{q}})g(\mathbf{q})g^*(\underline{\mathbf{q}})$$
$$\times \operatorname{Im}\left[M_{\mathbf{p}+\mathbf{q}-\underline{\mathbf{q}},\underline{\mathbf{q}};\mathbf{p},\mathbf{q}}\sum_{ij}\Gamma^{ij}\exp\left\{-i\phi_{ij}(\mathbf{q},\underline{\mathbf{q}})\right\}\right], \tag{2.54}$$

where the subscript 1 means we are thinking of this in first-order perturbation theory, and the phases are

$$\phi_{ij}(\mathbf{q},\underline{\mathbf{q}}) = -E^S_{\mathbf{q}}t + (\mathbf{x}-\mathbf{x}_i)\cdot\mathbf{q} + E^S_{\underline{\mathbf{q}}}t - (\mathbf{x}-\mathbf{x}_j)\cdot\underline{\mathbf{q}}. \tag{2.55}$$

Consider measuring the location of the outgoing A particle when t and $|\mathbf{x}-\mathbf{x}_i|$ are of the same order and large. Then the integral may be approximated by its stationary phase value, which here is given when

$$\mathbf{q} = \mathbf{q}_i = m_S\gamma_i v_i\widehat{\Delta\mathbf{x}_i}, \quad \underline{\mathbf{q}} = \mathbf{q}_j = m_S\gamma_j v_j\widehat{\Delta\mathbf{x}_j}, \tag{2.56}$$

where

$$\Delta\mathbf{x}_i = \mathbf{x}-\mathbf{x}_i, \quad \widehat{\Delta\mathbf{x}_i} = \frac{\Delta\mathbf{x}_i}{|\Delta\mathbf{x}_i|}, \quad v_i = \frac{|\Delta\mathbf{x}_i|}{t}, \quad \gamma_i = \frac{1}{\sqrt{1-v_i^2}}. \tag{2.57}$$

Note that $v_i = v_i(\mathbf{x},t)$ and likewise $\gamma_i = \gamma_i(\mathbf{x},t)$ depend on the point of observation $\mathbf{x}$ and the time t; we suppress this dependence in the formulas that follow. At these values for the momenta, we have that

$$E^S_{q_i} = m_S\gamma_i, \quad \phi_{ij} = -m_S t\left[\gamma_i^{-1} - \gamma_j^{-1}\right] = -\phi_{ji}. \tag{2.58}$$

In particular, we see that the LL and RR terms have zero phase, and thus give real contributions in (2.54) since our amplitude (2.37) is real at lowest order, so they do not contribute to the outgoing position distribution. The interference terms LR and RL do contribute, however, and we get

$$P_1(\mathbf{x},t) = A(\mathbf{x},t)\sin\left(\phi_{LR}(\mathbf{x},t)\right) \tag{2.59}$$

where at this point we have finally used the reality of our amplitude (2.37). The position-space amplitude is

$$A = \alpha\frac{2(2\pi)^4}{V}\gamma_L^{5/2}\gamma_R^{5/2}m_S^3t^{-3}g(\mathbf{q}_L)g(\mathbf{q}_R)\left[\delta_{LR}M_{LR} - \delta_{RL}M_{RL}\right] \tag{2.60}$$

where we defined for brevity

$$M_{ij} = M_{\mathbf{p}_0+\mathbf{q}_i-\mathbf{q}_j,\mathbf{q}_j;\mathbf{p}_0,\mathbf{q}_i}, \quad \delta_{ij} = \delta(E^A_{\mathbf{p}+\mathbf{q}_i-\mathbf{q}_j} + E^S_{\mathbf{q}_i} - E^A_{\mathbf{p}} - E^S_{\mathbf{q}_j}). \quad (2.61)$$

The delta-functions localize the distribution to the stationary-phase wavefronts, and are an artifact of the way we did the integrals. In reality, they should be smoothed out.

The key physics is in the sine term in (2.59), and the fact that A is linear in both the coupling λ and coherence parameter α. The amplitude A is a rather complicated function of $\mathbf{x}, t$, but the point is clear enough: if we arrange an array of particle detectors in a sphere around the origin, it will pick up the interference pattern given by the sine term in (2.59). The heights of the interference fringes, in turn, are set by the coherence α: in particular, if the system is initialized in a classical ensemble, $\alpha = 0$ and there are no fringes.

Physically, these are interferences between the process where no scattering occurs and the process where the apparatus scatters off one or the other system locations, see figure 2.3. Mathematically, this is in the $M\rho 1 - 1\rho M^\dagger$ terms in the action of the S-matrix on the density matrix. This is why the interference appears at order λ and not λ^2. This should be contrasted with momentum-space observables, which are only sensitive to the interference at λ^2: the position-space observable is sensitive to off-diagonal momentum-space density matrix elements, which are generated at lowest order in perturbation theory.

2.5 Conclusions

We have studied some prototypical examples of an apparatus particle scattering off a collection of system particles, applying the language of quantum measurement theory to a field-theoretic problem. Our general density matrix formalism allows for the computation of arbitrary apparatus observables at early and late times, and we showed how to compute the apparatus-system entanglement entropy generated during scattering.

Our scenario contrasts standard formulations of measurement theory in some significant ways. For one thing, our system and apparatus are relativistic and have continuous spectra. For another, we do not imagine that we can precisely engineer some interaction Hamiltonian; here we are just stuck with whatever our effective field theory happens to give us. Nonetheless we have found that it is straightforward to use standard measurement-theory techniques.

A potential application is detection of system properties at lower orders of perturbation theory than usually considered in scattering. For example, one often hears that $\lambda\phi^4$ scattering is only sensitive to λ^2 as opposed to λ, because the cross-section scales like λ^2. On the contrary, one can clearly do an interference measurement as described above to measure the coupling at order λ.

More theoretically, these kinds of calculations may help shed some light on certain aspects of black hole physics. In particular, a recent proposal is that the black hole information is radiated out to null infinity by soft bosonic modes.[24] This information should thus be quantified by precisely the kind of von Neumann entropy we have considered here. Implications of the soft boson theorems for the entropy calculations presented above will appear in a future article.

Chapter 3

Infrared Quantum Information

We discuss information-theoretic properties of low-energy photons and gravitons in the S-matrix. Given an incoming n-particle momentum eigenstate, we demonstrate that unobserved soft photons decohere nearly all outgoing momentum superpositions of charged particles, while the universality of gravity implies that soft gravitons decohere nearly all outgoing momentum superpositions of <u>all</u> the hard particles. Using this decoherence, we compute the entanglement entropy of the soft bosons and show that it is infrared-finite when the leading divergences are re-summed à la Bloch and Nordsieck.

3.1 Introduction

The massless nature of photons and gravitons leads to an infrared catastrophe, in which the S-matrix becomes ill-defined due to divergences coming from low-energy virtual bosons. The usual solution to this problem, originally given by Bloch and Nordsieck in electrodynamics [1] and extended to gravity by Weinberg [2], is to argue that an infinite number of low-energy bosons are radiated away during a scattering event; this leads to divergences which cancels the divergences from the virtual states, and physical predictions in terms of infrared-finite inclusive transition probabilities.

In this letter, we study quantum information-theoretic aspects of this proposal. Since each photon and graviton has two polarization states and three momentum degrees of freedom, one might suspect that the low-energy radiation produced during scattering could carry a huge amount of information. Here we demonstrate that, according to the methodology of [1, 2, 25], if the initial state is an incoming n-particle momentum eigenstate, the "soft" bosonic divergences can lead to complete decoherence of the momentum state of the outgoing "hard" particles. This decoherence is avoided only for superpositions of pairs of outgoing states for which an infinite set of angle-dependent currents match, see eq. (3.11). In simple examples like QED, this will be enough to get complete decoherence of all momentum superpositions. In less simple cases, one is still left with an extremely sparse density matrix dominated by its diagonal elements.

Having traced the radiation in this fashion, we obtain an infrared-finite, mixed reduced density matrix for the hard particles. In the simple cases when we get a completely diagonal matrix, we compute the entanglement entropy carried by the soft gauge bosons. The answer is finite and scales like the logarithm of the energy resolution E of a hypothetical soft boson detector.

While the tracing out of the soft radiation can be viewed as a physical statement about the energy resolution of a real detector, in this formalism, the trace is also forced on us by mathematical consistency: it is the only way to get well-defined transition probabilities from the infrared-divergent S-matrix. There is an alternative approach to the infrared catastrophe, in which one constructs an IR-finite S-matrix of transition amplitudes between "dressed" matter states.[3, 4, 26, 27] In such an approach, there are no divergences and so one is not forced to trace over any soft radiation. Whether the two formalisms lead to the same physical picture is an interesting question, and we leave a detailed comparison to future work.

Recently, the infrared structure of gauge theories has become a topic of much interest due to the proposal that soft radiation may encode information about the history of formation of a black hole.[24, 28, 29] We hope that our work can make this discussion more quantitatively grounded; we comment on black holes at the end of this letter. More generally, it is of interest to understand the information-theoretic nature of the infrared sector of quantum field theories, and our paper is intended to make some first steps in this direction.

3.2 Decoherence of the Hard Particles.

Fix a single-particle energy resolution E. We define soft bosons as those with energy less than E, and hard particles as anything else. Consider an incoming state $|\alpha\rangle_{in}$ consisting of hard particles, charged or otherwise, of definite momenta.[1] The S-matrix evolves this into a coherent superposition of states with hard particles β and soft bosons $b = \gamma, h$ (photons γ and gravitons h),

$$|\alpha\rangle_{in} = \sum_{\beta b} S_{\beta b,\alpha} \, |\beta b\rangle_{out} \, . \tag{3.1}$$

[1]Our field theory conventions follow [21]. Labels like α, β, b mean a list of free-particle quantum numbers, e.g. $|\alpha\rangle_{in} = |\mathbf{p}_1\sigma_1, \ldots\rangle_{in}$ listing momenta and spin of the incoming particles.

Hereafter we drop the subscript on kets, which will always be out-states. Tracing out the bosons $|b\rangle$, the reduced density matrix for the outgoing hard particles is

$$\rho = \sum_{\beta\beta' b} S_{\beta b,\alpha} S^*_{\beta' b,\alpha} |\beta\rangle \langle\beta'| \,. \tag{3.2}$$

Using the usual soft factorization theorems [2, 25, 30], we can write the amplitudes in terms of the amplitudes for $\alpha \to \beta$ multiplied by soft factors, one for each boson:

$$S_{\beta b,\alpha} = S_{\beta,\alpha} F_{\beta,\alpha}(\gamma) G_{\beta,\alpha}(h), \tag{3.3}$$

where the soft factors F, G are

$$\begin{aligned} F_{\beta,\alpha}(\gamma) &= \sum_{n\in\alpha,\beta} \sum_{\pm} \prod_{i\in\gamma} \frac{e_n \eta_n}{(2\pi)^{3/2} |\mathbf{k}_i|^{1/2}} \frac{p_n^\mu \epsilon^*_{\mu,\pm}(\mathbf{k}_i)}{p_n \cdot k_i - i\eta_n \epsilon} \\ G_{\beta,\alpha}(h) &= \sum_{n\in\alpha,\beta} \sum_{\pm} \prod_{i\in h} \frac{M_p^{-1} \eta_n}{(2\pi)^{3/2} |\mathbf{k}_i|^{1/2}} \frac{p_n^\mu p_n^\nu \epsilon^*_{\mu\nu,\pm}(\mathbf{k}_i)}{p_n \cdot k_i - i\eta_n \epsilon} . \end{aligned} \tag{3.4}$$

Here the index n runs over all the incoming and outgoing hard particles, i runs over the outgoing soft bosons; $\eta_n = -1$ for an incoming and $+1$ for an outgoing hard particle. The e_n are electric charges and $M_p = (8\pi G_N)^{-1/2}$ is the Planck mass, and the ϵ's are polarization vectors or tensors for outgoing soft photons and gravitons, respectively. By an argument identical to the one employed by Weinberg [2], and assuming we can neglect the total lost energy E_T compared to the energy of the hard particles, we can use this factorization to perform the sum over soft bosons in (3.2), and we find that

$$\begin{aligned} \sum_b S_{\beta b,\alpha} S^*_{\beta' b,\alpha} = S_{\beta,\alpha} S^*_{\beta',\alpha} &\left(\frac{E}{\lambda}\right)^{\tilde{A}_{\beta\beta',\alpha}} \left(\frac{E}{\lambda}\right)^{\tilde{B}_{\beta\beta',\alpha}} \\ &\times f\left(\frac{E}{E_T}, \tilde{A}_{\beta\beta',\alpha}\right) f\left(\frac{E}{E_T}, \tilde{B}_{\beta\beta',\alpha}\right) . \end{aligned} \tag{3.5}$$

Here $\lambda \ll E$ is an infrared regulator used to cut off momentum integrals which we will send to zero later; one can think of λ as a mass for the photon and graviton. The exponents are

$$\begin{aligned}
\tilde{A}_{\beta\beta',\alpha} &= -\sum_{\substack{n\in\alpha,\beta\\ n'\in\alpha,\beta'}} \frac{e_n e_{n'}\eta_n\eta_{n'}}{8\pi^2}\beta_{nn'}^{-1}\ln\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right] \\
\tilde{B}_{\beta\beta',\alpha} &= \sum_{\substack{n\in\alpha,\beta\\ n'\in\alpha,\beta'}} \frac{m_n m_{n'}\eta_n\eta_{n'}}{16\pi^2 M_p^2}\frac{1+\beta_{nn'}^2}{\beta_{nn'}\sqrt{1-\beta_{nn'}^2}}\ln\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right],
\end{aligned} \tag{3.6}$$

and f is a complicated function which can be found in [21]; for $E/E_T = \mathcal{O}(1)$ and for small A, f may be approximated as $f(1,A) \approx 1-\pi^2A^2/12+\mathcal{O}(A^4)$. In these formulas, $\beta_{nn'}$ is the relative velocity between particles n and n',

$$\beta_{nn'} = \sqrt{1-\frac{m_n^2 m_{n'}^2}{(p_n\cdot p_{n'})^2}},$$

For future use, we note that $0 \le \beta \le 1$, and both of the dimensionless functions of β appearing in (3.6) run over $[2,\infty)$ as β runs from 0 to 1. We have $\beta_{nm} = 0$ if and only if $p_n = p_m$.

The divergences as $\lambda \to 0$ in (3.5) come from summing over an infinite number of radiated, on-shell bosons. There are also infrared divergences inherent to the transition amplitude $S_{\beta,\alpha}$ itself coming from virtual bosons. Again following Weinberg, we can add these divergences up, and we have that

$$S_{\beta,\alpha} = S^{\Lambda}_{\beta,\alpha}\left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2}\left(\frac{\lambda}{\Lambda}\right)^{B_{\beta,\alpha}/2}, \tag{3.7}$$

where now $S^{\Lambda}_{\beta,\alpha}$ means the amplitude computed using only virtual bosons of energy above Λ, and

$$\begin{aligned}
A_{\beta,\alpha} &= -\sum_{n,m\in\alpha,\beta} \frac{e_n e_m\eta_n\eta_m}{8\pi^2}\beta_{nm}^{-1}\ln\left[\frac{1+\beta_{nm}}{1-\beta_{nm}}\right] \\
B_{\beta,\alpha} &= \sum_{n,m\in\alpha,\beta} \frac{m_n m_m\eta_n\eta_m}{16\pi^2 M_p^2}\frac{1+\beta_{nm}^2}{\beta_{nm}\sqrt{1-\beta_{nm}^2}}\ln\left[\frac{1+\beta_{nm}}{1-\beta_{nm}}\right].
\end{aligned} \tag{3.8}$$

An infrared-divergent "Coulomb" phase is suppressed in (3.7). We will see shortly that this phase cancels out of all the relevant density matrix elements.

