Second Order Relative Entropy in Holographic Theories

by

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Abstract

Recently, there has been growing recognition that the tools from quantum information theory might be well-suited to studying quantum gravity in the context of the gauge/gravity correspondence. It is exploring this connection that is the main motivation for the work in this thesis.

In particular, we focus on holographic field theories which possess classical spacetime duals. The aim is that certain conditions on the classical duals will narrow down the types of field theories that can be holographic. This will give a better understanding of the limitations and robustness of the gauge/gravity correspondence.

We do so by computing the canonical energy for general perturbations around anti-de Sitter spacetime, which is dual to quantum Fisher information in the field theory. We go on to prove the positivity of canonical energy and discuss the addition of matter fields. We further show that our result can be interpreted as an interaction between scalar fields living in an auxiliary de Sitter spacetime. We concluded with a summary of progress and future challenges for this program.

Preface

The results of Chapters 3 and 4 have appeared in the peer reviewed Journal of High Energy Physics, under the title “Entanglement entropy from one-point functions in holographic states” [1] and are based on research performed by myself in collaboration with Jaehoon H. Lee, Charles Riddeau and Mark Van Raamsdonk. All gravitational calculations were done by myself, and this thesis only includes such parts.

Figures 2.3 and 4.3 were created by myself and originally used in [1]. Section 3.1.4 is an edited version of an appendix in [1] which was also written by myself.

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Acknowledgements

Firstly, I would like to thank my supervisor Mark Van Raamsdonk for his constant guidance and inspirational passion for physics.

I would also like to thank Philippe Sabella Garnier, Charles Rabideau, and Jaehoon Lee for helpful conversations throughout this project. Without their help, much of this work would not of come to fruition.

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Dedication

To my Mother,

Chapter 1

Introduction

Modern fundamental physics has largely been guided by the reductionist philosophy; that is, to break matter apart and see 'what it is made of' on ever decreasing length scales. In quantum mechanics, the Heisenberg uncertainty principle demands that we need correspondingly higher energies in order to look at smaller distances. This has led to the creation of massive particle accelerators like the Large Hadron Collider, which is currently reaching energies of 13 TeV to probe lengths around 10^{-19} m, nine orders of magnitude past the atomic scale.

One of the remarkable discoveries of the reductionist program is that, sometimes, physics changes radically past a certain critical energy scale. For example, Fermi's theory of beta decay describes the weak interaction well at low energies, but it breaks down at energies near 290 GeV. In this case, the theory was replaced by electroweak theory which accommodates this drastic change by producing new particles, the W and Z bosons.

In the case of gravity, a critical energy scale emerges because concentrating very high energies in a small region results in the creation of microscopic black holes. A crude estimate, due to Planck, puts the energies of this scale at 10^{19} GeV, or equivalently, distances of 10^{-35} m. After this point increasing the energies only makes larger black holes and the reductionist program completely fails. It appears that quantum gravity is more than just a theory of "stuff" smaller than the Planck length. Indeed, such a length is not even well-defined since we cannot measure it. Although one might be tempted to argue that energies so incredibly large are irrelevant to physics, there are many important questions which can only be answered by understanding physics at the Planck scale; such questions include the nature of the Big Bang, black hole singularities, and the ultimate fate of the universe.

Despite the immense gap between current experiments and the Planck scale, there have been some remarkable hints as to what a theory of quantum gravity must look like. Perhaps the greatest clue was discovered by Bekenstein and Hawking around 1972, who were formulating an analogy between the laws of thermodynamics and the laws of black hole dynamics [2, 3, 4].

Their famous result says that the entropy of a black hole is proportional to its area through

$$S_{\mathrm{BH}} = \frac{A}{4\,G_N} \tag{1.1}$$

in natural units where $c = \hbar = 1$.

There are many remarkable things about the Bekenstein-Hawking formula. The most immediate observation is that black holes do indeed have a well-defined entropy (a measure of classical disorder) and hence obey certain thermodynamic relations. This suggests that gravity may emerge from a hitherto unknown microscopic theory, just as classical thermodynamic arises from statistical mechanics. Secondly, the entropy is proportional to the surface area of the system and not the volume (as is the case with most thermodynamic systems). This property has been displayed in certain ground states of condensed matter systems and suggests that the microscopic degrees of freedom are not defined in the full spacetime, but rather only a subspace of it. In a sense, spacetime contains redundant features which do not contribute to the true physics of quantum gravity.

The Bekenstein-Hawking formula also implies that the maximum amount of information that can be stored in any region of spacetime is proportional to the (surface) area of that region. Due to the universality of black holes, whatever the true microstates of quantum gravity are, they appear to live on the boundary of a spacetime region rather than in the region itself.

A similar clue comes from classical general relativity (GR) itself. It has long been known that, in GR, energy does not posses a proper local definition but rather it can only be defined for an observer at asymptotic infinity [5]. If energy is to be identified with the Hamiltonian (a natural starting point for quantum theory), then it is suggestive that the true gravitational Hamiltonian is only defined at infinity.

This idea is manifest in the *holographic principle*, which supposes that the complete description of the dynamics in some spacetime volume is completely encoded in the boundary of the region. Much like a 3d movie in theatres today, the two-dimensional screen contains enough information so that the polarized 3d glasses reconstruct the full 3d images for the viewer.

The gauge/gravity duality is a precise realization of a holographic theory. It conjectures that a complete theory of quantum gravity in $(d+1)$-dimensional spacetime is equivalent to a d-dimensional quantum conformal field theory (CFT) on the asymptotic boundary of that spacetime. This has been explicitly demonstrated for certain theories; the most celebrated of which is the so-called AdS/CFT duality between type-IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N}=4$ super-Yang-Mills theory in Minkowski spacetime **(author?)** [6]. Remarkably, this duality concretely relates a strongly coupled quantum theory, where calculations may be intractable, to a low energy classical gravitational theory where results may be possible. Exploiting this strong/weak coupling duality has yielded tremendous applications to far-reaching areas of physics; from superconductors to fluid dynamics [7, 8].

Although the original conjecture was specific to string theory, there is substantial evidence that the gauge/gravity duality holds for a much larger set of theories [9, 10, 11, 12, 13]. In fact, it may be that any consistent quantum gravity theory in asymptotically AdS space can be defined through a CFT on its boundary. Understanding precisely which CFTs can produces a dual spacetime is a very important question for exploring quantum gravity.

In recent years, it has been recognized that the language of quantum information theory might be well-suited for addressing questions about gauge/gravity. In particular, quantum entanglement in a quantum field theory (QFT), a key quantity in information theory, has been shown to be fundamentally linked to the geometry of the dual spacetime [14, 15, 16, 17]. In fact, without entanglement in the quantum theory, it has been argued there could be no dual spacetime [18, 19]. In a sense, entanglement is the glue that holds spacetime together.

The deep connection between entanglement and geometry was strengthen by the realization that certain conditions of entanglement in a CFT could be translated to restrictions on gravitational solutions in the bulk spacetime [20]. Remarkably, even Einstein's equations emerge (to first order) from the laws of entanglement [21, 22]. Further known properties of entanglement can be translated into constraints on gravitational physics, including the averaged null energy condition [20]. The entanglement structure of a CFT gives provides insight regarding which field theories can produce consistent gravitational spacetimes. It is understanding this restriction that is the main motivation for the work in this thesis.

The primary focus of this thesis is the holographic computation of the second order change in entanglement entropy of a CFT state when perturbed away from the vacuum. This change is known as quantum Fisher information within the literature and serves as a metric between states [23]. In calculating the bulk dual to this quantity, we establish that (to second order) the entanglement entropy of such states can be represented as a smearing functional over products of one-point functions, i.e. expectation values. We further explore connections between this result and a recent proposal for emergent dynamics in de Sitter spacetime [24].

In the remainder of this Chapter we review various aspects of holographic entanglement entropy. We start with a brief review of quantum entanglement and its useful properties. We then proceed to discuss the AdS/CFT dictionary and holographic entanglement entropy from Ryu-Takayangi surfaces. We conclude with an overview of subsequent Chapters and a summary of the main calculations of the thesis.

1.1 Entanglement

"I would not call that one, but rather the characteristic trait of quantum mechanics; the one that enforces its entire departure from classical lines of thought."

-Erwin Schrödinger on entanglement

Quantum entanglement is perhaps the most curious aspect of quantum mechanics. It allows for two particles to be perfectly correlated without any physical communication between them. This "spooky" action at a distance was one of the most conceptually troubling aspects of early quantum theory. Einstein never truly accepted this idea and, along with Podolsky and Rosen (EPR), he claimed that quantum mechanics could not be considered a "complete" description of reality **(author?)** [25]. This paradox loomed over theoretical physics for nearly 30 years until John Bell showed that any "complete theory", in the sense of EPR, must inherently be nonlocal **(author?)** [26]. Since locality is a cornerstone of fundamental physics, it seems that "spooky" action at a distance is here to stay. [1]

However bizarre entanglement is, it is an immensely important part of modern physics. It is paramount to modern applications in quantum cryptography, quantum teleportation, and very generally in quantum computing [27]. Recently, it has also found use in condensed matter physics where it may be used to characterize special quantum phase transitions where conventional order parameters fail. Such examples include superconducting phases and topological order **(author?)** [28, 29, 30].

Despite the widespread applications of entanglement entropy in modern physics, the main motivation for this thesis is of a fundamental nature. It has recently been established that, in the context of holography, entanglement entropy in a CFT is dual to a surface in the bulk with minimal area [14, 15, 31]. In this way, the geometric features of spacetime are intimately linked to the entanglement structure of a CFT. Furthermore, it has been argued that

[1] Of course this is unnecessary in certain interpretations of quantum mechanics, such as if one thinks of an entangled pair as "one particle" in two difference places.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

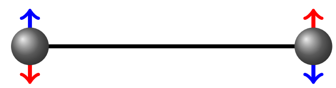


Figure 1.1: A traditional picture of two entanglement particles. If one particle is measured to have the red spin, then the other will also have the red spin.

without entanglement in a CFT there would be no dual spacetime [19, 18]. In this thesis we aim to better understand this connection.

In the following sections we define the entanglement entropy and list some of its properties. We then discuss the area law and give a simple example before moving on to the AdS/CFT correspondence.

1.1.1 Basics of entanglement entropy

A composite quantum state $|\Psi\rangle \in \mathcal{H}$, composed of substates $|\psi_i\rangle \in \mathcal{H}_i$, is said to be entangled if it cannot be factorized as a product of the substates,

$$|\Psi_{\text{entangled}}\rangle \neq \prod_i |\psi_i\rangle \,. \tag{1.2}$$

By the superposition principle, an arbitrary state is a linear combination of such product states $|\Psi_{\text{entangled}}\rangle = \sum_j c_j \prod_i |\psi_i\rangle$.

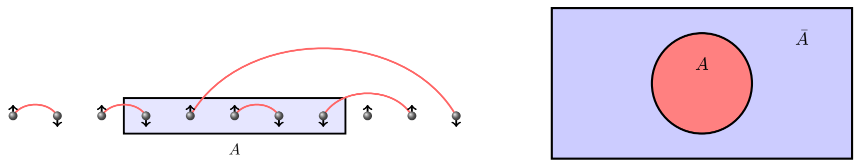
A more elegant way to characterize entanglement is through the density operator ρ, which is simply the outer product of the wavefunction,

$$\rho = |\Psi\rangle\langle\Psi|. \tag{1.3}$$

The density matrix is called *pure* if one of its eigenvalues is unity, otherwise it is called a *mixed* state. A pure state has no entanglement and represents a single quantum state, whereas a mixed state represents a statistical ensemble of states.

As a statistical ensemble, one can define an associated entropy

$$S = -\mathrm{tr}(\rho \log \rho) \tag{1.4}$$



(a) One-dimensional spin chain. (b) Quantum field theory in the continuum.

Figure 1.2: Examples of a bi-partitioned system. The subregion A is shown for each of the two examples: a) a spin chain where red lines represent entangled pairs. b) a continuous field theory.

called the von Neumann entropy. Since the density matrix has non-negative eigenvalues (due to requiring positive probabilities), the entropy is also non-negative, and vanishes if and only if ρ is a pure state. The von Neumann entropy quantifies the amount of uncertainty about which state the system is in. In the limit $h \to 0$, the von Neumann entropy reduces to the usual thermal entropy.

Now consider a quantum system in a pure state ρ. If we divide the system into two parts, A and B as in figure 1.2 then the total Hilbert space is simply the direct product between the two subspaces $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_B$. The state of the system A without any reference of B, is obtained by tracing out all degrees of freedom in B. This defines the reduced density matrix for A,

$$\rho_A = \sum_i \langle i_B|\rho|i_B\rangle = \mathrm{tr}_B(\rho). \tag{1.5}$$

An observer confined to the region A will only be aware of the effective density matrix ρ_A. This is a natural thing to consider in the context of black hole physics where the interior of the black hole is inaccessible to an outside observer.

The entanglement entropy of A is defined as the von Neumann entropy of the subsystem,

$$S_A = -\mathrm{tr}_A(\rho_A \log \rho_A). \tag{1.6}$$

This gives a direct way to characterize the entanglement between two regions. In the case of black holes, the entanglement entropy measures the amount of information hidden inside the black hole.

As per figure 1.2, the entanglement entropy depends on both the system and the imaginary entangling surface.

If the system was originally in a mixed state, the entanglement entropy would no longer measure only entanglement. Instead it would count both classical and quantum correlations. This is to be expected because in the high-temperature limit it will reproduce the classical thermal entropy which has no entanglement [32, 33, 34]. For this reason we will only consider globally pure states so that we isolate the quantum effects.

We now provide a simple example before discussing entanglement entropy in a more general QFT.

Example

A typical example of an entangled state is the generalized Bell pair

$$|\Psi\rangle = \sqrt{\alpha}|\uparrow\rangle_A|\downarrow\rangle_B + \sqrt{1-\alpha}|\downarrow\rangle_A|\uparrow\rangle_B \tag{1.7}$$

where $\alpha \in [0,1]$ enforces the normalization $\langle\Psi|\Psi\rangle = 1$. The corresponding reduced density matrix is

$$\rho_A = \mathrm{tr}_B(\rho) = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} \tag{1.8}$$

and the entanglement entropy is

$$S_A = -\alpha \log\alpha - (1-\alpha)\log(1-\alpha)\,. \tag{1.9}$$

As shown in figure 1.3 the entropy vanishes at $\alpha = 0{,}1$ and achieves a maximum at $\alpha = \frac{1}{2}$.

