The inertia operator and Hall algebra of algebraic stacks

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Abstract

We view the inertia construction of algebraic stacks as an operator on the Grothendieck groups of various categories of algebraic stacks. We are interested in showing that the inertia operator is (locally finite and) diagonalizable over for instance the field of rational functions of the motivic class of the affine line $q = [\mathbb{A}^1]$. This is proved for the Grothendieck group of Deligne-Mumford stacks and the category of quasi-split Artin stacks.

Motivated by the quasi-splitness condition we then develop a theory of linear algebraic stacks and algebroids, and define a space of stack functions over a linear algebraic stack. We prove diagonalization of the semisimple inertia for the space of stack functions. A different family of operators is then defined that are closely related to the semisimple inertia. These operators are diagonalizable on the Grothendieck ring itself (i.e. without inverting polynomials in q) and their corresponding eigenvalue decompositions are used to define a graded structure on the Grothendieck ring.

We then define the structure of a Hall algebra on the space of stack functions. The commutative and non-commutative products of the Hall algebra respect the graded structure defined above. Moreover, the two multiplications coincide on the associated graded algebra.

This result provides a geometric way of defining a Lie subalgebra of virtually indecomposables. Finally, for any algebroid, an ε-element is defined and shown to be contained in the space of virtually indecomposables. This is a new approach to the theory of generalized Donaldson-Thomas invariants.

Preface

This dissertation is the original work of the author, P. Ronagh, in collaboration with his PhD supervisor, Prof. Kai Behrend.

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Acknowledgments

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Chapter 1

Introduction

Donaldson-Thomas invariants of a Calabi-Yau 3-fold M should count the number of semistable coherent sheaves on it. They are mathematically interesting because they are invariant under continuous deformations of M and physically interesting because they count the BPS states of D-branes systems as predicted by string theory.

The conventional Donaldson-Thomas invariants [5, 49] are defined only when the semistable and stable coherent sheaves coincide. The generalized theories of Donaldson-Thomas invariants [29, 33] define these invariants without the mentioned restriction.

In [12], Bridgeland proposes a more conceptual and easy to understand approach to Joyce's motivic Hall Algebras [27] and in [11], he shows how the machinery can be applied to produce results on Donaldson-Thomas invariants of Calabi-Yau varieties.

In our correspondence with Bridgeland, he suggests the idea of viewing the construction of the inertia stack as an operator on a Grothendieck group of algebraic stacks and replace Joyce's virtual projections [27] with eigenprojections of this operator. Our goal is to reproduce Joyce's project [24–29] of generalized Donaldson-Thomas invariants in a more geometric framework.

1.1 Overview of the Donaldson-Thomas invariants

In this section, we will be working over the base field $\mathbb{C}$. Throughout, M will denote a fixed smooth complex projective Calabi-Yau threefold and by this we mean that the canonical bundle K_M is trivial and $H^1(M, \mathcal{O}_M) = 0$. $\mathsf{Coh(M)}$ will denote the abelian category of coherent sheaves on M and $\mathfrak{M}$ the moduli stack of objects of $\mathsf{Coh(M)}$. This means that the objects of $\mathfrak{M}$ over a scheme S are coherent sheaves on $S \times M$ that are flat over S. We will use the same notation for a scheme (if any) and the stack it represents.

Grothendieck group of the moduli

Let $K(\mathsf{Coh(M)})$ be the Grothendieck group of the category $\mathsf{Coh(M)}$. The bilinear form

$$\chi(E,F) = \sum_i (-1)^i \dim_{\mathbb{C}} \operatorname{Ext}^i(E,F)$$

is called the Euler form on $K(\mathsf{Coh(M)})$. By Serre duality, the sets of left and right orthogonal objects to $\mathsf{Coh(M)}$ with respect to the Euler form, i.e.

$$\{E : \chi(E,F) = 0 \text{ for all objects } F \text{ in } \mathsf{Coh(M)}\}$$

and

$$\{F : \chi(E,F) = 0 \text{ for all objects } E \text{ in } \mathsf{Coh(M)}\}$$

are the same subgroup $K(\mathsf{Coh(M)})^{\perp}$ and therefore the quotient

$$N(M) = K(\mathsf{Coh(M)})/\mathsf{K}(\mathsf{Coh(M)})^{\perp}$$

called the *numerical Grothendieck group* carries a well-defined non-degenerate bilinear form. There is a monoid $\Gamma \subset N(M)$ consisting of classes of sheaves. Fixing a class $\gamma \in \Gamma$ yields an open and closed substack $\mathfrak{M}^{\gamma} \subseteq \mathfrak{M}$ of objects of class γ.

Stability condition

Here we will fix our notion of stability condition to be that of Gieseker, although the theory should presumably work in more generality. Let $\mathcal{O}_M(1)$ be a fixed very ample line bundle on M. For a family $E \in \mathsf{Coh(M)}$ of coherent sheaves, the Hilbert polynomial P_E is the unique polynomial in $\mathbb{Q}[t]$ such that

$$P_E(n) = \dim H^0(E(n)) \quad \text{for all} \quad n \gg 0.$$

This polynomial only depends on the class of E in Γ. Thus we may write P_γ for $\gamma \in \Gamma$. We define $\tau(\gamma) = P_\gamma / r_\gamma$ where r_γ is the leading coefficient of P_γ. This means that τ associates to any class in Γ a monic polynomial of degree at most 3. If $p(t)$ and $p'(t)$ are two such polynomials we say

$$p \leq p', \quad \text{if} \quad \deg p > \deg p' \quad \text{and in case of equality} \quad p(t) \leq p'(t) \text{ for all } t \gg 0.[1]$$

This defines a notion of stability; E is stable if for all nonzero subobjects $S \subseteq E$, we have $\tau([S]) < \tau([E/S])$ and it is semistable if for all nonzero subobjects $S \subseteq E$, we have $\tau([S]) \leq \tau([E/S])$.

[1]The first condition on degrees serves to assure all semistable sheaves, E, are pure of dimension $\dim(\operatorname{supp} E)$.

The category $\mathsf{Coh}(\mathsf{M})$ equipped with this stability condition, has the properties one would expect from a stability condition (such as Harder-Narasimhan filtrations, Jordan-Holder filtrations, etc.)[23]. We write $\mathrm{SS}(\tau)$ for the open substack of $\mathfrak{M}$ consisting of semistable objects for τ. This itself consists of connected components $\mathrm{SS}^{\gamma}(\tau)$ of semistable objects of class γ. The latter stacks are all of finite type.

Conventional Donaldson-Thomas invariants

For a proper Deligne-Mumford stack X over $\mathbb{C}$, with a symmetric obstruction theory and virtual fundamental class $[X]^{vir} \in A_0(X)$, the *virtual count* of points on X is defined as the rational number

$$\#^{vir} = \int_{[X]^{vir}} 1 \in \mathbb{Q}.$$

On the other hand, by Behrend's theorem [6] this virtual count coincides with the Euler characteristic of X, weighted by the Behrend function ν_X of X:

$$\int_{[X]^{vir}} 1 = \chi(X, \nu_X).$$

Here is a few remarks on the concepts that appear in the above definition.

- The virtual fundamental class: Behrend and Fantechi [7], show that for any Deligne-Mumford stack with a perfect obstruction theory, the virtual class $[X]^{vir}$ is a well-defined class in $A_{\dim(X)}(X)$.

- The virtual count: If moreover, X is proper, the push-forward of this class over a point is well-defined and produces a rational number, $\int_{[X]^{vir}} 1$. This number is an integer if X is a scheme or an algebraic space.

  Let $X \to C$ be a family X_t of smooth projective varieties parametrized by $t \in C$, and consider the family of moduli spaces $\mathcal{M}_t$ of stable sheaves on the fiber X_t then the class $[\mathcal{M}_t]^{vir}$ is independent of t (provided that the *compact* moduli spaces and virtual fundamental classes we are talking about exist. This is for example the case if X_t are smooth projective 3-folds and $\mathcal{M}_t$ are moduli spaces of stable sheaves of fixed class in the numerical Grothendieck group and no strictly semistable object occurs [49, Cor. 3.53].

- Behrend function: This is a $\mathbb{Z}$-valued locally constructible functions are defined over schemes (as done by Behrend [6]), they can be generalized to all algebraic space, and algebraic stacks over $\mathbb{C}$, locally of finite type according to the property that for any smooth morphism $\varphi : W \to \mathfrak{X}$ of relative dimension n, we have $\varphi^*(\nu_{\mathfrak{X}}) = (-1)^n \nu_W$.

For Calabi-Yau 3-folds, Thomas [49] constructs a symmetric (in particular perfect) obstruction theory on $\mathrm{St}^{\mathcal{Y}}(\tau)$. If $\mathrm{SS}^{\mathcal{Y}}(\tau) = \mathrm{St}^{\mathcal{Y}}(\tau)$ then $\mathrm{St}^{\mathcal{Y}}(\tau)$ is proper as well which is the *conventional* case in which Donaldson-Thomas invariants are defined:

$$\mathrm{DT}^{\mathcal{Y}}(\tau) = \int_{[\mathrm{St}^{\mathcal{Y}}(\tau)]^{vir}} 1 = \chi(\mathrm{St}^{\mathcal{Y}}(\tau), \nu_{\mathrm{St}^{\mathcal{Y}}(\tau)}) = \sum_{n\in\mathbb{Z}} \chi({\nu^{-1}}_{\mathrm{St}^{\mathcal{Y}}(\tau)}(n)).$$

By the simplicity of the Behrend function, it is therefore suggestive to take the weighted Euler characteristic as the definition of Donaldson-Thomas invariants for moduli stacks. The problem with this naive approach is that (1) the Euler characteristic cannot be defined even for simplest Artin stacks such as $B_X\mathbb{G}_m$, and (2) the intersection theory tools that result deformation invariance are not in our disposal anymore. The goals of projects of Kontsevich and Soibelman [33], Joyce and Song [29] and that of ours is to find numbers $\mathrm{DT}^{\mathcal{Y}}(\tau) \in \mathbb{Q}$ that correctly count the semistable objects in $\mathrm{SS}^{\mathcal{Y}}(\tau)$ and are (1) invariant under deformations of M; and, (2) when $\mathrm{SS}^{\mathcal{Y}}(\tau) = \mathrm{St}^{\mathcal{Y}}(\tau)$ these numbers coincides with the conventional Donaldson-Thomas invariants above.

Dependence on Stability condition

Another caveat of these invariants is that they depend on the stability condition τ, which in turn depends on the choice of a very ample line bundle $\mathcal{O}_M(1)$. Explanation of how these numbers change subordinate to change of the stability condition is the content of the *wall-crossing* formulae in [33] and [29]. So another goal of ours would be to prove some wall-crossing formulae, giving $\mathrm{DT}^{\mathcal{Y}}(\tau)$ in terms of $\mathrm{DT}^{\mathcal{Y}}(\tau')$.

1.2 A break-down of our program

Local-finiteness and diagonalization of inertia operator

Let $\mathcal{S}$ be a base category of schemes. For the case of Deligne-Mumford stacks in §5 this would be the category of finite type schemes over a noetherian base scheme. For the cases of Artin stacks in §6 and §7 and all the rest of this work, $\mathcal{S}$ would be the category of varieties over the spectrum of an algebraically closed field in characteristic zero. The Grothendieck ring of $\mathcal{S}$, will be denoted by $K(\mathcal{S})$.

We then consider a category of algebraic stacks over $\mathcal{S}$. In §5 this category is that of the Deligne-Mumford stacks DM and in §6 it will be the category St of algebraic stacks with affine diagonals. In §7 we work with the category QS of quasi-split algebraic stacks and finally in §11 we consider the space of stack functions from algebroids to a fixed base linear stacks.

The Grothendieck ring of this category is defined as the commutative ring, K, generated by isomorphism classes of algebraic stacks modulo season relations. K is a unital associative $K(\mathcal{S})$-algebra in the obvious way. We will use the terms Grothendieck ring, Grothendieck module and Grothendieck algebra to stress the structure of our subject matter. For an object $\mathfrak{X}$ of K, the construction of the inertia stack $I\mathfrak{X}$ respects the equivalence and scissor relations and is $K(\mathscr{S})$-linear, and it hence induces a well-defined inertia operator on the $K(\mathscr{S})$-algebra K.

Other important variants of this operator are the semisimple and unipotent inertia operators, respectively I^{ss} and I^{u}, which are defined carefully in §3.4 but are roughly the semisimple and unipotent loci of $I\mathfrak{X}$ when viewed as a group object over $\mathfrak{X}$.

We implicitly always assume that K is tensored with $\mathbb{Q}$. Our main results on local finiteness and diagonalization of the endormophism $I : K \to K$ are as follows. Here $q = [\mathbb{A}^1]$ is our notation for the class of the affine line.

1. Corollary 5.11: In the case of Deligne-Mumford stacks the operator I is diagonalizable as a $K(\mathsf{Sch}/B)$-linear endomorphism, and the eigenvalue spectrum of it is equal to $\mathbb{N}$, the set of positive integers.

2. Corollary 6.19: In the case of Artin stacks, the unipotent inertia I^u is diagonalizable on $K(\mathsf{St})[q^{-1}, \{(q^k - 1)^{-1} : k \geq 1\}]$ and the eigenvalue spectrum of it is the set $\{q^k : k \geq 0\}$ of all power of q.

3. Theorem 7.10: In the case of quasi-split stacks, the operator I is diagonalizable as a $\mathbb{Q}(q)$-linear endomorphism and the eigenvalue spectrum of it is the set of all polynomials of the form
$$nq^u \prod_{i=1}^{k} (q^{r_i} - 1).$$

4. Theorem 7.14: In the case of quasi-split stacks the endormorphism I^{ss} is diagonalizable as a $\mathbb{Q}(q)$-linear operator and the eigenvalue spectrum of it is the set of all polynomials of the form
$$n \prod_{i=1}^{k} (q^{r_i} - 1).$$

5. Theorem 11.10: In the case of stacks functions of algebroids over a linear stack $\mathfrak{M}$, the operator $I^{ss} : K(\mathfrak{M})(q) \to K(\mathfrak{M})(q)$ is diagonalizable and the eigenvalue spectrum of it onsists of the set of all polynomials of the form
$$\prod_{i=1}^{k} (q^{r_i} - 1).$$

Eigenprojections and eigendecompositions

The fact that in Theorem 11.10, the semisimple inertia is diagonalizable only after inverting q is not convenient. However a technique that is represented in simpler cases in §4.3 and §5.4 for computing eigenprojections of the inertia of Deligne-Mumford stacks, hints to definition of a different set of operators, $\{E_n\}$, that are diagonalizable on $K(\mathfrak{M})$.

These operators are defined in terms of complete sets of mutually orthogonal idempotents in §11.3. They respect only a coarser filtration than the usual semisimple inertia (explained in §11.2) but they are simultaneously diagonalizable (Corollary 11.5). This creates an eigenvalue decomposition

$$K(\mathfrak{M}) = \bigoplus_{k\geq 0} K^k(\mathfrak{M})\,.$$

and a filtration by the order of vanishing of the inertia at $q = 1$,

$$K^{\leq n}(\mathfrak{M}) = \ker E_{n+1} = \bigoplus_{k\leq n} K^k(\mathfrak{M})\,,$$

also called the *order filtration*. Details are explained in §12.1.