Putting the above results together, we find that the reduced density matrix coefficient for $|\beta\rangle\langle\beta'|$ is given by

$$\begin{aligned}\rho_{\beta\beta'} = S^{\Lambda}_{\beta,\alpha} S^{\Lambda *}_{\beta',\alpha} \left(\frac{E}{\lambda}\right)^{\tilde{A}_{\alpha,\beta\beta'}} \left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2 + A_{\beta',\alpha}/2} \\ \times \left(\frac{E}{\lambda}\right)^{\tilde{B}_{\alpha,\beta\beta'}} \left(\frac{\lambda}{\Lambda}\right)^{B_{\beta,\alpha}/2 + B_{\beta',\alpha}/2} f(\tilde{A}_{\beta\beta',\alpha}) f(\tilde{B}_{\beta\beta',\alpha}).\end{aligned} \tag{3.9}$$

The question is how this behaves in the limit that the infrared regulator $\lambda \to 0$. The coefficient scales as $\lambda^{\Delta A + \Delta B}$, where

$$\begin{aligned}\Delta A_{\beta\beta',\alpha} &= \frac{A_{\beta,\alpha}}{2} + \frac{A_{\beta',\alpha}}{2} - \tilde{A}_{\beta\beta',\alpha} \\ \Delta B_{\beta\beta',\alpha} &= \frac{B_{\beta,\alpha}}{2} + \frac{B_{\beta',\alpha}}{2} - \tilde{B}_{\beta\beta',\alpha}.\end{aligned} \tag{3.10}$$

In the appendix, we prove that both of these exponents are positive-definite, $\Delta A_{\beta\beta',\alpha} \geq 0$ and $\Delta B_{\beta\beta',\alpha} \geq 0$. The density matrix components (3.9) which survive as the regulator $\lambda \to 0$ are those for which $\Delta A = \Delta B = 0$; all other density matrix elements will vanish.

To give necessary and sufficient conditions for $\Delta A = \Delta B = 0$, we define two currents for each spatial velocity vector $\mathbf{v}$. We assume for simplicity that only massive particles carry electric charge. For massive particles, there are electromagnetic and gravitational currents defined as

$$\begin{aligned} j^{EM}_{\mathbf{v}} &= \sum_i e^i a^{i\dagger}_{\mathbf{p}_i(\mathbf{v})} a^i_{\mathbf{p}_i(\mathbf{v})} \\ j^{GR}_{\mathbf{v}} &= \sum_i E_i(\mathbf{v}) a^{i\dagger}_{\mathbf{p}_i(\mathbf{v})} a^i_{\mathbf{p}_i(\mathbf{v})}.\end{aligned} \tag{3.11}$$

Here i labels particle species, e^i their charges and m^i their masses; the kinematic quantities $\mathbf{p}_i(\mathbf{v}) = m_i \mathbf{v}/\sqrt{1-\mathbf{v}^2}$ and $E_i(\mathbf{v}) = m_i/\sqrt{1-\mathbf{v}^2}$ are the momentum and energy of species i when it has velocity $\mathbf{v}$. For lightlike particles we have to separately define the gravitational current, since a velocity and species does not uniquely determine a momentum:

$$j^{GR,m=0}_{\mathbf{v}} = \sum_i \int_0^\infty d\omega\ \omega a^{i\dagger}_{\omega\mathbf{v}} a^i_{\omega\mathbf{v}}. \tag{3.12}$$

Momentum eigenstates of any number of particles are obviously eigenstates of these currents and we denote their eigenvalues $j_{\mathbf{v}}|\alpha\rangle = j_{\mathbf{v}}(\alpha)|\alpha\rangle$.

The photonic exponent $\Delta A_{\beta\beta',\alpha}$ is zero if and only if the charged currents in β are the same as those in β'; the gravitational exponent $\Delta B_{\beta\beta',\alpha}$ is zero if and only if *all* the hard gravitational currents in β are the same as those in β'. This is demonstrated in detail in the appendix. For any such pair of outgoing states $|\beta\rangle, |\beta'\rangle$, (3.9) becomes independent of the IR regulator λ and is thus finite as $\lambda \to 0$,

$$\rho_{\beta\beta'} = S^{\Lambda *}_{\beta'\alpha} S^{\Lambda}_{\beta\alpha} \mathcal{F}_{\beta\alpha}\left(E, E_T, \Lambda\right), \tag{3.13}$$

where

$$\mathcal{F}_{\beta\alpha} = f\left(\frac{E}{E_T}, A_{\beta\alpha}\right) f\left(\frac{E}{E_T}, B_{\beta\alpha}\right) \left(\frac{E}{\Lambda}\right)^{A_{\beta\alpha}+B_{\beta\alpha}}. \tag{3.14}$$

This is the case in particular for diagonal density matrix elements $\beta = \beta'$, for which we obtain the standard transition probabilities

$$\rho_{\beta\beta} = \left|S^{\Lambda}_{\beta\alpha}\right|^2 \mathcal{F}_{\beta\alpha}\left(E, E_T, \Lambda\right). \tag{3.15}$$

On the other hand, if there is even a single $\mathbf{v}$ for which one of the currents (3.11) or (3.12) does not have the same eigenvalue in $|\beta\rangle$ and $|\beta'\rangle$, then the density matrix coefficient decays as $\lambda^{\Delta A+\Delta B} \to 0$ as the regulator $\lambda \to 0$. We see that the unobserved soft bosons have almost completely decohered the momentum state of the hard particles. Only a very sparse subset of superpositions in which all the $j_{\mathbf{v}}(\beta) = j_{\mathbf{v}}(\beta')$ survive.

3.3 Examples

To get a feel for the results presented in the previous section, we consider a few examples. First, consider any scattering with a single incoming and outgoing charged particle, like potential or single Compton scattering. Let the incoming momentum be $\alpha = p$ and the outgoing momenta of the two branches $\beta = q, \beta' = q'$. We have either directly from the definition (3.10) or the theorem (B.1) that

$$\Delta A_{qq',p} = -\frac{e^2}{8\pi^2}\left[2 - \gamma_{qq'}\right], \tag{3.16}$$

where $\gamma_{qq'} = \beta^{-1}_{qq'} \ln\left[(1+\beta_{qq'})/(1-\beta_{qq'})\right]$. This ΔA is easily seen to equal zero if and only if $q = q'$. Thus other than the spin degree of freedom, the resulting density matrix for the charge is exactly diagonal in momentum space.

To see an example where the current-matching condition is non-trivially fulfilled, consider a theory with two charged particle species of charge e and $e/2$ and the same mass. Then we can get an outgoing superposition of a state $\beta = (e, q)$ and one with two half-charges $\beta' = (e/2, q_1') + (e/2, q_2')$. The differential exponent for such a superposition is

$$\Delta A_{\beta\beta',p} = -\frac{e^2}{8\pi^2}\left[3 + \frac{1}{2}\gamma_{q_1 q_2} - \gamma_{qq_1} - \gamma_{qq_2}\right], \tag{3.17}$$

which is zero if $q = q_1 = q_2$. In other words, the currents (3.11) cannot distinguish between a full charge of momentum q and two half-charges of the same momentum.

3.4 Entropy of the Soft Bosons

We have seen that the reduced density matrix for the outgoing hard particles is very nearly diagonal in the momentum basis. In a simple example like a theory with various scalar fields ϕ_i of different, non-zero masses m_i, the soft graviton emission causes complete decoherence into a diagonal momentum-space reduced density matrix for the hard particles. More generally, we may have a sparse set of superpositions, and in any case spin and other internal degrees of freedom are unaffected by the soft emission.

In a simple example with a purely diagonal reduced density matrix, it is straightforward to compute the entanglement entropy of the soft emitted bosons. The total hard + soft system is in a bipartite pure state, with the partition being between the hard particles and soft bosons, so the entanglement entropy of the bosons is the same as that of the hard particles. Following the calculation in [22, 31, 32], we can simply write down the entropy:

$$S = \sum_\beta \left|S^\Lambda_{\beta\alpha}\right|^2 \mathcal{F}_{\beta\alpha} \ln\left[\left|S^\Lambda_{\beta\alpha}\right|^2 \mathcal{F}_{\beta\alpha}\right]. \tag{3.18}$$

This sum is infrared-finite; again, $\mathcal{F}$ is given in (3.14), and the superscript Λ means the naive S-matrix computed with virtual bosons only of energies greater than Λ. Given the explicit form of $\mathcal{F}$, we see that the entropy scales like the log of the infrared detector resolution E.

3.5 Discussion

According to the solution of the infrared catastrophe advocated in [1, 2, 25], an infinite number of very low-energy photons and gravitons are produced during scattering events. We have shown that if taken seriously, considering this radiation as lost to the environment completely decoheres almost any momentum state of the outgoing hard particles. The basic idea is simple: the radiation is essentially classical, so any two scattering events are easy to distinguish by their radiation.

The physical content of this result is somewhat unclear. A conservative view is that the methodology of [1, 2, 25] is ill-suited to finding outgoing density matrices. As remarked earlier, in this formalism, one must trace the radiation to get well-defined transition probabilities. An alternative would be to use the infrared-finite S-matrix program [3, 4, 26, 27], in which no trace over radiation is needed at all. But then we need to understand where the physical low-energy radiation is within that formalism–since after all, a photon that is lost to the environment certainly does decohere the system.

The decoherence found here is for the momentum states of the particles: at lowest order in their momenta, soft bosons do not lead to decoherence of spin degrees of freedom. However, the sub-leading soft theorems [33–35] do involve the spin of the hard particles, so going to the next order in the soft particles would be interesting. We would also like to understand to what extent our answers depend on the infinite-time approximation used in the S-matrix approach.

To end, we comment on potential applications to the black hole information paradox. The idea advocated in [24, 28] is that correlations between the hard and soft particles mean that information about the black hole state can be encoded into soft radiation. In [29, 36, 37], the dressed-state formalism and soft factorization has been used to argue that the soft particles simply factor out of the S-matrix and thus contain no such information. In the approach used here, it is manifest that the outgoing hard state and outgoing soft state are highly correlated, leading to the decoherence of the hard state. The outgoing density matrix for the hard particles, while not completely thermal, has been mixed in momentum as much as possible while retaining consistency with standard QED/perturbative gravity predictions. It is tempting to conjecture that this generalizes to all asymptotically measurable quantum numbers.

At high center-of-mass energies $\sqrt{s}$, black holes should have production cross-sections given by their geometric areas $\sigma_{prod} \sim \pi r_h^2(\sqrt{s})$.[38] Using this in (3.18), one obtains a hard-soft entanglement entropy scaling like the black hole area times logarithmic soft factors. In this sense one might view the soft radiation as containing a significant fraction of the black hole entropy.

Chapter 4

Dressed Infrared Quantum Information

We study information-theoretic aspects of the infrared sector of quantum electrodynamics, using the dressed-state approach pioneered by Chung, Kibble, Faddeev-Kulish and others. In this formalism QED has an IR-finite S-matrix describing the scattering of electrons dressed by coherent states of photons. We show that measurements sensitive only to the outgoing electronic degrees of freedom will experience decoherence in the electron momentum basis due to unobservable photons in the dressing. We make some comments on possible refinements of the dressed-state formalism, and how these considerations relate to the black hole information paradox.

4.1 Introduction

There are two common methods for dealing with infrared divergences in quantum electrodynamics. One is to form inclusive transition probabilities, tracing over arbitrary low-energy photon emission states.[1, 2, 25] However, one may wish to retain an S-matrix description instead of working directly with probabilities. To this end, a long literature initiated by Chung, Kibble, and Faddeev-Kulish has advanced a program in QED where one forms an infrared-finite S-matrix between states of charges "dressed" by long-wavelength photon modes.[3, 4, 8, 26, 39–41] The extension to perturbative gravity in flat spacetime has been initiated in [27].

In the inclusive probability formalism, one is forced to trace out soft photons to get finite answers. In previous work, we showed that this leads to an almost completely decohered density matrix for the outgoing state after a scattering event.[42] This paper analyses the situation in dressed state formalisms, in which no trace over IR photons is needed to obtain a finite outgoing state. However, consider the measurement of an observable sensitive only to electronic and high-energy photonic degrees of freedom. We show that for such observables, there will be a loss of coherence identical to that obtained in the inclusive probability method. Quantum information is lost to the low-energy bremsstrahlung photons created in the scattering process.

The primary goal of this paper is to give concrete calculations exhibiting the dressed formalism and how it leads to decoherence. To this end, we work with the formulas from the papers of Chung and Faddeev-Kulish. The result of this calculation should carry over identically to any of the existing refinements of Chung's formalism. In section 4.4, we make a number of remarks on possible refinements to the basic dressing formalism, give an expanded physical interpretation of our results, and relate our work to literature in mathematical physics on QED superselection rules. In section 4.5 we make remarks on how this work fits into the recent literature on the black hole information paradox; in brief, we believe that our results are consistent with the recent proposal of Strominger [43], but not the original proposal of Hawking, Perry and Strominger.[24, 28]

4.2 IR-safe S-matrix Formalism

Following Chung, we study an electron with incoming momentum $\mathbf{p}$ scattering off a weak external potential. This $1 \to 1$ process is simple and sufficient to understand the basic point; at the end of the next section, we show how to generalize our results to n-particle scattering. The electron spin will be unimportant for us and we supress it in what follows. The standard free-field Fock state $|\mathbf{p}\rangle$ for the electron is promoted to a dressed state $\|\mathbf{p}\rangle\!\rangle$ as follows. For a given photon momentum $\mathbf{k}$ we define the soft factor

$$F_\ell(\mathbf{k}, \mathbf{p}) = \frac{p \cdot e_\ell(\mathbf{k})}{p \cdot k} \phi(\mathbf{k}, \mathbf{p}). \tag{4.1}$$

Here $\ell = 1, 2$ labels the photon polarization states, and $\phi(\mathbf{k}, \mathbf{p})$ is any function that smoothly goes to $\phi \to 1$ as $|\mathbf{k}| \to 0$. We introduce an IR regulator ("photon mass") λ and an upper infrared cutoff $E > \lambda$, which can be thought of as the energy resolution of a single-photon detector in our experiment. Let

$$R_{\mathbf{p}} = e \sum_{\ell=1}^{2} \int_{\lambda < |\mathbf{k}| < E} \frac{d^3\mathbf{k}}{\sqrt{2k}} \left[F_\ell(\mathbf{k}, \mathbf{p}) a_\ell^\dagger(\mathbf{k}) - F_\ell^*(\mathbf{k}, \mathbf{p}) a_\ell(\mathbf{k}) \right] \tag{4.2}$$

and define the single-electron dressing operator

$$\begin{aligned} W_{\mathbf{p}} &= \exp\{R_{\mathbf{p}}\} \\ &= N_{\mathbf{p}} \exp\left\{e\sum_{\ell=1}^{2}\int \frac{d^3\mathbf{k}}{\sqrt{2k}} F_\ell(\mathbf{p},\mathbf{k}) a_\ell^\dagger(\mathbf{k})\right\} \\ &\times \exp\left\{-e\sum_{\ell=1}^{2}\int \frac{d^3\mathbf{k}}{\sqrt{2k}} F_\ell^*(\mathbf{p},\mathbf{k}) a_\ell(\mathbf{k})\right\}, \end{aligned} \tag{4.3}$$

where in the second line, we have put this coherent-state displacement operator into its normal-ordered form, with normalization factor [1]

$$N_{\mathbf{p}} = \exp\left\{-\frac{e^2}{2}\sum_{\ell=1}^{2}\int \frac{d^3\mathbf{k}}{2k} \left|F_\ell(\mathbf{p},\mathbf{k})\right|^2\right\}. \tag{4.4}$$

Here and in the following all momentum-space integrals are evaluated in the shell $\lambda < |\mathbf{k}| < E$. The dressed single-electron state $\|\mathbf{p}\rangle\!\rangle$ is then defined as

$$\|\mathbf{p}\rangle\!\rangle = W_{\mathbf{p}} |\mathbf{p}\rangle . \tag{4.5}$$

This consists of the electron and a coherent state of on-shell, transversely-polarized photons.

Consider now an incoming dressed electron scattering into a superposition of outgoing dressed electron states. The outgoing state is, to lowest order in perturbation theory in the electric charge,

$$|\psi\rangle = \int d^3\mathbf{q} \mathbb{S}_{\mathbf{qp}} \|\mathbf{q}\rangle\!\rangle. \tag{4.6}$$

At higher orders there will be additional photons in the outgoing state; as explained in the next section, these will not affect the infrared behavior studied here, so we ignore them for now. Here the S-matrix is just the standard Feynman-Dyson time evolution operator, evaluated between dressed states. That is,

$$\mathbb{S}_{\mathbf{qp}} = \langle\!\langle \mathbf{q}|S|\mathbf{p}\rangle\!\rangle, \tag{4.7}$$

[1] This factor diverges, so these states have zero norm. In this sense, the dressed-state formalism simply re-organizes the calculations such that the divergences are in the definitions of the states $\|\mathbf{p}\rangle\!\rangle$ instead of the S-matrix elements. We view this is a major difficulty with these formalisms, and understanding this better would be very useful. See eg. [44] for some ideas in this direction.

with $S = T\exp\left(-i\int_{-\infty}^{\infty} V(t)dt\right)$ as usual.[21] As calculated by Chung, the dressed $1 \to 1$ elements of this matrix are independent of the IR regulator λ and thus infrared-finite as we send $\lambda \to 0$. We can write the matrix element

$$\mathbb{S}_{\mathbf{qp}} = \left(\frac{E}{\Lambda}\right)^{A} S^{\Lambda}_{\mathbf{qp}} \tag{4.8}$$

where

$$A = -\frac{e^2}{8\pi^2}\beta^{-1}\ln\left[\frac{1+\beta}{1-\beta}\right], \quad \beta = \sqrt{1-\frac{m^4}{(p\cdot q)^2}}. \tag{4.9}$$

The undressed S-matrix element on the right side means the amplitude computed by Feynman diagrams with photon loops evaluated only with photon energies above Λ and evaluated between undressed electron states, that is, with no external soft photons. By definition, this quantity is infrared-finite; moreover, the dependence on the scale Λ cancels between the prefactor and S^{Λ}.