Area law

In a field theory the entanglement entropy is generally divergent. This is due to the infinite number of degrees of freedom in a field. Fortunately, understanding the leading order divergences in the entanglement entropy can give profound insight into the behaviour of the underlying field theory.

Consider a d-dimensional QFT on flat spacetime. For simplicity, we will work on a constant time slice $t = t_0$ which defines the spacelike $(d-1)$-dimensional submanifold $\mathcal{N}$. The entanglement entropy of a subregion $A \subset \mathcal{N}$ can be computed through (1.6). To regularize the entropy, we introduce an ultraviolet (UV) cutoff parameter ϵ, i.e. the lattice spacing. The entanglement entropy of a region A is then

$$S_A = \gamma\frac{\mathrm{Area}\,(\partial A)}{\epsilon^{d-2}} + (\text{subleading terms}) \tag{1.10}$$

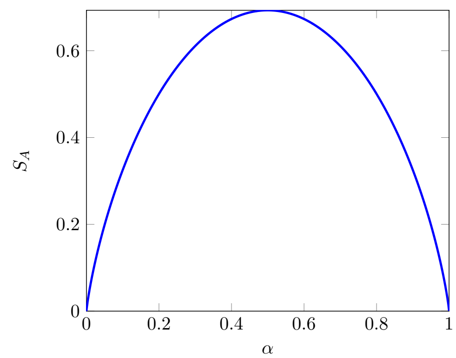


Figure 1.3: The entanglement entropy for a generalized Bell pair described by equation (1.7).

where γ is a constant that depends on the system in question [35, 36]. This is known as the area law. The fact that the entanglement is dominated by the contribution from the boundary ∂A comes from the intuitive picture that the entanglement between A and $\bar{A}$ is strongest *near* the boundary. This must be the case because the interactions are governed by local Hamiltonian dynamics, and hence entanglement between A and $\bar{A}$ is predominately generated by local effects near ∂A.

There are some systems which violate the area law. For example, any metal or material with a Fermi surface **(author?)** [28]. Another common example is for a 2d CFT, where the entanglement entropy for region of length L in an infinitely long spin chain with lattice spacing ϵ is

$$S_A = \frac{c}{3} \log \frac{L}{\epsilon}, \tag{1.11}$$

where c is the central charge of the CFT [37, 33]. Whereas this violates the naive area law, it acquires a simple geometric meaning in holography.

In the next section we cover the essentials of AdS/CFT which will be used later in the thesis.

1.2 AdS/CFT correspondence

The AdS/CFT (anti-de Sitter/conformal field theory) conjectures that a quantum theory of gravity in $(d+1)$-dimensional spacetime is equivalent to a quantum field theory with conformal symmetry in d dimensions. Maldacena originally formulated the duality between superstring theory on $\mathrm{AdS}_5 \times S^5$

and strongly coupled $\mathcal{N}$=4 $SU(N)$ super-Yang-Mills theory in Minkowski spacetime **(author?)** [6]. Since then, a substantial amount of evidence has demonstrated that it is in fact more general [9, 10, 11, 12, 13]. In practice, exploiting this duality has led to novel insights about strongly coupled field theories from gravity, as well as new hints towards a complete theory of quantum gravity.

It is all but impossible to describe all the ramifications of the AdS/CFT correspondence as, at present, the original paper by Maldacena has over 11,000 citations. With applications in condensed matter [33, 38, 39, 8, 40], numerical relativity, and quantum information [41, 42, 17, 43, 44, 45, 16]; the AdS/CFT has proven to be one of the most significant theoretical discoveries in the last 20 years.

In the following section, we will discuss AdS spacetime and the holographic dictionary between two dual theories.

Anti-de Sitter spacetime

Anti-de Sitter (AdS) is a maximally symmetric spacetime with constant negative curvature. As such, it is also a solution to Einstein's equations with a negative cosmological constant. It is the Lorentzian version of hyperbolic space, just as Minkowski space is to Euclidean space.

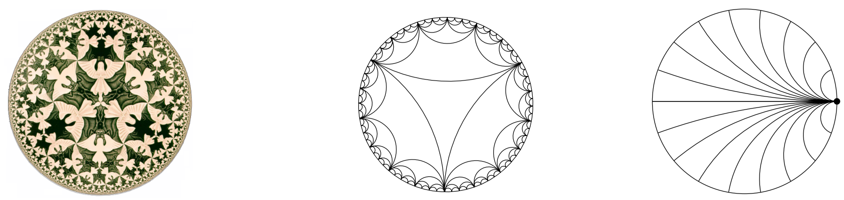
The asymptotic symmetry group of AdS$_{d+1}$ is $SO(2, d)$, which happens to be isomorphic to the conformal group in d dimensions [46]. This is one of the many reasons which suggest that AdS may be related to a lower-dimensional theory with conformal symmetry.

AdS in three dimensions can be visualized as a cylinder as shown in figure 1.6. A constant time slice of three-dimensional AdS is referred to as the hyperbolic disk, which was made famous by the artwork of Escher [47]. Figure 1.4 compares Escher's artwork with spatial geodesics on the disk.

One coordinate patch for AdS$_{d+1}$ is

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_\mu dx^\mu) \tag{1.12}$$

with the index $\mu = 0, 1, ..., d$. The dual CFT is said to live on the boundary of AdS, that is the limiting surface as $z \to 0$ (the boundary of the cylinder). This is not a literal statement, as the two theories are distinct (and yet are supposedly different manifestations of the same theory). The radial coordinate z can be interpreted as the length scale of the CFT in the sense of renormalization group (RG) flows (or energy scale). Small z corresponds to high energies in the field theory. Likewise, physics deep in the bulk (large



(a) Escher's disk. (b) Geodesic tesselations. (c) Geodesics starting at one point.

Figure 1.4: Comparison between Escher's artwork and proper geodesics in the hyperbolic disk. To understand a spatial slice of AdS_3, imagine that the bats/angels in Escher's disk are have the same area on. It is merely a property of the disk that more bats fit near the boundary. The geodesics are perpendicular to the boundary in b) and c).

z) is related to the low energy infrared (IR) properties of the CFT. To deal with the divergence in the metric at $z = 0$, one usually imposes a cutoff at $z = \epsilon$. This amounts to the CFT having a lattice spacing (or UV cutoff) of ϵ.

Holographic dictionary

For two theories to be considered dual to each other, there must be a map between the physical spectra of each theory. Bulk quantities in AdS should be able to be written in terms of field theory observables and vice versa. In QFT, all properties of the theory are contained within the generating functional (partition function) Z. In this way, the AdS/CFT correspondence can be neatly phrased as the equivalence between the partition functions of both theories,

$$Z_{\text{AdS}} = Z_{\text{CFT}} \,. \tag{1.13}$$

For strongly coupled field theories, the gravity action can be approximated by low energy gravity (classical gravity), so that

$$Z_{\text{AdS}} \approx e^{S_0} \,, \tag{1.14}$$

where S_0 is the classical action for the gravitational theory.

For there to be an equivalence between observables, every classical field ϕ^α in the gravitational side must be associated to an operator $\mathcal{O}_\alpha$ in the CFT. More precisely, the boundary value of the field $\phi^\alpha(z=0) = \phi_0^\alpha$, couples to $\mathcal{O}_\alpha$ as a source term. That is, $Z_{\text{CFT}} = \langle e^{\int \phi_0^\alpha \mathcal{O}_\alpha} \rangle$. In this sense, the operators $\mathcal{O}_\alpha$ source the bulk fields ϕ^α at the boundary.

We can then compute the connected n-point correlation functions of an operator $\mathcal{O}_\alpha$ from

$$\begin{aligned}
\langle \mathcal{O}_{\alpha_1}(x_1) ... \mathcal{O}_{\alpha_n}(x_n) \rangle &= \frac{\delta}{\delta \phi_0^{\alpha_1}} \cdots \frac{\delta}{\delta \phi_0^{\alpha_n}} \log Z_{\text{CFT}} &(1.15)\\
&= \frac{\delta}{\delta \phi_0^{\alpha_1}} \cdots \frac{\delta}{\delta \phi_0^{\alpha_n}} S_0 . &(1.16)
\end{aligned}$$

It is convenient that, in order to compute quantum correlation functions, one only needs to take derivatives of a classical action. Although this formal equation gives a powerful computation tool, it does not address some of the conceptual aspects of a bulk/boundary correspondence. In the next section we will explore a more recent entry in the holographic dictionary which has a more direct link to geometry.

1.3 Holographic entanglement entropy

In this section we introduce a generalization of the Bekenstein-Hawking formula due to Ryu and Takayangi **(author?)** [14]. Motivated by black hole entropy, they proposed that the area of a certain minimal area surface (of codimension 2) is exactly the entanglement entropy of region in the dual CFT. In taking the appropriate limits, the Bekenstein-Hawking entropy may be interpreted as such a surface [11]. This provides a method to compute entanglement entropy in a CFT by calculating the area of an extremal surface in AdS spacetime. We will only present a heuristic argument, and a more formal derivation can be found in **(author?)** [15].

We would like a way to holographically calculate the entanglement entropy on a time slice of a CFT between a subregion A and its compliment $\bar{A}$, who share a boundary ∂A. Firstly, we imagine extending the division between A and $\bar{A}$ to regions in the bulk spacetime with an imaginary surface γ_A. The boundary $\partial\gamma_A$ exactly matches ∂A on the boundary as shown in figure 1.5. Inspired by the Bekenstein-Hawking formula, we hope that the area of such a surface will reproduce the entanglement entropy of A. Of course there are infinitely many choices for γ_A with very different areas, and so we need a method to choose a single surface. Ryu and Takayanagi proposed

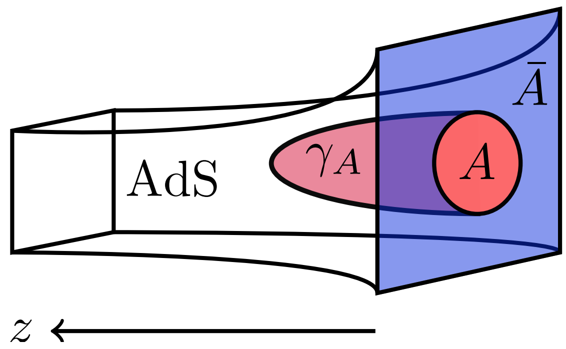


Figure 1.5: The minimal Ryu-Takayanagi surface stretching into the bulk spacetime. The surface γ_A associated to the region A has minimal area with respect to any other surfaces anchored to A.

that the proper surface, $\tilde{A}$ should be the surface possessing minimal area as will be argued later. The surface with the minimal area which is anchored to ∂A is called the Ryu-Takayangi surface and is denoted by $\tilde{A}$.

The entanglement entropy S_A of the CFT is then hypothesized to be given by

$$S_A = \frac{\text{Area}(\tilde{A})}{4\,G}\,. \tag{1.17}$$

In AdS spacetime, the leading order contribution to the area in (1.17) comes from the boundary. In general, (1.17) diverges as

$$S_A = \frac{1}{4G_N}\frac{\text{Area}(\partial A)}{\epsilon^{d-2}} + \cdots \tag{1.18}$$

which is consistent with the area law in (1.10). In this way, the Ryu-Takayangi claims to describe the entanglement entropy of a region to all orders.

The Ryu-Takayangi formula is only applicable to static spacetimes or constant time slices. In the dynamic case, there is a modified prescription by Hubeny-Rangamany-Takayanagi [31]. In this thesis, we will only be concerned with static spacetimes so the simpler Ryu-Takayangi method will suffice. There have also been extensions to include quantum corrections [48], as well as a formal proof of (1.17) [[49]].

The elegance of calculating entanglement entropies from minimal area surfaces is best illustrated with an example. In the following section, we

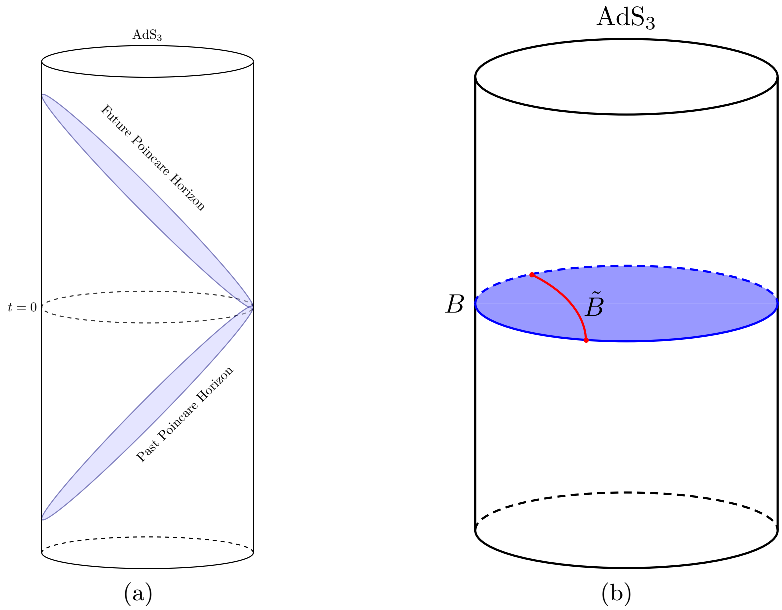


Figure 1.6: (a) The Poincare patch of the AdS_3 cylinder. The time t is represented by the height of the cylinder and z the radius. (b) AdS_3 spacetime as a cylinder. The red line represents the Ryu-Takayangi surface γ_A which has minimal area and is anchored to the boundary region A.

compute the entanglement entropy of a region in a 2d CFT which is dual to pure AdS spacetime.

1.3.1 Example in $\mathrm{AdS}_3/\mathrm{CFT}_2$

As a simple example, let us consider calculating the entanglement entropy for an interval $x \in [-R, R]$ of a two-dimensional CFT using the Ryu-Takayanagi formula. The dual theory is pure AdS_3 spacetime, which makes it is easy to calculate the area of extremal surfaces. It is convenient to work in the Poincare patch of AdS_3 as illustrated in figure 1.6. The line element in these coordinates is

$$ds^2 = \frac{1}{z^2}(dz^2 + dt^2 - dx^2). \tag{1.19}$$

As required for the Ryu-Takayangi prescription, we take a constant time slice $t = 0$. The geodesic equation for $z = z(x)$ is then

$$z''\, z + z'^2 + 1 = 0 \tag{1.20}$$

which has solutions

$$z^2 = R^2 - x^2\,. \tag{1.21}$$

From this we see that the geodesics in AdS_3 are circles of radius R, as shown previously in figure 1.4. The area (or in this case length) of this minimal geodesic surface is given by

$$A = \int dx\, \sqrt{-g} = 2 \int_\epsilon^R dz\, \frac{\sqrt{1 + z'}}{z} = 2 \ln\left(\frac{2\,R}{\epsilon}\right) + \mathcal{O}(1) \tag{1.22}$$

where ϵ is a cutoff near the $z = 0$ boundary. The entropy then precisely matches the result for a 2d CFT in (1.11) with $L = 2R$ and a central charge of

$$c = \frac{3}{2G_N}\,. \tag{1.23}$$

The elegance of this approach is remarkable. By merely minimizing the surface area for a anchored surface in AdS spacetime, we have computed the entanglement entropy for a region in a CFT.