Hall algebra

A noncommutative product denoted by $*$, is then defined in §12.1 on $K(\mathfrak{M})$ in the usual way of defining Hall algebras. The main result of §12 is that the commutative product (defined by the cartesian product of stacks) and the new convolution product respect the order filtration and on the associated graded they coincide (Theorem 12.3).

The Hall product is known to be associative (cf. [12]) and induces the structure of a Lie algebra on $K(\mathfrak{M})$. In particular $K_{\mathrm{ind}} = K^{\leq 1}(\mathfrak{M})$ is a Lie subalgebra of $K(\mathfrak{M})$ which is our counterpart for the space of virtually indecomposables of Joyce and Song [29].

In §12.2 we introduce the elements ε_k associated to a stack function and show in Corollary 12.9 that it lives in $K^{\leq k}(\mathfrak{M})$. In particular, $\varepsilon_1\,\mathrm{SS}^{\gamma}(\tau)$ is a virtually indecomposable, and can be viewed as a logarithm. This is the key point in defining invariants similar to that of [29], details of which are to be published by the author and K. Behrend.

Part I

Preliminaries

Chapter 2

Stratification of group schemes

Throughout, Sch/B, denotes the big étale site of schemes of finite type over a fixed noetherian base scheme B. St/B denotes the category of algebraic stacks over Sch/B with affine diagonals. This in particular means that for an S-point $s : S \to \mathfrak{X}$ of an algebraic stack $\mathfrak{X}$, the sheaf of automorphisms $\underline{\mathrm{Aut}}(s) \to S$ is an affine group scheme.

2.1 Stratification of group spaces

By a *group space* $G \to X$, we mean a group object in the category St/B. In this section we will see that we can always stratify such objects by nicely-behaved group schemes. The results of this section will be used in §6 and §7.

Let G be a finitely presented group scheme over a base scheme X. Recall that the *connected component of unity*, G^0, is a priori defined in [4, Exp. VI(B), Def. 3.1] as the subfunctor of G that assigns to any morphism $T \to X$, the set

$$G^0(T) = \{u \in G(T) : \forall x \in X, u_x(T_x) \subset G^0_x\}.$$

Here G^0_x is the connected component of unity of the algebraic group $G_x = G \otimes_X \kappa(x)$. By [4, Exp. VI(B), Thm. 3.10], if G is smooth over X, this functor is representable by a unique open subgroup scheme of G. Also note that in this case, G^0 is smooth and finitely presented and is preserved by base change [4, Exp. VI(B), Prop. 3.3]. The following lemma is essentially stated through [8, §5], but we will reframe it for further reference:

Lemma 2.1. *Let $G \to X$ be a smooth group space of finite type, and assume X is a noetherian scheme. Then X can be written as a disjoint union of a finite family $\{X_\alpha\}_{\alpha \in \mathsf{A}}$ of reduced, locally closed subschemes, such that for each $\alpha \in \mathsf{A}$, $G|_{X_\alpha} \to X_\alpha$ is a group scheme and the functor of connected component of the identity is representable.*

Remark 2.2. The above result holds without the smoothness condition if X is a scheme of finite type over a field of characteristic zero. This is a consequence of combing this result with generic smoothness theorem [50, Thm. 25.3.1] in characteristic zero.

When G is finitely presented and smooth, the quotient space G/G^0 exists as a finitely presented and étale algebraic space over X. For sheaf theoretic reasons, the formation of this quotient is also preserved by base change.

G^0 is not closed in general (interesting examples can be found in [40, §7.3 (iii)] and [4, Exp. XIX, §5]), however we have the following

Lemma 2.3. *Let $G \to X$ be a smooth group scheme and assume that $\overline{G} = G/G^0$ is finite and étale. Then G^0 is a closed subscheme of G.*

Proof. $\overline{G} \to X$ is finite, hence proper and consequently universally closed. Thus in the cartesian diagram

$$\begin{array}{ccc} G^0 \times_X G & \xrightarrow{\varphi} & G \times_X G \\ \downarrow & \square & \downarrow \\ \overline{G} & \longrightarrow & \overline{G} \times_X \overline{G} \end{array}$$

the morphism $\varphi : (h,g) \mapsto (hg,g)$ is a closed immersion. The property of being a closed immersion is local in the fppf topology and thus by the cartesian diagram

$$\begin{array}{ccc} G^0 \times_X G & \longrightarrow & G \times_X G \\ \downarrow & \square & \downarrow \\ G^0 & \longrightarrow & G \end{array}$$

the embedding of G^0 in G is also a closed immersion. The vertical right hand arrow is described by $(g,h) \mapsto gh^{-1}$. □

Corollary 2.4. *Let G be a smooth finitely presented group scheme over X. There exists a stratification of X by finitely many locally closed subschemes $X = \{X_\alpha\}_{\alpha \in \mathsf{A}}$ such that for all $\alpha \in \mathsf{A}$ and all group schemes $G_\alpha = G|_{X_\alpha}$, (1) G_α^0 is closed and (2) G_α/G_α^0 is finite and étale over X_α.*

Proof. Let G^0 be the connected component of identity. Then G/G^0 is a group object in the category of algebraic spaces over X and by Lemma 2.1 we may without loss of generality assume that G/G^0 is a group scheme over X. By [46, Lem. 03I1] we can further assume that each $(G/G^0)|_{X_\alpha}$ is finite over X_α. By the above remarks on the base change this means that with the notation $G_\alpha = G|_{X_\alpha}$, each $\overline{G_\alpha} = G_\alpha/G_\alpha^0$ is a finite étale group scheme over X_α. The assertion now follows from Lemma 2.3. □

For a group scheme $G \to X$ and a closed subscheme Y of it, the *functorial centralizer* $Z_G(Y)$ is defined as in [13, §2.2]. It is not generally true that this functor is representable by a scheme. However by what we have proved so far, we may derive the following

Corollary 2.5. *Let G be a smooth group scheme over an integral base scheme X. There exists a stratification of the base $X = \{X_\alpha\}_{\alpha \in \mathsf{A}}$ such that each restricted group scheme $G|_{X_\alpha}$ has a closed subscheme representing its centre.*

Before stating a proof, we recall a Galois theoretic fact about finite étale covers of schemes:

Remark 2.6. If $C \to X$ is a connected degree d étale cover and L/K is a separable extension of the residue field of the generic point of X, then $C_{X_L} \to X_L$ is the union of d degree one coverings. More generally let $\{C_i\}_{i=1,\dots,k}$ be all the connected components of C with corresponding generic points $\{\eta_i\}$ and residue fields $\{\kappa(\eta_i)\}$. Let L be a common separable closure of the latter. Then $X_L \to X$ is a finite étale covering and C_{X_L} is a union of connected components all of which are isomorphically mapped to X_L.

PROOF. By Corollary 2.4 we may assume that G^0 is a closed and open connected subscheme and $\overline{G} = G/G^0$ is a finite étale group scheme over X. We will find a finite étale cover $\widetilde{X} \to X$ such that $G|_{\widetilde{X}}$ has a scheme theoretic centre and then use affine descent.

By the above remark we may take a covering $\widetilde{X} \to X$ such that $\overline{G} \times_X \widetilde{X} \to \widetilde{X}$ is a union of connected components all of which are isomorphically mapped to $\widetilde{X}$. Moreover G is a torsor for a connected group over $\overline{G}$, and thus so is $G|_{\widetilde{X}}$ over $\overline{G}|_{\widetilde{X}}$.

$$\begin{array}{ccccccc}
G^0|_{\widetilde{X}} & \rightrightarrows & G|_{\widetilde{X}} & \longrightarrow & \overline{G}|_{\widetilde{X}} & \longrightarrow & \widetilde{X} \\
 & & \downarrow & & \downarrow & \square & \downarrow \\
G^0 & \rightrightarrows & G & \longrightarrow & \overline{G} & \longrightarrow & X
\end{array}$$

Therefore the connected components of the source and target correspond bijectively. But every connected component of $\overline{G}|_{\widetilde{X}}$ maps isomorphically to $\widetilde{X}$, thus each connected component of $G|_{\widetilde{X}}$ is isomorphic to G^0. Now by [13, Lem. 2.2.4], the centralizer of each connected component exists over $\widetilde{X}$ and their intersection is the centre of $G|_{\widetilde{X}}$. Finally $\widetilde{X} \to X$ is étale and in particular an fpqc covering and the centre of $G|_{\widetilde{X}}$ is affine over $\widetilde{X}$, so affine descent finishes the proof. □

2.2 Groups of multiplicative type

2.2.1 Quasi-split tori

A commutative group scheme $T \to X$ is said to be of multiplicative type if it is locally diagonalizable over X in the fppf topology (and therefore in the étale topology [4, Exp. X, Cor. 4.5]). For the general theory of group schemes of multiplicative type we refer the reader to [4, Ch. IIIV–X], however we recall a few preliminary facts here. Associated to T there exists [4, Exp. X, Prop. 1.1] a locally constant étale abelian sheaf

$$T \mapsto M = \underline{\mathrm{Hom}}_{X-\mathrm{gp}}(T, \mathbb{G}_m), \tag{2.1}$$

and T is the scheme representing the sheaf $\underline{\mathrm{Hom}}_{X-\mathrm{gp}}(M, \mathbb{G}_m)$. This is an anti-equivalence of the categories of X-group schemes of multiplicative type and locally constant étale sheaves on X whose geometric fibers are finitely generated abelian groups.

We will restrict ourselves to *isotrivial* group schemes, i.e. ones that trivialize by a finite étale cover of X (which can be assumed to be connected and Galois by [39, Proposition 6.18]). The reason this does not limit us is the following lemma:

Lemma 2.7. *Let T be a group of multiplicative type over an integral base scheme X. There exists an open subset $U \subseteq X$ such that after pulling back T_U along a finite étale morphisms $\tilde{U} \to U$, $T_{\tilde{U}}$ is isomorphic to $H_{\tilde{U}} \times \mathbb{G}^r_{m,\tilde{U}}$ where H is a finite commutative group.*

Proof. This immediately follows from the fact that every étale morphism is Zariski locally quasi-finite and every quasi-finite morphism is Zariski locally finite [46, Lem. 03I1]. □

So we may associate to T a Galois cover $X' \to X$ with group Γ such that $T_{X'}$ is isomorphic to $\underline{\mathrm{Hom}}_{X'-\mathrm{gp}}(M_{X'}, \mathbb{G}_{m,X'})$ where M is a finitely generated abelian group. The action of Γ on X' induces an action of it on $M_{X'}$.

An X-torus, T, is an X-group scheme which is fppf locally isomorphic to $\mathbb{G}^r_{m,X}$. This is equivalent to asking for M to be torsion-free. In this case we say T *splits* over X' if $T_{X'} = \mathbb{G}_{m,X'}$ and we denote $M_{X'}$ by χ_T. Our anti-equivalence of categories is now between isotrivial X-tori that split over X' and Γ-lattices (i.e. a finitely generated torsion free abelian groups equipped with the structure of some Γ-module of finite type) given by

$$T \mapsto H^0(X', \underline{\mathrm{Hom}}_{X'-\mathrm{gp}}(X' \times_X T, \mathbb{G}_m))$$

and by

$$A \mapsto \underline{\mathrm{Hom}}_{X-\mathrm{gp}}(X' \times A/\Gamma, T)$$

in the reserve direction where A is a Γ-lattice.

In view of 2.1, the Γ-module $M_{X'}$ is called the *character lattice* of T and also denoted by χ_T. Note that, if ρ is the Galois action of Γ on $T_{X'}$, the induced action on χ_T is via pre-composition: $\rho_\gamma(m) = m \circ \rho_\gamma$.

Definition 2.8. Let T be an isotrivial X-torus splitting on the Galois cover $X' \to X$ with Galois group Γ. T is called a *quasi-split* torus if χ_T is a permutation Γ-lattice (i.e. the action of Γ on χ_T is by permutation of the elements of a $\mathbb{Z}$-basis).

2.2.2 Maximal tori of group schemes

We recall [4, Exposé IXX, Def. 1.5] that the *reductive rank*, of algebraic k-group G, is the rank of a maximal torus T of $G_{\overline{k}}$ where $\overline{k}$ is the algebraic closure of k:

$$\rho_r(G) = \dim_{\overline{k}} T.$$

Likewise, the unipotent rank of G is the dimension of the unipotent radical U of $G_{\overline{k}}$ and denoted by

$$\rho_u(G) = \dim_{\overline{k}} U.$$

In this section we show that the structure theory of commutative algebraic groups extend to a non-empty Zariski open neighborhood of the generic point of X.

For an affine smooth group scheme G over a base scheme X, the above integers can be more generally considered as functions on X, that assign to every point $x \in X$, the corresponding

$$\rho_r(x) = \rho_r(G_x), \quad \text{and} \quad \rho_u(x) = \rho_u(G_x).$$

The function ρ_r is lower semi-continuous in the Zariski topology [4, Exposé XII, Thm. 1.7]. Moreover, the condition of ρ_r being a locally constant function in the Zariski topology, is equivalent to existence of a global maximal X-torus for G in the étale topology by [4, Exposé XII, Thm. 1.7]. If G is commutative, then this is furthermore equivalent to existence of a global maximal X-torus for G in the Zariski topology [4, Exposé XII, Cor. 1.15]. This immediately implies the following

Proposition 2.9. *Let G be an affine smooth group scheme over a noetherian base scheme X. Then there exists a stratification of X by finitely many locally closed Zariski subschemes $\{X_i\}$ such that each group scheme $G|_{X_i}$ admits an isotrivial maximal torus. If G is moreover commutative, $G|_{X_i}$ admits a maximal torus in Zariski topology.*

Proof. Since ρ_r is lower semi-continuous and integer valued there exists a stratification by locally closed subspaces on which ρ_r is constant. We can further refine such a stratification by Lemma 2.7 for the global tori to be trivial after finite étale base change. □

2.3 Spreading out arguments

We say a group scheme $H_\eta \to X$ (or a property of it with respect to another group scheme G_η) *spreads out* to a neighborhood of the generic point $\eta \in X$ if there exists a dense open subset $U \subset X$ over which there is a U-group scheme $H|_U$ pulling back to the prior one (and satisfying the same property with respect to a spreading out $G|_U$ of G_η).

Lemma 2.10. *Let X be an integral scheme and G a finitely presented affine group scheme over X. Then closed subgroups of the generic fiber spread out: i.e. let η be the generic point of X and H_η a closed subgroup of G_η. Then there exists a non-empty open set $U \subseteq X$ such that $G|_U$ contains a subgroup scheme H_U fitting in the commutative diagram*

$$\begin{array}{ccc} H_\eta & \longrightarrow & H_U \\ \downarrow & & \downarrow \\ G_\eta & \longrightarrow & G_U \end{array}$$

where the horizontal arrows are pull-back morphisms and the vertical arrows are monomorphisms of group schemes.

Proof. It suffices to consider the case where X is an affine scheme $X = \operatorname{Spec} R$. Let K be the function field of R with the canonical homomorphism $R \to K$ corresponding to the inclusion of the generic point $\eta \to X$. Then $G = \operatorname{Spec} S$, where $S = R[x_1,\dots,x_k]/\mathfrak{I}$ for some finitely generated ideal $\mathfrak{I} = \langle p_1,\dots,p_\ell\rangle \subset R[x_1,\dots,x_k]$ and therefore $G_\eta = \operatorname{Spec} K \otimes_R S$. With this notation, H_η is cut out as a subscheme by a finitely generated ideal $\mathfrak{p} \subseteq K \otimes_R S = K[x_1,\dots,x_k]/\mathfrak{I}K$. Thus each generator of $\mathfrak{p}$ can be considered as a polynomial with coefficients in K. Since K is the inverse limit of localizations of R in its elements, there exists $f \in R$ such that all elements of $\mathfrak{p}$ are defined with coefficients in R_f. This defines a subscheme H_U of G_U satisfying the commutativity of the above diagram, if we set $U = \operatorname{Spec} R_f$.