4.3 Soft Radiation and Decoherence

The state (4.6) is a coherent superposition of states, each containing a bare electron and its corresponding photonic dressing. The presence of hard photons in the outgoing state will not change our conclusions below, so for simplicity we ignore them. In particular, the density matrix formed from this state has off-diagonal elements of the form

$$\mathbb{S}^*_{\mathbf{q'p}}\mathbb{S}_{\mathbf{qp}}\|\mathbf{q}\rangle\!\rangle\langle\!\langle\mathbf{q'}\|. \tag{4.10}$$

These states have highly non-trivial photon content. However, if one is doing a measurement involving only the electron degree of freedom, then these photons are unobserved, and we can make predictions with the reduced density matrix of the electron, obtained by tracing the photons out. The resulting electron density matrix has coefficients damped by a factor involving the overlap of the photon states, namely

$$\rho_{electron} = \int d^3\mathbf{q}d^3\mathbf{q'}\mathbb{S}^*_{\mathbf{q'p}}\mathbb{S}_{\mathbf{qp}}D_{\mathbf{qq'}}\|\mathbf{q}\rangle\!\rangle\langle\!\langle\mathbf{q'}\| \tag{4.11}$$

where the dampening factor is given by the photon-vacuum expectation value

$$D_{\mathbf{qq'}} = \langle 0|W^\dagger_{\mathbf{q'}}W_{\mathbf{q}}|0\rangle\,. \tag{4.12}$$

Straightforward computation gives this factor as

$$\begin{aligned} D_{\mathbf{q}\mathbf{q}'} &= \exp\left\{-\frac{e^2}{2}\sum_{\ell=1}^{2}\int\frac{d^3\mathbf{k}}{2k}\left|F_\ell(\mathbf{q}) - F_\ell^*(\mathbf{q}')\right|^2\right\} \\ &= \exp\left\{-e^2\int\frac{d^3\mathbf{k}}{2k}\frac{(q-q')^2}{(q\cdot k)(q'\cdot k)}\right\}. \end{aligned} \tag{4.13}$$

In this integrand, since q and q' are two timelike vectors with the same temporal component, we have that the numerator is positive definite and the denominator is positive. It is therefore manifest that we have $D = 1$ if $q = q'$ and $D = 0$ otherwise, since the integral over $d^3\mathbf{k}$ diverges in its lower limit. Thus, tracing the photons leads to an electron density matrix that is completely diagonalized in momentum space.

It is noteworthy that the factor (4.13) depends only on properties of the outgoing superposition; it has no dependence on the initial state. This may seem surprising since we are tracing over outgoing radiation, the production of which depends on both the initial and final electron state. The point is that the damping factor measures the distinguishability of the radiation fields given the processes $\mathbf{p} \to \mathbf{q}$ and $\mathbf{p} \to \mathbf{q}'$. The radiation field for a scattering process consists of two pieces added together: a term $A_\mu \sim p_\mu / p \cdot k$ peaked in the direction of the incoming electron and a term $A_\mu \sim q_\mu / q \cdot k$ peaked in the direction of the outgoing electron. The outgoing radiation fields with outgoing electrons q, q' are then only distinguishable by the second terms here, since both radiation fields will have the same pole in the incoming direction.

The damping factor (4.13) is precisely what was found in [42], reduced to the problem of $1 \to 1$ scattering. The mechanism is the same: physical, low-energy photon bremsstrahlung is emitted in the scattering. These photons are highly correlated with the electron state and thus, if one does not observe them jointly with the electron, one will measure a highly-decohered electron density matrix. The only difference is bookkeeping: in the dressed formalism, the bremsstrahlung photons are folded into the dressed electron states (the incoming/outgoing parts of the bremsstrahlung in the incoming/outgoing dressing, respectively). However, referring to "an electron" as a state including these soft photons is just an abuse of semantics. In an actual measurement of the electron momentum, one does not measure these soft photons.

The dressed states are not energy eigenstates, and in fact contain states of arbitrarily high total energy. This should be contrasted with the inclusive-probability treatment used by Weinberg, which has a cutoff on both the single-photon energy E and the total outgoing energy contained by all the photons $E_T \geq E$ in the outgoing state [2]. This additional parameter, however, appears only in the ratio E_T/E in Weinberg's probability formulas, and one finds that the dependence on E_T vanishes as $E_T \to \infty$. This can be understood because what is important is the very low-energy behavior of the photons, so moving an upper cutoff has limited impact.

We note that (4.6) does not include effects from the bremsstrahlung of additional soft photons beyond those in the dressing. There is no kinematic reason to exclude such photons, so the outgoing state should properly be written as

$$|\psi\rangle = \sum_{n=0}^{\infty} \sum_{\{\ell\}} \int d^3\mathbf{q} d^{3n}\mathbf{k} \mathbb{S}_{\mathbf{q}\{\mathbf{k}\ell\};\mathbf{p}} \|\mathbf{q}\rangle\!\rangle. \tag{4.14}$$

Here $\{\mathbf{k}\ell\} = \{\mathbf{k}_1\ell_1, \ldots, \mathbf{k}_n\ell_n\}$ is a list of n photon momenta and polarizations. By the dressed version of the soft photon factorization theorem (see appendix), we have that

$$\mathbb{S}_{\mathbf{q}\mathbf{k}\ell;\mathbf{p}} = \mathbb{S}_{\mathbf{q}\mathbf{p}} \times e\mathcal{O}\left(|\mathbf{k}|^0\right), \tag{4.15}$$

or in other words $\lim_{|\mathbf{k}|\to 0} |\mathbf{k}|\mathbb{S}_{\mathbf{q}\mathbf{k}\ell;\mathbf{p}} = 0$. Thus, when we take a trace over n-photon dressed states in (4.14), we obtain a sum of additional decoherence factors of the form

$$\begin{aligned} D^{nm}_{\mathbf{q}\mathbf{q}'} = e^{n+m}\mathcal{O}\left(|\mathbf{k}|^0\right) \times \sum_{\ell_1,\ldots,\ell_n} \sum_{\ell'_1,\ldots,\ell'_m} \int d^{3n}\mathbf{k} d^{3m}\mathbf{k}' \\ \langle 0| a_{\ell'_m}(\mathbf{k}'_m) \cdots a_{\ell'_1}(\mathbf{k}'_1) W^\dagger_{\mathbf{q}'} W_{\mathbf{q}} a^\dagger_{\ell_1}(\mathbf{k}_1) \cdots a^\dagger_{\ell_n}(\mathbf{k}_n)|0\rangle\,. \end{aligned} \tag{4.16}$$

Evaluating the inner product using (4.3), one finds

$$D^{nm}_{\mathbf{q}\mathbf{q}'} \sim \left[\sum_{\ell=1}^{2} \int \frac{d^3\mathbf{k}}{\sqrt{2k}} \mathrm{Re}\left(F_\ell(\mathbf{q}) - F_\ell(\mathbf{q}')\right)\right]^{n+m}, \tag{4.17}$$

which is infrared-finite. Summing these contributions, which exponentiate, will not change the conclusion that (4.13) leads to vanishing off-diagonal electron density matrix elements.

Finally, we explain the generalization to n-electron states. We will find that the same decoherence is found in the dressed formalism as in the inclusive formalism.[42] Following Faddeev-Kulish [4], we write the multi-particle dressing operator by replacing (4.3) with

$$
\begin{aligned}
R_p \to e \sum_{l=1}^{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{\sqrt{2k}} \\
\left[F_l(\mathbf{k},\mathbf{p}) a_l^\dagger(\mathbf{k}) - F_l^*(\mathbf{k},\mathbf{p}) a_l^\dagger(\mathbf{k})\right] \rho(\mathbf{p}),
\end{aligned}
\tag{4.18}
$$

where we have introduced an operator which counts charged particles with momentum $\mathbf{p}$.

$$
\rho(\mathbf{p}) = \sum_s \left(b_{\mathbf{p},s}^\dagger b_{\mathbf{p},s} - d_{\mathbf{p},s}^\dagger d_{\mathbf{p},s} \right), \tag{4.19}
$$

and the b and d are electron and positron operators, respectively.[2] As in the one-particle case, additional photons do not affect the IR behaviour of scattering amplitudes. Hence, we will ignore them and only consider the case where the out-state is a linear superposition of dressed electron states. In that case we have to replace the outgoing momentum by list of momenta, $\mathbf{q} \to \beta = \{\mathbf{q}_1, \mathbf{q}_2, \ldots\}$ and similarly $\mathbf{q}' \to \beta' = \{\mathbf{q}_1', \mathbf{q}_2', \ldots\}$. This results to a replacement in (4.13) of

$$
\begin{aligned}
F_\ell(\mathbf{q}) &\to \sum_{n\in\beta} F_\ell(\mathbf{q}_n) \\
F_\ell^*(\mathbf{q}') &\to \sum_{m\in\beta'} F_\ell^*(\mathbf{q}_m').
\end{aligned}
\tag{4.20}
$$

Using the explicit form of F in the limit $\mathbf{k} \to 0$, the damping factor (4.13) then then becomes

$$
D_{\beta\beta'} = \exp\left[-e^2 \int \frac{d^3\mathbf{k}}{2k} \sum_{m,n\in\beta,\beta'} \frac{\eta_m \eta_n p_m \cdot q_n}{(q_m \cdot k)(q_n \cdot k)} \right]. \tag{4.21}
$$

In this equation the labels m, n both run over the full set $\beta \cup \beta'$, and $\eta_n = +1$ if $n \in \beta$ while $\eta_n = -1$ if $n \in \beta'$. This is precisely the quantity $\Delta A_{\beta\beta',\alpha}$ defined in [42], so we see that the results of that paper carry over to the dressed formalisms used here.

[2]Note that in the multi-particle case there is an infinite phase factor which needs to be included in the definition of the S-matrix. Since this phase factor does not affect our discussion, we ignore it in the following.

4.4 Physical Interpretation

Dressed-state formalisms are engineered to provide infrared-finite transition amplitudes, as opposed to inclusive probabilities constructed in the traditional approach studied in [42]. In the dressed formalism, the outgoing state (4.6) is a coherent superposition of states $\|\mathbf{p}\rangle\!\rangle$ consisting of electrons plus dressing photons. However, if one does a measurement of an observable sensitive only to the electron state, the measurement will exhibit decoherence because the unobserved dressing photons are highly correlated to the electron state. We have given a concrete calculation showing that the damping factor (4.21) is identical in either the dressed or undressed formalism.

The physical relevance of this calculation rests on the idea that the basic observable is a simple electron operator in Fock space. What would be much better would be to use a dressed LSZ reduction formula to understand the asymptotic limits of electron correlation functions. [41, 44] Nevertheless, the basic physical picture seems clear: in a scattering experiment, one does not measure an electron plus a finely-tuned shockwave of outgoing bremsstrahlung photons, just the electron on its own. This is responsible for well-measured phenomena like radiation damping.

QED has a complicated asymptotic Hilbert space structure which is still somewhat poorly understood. For example, although Faddeev-Kulish try to define a single, separable Hilbert space $\mathcal{H}_{as}$ [4, 44] other authors have argued that one needs an uncountable set of separable Hilbert spaces.[26, 41] Formally, this is related to the fact that the dressing operator (4.3) does not converge on the usual Fock space. A related idea is that one can argue that QED has an infinite set of superselection rules based on the asymptotic charges

$$Q(\Omega) = \lim_{r\to\infty} r^2 E_r(r,\Omega) \tag{4.22}$$

defined by the radial electric field at infinity.[45, 46] We believe that the calculations presented here and in [42] demonstrate the physical mechanism for enforcing such a superselection rule. The charges (4.22), the currents defined in our previous work [42], and the large-$U(1)$ charges defined in [7, 47] are presumably closely related, and working out the precise relations is an interesting line of inquiry.

4.5 Black Hole Information

The recent resurgence of interest in infrared issues in QED and gravity was sparked largely by a proposal of Hawking, Perry and Strominger suggesting that information apparently "lost" in the process of black hole formation and evolution could be encoded in soft radiation.[24, 28] The original proposal was that there are symmetries which relate "hard" scattering (like the black hole process) to soft scattering and thus led to constraints on the S-matrix. As emphasized by a number of authors, this is not true in the dressed state approach.[29, 36, 37, 48] As we review in the appendix, soft modes decouple from the dressed hard scattering event at lowest order, in the sense that $\lim_{\omega\to 0}[a_\omega, S_{dressed}] = 0$. Dropping a soft boson into the black hole will not yield any information about the black hole formation and evaporation process.

However, a more recent proposal due to Strominger is to simply posit that outgoing soft radiation purifies the outgoing Hawking radiation.[43] That is, the state after the black hole has evaporated is of the form $|\psi\rangle = \sum_a |a\rangle_{Hawking} |a\rangle_{soft}$, such that the Hawking radiation is described by a thermal density matrix, i.e. $\rho_{Hawking} = \mathrm{tr}_{soft} |\psi\rangle \langle\psi| \approx \rho_{thermal}$. We believe that both the results presented here and those in our previous work are consistent with this proposal. In either the inclusive or dressed formalism, the final state of any scattering process contains soft radiation which is highly correlated with the hard particles because the radiation is created due to accelerations in the hard process. The open issue is to explain why the hard density matrix coefficients behave thermally, which likely relies on details of the black hole S-matrix.

4.6 Conclusions

When charged particles scatter, they experience acceleration, causing them to radiate low-energy photons. If one waits an infinitely long time (as mandated by an S-matrix description), these photons cause severe decoherence of the charged particle momentum state. This was first seen in [42] in the standard formulation of QED involving IR-finite inclusive cross section, and here we have shown the same conclusion holds in IR-safe, dressed formalisms of QED; they should carry over in a simple way to perturbative quantum gravity. These results constitute a sharp and robust connection between the infrared catastrophe and quantum information theory, and should provide guidance in problems related to the infrared structure of gauge theories.

Chapter 5

On the Need for Soft Dressing

In order to deal with IR divergences arising in QED or perturbative quantum gravity scattering processes, one can either calculate inclusive quantities or use dressed asymptotic states. We consider incoming superpositions of momentum eigenstates and show that in calculations of cross-sections these two approaches yield different answers: in the inclusive formalism no interference occurs for incoming finite superpositions and wavepackets do not scatter at all, while the dressed formalism yields the expected interference terms. This suggests that rather than Fock space states, one should use Faddeev-Kulish-type dressed states to correctly describe physical processes involving incoming superpositions. We interpret this in terms of selection rules due to large $U(1)$ gauge symmetries and BMS supertranslations.

5.1 Introduction

Quantum electrodynamics and perturbative quantum gravity are effective quantum field theories which describe the two long-ranged forces seen in nature. They also both suffer from infrared divergences coming from virtual boson loops in Feynman diagrams in the perturbative computation of the S-matrix. These divergences exponentiate when resummed and set the amplitude for any process between a finite number of interacting particles to zero. This is known as the infrared catastrophe.

One proposed resolution of the infrared catastrophe is to consider only inclusive quantities, for example soft-inclusive transition probabilities in the context of scattering theory, which are defined by summing over the production of any number of soft photons and gravitons. In the case of electrodynamics, this resolution dates back to Bloch and Nordsieck [1, 25] and, in perturbative quantum gravity, it was developed by Weinberg [2]. The contributions from emitted soft bosons cancel the IR divergences from the virtual loops. An upshot of this solution of the infrared problem is the fact that, in QED, any non-trivial scattering process involving charged particles inevitably produces a cloud of an infinite number of arbitrarily soft photons. In the case of quantum gravity, soft gravitons are produced, and, since all particles carry gravitational charge, IR divergences arise in any scattering process. The use of inclusive probabilities is justified by the assumption that the softest photons and gravitons must escape detection. These bosons carry very little energy and have a negligible effect on the kinematics of the process. However, it was recently shown that they carry a lot of information in the sense that their quantum states are highly entangled with those of the charged particles. The loss of the soft particles results in decoherence of the final state of the hard particles, where the momentum eigenstates for electrically or gravitationally charged particles are the pointer basis [42, 49]. See refs. [50–52] for related work.