1.4 Overview

As demonstrated by the simplicity of the Ryu-Takayangi formula, the geometric nature of entanglement is profoundly powerful. The bulk spacetime encodes information about the nonlocal entanglement properties of a CFT state. We can then hope to ask which CFT states have entanglement structure that is consistent with having a holographic dual. It is clear that in general, the space of all entanglement entropies for spherical regions, $\mathcal{S} = \big\{S(R, x^\mu) \forall x^\mu \in \mathbb{R}^d, R > 0\big\}$, is much larger than the space of all asymptotically AdS metrics. It is then plausible that there exists a subset of $\mathcal{S}$ which represents the entanglement entropies which can be possessed by a holographic CFT.

One method to characterizing these subsets is to study arbitrary perturbations to asymptotically AdS which are dual to state perturbations of the CFT vacuum. We know from the standard AdS/CFT correspondence that bulk fields will be sourced by expectation values of CFT operators $\langle \mathcal{O}_\alpha \rangle$. Propagating these into the bulk should determine the full fields and metric

$\phi_\mu(z,x)$ at least to some finite distance into the bulk. From this, the entanglement entropy of any region can be constructed from the Ryu-Takayanagi formula. Following this logic, we see that the entanglement of regions in a CFT can be determined merely by knowledge of the one-point functions $\langle \mathcal{O}_\alpha \rangle$. This provides a stringent test on whether a CFT possesses a classical gravitational dual.

In this thesis we present some explicit results for the entanglement entropies allowed for theories with a classical dual. For states near the vacuum, the first order result is known to hold universally for all CFT. In this case holography does not place a constraint on the allowed entanglement entropies. Continuing to second order is it less clear if there are any additional constraints coming from holography. To find an explicit condition, we make use of the recently recognized equivalence between quantum Fisher information and bulk canonical energy [50].

The remainder of this thesis is organized as follows. In Chapter 2, we introduce relative entropy, a natural extension of entanglement entropy, which has some useful properties for studying quantum gravity. We then discuss the gravitational dual to relative entropy, known as canonical energy. Following that, Chapter 3 provides the main calculation of the thesis, that is, the second order relative entropy for a CFT state perturbed from the vacuum. We go on in Chapter 4 to explore connections between this result and emergent dynamics in de Sitter space. We conclude in Chapter 5 with a discussion of open questions and future directions for research.

Chapter 2

Background

This chapter introduces the technical aspects of calculating the quantum Fisher information in a CFT. We first review the fundamental features of entanglement for a spherical region and its physical interpretations. We motivate the first law of entanglement and introduce quantum Fisher information. Furthermore, we discuss the gravitational dual to holographic states, and define the canonical energy. Finally, we review the holographic dictionary between Fisher information and canonical energy.

2.1 Holographic relative entropy

2.1.1 Relative entropy

This section is concerned with formally defining the entanglement, and relative, entropy for a holographic state.

Consider a d-dimensional CFT which possesses a state $|\Psi\rangle$. For any spherical region B, the reduced density matrix is obtained by tracing out the degrees of freedom associated to the compliment of B,

$$\rho_B = \mathrm{tr}_{\bar{B}} \left(|\Psi\rangle\langle\Psi| \right) .$$

The density matrix ρ must be positive semi-definite and Hermitian so that it has non-negative probabilities. Any such operator can be written as a exponential of another Hermitian operator through

$$\rho_B^{\mathrm{vac}} = \frac{1}{Z} e^{-H_B} \tag{2.1}$$

where H_B is the modular Hamiltonian and $Z = \mathrm{tr}(e^{-H_B})$ is the usual partition function.[2] The normalization constant Z could have been absorbed into H_B through an additive constant and ensures that $\mathrm{tr}(\rho) = 1$. Throughout the rest of this chapter we will set $Z = 1$ without loss of generality.

[2]The term modular Hamiltonian originates from axiomatic quantum field theory, while in condensed matter it is often called the entanglement Hamiltonian.

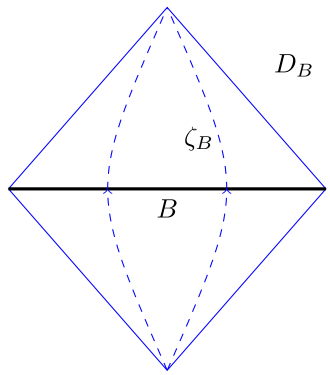


Figure 2.1: The domain of dependence D_B of the ball-shaped region B. The Killing vector ζ_B is the timelike flow through D_B.

The explicit form of the modular Hamiltonian is only known in special local examples. In the case that B is a spherical spatial region of radius R, the modular Hamiltonian is given by the well-known expression

$$H_B = 2\pi \int_B d^{d-1}x \, \frac{R^2 - x^2}{2\,R} T_{tt}(x) \tag{2.2}$$

where $\langle T_{tt}(x)\rangle$ is the CFT stress tensor.

The modular Hamiltonian generates a conformal Killing vector ζ, which acts within the causal diamond D_B (the causal past and future of B) as shown in figure 2.1. For a circular region, the Killing vector is explicitly

$$\zeta = \frac{\pi}{R}\left[\left(R^2 - (t-t_0)^2 + |x - x_0|^2\right)\partial_t - 2(t-t_0)(x^i - x_0^i)\partial_i\right] \tag{2.3}$$

where i runs over spatial indices. Using ζ, a covariant way to write the modular Hamiltonian is

$$H_B = \int_B \zeta^\mu \langle T_{\mu\nu}\rangle \boldsymbol{\epsilon}^\nu \tag{2.4}$$

where $\boldsymbol{\epsilon}^\nu$ is a differential form related to the Levi-Civita tensor through

$$\boldsymbol{\epsilon}_\nu = \frac{1}{(d-1)!}\epsilon_{\nu_1 \ldots \nu_{d-1}} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{d-1}}\,. \tag{2.5}$$

One can easily check that with the Killing vector in (2.3), the covariant expression in (2.4) reproduces (2.2).

For a state $|\Psi\rangle$ perturbed away from the vacuum state $|0\rangle$, we define the difference in entanglement entropy as

$$\Delta S_B = S(\rho_B) - S(\rho_B^{\mathrm{vac}})\,.$$

While the entanglement entropy is typically divergent, the entropy difference has the benefit of remaining finite. Similarly we write the difference in the expectation value of the modular Hamiltonian as

$$\Delta\langle H_B\rangle = \mathrm{tr}(\rho_B H) - \mathrm{tr}(\rho_B^{\mathrm{vac}} H_B)\,.$$

Another quantity of great importance is the relative entropy between two quantum states. Considering a state ρ and a reference state σ, the relative entropy is defined as

$$S(\rho||\sigma) = \mathrm{tr}(\rho\log\rho) - \mathrm{tr}(\rho\log\sigma).$$

Notice that this definition is not symmetric in ρ and σ; however, it has many useful properties. It can be shown that in general the relative entropy is non-negative;

$$S(\rho||\sigma) \geq 0, \tag{2.6}$$

where the equality holds if and only if the states are identical ($\rho = \sigma$). Furthermore, as the region on which ρ is defined increases in size, the relative entropy increases; that is to say, it is monotonically increasing with volume. The positivity and monotonicity properties make the relative entropy a particular useful quantity to study.

Relative entropy can be physically interpreted in the context of thermodynamics. For a thermal density matrix $\rho_{\mathrm{th}} = e^{-H/T}/Z$, the relative entropy between any state ρ_1 and the thermal one is the difference between the free energies

$$S(\rho_1||\rho_{\mathrm{th}}) = \frac{1}{T}\left(F(\rho_1) - F(\rho_{\mathrm{th}})\right), \tag{2.7}$$

where as usual $F = \langle E\rangle - TS$. The temperature in this expression is only that of the initial thermal state so ρ_1 can be arbitrary (it need not be thermal).

In the case where we know the reduced density matrix ρ explicitly, it is useful to write the modular Hamiltonian as

$$H_B = -\log\rho \tag{2.8}$$

which coincides with the definition from (2.1) up to a normalization constant, $Z = 1$. The relative entropy then reduces to

$$\begin{aligned}
S(\rho||\sigma) =& \operatorname{tr}(\rho \log \rho) - \operatorname{tr}(\rho H_B) \\
&+ \operatorname{tr}(\sigma \log \sigma) - \operatorname{tr}(\sigma \log \rho) \\
=& - S(\rho) - \langle H_B \rangle_{\rho_{B(\lambda)}} + \langle H_B \rangle_{\rho_{B(0)}} + S(\sigma) \\
=& \Delta \langle H_B \rangle - \Delta S.
\end{aligned}$$

The positivity of relative entropy then immediately implies the constraint

$$\Delta \langle H_B \rangle - \Delta S \geq 0. \tag{2.9}$$

For a sufficiently small perturbation of the vacuum state, this implies to first order

$$\delta^{(1)} S(\rho||\sigma) = 0, \tag{2.10}$$

which gives the so-called first law of entanglement

$$\delta^{(1)} S = \delta^{(1)} \langle H_B \rangle. \tag{2.11}$$

Adding the temperature back into this expression yields the usual thermodynamic equation $T\delta S = \delta E$, which applies to states near equilibrium.

The second order, $\delta^{(2)} S(\rho||\sigma)$ is not symmetric under exchanging ρ and σ.

If we consider a state $\rho = \sigma + \delta\rho_1 + \delta\rho_2$, where $\delta\rho_1$ and $\delta\rho_2$ are independent perturbations, may define an inner product

$$2\langle \delta\rho_1, \delta\rho_2 \rangle_\sigma = \delta^{(2)} S(\sigma + \delta\rho_1 + \delta\rho_2||\sigma) - \delta^{(2)} S(\sigma + \delta\rho_1||\sigma) - \delta^{(2)} S(\sigma + \delta\rho_2||\sigma)\,, \tag{2.12}$$

which is symmetric under ($\delta\rho_1$,$\delta\rho_2$). Notice that in the case that $\delta\rho_1 = \delta\rho_2 = \frac{1}{2}\delta\rho$, we recover

$$\langle \delta\rho, \delta\rho \rangle_\sigma = \delta^{(2)} S(\sigma + \delta\rho||\sigma)\,. \tag{2.13}$$

The inner product in (2.12) is clearly non-negative, symmetric, and vanishes only if $\delta\rho_1 = \delta\rho_2 = 0$. We can think of it as a metric on the space of states perturbed away from the reference state σ. The metric lives on the tangent space to σ and is the known as quantum Fisher information. It is also sometimes called the Bures, or Helstrom metric; However, we will stick to the general terminology of quantum Fisher information.

Our principle motivation to study Fisher information comes from the properties of relative entropy, although there are many other reasons it is an significant quantity. For example, the quantum Fisher information plays an important role in quantum state estimation, that is, how to approximate the state ρ given a set of measurements on n copies of the quantum state [23, 51].

2.1.2 Quantum Fisher information

Consider a one-parameter family of states $|\Psi(\lambda)\rangle$ where $\lambda = 0$ denotes the vacuum state. Such an expansion is always possible for well-behaved states [50]. We consider a perturbative expansion around the vacuum state in λ:

$$\rho(\lambda) = \sigma + \lambda\,\delta\rho \tag{2.14}$$

where $\rho(0) = \sigma$. The relative entropy is then

$$S(\rho||\sigma) \quad = \quad \mathrm{tr}((\sigma + \delta\rho)\log(\sigma + \delta\rho)) - \mathrm{tr}((\sigma + \delta\rho)\log\sigma)$$

We expand the logarithm using a Taylor series;

$$\log(\sigma + \delta\rho) = \log(\sigma_0) + \lambda\frac{d}{d\lambda}\log\rho(\lambda)\big|_{\lambda=0} + \frac{\lambda^2}{2}\frac{d^2}{d\lambda^2}\log\rho(\lambda)\big|_{\lambda=0} + \mathcal{O}(\lambda^3)$$

so that the relative entropy becomes

$$S(\rho||\sigma) = \lambda\mathrm{tr}\left[\sigma\frac{d}{d\lambda}\log\rho\big|_{\lambda=0}\right] + \lambda^2\mathrm{tr}\left[\delta\rho\frac{d}{d\lambda}\log\rho\big|_{\lambda=0} + \frac{\sigma}{2}\frac{d^2}{d\lambda^2}\log\rho\big|_{\lambda=0}\right] + \mathcal{O}(\lambda^3) \tag{2.15}$$

The first term vanishes since $\mathrm{tr}\left[\sigma\frac{d}{d\lambda}\log\rho\big|_{\lambda=0}\right] \sim \mathrm{tr}\left[\sigma\sigma^{-1}\delta\rho\right] = \mathrm{tr}\left[\delta\rho\right] = 0$. Now consider the related quantity $\frac{d}{d\lambda}\mathrm{tr}\left[\rho\frac{d}{d\lambda}\log\rho\right]$ which is also identically zero for any λ since $\frac{d}{d\lambda}\log\rho$ acts as the formal inverse operator to ρ. We then evaluate

$$\begin{aligned} \frac{d}{d\lambda}\mathrm{tr}\left[\rho\frac{d}{d\lambda}\log\rho\right]\big|_{\lambda=0} &= \mathrm{tr}\left[\left(\frac{d}{d\lambda}\rho\right)\frac{d}{d\lambda}\log\rho\right]\big|_{\lambda=0} + \mathrm{tr}\left[\rho\frac{d^2}{d\lambda^2}\log\rho\right]\big|_{\lambda=0} \\ &= \mathrm{tr}\left[\delta\rho\frac{d}{d\lambda}\log\rho\big|_{\lambda=0}\right] + \mathrm{tr}\left[\sigma\frac{d^2}{d\lambda^2}\log\rho\big|_{\lambda=0}\right]. \end{aligned}$$

This is nearly the second term in (2.15).