Now we put a group scheme structure on H_U by shrinking U further. Let $i : G \to G$ and $m : G \times_X G \to G$, respectively be the inversion and multiplication morphisms on G. Considering the inversion morphism, existence of group structure on H_η means that in level of coordinate rings, we are given a commutative diagram

$$\begin{array}{ccccc} R_f[x_1,\dots,x_k]/\mathfrak{I}R_f & \xrightarrow{i^\#} & R_f[x_1,\dots,x_k]/\mathfrak{I}R_f & & \\ \downarrow & & \downarrow & & \\ K[x_1,\dots,x_k]/\mathfrak{I}K & \xrightarrow{\widetilde{i^\#}} & K[x_1,\dots,x_k]/\mathfrak{I}K & \xrightarrow{q} & K[x_1,\dots,x_k]/\mathfrak{p}\mathfrak{I}K \end{array}$$

and the composition of the induced morphism $\widetilde{i^\#}$ and the quotient map q has precisely

$\mathfrak{p}\mathfrak{I}K$ as its kernel. Hence by a similar argument, there exists some $g \in R_f$ lifting this composition as in the cartesian diagram

$$\begin{array}{ccc} R_{fg}[x_1,\dots,x_k]/\mathfrak{p}\mathfrak{I}R_{fg} & \longrightarrow & R_{fg}[x_1,\dots,x_k]/\mathfrak{p}\mathfrak{I}R_{fg} \\ \downarrow & & \downarrow \\ K[x_1,\dots,x_k]/\mathfrak{p}\mathfrak{I}K & \longrightarrow & K[x_1,\dots,x_k]/\mathfrak{p}\mathfrak{I}K. \end{array}$$

The case of multiplication morphism is similar. So by shrinking further we may assume that H_U is a U-group scheme. Commutativity of the diagram

$$\begin{array}{ccc} H_U \times H_U & \longrightarrow & H_U \\ \downarrow & & \downarrow \\ G_U \times G_U & \longrightarrow & G_U \end{array}$$

where the horizontal arrows are morphisms $(x,y) \mapsto xy^{-1}$ is now obvious. Thus H_U is the desired subgroup scheme of G_U. □

Lemma 2.11. *Group homomorphisms (repectively isomorphisms) spread out. Let $G \to X$ and $\eta \in X$ be as in previous lemma. If $G' \to X$ is another group scheme and $\varphi_\eta : G_\eta \to G'_\eta$ is a group scheme homomorphism (resp. isomorphism), then there exists a non-empty $U \subseteq X$ and a homomorphism (resp. isomorphism) $\varphi : G'_U \to G_U$ such that $\varphi|_\eta = \varphi_\eta$.*

Proof. The proof is by similar arguments as in previous lemma. □

2.4 Tori and unipotent group schemes

Lemma 2.12. *The property of being a quasi-split torus spreads out. Let $G \to X$ and $\eta \in X$ be as in the previous lemmas. If G_η is a quasi-split torus, then there exists a non-empty $U \subseteq X$ such that G_U is a quasi-split torus.*

Proof. Since G_η is isotrivial, by Lemma 2.7 we may assume that G is also isotrivial. We recall that the character lattice of a torus is expressed in terms of the (étale locally constant) sheaf $\chi(G) = \underline{\mathrm{Hom}}_{U-\mathrm{gp}}(G, \mathbb{G}_{m,U})$. Restriction from U to η induces a homomorphism of finitely generated $\mathbb{Z}$-modules

$$\underline{\mathrm{Hom}}_{U-\mathrm{gp}}(G, \mathbb{G}_{m,U}) \to \underline{\mathrm{Hom}}_{K-\mathrm{gp}}(G_\eta, \mathbb{G}_{m,\eta})$$

and by Lemma 2.11 we may assume that this is an isomorphism of $\mathbb{Z}$-modules. We also note that if $\widetilde{U} \to U$ is a finite étale covering that trivializes G_U and restricts to the finite separable extension $L/\kappa(\eta)$, then $\Gamma = \mathrm{Gal}(L/K)$ is at the same time the fundamental group of this covering and the associated action of Γ on $\chi(G_\eta)$ induces same action of this group on $\chi(G)$. □

Now we analyze how unipotency behaves with respect to stratifications. We first clarify what we mean by a unipotent group scheme over a scheme X and then show that unipotency spreads out from the generic fiber to a Zariski open neighborhood.

Definition 2.13. An affine group scheme $Z \to X$ is said to be a unipotent group scheme if it is unipotent over each geometric fiber.

Lemma 2.14. *Let G be unipotent group scheme over an integral base scheme X. Then there exists a non-empty Zariski open set $U \subset X$ such that G_U has a filtration in subgroups $1 \subset G_1 \subset \ldots \subset G_{r-1} \subset G_r = G$ with all factors $G_\ell / G_{\ell-1}$ isomorphic to the constant U-group scheme $\mathbb{G}_{a,U}$.*

Proof. Let H_η be a subgroup of the generic fiber G_η. By Lemma 2.10 the property of being a subgroup spreads out to a non-empty open in X. We also observe that the property of being isomorphic to $\mathbb{G}_a$ spreads out. That is, if H_η is isomorphic to $\mathbb{G}_{a,\eta}$, then there exists a non-empty open $U \subseteq X$ such that H_η spreads out over it to the constant group scheme $\mathbb{G}_{a,U}$. This is another straightforward spreading out argument: let K be the field of fraction of an integral domain R. Let S be a finitely presented R-algebra generated by $x_1, \ldots, x_\ell$ and that there exists an R-algebra isomorphism $\varphi : K \otimes_R S \to K[t]$. It is easy to check that there exists a localization R_f of R that extends φ to an isomorphism $\tilde{\varphi} : R_f \otimes_R S \to R_f[t]$. The claim now follows by induction on the quotient scheme $G_U/\mathbb{G}_{a,U}$.[1] □

Corollary 2.15. *Let X be an integral scheme and G a finitely presented smooth affine commutative group scheme over X. Let η be the generic point of X. Then the decomposition of G_η as $T_\eta \times U_\eta$ to a maximal torus and the unipotent radical spreads out; i.e. there exists a non-empty Zariski open $V \subseteq X$ such that G_V is isomorphic to $T_V \times U_V$ where T_V is a maximal torus with T_η as generic fiber and U_V is a unipotent V-group scheme with U_η as generic fiber. Moreover, if T_η is quasi-split we may assume T_V is also quasi-split.*

Proof. This is now obvious by spreading the maximal torus of the generic fiber out by Proposition 2.9, and spreading the unipotent radical out by Lemma 2.14, and observing that the group structure of $T_\eta \times U_\eta$ also spreads out by Lemma 2.11. □

[1] A relevant note is that $\mathbb{A}_1$-fibrations are always Zariski locally trivial (cf. [30]).

Chapter 3

Inertia operator of K-groups

3.1 K-groups

Recall Sch/B, the big étale site of schemes of finite type over a fixed noetherian base scheme B. The Grothendieck ring of Sch/B, denoted by $K(\mathsf{Sch}/B)$ is the free abelian group of isomorphism classes of such schemes, modulo the scissor relations,

$$[X] = [Z] + [X \setminus Z], \text{ for } Z \subset X \text{ a closed subscheme,}$$

equipped with structure of a commutative unital ring according to the fiber product in Sch/B,

$$[X] \cdot [Y] = [X \times Y].$$

We will always tensor this ring with $\mathbb{Q}$. Hence throughout, any group, ring or algebra denoted as K is assumed to be a $\mathbb{Q}$-vector space.

The Grothendieck ring of the category St/B of algebraic stacks over Sch/B, is defined similar to above. As an abelian group, $K(\mathsf{St}/B)$ is generated by isomorphism classes of algebraic stacks modulo similar relations; i.e. for any closed immersion $\mathfrak{Z} \hookrightarrow \mathfrak{X}$ of algebraic stacks we have $[\mathfrak{X}] = [\mathfrak{Z}] + [\mathfrak{X} \setminus \mathfrak{Z}]$. And the fiber product over the base category turns $K(\mathsf{St}/B)$ into a commutative ring. Hence for any algebraic stack $\mathfrak{Y}$, isomorphic to a fiber product $\mathfrak{X} \times \mathfrak{Z}$, we have $[\mathfrak{Y}] = [\mathfrak{X}][\mathfrak{Z}]$.

Moreover, $K(\mathsf{St}/B)$ is a unital associative $K(\mathsf{Sch}/B)$-algebra in the obvious way. We will use the terms *Grothendieck ring*, *Grothendieck module* and *Grothendieck algebra* to stress the structure of our subject matter.

3.2 Inertia stacks

For any algebraic stack, $\mathfrak{X}$, the inertia stack $I\mathfrak{X}$ is the fiber product

$$\begin{array}{ccc} I\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & \square & \downarrow \Delta \\ \mathfrak{X} & \xrightarrow[\Delta]{} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

where Δ is the diagonal morphism. $I\mathfrak{X}$ is isomorphic to the stack of objects (x, f), where x is an object of $\mathfrak{X}$ and $f : x \to x$ is an automorphism of it. Here a morphism $h : (x, f) \to (y, g)$ is an arrow $h : x \to y$ in $\mathfrak{X}$ satisfying $g \circ h = h \circ f$.

This construction respects the equivalence and scissor relations and is $K(\mathsf{Sch}/B)$-linear, hence inducing a well-defined *inertia operator* on the $K(\mathsf{Sch}/B)$-algebra $K(\mathsf{St}/B)$, and in particular an *inertia endomorphism* on the $K(\mathsf{Sch}/B)$-module $K(\mathsf{St}/B)$.

We use the notation $I^{(k)}\mathfrak{X}$ for k-times application of the inertia construction on the stack $\mathfrak{X}$. We may think of the objects of $I^{(k)}\mathfrak{X}$ as tuples $(x, f_1, \cdots, f_k)$ of an object x in $\mathfrak{X}$ and pairwise commuting automorphisms $f_1, \cdots, f_k$. A morphism $(x, f_1, \cdots, f_k) \to (y, g_1, \cdots, g_k)$ is an arrow $h : x \to y$ of $\mathfrak{X}$ satisfying $h \circ f_i = g_i \circ h$ for all $i = 1, \cdots, k$.

3.3 Central band of a gerbe

We recall that to any algebraic stack $\mathfrak{X}$ we can associate an fppf coarse moduli sheaf X of isomorphism classes of objects of $\mathfrak{X}$ [34, Rmk. 3.19]. The morphism of stacks $\mathfrak{X} \to X$ is always an fppf (in particular, étale) gerbe.

Note that X is not generally represented by a scheme. For example, working over $\mathbb{C}$, consider the quotient stack $\mathfrak{D} = [\mathbb{A}^1/\mathbb{G}_m]$, of the affine line $\mathbb{A}^1$ by the natural action of the multiplicative group $\mathbb{G}_m$ on it. Over any scheme S, the objects of this stack are pairs $(L \to S, s)$ of a line bundle L on S with a section $s : S \to L$. This quotient stack plays a key role in logarithmic geometry and the coarse moduli sheaf of it is the classifying space of generalized cartier divisors (cf. [1] and [46, Tag 02T7]). We now show that $\mathfrak{D}$ does not admit a coarse moduli scheme. Suppose to the contrary that D is a coarse moduli scheme with the universal map $m : \mathfrak{D} \to D$. The quotient morphism $p : \mathbb{A}^1 \to \mathfrak{D}$ maps every morphism $f : S \to \mathbb{A}^1$, to the pair $(S \times \mathbb{A}^1, \mathrm{id} \times f)$. Restricting p to $\mathbb{G}_m \subset \mathbb{A}^1$, we the get a morphism $p : \mathbb{G}_m \to [\mathbb{G}_m/\mathbb{G}_m]$ where the latter quotient stack is a point. We conclude that $m \circ p : \mathbb{A}^1 \to D$ is a constant map from the affine line to a point. By universality of m, this map has to be surjective therefore $D = \operatorname{Spec} \mathbb{C}$. However, for D to be a moduli space, the isomorphism classes of $\mathbb{C}$-points of $\mathfrak{D}$ and D need to be in one-to-one bijection but $\mathfrak{D}(\mathbb{C})$ consists of two isomorphic classes of objects and this is a contradiction.

Other examples of quotient stacks which do not admit coarse moduli spaces are studied for instance in [2], [3] and [44]. We also refer the reader to [31] for a proof of existence of coarse moduli spaces for separated Deligne-Mumford stacks.

Proposition 3.1. *Let $\mathfrak{X} \to X$ be an étale gerbe. Then there exists a sheaf of abelian groups $Z \to X$ and a morphism of sheaves of groups $\varphi : Z \times_X \mathfrak{X} \to I\mathfrak{X}$ such that*

1. *For every $s : S \to \mathfrak{X}$, the induced morphism of sheaves of groups $s^*\varphi : Z|_S \to \underline{\mathrm{Aut}}(s)$ identifies $Z|_S$ with the centre of the sheaf of groups $\underline{\mathrm{Aut}}(s)$; and,*
2. *The pair (Z, φ) is unique, up to isomorphism of sheaves of groups over X.*

Proof. This is explained in [18, Ch. IV, §1.5], specifically refer to [18, Ch. IV, §1.5.3.2] for existence of the sheaf and to [18, Ch. IV, Cor. 1.5.5] for the properties of it. □

Definition 3.2. In the above setting, Z is called the *central band* associated to $\mathfrak{X}$ and if it is a scheme we call it the *central group scheme*. The *central inertia* of $\mathfrak{X}$ is defined to be the fiber product

$$\begin{array}{ccc} I^Z\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & \square & \downarrow \\ Z & \longrightarrow & X. \end{array}$$

Lemma 3.3. *The map $\varphi : I^Z\mathfrak{X} \to I\mathfrak{X}$ as in Proposition 3.1 is representable and identifies $I^Z\mathfrak{X}$ with a closed (resp. open and closed) substack of $I\mathfrak{X}$, if the following condition is satisfied: For any object $s \in \mathfrak{X}$, $\underline{\mathrm{Aut}}(s)$ is a group scheme and its functorial centralizer as defined in [13, §2.2] is representable by a closed (resp. open and closed) subscheme.*

Proof. For any scheme U, any U-point $u \in I\mathfrak{X}(U)$ is determined by a pair $(\underline{u}, \underline{\sigma})$ with $\underline{u} \in \mathfrak{X}(U)$ and $\underline{\sigma} \in \underline{\mathrm{Aut}}(\underline{u})$. Thus the S-points (for any scheme S) of the fiber product $U \times_{I\mathfrak{X}} I^Z\mathfrak{X}$ is given by objects

$$\langle f \in U(S), \underline{s} \in \mathfrak{X}(S), \underline{\tau} \in Z(\underline{\mathrm{Aut}}(\underline{s})), \iota : f^*\underline{u} \xrightarrow{\cong} \underline{s} \rangle \quad \text{such that} \quad \iota \circ f^*\underline{\sigma} = \underline{\tau} \circ \iota.$$

This stack is then 1-isomorphic to the stack of objects

$$f \in U(S), \quad \text{such that} \quad f^*\underline{\sigma} \in Z(\underline{\mathrm{Aut}}(f^*\underline{u})).$$

The group sheaf $\underline{\mathrm{Aut}}(\underline{u})$ is represented by an affine U-group scheme G and $\sigma = \underline{\sigma}(U) : U \to G$ is a section of the structure morphism. Thus the stack of the objects above, is represented by the fiber product $Z(G)\ {}_j{\times}_{G,\sigma} U$ where j is the closed immersion of the centre, $Z(G) \hookrightarrow G$. □

Discrete central inertia

Suppose we are in the case that $Z \to X$ is a group scheme and consider the open subgroup scheme Z^0 and the quotient algebraic space Z/Z^0 over X. Pulling back to $\mathfrak{X}$, we define the *connected component of the central inertia*, $I^{z,0}\mathfrak{X}$ as the sub-group space, and the *discrete central inertia* as the quotient group space, which are respectively given by the following fiber products

$$\begin{array}{ccc} I^{z,0}\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & \square & \downarrow \\ Z^0 & \longrightarrow & X \end{array} \quad \text{and} \quad \begin{array}{ccc} I^{z/z,0}\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & \square & \downarrow \\ Z/Z^0 & \longrightarrow & X. \end{array}$$

Here $Z/Z^0 \to X$ is called the *discrete central band* of $\mathfrak{X} \to X$.