The infrared catastrophe can be traced back to the long-ranged nature of the interactions which is in conflict with the assumption of asymptotic decoupling needed to formulate scattering theory [53]. An approach to the infrared problem, alternative to using inclusive probabilities, is to use dressed states which are defined by including the aforementioned clouds of soft photons and gravitons with the asymptotic states [3, 4, 26, 27, 41, 44, 54–56]. Faddeev and Kulish argued that such an approach diagonalizes the correct asymptotic Hamiltonian and therefore yields the asymptotic decoupling which is necessary for a satisfactory formulation of scattering theory. The detailed structure of the coherent states can be adjusted so as to cancel the infrared divergences in the S-matrix, providing an IR-finite S-matrix and scattering probabilities. However, the out-going states still contain particles accompanied by soft photon and graviton clouds. One can ask the same question: given these infrared safe states, what is the nature of the state of the outgoing hard particles? The answer is that precisely the same decoherence is found to occur in either the inclusive or dressed approaches [57], i.e. there is still a lot of information in the entanglement between the hard particles and the radiation.[1]

Both the dressed and inclusive formalisms are designed to give the same predictions for the probability of scattering from an incoming set of momenta $\mathbf{p}_1, \ldots, \mathbf{p}_n$ into an outgoing set of momenta $\mathbf{p}'_1, \ldots, \mathbf{p}'_m$. The measurement of observables which only depend on the hard particles should be predictable from the reduced density matrix obtained by tracing over soft bosons, which are invisible to a finite size detector. Given an incoming momentum eigenstate the two formalisms agree. Thus, one might naively think for calculating cross-sections it does not matter which formalism one chooses. We show in this paper that this is not the case: the two approaches differ in their treatment of incoming superpositions. Consider a simple superposition of two

[1]Note, there are also other proposals for how to define an IR finite density matrix [58], which we will not discuss here.

momentum eigenstates for a single charged particle

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\mathbf{p}\rangle + |\mathbf{q}\rangle), \tag{5.1}$$

scattering off of a classical potential. We expect the out-state to be described by a density matrix of the form

$$\rho = \frac{1}{2} S \left(|\mathbf{p}\rangle \langle \mathbf{p}| + |\mathbf{p}\rangle \langle \mathbf{q}| + |\mathbf{q}\rangle \langle \mathbf{p}| + |\mathbf{q}\rangle \langle \mathbf{q}| \right) S^{\dagger}. \tag{5.2}$$

Here S is the scattering operator and we have performed a trace over the soft radiation, hence ρ is the density matrix for the hard particles. If $|\mathbf{p}\rangle, |\mathbf{q}\rangle$ are correctly dressed states, this expectation is indeed correct. In the inclusive formalism, however, where $|\mathbf{p}\rangle$, $|\mathbf{q}\rangle$ are Fock space momentum eigenstates, there is no interference between the different momenta as opposed to the diagonal terms of (5.2). We find that the diagonal entries of the density matrix which encode the cross-sections are of the form

$$\sigma_{\psi \to out} \propto \langle out| \rho^{\text{incl}} |out\rangle = \frac{1}{2} \langle out| \; S \left(|\mathbf{p}\rangle \langle \mathbf{p}| + |\mathbf{q}\rangle \langle \mathbf{q}| \right) S^{\dagger} |out\rangle . \tag{5.3}$$

In other words, the cross-section behaves as if we had started with a classical ensemble of states with momenta $\mathbf{p}$ and $\mathbf{q}$. The entire scattering history is decohered by the loss of the soft radiation. This appears to contrast starkly with any realistic experiment.

Moreover, as we will show, repeating the analysis for wavepackets, e.g. $|\psi\rangle = \int d\mathbf{p} f(\mathbf{p}) |\mathbf{p}\rangle$, leads to the nonsensical conclusion that a wave-packet is not observed to scatter at all. However, in the dressed state formalism of Faddeev-Kulish the interference appears as in equation (5.2). This strongly suggests that scattering theory in quantum electrodynamics and perturbative quantum gravity should really not be formulated in terms of standard Fock states of charged particles. Formulating the theories using dressed states seems to be a good alternative.

Dressed states also arise naturally in the recent discussions of asymptotic gauge symmetries [7, 28, 47, 59–61], which imply the existence of selection sectors[8, 10, 36, 62]. See also [63, 64] for work on soft charges and dressing in holography. Our findings have a nice interpretation in the language of this program: only superpositions of states within the same selection sector can interfere. This explains the failure of the undressed approach. In the inclusive formalism, essentially any pair of momentum eigenstates live in different charge sectors. In contrast, the Faddeev-Kulish formalism is designed so that all of the dressed states live within the same charge sector.

Our results can also be viewed in the context of the black hole information problem [13, 65]. In particular, Hawking, Perry, and Strominger [9] and Strominger [14] have recently suggested that black hole information may be encoded in soft radiation. In black hole thought experiments, one typically imagines preparing an initial state of wavepackets organized to scatter with high probability to form an intermediate black hole. Our results suggest then that one needs to use dressed initial states to study this problem. See also [29, 37] for some remarks on the use of dressed or inclusive formalisms for studying black hole information.

The rest of the paper is organized as follows. We start by presenting the calculations showing that the dressed and undressed formalism disagree in section 5.2 for discrete superpositions and in section 5.3 for wavepackets. The discussion and interpretation of the results takes place in section 5.4. There, we will argue why our findings imply that dressed states are better suited to describe scattering than the inclusive Fock-space formalism. We will give a new very short argument for the known result of [10] that the dressing operators and the S-matrix weakly commute and argue for a more general form of dressing beyond Faddeev-Kulish. We will then interpret our results in terms of asymptotic symmetries and selection sectors before concluding in section 5.5. The appendix contains proofs of certain statements in sections 5.2 and 5.3.

5.2 Scattering of Discrete Superpositions

In this and the next section we generalize the results of [42] to the case of incoming superpositions of momentum eigenstates. We begin in this section by studying discrete superpositions $|\psi\rangle = |\alpha_1\rangle + \cdots + |\alpha_N\rangle$ of states with various momenta $\alpha = \mathbf{p}_1, \mathbf{p}_2, \ldots$. We will see that the dressed and inclusive formalisms give vastly different predictions for the probability distribution of the outgoing momenta: dressed states will exhibit interference between the α_i whereas undressed states do not.

5.2.1 Inclusive Formalism

Consider scattering of an incoming superposition of charged momentum eigenstates

$$|in\rangle = \sum_i^N f_i \, |\alpha_i\rangle \,, \tag{5.4}$$

with $\sum_i |f_i|^2 = 1$. The outgoing density matrix vanishes due to IR divergences in virtual photon loops. However, we can obtain a finite result if we trace over outgoing radiation [1, 2, 25, 42]. The resulting reduced density matrix of the hard particles takes the form

$$\rho = \sum_b \sum_{i,j}^N \iint d\beta \; d\beta' f_i f_j^* S_{\beta b,\alpha_i} S^*_{\beta' b,\alpha_j} \, |\beta\rangle \, \langle\beta'| \,, \tag{5.5}$$

where β and β' are lists of the momenta of hard particles in the outgoing state, and the sum over b denotes the trace over soft bosons. We will be interested in the effect of infrared divergences on this expression.

The sum over external soft boson states b produces IR divergences which cancel those coming from virtual boson loops. We can regulate these divergences by introducing an IR cutoff (e.g. a soft boson mass λ). Following the standard soft photon resummation techniques [2], one finds that the total effect of these divergences yields reduced density matrix elements of the form

$$\rho_{\beta\beta'} = \sum_{i,j}^N f_i f_j^* S^{\Lambda}_{\beta\alpha_i} S^{\Lambda *}_{\beta'\alpha_j} \lambda^{\Delta A_{\beta\beta',\alpha_i\alpha_j} + \Delta B_{\beta\beta',\alpha_i\alpha_j}} \mathcal{F}_{\beta\beta',\alpha_i\alpha_j}(E, E_T, \Lambda). \tag{5.6}$$

Here we have introduced "UV" cutoffs Λ, E on the virtual and real soft boson energies, so S^{Λ} are S-matrix elements with the soft boson loops cut off below Λ and we only trace over outgoing bosons with individual energies up to E and total energy E_T. The explicit form of the Sudakov rescaling function $\mathcal{F}$ can be found in [42]. What concerns us here is the behavior of this expression in the limit where we remove the IR regulator $\lambda \to 0$, which is controlled by the exponents

$$\Delta A_{\beta\beta',\alpha\alpha'} = -\frac{1}{2} \sum_{n,n' \in \alpha,\bar{\alpha}',\beta,\bar{\beta}'} \frac{e_n e_{n'} \eta_n \eta_{n'}}{8\pi^2} \beta_{nn'}^{-1} \ln \left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}} \right],$$
$$\Delta B_{\beta\beta',\alpha\alpha'} = -\frac{1}{2} \sum_{n,n' \in \alpha,\bar{\alpha}',\beta,\bar{\beta}'} \frac{m_n m_{n'} \eta_n \eta_{n'}}{16\pi^2 M_p^2} \beta_{nn'}^{-1} \frac{1+\beta_{nn'}^2}{\sqrt{1-\beta_{nn'}^2}} \ln \left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}} \right]. \tag{5.7}$$

The factor η_n is defined as $+1$ (-1) if particle n is incoming (outgoing). The quantities $\beta_{nn'} = \sqrt{1 - \frac{m_n^2 m_{n'}^2}{(p_n \cdot p_{n'})^2}}$ are the relative velocities between pairs of particles and a bar interchanges incoming states for outgoing and vice versa. The expressions for ΔA and ΔB come from contributions of soft photons and gravitons, respectively. The question now is which terms survive.

The special case of no superposition, $\alpha_i = \alpha_j = \alpha$, was discussed in [42]. There it was shown that $\Delta A_{\beta\beta',\alpha\alpha} \geq 0$ and $\Delta B_{\beta\beta',\alpha\alpha} \geq 0$, so that in the limit $\lambda \to 0$, all of the terms in the sum except those with $\Delta A = \Delta B = 0$ will vanish. The equality holds if and only if the out states β and β' contain particles such that the amount of electrical charge and mass carried with any choice of velocity agrees for β and β'. This can be phrased in terms of an infinite set of operators which measure charges flowing along a velocity $\mathbf{v}$. These are defined as

$$\begin{aligned} \hat{j}^{em}_{\mathbf{v}} &= \sum_i e_i a^\dagger_{i,\mathbf{p}_i(\mathbf{v})} a_{i,\mathbf{p}_i(\mathbf{v})}, \\ \hat{j}^{gr}_{\mathbf{v}} &= \sum_i E_i(\mathbf{v}) a^\dagger_{i,\mathbf{p}_i(\mathbf{v})} a_{i,\mathbf{p}_i(\mathbf{v})}, \\ \hat{j}^{gr,0}_{\mathbf{v}} &= \sum_i \int d\omega \; \omega a^\dagger_{i,\mathbf{v}\omega} a_{i,\mathbf{v}\omega}, \end{aligned} \tag{5.8}$$

for charged particles, massive particles and hard massless particles, respectively. The sum runs over all particle species. Clearly, momentum eigenstates are also eigenstates of these operators. Using these operators, the equality of currents can be expressed as

$$\hat{j}_{\mathbf{v}} \left|\beta\right\rangle \sim \hat{j}_{\mathbf{v}} \left|\beta'\right\rangle, \tag{5.9}$$

where the tilde means that the eigenvalues of the states are the same on both sides for all velocities. In appendix D, we show that the more general exponents $\Delta A_{\beta\beta',\alpha\alpha'}$ and $\Delta B_{\beta\beta',\alpha\alpha'}$ are positive. Similarly to the argument in [57], one can show that ΔA and ΔB are non-zero if and only if

$$\hat{j}_{\mathbf{v}} |\alpha_i\rangle + \hat{j}_{\mathbf{v}} |\beta'\rangle \sim \hat{j}_{\mathbf{v}} |\alpha_j\rangle + \hat{j}_{\mathbf{v}} |\beta\rangle , \tag{5.10}$$

that is if the list of hard currents in states $|\alpha\rangle$ and $|\beta'\rangle$ is the same as the list of hard currents in states $|\alpha'\rangle$ and $|\beta\rangle$. An easy way to understand the form of equation (5.10) is by looking at equation (5.7). There, the bar over α' (which corresponds to α_j) indicates that it should be treated as an outgoing particle, i.e. similarly to β. On the other hand $\bar{\beta}'$ should be treated similarly to α. Hence, we obtain equation (5.10) from (5.9) by replacing $\alpha_i \to \alpha_i + \beta'$ and $\alpha_j \to \alpha_j + \beta$. On the other hand it is clear that in the case of $|\alpha_i\rangle = |\alpha_j\rangle = |\alpha\rangle$ equation (5.10) reduces to equation (5.9).

Armed with these results, we can calculate the cross-sections given an incoming superposition. These are proportional to the diagonal elements $\beta = \beta'$ of the density matrix; for simplicity we ignore forward scattering terms. The diagonal terms of the density matrix (5.6) are proportional to $\lambda^{\Delta A + \Delta B}$. This factor reduces to unity if $\hat{j}_{\mathbf{v}} |\alpha_i\rangle \sim \hat{j}_{\mathbf{v}} |\alpha_j\rangle$ for all of the currents (5.8) and is zero otherwise. For a generic superposition, this implies that only terms with $i = j$ contribute and we find

$$\sigma_{\text{in}\to\beta} \propto \rho_{\beta\beta} = \sum_{i,j}^{N} f_i f_j^* \mathcal{F}_{\beta\beta,\alpha_i\alpha_j} S^{\Lambda}_{\beta\alpha_i} S^{\Lambda *}_{\beta\alpha_j} \delta_{\alpha_i\alpha_j} = \sum_{i}^{N} |f_i|^2 |S^{\Lambda}_{\beta,\alpha_i}|^2 \mathcal{F}_{\beta\beta,\alpha_i\alpha_i}. \tag{5.11}$$

As we see, no interference terms between incoming states are present. Instead, the total cross-section is calculated as if the incoming states were part of a classical ensemble with probabilities $|f_i|^2$. The reason is that in the inclusive approach the information about the interference is carried away by unobservable soft radiation. To define the scattering cross-section, however, we need to trace out the soft radiation and we obtain the above prediction, which is at odds with the naive expectation, equation (5.2).

5.2.2 Dressed Formalism

The calculation above was done using the usual, undressed Fock states of hard charges, which required to calculate inclusive cross-sections. The alternative approach we will now turn to is to consider transitions between dressed states. For concreteness, we will follow the dressing approach of

Chung and Faddeev-Kulish[2], which contains charged particles accompanied by a cloud of real bosons which radiate out to lightlike infinity [3, 4, 27]. For a given set of momenta $\alpha = \mathbf{p}_1, \mathbf{p}_2, \ldots$, we write the dressed state as[3]

$$\|\alpha\rangle\!\rangle \equiv W_\alpha \, |\alpha\rangle \, . \tag{5.12}$$

The operator W_α equips the state $|\alpha\rangle$ with a cloud of photons/gravitons. For QED, W_α is the unitary operator (with a finite IR cutoff λ)

$$W_\alpha \equiv \exp\left\{ e \sum_{l=1}^{2} \int_\lambda^E \frac{d^3\mathbf{k}}{\sqrt{2k}} \left(F_l(\mathbf{k}, \alpha) a_{\mathbf{k}}^{l\dagger} - F_l^*(\mathbf{k}, \alpha) a_{\mathbf{k}}^{l} \right) \right\} , \tag{5.13}$$

where $a_{\mathbf{k}}^{l\dagger}$ creates a photon in the polarization state l and the soft factor

$$F_l(\mathbf{k}, \alpha) = \sum_{\mathbf{p} \in \alpha} \frac{\epsilon_l \cdot p}{k \cdot p} \phi(\mathbf{k}, \mathbf{p}) \tag{5.14}$$

depends on the polarization vectors ϵ_l and some smooth, real function $\phi(\mathbf{k}, \mathbf{p})$ which goes to 1 as $|\mathbf{k}| \to 0$. Letting W act on Fock space states for $\lambda = 0$ gives states with vanishing normalization, hence in the strict $\lambda \to 0$ limit W is no good operator on Fock space. Thus, as before, we will do calculations with finite λ and only at the end we will take $\lambda \to 0$.[4]

The Faddeev-Kulish construction was adapted to perturbative quantum gravity in [27]. In this case the dressing has the same form as equation (5.13), the only difference being that a ($a^\dagger$) is now a graviton annihilation (creation) operator and the functions F depend on the polarization tensor $\epsilon_{\mu\nu}$ [27],

$$F_l^{gr}(\mathbf{k}, \alpha) = \sum_{\mathbf{p} \in \alpha} \frac{p_\mu \epsilon_l^{\mu\nu} p_\nu}{k \cdot p} \phi(\mathbf{k}, \mathbf{p}). \tag{5.15}$$

S-matrix elements taken between dressed states

$$\mathbb{S}_{\beta\alpha} \equiv \langle\!\langle \beta \| S \| \alpha \rangle\!\rangle = \langle \beta | \, W_\beta^\dagger S W_\alpha \, | \alpha \rangle \tag{5.16}$$

[2]Recently, a generalization of Faddeev-Kulish states was suggested [8]. We will extend our discussion to those states in section 5.4.