Using this expression to simplify (2.15), we have

$$S(\rho||\sigma) = -\frac{\lambda^2}{2}\,\mathrm{tr}\left[\sigma\frac{d^2}{d\lambda^2}\log\rho\big|_{\lambda=0}\right] + \mathcal{O}(\lambda^3) \tag{2.16}$$

or

$$S(\rho||\sigma) = \frac{\lambda^2}{2}\,\mathrm{tr}\left[\delta\rho\frac{d^2}{d\lambda^2}\log\rho\big|_{\lambda=0}\right] + \mathcal{O}(\lambda^3)\,. \tag{2.17}$$

We define the right-hand-side of this expression at order λ^2 as the quantum Fisher information

$$\mathcal{F}_\sigma(\delta\rho, \delta\rho) = \frac{\langle\delta\rho, \delta\rho\rangle_\sigma}{\lambda^2} = \frac{1}{2}\mathrm{tr}\left[\delta\rho\frac{d^2}{d\lambda^2}\log\rho\big|_{\lambda=0}\right], \tag{2.18}$$

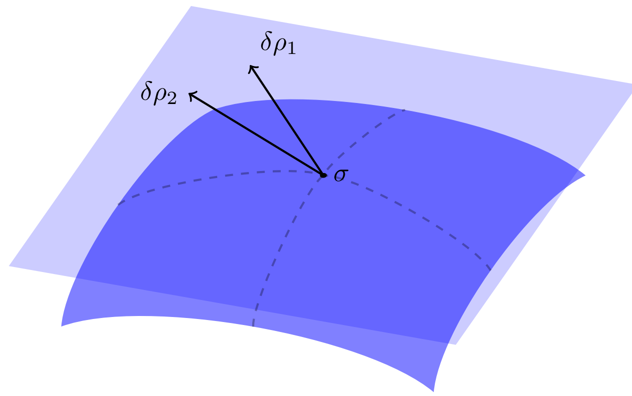


Figure 2.2: The tangent plane to a density matrix σ.

which we recognize as the inner product from (2.12). To get a better sense of what (2.18) means, we naively disregard the ordering so that

$$\mathcal{F}_\sigma(\delta\rho, \delta\rho) = \frac{1}{2}\mathrm{tr}(\delta\rho\sigma^{-1}\delta\rho)\,. \tag{2.19}$$

This is a much more illuminating expression. We see that $\mathcal{F}_\sigma$ is quadratic in the perturbation $\delta\rho$, and as such it is clearly positive as required the positivity of relative entropy. Note that because our constraint in (**??**) is limited to second order, the Fisher information is precisely the $\mathcal{O}(\lambda^2)$ term in the relative entropy.

$$\delta^{(2)}S(\sigma + \delta\rho||\sigma) = \lambda^2\mathcal{F}_\sigma(\delta\rho, \delta\rho) \tag{2.20}$$

Using the above results, the total entanglement entropy of a region B in a field theory with a state expanded as (2.14) is

$$S_B = S_B^{\mathrm{vac}} + \lambda\int_B \zeta^\mu\langle T_{\mu\nu}\rangle\boldsymbol{\epsilon}_\nu - \lambda^2\mathcal{F}_\sigma(\delta\rho, \delta\rho) + \mathcal{O}(\lambda^3)\,. \tag{2.21}$$

This expression is exact for any CFT, however, it is expected that the structure of the $\mathcal{O}(\lambda^2)$ term contains information about whether the CFT is holographic [1]. Whereas the linear term is given by the vacuum expectation value of the stress tensor, the Fisher term generally may not have a local VEV.

2.1.3 Gravitational relative entropy

In this section we discuss the gravitation dual of relative entropy.

Consider a one-parameter family of geometries $\mathcal{M}(\lambda)$ which are dual to states $|\Psi(\lambda)\rangle$ in a holographic CFT. We take the origin $\lambda = 0$, to be pure AdS spacetime with the dual vacuum state $|\Psi(0)\rangle$. Now consider a spherical region B, which lives on a spatial slice of the boundary of the unperturbed spacetime. The Rindler wedge R_B associated with B is defined as the intersection of the causal past and causal future of D_B; the causal diamond. The boundary of the Rindler wedge is a bulk surface, $\tilde{B}$, which possesses a minimal area in the sense of Ryu and Takayanagi, that is $\partial\tilde{B} = \partial B$. The boundary Killing vector ζ can be extended into the bulk wedge such that it is null on the (bulk) boundary of the wedge as illustrated in figure 2.3; we denote this new vector by ξ.

On the boundary, ζ generates a notion of time within the causal diamond. Likewise, the bulk ξ gives a notion of time to the full Rindler wedge. By a diffeomorphism, the wedge R_B can be mapped to the exterior of a Schwarzschild-AdS black hole where $\tilde{B}$ acts as the horizon. This fact will be useful (since there exist many powerful tools) to understand black hole spacetimes.

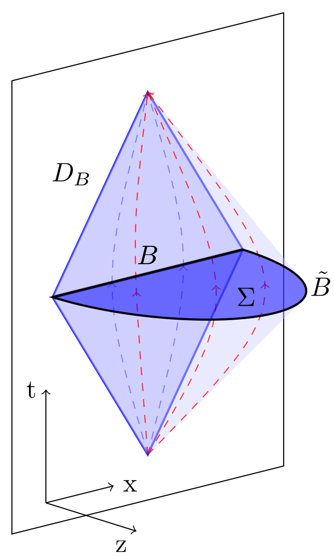


Figure 2.3: The Rindler wedge R_B associated with the ball-shaped boundary region B. The blue lines indicate the flow of ζ, and the red lines ξ. The surface Σ lies between B and the extremal surface $\tilde{B}$.

The extension to a perturbed spacetime with $\lambda \neq 0$ spacetime is clear; for each perturbed region $B(\lambda)$ we associate the Ryu-Takayanagi surface $\tilde{B}(\lambda)$

in $\mathcal{M}(\lambda)$ as the extremal surface which satisfies $\partial\tilde{B}(\lambda) = \partial B(\lambda)$. We may now relate each perturbed CFT state to a geometry which is perturbed away from a hyperbolic black hole in AdS.

In analogy with the CFT case, we wish to compute the relative entropy, $S^{\text{grav}}(g(\lambda)||g(0))$, between the perturbed metric and pure AdS. Motivated by the CFT result we define

$$S^{\text{grav}}(g_1||g_0) = \Delta E^{\text{grav}} - \Delta S^{\text{grav}}\,, \tag{2.22}$$

where ΔE^{grav} is the difference in gravitational energy between $\mathcal{M}(0)$ and $\mathcal{M}(\lambda)$, and ΔS^{grav} is the entropy difference.

The notion of gravitational entropy can be identified with the area of an extremal area surface via the Ryu-Takayanagi formula, so that

$$S^{\text{grav}}(\lambda) \equiv \frac{1}{4G_N}\text{Area}\left[\tilde{B}(\lambda)\right]\,, \tag{2.23}$$

and hence $\Delta S^{\text{grav}} = S^{\text{grav}}(\lambda) - S^{\text{grav}}(0)$.

We may also define the gravitational energy E^{grav}, the analogous quantity to $\langle H_B\rangle$, by utilizing some key results from holography. For a $(d+1)-$dimensional spacetime, an asymptotic expansion near the boundary can be parameterized in Fefferman-Graham coordinates as

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + dx_\mu dx^\mu + z^d\,\Gamma_{\mu\nu}(z,x)dx^\mu dx^\nu\right) \tag{2.24}$$

where $\Gamma_{\mu\nu}(z,x)$ remains finite in the $z \to 0$ limit. In the remainder of this section we set the AdS scale to unity, $\ell = 1$. The expectation value of the CFT stress tensor is related to the asymptotic behaviour of the metric through

$$\langle T_{\mu\nu}\rangle = \frac{d}{16\pi G_N}\Gamma_{\mu\nu}(z=0,x)\,. \tag{2.25}$$

Thus, from the first law of entanglement for a spherical region, we have that the change in the modular Hamiltonian in B is

$$\Delta\langle H_B\rangle = \frac{d}{8G_N}\int_{|x|<R} d^{d-1}x\,\frac{R^2-|x|^2}{2\,R}\Gamma_{tt}(z=0,x)\,, \tag{2.26}$$

since $\langle H_B\rangle = 0$ for pure AdS. The right hand side of (2.26) only depends on bulk quantities, so we define it as ΔE^{grav}. Then for holographic states, we may write the relative entropy between two metrics $g(\lambda), g(0)$ completely in terms of bulk quantities through (2.23) and (2.26).

The gravitational relative entropy is also important from the prescriptive of black hole thermodynamics. To first order, it gives the first law of black hole mechanics $\delta E^{\text{grav}} = \delta S^{\text{grav}}$. The second order contribution was recently discussed in [52] to understand the stability of black branes and black holes.

We now have the tools to compute the relative entropy of a holographic state $|\Psi(\lambda)\rangle$ near the vacuum from the gravitational side. In the next section we will focus specifically on the second order contribution to the relative entropy known as canonical energy.

2.2 Canonical energy

In this section we review the very basics of Wald's canonical energy as presented in [52]. We do not attempt to go into full mathematical detail; rather we only to introduce the general picture.

Consider a smooth one-parameter family of geometries with a metric $g(\lambda)$ perturbed away from pure AdS. It was shown in [50] that to first order in the perturbation $\gamma = dg/d\lambda|_{\lambda=0}$, the gravitational relative entropy obeys

$$\frac{d^2}{d\lambda^2} S^{\text{grav}}\left(g(\lambda)||g(0)\right)\Big|_{\lambda=0} = W_\Sigma(g, \gamma, \pounds_\xi \gamma) + 2\int_\Sigma \xi^a \boldsymbol{\epsilon}^b \, \frac{d^2}{d\lambda^2} \hat{E}_{\mu\nu} \tag{2.27}$$

where $\boldsymbol{\epsilon}^b$ is defined in (2.5) and $\hat{E}_{\mu\nu}$ vanish for a solution of the full non-linear Einstein equations of motion coupled to matter. Following [52] we rename $W_\Sigma(g, \gamma, \pounds_\xi \gamma)$ to the canonical energy $\mathcal{E}$ because it plays the role of a Rindler energy associated with the Killing vector ξ.

There are a number of important properties of $\mathcal{E}$. Firstly, it is independent of the Cauchy slice Σ, which makes it a conserved quantity analogous to an energy. For non-trivial perturbations, the canonical energy $\mathcal{E}$ is non-degenerate and positive definite.[3] If $\mathcal{E}$ is negative, it implies that the spacetime is unstable with respect to the perturbations. These properties were used in [52] to probe the stability of black hole and black branes.

The canonical energy is quadratic in the perturbative bulk fields and is given explicitly by as integral over a Cauchy slice Σ. It can be decomposed into a geometric term and a matter term via

[3] A notable exception is for a mass perturbation of a Schwarzchild black hole. It is known to be stable, and yet the canonical energy is negative.

$$\begin{aligned}
\mathcal{E}(\delta g_1, \delta g_1) &= W_\Sigma(\delta g_1, \pounds_\xi \delta g_1) \\
&= \int_\Sigma \omega^{\text{full}}(\delta g_1, \pounds_\xi \delta g_1) \\
&= \int_\Sigma \omega^{\text{grav}}(\delta g_1, \pounds_\xi \delta g_1) + \int_\Sigma \omega^{\text{matter}}(\delta \phi_1, \pounds_\xi \delta \phi_1) \\
&= \int_\Sigma \omega^{\text{grav}}(\delta g_1, \pounds_\xi \delta g_1) - \int_\Sigma \xi^a T^{(2)}_{ab}, \qquad (2.28)
\end{aligned}$$

where $\delta\phi_1$ represents the perturbation to all matter fields.

In order to generalize these arguments to non-homogeneous spacetime, it will be necessary to promote the canonical energy to a bilinear form as

$$\mathcal{E}(\delta g_1, \delta g_2) = W_\Sigma(g, \delta g_1 \pounds_\xi \delta g_2)\,, \qquad (2.29)$$

for independent metric variations δg_1, δg_2. It can be shown that $\mathcal{E}(\delta g_1, \delta g_2)$ is always non-negative and only vanishes if and only if $\delta g_1 = \delta g_2 = 0$ [52]. In this sense, the canonical energy serves as a natural metric on the space of perturbations to a metric. It is then not surprising that its dual is quantum Fisher information, a metric on the space of density matrices. This correspondence further strengthens the idea that the geometry of spacetime is fundamentally related to the entanglement structure of a quantum theory.

2.2.1 Definitions

The full derivation of the canonical energy is fairly involved and is reviewed in [50, 52]. Here we only present the basics needed to understand the computation of $\mathcal{E}$ in practice.

Implicitly, the above analysis depended on the extremal surface remaining in the same coordinate location as we vary λ. Fortunately, it was shown in [52] that it is always possible to choose coordinates such that the extremal surface $\tilde{B}$ remains in the same coordinate location and that the Killing vector ξ satisfies the gauge conditions

$$\xi|_{B(\lambda)} = \zeta, \qquad (2.30)$$

$$\xi|_{\tilde{B}(\lambda)} = 0, \qquad (2.31)$$

$$\pounds_\xi g(\lambda)|_{\tilde{B}(\lambda)} = 0. \qquad (2.32)$$

The third equation merely states the ξ continues to be a Killing vector on the extremal surface $\tilde{B}$. The explicit formula for the bulk Killing vector is

$$\xi = \frac{\pi}{R}\left(R^2 - (t-t_0)^2 + |x-x_0|^2\right)\partial_t - \frac{2\pi}{R}(t-t_0)z\,\partial_z - \frac{2\pi}{R}(t-t_0)(x^i - x_0^i)\,\partial_i \tag{2.33}$$

where i runs over spatial indices.