3.4 The semisimple and unipotent inertia

Recall that if G is an affine group scheme of finite type on base scheme S, an element $g \in G(S)$ is defined to be semisimple if for all scheme points $s \in S$ given as spectrum of a field, g_s is semisimple in fiber G_s.

Definition 3.4. We define the *semisimple inertia* of an algebraic stack $\mathfrak{X}$ to be the strictly full subcategory $I^{ss}\mathfrak{X}$ of the inertia stack $I\mathfrak{X}$ consisting over a base S of those objects (x,φ) such that $\varphi \in \mathrm{Aut}(x)$ is a semisimple element of the S-group scheme $\mathrm{Aut}(x)$.

According to [34, 3.5.1] in order to check that $I^{ss}\mathfrak{X}$ is a substack of $I\mathfrak{X}$ we only need to observe that if $f: U \to S$ is an étale surjection and $\varphi \in \mathrm{Aut}(x)$ is an automorphism of an object x over S where $f^*\varphi \in \mathrm{Aut}(f^*x)$ is semisimple then f is also semisimple. But this follows from the above definition and the fact that being semisimple is preserved along field extensions. That is, given a group scheme $G \to S$, if g is a k-valued point of G_k, and K/k is an algebraically closed extension, and g' the K-valued point of G_K obtained by pullback, then g is semisimple if and only if g' is [4, Exposé XII].

Let $u: U \to \mathfrak{X}$ be an étale covering of $\mathfrak{X}$ by an algebraic space. We note that $\mathrm{Aut}^{ss}(u)$ may fail to be an algebraic space, however it is a locally constructible space by [4, Exposé XII, Proposition 8.1]. On the other hand, the diagonal of $I^{ss}\mathfrak{X}$ is easily seen to be representable, separated and quasi-compact. Therefore $I^{ss}\mathfrak{X}$ can be written as a well-defined element in $K(\mathsf{St})$ even though it is not necessarily an algebraic stack.

We use the notation $I^{ss,z}\mathfrak{X}$ for the locus $I^{ss}\mathfrak{X} \cap I^{z}\mathfrak{X}$. We may also write this is a fiber product

$$\begin{array}{ccc} I^{ss,z}\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ Z^{ss} & \longrightarrow & X \end{array}$$

which is a relative group scheme, since the set of semisimple elements Z^{ss} of the central band $Z \to X$ form a group scheme of multiplicative type. Equivalently,

$$\begin{array}{ccc} I^{ss,z}\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ Z/U & \longrightarrow & X \end{array}$$

where U is the unipotent radical of Z.

Likewise the *unipotent inertia* of $\mathfrak{X}$ is the strictly full subcategory $I^{u}\mathfrak{X}$ consisting of those objects (x,φ) such that φ is a unipotent element of $\mathrm{Aut}(x)$. It is easy to check that $I^{u}\mathfrak{X} \subset I\mathfrak{X}$ is a closed substack of the inertia.

Part II

Diagonalization of the inertia

Chapter 4

Groupoids

Let $K(\text{gpd})$ be the $\mathbb{Q}$-vector space generated by finite groupoids, modulo equivalence and scissor relations. It is easy to verify that the vector space $K(\text{gpd})$ is generated by $[BG]$, for finite groups G.

4.1 Filtration by central number

The vector space $K(\text{gpd})$ has two natural gradings, which will be important for us. First, there is the grading by size of the automorphism group, denoted by upper indices, and then there is the grading by size of the centre of the automorphism group, denoted by lower indices. Thus $[BG]$ is in $K_i^n(\text{gpd})$, if $\#G = n$, and $\#Z(G) = i$. We have

$$K(\text{gpd}) = \bigoplus_{n=1}^{\infty} \bigoplus_{i=1}^{\infty} K_i^n(\text{gpd}) .$$

Clearly, $K^n(\text{gpd})$ is finite-dimensional, for every n, but $K_i(\text{gpd})$ is infinite-dimensional, for all i. This grading defines an ascending filtration $K^{\leq n}(\text{gpd})$ and a descending filtration $K_{\geq i}(\text{gpd})$.

Denote by $I : K(\text{gpd}) \to K(\text{gpd})$ the endomorphism sending $[X]$ to $[IX]$, where IX is the inertia groupoid of X. Note that inertia is compatible with equivalence and scissor relations, so that I is well-defined.

Lemma 4.1. *The endomorphism I preserves the associated filtrations $K^{\leq n}(\text{gpd})$ and $K_{\geq i}(\text{gpd})$. Moreover, on the associated graded, $K_{\geq i}/K_{>i}(\text{gpd})$, the endomorphism I is multiplication by i.*

Proof. Recall that

$$I(BG) \cong \coprod_{g \in C(G)} BZ_G(g) ,$$

where $C(G)$ denotes the set of conjugacy classes of G, and $Z_G(g)$ is the centralizer of g in G. Thus,

$$\begin{aligned} I[BG] &= \sum_{g \in C(G)} [BZ_G(g)] \\ &= \#Z(G)[BG] + \sum_{g \in C(G)^*} [BZ_G(g)], \end{aligned}$$

where $C(G)^*$ denotes the set of non-central conjugacy classes. Now we note that for non-central g we have strict inequalities

$$\#Z(G) < \#Z_G(g) < \#G.$$

This is enought to prove the claim. □

4.2 Local finiteness and diagonalization

Proposition 4.2. *The endomorphism* $I : K(\text{gpd}) \to K(\text{gpd})$ *is diagonalizable, with spectrum of eigenvalues equal to the positive integers.*

Proof. Every subspace $K^{\leq n}(\text{gpd})$ is finite dimensional, and preserved by I. On this finite dimensional subspace, I is triangular, with respect to the lower grading, and with *different* eigenvalues on the diagonal. This proves that I is diagonalizable when restricted to $K^{\leq n}(\text{gpd})$ for all n. □

Thus $K(\text{gpd})$ has another natural grading, namely the grading induced by the direct sum decomposition into eigenspaces under I, also called the *grading by virtual size of centre.* Denote the corresponding projection operators by π_n.

Example 1. If A is a finite abelian group, then $[BA]$ is an eigenvector for I, with eigenvalue $\#A$. Thus $\pi_n[BA] = [BA]$, if A had n elements, and $\pi_n[BA] = 0$, otherwise.

Example 2. We have

$$I[BS_3] = [BS_3] + [B\mathbb{Z}_3] + [B\mathbb{Z}_2],$$

where we have commited the abuse of notation of writing $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$. From this, and the previous example, we can see that

$$[BS_3] - [B\mathbb{Z}_2] - \tfrac{1}{2}[B\mathbb{Z}_3]$$

is an eigenvector for I, with eigenvalue 1. Thus

$$\pi_n[BS_3] = \begin{cases} [BS_3] - [B\mathbb{Z}_2] - \frac{1}{2}[B\mathbb{Z}_3] & \text{if } n = 1 \\ [B\mathbb{Z}_2] & \text{if } n = 2 \\ \frac{1}{2}[B\mathbb{Z}_3] & \text{if } n = 3 \\ 0 & \text{otherwise}. \end{cases}$$

Example 3. For the dihedral group D_4 with eight elements, we have

$$I[BD_4] = 2[BD_4] + [B\mathbb{Z}_4] + 2[BD_2].$$

Hence

$$[BD_4] - \frac{1}{2}[B\mathbb{Z}_4] - [BD_2]$$

is an eigenvalue of I with eigenvalue 2. It follows that

$$\pi_n[BD_4] = \begin{cases} [BD_4] - \frac{1}{2}[B\mathbb{Z}_4] - [BD_2] & \text{if } n = 2 \\ \frac{1}{2}[B\mathbb{Z}_4] + [BD_2] & \text{if } n = 4 \\ 0 & \text{otherwise}. \end{cases}$$

4.3 The operators I_r and eigenprojections

Let $I_r BG$ be the stack of tuples $(s_1, \dots, s_r)$ where s_i are r distinct pairwise commuting elements of G:

$$I_r(BG) = [(G^{\times r})^* / G],$$

where the brackets are used as the notation for quotient algebroids. In $K(\text{gpd})$ we write

$$I_r[BG] = [(G^{\times r})^* / G],$$

where bracket stands for the element in the K-group and the quotient notation is omitted.

This defines another family of operators on $K(\text{gpd})$. For $r = 0$, I_0 is identity on all BG and I_1 is the usual inertia operator.

Theorem 4.3. *The operators I_r, for all $r \geq 0$, preserve the filtration $K_{\geq k}(\text{gpd})$. On the quotient $K_{\geq k}(\text{gpd})/K_{>k}(\text{gpd})$, the operator I_r acts as multiplication by $r!\binom{k}{r}$.*

Proof. Let n be the size of the group G and k the size of its centre. Notice that there are $r!\binom{k}{r}$ ways of choosing the r sections so that they are all in the centre. Thus, we conclude,

$$I_r[BG] = r!\binom{k}{r}[BG] + \sum_{\substack{S\in (G^{\times r})^* \\ S\notin Z(G)}} [BZ_G(S)]\,.$$

□

Corollary 4.4. *The operators I_r, for $r \geq 0$ are simultaneously diagonalizable. The common eigenspaces form a family $\Pi_k(\mathrm{gpd})$ of subspaces of $K(\mathrm{gpd})$ indexed by positive integers $k \geq 0$, and*

$$K(\mathrm{gpd}) = \bigoplus_{k\geq 0} \Pi_k(\mathrm{gpd})\,.$$

Let π_k denote the projection onto $\Pi_k(\mathrm{gpd})$. We have

$$I_r \pi_k = r!\binom{k}{r}\pi_k\,,$$

for all $r, k \geq 0$.

Corollary 4.5. *For $r \geq 0$, we have*

$$\ker I_r = \bigoplus_{k<r} \Pi_k(\mathrm{gpd})\,.$$

Corollary 4.6. *For every $k \geq 0$, we have*

$$\pi_k = \sum_{r=k}^{\infty} \frac{(-1)^{r+k}}{r!}\binom{r}{k} I_r\,.$$

In particular, $\pi_0 = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} I_r$, and $\pi_1 = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!} I_r$.

Proof. We now use the "beautiful identity" [43]

$$\sum_r (-1)^{r+k}\binom{\ell}{r}\binom{r}{k} = \delta_{\ell k}$$

to find the projections. We have

$$\mathrm{id} = \sum_{\ell\geq 0} \pi_\ell\,,$$

and hence

$$I_r = \sum_{\ell\geq 0} I_r \pi_\ell = \sum_{\ell \geq 0} r!\binom{\ell}{r}\pi_\ell\,,$$

and therefore

$$\sum_{r\geq 0} \frac{(-1)^{r+k}}{r!}\binom{r}{k} I_r = \sum_{r\geq 0} \frac{(-1)^{r+k}}{r!}\binom{r}{k}\left(\sum_{\ell\geq 0} r!\binom{\ell}{r}\pi_\ell\right)$$

$$= \sum_{\ell\geq 0}\left(\sum_{r\geq 0}(-1)^{r+k}\binom{r}{k}\binom{\ell}{r}\right)\pi_\ell = \sum_{\ell\geq 0}\delta_{\ell,k}\pi_\ell = \pi_k\,.$$

□

Example 4. We previously showed that the projections for the inertia operator are as follows:

$$\pi_n[BS_3] = \begin{cases} [BS_3] - [B\mathbb{Z}_2] - \frac{1}{2}[B\mathbb{Z}_3] & \text{if } n = 1 \\ [B\mathbb{Z}_2] & \text{if } n = 2 \\ \frac{1}{2}[B\mathbb{Z}_3] & \text{if } n = 3 \\ 0 & \text{otherwise}\,. \end{cases}$$

Notice that we have

$$I_r[BS_3] = \begin{cases} [BS_3] + [B\mathbb{Z}_2] + [B\mathbb{Z}_3] & \text{if } r = 1 \\ 2[B\mathbb{Z}_2] + 3[B\mathbb{Z}_3] & \text{if } r = 2 \\ 3[B\mathbb{Z}_3] & \text{if } r = 3 \\ 0 & \text{otherwise}\,. \end{cases}$$

This gives us a way of computing

$$\pi_n[BS_3] = \begin{cases} I_1 - I_2 + \frac{1}{2}I_3 = [BS_3] - [B\mathbb{Z}_2] - \frac{1}{2}[B\mathbb{Z}_3] & \text{if } n = 1 \\ \frac{1}{2}I_2 - \frac{1}{2}I_3 = [B\mathbb{Z}_2] & \text{if } n = 2 \\ \frac{1}{6}I_3 = \frac{1}{2}[B\mathbb{Z}_3] & \text{if } n = 3 \\ 0 & \text{otherwise}\,. \end{cases}$$

Chapter 5

Inertia endomorphism on Deligne-Mumford stacks

For the case of Deligne-Mumford stacks we fix a noetherian scheme B and work with categories of objects defined over B. We let DM/B be the full subcategory of St/B of all Deligne-Mumford stacks over B. We shortly write DM for this category and shortly write Sch for the category, Sch/B, of B-schemes of finite type. The inertia operator of St/B, induces an operator on DM. The Grothendieck ring of it, $K(\mathsf{DM})$, also inherits the structure of a $K(\mathsf{Sch})$-algebra. Therefore we have an induced inertia endomorphism on $K(\mathsf{DM})$. In this chapter we prove our main results (local finiteness and diagonalization) for the $K(\mathsf{Sch})$-module $K(\mathsf{DM})$.

5.1 Stratification of Deligne-Mumford stacks

Definition 5.1. An *irreducible gerbe* $\mathfrak{X}$ is a connected Deligne-Mumford stack, with finite étale inertia, $I\mathfrak{X} \to \mathfrak{X}$.