[3]The double bracket notation is due to [29]. The previous paper of the authors [57] used $|\tilde{\alpha}\rangle$ to denote dressed states. The authors regret this life decision.

[4]Note that as argued in [4], a proper definition of W in the limit $\lambda \to 0$ should be possible on a von Neumann space.

are independent of λ and thus finite as $\lambda \to 0$. The Sudakov factor $\mathcal{F}$ is contained in the dressed S-matrix elements.[5]

Consider now an incoming state consisting of a discrete superposition of such dressed states,

$$\|in\rangle\!\rangle = \sum_i f_i \|\alpha_i\rangle\!\rangle. \tag{5.17}$$

The outgoing density matrix is then

$$\rho = \sum_{i,j} \iint d\beta d\beta' f_i f_j^* \mathbb{S}_{\beta\alpha_i} \mathbb{S}^*_{\beta'\alpha_j} \|\beta\rangle\!\rangle\langle\!\langle\beta'\|. \tag{5.18}$$

This density matrix is formally unitary, however, every experiment should be able to ignore soft radiation. Following [42], we treat the soft modes as unobservable and trace them out. This yields the reduced density matrix for the outgoing hard particles,

$$\rho^{hard}_{\beta\beta'} = \sum_{i,j} f_i f_j^* \mathbb{S}_{\beta\alpha_i} \mathbb{S}^*_{\beta'\alpha_j} \langle 0|W^\dagger_\beta W_{\beta'}|0\rangle \,. \tag{5.19}$$

The last term is the photon vacuum expectation value of the out-state dressing operators. This factor reduces to one or zero as shown in [42]; one if $\hat{j}(\beta) \sim \hat{j}(\beta')$ and zero otherwise. This is responsible for the decay of most off-diagonal elements in (5.19). However, if we are interested in the cross-section for a particular outgoing state β, this is again given by a diagonal density matrix element,

$$\sigma_{\text{in}\to\beta} \propto \rho_{\beta\beta} = \sum_{i,j} f_i f_j^* \mathbb{S}_{\beta\alpha_i} \mathbb{S}^*_{\beta\alpha_j}. \tag{5.20}$$

In stark contrast to the result obtained in the inclusive formalism, equation (5.11), this cross-section exhibits the usual interference between the various incoming states, c.f. equation (5.2). The reason for this is that in the dressed formalism, the outgoing radiation is described by the dressing which only depends on the out-state and not on the in-state. We will discuss this in more detail in section 5.4. This establishes that the inclusive and dressed formalism are not equivalent but yield different predictions for cross-sections

[5] The actual definition of the S-matrix should also contain a term to cancel the infinite Coulomb phase factor. Since this is immaterial to the current discussion we neglect this subtlety.

of finite superpositions.

5.3 Wavepackets

We will now proceed to look at scattering of wavepackets and find that the result is even more disturbing. After tracing out infrared radiation in the undressed formalism, no indication of scattering is left in the hard system. On the contrary, once again we will see that with dressed states, one gets the expected scattering out-state.

5.3.1 Inclusive Formalism

We consider incoming wavepackets of the form

$$|in\rangle = \int d\alpha f(\alpha) |\alpha\rangle \,, \tag{5.21}$$

normalized such that $\int d\alpha |f(\alpha)|^2 = 1$. The full analysis of the preceding section still applies, provided we replace $\sum_{\alpha_i} \to \int d\alpha$, $\alpha_i \to \alpha$, $f_i \to f(\alpha)$ and similarly for $a_j \to \alpha'$. The only notable exception is the calculation of single matrix elements as in equation (5.11), which now reads

$$\rho_{\beta\beta} = \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') S^{\Lambda}_{\beta,\alpha} S^{\Lambda *}_{\beta,\alpha'} \delta_{\alpha\alpha'}. \tag{5.22}$$

Note that here, by the same argument as before, the λ-dependent factor is turned into a Kronecker delta, which now reduces the integrand to a measure zero subset on the domain of integration. The only term that survives the integration is the initial state, which is acted on with the usual Dirac delta $\delta(\alpha - \beta)$, i.e. the "1" term in $S = 1 - 2\pi i \mathcal{M}$. The detailed argument can be found in appendix E. Thus we conclude that

$$\rho^{out}_{\beta\beta'} = f(\beta) f^*(\beta') = \rho^{in}_{\beta\beta'}. \tag{5.23}$$

The hard particles show no sign of a scattering event.

5.3.2 Dressed Wavepackets

The dressed formalism has perfectly reasonable scattering behavior. Consider wavepackets built from dressed states

$$\|in\rangle\!\rangle = \int d\alpha\ f(\alpha)\|\alpha\rangle\!\rangle, \tag{5.24}$$

with $\|\alpha\rangle\!\rangle$ a dressed state in the same notation as in equation (5.12). The S-matrix applied on dressed states is infrared-finite and the outgoing density matrix can be expressed as

$$\rho = \iint d\beta d\beta' \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathbb{S}_{\beta\alpha} \mathbb{S}^*_{\beta'\alpha'} \|\beta\rangle\!\rangle \langle\!\langle \beta'\|. \tag{5.25}$$

Tracing over soft modes, we find

$$\rho_{\beta\beta'} = \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathbb{S}_{\beta\alpha} \mathbb{S}^*_{\beta'\alpha'} \langle W^\dagger_\beta W_{\beta'}\rangle\,. \tag{5.26}$$

Again the expectation value is taken in the photon vacuum. The crucial point here is that this factor is independent of the initial states α. Upon sending the IR cutoff λ to zero, the expectation value for $W^\dagger W$ takes only the values 1 or 0, leading to decoherence in the outgoing state, but the cross-sections still exhibit all the usual interference between components of the incoming wavefunction,

$$\rho_{\beta\beta} = \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathbb{S}_{\beta\alpha} \mathbb{S}^*_{\beta\alpha'}, \tag{5.27}$$

unlike in the inclusive formalism.

5.4 Implications

In this section we will discuss the implications of our results and generalize and re-interpret our findings in particular in view of asymptotic gauge symmetries in QED and perturbative quantum gravity.

5.4.1 Physical Interpretation

The reason for the different predictions of the inclusive and dressed formalism is the IR radiation produced in the scattering process. The key idea is that accelerated charges produce radiation fields made from soft bosons. In the far infrared, the radiation spectrum has poles as the photon frequency $k^0 \to 0$ of the form $p_i/p_i \cdot k$, where p_i are the hard momenta. These poles reflect the fact that the radiation states are essentially classical and are completely distinguishable for different sets of asymptotic currents $\hat{j}_{\mathbf{v}}$.

In the inclusive formalism, we imagine incoming states with no radiation, and so the outgoing radiation state has poles from both the incoming hard particles α and the outgoing hard particles β. In the dressed formalism, the incoming part of the radiation is instead folded into the dressed state $\|\alpha\rangle\rangle$, which in the Faddeev-Kulish approach is designed precisely so that the outgoing radiation field only includes the poles from the outgoing hard particles. Thus if we scatter undressed Fock space states, a measurement of the radiation field at late times would completely determine the entire dynamical history of the process $\alpha \to \beta$, leading to the classical answer (5.11). If we instead scatter dressed states, the outgoing radiation has incomplete information about the incoming charged state, which is why the various incoming states still interfere in (5.20). Given that this type of interference is observed all the time in nature, this seems to strongly suggest that the dressed formalism is correct for any problem involving incoming superpositions of momenta.

Based on the result of section 5.2, one might argue that equation (5.11) perhaps is the correct answer and one would have to test experimentally whether or not interference terms appear if we give a scattering process enough time so that the decoherence becomes sizable. After all, the inclusive and dressed approach to calculating cross-sections are at least in principle distinguishable, although maybe not in practice due to very long decoherence times. However, we have demonstrated in section 5.3 that the inclusive formalism predicts an even more problematic result for continuous superpositions, namely that no scattering is observed at all. We thus propose that using the dressed formalism is the most conservative and physically sensible solution to the problem of vanishing interference presented in this paper.

5.4.2 Allowed Dressings

Dressing operators weakly commute with the S-matrix

It was conjectured in [8] and proven in [10] that the far IR part of the dressing weakly commutes with the S-matrix to leading order in the energy of the bosons contained in the dressing. In particular, this means that the amplitudes

$$\langle\beta|\, W_\beta^\dagger S W_\alpha \,|\alpha\rangle \sim \langle\beta|\, W_\beta^\dagger W_\alpha S \,|\alpha\rangle \sim \langle\beta|\, S W_\beta^\dagger W_\alpha \,|\alpha\rangle \tag{5.28}$$

are all IR finite, while they might differ by a finite amount. A short proof of this in QED, complementary to [10], can be given as follows (the gravitational case follows analogously). Recall that Weinberg's soft theorem for QED states that to lowest order in the soft photon momentum $\mathbf{q}$ of outgoing soft photons

$$\langle \epsilon_{l_1} a_{\mathbf{q}_1}^{l_1} \ldots \epsilon_{l_N} a_{\mathbf{q}_N}^{l_N} S\rangle \sim \prod_{i=1}^{N} \left(\sum_{j}^{M} \eta_j e_j \frac{\epsilon_{l_i} \cdot p_j}{q_i \cdot p_j} \right) \langle S \rangle \,. \tag{5.29}$$

A similar argument holds for incoming photons. For incoming photons with momentum $\mathbf{q}$ we find that

$$\langle S \epsilon_{l_1}^* a_{\mathbf{q}_1}^{l_1 \dagger} \ldots \epsilon_{l_N}^* a_{\mathbf{q}_N}^{l_N \dagger}\rangle \sim \prod_{i=1}^{N} \left(-\sum_{j}^{M} \eta_j e_j \frac{\epsilon_{l_i}^* \cdot p_j}{q_i \cdot p_j} \right) \langle S \rangle \,. \tag{5.30}$$

The reason for the relative minus sign is that incoming photons add energy-momentum to lines in the diagram instead of removing it. That means that the momentum in the denominator of the propagator changes $(p-q)^2+m^2 \to (p+q)+m^2$ and vice versa. For small momentum, the denominator becomes $-2pq \to 2pq$. From this it directly follows that for general dressings at leading order in the IR divergences,

$$\begin{aligned} \langle SW \rangle = \langle S e^{\int d^3k (F_l(\mathbf{k}) a_{\mathbf{k}}^{l\dagger} - F_l^*(\mathbf{k}) a_{\mathbf{k}}^{l})} \rangle &\sim \mathcal{N} \, \langle S e^{\int d^3k F_l(\mathbf{k}) a_{\mathbf{k}}^{l\dagger}} \rangle \\ &\sim \mathcal{N} \, \langle e^{-\int d^3k F_l^*(\mathbf{k}) a_{\mathbf{k}}^{l}} S \rangle \\ &\sim \langle e^{\int d^3k (F_l(\mathbf{k}) a_{\mathbf{k}}^{l\dagger} - F_l^*(\mathbf{k}) a_{\mathbf{k}}^{l})} S \rangle = \langle W S \rangle \,. \end{aligned} \tag{5.31}$$

In the first and third step we have split the exponential using the Baker-Campbell-Hausdorff formula ($\mathcal{N}$ is the normalization which is finite for finite λ) and in the second equality we have used Weinberg's soft theorem for outgoing and incoming particles.

Dressings cannot be arbitrarily moved between in- and out-states

This opens up the question about the most general structure of a consistent Faddeev-Kulish-like dressing. For example, one could ask whether one can consistently define S-matrix elements with the dressing only acting on the out-state. To answer this question, we assume that the dressing of the out-state has the same IR structure as equation (5.13), but is more general in that it may also include the momenta of (some) particles of the in-state, i.e. $W_\beta \to W_\beta W_{\tilde{\alpha}}$ or any other momenta which might not even appear in the process, $W_\beta W_{\tilde{\alpha}} \to W_\beta W_{\tilde{\alpha}} W_\zeta$. The IR structure of the in-dressing is then fixed by the requirement that the S-matrix element is finite. In addition to the requirement of IR-finiteness we ask that the so defined S-matrix elements give rise to the correct rules for superposition and the correct scattering for wavepackets, even after tracing out soft radiation.

Applying the logic of the previous sections and [57], one finds that tracing over the soft bosons yields for a diagonal matrix element $\rho_{\beta\beta}$

$$\rho_{\beta\beta}^{hard} = \sum_{i,j} f_i f_j^* \mathbb{S}_{\beta\alpha_i} \mathbb{S}^*_{\beta'\alpha'_j} \langle 0| W^\dagger_{\tilde{\alpha}'} W_{\tilde{\alpha}} |0\rangle \tag{5.32}$$

and

$$\rho_{\beta\beta}^{hard} = \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathbb{S}_{\beta\alpha} \mathbb{S}^*_{\beta'\alpha'} \langle 0| W^\dagger_{\tilde{\alpha}'} W_{\tilde{\alpha}} |0\rangle \tag{5.33}$$

for finite and continuous superpositions, respectively. Here, we have used that

$$\langle W^\dagger_{\tilde{\alpha}'} W^\dagger_{\beta'} W_\beta W_{\tilde{\alpha}} \rangle \Big|_{\beta=\beta'} = \langle W^\dagger_{\tilde{\alpha}'} W_{\tilde{\alpha}} \rangle \, . \tag{5.34}$$

The expectation value is taken in the soft boson Fock space. The expression in the case of $\tilde{\alpha} = \alpha$ and $\tilde{\alpha}' = \alpha'$ was already encountered in sections 5.2 and 5.3 in the context of inclusive calculations, where it was responsible for the unphysical form of the cross-sections. By the same logic it follows that even in the case where $\tilde{\alpha}$ is a proper subset of α, we will obtain a Kronecker delta which sets $\tilde{\alpha} = \tilde{\alpha}'$ and we again do not obtain the expected form of the cross-section. Instead, particles from the subset $\tilde{\alpha}$ will cease to interfere. We thus conclude that the dressing of the out-states must be independent of the in-states and it is not consistent to build superposition of states which are dressed differently. This means that building superpositions from hard and charged Fock space states is not meaningful. In particular, we cannot use undressed states to span the in-state space by simply moving all dressings to the out-state.

Generalized Faddeev-Kulish states

However, it would be consistent to define dressed states by acting with a constant dressing operator W_ζ for fixed ζ on states $\|\alpha\rangle\!\rangle$,

$$\|\alpha\rangle\!\rangle_\zeta \equiv W_\zeta^\dagger W_\alpha \, |\alpha\rangle \, . \tag{5.35}$$

Physically this corresponds to defining all asymptotic states on a fixed, coherent soft boson background, defined by some momenta ζ. This state does not affect the physics since soft modes decouple from Faddeev-Kulish amplitudes [29] and thus this additional cloud of soft photons will just pass through the scattering process. The difference between the Faddeev-Kulish dressed state $\|\alpha\rangle\!\rangle$ and the generalized states of the form $\|\alpha\rangle\!\rangle_\zeta$ is that the state $\|\zeta\rangle\!\rangle_\zeta = W_\zeta^\dagger W_\zeta \, |\zeta\rangle = |\zeta\rangle$ does not contain additional photons. This also explains why QED calculations using momentum eigenstates without any additional dressing give the correct cross-sections once we trace over soft radiation. Such a calculation can be interpreted as happening in a set of dressed states defined by

$$\|\alpha\rangle\!\rangle_{in} = W_{in}^\dagger W_\alpha \, |\alpha\rangle \, , \tag{5.36}$$

such that the in-state $\|in\rangle\!\rangle_{in}$ does not contain photons and looks like a standard Fock-space state.

Localized particles are accompanied by radiation

We also conclude from the previous sections that there are no charged, normalizable states which do not contain radiation. The reason is that within each selection sector there is only one non-normalizable state which does not contain radiation. Thus building a superposition to obtain a normalizable state will necessarily include dressed states which by definition contain soft bosons. A nice argument which makes this behavior plausible was given by Gervais and Zwanziger [45], see figure 5.1.