Consider the pure AdS metric $g(0)$ and a perturbation h which satisfies the linearized equations of motion about g. We require that a new perturbation γ satisfies the gauge condition (2.32) and is related to h through the gauge transformation

$$\gamma = h + \pounds_V g. \tag{2.34}$$

Our main task now is to determine the gauge field V which leaves the surface $\tilde{B}$ extremal. We choose coordinates along the geodesic such that the index i runs over dynamic coordinates and the index A runs over static, constant coordinates,

$$X^i = \sigma^i, \qquad X^A = X_0^A\,. \tag{2.35}$$

The metric perturbation has to satisfy two equations: the condition that ξ is a Killing vector, and that the first order variation of $A(X+\delta X, g+h)$ vanishes. These give

$$\pounds_\xi h|_{\tilde{B}(\lambda)} = 0\,, \tag{2.36}$$

$$\nabla_i \gamma^i_A - \frac{1}{2}\nabla_A \gamma^i_i|_{\tilde{B}(\lambda)} = 0\,, \tag{2.37}$$

evaluated on the extremal surface $\tilde{B}(\lambda)$. These can be further broken down into three equations

$$\gamma^i_A = 0\,, \tag{2.38}$$

$$\gamma^A_B - \delta^A_B \gamma^C_C = 0\,, \tag{2.39}$$

$$\nabla_i \gamma^i_A - \frac{1}{2}\nabla_A \gamma^i_i = 0\,. \tag{2.40}$$

These three equations represent the differential equations for V, given a specific metric perturbation h. Explicitly, they can also be written as

$$\left(h_{iA} + \nabla_i V_A + \nabla_A V_i\right)|_{\tilde{B}(\lambda)} = 0\,, \tag{2.41}$$

$$\left(h^A_D - \frac{1}{2}\delta^A_D h^C_C + \nabla^A V_D + \nabla_D V^A - \delta^D_D \nabla_C V^C_C\right)\Big|_{\tilde{B}(\lambda)} = 0\,. \tag{2.42}$$

The bilinear canonical energy is an integral of a form $\omega(g, \gamma_1, \gamma_2)$ over the region Σ between B and $\tilde{B}$. Under the gauge transformation $\gamma \to h$ which continues to satisfy the equations of motion, the integrand of the canonical energy only changes by a derivative of a form $\rho(h, V)$ defined in (2.46),

$$\omega(g, \gamma, \pounds_\xi \gamma) = \omega(g, h, \pounds_\xi h) + d\rho(h, V)\,. \tag{2.43}$$

The integral over $d\rho$ is equivalent to a boundary integral through Stokes's theorem. Moreover, the only contribution is from $\tilde{B}$ since the field V vanishes on B. The end result is that the canonical energy reduces to an integral over a two-form $\omega(g, h, \pounds_\xi h)$ and a one-form $\rho(h, V)$,

$$\mathcal{E}(h, h) = \int_\Sigma \omega(g, h, \pounds_\xi h) + \int_{\tilde{B}} \rho(h, V)\,, \tag{2.44}$$

the details of which can be found in [50].

In summary, for some metric perturbation h in an arbitrary gauge, we first must find a field V which solves the conditions (2.41). We can then calculate $\mathcal{E}$ in a the Rindler wedge R_B. Because the integral over ρ is evaluated at $\tilde{B}$, we only need to compute V on the surface and not throughout the entire region Σ.

As per (2.44), the ω and ρ tensors are defined through

$$\omega(g, h, \pounds_\xi h) = \frac{1}{16\pi G_N} \boldsymbol{\epsilon}_a P^{abcdef} \left(\pounds_\xi h_{bc} \nabla_d h_{ef} - h_{bc} \nabla_d \pounds_\xi h_{ef}\right), \tag{2.45}$$

$$\rho(h, V) = \chi(\gamma_h, [\xi, V]) - \chi(\pounds_\xi h, V), \tag{2.46}$$

with

$$\begin{aligned}
\chi(A, B) &= \frac{1}{16\pi G_N} \boldsymbol{\epsilon}_{ab} \\
&\qquad \left(A^{ac} \nabla_c B^b - \frac{1}{2} A_c^c \nabla^a B^b + \nabla^b A_c^a B^c - \nabla_c A^{ac} B^b + \nabla^a A_c^c B^b\right), \\
P_{abcdef} &= g^{ae} g^{fb} g^{cd} - \frac{1}{2} g^{ad} g^{be} g^{fc} - \frac{1}{2} g^{ab} g^{cd} g^{ef} - \frac{1}{2} g^{bc} g^{ae} g^{fd} + \frac{1}{2} g^{bc} g^{ad} g^{ef}, \\
[\xi, V]^a &= \xi^b\, \partial_b V^a - V^b\, \partial_b \xi^a\,,
\end{aligned}$$

and the $\boldsymbol{\epsilon}$ tensors are defined as in (2.5).

2.2.2 Example for BTZ black hole

In this section we provide a sample calculation for the canonical energy of a planar black hole geometry. We show this to become familiar with the steps involved before generalizing to a more complicated spacetime in later chapters.

Consider a 2-dimensional CFT which is dual to an asymptotically AdS_3 spacetime given by the metric

$$ds^2 = \frac{1}{z^2}\left(dz^2 + (1+\lambda z^2/2)\,dx^2 - (1-\lambda z^2/2)\,dt^2\right) . \tag{2.47}$$

This represents the first order approximation to a planar black hole geometry. In these planar coordinates the perturbation is simply, $H_{tt} = \lambda$, $H_{xx} = \lambda$; However, these coordinates are ill-suited to working with the geodesic surface $\tilde{B}$. Instead, we opt for polar coordinates $x = r\,\sin\theta, z = r\,\cos\theta$ since the extremal surface is easily parameterized by constant R and $\theta \in [0, \pi/2)$. The perturbation $H_{\mu\nu}$ then takes the form

$$\begin{aligned} h_{tt} =& \lambda\,, \\ h_{rr} =& \lambda \sin^2\theta\,, \\ h_{tr} =& \lambda\, r\,\sin\theta\cos\theta\,, \\ h_{\theta\theta} =& \lambda\, r^2\cos^2\theta\,, \end{aligned}$$

with all other components equal to zero. Since we are considering the $t = 0$ surface, we may choose $V_t = 0$ to simplify the calculation. The six V equations from (2.41) then reduce to three independent equations

$$\partial_\theta^2 V_r - 3\tan\,\partial_\theta V_r - 2\,V_r - 2\,r\,\sin^2\theta \quad = \quad 0\,, \tag{2.48}$$
$$\partial_r V_\theta + \partial_\theta V_r - 2\tan\,V_r + \,r\,\sin\theta\cos\theta \quad = \quad 0\,, \tag{2.49}$$
$$2\,\partial_r V_r + \frac{4}{r}\,V_r + 2 - \cos^2\theta \quad = \quad 0\,. \tag{2.50}$$

The final equation is an ordinary differential equation for V_r, so it can be solved immediately as

$$V_r = \frac{r(\cos^2\theta - 2)}{3} + \frac{C_1}{r^2\,\cos^2\theta} + \frac{C_2\,\sin\theta}{r^2\cos^2\theta}\,. \tag{2.51}$$

The undetermined coefficients C_1, C_2, must vanish to keep V_r finite on then boundary $\theta = 0, \pi/2$. The remaining components can now easily be solved from (2.49). The result is

$$V_\theta = -\frac{r^2\,\sin\theta\,(2+\cos^2\theta)}{3\,\cos\theta} + \frac{C_2}{r\,\cos\theta} + F_1(\theta) \tag{2.52}$$

where $F_1(\theta)$ is an undetermined function which will thankfully not enter the canonical energy calculation.

Making use of the definitions (2.45), the two-form contribution is

$$\omega(g, H, \pounds_\xi H) = -\frac{r^4 \cos^3\theta}{2RG_N}\, dr \wedge d\theta, \tag{2.53}$$

which integrates to

$$\int_\Sigma \omega(g, H, \pounds_\xi H) = -\frac{1}{2\,R}\int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^4 \cos^3\theta\, d\theta\, dr \; = -\frac{2\,R^4}{15}\,. \tag{2.54}$$

The evaluation of $\rho = \rho_r dr + \rho_\theta d\theta$ only requires knowledge of ρ_θ, since we are concerned with the constant R surface. Explicitly evaluating ρ_θ at $r = R$ gives

$$\rho_\theta|_{\tilde{B}} = -\frac{R^4}{12G_N}\cos^3\theta(2\cos^2\theta - 3) \tag{2.55}$$

which upon integration yields

$$\int_{\tilde{B}} \rho = \frac{7}{45G_N} R^4\,. \tag{2.56}$$

Summing the contributions from $\int \omega$ and $\int \rho$ gives the total canonical energy

$$\mathcal{E} = \frac{R^4}{45G_N}\,. \tag{2.57}$$

For pure gravity, (2.57) represents the sole contribution to the canonical energy associated with the Rindler wedge R_B. This surprisingly simply result agrees with the requirement that the relative entropy is positive. It was also confirmed in [50] that this matches the CFT result for Fisher information in two dimensions

2.3 Summary

In this chapter we introduced the concept of relative entropy in both the field theory and gravitational theory. By considering only the second order change relative entropy for a state perturbed from the vacuum, we defined the quantum Fisher information which acts as a metric on the space of perturbations to a density matrix. On the gravity side, we defined the canonical energy, a natural sympletic form, which represents a metric on the space of perturbations to a black hole. We provided an example calculation for a BTZ black hole and the result matches that expected in [50]

In the next chapter, we will use the canonical energy formalism to compute the quantum Fisher information for a general state perturbed from the vacuum.

Chapter 3

Relative entropy of general perturbations

In this Chapter we describe how to find the second order relative entropy for a state which is perturbed from the vacuum by solving the dual gravitational problem. As discussed in [1], such states can be attained from a local conformal transformation of the vacuum. On the gravitational side, this corresponds to a general linear perturbation of AdS spacetime.

This Chapter is broken down into four main parts. First, we derive the most general perturbations of AdS_3 which are consistent with a 2d CFT dual. We then apply the procedure from Chapter 2 to compute the canonical energy for such a spacetime. Following that, we prove the positivity of this expression and perform some consistency checks. Lastly, we go on to calculate the matter contribution to the canonical energy from a scalar field perturbation.

3.1 Gravitational contribution to relative entropy

3.1.1 General perturbation of AdS_3

Let us consider the constraints on the boundary data of AdS_3 arising from the consistancy of the stress tensor in a two-dimensional CFT. For any state in a CFT, the stress tensor must be traceless and conserved,

$$\begin{aligned}\langle T^{\mu}_{\mu} \rangle &= 0\,, &(3.1)\\ \langle \partial_\mu T^{\mu\nu} \rangle &= 0\,. &(3.2)\end{aligned}$$

In our two-dimensional case, we can express the constraints most simply in terms of light-cone coordinates, $x^{\pm} = x \pm t$, so that

$$\begin{aligned}\langle T_{+-} \rangle &= 0\,, &(3.3)\\ \partial_+ \langle T_{--} \rangle = \partial_- \langle T_{++} \rangle &= 0\,. &(3.4)\end{aligned}$$

Thus, a general 2d CFT stress tensor can be described solely by the two functions, $\langle T_{++}(x^+)\rangle$ and $\langle T_{--}(x^-)\rangle$. These can to be identified with the holomorphic and antiholomorphic components.

Assuming the CFT is holographic, there exists a gravitational dual with the metric given by

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dx^\mu dx_\mu + z^2\Gamma_{\mu\nu}dx^\mu dx^\nu\right) . \tag{3.5}$$

The CFT stress tensor is related to the asymptotic behaviour of the AdS metric through (2.25), so that

$$\Gamma_{++}(x,0) = 8\pi\frac{G_N}{\ell_{AdS}}\langle T_{++}(x^+)\rangle\,, \qquad \Gamma_{--}(x,0) = 8\pi\frac{G_N}{\ell_{AdS}}\langle T_{--}(x^-)\rangle\,. \tag{3.6}$$

Now consider a a small perturbation to the CFT vacuum governed by λ. To first order we have

$$\Gamma_{++}(x,0) \equiv \lambda h_+(x^+)\,, \qquad \Gamma_{--}(x,0) \equiv \lambda h_-(x^-)\,. \tag{3.7}$$

The z behaviour of $\Gamma_{\pm\pm}(x,z)$ is then determined by the Einstein equations and the initial data from (3.7). To first order in λ, the radial component of the linearized Einstein equations becomes

$$\frac{1}{z^3}\partial_z(z^3\partial_z\Gamma_{\mu\nu}) + \partial_\rho\partial^\rho\Gamma_{\mu\nu} = 0\,. \tag{3.8}$$

The first order solution of (3.8) is simply given by (3.7). Together these results give the most general form for an asymptotically AdS metric which has a consistent CFT on the boundary. In terms of the functions $h_\pm$, the metric is

$$\begin{aligned} ds^2 &= \frac{1}{z^2}\left(dz^2 + (1+\lambda z^2 h_+)(dx^+)^2 + (1+\lambda z^2 h_-)(dx^-)^2\right) \\ &= \frac{1}{z^2}\Big(dz^2 - (1-\lambda z^2 h_+)(1-\lambda z^2 h_-)\,dt^2 + (1+\lambda z^2 h_+)(1+\lambda z^2 h_-)\,dx^2 \\ &\qquad +2\lambda z^2(h_+ - h_-)\,dt\,dx\Big) \end{aligned} \tag{3.9}$$

Notice that if $h_+ = h_- = \frac{1}{2}$ we recover the BTZ black hole metric in (2.47).

3.1.2 Gauge-fixed coordinates

Now that we know the most general form of the metric, we can compute the canonical energy of a Rindler wedge. We will use

$$\mathcal{E}(h,h) = \int_\Sigma \omega(g,h,\pounds_\xi h) + \int_{\tilde{B}} \rho(h,V) \tag{3.10}$$

as per (2.44). This formula assumes a gauge condition in which the coordinate location of the minimal surface $\tilde{B}$ remains fixed. These differ from the Fefferman-Graham gauge conditions we have been using so far. Thus, we must first find a gauge transformation, V, to bring the metric perturbation to the appropriate form. Finding such a gauge for an arbitrary h is hopeless, instead we opt to decompose h into its Fourier components and find the corresponding $\hat{V}_k$ for each mode of h. To do so, we first note that any solution to the linearized Einstein's equations around AdS can be written as a Fourier integral

$$h_{\mu\nu}(t,x,z) = \lambda \int \left[\delta_\mu^+ \delta_\nu^+ \hat{h}_+(k) e^{ikx^+} + \delta_\mu^- \delta_\nu^- \hat{h}_-(k) e^{ikx^-}\right] dk\,, \tag{3.11}$$

with the gauge choice $h_{az}(t,x,z) = 0$. Note that Greek indices run over the CFT dimensions while Latin indices run over the full AdS spacetime dimensions.