Proposition 5.2. *Every noetherian Deligne-Mumford stack can be stratified into finitely many locally closed irreducible gerbes.*

Proof. Using [41, Prop. 5.7.6] we may assume that $\mathfrak{X}$ is an integral Deligne-Mumford stack. Flatness is an fppf-local property hence by generic flatness [21, Thm. 6.9.1] there exists an open substack of $\mathfrak{X}$ such that $I\mathfrak{X} \to \mathfrak{X}$ is flat. Since $I\mathfrak{X} \to \mathfrak{X}$ is unramified, it is étale as well. A quasi-finite morphism of schemes is generically finite [46, Lem. 03I1]. Therefore by fpqc-descent of finiteness on base [46, Lem. 02LA], $I\mathfrak{X} \to \mathfrak{X}$ is finite on an open substack of $\mathfrak{X}$. This means that we can stratify $\mathfrak{X}$ into finitely many locally closed substacks $\{\mathfrak{X}_i\}_{i \in \mathsf{A}}$ such that each pullback morphism, $I\mathfrak{X}_i \to \mathfrak{X}_i$ is finite étale. □

Lemma 5.3. *If $\mathfrak{X}$ is an irreducible gerbe, then $I^z\mathfrak{X} \to \mathfrak{X}$ is finite étale.*

PROOF. It suffices to show that the inclusion $I^z\mathfrak{X} \to I\mathfrak{X}$ is a closed immersion. Let $U \to \mathfrak{X}$ be an étale cover of $\mathfrak{X}$ by a scheme U such that $\mathfrak{X}_U$ is the neutral gerbe $B_U G$ for an étale U-group scheme G. In fact according to Remark 2.6, we may assume that G splits to finite union of copies of connected component of unity and hence has the structure of a constant U-group scheme: $G = U \times H$ where H is a finite group. The centre of $\widetilde{G}_j$ is obviously closed and open, and so is $I^z\mathfrak{X} \to I\mathfrak{X}$ by descent. □

Remark 5.4. If $\mathfrak{X}$ is an irreducible gerbe, then each connected component $\mathfrak{Y}$ of $I\mathfrak{X}$ is an irreducible gerbe. This is clear since $\mathfrak{Y} \to \mathfrak{X}$ is finite étale and therefore so is $I\mathfrak{Y} \to I\mathfrak{X}$ by definition of the inertia stack. In other words, an inertia stack of an irreducible gerbe has a canonical stratification into irreducible gerbes by its connected components.

5.2 Filtration by split central number

Recall [21, Cor. 17.9.3] that if $\varphi : Y \to X$ is a separated étale morphism over a connected base scheme X, there is a one-to-one correspondence between the sections of φ and the number of connected components of Y isomorphic to X. Thus for a finite étale covering, the number of such sections is an indication of how close Y is to being a trivial degree n covering, $\bigsqcup_n X \to X$.

Definition 5.5. For an irreducible gerbe $\mathfrak{X}$ we define the *split central number*, $\nu(\mathfrak{X})$, to be the number of sections of $I^z\mathfrak{X} \to \mathfrak{X}$.

We define an ascending filtration of $K(\mathsf{DM})$ by declaring $[\mathfrak{X}]$, for an irreducible gerbe $\mathfrak{X}$, to belong to $K_{\geq n}$ if $\nu(\mathfrak{X}) \geq n$. Thus a linear combination of irreducible gerbes is in $K_{\geq n}$ if the split central number is at least n for all terms.

Proposition 5.6. *The inertia endomorphism on $K(\mathsf{DM})$ preserves the filtration $K_{\geq \bullet}$. Furthermore, on the associated graded piece $K_{\geq n}/K_{>n}$, the inertia endomorphism induces multiplication by the integer n.*

PROOF. Consider an irreducible gerbe $\mathfrak{X}$, with split central number n and $\{\mathfrak{Y}_\alpha\}_{\alpha \in \mathsf{A}}$ be the stratification of $I\mathfrak{X}$ by connected components (hence irreducible gerbes). There are precisely n of the $\mathfrak{Y}_\alpha$ which are contained in $I^z\mathfrak{X}$ and map isomorphically to $\mathfrak{X}$ (i.e. are degree one connected étale covers of $\mathfrak{X}$). It suffices to show that any other strata $\mathfrak{Y}$ has split central number strictly larger than n.

There exists a diagram

$$\begin{array}{ccccc} I^z(I\mathfrak{X}) & \overset{j}{\hookleftarrow} & I\mathfrak{X}\times_{\mathfrak{X}} I^z\mathfrak{X} & \longrightarrow & I^z\mathfrak{X} \\ & \searrow^{\pi_3} & \downarrow{\scriptstyle \pi_2} & \square & \downarrow{\scriptstyle \pi_1} \\ & & I\mathfrak{X} & \longrightarrow & \mathfrak{X} \end{array} \tag{5.1}$$

where the square is cartesian. For any object x of $\mathfrak{X}$, elements of $I\mathfrak{X}$ over x are pairs (x,φ) such that $\varphi \in \mathrm{Aut}(x)$ and objects of $I^z\mathfrak{X}$ over x are pairs (x,ψ) where $\psi \in Z(\mathrm{Aut}(x))$. The fibered product $I\mathfrak{X}\times_{\mathfrak{X}} I^z\mathfrak{X}$ is hence the stack of triples (x,φ,ψ) with x,φ and ψ as above. On the other hand, $I^z(I\mathfrak{X})$ is the stack of the objects (x,φ,ψ) such that $\varphi \in \mathrm{Aut}(x), \psi \in Z(Z_{\mathrm{Aut}(x)}(\varphi))$. Hence there is an embedding of the fibered product into $I^z(I\mathfrak{X})$. Restricting to a substack $\mathfrak{Y} \subset I\mathfrak{X}$ we get the following diagram.

$$\begin{array}{ccccc} I^z\mathfrak{Y} & \overset{j}{\hookleftarrow} & \mathfrak{Y}\times_{\mathfrak{X}} I^z\mathfrak{X} & \longrightarrow & I^z\mathfrak{X} \\ & \searrow^{\pi_3} & \downarrow{\scriptstyle \pi_2} & \square & \downarrow{\scriptstyle \pi_1} \\ & & \mathfrak{Y} & \longrightarrow & \mathfrak{X} \end{array} \tag{5.2}$$

In diagram 5.2 the embedding j is necessarily a union of connected components, because all vertical and diagonal maps in the diagram are representable finite étale covering maps. Note also that there is a canonical section, δ, to $\pi_3 : I(\mathfrak{Y}) \to \mathfrak{Y}$ via the diagonal $\mathfrak{Y} \to \mathfrak{Y}\times_{\mathfrak{X}} \mathfrak{Y}$ since any automorphism of an object x in $\mathfrak{X}$ is in its own centralizer. It is obvious that any section of π_1 pullback to a (distinct) section of π_2 and gives a (distinct) section of π_3. This shows that inertia endomorphism preserves the filtration $K_{\geq \bullet}$.

For the action of inertia on the graded piece $K_{\geq n}/K_{>n}$ we show that if $\mathfrak{Y}$ is a component of $I\mathfrak{X}$ which is not a section of π_1, then the associated section δ is not induced by pulling back sections of π_1. In fact, if $\mathfrak{Y}$ is not contained in $I^z\mathfrak{X}$ then δ does not lift to π_2 and we are done. Otherwise, (when $\mathfrak{Y}$ is completely contained in $I^z\mathfrak{X}$), δ lifts to a section of π_2 but the image of this section in $I^z\mathfrak{X}$ is $\mathfrak{Y}$ itself, which is not a degree one cover of $\mathfrak{X}$. □

5.3 Local finiteness and diagonalization

Before we can deduce that $I : K(\mathsf{DM}) \to K(\mathsf{DM})$ is diagonalizable, we need to prove that for every irreducible gerbe $\mathfrak{X}$ the class $[\mathfrak{X}]$ is contained in a finite dimensional subspace of $K(\mathsf{DM})$, which is preserved by the inertia endomorphism. In this section we use the notation $\prod_{\mathfrak{X}}^k \mathfrak{Y}$ to denote the k-fold fiber product of a stack $\mathfrak{Y}$ by itself over $\mathfrak{X}$.

Lemma 5.7. *Let $\mathfrak{Y} \to \mathfrak{X}$ be a finite étale representable morphism of algebraic stacks. Then the following family is a finite set up to isomorphism of stacks.*

$$C(\mathfrak{Y} \to \mathfrak{X}) = \{\mathfrak{W} : \mathfrak{W} \textit{ is a connected component of } \prod_{\mathfrak{X}}^{k} \mathfrak{Y} \textit{ for some } k \geq 0\}$$

Proof. This is trivial since the Galois closure of $\mathfrak{Y}$ with respect to $\mathfrak{X}$ is a finite étale $\mathfrak{X}$-stack $\overline{\mathfrak{Y}} \to \mathfrak{X}$. And every element in the above family is isomorphic to an intermediate cover, in between $\overline{\mathfrak{Y}}$ and $\mathfrak{Y}$. □

Corollary 5.8. *Let $\mathfrak{Y}_1, \cdots, \mathfrak{Y}_s$ be finitely many algebraic stacks, finite étale over $\mathfrak{X}$. The the following family is finite up to isomorphism*

$$\{\mathfrak{W} : \mathfrak{W} \textit{ is a connected component of } \prod_{\mathfrak{X}}^{k_1} \mathfrak{Y}_1 \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \prod_{\mathfrak{X}}^{k_s} \mathfrak{Y}_s \textit{ for some } k_1, \cdots, k_s \geq 0\}$$

Proof. There are s projection maps

$$p_\ell : \prod_{\mathfrak{X}}^{k_1} \mathfrak{Y}_1 \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \prod_{\mathfrak{X}}^{k_s} \mathfrak{Y}_s \to \prod_{\mathfrak{X}}^{k_\ell} \mathfrak{Y}_\ell, \quad \ell = 1, \cdots, s$$

which are all finite étale and in particular closed and open. The immersion

$$i : \prod_{\mathfrak{X}}^{k_1} \mathfrak{Y}_1 \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \prod_{\mathfrak{X}}^{k_s} \mathfrak{Y}_s \to \prod_{\mathfrak{X}}^{k_1} \mathfrak{Y}_1 \times \cdots \times \prod_{\mathfrak{X}}^{k_s} \mathfrak{Y}_s$$

is similarly closed and open. Hence $\mathfrak{W}$ is isomorphic to its image $i(\mathfrak{W})$ which is a connected component of $\pi_1(\mathfrak{W}) \times \cdots \times \pi_s(\mathfrak{W})$. However any fiber product $\mathfrak{W}_1 \times \cdots \times \mathfrak{W}_s$ where $\mathfrak{W}_i \in C(\mathfrak{Y}_i \to \mathfrak{X})$ has finitely many connected components and by Lemma 5.7 there are only finitely many such fiber products. □

Corollary 5.9. *Let $\mathfrak{X}$ be an irreducible gerbe. Then the following family is finite up to isomophism.*

$$\{\mathfrak{W} : \mathfrak{W} \textit{ is a connected component of } I^{(m)}\mathfrak{X} \textit{ for some } m \geq 0\}$$

Proof. For an irreducible gerbe $I\mathfrak{X} \to \mathfrak{X}$ is finite étale, hence closed and open and therefore the inertia stratifies to finitely many connected components $\mathfrak{Y}_1, \cdots, \mathfrak{Y}_s$ which are finite étale over $\mathfrak{X}$. In the commutative diagram

$$\begin{array}{ccc} I^{(m)}\mathfrak{X} & \overset{j}{\hookrightarrow} & \prod_{\mathfrak{X}}^{m} I\mathfrak{X} \\ & \searrow \quad \swarrow & \\ & I\mathfrak{X} & \end{array}$$

the downward arrows are finite étale and hence so is the inclusion j. Consequently j is open and closed, and therefore any connected component of $I^{(m)}\mathfrak{X}$ is a stratum of some substack

$$\mathfrak{Y}_{i_1} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \mathfrak{Y}_{i_m} \subset \prod_{\mathfrak{X}}^{m} I\mathfrak{X}$$

for a choice of $i_1, \cdots, i_m \in \{1, \cdots, s\}$. The claim now follows from Corollary 5.8. □

This completes the proof of our main results for Deligne-Mumford stacks:

Theorem 5.10 (Local finiteness). *Let $\mathfrak{X}$ be a noetherian Deligne-Mumford B-stack and $\{\mathfrak{X}_i\}_{i\in\mathsf{A}}$, the stratification of it by irreducible gerbes. Then the $K(\mathsf{Sch})$-submodule of $K(\mathsf{DM})$ generated by the set of motivic classes of all $\mathfrak{X}_i$ and all intermediate Galois covers between $\overline{I\mathfrak{X}_i} \to I\mathfrak{X}_i$ is finitely generated, invariant under inertia endomorphism, and contains $[\mathfrak{X}]$.*

Corollary 5.11 (Diagonalization). *The endomorphism $I : K(\mathsf{DM}) \to K(\mathsf{DM})$ is diagonalizable, with eigenvalue spectrum equal to $\mathbb{N}$, the set of positive integers.*

5.4 The operators I_r and eigenprojections

Let $I_r\mathfrak{X}$ be the stack of tuples $(x, s_1, \ldots, s_r)$ where s_i are distinct pairwise commuting automorphisms of x. By this we mean that of $x : X \to \mathfrak{X}$ is an X-point of $\mathfrak{X}$, and $G = \mathrm{Aut}(x)$ is the X-group scheme of automorphisms of x, then s_i are sections of $G \to X$ and not any two of them are identical sections. This definition applies also to $r = 0$. The stack $I_0\mathfrak{X}$ is just $\mathfrak{X}$. For $r = 1$, $I_1\mathfrak{X}$ is the usual inertia. Hence I_1 is diagonalizable with integer eigenvalues.

Note that I_r is closely related to the k-fold inertia operators $I^{(k)}$ of §3.2. In fact it is easy to see that by an inclusion-exclusion argument that they satisfy the following identity,

$$I_r = \sum_{k=1}^{r} s(r,k) I^{(k)},$$

where $s(r,k)$ are the signed Stirling number of the first kind.

We use the notation $ZI_r\mathfrak{X}$ for the substack of $I_r\mathfrak{X}$ consisting of objects $(x, s_1, \dots, s_r)$ such that all s_i are in the centre of $\mathrm{Aut}(x)$. The complement locus will be denoted by $NZI_r\mathfrak{X}$.