5.4.3 Selection Sectors

Everything said so far has a nice interpretation in terms of the charges $\mathcal{Q}_\varepsilon^\pm$ of large gauge transformations (LGT) for QED and supertranslation for perturbative quantum gravity. For a review see [61]. Large gauge transfor-

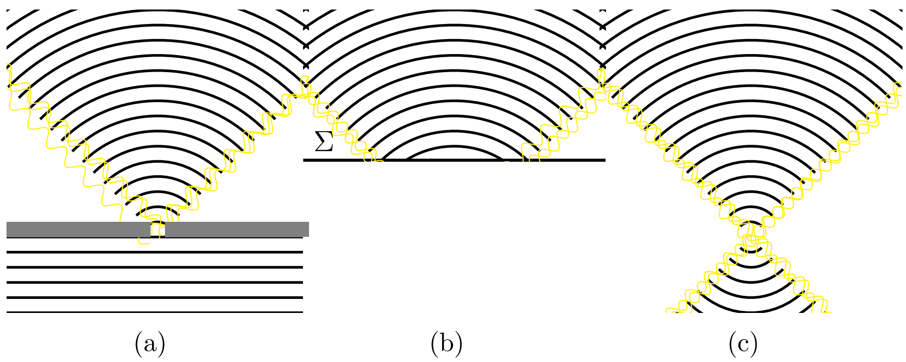


Figure 5.1: **(a)** A plane wave goes through a single slit and emerges as a localized wavepacket. The scattering of the incoming wavepacket results in the production of Bremsstrahlung. **(b)** We can also define some Cauchy slice Σ and create the state by an appropriate initial condition. **(c)** Evolving this state backwards in time while forgetting about the slit results in an incoming localized particle which is accompanied by a radiation shockwave.

mations in QED are gauge transformations which do not die off at infinity. They are generated by an angle-dependent function $\varepsilon(\phi, \theta)$. Similarly, supertranslations in perturbative quantum gravity are diffeomorphisms which do not vanish at infinity. They are constrained by certain falloff conditions. The transformations are generated by an infinite family of charges $\mathcal{Q}^{\pm}_{\varepsilon}$ at future and past lightlike infinity, parametrized by a functions $\varepsilon(\phi, \theta)$ on the celestial sphere. The charges split into a hard and a soft part

$$\mathcal{Q}^{\pm}_{\varepsilon} = \mathcal{Q}^{\pm}_{H,\varepsilon} + \mathcal{Q}^{\pm}_{S,\varepsilon}. \tag{5.37}$$

The soft charge generates the transformation on zero frequency photons or gravitons and leaves undressed particles invariant, while the hard charge generates LGT or supertranslations of charged particles, i.e. electrons in QED and all particles in perturbative quantum gravity. The action on particles can be found in [7, 36, 47, 62].

The charges $\mathcal{Q}^{\pm}_{\varepsilon}$ are conserved during time evolution (and in particular in any scattering process) and thus give rise to selection sectors of QED and gravity. These selection sectors give a different perspective on the IR catastrophe: Fock states of different momenta are differently charged under $\mathcal{Q}^{\pm}_{\varepsilon}$ and thus cannot scatter into each other. For dressed states, the situation is different: It was shown in [8, 10, 36] that for QED and gravity, Faddeev-Kulish dressed states $\|\alpha\rangle\rangle$ are eigenstates of $\mathcal{Q}^{\pm}_{\varepsilon}$ with an eigenvalue independent of α.

It turns out that also our generalized version of Faddeev-Kulish states $\|\alpha\rangle\!\rangle_\zeta$, equation (5.35), are eigenstates of the generators $\mathcal{Q}^\pm_\varepsilon$ with eigenvalues which depend on ζ. To see this note that [36]

$$[\mathcal{Q}^\pm_\varepsilon, W^\dagger_\zeta] = [\mathcal{Q}^\pm_{S,\varepsilon}, W^\dagger_\zeta] \propto \int_{S^2} d\hat{\mathbf{q}}\, \frac{\zeta^2}{\zeta \cdot \hat{q}}\, \varepsilon(\phi, \theta), \tag{5.38}$$

and similarly for gravity [10]. Thus the generalized Faddeev-Kulish states span a space of states which splits into selection sectors parametrized by ζ. The statement that we can build physically reasonable superpositions using generalized Faddeev-Kulish states translates into the statement that superpositions can be taken within a selection sector of the LGT and supertranslation charges $\mathcal{Q}^\pm_\varepsilon$.

In the context of these charges, zero energy eigenstates of $\mathcal{Q}^\pm_{S,\varepsilon}$ are often interpreted as an infinite set of vacua. Note that the name vacuum might be misleading as states in a single selection sector are in fact built on different vacua. Our results also raise doubt on whether physical observables exist which can take a state from one selection sector into another. If they did we could use them to create a superpositions of states from different sectors. But as we have seen above, in this case interference would not happen, which is in conflict with basic postulates of quantum mechanics.

5.5 Conclusions

Calculating cross-sections in standard QED and perturbative quantum gravity forces us to deal with IR divergences. Tracing out unobservable soft modes seems to be a physically well-motivated approach which has successfully been employed for plane-wave scattering. However, as we have shown this approach fails in more generic examples. For finite superpositions it does not reproduce interference terms which are expected; for wavepackets it predicts that no scattering is observed. We have demonstrated in this paper that dressed states à la Faddeev-Kulish (and certain generalizations) resolve this issue, although it is not clear if the inclusive and dressed formalism are the only possible resolutions. Importantly, we have shown that predictions of different resolutions can disagree, making them distinguishable.

Superpositions must be taken within a set of states with most of the states dressed by soft bosons. The corresponding dressing operators are only well-defined on Fock space if we use an IR-regulator which we only remove at the end of the day. In the strict $\lambda \to 0$ limit, the states are not in Fock space but rather in the much larger von Neumann space which allows for any photon content, including uncountable sets of photons [41, 54]. This suggests an interesting picture which seems worth investigating. The Hilbert space of QED is non-separable but has separable subspaces which are stable under action of the S-matrix and form selection sectors. These subspaces are not the usual Fock spaces but look like the state spaces defined by Faddeev and Kulish [4], in which almost all charged states are accompanied by soft radiation. It would be an interesting task to make these statements more precise.

Our results may have implications for the black hole information loss problem. Virtually all discussions of information loss in the black hole context rely on the possibility of localizing particles – from throwing a particle into a black hole to keeping information localized. We argued above that normalizable (and in particular localized) states are necessarily accompanied by soft radiation. It is well known that the absorption cross-section of radiation with frequency ω vanishes as $\omega \to 0$ and therefore it seems plausible that, whenever a localized particle is thrown into a black hole, the soft part of its state which is strongly correlated with the hard part remains outside the black hole. If this is true a black hole geometry is always in a mixed state which is purified by radiation outside the horizon.

Chapter 6

Conclusion

In this dissertation we investigated quantum information properties of relativistic scattering theory with an emphasis on the infrared behavior of massless gauge field theories. To do so, we used a density matrix approach mixed with the S-matrix machinery of quantum field theory to uncover long time properties of interacting QFTs.

6.1 Scattering with Partial Information

In chapter 2, we analyzed interacting quantum field theories where an observer only has partial access to the full state of the system. We considered the case where a set of apparatus particles interact with an unobserved collection of system particles and used the more general framework of density matrices to express the global state. Incoming states were time evolved into outgoing ones using the S-matrix formalism well known to field theorists. Outgoing states described in such a way were however still comprised of apparatus and system particles. At asymptotic times, we expect the full Hilbert space to decompose into a tensor product between the apparatus Hilbert space and the system Hilbert space allowing us to trace out the collection of unobserved particles. This procedure left us with a density matrix describing the precise state an observer could measure at late times. We could then compute various quantities like the late time expectation value of apparatus observable but more notably we were able to produce an exact formula for the von Neumann entanglement entropy between the apparatus and the system states. The entanglement entropy in this case quantifies the amount of information of the total state that has been lost by not being able to observe the system particles.

For the simple example of a pair of interacting massive scalar fields with coupling of the form $\lambda\phi_A^2\phi_S^2$, we could give a perturbative expression for the entanglement entropy which scaled like the total cross-section of the process integrated over time against the flux of incoming particles. We also used this example to suggest a way to directly measure the parameter λ while the literature usually suggests the theory is only sensible to λ^2. Starting with the system scalar field in a spatial superposition of states, One should observe interference fringes at specific locations on a spherical detector surrounding the scattering. We demonstrated that the amplitudes of these interference fringes depend on λ at lowest order in perturbation theory as the position space observable depends on the off-diagonal elements of the reduced density matrix.

6.2 Decoherence from Infrared Photons and Gravitons

In chapters 3, we specialize to the case of QED and gravity where scattering of charged particles radiates away an infinite number of low energy gauge bosons. Using again our density matrix based approach to scattering theory we were able to extend Weinberg's soft theorem to off-diagonal elements of the density matrix. Applying the S-matrix on both sides of the incoming density matrix we could obtain a late time density matrix as usual. Starting by tracing out the outgoing radiation in the standard way, we obtained divergences in the photon mass of the form $\lambda^{-\tilde{A}_{\beta\beta',\alpha}}$, with an explicit formula for $\tilde{A}$. We then had to account for the divergences arising from soft boson loop diagrams in the usual Weinberg fashion providing factors of $\lambda^{A_{\beta,\alpha}/2+A_{\beta',\alpha}/2}$. The limit when the infrared regulator λ is taken to zero told us that the final infrared divergences in the density matrix would depend on the exponent

$$\Delta A_{\beta\beta',\alpha} = \frac{A_{\beta,\alpha}}{2} + \frac{A_{\beta',\alpha}}{2} - \tilde{A}_{\beta\beta',\alpha}. \tag{6.1}$$

We were able to prove the positivity of the exponent, meaning that the $\lambda \to 0$ limit would leave every term in the density matrix to be either 0 or finite. Then, we showed that the exponent $\Delta A_{\beta\beta',\alpha}$ is exactly 0 if and only if the sum of the charged currents in states β, β' are identical. This has the effect of sending nearly all off-diagonal terms in the reduced outgoing density matrix to zero. In the case of QED, we are left with some coherence in the state from the inclusion of uncharged matter. However, in gravity all particles are charged leaving us with a completely decohered density matrix. We therefore arrive to the conclusion that after waiting for an infinite amount of time after scattering, long wavelength photons and gravitons are sufficient to provide nearly complete decoherence of the outgoing states: Instead of leaving us with an entangled density matrix as one could have expected, at late times we observe a classical ensemble of states with probabilities given by Weinberg's inclusive cross-sections approach.

In chapter 4, we addressed the same question from the Faddeev-Kulish dressing perspective. This implies considering scattering between states of charged particles accompanied by clouds of finely tuned soft bosons. There, the S-matrix elements are finite due to the cancellation of divergence arising from cloud interactions. This means the S-matrix is well defined for any values of the infrared regulator λ even as it is taken to zero. However, we could still make the argument that an observer with a finite sized detector would not have access to those soft photon clouds. Tracing out the outgoing soft radiation, we found that the terms in the reduced outgoing density matrix depended on the dampening factor

$$D_{\beta,\beta'} = \langle 0|W^{\dagger}_{\beta'} W_{\beta}|0\rangle \,, \tag{6.2}$$

made from the expectation value of dressing operators. A detailed analysis of the dampening factor allowed us to conclude the same result as in the Weinberg formalism: The dampening factor turned out to be exactly 0 for every pair of states β, β' that did not contain the same charged current content. In the case where the charged currents agreed, the dampening factor was simply 1. Our result indicated that whichever formalism you chose to use, late-time decoherence would inevitably happen.

6.3 Wavepacket Scattering and the Need for Soft Dressing

In chapter 5, we gave an original argument advocating for the use of the dressing approach. While the recent literature has provided some evidence that the dressing approach is intertwined with the topic of asymptotic symmetries of massless gauge field theories, no specific examples were found where the two approaches would give different results. We investigated the general case of scattering of an entangled superposition of particles and found disturbing results. In the inclusive formalism, we demonstrated that after scattering no trace of interference between incoming states were present. Scattering of an entangled superposition would therefore be the same as scattering a classicle ensemble of particles which certainly goes against known observational evidence. From the dressing perspective however, we find a more natural answer. As the outgoing dressing contains no information about the incoming dressing, the reduced outgoing density matrix does not destroy interference between incoming superpositions. While tracing out the outgoing radiation leads to decoherence, it still maintains the initial entanglement of the system.

Moving on to wavepacket scattering, we showed that the continuous superposition case is even more dramatic in the inclusive approach: There, no scattering can happen. The decoherence condition forces the reduced outgoing density matrix to an integral where the integrand is a subset of measure zero of the domain of integration. The only portion of the S-matrix which remains is the identity factor $\delta(\alpha-\beta)$, leaving the outgoing state to be precisely the same as the outgoing state. As we certainly live in a universe where wavepacket scattering is possible, the inclusive formalism must be rejected. However, wavepacket scattering from the dressed picture still yields the expected results where outgoing states exhibit the usual interference between components of the incoming wavefunction.

Advocating for the use of the Faddeev-Kulish dressing, we used the new developments from asymptotic symmetries of QED and gravity to interpret our results. Decoherence was explained in terms of conserved charges. Performing the trace over soft photons forced the soft charge to agree for the kets and bras of the reduced density matrix elements. Total charge conservation then required that the hard charges must also agree, leaving us with a decohered state. However, allowing for different outgoing soft vacuums restored the coherence of the hard data. We could also explain in the language of soft charges why the inclusive formalism gave the correct results for non-entangled incoming states while it failed for entangled ones. In the inclusive formalism, one makes a very specific choice of dressing where the incoming vacuum contains no soft photons. While this was possible to make such a choice for non-entangled state, entangled states cannot accomodate this restriction. There, every entangled parts are found within different selection sectors and cannot interact together.

6.4 Closing Remarks

The primary focus of this thesis was to investigate quantum information properties of scattering theory with a focus on infrared divergent effects found in massless gauge field theories. We were able to extract meaningful quantities related to real observations where soft particles are lost to the environment. For QED and gravity, we postulated the existence of a late time loss of coherence effect on hard data driven by soft photons and gravitons. We gave new evidence in favor of the use of dressed charged states and expressed our results in terms of the new language of asymptotic symmetries of QED and gravity. Finally, we believe our work has implications towards the resolution of the black hole information paradox: The dressing of charged particles could potentially encode the information otherwise lost in the formation and evaporation process of black holes.

Bibliography

[1] F. Bloch and A. Nordsieck, *Note on the Radiation Field of the Electron*, Phys. Rev. **52** (1937) 54–59.

[2] S. Weinberg, *Infrared Photons and Gravitons*, Phys. Rev. B **140** (1965), no. 2B 516–524.

[3] V. Chung, *Infrared Divergence In Quantum Electrodynamics*, Phys. Rev. B **140** (1965), no. 4 1122.

[4] P. P. Kulish and L. D. Faddeev, *Asymptotic Conditions and Infrared Divergences in Quantum Electrodynamics*, Teoreticheskaya i Mathematicheskaya Fizika **4** (1970), no. 2 153–170.

[5] T. He, P. Mitra, A. P. Porfyriadis, and A. Strominger, *New Symmetries of Massless QED*, JHEP **10** (2014) 112, [`arXiv:1407.3789`].

[6] T. He, V. Lysov, P. Mitra, and A. Strominger, *BMS supertranslations and Weinberg's soft graviton theorem*, JHEP **05** (2015) 151, [`arXiv:1401.7026`].

[7] D. Kapec, M. Pate, and A. Strominger, *New Symmetries of QED*, .

[8] D. Kapec, M. Perry, A. M. Raclariu, and A. Strominger, *Infrared divergences in QED revisited*, Phys. Rev. D **96** (2017), no. 8.

[9] S. W. Hawking, M. J. Perry, and A. Strominger, *Soft Hair on Black Holes*, Phys. Rev. Lett. **116** (2016), no. 23 1–9.

[10] S. Choi and R. Akhoury, *BMS Supertranslation Symmetry Implies Faddeev-Kulish Amplitudes*, .

[11] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, *Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems*, Proc. Roy. Soc. Lond. **A269** (1962) 21–52.

[12] R. K. Sachs, *Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times*, Proc. Roy. Soc. Lond. **A270** (1962) 103–126.

[13] S. W. Hawking, *Particle creation by black holes*, Commun. Math. Phys. **43** (1975), no. 3 199–220.

[14] A. Strominger, *Black Hole Information Revisited*, .

[15] J. von Neumann, Mathematical Foundations of Quantum Mechanics. Princeton University Press, 1955.

[16] J. Preskill, Caltech lecture notes on quantum computation. 1998.

[17] P. Calabrese and J. L. Cardy, *Entanglement entropy and quantum field theory*, J. Stat. Mech. **0406** (2004) P06002, [hep-th/0405152].