Due to the linearity of the problem, V can also be decomposed into left and right k-modes as

$$V_a(t,x,z) = \lambda \int \left[\hat{V}_a^+(k) e^{ikx^+} + \hat{V}_a^-(k) e^{ikx^-}\right] dk\,. \tag{3.12}$$

With this parameterization, the gauge functions $\hat{V}_a^\pm$ can be directly solved through the differential equations in (2.41). In terms of polar coordinates ($x = r\sin\theta, z = r\cos\theta$), these become

$$-2\tan\theta\,\hat{V}_r + \partial_\theta \hat{V}_r + \partial_r \hat{V}_\theta = -\hat{h}_{\theta r}\,, \tag{3.13}$$

$$-2\tan\theta\,\hat{V}_t + \partial_\theta \hat{V}_t + \partial_t \hat{V}_\theta = -\hat{h}_{\theta t}\,, \tag{3.14}$$

$$r\partial_r \hat{V}_t + r\partial_t \hat{V}_r + 2\hat{V}_t = -r\hat{h}_{rt}\,, \tag{3.15}$$

$$2r\partial_r \hat{V}_r + 2r\partial_t \hat{V}_t + 4\hat{V}_r = -r\left(\hat{h}_{rr} + \hat{h}_{tt}\right)\,, \tag{3.16}$$

$$\partial_\theta^2 \hat{V}_r - 3\,\tan\theta\,\partial_\theta \hat{V}_r - 2\,\hat{V}_r = \tan\theta\,\hat{h}_{r\theta}\,, \tag{3.17}$$

$$\partial_\theta^2 \hat{V}_t - 3\,\tan\theta\,\partial_\theta \hat{V}_t - 2\,\hat{V}_t = \tan\theta\,\hat{h}_{t\theta}\,, \tag{3.18}$$

where $\hat{V}_a = \hat{V}_a^+(k)e^{ikx^+} + \hat{V}_a^-(k)e^{ikx^-}$. Due to symmetry, the mode solutions are related through $V_r^+(\theta,r,t) = V_r^-(\theta,r,-t)$ and $V_t^-(\theta,r,t) = -V_t^+(\theta,r,-t)$, and so we need only focus on the right-moving component x^-.

Equations (3.17) and (3.18) are ordinary differential equations for $\hat{V}_t(\theta)$ and $\hat{V}_r(\theta)$ which can be solved directly. The solutions for the right-moving part are

$$V_t^-(t,r,\theta) = \frac{F_2(t,r)}{\cos^2\theta} + \frac{F_1(t,r)\sin\theta}{\cos^2\theta} - \frac{i\,e^{ikx^-}\left(k^2r^2\cos^2\theta - 2\right)}{2k^3r^2\cos^2\theta},$$
$$V_r^-(t,r,\theta) = \frac{F_4(t,r)}{\cos^2\theta} + \frac{F_3(t,r)\sin\theta}{\cos^2\theta} - \frac{e^{-ikx^-}\left(ik^2r^2\sin\theta\cos^2\theta - 2kr\cos^2\theta + 2\,i\sin\theta\right)}{2\,k^3r^2\cos^2\theta},$$

where $F_i(t,r)$ are undetermined functions.

To find the canonical energy, we also need the derivatives $\partial_r V_\theta, \partial_t V_\theta$. These can then be obtained from (3.13), (3.14) respectively,

$$\partial_t V_\theta^- = \frac{e^{ikx^-}}{2\,k^2\,r\cos\theta}\left(2 + k^2r^2\cos^2\theta - 2\,ikr\,\sin\theta - 2F_1(t,r)e^{-ikx^-}\right)$$
$$\partial_r V_\theta^- = \frac{e^{ikx^-}}{k^3r^2\cos\theta}\Big(2kr\sin\theta + r^3k^3\sin\theta\cos^2\theta + i\left(r^2k^2\cos^2\theta - kr^2 + 2\right) - 2k^3r^2e^{-ikx^-}F_3(t,r)\Big)\ .$$

Now the solutions for V_t and V_r diverge at the boundary $\theta = \pm\frac{\pi}{2}$ unless we choose $F_i(t,r)$ appropriately. Requiring these to be finite at the boundary, we obtain

$$F_1(t,r) = F_4(t,r) = \frac{\sin(kr)}{r^2\,k^3}\,e^{-i\,k\,t}\,, \tag{3.19}$$

$$F_2(t,r) = F_3(t,r) = -\,i\,\frac{\cos(kr)}{r^2\,k^3}e^{-i\,k\,t}\,. \tag{3.20}$$

Now that we have V, we can compute the canonical energy by using (2.45), (2.46) and (3.10). Such calculations are best handled with symbolic algebra packages.

3.1.3 Results

The canonical energy as a momentum space integral in terms of the boundary stress tensor neatly splits into the sum of two terms

$$\mathcal{E} = \int dk_1 dk_2\,\hat{K}^{(2)}(k_1,k_2)\,\langle\hat{T}_{++}(k_1)\rangle\langle\hat{T}_{++}(k_2)\rangle\;+\;\{+\leftrightarrow-\} \tag{3.21}$$

with the kernel

$$\hat{K}^{(2)}(k_1,k_2) = \frac{256\pi^2R^4G_N}{K^3(K-\kappa)^3(K+\kappa)^3}\Big((K^5 - 2(\kappa^2+4)K^3 + \kappa^4K)\cos K - (5K^4 - 6K^2\kappa^2 + \kappa^4)\sin K + 8K^3\cos\kappa\Big) \tag{3.22}$$

where $K \equiv R(k_1+k_2)$ and $\kappa \equiv R(k_1-k_2)$. The canonical energy splits into a left-moving part and a right-moving part because in the dual CFT there is no mixing between the holomorphic and antiholomorphic components. This is because the conformal group in two dimensions, $SO(2,2)$, factorizes into $SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

Taking the inverse Fourier transformation, we obtain the position space kernel

$$\mathcal{E} = \int_B dx_1^+ dx_2^+ \, K^{(2)}(x_1^+, x_2^+) \, \langle T_{++}(x_1^+)\rangle\langle T_{++}(x_2^+)\rangle \; + \{+ \leftrightarrow -\} \tag{3.23}$$

where the kernel $K^{(2)}$ is given by

$$K^{(2)}(x_1^\pm, x_2^\pm) = \frac{4\pi^2 G_N}{R^2} \begin{cases} (R-x_1^\pm)^2(R+x_2^\pm)^2, & x_1^\pm \geq x_2^\pm \\ (R+x_1^\pm)^2(R-x_2^\pm)^2, & x_1^\pm < x_2^\pm \end{cases} . \tag{3.24}$$

Notice that the kernel is symmetric under exchange of $x_1^\pm$,$x_2^\pm$ and vanishes for $|x_i| > R$.

Using $c = \frac{3\ell_{\text{AdS}}}{2G_N}$, we can write (3.23) as the second order relative entropy of a CFT state. Like the leading order result from the entanglement first law, the integrals in (3.23) can be taken over any surface B with boundary ∂B. The fact that we only need the stress tensor on a Cauchy surface for D_B is special to the stress tensor in two dimensions since the conservation relations allow us to find the stress tensor expectation value everywhere in D_B from its value on a time slice. For other operators, or in higher dimensions, the result will involve integrals over the full domain of dependence. This limits what we can say about the relative entropy in this case. We will see an explicit example of such in the next section.

As a consistency check, we can plug in the homogeneous BTZ metric where $k = 0$ and $h_\pm(0) = \frac{1}{2}$. From (3.6) we have that $\langle T_{\pm\pm}(k_i)\rangle = \frac{\lambda}{16\pi G_N}$ and so it only remains to evaluate (3.22) at $k_1 = k_2 = 0$. In taking the limit, the kernel evaluates to $512\pi^2 R^4 G_N/90$. The total canonical energy is then

$$\mathcal{E} = \left(\frac{\lambda}{16\pi G_N}\right)^2 \left(\frac{512\pi^2 R^4 G_N}{90}\right) \tag{3.25}$$

$$= \lambda^2 \frac{R^4}{45 G_N} \tag{3.26}$$

which agrees with the expected result from (2.57).

Another important consistency check will be to demonstrate that the canonical energy in (3.23) is explicitly positive as required by the positivity of relative entropy. In the next section we give an elementary proof of the positivity.

3.1.4 Positivity of canonical energy

Let us test the positivity of relative entropy from our result in (3.23). Consider the left-moving part of the perturbation $h_+(x^+) \propto \langle T_{++}(x^+)\rangle$ (as an identical analysis will follow for $h_-(x^-)$). The function $h_+(x)$ must be real valued for a perturbation of AdS_3 so we can expand $h_+(x)$ in a Taylor series $h_+(x) = \sum_{n=0}^{\infty} a_n x^n$. The canonical energy is then given by

$$\mathcal{E} \sim \sum_n \sum_m a_n a_m \int_B \int_B dx_1 dx_2 \, x_1^n x_2^m K^{(2)}(x_1, x_2)\,. \tag{3.27}$$

Whereas it might seem convenient to use an orthogonal basic such as the Legendre or Chebyshev polynomials, this integral is only analytically tractable for the polynomial basis chosen. The result is

$$\mathcal{E} \sim \sum_n \sum_m a_n a_m R^{4+n+m} \mathcal{A}_{n,m} \tag{3.28}$$

where the proportionality factor is up to a positive constant and

$$\mathcal{A}_{n,m} = \frac{1}{(n+m+3)(n+m+1)} \begin{cases} 0, & \text{if } n+m \text{ odd} \\ \frac{1}{(n+1)(m+1)}, & \text{if } n,m \text{ even} \\ \frac{nm+n+m+3}{nm(n+2)(m+2)}, & \text{if } n,m \text{ odd} \end{cases} \tag{3.29}$$

which is clearly non-negative and symmetric in n, m.

To show that the canonical energy is positive, we need to show that the matrix M with entries given by $A_{n,m} = \mathcal{A}_{n-1,m-1}$ [4] is positive semidefinite. To do so, we will use proof by induction and Sylvester's criterion which states that a square matrix M is positive semidefinite if and only if it has a positive determinant and all the upper-left sub-matrices also have positive determinants.

Proof by induction

Suppose that the $N \times N$ matrix M_N whose components are given by $A_{n,m}$ is positive semidefinite. Then consider the block matrix constructed as

$$M_{N+1} = \begin{pmatrix} \tilde{M}_N & \tilde{B} \\ \tilde{B}^T & A_{N+1,N+1} \end{pmatrix} = A_{N+1,N+1} \begin{pmatrix} M_N & B \\ B^T & 1 \end{pmatrix} \tag{3.30}$$

[4] The inelegant notation change is due to conventional matrix notation starting at $n = 1$, while the Taylor series starts at $n = 0$.

where B is an N column vector with entries given by $A_{i,n+1}$. Since M_N is positive semidefinite, it has a positive determinant and all the upper-left sub-matrices of M_N also have positive determinants by Sylvester's criterion. To show that M_{N+1} is positive semidefinite, we need only show it has a positive determinant since all the upper-left sub-matrices are already known.

The determinant of M_{N+1} may be evaluated using the formula

$$\det(M_{N+1}) = A_{n+1,n+1}\left[2\det(M_N) - \det(M_N + B^T B)\right] \tag{3.31}$$

so it is sufficient to show

$$\det(M_N + B^T B) < 2\det(M_N)\,. \tag{3.32}$$

We denote the eigenvalues of $M_N + B^T B$ by λ_i^{M+B} where they are ordered from largest to smallest $\lambda_1^{M+B} \geq \lambda_2^{M+B} \geq ... \geq \lambda_N^{M+B}$. Since $B^T B$ is a rank-one matrix, the sole non-zero eigenvalue is given by $\beta = \mathrm{Tr}(B^T B) = \sum_{i=1}^N A_{i,N+1} \geq 0$. Since $B^T B$ is positive semidefinite, there exists an upper bound on $\det(M_N + B^T B)$ given by the Weyl inequality $\lambda_i^{M+B} \leq \lambda_i^M + \beta_i$ where λ_i^M are the eigenvalues of M_N in order from largest to smallest $\lambda_1^M \geq \lambda_2^M \geq ... \geq \lambda_N^M$ [53]. We then expand the determinant as

$$\begin{aligned}\det(M_N + B^T B) &= \prod_{i=1}^N \lambda_i^{M+B} \\ &\leq \frac{\lambda_1^{M+B}}{\lambda^M}\prod_{i=1}^N \lambda_i^M = \left(1 + \frac{\beta}{\lambda_1^M}\right)\det(M_N)\,.\end{aligned}$$

So it remains to show that $\lambda_1^M - \beta_B \geq 0$ to complete the proof. The maximum eigenvalue λ_1^M is bounded from below by the minimum sum of a column of M_N through the Perron-Frobenius theorem (equivalently Gershgorin circle theorem) [53]. For the matrix M_N, the minimum sum of a column vector is simply the sum of the Nth column $\sum_{i=1}^N A_{i,N}$. Therefore, it remains to show that

$$\sum_{i=1}^N \left(A_{i,N} - A_{i,N+1}^2\right) \geq 0\,. \tag{3.33}$$

We split this sum up into two cases. The first case is if N is even. Then we have

$$\sum_{i=1}^{N/2} A_{2i,N} - \sum_{i=1}^{(N+1)/2} A_{2i-1,N+1}^2 = \sum_{i=1}^{N/2}\left(A_{2i,N} - A_{2i-1,N+1}^2\right) \tag{3.34}$$

since the final term in $\sum_{i=1}^{(N-1)/2} A^2_{2i-1,N+1}$ is zero. Explicitly analyzing the coefficients, we see that $\left(A_{2i,N} - A^2_{2i-1,N+1}\right)$ is always positive for all $i \in \{1..N/2\}$, so clearly the entire sum is positive.

In the case of odd N, the sum becomes

$$\sum_{i=1}^{(N+1)/2} A_{2i-1,N} - \sum_{i=1}^{N/2} A^2_{2i,N+1} = A_{N,N} + \sum_{i=1}^{N/2} \left(A_{2i-1,N} - A^2_{2i,N+1}\right) . \tag{3.35}$$

Each term in this sum is also positive, so we have shown $\lambda_1^M - \beta_B \geq 0$.

The expressions in (3.34) and (3.35) are not obviously positive, but they reduce to some (tractable, but unattractive) polynomial equations which can be shown to be positive. Therefore we have shown $\det(M_N + B^T B) < 2\det(M_N)$, and thus M_{N+1} is positive semidefinite given that M_N is. Since M_1 is positive semidefinite by induction, so too is M_N for all N. Therefore, the canonical energy is explicitly positive semidefinite as expected by the positivity of relative entropy.