Let $\mathfrak{X}$ be an irreducible gerbe with $I\mathfrak{X} \to \mathfrak{X}$ an étale morphism of degree n. Then there exists a Galois covering $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ of $\mathfrak{X}$ such that $I^z\mathfrak{X}|_{\widetilde{\mathfrak{X}}}$ is a disjoint union of n copies of $\widetilde{\mathfrak{X}}$. So we have

$$[ZI_r\mathfrak{X}|_{\widetilde{\mathfrak{X}}}] = r!\binom{n}{r}[\widetilde{\mathfrak{X}}].$$

We use the notation $\mathrm{Inj}(\underline{r}, \underline{n})$ for the set of injections from a set of cardinality r to a set of of cardinality n. So

$$\#\mathrm{Inj}(\underline{r}, \underline{n}) = r!\binom{n}{r}.$$

Theorem 5.12. *The operators I_r, for all $r \geq 0$, preserve the filtration $K_{\geq k}(\mathsf{DM})$ by split central number. On the quotient $K_{\geq k}(\mathsf{DM})/K_{>k}(\mathsf{DM})$, the operator I_r acts as multiplication by $r!\binom{k}{r}$.*

Proof. Let $\mathfrak{X}$ be an irreducible gerbe with split central number k and $I^z\mathfrak{X} \to \mathfrak{X}$ be of degree n. Let $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ be a splitting cover for $I^z\mathfrak{X} \to \mathfrak{X}$ with Galois group Γ. Hence Γ acts on $\underline{n}$.

$$\begin{array}{ccc} I^z\mathfrak{X}|_{\widetilde{\mathfrak{X}}} & \xrightarrow{\iota} & I^z\mathfrak{X} \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{X}} & \longrightarrow & \mathfrak{X} \end{array}$$

Then $I^z\mathfrak{X}|_{\widetilde{\mathfrak{X}}}$ is the disjoint union of n copies of $\widetilde{\mathfrak{X}}$ and Γ acts on it by permuting these copies. Let us rename the i-th copy to $\widetilde{\mathfrak{Y}}$ and the image to $\mathfrak{Y}$. The integer $i \in \underline{n}$ is fixed under the action of Γ precisely when $\mathfrak{Y} \cong \widetilde{\mathfrak{Y}}/\Gamma$ via the horizontal morphism. Since $\widetilde{\mathfrak{X}}/\Gamma \cong \mathfrak{X}$, the above happens precisely when $\mathfrak{Y}$ is isomorphic to a copy of $\mathfrak{X}$ by the vertical morphism. By Proposition 5.6 this only is the case if $\mathfrak{Y}$ is one of the k copies of $\mathfrak{X}$ contributing to the split central number of $\mathfrak{X}$. Hence the set of fixed points of Γ is of size k. Also,

$$\widetilde{\mathfrak{X}} \times \mathrm{Inj}(\underline{r}, \underline{n}) \xrightarrow{\simeq} ZI_r\mathfrak{X}|_{\widetilde{\mathfrak{X}}},$$

and the action of Γ on $\underline{n}$ induces an action of it on $\mathrm{Inj}(\underline{r}, \underline{n})$. A morphism $\varphi : \underline{r} \hookrightarrow \underline{n}$ is invariant under this action if every element in the image of φ is so. Therefore the number of fixed points of $\mathrm{Inj}(\underline{r},\underline{n})$ is $r!\binom{k}{r}$. We may hence calculate as follows:

$$\begin{aligned} ZI_r[\mathfrak{X}] &= [\tilde{\tilde{\mathfrak{X}}} \times_\Gamma \mathrm{Inj}(\underline{r},\underline{n})] \\ &= \sum_{\varphi \in \mathrm{Inj}(\underline{r},\underline{n})/\Gamma} [\tilde{\tilde{\mathfrak{X}}}/\operatorname{Stab}_\Gamma \varphi] \\ &= \sum_{\varphi \in \mathrm{Inj}(\underline{r},\underline{n})^\Gamma} [\mathfrak{X}] + \sum_{\substack{\varphi \in \mathrm{Inj}(\underline{r},\underline{n})/\Gamma \\ \operatorname{Stab}_\Gamma \varphi \neq \Gamma}} [\tilde{\tilde{\mathfrak{X}}}/\operatorname{Stab}_\Gamma \varphi] \end{aligned}$$

Thus, we conclude,

$$ZI_r[\mathfrak{X}] = r!\binom{k}{r}[\mathfrak{X}] + \sum_{\substack{\varphi \in \mathrm{Inj}(\underline{r},\underline{n})/\Gamma \\ \operatorname{Stab}_\Gamma \varphi \neq \Gamma}} [\tilde{\tilde{\mathfrak{X}}}/\operatorname{Stab}_\Gamma \varphi].$$

Finally note that each intermediate cover $\mathfrak{Y} = \tilde{\tilde{\mathfrak{X}}}/\operatorname{Stab}_\Gamma \varphi$ has a strictly larger split central number k. In fact, $I^z\mathfrak{Y} = I^z\mathfrak{X}|_\mathfrak{Y}$ so every section of $I^z\mathfrak{X} \to \mathfrak{X}$ pulls back to a section of $I^z\mathfrak{Y} \to \mathfrak{Y}$ but also $I^z\mathfrak{Y} \to \mathfrak{Y}$ has sections induced by φ that do not descend to $I^z\mathfrak{X} \to \mathfrak{X}$.

Finally for every irreducible gerbe $\mathfrak{Y} \subseteq NZI_r\mathfrak{X}$, the split central rank is strictly larger than n, because at least one of the sections s_i is noncentral. □

Corollary 5.13. *The operators I_r, for $r \geq 0$ are simultaneously diagonalizable. The common eigenspaces form a family $\Pi_k(\mathsf{DM})$ of subspaces of $K(\mathsf{DM})$ indexed by non-negative integers $k \geq 0$, and*

$$K(\mathsf{DM}) = \bigoplus_{k \geq 0} \Pi_k(\mathsf{DM}).$$

Let π_k denote the projection onto $K^k(\mathsf{DM})$. We have

$$I_r \pi_k = r!\binom{k}{r}\pi_k,$$

for all $r \geq 0$, $k \geq 0$.

Corollary 5.14. *For $r \geq 1$, we have*

$$\ker I_r = \bigoplus_{k<r} \Pi_k(\mathsf{DM}).$$

Corollary 5.15. *For every $k \geq 0$, we have*

$$\pi_k = \sum_{r=k}^{\infty} \frac{(-1)^{r+k}}{r!}\binom{r}{k} I_r.$$

In particular, $\pi_0 = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} I_r$*, and* $\pi_1 = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!} I_r$*.*

The proof is similar to that of Corollary 4.6.

Chapter 6

Inertia endomorphism of algebraic stacks

6.1 Stratification of stacks in characteristic zero

From now on we need to work over a field k of characteristic zero and every stack is of finite-type. The categories of k-schemes, Sch/k would be shortened to Sch, and that of k-algebraic stacks St/k would be written as St. We will show that by stratification the Grothendieck group of St is generated by nicely behaving algebraic stacks which will be named *clear gerbes*.

Definition 6.1. Let $\mathfrak{X}$ be a noetherian algebraic stack over the associated coarse moduli space X. We say $\mathfrak{X}$ is a *clear gerbe* over the algebraic space X, if:

(C1) $\mathfrak{X} \to X$ is an étale gerbe with faithfully flat structure morphism of finite type;
(C2) the projection $I\mathfrak{X} \to \mathfrak{X}$ is a representable smooth morphism of finite type;
(C3) X is a k-variety (i.e. a reduced, separated, k-scheme of finite type);
(C4) the central inertia is a closed substack of the inertia stack;
(C5) its central band is a smooth commutative X-group scheme;
(C6) the discrete central band is an étale finite X-group scheme;
(C7) the central band admits a maximal torus.

Remark 6.2. Note that the above properties assure that $I\mathfrak{X}$ is a noetherian group object in the category of algebraic spaces over $\mathfrak{X}$ [46, Lem. 01T6]. The condition of affine diagonal implies further that $I\mathfrak{X}$ is an affine group space over $\mathfrak{X}$.

We start with a modification of the stratification in [34, Thm. 11.5].

Lemma 6.3. *Let $\mathfrak{X}$ be a noetherian algebraic stack of finite type. There exists a finite family $\{\mathfrak{X}_\alpha\}_{\alpha\in \mathsf{A}}$ of locally closed substacks of $\mathfrak{X}$ such that $\mathfrak{X}$ is the disjoint union of $\mathfrak{X}_\alpha$'s and for each index $\alpha \in \mathsf{A}$, the stack $\mathfrak{X}_\alpha$ satisfies (C1), (C2), (C3) over an associated coarse moduli space X_α.*

Proof. By replacing generic smoothness instead of generic flatness in the proof of [34, Thm. 11.5] we may assume (C1) and (C2) are already satisfied and the coarse moduli sheaf X, is a noetherian algebraic space of finite type and in particular quasi-compact and quasi-separated. Now we stratify X into k-varieties by [14, Thm. 3.1.1]. □

Theorem 6.4. *Every noetherian algebraic stack of finite type with affine diagonal has a stratification into a disjoint union of locally closed clear gerbes.*

Proof. By previous lemma we already assume that $\mathfrak{X} \to X$ satisfies (C1), (C2) and (C3). Now we stratify it such that for each stratum the central inertia descends to a smooth group scheme over the base coarse moduli space.

First we stratify $\mathfrak{X} \to X$ on the target so that the central band (which is a sheaf of groups) associated to each stratum is an algebraic space and its central inertia is a closed substack of inertia stack.

Let Z be the central band of $\mathfrak{X} \to X$. It is easy to check that the diagonal morphism $Z \to Z \times_X Z$ is representable. So let $U \to X$ be an étale covering of X which trivializes $\mathfrak{X}$ to the trivial G-gerbe for some U-group scheme G. By generic smoothness we stratify X to a finite family $X = \{X_i\}_{i\in \mathsf{I}}$ such that each $G_i = G|_{U_i}$ is smooth over X_i. By descent of smoothness [21, Cor. 17.7.3], we conclude that G_i is a smooth U_i-group scheme. By Corollary 2.5 we may further stratify each X_i by $\{X_{ij}\}_{\beta\in \mathsf{J}}$ such that each $G|_{X_{ij}}$ admits its centre Z_{ij} as a scheme. Then Z_{ij} is an étale cover of the restricted sheaf of groups $Z|_{X_{ij}}$. The latter is hence an algebraic space.

Notice also that the morphisms $I^z\mathfrak{X}_{ij} \to I\mathfrak{X}_{ij}$ are now closed immersions according to Lemma 3.3. Moreover $Z|_{X_{ij}}$ is noetherian, and smooth over X_{ij} of finite type. We now use Lemma 2.1 to stratify each X_{ij} further via $\{X_{ijk}\}_{k\in \mathsf{K}}$ such that each pullback $Z|_{ijk}$ is a group scheme over X_{ijk}.

Finally, if $\mathfrak{X} \to X$ has smooth central group scheme $Z \to X$, the quotient space Z/Z^0 is a finitely presented, étale algebraic space over X. By Lemma 2.1 we may assume that Z/Z^0 is a scheme. Then we stratify further using generic finiteness [46, Lem. 03I1].

Recall that a smooth commutative algebraic group over a perfect field has a decomposition $T \times U$ where U is the unipotent radical of the algebraic group and T is a group of multiplicative type. If the group is connected then so is T, in which case T is a torus [37, XIV, Theorem 2.6].

Let η be a generic point of X. Then the mentioned decomposition holds for the generic fiber Z_η of central band over X. We now use the machinery of §2.2 on how the structure of a commutative algebraic groups spread out by Corollary 2.15. □

Definition 6.5. For a clear gerbe $\mathfrak{X} \to X$, the *split central number* of $\mathfrak{X}$ is defined to be the number of sections of $I^{z/z,0}\mathfrak{X} \to \mathfrak{X}$ and is denoted by $\nu(\mathfrak{X})$. The *discrete degree of the centre* of $\mathfrak{X}$, is the degree of $I^{z/z,0}\mathfrak{X} \to \mathfrak{X}$ as a finite étale map and is denoted by $\deg^z(\mathfrak{X})$. The non-negative integer $\tau(\mathfrak{X}) = \deg^z(\mathfrak{X}) - \nu(\mathfrak{X})$, is called the *central twistedness* of $\mathfrak{X}$.

Definition 6.6. Let $\mathfrak{X} \to X$ be a clear gerbe. The relative dimensions $\dim_{\mathfrak{X}}(I\mathfrak{X})$ and $\dim_{\mathfrak{X}}(I^z\mathfrak{X})$ are well-defined. The latter is called the *central rank* of $\mathfrak{X}$, and is denoted by $\rho(\mathfrak{X})$. This is also the rank of the associated commutative group scheme $Z^0 \to X$. The non-negative integer $\mathrm{corank}(\mathfrak{X}) = \dim_{\mathfrak{X}}(I\mathfrak{X}) - \dim_{\mathfrak{X}}(I^z\mathfrak{X})$ is called the co-central rank of $\mathfrak{X}$. Let $c(\mathfrak{X}) \geq 1$ will be the maximum number of geometrically connected components of the fibers of $I\mathfrak{X} \to \mathfrak{X}$. This number is well-defined and finite according to [46, Lem. 055Q].

Definition 6.7. For a clear gerbe $\mathfrak{X} \to X$ the unipotent and reductive ranks of the central band are constant over the coarse moduli space. We will denote these integers respectively by $\rho_u(\mathfrak{X})$ and $\rho_r(\mathfrak{X})$.

We will define two types of filtration on $K(\mathsf{St})$ now. One is an descending double filtration $K_{\geq(\cdot,\cdot)}$ and the other one is a ascending double filtration $K^{\leq(\cdot,\cdot,\cdot)}$. We will show that the inertia operator preserves both filtrations. The latter filtration is used to prove local finiteness whereas the former filtration serves in proving diagonalizability results in the next chapter.

6.2 Filtration by central rank and split central number

Let $K_{\geq(r,n)}$ be the sub-module of $K(\mathsf{St})$ spanned by clear gerbes $\mathfrak{X} \to X$, for which the central rank, is at least r and if this rank is exactly r then the split central number is at least n.

First, we prove a property of monomorphisms of group schemes of same dimension. If $u : G \to H$ is any homomorphism of group schemes, it follows from the definition of the functor of connectedness component of identity [4, Exp. VI, Def. 3.1] that G^0 maps to H^0. The following lemma shows a stronger fact in case of monomorphisms of group schemes of same dimension:

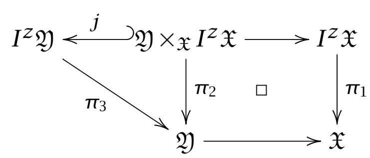
Lemma 6.8. *Let $u : G \to H$ be a monomorphism of smooth group schemes, locally of finite type over base scheme S. If G and H are of same dimension over S, the induced map $u^0 : G^0 \to H^0$ is surjective.*[1]

PROOF. First suppose S is the spectrum of a local artin ring, A. Then G^0 is of finite type over A [4, Exp. VI(A), Prop. 2.4] hence noetherian, and therefore u^0 is quasi-compact and consequently $u^0(G^0)$ is closed in H^0 [4, VI(B), Prop. 1.2]. But H^0 is irreducible [4, VI(A), Cor. 2.4.1] and $u^0(G^0)$ is of same dimension as H^0 hence it has to be all of H^0. For a general base scheme, S, we now have that for any point $s \in S$, the induced morphism $u^0_s : G^0_s \to H^0_s$ is surjective, hence u^0 is surjective. □

Let $\mathfrak{X}$ is a clear gerbe. We fix a stratification of $I\mathfrak{X}$ by clear gerbes and let $\mathfrak{Y}$ be one such a stratum. We prove that if $\mathfrak{Y}$ is not contained in $I^z\mathfrak{X}$ then either $\rho(\mathfrak{Y}) > \rho(\mathfrak{X})$ or $\nu(\mathfrak{Y}) > \nu(\mathfrak{X})$. And if $\mathfrak{Y}$ is contained in $I^z\mathfrak{X}$, and maps to a component of $I^{z/z,0}\mathfrak{X}$, *not* of degree 1 over $\mathfrak{X}$, then $\rho(\mathfrak{X}) = \rho(\mathfrak{Y})$ but $\nu(\mathfrak{Y}) > \nu(\mathfrak{X})$.