[18] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, Phys. Rev. Lett. **96** (2006) 181602, [hep-th/0603001].

[19] V. Balasubramanian, M. B. McDermott, and M. Van Raamsdonk, *Momentum-space entanglement and renormalization in quantum field theory*, Phys. Rev. **D86** (2012) 045014, [arXiv:1108.3568].

[20] J. M. Maldacena, *Eternal black holes in anti-de Sitter*, JHEP **04** (2003) 021, [hep-th/0106112].

[21] S. Weinberg, The Quantum Theory of Fields. Vol. 1: Foundations. Cambridge University Press, 2005.

[22] S. Seki, I. Y. Park, and S.-J. Sin, *Variation of Entanglement Entropy in Scattering Process*, Phys. Lett. **B743** (2015) 147–153, [arXiv:1412.7894].

[23] R. Peschanski and S. Seki, *Entanglement Entropy of Scattering Particles*, Phys. Lett. **B758** (2016) 89–92, [arXiv:1602.00720].

[24] S. W. Hawking, M. J. Perry, and A. Strominger, *Soft Hair on Black Holes*, arXiv:1601.00921.

[25] D. R. Yennie, S. C. Frautschi, and H. Suura, *The Infrared Divergence Phenomena and High-Energy Processes*, Annals Phys. **13** (1961) 379–452.

[26] T. W. B. Kibble, *Coherent Soft-Photon States and Infrared Divergences. IV. The Scattering Operator*, Phys. Rev. **175** (1968), no. 5 1624–1640.

[27] J. Ware, R. Saotome, and R. Akhoury, *Construction of an asymptotic S matrix for perturbative quantum gravity*, JHEP **10** (2013).

[28] S. W. Hawking, M. J. Perry, and A. Strominger, *Superrotation charge and supertranslation hair on black holes*, JHEP **05** (2017).

[29] M. Mirbabayi and M. Porrati, *Dressed hard states and black hole soft hair*, Phys. Rev. Lett. **117** (2016), no. 21 1–5.

[30] F. E. Low, *Bremsstrahlung of very low-energy quanta in elementary particle collisions*, Phys. Rev. **110** (1958) 974–977.

[31] D. Carney, L. Chaurette, and G. Semenoff, *Scattering with partial information*, `arXiv:1606.03103`.

[32] G. Grignani and G. W. Semenoff, *Scattering and momentum space entanglement*, Phys. Lett. **B772** (2017) 699–702, [`arXiv:1612.08858`].

[33] F. E. Low, *Scattering of light of very low frequency by systems of spin 1/2*, Phys. Rev. **96** (1954) 1428–1432.

[34] M. Gell-Mann and M. L. Goldberger, *Scattering of low-energy photons by particles of spin 1/2*, Phys. Rev. **96** (1954) 1433–1438.

[35] V. Lysov, S. Pasterski, and A. Strominger, *Low's Subleading Soft Theorem as a Symmetry of QED*, Phys. Rev. Lett. **113** (2014), no. 11 111601, [`arXiv:1407.3814`].

[36] B. Gabai and A. Sever, *Large gauge symmetries and asymptotic states in QED*, JHEP **12** (2016).

[37] R. Bousso and M. Porrati, *Soft Hair as a Soft Wig*, Class. Quant. Grav. **34** (2017).

[38] T. Banks and W. Fischler, *A Model for high-energy scattering in quantum gravity*, `hep-th/9906038`.

[39] P. A. M. Dirac, *Gauge invariant formulation of quantum electrodynamics*, Can. J. Phys. **33** (1955) 650.

[40] J. Dollard, *Asymptotic Convergence and the Coulomb Interaction*, J. Math. Phys. (1964) 729.

[41] D. Zwanziger, *Scattering Theory for Quantum Electrodynamics. 1. Infrared Renormalization and Asymptotic Fields*, Phys. Rev. D **11** (1975) 3481.

[42] D. Carney, L. Chaurette, D. Neuenfeld, and G. W. Semenoff, *Infrared quantum information*, Phys. Rev. Lett. **119** (6, 2017) 1–5.

[43] A. Strominger, *Black Hole Information Revisited*, arXiv:1706.07143.

[44] W. Dybalski, *From Faddeev-Kulish to LSZ. Towards a non-perturbative description of colliding electrons*, Nucl. Phys. B **925** (2017) 1–12.

[45] J. L. Gervais and D. Zwanziger, *Derivation from first principles of the infrared structure of quantum electrodynamics*, Phys. Lett. B **94** (1980), no. 3 389–393.

[46] D. Buchholz, *The Physical State Space of Quantum Electrodynamics*, Commun. Math. Phys. **85** (1982) 49–71.

[47] M. Campiglia and A. Laddha, *Asymptotic symmetries of QED and Weinberg's soft photon theorem*, JHEP **7** (2015).

[48] C. Gomez and M. Panchenko, *Asymptotic dynamics, large gauge transformations and infrared symmetries*, arXiv:1608.05630.

[49] W. H. Zurek, *Pointer Basis of Quantum Apparatus: Into What Mixture Does the Wave Packet Collapse?*, Phys. Rev. D **24** (1981) 1516–1525.

[50] H. P. Breuer and F. Petruccione, *Destruction of quantum coherence through emission of bremsstrahlung*, Phys. Rev. A **63** (2001), no. 3 1–18.

[51] G. Calucci, *Loss of coherence due to bremsstrahlung*, Phys. Rev. A **67** (2003), no. 4 1–6.

[52] G. Calucci, *Graviton emission and loss of coherence*, Class. Quant. Grav. **21** (2004) 2339–2349.

[53] J. D. Dollard, *Asymptotic Convergence and the Coulomb Interaction*, J. Math. Phys. **5** (1964) 729.

[54] T. W. B. Kibble, *Coherent Soft-Photon States and Infrared Divergences. I. Classical Currents*, Journal of Mathematical Physics **9** (1968) 315–324.

[55] T. W. B. Kibble, *Coherent Soft-Photon States and Infrared Divergences. II. Mass-Shell Singularities of Green's Functions*, Phys. Rev. **173** (1968), no. 5 1527–1535.

[56] T. W. B. Kibble, *Coherent Soft-Photon States and Infrared Divergences. III. Asymptotic States and Reduction Formulas*, Phys. Rev. **174** (1968), no. 5 1882–1901.

[57] D. Carney, L. Chaurette, D. Neuenfeld, and G. W. Semenoff, *Dressed infrared quantum information*, Phys. Rev. D **97** (6, 2018).

[58] C. Gomez, R. Letschka, and S. Zell, *Infrared Divergences and Quantum Coherence*, .

[59] A. Strominger, *On BMS invariance of gravitational scattering*, JHEP **2014** (2014), no. 7 0–23.

[60] M. Campiglia and A. Laddha, *Asymptotic symmetries of gravity and soft theorems for massive particles*, JHEP **12** (2015).

[61] A. Strominger, Lectures on the Infrared Structure of Gravity and Gauge Theory. Princeton University Press, 2018.

[62] S. Choi, U. Kol, and R. Akhoury, *Asymptotic Dynamics in Perturbative Quantum Gravity and BMS Supertranslations*, JHEP **1** (2018).

[63] H. Afshar, D. Grumiller, and M. M. Sheikh-Jabbari, *Near horizon soft hair as microstates of three dimensional black holes*, Phys. Rev. D **96** (2017), no. 8 84032.

[64] R. K. Mishra and R. Sundrum, *Asymptotic Symmetries, Holography and Topological Hair*, JHEP **01** (2018) 14.

[65] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, *Black holes: Complementarity or firewalls?*, JHEP **2** (2013) 1–19.

Appendix A

Optical Theorem

Here we repeat Weinberg's proof of the optical theorem, for completeness, because the same techniques appear repeatedly in the above. In particular, we explain how unitarity of the density matrices used in scattering is directly related to the optical theorem.

Our scattering states are supposed to be continuum-normalized

$$\langle \alpha'^{\pm} | \alpha^{\pm} \rangle = \delta(\alpha - \alpha'), \tag{A.1}$$

where the right hand side as usual means a product of Dirac deltas on the spatial momenta. Now, this equation needs to be consistent with the unitarity of the S-matrix, i.e. we should have

$$\delta(\alpha - \alpha') = \langle \alpha'^{+} | \alpha^{+} \rangle = \int d\beta d\beta' S_{\beta\alpha} S^{*}_{\beta'\alpha'} \langle \beta'^{-} | \beta^{-} \rangle = \int d\beta S_{\beta\alpha} S^{*}_{\beta\alpha'}. \tag{A.2}$$

Writing the usual decomposition of S as in (2.11) and doing some of the integrals, we see that we need

$$\begin{aligned} &2\pi i \left[M_{\alpha'\alpha} \delta^4(p_\beta - p_\alpha) - M^{*}_{\alpha\alpha'} \delta^4(p_\beta - p_{\alpha'}) \right] \\ &= (2\pi)^2 \int d\beta M_{\beta\alpha} M^{*}_{\beta\alpha'} \delta^4(p_\beta - p_\alpha) \delta^4(p_\beta - p_{\alpha'}). \end{aligned} \tag{A.3}$$

Specialize to the case $\alpha = \alpha'$. We obtain the optical theorem

$$\mathrm{Im} M_{\alpha\alpha} = -\pi \int d\beta \left| M_{\beta\alpha} \right|^2 \delta^4(p_\beta - p_\alpha). \tag{A.4}$$

Consider scattering an initial state $|\alpha^{+}\rangle^{box}$, now in a finite spacetime box as described in the main text. Then our density matrix should have unit trace. Writing this after applying the S-matrix and doing the trace using out-states $|\beta^{-}\rangle^{box}$, we have

$$1 = \mathrm{tr}\, \rho = \sum_{\beta} \left| S^{box}_{\beta\alpha} \right|^2 . \tag{A.5}$$

If we expand the S-matrix as in (2.16), then the delta-squared term on the right hand side will give exactly the 1 on the left-hand side in our trace norm condition here. So then the remaining three terms will have to cancel amongst themselves, which is exactly the case when (A.4) holds.

Note that in perturbation theory in some weak coupling λ, the optical theorem mixes orders of λ. For our purposes above, for example to get the entanglement entropy to $\mathcal{O}(\lambda^2)$, we need to ensure that we normalize the density matrix to $\operatorname{tr}\rho = 1 + \mathcal{O}(\lambda^3)$. But then we need the scattering matrix elements appearing in (A.4) to cancel on the two sides of the equation up to $\mathcal{O}(\lambda^2)$. In other words, to explicitly check the normalization of the density matrix in perturbation theory at this order, we need to include the lowest-order loop diagram for forward scattering in computing the scattering amplitudes.

Appendix B

Positivity of A, B Exponents

Appendix. Here, we show that the exponents $\Delta A, \Delta B$ controlling the infrared divergences are always positive or zero, and give necessary and sufficient conditions for these exponents to vanish.

The first step is to notice that the expressions for the differential exponents (3.10) between the processes $\alpha \to \beta$ and $\alpha \to \beta'$ are the same as the exponents (3.8) for the divergences in the process $\beta \to \beta'$, that is

$$\begin{aligned} \Delta A_{\beta\beta',\alpha} &= A_{\beta',\beta}/2, \\ \Delta B_{\beta\beta',\alpha} &= B_{\beta',\beta}/2. \end{aligned} \tag{B.1}$$

To see this, note from the definitions (3.6),(3.8), and (3.10) that there are terms in each of $A_{\beta,\alpha}, A_{\beta',\alpha}$, and $\tilde{A}_{\beta\beta',\alpha}$ coming from contractions between pairs of incoming legs, pairs of an incoming and outgoing leg, and pairs of outgoing legs. One can easily check that the in/in and in/out terms cancel pairwise between the A and $\tilde{A}$ terms in ΔA. The remainder is the terms involving contractions between pairs of outgoing legs:

$$\Delta A_{\beta\beta',\alpha} = \frac{1}{2}\sum_{p,p'\in\beta}\gamma_{pp'} + \frac{1}{2}\sum_{p,p'\in\beta'}\gamma_{pp'} - \sum_{p\in\beta,p'\in\beta}\gamma_{pp'} \tag{B.2}$$

where we defined $\gamma_{pp'} = e_p e_{p'} \beta_{pp'}^{-1} \ln[(1+\beta_{pp'})/(1-\beta_{pp'})]$. We have used the fact that every η_p that would have been in (B.2) is a -1 since every line being summed is an outgoing particle, cf. (3.3). But then we have a relative minus sign and factor of 2 between the first two terms and the third; this is precisely the same factor that would have come from the relative $\eta_{in} = -1$ and $\eta_{out} = +1$ terms in exponent for the process $\beta \to \beta'$, namely

$$A_{\beta',\beta} = \sum_{p,p'\in\beta}\gamma_{pp'} + \sum_{p,p'\in\beta'}\gamma_{pp'} - 2\sum_{p\in\beta,p'\in\beta'}\gamma_{pp'}. \tag{B.3}$$

This proves (B.1) for ΔA; an identical combinatorial argument shows that the gravitational exponent obeys the analogous relation, $\Delta B_{\beta\beta',\alpha} = B_{\beta',\beta}/2$.

Now we prove that for the process $\alpha \to \beta + (soft)$ the exponent $A_{\beta\alpha}$ is always greater or equal to zero with equality if and only if the in and outgoing currents agree; we can then take $\alpha = \beta'$ to get the results quoted in the text. Referring to Weinberg's derivation [2], we can write $A_{\beta\alpha}$ as

$$A_{\beta\alpha} = \frac{1}{2(2\pi)^3} \int_{S^2} d\hat{q}\ t^\mu(\hat{q}) t_\mu(\hat{q}). \tag{B.4}$$

Here,

$$t^\mu(\hat{q}) \equiv \sum_n \frac{e_n \eta_n p_n^\mu}{p_n \cdot q} = c(q) q^\mu + c_i(q)(q_\perp^i)^\mu. \tag{B.5}$$

In this equation, we have defined a lightlike vector $q^\mu = (1, \hat{q})$ and $q_\perp^i$, $i = 1, 2$ are two unit normalized, mutually orthogonal, purely spatial vectors perpendicular to q^μ. The sum on $n \in \alpha, \beta$ runs over in- and out-going particles. By charge conservation, $t \cdot q = 0$, which justifies the decomposition in the second equality in (B.5). With this decomposition we may write

$$A_{\beta\alpha} = \frac{1}{2(2\pi)^3} \int_{S^2} d\hat{q}(c_1^2(q) + c_2^2(q)) \geq 0, \tag{B.6}$$

which immediately proves the statement that $A_{\beta\alpha} \geq 0$.

Now it remains to be shown that equality holds if and only if all of the in- and out-going currents match. From the previous paragraph we know that $A_{\beta\alpha}$ vanishes if and only if both $c_i(q) = 0$ for all q, that is if and only if $t \cdot q_\perp^i = 0$. Assume that $A_{\beta\alpha} = 0$, so that $q_\perp \cdot t(q) = 0$. Now suppose also that $j_{\mathbf{v}_0}(\alpha) \neq j_{\mathbf{v}_0}(\beta)$ for some $\mathbf{v}_0$, where these are the eigenvalues of $j_{\mathbf{v}} |\alpha\rangle = j_{\mathbf{v}}(\alpha) |\alpha\rangle$ and similarly for β. We derive a contradiction. For any finite set of velocities, the functions $f_{\mathbf{v}}(\hat{\mathbf{q}}) = (\mathbf{v} \cdot \mathbf{q}_\perp)/(1 - \mathbf{v} \cdot \hat{\mathbf{q}})$ are linearly independent. Therefore the terms in

$$0 = t \cdot q_\perp = \sum_n \frac{e_n \eta_n v_n \cdot q_\perp}{v_n \cdot q} \tag{B.7}$$

must cancel separately for each velocity in the list of $\mathbf{v}_n$. Consider in particular the term for $\mathbf{v}_0$. For this to vanish, the sum of the coefficients must vanish, i.e.

$$0 = \sum_{n | v_n = v_0} e_n \eta_n = [j_{v_0}(\alpha) - j_{v_0}(\beta)], \tag{B.8}$$

the relative minus coming from the η factors. But this contradicts our assumption that $j_{\mathbf{v}_0}(\alpha) \neq j_{\mathbf{v}_0}(\beta)$. This completes the proof for A.