3.2 Scalar field in AdS$_3$

As an extension of the previous results, we could consider adding a simple scalar field to the AdS spacetime. The canonical energy in this case is given by the sum of the gravitation and matter contribution

$$\mathcal{E} = \mathcal{E}_{\text{grav}} + \int_{\Sigma_B} \xi^a T^{\text{matter}}_{ab} \, d\Sigma^b \tag{3.36}$$

where T^{matter}_{ab} is the stress tensor for the scalar field. Explicitly taking z as the bulk coordinate, the matter contribution to the canonical energy is

$$\mathcal{E}_{\text{matter}} = \int_{\Sigma_B} dz dx x \, \frac{\pi(R^2 - z^2 - x^2)}{Rz} T^{\text{matter}}_{tt} . \tag{3.37}$$

In the holographic context, one considers excited states of a CFT which are induced by normalizable modes of bulk scalar fields in AdS. The bulk fields correspond to non-zero one-point functions of the dual operator $\mathcal{O}$. As in the gravitational case, we wish to express the canonical energy in terms of a product of one-point functions; this time of a general operator $\mathcal{O}$, rather than the stress tensor $T_{\pm\pm}$:

$$\delta^{(2)}_{\mathcal{O}} S = \int dx_1 dx_2 \, K^{(2)}_{\mathcal{O}}(x_1,x_2) \langle \mathcal{O}(x_1) \rangle \langle \mathcal{O}(x_2) \rangle . \tag{3.38}$$

The total change in the entanglement entropy up to second order in the perturbations is then

$$\Delta S = \delta^{(1)} S + \delta^{(2)} S + \delta^{(2)}_{\mathcal{O}} S\,. \tag{3.39}$$

The $\delta^{(1)}_{\mathcal{O}} S$ term vanishes for a perturbation around the vacuum so the only linear contribution to the entanglement entropy is from the first law.

In the remainder of this Chapter we focus on explicitly evaluating the second order contribution from a field, $\delta^{(2)} S_{\mathcal{O}}$.

3.2.1 General scalar field

Consider the case of a scalar field ϕ in AdS which is dual to the operator $\mathcal{O}$. The equation of motion of the scalar field ϕ, in AdS spacetime is

$$\frac{1}{z}\partial_z\left(z\partial_z\phi\right) + \partial_\mu\partial^\mu\phi - \frac{m^2}{z^2}\phi = 0\,, \tag{3.40}$$

where μ run over the CFT dimensions.

The solution of (3.40) may be written in term of the Fourier transform of ϕ,

$$\phi(z,t,x) = \int_{\omega^2-k^2>1} d\omega dk\, e^{i(kx-\omega t)}\phi_{\omega,k}(z) \tag{3.41}$$

so that the equation of motion becomes

$$\left[z^2(\omega^2-k^2) + z^2\partial_z^2 - z\partial_z - m^2\right]\phi_{\omega,k}(z) = 0\,. \tag{3.42}$$

The solution for each Fourier component is then given by a sum of Bessel functions

$$\phi_{\omega,k}(z) = C_1 z J_\nu(qz) + C_2 z Y_\nu(qz) \tag{3.43}$$

where $\nu = \sqrt{1+m^2} = \Delta - 1$ and $q = \sqrt{\omega^2-k^2}$. Near the boundary, the field must behave as

$$\lim_{z\to 0}\phi(z,t,x) = z^{\Delta}\phi_1(t,x) + z^{2-\Delta}\phi_0(t,x) \tag{3.44}$$

where ϕ_0 and ϕ_1 are the normalizable and the non-normalizable modes respectively. These modes can be related to the Bessel function solutions through the holographic dictionary in (1.16). We thus have

$$\langle\mathcal{O}(t,x)\rangle = (2\Delta-2)\phi_1(t,x)\,, \qquad \phi_0(t,x) = 0\,. \tag{3.45}$$

The constants C_1, C_2, can be determined by taking the $z \to 0$ limit of the Bessel function so that

$$C_1 = \frac{2^\Delta \Gamma(\Delta)}{\Delta - 1} q^{-\nu/2} \langle \mathcal{O}_{\omega,k}(z) \rangle \,, \qquad C_2 = 0 \,. \tag{3.46}$$

Therefore, the scalar field takes the form

$$\phi(z,t,x) = \frac{2^\Delta \Gamma(\Delta)}{\Delta - 1}\, z \int_{\omega^2 > k^2} d\omega dk\, e^{i(kx - \omega t)} q^{-\nu/2} J_\nu(qz) \langle \mathcal{O}_{\omega,k}(z) \rangle \,. \tag{3.47}$$

This is a fairly well-known result and can be found in [54, 55]

We can now compute the stress tensor from

$$T^{(2)}_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} \left(\partial^a \phi \partial^b \phi + m^2 \phi^2 \right) \,. \tag{3.48}$$

The evaluation (3.48) is fairly involved, but with the aid of symbolic computation, the result is

$$T^{(2)}_{ab} = \int_{\omega_1^2 > k_1^2} \int_{\omega_2^2 > k_2^2} dw_1 dk_1 dw_2 dk_2\, \tilde{T}^{(2)}_{tt}(z, w_1, w_2, k_1, k_2) e^{i\,(k_1 + k_2) x} \langle \mathcal{O}_{w_1, k_2} \rangle \langle \mathcal{O}_{w_2, k_2} \rangle \,.$$

For the canonical energy evaluated on a Cauchy slice Σ_B, will only need the $T^{(2)}_{tt}$ component of this equation. This is given by

$$\begin{aligned} \tilde{T}^{(2)}_{tt}(z, w_1, w_2, k_1, k_2) \;&=\; \frac{1}{4} \left(\frac{2^\Delta \Gamma(\Delta)}{\Delta - 1} \right)^2 q_1^{-\nu} q_2^{-\nu} \\ &\times \Bigg[\left(-z^2 (w_1 w_2 + k_1 k_2) + 2\nu(\nu - 1) \right) J_\nu(z q_1) J_\nu(z q_2) \\ &\quad - (\nu - 1) \left(q_1 z J_{\nu-1}(q_1 z) J_\nu(q_2 z) + q_2 z J_{\nu-1}(q_2 z) J_\nu(q_1 z) \right) \\ &\quad + q_1 q_2 z^2 J_{\nu-1}(q_1 z) J_{\nu-1}(q_2 z) \Bigg] \,. \end{aligned}$$

Although this formula is not very transparent, it can be simplified in certain cases. For $m = 0$, the stress tensor kernel reduces to

$$\tilde{T}^{(2)}_{tt}(z, w_1, w_2, k_1, k_2)|_{m=0} = \frac{4 z^2}{q_1 q_2} \Bigg(q_1 q_2 J_0\left(z q_1\right) J_0\left(z q_2\right) - \left(k_1 k_2 + w_1 w_2\right) J_1\left(z q_1\right) J_1\left(z q_2\right) \Bigg) \,.$$

The entanglement entropy contribution from the scalar field is then

$$\begin{aligned} \delta^{(2)} S_{\mathcal{O}} \;&=\; \int_{\Sigma_B} \int_{\omega_1^2 > k_1^2} \int_{\omega_2^2 > k_2^2} dz\, dx\, dw_1 dk_1 dw_2 dk_2 \\ &\times \frac{\pi (R^2 - z^2 - x^2) e^{i(k_1 + k_2) x}}{R z}\, \tilde{T}^{(2)}_{tt}(z, w_1, w_2, k_1, k_2) \langle \mathcal{O}_{w_1, k_2} \rangle \langle \mathcal{O}_{w_2, k_2} \rangle \,. \end{aligned} \tag{3.49}$$

3.2.2 Constant scalar field

asdasd

To simplify this expression even more, consider the case where $\langle\mathcal{O}\rangle$ is constant so that the solution for ϕ is $\phi(x,z) = \gamma\langle\mathcal{O}\rangle z^{\Delta}$ with some normalization constant γ. In this case, the integrals in (3.49) are calculable and the contribution to the entanglement entropy is explicitly

$$\delta^{(2)}S_{\mathcal{O}} = -\frac{\pi^{3/2}}{4}\gamma^2\langle\mathcal{O}\rangle^2 R^{2\Delta}\Omega_{d-2}\frac{\Delta\Gamma(\Delta)}{\Gamma(\Delta+\frac{3}{2})}\,. \tag{3.50}$$

It is actually possible to carry out the constant scalar field calculation in d dimensions [1]. In this case, the formula generalizes to

$$\delta^{(2)}S_{\mathcal{O}} = -\frac{\pi\ell_{AdS}^{d-1}}{4}\gamma^2\langle\mathcal{O}\rangle^2 R^{2\Delta}\Omega_{d-2}\frac{\Delta\Gamma(\frac{d}{2}-\frac{1}{2})\Gamma(\Delta-\frac{d}{2}+1)}{\Gamma(\Delta+\frac{3}{2})}\,. \tag{3.51}$$

This result agrees with previous calculations in [56, 57].

3.3 Summary

This Chapter has covered a lot of material. Firstly, we derived the most general perturbation to AdS spacetime from the CFT requirements on $T_{\mu\nu}$. The result agreed with the expectation that the two-dimensional CFT dual factorizes into holomorphic and antiholomorphic parts.

We then derived the canonical energy for such a spacetime with the main result in (3.23). This result reduces to the BTZ black hole case discussed in Chapter 2 as well as [50]. Furthermore, we proved the positivity directly, by using some powerful theorems from linear algebra.

Lastly, we found the contrition to the entanglement entropy from the addition of an operator $\mathcal{O}$ in the CFT. Our calculation makes use of the matter contribution canonical energy via (3.37). Although the formal expression is almost unwieldy, in the simple case of constant $\langle\mathcal{O}\rangle$, the expression reduces nicely to (3.51).

Overall, these methods contribute to the understanding how to reconstruct bulk geometry from the entanglement structure of a CFT. In particular, we find that (to second order), the pure gravitational components may be written as the product of CFT one-point functions. This remains possible for scalar perturbations, albeit more complicated.

Chapter 4

Emergent de Sitter spacetime from entanglement

In a recent paper by de Boer et al. [24] it was recognized that, to first order, the entanglement entropy of a spatial ball-shaped region obeys the Klein-Gordon equation in de Sitter (dS) spacetime. In this construction, the size of the ball directly determines how far into the future/past a field propagates. Causality in the bulk dS is then equivalent to an ordering of the spheres based on their size. This result holds for an arbitrary number of dimensions and is independent of the standard AdS/CFT correspondence.

In this Chapter, we first review the results of [24] for the first order entanglement entropy. We then show how the second order entanglement entropy calculated from the canonical energy formalism is related to dynamical fields in dS. We find that the total entanglement entropy looks like a scalar field theory with a cubic interaction in dS spacetime. Although the original work in [24] was independent of the number of dimension, our result is only for $d = 2$.

4.1 Emergent de Sitter dynamics from entanglement

In this section we review the argument of de Boer et al. [24] regarding the entanglement structure in a CFT and it's implications for dynamics in de Sitter spacetime.

In the case of a flat d-dimensional CFT the vacuum subtracted entanglement entropy of a spherical region B, which is centred at x, with radius R, is

$$\delta^{(1)}S(R,x) \quad = \quad 2\pi \int_B d^{d-1}x' \, \frac{R^2 - |x-x'|^2}{2R} \langle T_{tt}(x') \rangle \qquad (4.1)$$

where $\delta^{(1)}S$ denotes the entanglement entropy to first order in T_{tt}, the energy density operator. As discussed in Chapter 2 this comes from the first law of entanglement $\delta^{(1)}S = \delta^{(1)}\langle H_B\rangle$.

It was noticed by the authors of [24] that the integration kernel in (4.1) is the bulk-to-boundary propagator in d-dimensional de Sitter spacetime with the metric

$$ds^2 = \frac{L^2}{R^2}(-dR^2 + dx^2). \tag{4.2}$$

The scale L is a free parameter in this theory.

The entropy $\delta^{(1)}S(R,x)$ may be interpreted as a solution to the free scalar wave equation in dS spacetime

$$(\nabla^2_{\mathrm{dS}} - m^2)\delta^{(1)}S(R,x) = 0\,, \tag{4.3}$$

with the mass parameter set by $m^2 = -d/L^2$. Notice the mass is tachyonic. Fortunately, we will show that the boundary conditions imposed will remove the unstable modes associated with the tachyon.

The CFT can be interpreted as living on either future or past infinity $\mathcal{I}^{\pm}$. In this case, we will take $\mathcal{I}^{+}$ so that a future direction light cone of a point (R,x) in dS intersects $\mathcal{I}^{+}$on the region with entanglement entropy $\delta^{(1)}(R,x)$. The diagrammatic interpretation is illustrated in figure 4.1.

The general solution to (4.3) has two independent asymptotic solutions,

$$\lim_{R\to 0}\delta^{(1)}S(R,x) = \frac{F(x)}{R} + R^d f(x)\,. \tag{4.4}$$

Through the holographic dictionary, the first term corresponds to a state with conformal weight $\Delta = -1$, and the second to a weight $\Delta = d$. Requiring the solution to be regular at $R = 0$, we have

$$F(x) = 0, \qquad f(x) = \frac{\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+3}{2})}\langle T_{tt}(x)\rangle\,. \tag{4.5}$$

The absence of the $F(x)$ precisely corresponds to removing the unstable (non-normalizable) modes of the tachyon [24].

It can be interpreted that the energy density specifies the entanglement entropy at small scales ($R\to 0$), while at larger scales the entanglement entropy is determined through the Lorentzian propagation into the bulk de Sitter geometry. In other words, the entanglement entropy for small regions is determined by (4.1) and inputting this as an initial condition, the entanglement entropy for large regions is determined by (4.3). In contrast

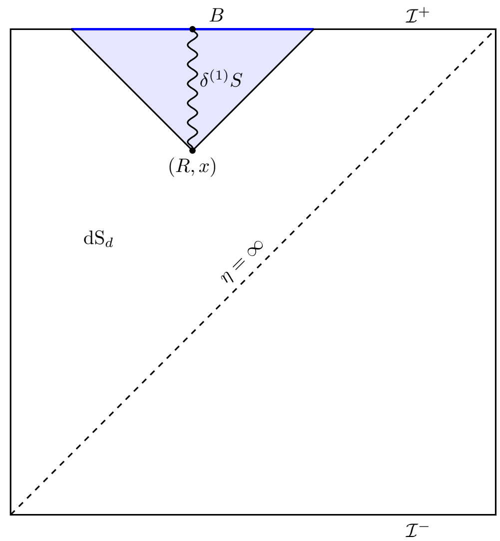


Figure 4.1: Penrose diagram for dS_d spacetime. The entangling region B is in blue with the light cone reaching out to the unique bulk point (R, x_i). The wavy line represents the field $\delta^{(1)}S$ propagating from the bulk to the boundary.

to the usual route in holography of computing holographic entanglement entropies, this method demonstrates that entanglement entropies themselves gives hints at constructing dS spacetime holographically.

In this picture, the radius of the ball in the CFT is the time coordinate in dS. There exists a specific map between the time in de Sitter spacetime and the size of the ball in a CFT. A ball that is contained in another ball is considered timelike separated, while two disjoint balls are spacelike separated as shown in figure 4.2. This gives a natural ordering to ball-shaped regions as was suggested in [58].