Proposition 6.9. *Let $\mathfrak{Y}$ be a stratum of $I\mathfrak{X}$ not (completely) contained in $I^z\mathfrak{X}$. Then $\rho(\mathfrak{Y}) \geq \rho(\mathfrak{X})$ and if $\rho(\mathfrak{Y}) = \rho(\mathfrak{X})$ then $\nu(\mathfrak{Y}) > \nu(\mathfrak{X})$.*

PROOF. We use diagram 5.2 here again.

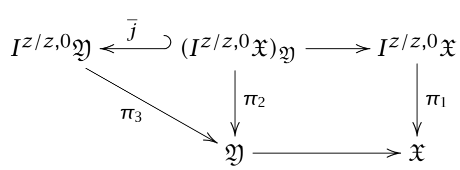


From this diagram, it is obvious that $\rho(\mathfrak{Y}) \geq \rho(\mathfrak{X})$.

Now suppose $\rho(\mathfrak{Y}) = \rho(\mathfrak{X})$. Here j is a monomorphism of commutative group spaces over $\mathfrak{Y}$ thus by the previous lemma we have an induced monomorphism of discrete central group schemes

$$\iota : (I^z\mathfrak{X})_{\mathfrak{Y}} / (I^z\mathfrak{X})^0_{\mathfrak{Y}} \hookrightarrow I^z\mathfrak{Y} / (I^z\mathfrak{Y})^0.$$

Since the connectedness component of identity and the quotient by it, are preserved by base change, we have the commutative diagram



(6.1)

[1] Also cf. [4, Exp. VI(B), Cor. 1.3.2].

in level of stacks. The morphism π_3 has a canonical section induced by the diagonal morphism $\mathfrak{Y} \to \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ (explicitly $(x,\varphi) \mapsto (x,\varphi,[\varphi])$, where x is an object of $\mathfrak{X}$, $\varphi \in \mathrm{Aut}(x)$ and $[\varphi]$ is the orbit of φ by the action of connectedness component of unity). Since $\mathfrak{Y}$ is ultra-central, this section is not in the image of $\tilde{\iota}$. Therefore $\nu(\mathfrak{Y}) > \nu(\mathfrak{X})$. □

When $\mathfrak{X}$ is a clear gerbe, the connectedness components of the central inertia $I^z\mathfrak{X} \to \mathfrak{X}$ are also clear gerbes; this yields a canonical stratification of the central inertia by clears gerbes. The next two propositions pertain to these strata.

Proposition 6.10. *The connectedness components of the central inertia $I^z\mathfrak{X} \to \mathfrak{X}$ satisfy the following properties: (1) they all have the same central rank as that of $\mathfrak{X}$; and (2) there is a one-to-one correspondence between connectedness components of $I^z\mathfrak{X}$ and that of $I^{z/z,0}\mathfrak{X}$ and each compotent $\mathfrak{Y} \subseteq I^z\mathfrak{X}$ is an $(I^{z,0}\mathfrak{X})$-torsor over the associated connected component $\mathfrak{Y}' \subset I^{z/z,0}\mathfrak{X}$.*

Proof. It is easy to verify the isomorphism $I^z(I^z\mathfrak{X}) \cong I^z\mathfrak{X} \times_{\mathfrak{X}} I^z\mathfrak{X}$ of stacks which also fit in the commutative diagram

$$\begin{array}{ccc} I^z(I^z\mathfrak{X}) & \longrightarrow & I^z\mathfrak{X} \\ \downarrow & \square & \downarrow \\ Z \times_X Z & \longrightarrow & Z \end{array}$$

Thus for any locally closed substack $\mathfrak{Y}$ is $I^z\mathfrak{X}$ which descends to a locally closed subspace Y of Z, we have

$$\begin{array}{ccc} I^z(\mathfrak{Y}) & \longrightarrow & \mathfrak{Y} \\ \downarrow & \square & \downarrow \\ Z \times_X Y & \longrightarrow & Y. \end{array} \tag{6.2}$$

In particular the central Y-group scheme associated to $\mathfrak{Y}$ is the pull-back $Z|_Y$. (1) is now obvious form diagram 6.2. For (2) notice that the morphism $I^z\mathfrak{X} \longrightarrow I^{z/z,0}\mathfrak{X}$ is a principal bundle for the connected group scheme $I^{z,0}\mathfrak{X}$ over $\mathfrak{X}$, and therefore there is a bijection between connectedness components of the source and the target of this morphism. Passing to a component gives us the cartesian diagram

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \mathfrak{Y}' \\ \downarrow & & \downarrow \\ I^z\mathfrak{X} & \longrightarrow & I^{z/z,0}\mathfrak{X} \end{array}$$

together with a finite étale mapping $\mathfrak{Y}' \to \mathfrak{X}$ proving the lemma. □

Proposition 6.11. *Let $\mathfrak{Y}$ be a connected component of $I^z\mathfrak{X}$. We always have $\nu(\mathfrak{Y}) \geq \nu(\mathfrak{X})$, with equality happening if and only if the image of $\mathfrak{Y}$ in $I^{z/z,0}\mathfrak{X}$ maps down isomorphically to $\mathfrak{X}$.*

Proof. By last lemma, $\mathfrak{Y}$ sits over a connectedness component $\mathfrak{Y}' \subseteq I^{z/z,0}\mathfrak{X}$. The homomorphism of commutative group schemes $(I^z\mathfrak{X})_{\mathfrak{Y}} \to I^z\mathfrak{Y}$ over $\mathfrak{Y}$ is an isomorphism giving the left cartesian square of homomorphisms of commutative group schemes and inducing the right hand one:

$$\begin{array}{ccc} I^z\mathfrak{Y} & \longrightarrow & I^z\mathfrak{X} \\ {\scriptstyle \pi_2}\downarrow & \Box & \downarrow{\scriptstyle \pi_1} \\ \mathfrak{Y} & \longrightarrow & \mathfrak{X} \end{array} \qquad \begin{array}{ccc} I^{z/z,0}\mathfrak{Y} & \longrightarrow & I^{z/z,0}\mathfrak{X} \\ {\scriptstyle \overline{\pi}_2}\downarrow & \Box & \downarrow{\scriptstyle \overline{\pi}_1} \\ \mathfrak{Y} & \longrightarrow & \mathfrak{X} \end{array} \tag{6.3}$$

So distinct sections of $\overline{\pi}_1$ pull back to distinct sections of $\overline{\pi}_2$, therefore $\nu(\mathfrak{Y}) \geq \nu(\mathfrak{X})$.

Now suppose $\mathfrak{Y}'$ is not isomorphic to $\mathfrak{X}$. Then, as the structure map $\mathfrak{Y} \to \mathfrak{X}$ factors through $\mathfrak{Y}'$, and yields a section of $\overline{\pi}_2$ that is not induced by $\overline{\pi}_1$. In this case we have $\nu(\mathfrak{Y}) > \nu(\mathfrak{X})$. If $\mathfrak{Y}'$ maps isomorphically to $\mathfrak{X}$, the structure map $\mathfrak{Y} \to \mathfrak{X}$ is an $I^{z,0}\mathfrak{X}$-torsor over $\mathfrak{X}$. Hence, the upper horizontal map in the left hand diagram of (6.3) is a torsor for a connected group scheme, and therefore we can push the sections forward. In this case, $\nu(\mathfrak{Y}) = \nu(\mathfrak{X})$. □

Remark 6.12. The $I^{z,0}\mathfrak{X}$-torsors $\mathfrak{Y}$ in Proposition 6.10 and 6.11 all come from scheme torsors. In fact, the $I^{z,0}\mathfrak{X}$-principal bundle $I^z\mathfrak{X} \to I^{z/z,0}\mathfrak{X}$ is the pull-back of the Z^0-principal bundle $Z \to Z/Z^0$. Passing to a strata $\mathfrak{Y}$, we likewise observe that $\mathfrak{Y} \to \mathfrak{Y}'$ is the pull back of a Z^0-torsor.

6.3 An ascending filtration and local finiteness

Let $K^{\leq(d,c,t)}$ be the sub-module of $K(\mathsf{St})$ spanned by clear gerbes $\mathfrak{X} \to X$, for which the central corank, is at most d, and if this number is exactly d then the maximum number of geometrically connected components of $I\mathfrak{X} \to \mathfrak{X}$ is at most c, and if this number is exactly c, then the central twistedness of $\mathfrak{X}$ is at most t.

Proposition 6.13. *Let $\mathfrak{Y}$ be a stratum of $I\mathfrak{X}$ not completely contained in $I^z\mathfrak{X}$. Then* $\operatorname{corank}(\mathfrak{Y}) \leq \operatorname{corank}(\mathfrak{X})$ *and equality happens only if $c(\mathfrak{Y}) < c(\mathfrak{X})$.*

Proof. We consider a different commutative diagram

$$\begin{array}{ccccc} I\mathfrak{Y} & \hookrightarrow & I\mathfrak{X}|_{\mathfrak{Y}} & \longrightarrow & I\mathfrak{X} \\ & \searrow & \downarrow & & \downarrow \\ & & \mathfrak{Y} & \longrightarrow & \mathfrak{X} \end{array} \tag{6.4}$$

The injective morphism is given on the fiber of any point $x \in \mathfrak{X}$ by mapping of subgroup $Z_G(g) \hookrightarrow G$ where $G = \mathrm{Aut}(x)$ and $g \in G \setminus Z(G)$. Thus we always have $\dim_{\mathfrak{X}} I\mathfrak{X} \geq \dim_{\mathfrak{Y}} I\mathfrak{Y}$. This together with Proposition 6.9 show that $\mathrm{corank}(\mathfrak{Y}) \leq \mathrm{corank}(\mathfrak{X})$.

Now suppose $\mathrm{corank}(\mathfrak{Y}) = \mathrm{corank}(\mathfrak{X})$. In particular this means that $\dim_{\mathfrak{Y}}(I\mathfrak{Y}) = \dim_{\mathfrak{X}}(I\mathfrak{X})$. The inclusion in diagram 6.4 is given by $Z_G(g) \hookrightarrow G$ for any object x of $\mathfrak{X}$ with $G = \mathrm{Aut}(x)$, where g is not contained in the centre. Therefore $Z_G(g)$ is a subgroup of G not equal to it. By [4, VI(A), Cor. 2.4.1], the closed immersion $Z_G(g) \hookrightarrow G$ maps connected components of $Z_G(g)$ to that of G. Thus the number of connected components of $Z_G(g)$ in strictly less than that of G. □

Proposition 6.14. *Let $\mathfrak{Y}$ be a connected component of $I^z\mathfrak{X}$. Then* $\mathrm{corank}(\mathfrak{Y}) = \mathrm{corank}(\mathfrak{X})$ *and $c(\mathfrak{X}) = c(\mathfrak{Y})$. Also $\tau(\mathfrak{Y}) \leq \tau(\mathfrak{X})$ with equality happening if and only if $\mathfrak{Y}$ is isomorphic to $I^{z,0}\mathfrak{X}$.*

Proof. It is easy to verify the isomorphism $I(I^z\mathfrak{X}) \cong I\mathfrak{X} \times_{\mathfrak{X}} I^z\mathfrak{X}$. But from Proposition 6.10 we already know that $\rho(\mathfrak{Y}) = \rho(\mathfrak{X})$. So we also conclude that $\mathrm{corank}(\mathfrak{Y}) = \mathrm{corank}(\mathfrak{X})$. It is also obvious from the same isomorphism that $c(\mathfrak{X}) = c(\mathfrak{Y})$. The last claim follows from the fact that the commutative diagrams 6.3 also show that $\deg^z(\mathfrak{Y}) = \deg^z(\mathfrak{X})$. The claim is then clear from Proposition 6.11.□

Corollary 6.15. *The filtration $K^{\leq(\cdot,\cdot,\cdot)}$ is preserved by the inertia operator of $K(\mathsf{St})$.*

We see that even proving local finiteness in the general case of algebraic stacks is a challenge. However the above ascending filtration suggests sufficient conditions for local finiteness.

Recall that a smooth commutative algebraic group Z over a perfect field has a decomposition $Z = Z^{ss} \times U$ where U is the unipotent radical of Z and Z^{ss} is a group of multiplicative type. If Z is connected then so is Z^{ss}, in which case the latter is a torus [37, XIV].

Corollary 6.16. *Let $Z \to X$ be a smooth connected commutative group scheme of dimension r with constant reductive rank t and maximal torus T. Then $[Z] = q^{r-t}[T]$.*

Proof. Let $\eta \in X$ be the generic point with residue field K. Over the algebraic closure, we have a decomposition $Z_{\overline{K}} = T_{\overline{K}} \times U_{\overline{K}}$ where $U_{\overline{K}}$ is the unipotent radical and descents on K to the quotient group scheme Z_η/T_η. By [36, Cor. 15.10] the latter is a unipotent algebraic group as well. Now we use Lemma 2.14 and Corollary 2.15 to spread out the short exact sequence

$$1 \to T_\eta \to Z_\eta \to Z_\eta/T_\eta \to 1$$

to an open subscheme U of X. This implies that $[Z_U] = q^{r-t}[T_U]$ since the quotient Z_U/T_U has a composition series with $r-t$ line bundle factors. The claim now follows by noetherian induction. □

Let $\mathfrak{X}$ be a clear gerbe in $K^{\leq(d,c,t)}$. Then $I\mathfrak{X}$ can be stratified by finitely many clear gerbes. Let $\mathfrak{Y}$ be such an strata. If $\mathfrak{Y}$ is not contained in $I^z\mathfrak{X}$ or is contained in $I^z\mathfrak{X}$ but is not an $I^{z,0}\mathfrak{X}$-torsor then it is in $F^{<(d,c,t)}$. If $\mathfrak{Y}$ is an $I^{z,0}\mathfrak{X}$-torsor then $[\mathfrak{Y}] = [Z^0 \times_X \mathfrak{X}]$ by Remark 6.12. Alternatively by condition (C7) we may write

$$[\mathfrak{Y}] = q^{\rho_u(\mathfrak{X})}[T \times_X \mathfrak{X}]$$

where T is the maximal torus of the central band of $\mathfrak{X}$. So by an inductive argument to prove a local finiteness result, we only need to deal with the computation of motivic classes of such tori over $\mathfrak{X}$ in terms of $\mathfrak{X}$.

6.4 Spectrum of the unipotent inertia

Let $K_{\geq r}$ be the sub-module of $K(\mathsf{St})$ spanned by clear gerbes $\mathfrak{X} \to X$, for which the unipotent central rank, is at least r. This defines an descending filtration $K_{\geq \cdot}$ on $K(\mathsf{St})$ and we show that it is preserved by the unipotent inertia. We also define an ascending filtration $K^{\leq \cdot}$ by declaring a clear gerbe $\mathfrak{X} \to X$ to be in $K^{\leq s}$ if the *unipotent co-rank* defined as the difference $\dim_{\mathfrak{X}} I\mathfrak{X} - \rho_u(\mathfrak{X})$ is at most s.