The proof for gravitons goes similarly. Again referring to Weinberg we write B as

$$B_{\beta\alpha} = \frac{G}{4\pi^2}\int_{S^2} d\hat{q} t^{\mu\nu} D_{\mu\nu\rho\sigma} t^{\rho\sigma}. \tag{B.9}$$

Here, $D_{\mu\nu\rho\sigma} = \eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}$ is the numerator of the graviton propagator, and

$$t^{\mu\nu} = \sum_n \frac{\eta_n p_n^\mu p_n^\nu}{p_n \cdot q} = c q^{(\mu} q^{\nu)} + c^i q^{(\mu} q^{\nu)}_{\perp,i} + c^{ij} q^{(\mu}_{\perp,i} q^{\nu)}_{\perp,j}. \tag{B.10}$$

This symmetric tensor obeys $t^{\mu\nu} q_\nu = 0$ by energy-momentum conservation, which justifies the decomposition in the second equality. Using this we have

$$t^{\mu\nu} D_{\mu\nu\rho\sigma} t^{\rho\sigma} = 2 c^i_j c^j_i - \left(c^i_i\right)^2 = (\lambda_1 - \lambda_2)^2 \tag{B.11}$$

with $\lambda_{1,2}$ the two eigenvalues of the matrix c^{ij}. Plugging this into (B.9) we immediately see that $B \geq 0$. The condition for vanishing of $B_{\beta\beta'}$ is that the eigenvalues are equal $\lambda_1 = \lambda_2$, which means that c^{ij} is proportional to the identity matrix. Hence, if B vanishes we have that

$$0 = t^{\mu\nu} q^{\perp,1}_\mu q^{\perp,2}_\nu = \sum_n \eta_n E_n \frac{(v_n \cdot q^1_\perp)(v_n \cdot q^2_\perp)}{v_n \cdot q}. \tag{B.12}$$

As before, any finite set of functions $g_v(q) = (v \cdot q^1_\perp)(v \cdot q^2_\perp)/(v \cdot q)$ are linearly independent functions of q, and so by direct analogy with the previous proof, $B = 0$ if and only if $j^{grav}_{\mathbf{v}}(\alpha) = j^{grav}_{\mathbf{v}}(\beta)$ for every $\mathbf{v}$.

Appendix C

Dressed Soft Factorization

The soft photon theorem looks somewhat different in dressed QED. In standard, undressed QED, the theorem says that the amplitude for a process $\mathbf{p} \to \mathbf{q}$ accompanied by emission of an additional soft photon of momentum $\mathbf{k}$ and polarization ℓ has amplitude

$$S_{\mathbf{qk}\ell,\mathbf{p}} = e\left[\frac{q \cdot e_\ell^*(\mathbf{k})}{q \cdot k} - \frac{p \cdot e_\ell^*(\mathbf{k})}{p \cdot k}\right] S_{\mathbf{q},\mathbf{p}}. \tag{C.1}$$

This is singular in the $k \to 0$ limit. On the other hand, in the dressed formalism of QED, the statement is that

$$\tilde{S}_{\mathbf{qk}\ell,\mathbf{p}} = ef(\mathbf{k})\tilde{S}_{\mathbf{q},\mathbf{p}}, \tag{C.2}$$

where $f(\mathbf{k}) \sim \mathcal{O}(|\mathbf{k}|^0)$, so that the right-hand side is finite as $k \to 0$. We can see this by straightforward computation. In computing the matrix element (C.2), there will be four Feynman diagrams at lowest order in the charge. We will get the usual pair of Feynman diagrams coming from contractions of the interaction Hamiltonian with the external photon state, leading to the poles (C.1). Moreover we will get a pair of terms coming from contractions of the interaction Hamiltonian with dressing operators. These contribute a factor

$$\begin{aligned} &[F_\ell^*(\mathbf{k},\mathbf{p}) - F_\ell^*(\mathbf{k},\mathbf{q})] \\ &\to \left[\frac{q \cdot e_\ell^*(\mathbf{k})}{q \cdot k} - \frac{p \cdot e_\ell^*(\mathbf{k})}{p \cdot k}\right] + \mathcal{O}(|\mathbf{k}|^0), \end{aligned} \tag{C.3}$$

times $-e$, where the limit as $k \to 0$ follows from the definition (4.1). This extra contribution precisely cancels the poles in (C.1), leaving only the order $\mathcal{O}(|\mathbf{k}|^0)$ term.

Appendix D

Proof of Positivity of $\Delta A, \Delta B$

The exponent that is responsible for the decoherence of the system is defined as

$$\Delta A_{\beta\beta',\alpha\alpha'} = \frac{1}{2}A_{\beta,\alpha} + \frac{1}{2}A_{\beta',\alpha'} - \tilde{A}_{\beta\beta',\alpha\alpha'}. \tag{D.1}$$

The factor in the first two terms, $A_{\beta,\alpha}$, is defined as in [2]

$$A_{\beta,\alpha} = \frac{1}{2(2\pi)^3}\int_{S^2} d\hat{\mathbf{q}} \left(\sum_{n\in\beta}\frac{e_n\eta_n p_n^\mu}{p_n\cdot\hat{q}}\right) g_{\mu\nu} \left(\sum_{m\in\alpha}\frac{e_m\eta_m p_m^\mu}{p_m\cdot\hat{q}}\right). \tag{D.2}$$

Performing the integral over $\hat{\mathbf{q}}$ yields

$$A_{\beta,\alpha} = -\sum_{n,n'\in\alpha,\beta}\frac{e_n e_{n'}\eta_n\eta_{n'}}{8\pi^2}\beta_{nn'}\ln\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right]. \tag{D.3}$$

Similarly $\tilde{A}_{\beta\beta',\alpha\alpha'}$ can be written as

$$\tilde{A}_{\beta\beta',\alpha\alpha} = -\sum_{\substack{n\in\alpha,\beta \\ n'\in\alpha'\beta'}}\frac{e_n e_{n'}\eta_n\eta_{n'}}{8\pi^2}\beta_{nn'}\ln\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right]. \tag{D.4}$$

We rearrange the terms such that ΔA can be written as

$$\Delta A_{\beta\beta',\alpha\alpha'} = -\frac{1}{2}\sum_{n,n'\in\alpha,\bar{\alpha}',\beta,\bar{\beta}'}\frac{e_n e_{n'}\eta_n\eta_{n'}}{8\pi^2}\beta_{nn'}^{-1}\ln\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right], \tag{D.5}$$

where a bar means incoming particles are taken to be outgoing and vice versa (or equivalently, $\eta_{\bar{\alpha}'} = -\eta_{\alpha'}$). From equation (D.5), it is clear that incoming particles are found within the set $\{\alpha, \beta'\}$ while the outgoing particles are part of $\{\alpha', \beta\}$. Let us rename those sets σ and σ' respectively. ΔA now takes the form

$$\Delta A_{\beta\beta',\alpha\alpha'} = -\frac{1}{2}\sum_{n,n'\in\sigma,\sigma'}\frac{e_n e_{n'}\eta_n\eta_{n'}}{8\pi^2}\beta_{nn'}^{-1}\ln\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right] = \frac{1}{2}A_{\sigma\sigma'} \geq 0, \tag{D.6}$$

as was proven in [42]. This shows that $\Delta A_{\beta\beta',\alpha\alpha'} \geq 0$. The same proof goes through for $\Delta B_{\beta\beta',\alpha\alpha'}$.

Appendix E

The out-Density Matrix of Wavepacket Scattering

In this part of the appendix we flesh out the argument in section 5.3, namely that after tracing out soft radiation, the only contribution to the out-density matrix is coming from the identity term in the S-matrix. We will focus on the case of QED.

E.0.1 Contributions to the out-Density Matrix

First, let us decompose the IR regulated S-matrix into its trivial part and the $\mathcal{M}$-matrix element. For simplicity we ignore partially disconnected terms, where only a subset of particles interact. Then,

$$S^{\Lambda}_{\alpha\beta} = \delta(\alpha - \beta) - 2\pi i \mathcal{M}^{\Lambda}_{\alpha\beta}\delta^{(4)}(p^{\mu}_{\alpha} - p^{\mu}_{\beta}), \tag{E.1}$$

where the first term is the trivial LSZ constribution to forward scattering. This trivial part does not involve any divergent loops and therefore exhibits no Λ-dependence. However, the factorization of the S-matrix into a cutoff dependent term times some power of λ/Λ remains valid since all exponents of the form $A_{\alpha,\beta}$ vanish identically for forward scattering. This decomposition of the S-matrix gives rise to three different terms for the outgoing density matrix, containing different powers of $\mathcal{M}$.

"No scattering"-term

The case where both S-matrices contribute the delta function term results – unsurprisingly – in the well-defined outgoing density matrix

$$\rho^{(I)}_{\beta\beta'} = \int d\alpha d\alpha' f(\alpha) f(\alpha')^{*} \delta(\alpha - \beta)\delta(\alpha' - \beta')\delta_{\alpha\alpha'} = f(\beta) f^{*}(\beta'). \tag{E.2}$$

Contribution from forward scattering

We would now expect to find an additional contribution to the density matrix reflecting the non-trivial scattering processes, coming from the cross-terms

$$-2\pi i \left(\delta(\alpha-\beta)\mathcal{M}^{\Lambda}_{\alpha'\beta}\delta^{(4)}(p^{\mu}_{\alpha'}-p^{\mu}_{\beta}) - \delta(\alpha'-\beta)\mathcal{M}^{\dagger\Lambda}_{\alpha\beta}\delta^{(4)}(p^{\mu}_{\alpha}-p^{\mu}_{\beta})\right). \quad \text{(E.3)}$$

For simplicity, let us focus solely on the case in which S^* contributes the delta function and S contributes the connected part

$$\rho^{(II)}_{\beta\beta'} = -2\pi i f^*(\beta')\int d\alpha f(\alpha)\mathcal{M}^{\Lambda}_{\beta\alpha}\delta^{(4)}(p^{\mu}_{\alpha}-p^{\mu}_{\beta})\lambda^{\Delta A_{\alpha,\beta}}\mathcal{F}(E,E_T,\Lambda)_{\beta,\alpha}+\dots, \quad \text{(E.4)}$$

where the ellipsis denotes the contribution coming from the omitted term of (E.3). The exponent of λ only vanishes if the currents in α and β agree. We will show in appendix E.0.2 that we can take the limit $\lambda \to 0$ before doing the integrals. Taking this limit, $\lambda^{\Delta A_{\alpha,\beta}}$ gets replaced by

$$\delta_{\alpha\beta} = \begin{cases} 1, & \text{if charged particles in } \alpha \text{ and } \beta \text{ have the same velocities} \\ 0, & \text{otherwise,} \end{cases} \quad \text{(E.5)}$$

which is zero almost everywhere. If the integrand was regular, we could conclude that the integrand is a zero measure subset and integrates to zero and thus

$$\rho^{(II)}_{\beta\beta'} = 0. \quad \text{(E.6)}$$

However, the integrand is not well-behaved. Singular behavior can come from the delta function or the matrix element, so let's consider the two possibilities.

The singular nature of the Dirac delta does not affect our conclusion: for n incoming particles, the measure $d\alpha$ runs over $3n$ momentum variables while the delta function constrains 4 of them, leaving us with $3n-4$ independent ones. If we managed to find a configuration for which $\Delta A_{\beta\alpha}=0$, any infinitesimal variation of the momenta in α along a direction that conserves energy and momentum would modify the eigenvalue of the current operator $\hat{j}_v(\alpha)-\hat{j}_v(\beta)$ and make $\Delta A_{\beta\alpha}$ non-zero. Therefore, the integrand would still be a zero-measure subset for the remaining integrals.

What could still happen is that $\mathcal{M}^{\Lambda}_{\beta\alpha}$ is so singular that it gives a contribution. For this to happen it would need to have contributions in the form of Dirac delta functions. However, also this does not happen, for example for Compton scattering which scatters into a continuum of states. Additional IR divergences also do not appear as guaranteed by the Kinoshita-Lee-Nauenberg theorem. We will not give a general proof since for our purposes it is problematic enough to know that no scattering is observed for some physical process.

The scattering term

It is evident that a similar argument goes through for the $\mathcal{M}^2$ term. One finds

$$\rho^{(III)}_{\beta\beta'} = -4\pi^2 \int d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathcal{M}^{\Lambda}_{\beta\alpha} \mathcal{M}^{\Lambda *}_{\alpha'\beta'} \lambda^{\Delta A_{\alpha\alpha',\beta\beta'}} \tag{E.7}$$

$$\times \mathcal{F}(E, E_T, \Lambda)_{\beta\beta',\alpha\alpha'} \delta^{(4)}(p^{\mu}_{\alpha} - p^{\mu}_{\beta}) \delta^{(4)}(p^{\mu}_{\alpha'} - p^{\mu}_{\beta'}). \tag{E.8}$$

The analysis boils down the the question whether the term

$$\int d\alpha d\alpha' \lambda^{\Delta A_{\alpha\alpha',\beta\beta'}} \delta^{(4)}(p^{\mu}_{\alpha} - p^{\mu}_{\beta}) \delta^{(4)}(p^{\mu}_{\alpha'} - p^{\mu}_{\beta'}). \tag{E.9}$$

vanishes. As soon as there is at least one particle with charge, we need to obey the condition that the charged particles in α and β' agree with those in β and α' for the exponent of λ to vanish. Infinitesimal variations of α and α' that preserve the eigenvalue of the current operator $\hat{j}_v(\alpha) - \hat{j}_v(\alpha')$ form a zero-measure subset of the $6n - 8$ directions that preserve momentum and energy, forcing us to conclude that the integration runs over a zero measure subset and the only contribution to the reduced density matrix comes from the trivial part of the scattering process. This means that

$$\rho^{out,red.}_{\beta\beta'} = f(\beta) f^*(\beta') = \rho^{in}_{\beta\beta'}, \tag{E.10}$$

or in other words it predicts that a measurement will not detect scattering for wavepackets. This is clearly in contradiction with reality and suggests that the standard formulation of QED and perturbative quantum gravity which relies on the existence of wavepackets is invalid.

E.0.2 Taking the Cutoff $\lambda \to 0$ vs. Integration

One might be concerned that the limit $\lambda \to 0$ and the integrals do not commute. In this part of the appendix, we will check the claim made in the preceding subsection, i.e. we will show that one can explicitly check that the integration and taking the IR regulator λ to zero commute. We assume in the following that we talk about QED with electrons and muons in the non-relativistic limit, which again is good enough as it is sufficient to show that we can find a limit in which no sign of scattering exists in the outgoing hard state. The wave packets are chosen to factorize for every particle and to be Gaussians in velocity centered around $v = 0$,

$$f(v) = \left(\frac{2}{\pi\kappa}\right)^{3/4} \exp\left(-\frac{v^2}{\kappa}\right). \tag{E.11}$$

In order to stay in the non-relativistic limit, κ must be sufficiently small. They are normalized such that

$$\int d^3v |f(v)|^2 = 1. \tag{E.12}$$

In the exponent of λ we set $\alpha' = \beta'$ for simplicity, i.e. we consider the case of forward scattering. In the non-relativistic limit, we can expand the exponent of λ into

$$\Delta A_{\alpha\beta} = \frac{e^2}{24\pi^2} \sum_{n,m\in\alpha,\beta} (v_\alpha - v_\beta)^2. \tag{E.13}$$

Thus, $\lambda^{\Delta A}$ has the form

$$\lambda^{\Delta A} \propto \exp\left(-\frac{1}{2}\gamma \sum_{n,m\in\alpha,\beta} (v_\alpha - v_\beta)^2\right), \tag{E.14}$$

where taking the cutoff λ to zero corresponds to $\gamma \propto -\log(\lambda) \to \infty$. The state α consists of a muon with well defined momentum and one electron with momentum mv, where v is centered around 0. The state β consists of the same muon (we assume it was not really deflected) and one electron with momentum mv'. To obtain the contribution to forward scattering, we have to perform the integral

$$\propto \int d^3v \left(\frac{2}{\pi\kappa}\right)^{3/4} \exp\left(-\frac{v^2}{\kappa}\right) \exp\left(-\gamma(v-v')^2\right) \cdot (\text{other terms}). \quad \text{(E.15)}$$

Here, we assumed that the other terms which include the matrix element in the regime of interest is finite and approximately independent of v. The integral yields

$$\left(\frac{2\pi\kappa}{(1+\gamma\kappa)^2}\right)^{3/4} \exp\left(-\frac{\gamma v'^2}{1+\gamma\kappa}\right). \quad \text{(E.16)}$$

Taking the limit $\gamma \to \infty$, it is clear that this expression vanishes. If we want to consider an outgoing wave packet we have to integrate this over $f(v' - v_{out})$. The result is proportional to

$$\left(\frac{2\pi\kappa}{(1+2\gamma\kappa)^2}\right)^{3/4} \exp\left(-\frac{\gamma v_{out}^2}{1+2\gamma\kappa}\right) \quad \text{(E.17)}$$

and still vanishes if we remove the cutoff, $\gamma \to \infty$.