4.2 Interactions in de Sitter

The view presented above is remarkably simple and begs the question; what do higher order terms of the entanglement entropy terms correspond to in de Sitter? Since in Chapter 3 we computed the second order correction to the entanglement entropy in a 2d CFT, we can at least answer this question for $\delta^{(2)}S$ and $d = 2$.

Recall from Section 2.2 that the second order entanglement entropy is

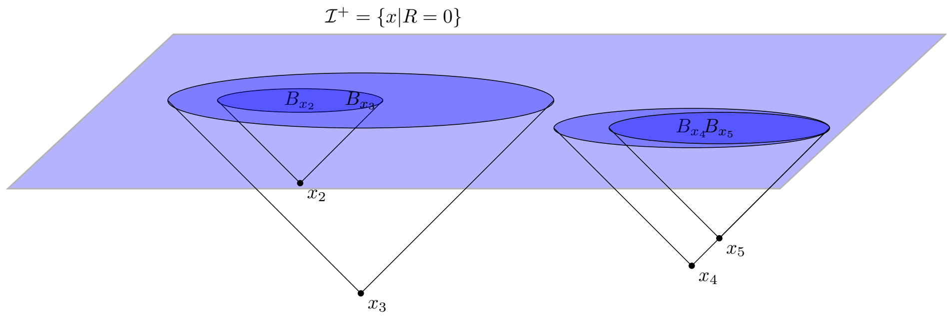


Figure 4.2: The size of the entangling region determines the bulk depth in the dS geometry. There is a one-to-one mapping between bulk points and spherical regions on the asymptotic future boundary $\mathcal{I}^+$. The light cone from a bulk point reaches the boundary on a spherical region.

given by the sum of two terms

$$\delta^{(2)}S(R,x) = \delta^{(2)}S_+ + \delta^{(2)}S_- \,, \tag{4.6}$$

where we suppress the (R,x) dependence of $\delta^{(2)}S_\pm$ for convenience. Acting with the Klein-Gordon equation for de Sitter spacetime on $\delta^{(2)}S_\pm$ gives

$$(\nabla^2_{\mathrm{dS}} - m^2)\delta^{(2)}S_\pm \;=\; \int dx_1 \int dx_2 \, (\nabla^2_{\mathrm{dS}} - m^2)K^{(2)}(R,x;x_1,x_2)\langle T_{\pm\pm}(x_1)\rangle\langle T_{\pm\pm}(x_2)\rangle \,.$$

Since only $K^{(2)}$ depends on x, we are only concerned with the action of the wave equation of $K^{(2)}$. Evaluating this explicitly gives

$$(\nabla^2_{\mathrm{dS}} - m^2)K^{(2)}(R,x;x_1;x_2) = \frac{12}{cL^2}K^{(1)}(R,x;x_1)K^{(1)}(R,x;x_2) \tag{4.7}$$

where $K^{(1)}(R,x;x_1) = 2\pi\frac{R^2-|x-x_1|^2}{2R}$ as per (4.1) and $K^{(2)}$ is given by (3.24). Reverting back to entropies, the equation of motion reads

$$(\nabla^2_{\mathrm{dS}} - m^2)\delta^{(2)}S_\pm(R,x) = \frac{12}{cL^2}\left(\delta^{(1)}S_\pm(R,x)\right)^2 \,. \tag{4.8}$$

Remarkably, the second order term has factored into an interaction term between two bulk-to-boundary propagators. If the first order term $\delta^{(1)}S$ corresponds to a free scalar field in de Sitter spacetime, then the second order term $\delta^{(2)}S_\pm$ corresponds to adding an cubic interaction for this field.

Equation (4.8) can be derived from the Lagrangian

$$\mathcal{L}_\pm = \frac{1}{2}(\nabla_a\delta S_\pm)^2 + \frac{1}{2}m^2\delta S_\pm^2 + \frac{4}{cL^2}(\delta S_\pm)^3 \tag{4.9}$$

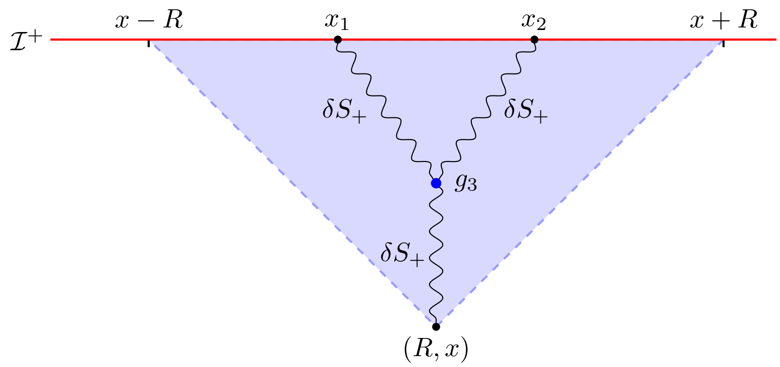


Figure 4.3: A visual interpretation of (4.7) as a Feynman diagram. The $\delta S_\pm$ field propagates from a point on asymptotic past infinity ($\mathcal{I}^-$), interacts with another $\delta S_\pm$ with a vertex given by $g_3 = \frac{12}{cL^2}$ and produces another $\delta S_\pm$ field which reaches the bulk point (R, x).

where the $\pm$ fields are independent and the total scalar field in dS is $\delta S_\pm = \delta^{(1)} S_\pm + \delta^{(2)} S_\pm + \mathcal{O}\left(\langle T \rangle^3\right)$. In this way the cubic interaction term in the Lagrangian precisely produces the diagram in figure 4.3. It is interesting that static entanglement entropy of regions produce dynamics (a classical field theory) in dS spacetime.

The past R evolution of $\delta S_\pm$ can be described by a past-directed dS Green's function

$$\delta S_\pm(R, x) = \int dx' \, G^{(1)}_{\mathrm{dS}}(R, x; x') \langle T_{\pm\pm}(x') \rangle \, . \tag{4.10}$$

This function propagates the field from a location on future infinity $\mathcal{I}^+$ to a point inside the bulk dS space. Likewise, the full bulk-to-bulk Green's function is required for the second order contribution

$$\delta^{(2)} S_\pm(R, x) = \frac{12}{c} \int_{\mathrm{dS}} dR' dx' \, \sqrt{|g_{\mathrm{dS}}|} G_{\mathrm{dS}}(R, x; R', x') \left(\delta S_\pm(R', x')\right)^2 \, . \tag{4.11}$$

The bulk-to-boundary function $G^{(1)}_{\mathrm{dS}}(R, x_0; x)$ can be obtained by taking the $R \to 0$ limit of the bulk-to-bulk propagator

$$G_{\mathrm{dS}}(R, x; R', x') = -4\pi \frac{R^2 + R'^2 - (x - x')^2}{4RR'} \tag{4.12}$$

so that

$$G^{(1)}_{\mathrm{dS}}(R, x; x') = 2\pi \frac{R^2 - (x - x')^2}{2R}, \tag{4.13}$$

which matches (4.1) from the first law of entanglement.

These propagators are defined to be non-zero only within the past directed light-cone, which is important in reproducing both the support and the exact form of $K^{(2)}(x_1, x_2)$ from (3.24).

Positivity check

The emergent de Sitter realization provides a very quick way to check the positivity of Fisher information. By directly inserting the propagator (4.12) into expression (4.11), we have

$$\delta^{(2)} S_{\pm}(R, x_0) = -\frac{12}{cL^2} \int_{\mathrm{dS}} dR' dx' \sqrt{|g_{\mathrm{dS}}|} \frac{R^2 + R'^2 - (x - x')^2}{RR'} \left(\delta S_{\pm}(R', x')\right)^2 . \tag{4.14}$$

The determinant of the metric is positive, along with the square of $\delta S_{\pm}$. The middle term is also positive for $(x' - x)^2 \leq (R - R')^2$ which lies completely in the range of integration. Thus it is evident that $\delta^{(2)} S_{\pm}$ is strictly non-negative as required by the positivity of relative entropy.

Thermal state

The analysis thus far has assumed the entropy is that of a state perturbed from the vacuum. Using the CFT method in [1], it is straightforward to generalize to the entropy of a state perturbed from a *thermal* state. The result is given by the propagation of a scalar field in a spacetime with the metric

$$ds^2_{\mathrm{thermal}} = \frac{4\pi^2 L_{dS}^2}{\beta \sinh^2\left(\frac{2\pi R}{\beta}\right)} \left(-dR^2 + dx^2\right) . \tag{4.15}$$

The bulk-to-boundary propagator in this spacetime is

$$K_{\beta}^{(1)}(R, x; x') = \frac{2\beta}{\sinh\left(\frac{2\pi R}{\beta}\right)} \sinh\left(\frac{\pi\left(R - x + x'\right)}{\beta}\right) \sinh\left(\frac{\pi\left(R + x - x'\right)}{\beta}\right) ,$$

with the entanglement entropy

$$\delta^{(1)} S_{\beta}(R, x) = 2\pi \int dx' \, K_{\beta}^{(1)}(R, x; x') \langle T_{tt}(x') \rangle . \tag{4.16}$$

The kernel for the second order entanglement entropy, or the bulk-to-bulk propagator, is

$$
\begin{aligned}
K_\beta^{(2)}(R,x;x_1,x_2) &= \frac{24\beta^2}{c\sinh^2\left(\frac{2\pi R}{\beta}\right)} \\
&\quad \times \begin{cases} \sinh^2\left(\frac{\pi(R-x_1+x)}{\beta}\right)\sinh^2\left(\frac{\pi(R+x_2-x')}{\beta}\right)\,, & x_2 \le x_1 \\ \sinh^2\left(\frac{\pi(R+x_1-x')}{\beta}\right)\sinh^2\left(\frac{\pi(R-x+x')}{\beta}\right)\,, & x_1 \le x_2 \\ 0\,, & |x_i| > R \end{cases} .
\end{aligned}
$$

From the entropy, we see that states perturbed from a background state (thermal or vacuum) correspond to a dynamical scalar field in a background spacetime. Whether this spacetime could be dynamical itself is an interesting open question.

4.3 Summary

In this Chapter we have reviewed the previous work of [24] which relates the entanglement entropy of a CFT to tachyonic fields in an auxiliary de Sitter spacetime. Using the results from Chapter 3 we extended this interpretation to include interacting fields in de Sitter. This construction provides a simple way to confirm the positivity of the canonical energy in Section 3.1.4. Additionally, we confirmed the result also holds for states perturbed from a thermal background.

Chapter 5

Conclusion

Firstly, let us comment on the canonical energy procedure from Chapter 3. While the Ryu-Takayangi formula gives a simple means to compute the exact entanglement entropy of a CFT, due to the difficulty of a direct CFT computation, it is impossible to verify in complex systems. In general, the first order contribution to the Ryu-Takayanagi entropy is given by the first law of entanglement, a feature that is universal to all CFTs. At what order does the Ryu-Takayangi formula differentiate holographic theories from other theories?

In this thesis we have suggested that the second order contribution might be appropriate to answer such a question. Unfortunately, in the case studied in this thesis, the result remained universal for all two-dimensional CFTs, and hence did not restrict the number of possible holographic theories. We did find however, that this contribution manifests itself as a cubic interaction term within the emergent de Sitter formalism of [24]. This is an exciting result that may yield insight into the structure of holographic theories and how more realistic models of our universe (such as de Sitter spacetime) may emerge.

There are many possible future directions for the line of inquiry pursued in Chapter 4 as summarized in figure 5.1. Firstly, it is possible to extend this work to arbitrary order in $\langle T \rangle$ which has recently been done in [59]. In this case, the cubic potential in (4.9) is replaced with an exponential of δS. It is doubtful that this expansion could be attained in higher dimensions since the result depends on the infinite Virasoro symmetry in 2d CFTs. It is also doubtful that these results could be derived from the gravitational approach since it would require new quantities beyond the canonical energy.

Another reason the de Sitter interpration might be unique to $d = 2$ CFTs is the fact that the conformal group in two-dimensions factorizes as $SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. It is clear from this why we obtained two copies of de Sitter space, one for each of the left- and right-moving components. Such a factorization does not hold in higher dimensions, so the emergent spacetime might be radically more complicated.

We could also imagine adding additional operators $\mathcal{O}_i$ in the CFT to

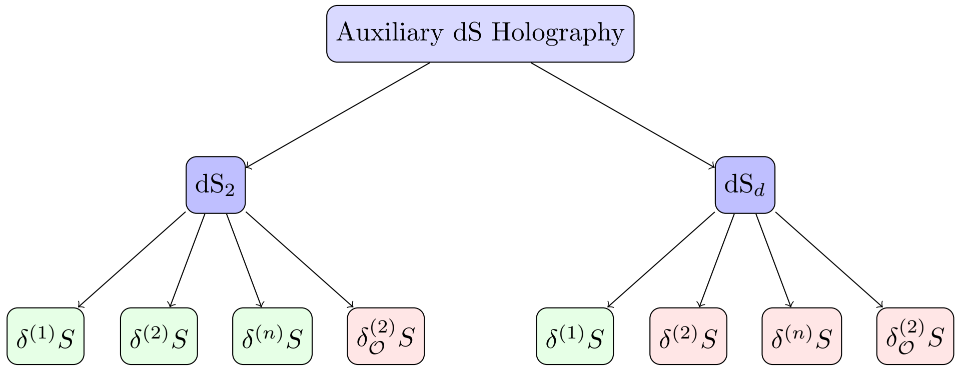


Figure 5.1: A summary of progress and future directions for the auxiliary de Sitter approach. Green boxes indicate solved problems, and red boxes indicate present challenges.

see what they correspond to within the auxiliary dS space. However, unlike the contribution from the stress tensor, the entropy from these operators involves integration of one-point functions over the full domain of dependence D_B. Rather than the space of entanglement entropies of spheres, one might consider the space of causal diamonds, to adequately generalize the auxiliary de Sitter interpretation.

It would also be interesting to see how this prescription works in higher dimensions. The first order term is already known to hold for arbitrary dimensions. Unfortunately, pure gravity in higher dimensions faces the same problem as additional operators in the CFT. The entanglement entropy depends on the entire causal region D_B, requiring a more advanced version of de Sitter holography.

Since the future boundary of the auxiliary de Sitter space does not include the time direction of the CFT, any extension of these results to dynamical entropies will also be problematic. It may be possible to circumvent this problem by using the canonical energy calculation in higher dimensions. This would involve solving coupled partial differential equations, which might not admit an analytic solution.

The work presented in this thesis has opened many new directions for future research exploring the interplay between quantum information and holography. With the insight gained from these calculations, further studies may undertake a more ambitious understanding of high-dimensional AdS spacetime and the role of de Sitter spacetime in understanding the structure of holographic theories.

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