Proposition 6.17. *Let $\mathfrak{X}$ is a clear gerbe which unipotent rank r and counipotent rank s. Let $\mathfrak{Y}$ be a clear gerbe contained in $I^u\mathfrak{X} \setminus I^{u,z}\mathfrak{X}$. Then $\mathfrak{Y}$ is contained in $K_{>r}$ and in $K^{<s}$.*

Proof. The diagram 5.2 maps unipotent automorphisms to unipotent ones. Therefore we have

$$\begin{array}{ccccc} I^{z,u}\mathfrak{Y} & \xleftarrow{j} & \mathfrak{Y} \times_{\mathfrak{X}} I^{u,z}\mathfrak{X} & \longrightarrow & I^{z,u}\mathfrak{X} \\ & \searrow^{\pi_3} & \downarrow{\scriptstyle \pi_2} & \square & \downarrow{\scriptstyle \pi_1} \\ & & \mathfrak{Y} & \longrightarrow & \mathfrak{X} \end{array}$$

From this diagram, it is obvious that $\rho(\mathfrak{Y}) \geq \rho(\mathfrak{X})$. But if $\mathfrak{Y}$ is not central then $I^{u,z}\mathfrak{Y} \to \mathfrak{Y}$ has a trivial section that does not lift to π_2. So the unipotent part of the central band of $\mathfrak{Y}$ is strictly larger than that of $\mathfrak{X}$. Hence $\rho^u(\mathfrak{Y}) > \rho^u(\mathfrak{X})$.

Now the diagram 6.4 showed that $\dim_{\mathfrak{Y}}(I\mathfrak{Y}) \leq \dim_{\mathfrak{X}}(I\mathfrak{X})$. Therefore the unipotent corank of $\mathfrak{Y}$ is strictly less that that of $\mathfrak{X}$, proving the second claim. □

Corollary 6.18. *The endomorphism $I^u : K(\mathsf{St}) \to K(\mathsf{St})$ is locally finite and triangularizable and the eigenvalue spectrum of it is the set of all monomials q^u for $u \geq 0$.*

PROOF. By Remark 6.12 we have $I^{u,z}[\mathfrak{X}] = q^{\rho_u(\mathfrak{X})}[\mathfrak{X}]$. The rest is clear. □

Corollary 6.19. *The endomorphism I^u is diagonalizable on $K(\mathsf{St})[q^{-1}, \{q^k - 1 : k \geq 1\}]$.*

Chapter 7

Quasi-split stacks

In this section we present a criteria that if satisfied guarantees the inertia endomorphism is locally finite and diagonalizable. This criteria restricts the central group schemes of the clear gerbes to fit in particular exact sequences of nicely-behaved commutative group schemes in lines with Chevalley's structure theorem. For some preliminaries on group schemes of multiplicative type and unipotent group schemes we refer the reader to §2.2.

7.1 Motivic classes of quasi-split tori

Let Γ be a finite group acting on the finite set $\underline{r}$ with orbit space

$$\underline{r}/\Gamma = \{O_1, \cdots, O_\ell\}.$$

The polynomial $\prod_{i=1}^{\ell}(q^{|O_i|} - 1)$ depends only on the sizes of the orbits. In fact, to Γ we may associate a partition $\lambda = (\lambda_i)_{i\geq 1}$ of the integer r with declaring λ_i to be the number of elements of $\underline{r}/\Gamma$ of size i. The the above polynomial is identical to

$$\mathcal{Q}_\lambda = \prod_{i=1}^{\ell}(q^i - 1)^{\lambda_i}.$$

The reason quasi-split tori are interesting to us is the following computation of their motivic classes.

Proposition 7.1. *Let T be an isotrivial quasi-split torus over the integral scheme X and $\overline{X}$ be the minimal splitting Galois cover of it, with Galois group Γ. Let $\Lambda = \{b_1, \cdots, b_r\}$ be a choice of basis for χ_T permuted by the Γ-action. Then the motivic class of T is given* *by*

$$[T] = \mathcal{Q}_\lambda(q)[X] + \sum_{\substack{I_\bullet \in F(\underline{r})/\Gamma \\ \mathrm{Stab}_\Gamma(I) \subsetneq \Gamma}} (-1)^{\ell(I_\bullet)} q^{|I_{\max}|} [\overline{X}/\,\mathrm{Stab}_\Gamma(I)]. \tag{7.1}$$

where $\lambda \vdash r$ *is the partition of integer* r *induced by the action of* Γ *on* Λ*.*

Here, for a subset $I \subset \underline{r}$, we denote by $\mathbb{A}^I \subset A^r$ the subset of all $(x_1, \dots, x_r)$ such that $x_i = 0$, for $i \notin I$, and by $\mathbb{G}_m^I \subset \mathbb{A}^I$, the set of all $(x_1, \dots, x_r) \in \mathbb{A}^I$, such that $x_i \neq 0$, for all $i \in I$. For every subset $I \subset \underline{r}$, we have a Γ-equivariant splitting

$$\mathbb{A}^I = \bigsqcup_{J \subset I} \mathbb{G}_m^J,$$

and hence,

$$[\mathbb{G}_m^I] = [\mathbb{A}^I] - \sum_{J \subsetneq I} [\mathbb{G}_m^J].$$

By induction, we get an equivariant inclusion-exclusion principle

$$[\mathbb{G}_m^r] = \sum_{k \geq 0} (-1)^k \sum_{I_\bullet \in F^k(\underline{r})} [\mathbb{A}^{I_k}].$$

Here $F^k(\underline{r})$ is the set of all flags $I_\bullet = I_k \subsetneq \ldots \subsetneq I_1 \subsetneq I_0 = \underline{r}$ in $\underline{r}$. We denote the length of a flag $I_\bullet$ by $k = \ell(I_\bullet)$, the maximal index by $k = \max$, and the set of all flags, regardless of length, by $F(\underline{r})$.

Proof. Γ acts on the split torus $T_{\overline{X}} = \mathrm{Spec}(\mathcal{O}_{\overline{X}}[\chi_T])$ and by [35, Prop. 5.21] we can revive T from this pull-back as $T \cong (\overline{X} \times_X T)/\Gamma$. Then the surjection of $\mathbb{Z}$-module, $\bigoplus_{i=1}^r b_i\mathbb{Z} \to \chi_T$, induces a sheaf homomorphism $\mathcal{O}_{\overline{X}}[b_1, \cdots, b_r] \to \mathcal{O}_{\overline{X}}[\chi_T]$ and consequently an open immersion $\mathbb{G}_{m,\overline{X}}^r \cong T_{\overline{X}} \to \mathbb{A}_{\overline{X}}^r$, which is equivariant for the Γ-action. Hence we may pass to quotient schemes and get

$$T = \overline{X} \times_\Gamma \mathbb{G}_m^r \hookrightarrow \overline{X} \times_\Gamma \mathbb{A}^r.$$

Thus we have

$$
\begin{aligned}
[T] &= [\overline{X} \times_\Gamma \mathbb{G}_m^r] \\
&= \sum_{k\geq 0} (-1)^k [\overline{X} \times_\Gamma \bigsqcup_{I_\bullet \in F^k(\underline{r})} \mathbb{A}^{I_k}] \\
&= \sum_{k\geq 0} (-1)^k \sum_{I_\bullet \in F^k(\underline{r})/\Gamma} [\overline{X} \times_{\mathrm{Stab}_\Gamma(I_\bullet)} \mathbb{A}^{I_k}] \\
&= \sum_{I_\bullet \in F(\underline{r})/\Gamma} (-1)^{\ell(I_\bullet)} q^{|I_{\max}|} [\overline{X}/\operatorname{Stab}_\Gamma(I_\bullet)] \\
&= \sum_{I_\bullet \in F(\underline{r})^\Gamma} (-1)^{\ell(I_\bullet)} q^{|I_{\max}|} [X] \\
&\qquad + \sum_{\substack{I_\bullet \in F(\underline{n})/\Gamma \\ \mathrm{Stab}_\Gamma(I) \subsetneq \Gamma}} (-1)^{\ell(I_\bullet)} q^{|I_{\max}|} [\overline{X}/\operatorname{Stab}_\Gamma(I)] \\
&= \mathcal{Q}_\Omega(q)[X] + \sum_{\substack{I_\bullet \in F(\underline{r})/\Gamma \\ \mathrm{Stab}_\Gamma(I) \subsetneq \Gamma}} (-1)^{\ell(I_\bullet)} q^{|I_{\max}|} [\overline{X}/\operatorname{Stab}_\Gamma(I)].
\end{aligned}
$$

Note that all forms of affine spaces occurring in this computation are vector bundles over their base by Hilbert's Theorem 90. This is the reason for the appearance of the terms $q^{|I_k|}$ in the calculation. □

7.2 Quasi-split stacks

Definition 7.2. An algebraic stack $\mathfrak{X}$, is called quasi-split if for any point of its coarse moduli space, $x \in X$, the central band $Z|_x$ admits a quasi-split maximal torus.

The category of quasi-split algebraic stacks is the full subcategory of St consisting of all quasi-split algebraic stacks is denoted by QS. It is easy to see that QS is closed under ienrtia and contains all its products and open and closed substacks, thus there is a well-defined induced $K(\mathsf{Sch})$-linear inertia endomorphism of the algebra $K(\mathsf{QS})$.

Remark 7.3. The results below are all about this subcategory of St however in §9, §10, and §11, we will be working with certain subcategories of QS. All results below hold true, for any full subcategory of QS that is (1) closed under inertia; and, (2) contains products and all closed and open subobjects of it.

Quasi-split algebraic stacks, can be stratified further to nicely-behaving clear gerbes as far as motivic computations are concerned.

Definition 7.4. A clear gerbe $\mathfrak{X} \to X$ with central group scheme $Z \to X$, is called quasi-split if Z admits a maximal torus T which is quasi-split.

Theorem 7.5. *A quasi-split stack can be stratified by finitely many quasi-split clear gerbes.*

Proof. In view of Theorem 6.4 we only need to prove the result for a quasi-split clear gerbe $\mathfrak{X} \to X$. Let $\eta = \operatorname{Spec} K$ be the generic point of X. By definition, the generic fiber $\mathfrak{X}_K$, is then a quasi-split K-stack. Therefore the central group scheme $Z \to X$, pulls back over η to the algebraic K-group Z_η where the connected component of unity has a decomposition as $Z^0 = T_\eta \times U_\eta$. Here T_η is a quasi-split maximal torus of Z and U_η is a unipotent algebraic K-group. It remains to observe that this decomposition spreads out to an open neighborhood of η in X by Corollary 2.15. □

Let $\mathfrak{X}$ be a quasi-split clear gerbe. The $I^{z,0}\mathfrak{X}$-torsors $\mathfrak{Y}$ in Proposition 6.10 are essential in our motivic computations and therefore we finish this section by finding the motivic class of such torsors in terms of the motivic class of $\mathfrak{X}$.

Proposition 7.6. *Let $\mathfrak{X}$ be a quasi-split clear gerbe with coarse moduli space X and central X-group scheme Z of rank r and reductive rank t. Let $Y \to X$ be a Z^0-torsor and $\mathfrak{Y}$ be the pullback $I^{z,0}\mathfrak{X}$-torsor over $\mathfrak{X}$. Then*

$$[\mathfrak{Y}] = q^{r-t}\mathfrak{Q}_{\lambda\vdash t}(q)[\mathfrak{X}] + \sum_{\substack{I_\bullet \in F(\underline{r})/\Gamma \\ \operatorname{Stab}_\Gamma(I)\subsetneq\Gamma}} (-1)^{\ell(I_\bullet)} q^{r-t+|I_{\max}|}[\overline{\mathfrak{X}}/\operatorname{Stab}_\Gamma(I)], \tag{7.2}$$

for some finite étale covering $\overline{X} \to X$ and an action of $\Gamma = \pi_1(\overline{X}/X)$ on the set $\{1,\cdots,t\}$. Here $\lambda \vdash t$ is the integer partition type of the orbit space $\underline{t}/\Gamma$ in the sense of §7.1 and $\overline{\mathfrak{X}} = \mathfrak{X}|_{\overline{X}}$.

Proof. By [10, Prop 2.2] any torsor for a quasi-split torus is Zariski locally trivial and therefore the assertion follows from Corollary 6.16 and Proposition 7.1 and pulling back along $\mathfrak{X} \to X$. □

7.3 An ascending filtration and local finiteness

Recall the ascending filtration of §6.3 by central corank, maximum number of geometrically connected components, and then central twistedness. Recall that in the same section we proved for any clear gerbe $\mathfrak{X} \to X$, with central corank d, maximum number of geometrically connected components c, and central twistedness t and a stratification of $I\mathfrak{X}$ by clear gerbes, every stratum is $\mathfrak{Y} \subseteq I\mathfrak{X}$ is contained in $K^{<(d,c,t)}$ unless $\mathfrak{Y}$ is an $I^{z,0}\mathfrak{X}$ torsor. We denote the associated graded piece by $K^{\le(d,c,t)}/K^{<(d,c,t)}$.

Proposition 7.7. *The endomorphism $I : K(\mathsf{QS}) \to K(\mathsf{QS})$ is locally finite. For every element $\mathfrak{X}$ in QS there exists a finite-dimensional $\mathbb{Z}[q]$-module of $K(\mathsf{QS})$ that is invariant under inertia and contains the motivic class $[\mathfrak{X}]$.*

Proof. Let $\mathfrak{X}$ and Γ be defined as in Proposition 7.6. Consider the finite-dimensional $\mathbb{Z}[q]$-submodule, L generated by the finitely many intermediate covers $\overline{\mathfrak{X}}/\operatorname{Stab}_\Gamma(I)$. Then the results of §6.3 and Proposition 7.6 show that

$$I(L) \subseteq L \mod K^{<(d,c,t)}.$$

The claim is now clear by induction. □

7.4 A descending filtration and diagonalization

Let $\mathfrak{X} \to X$ be a quasi-split clear gerbe with a central X-group scheme Z of reductive rank t. The Galois group $\Gamma = \pi_1(\overline{X}/X)$ of the minimal splitting Galois cover is then a subgroup of S_t, the group of permutations of t letters. This action induces an integer partition $\lambda \vdash t$ as explained in §7.1.

Definition 7.8. For a quasi-split clear gerbe $\mathfrak{X}$, the partition λ of integer t constructed as above is called the *twist type* of it.

The double-filtration in section 6.2 induces a double filtration of $K(\mathsf{QS})$ which will again be denoted by $K_{\geq(\cdot,\cdot)}$. We now introduce a refinement, $K_{\geq(\cdot,\cdot,\cdot)}$ of the former filtration and show that it is preserved by inertia endomorphism.

We first impose a well-ordering on the set of all integer partitions: for a given integer $t \geq 0$, we put the lexicographic ordering on the set of all partitions of t. And for any two integers $t < s$, we assume all partitions t are smaller than all partitions of s. In fact any well-ordering that satisfies the following two conditions would work for us

1. if $\lambda \vdash t$ and $\mu \vdash s$ and $t < s$ then $\lambda < \mu$; and
2. if $\lambda, \mu \vdash t$ and $b(\lambda) < b(\mu)$ then $\lambda < \mu$.

Now, given integers $r, n \geq 0$ and an integer partition $\lambda \vdash t$, the submodule $K_{\geq(r,n,\lambda)}$ is generated by those quasi-split clear gerbes in $K_{\geq(r,n)}$ that have twist type is at least λ.

Lemma 7.9. *The inertia endomorphism of* $K(\mathsf{QS})$ *respects the filtration* $K_{\geq(\cdot,\cdot,\cdot)}$.

Proof. In view of results of section 6.2 it suffices to consider a tuple (r,n,λ), and a quasi-split clear gerbe $\mathfrak{X} \to X$ with central rank r, discrete split central number n, and twist type λ. Also from Propositions 6.9 and 6.10 we only need to consider a central stratum $\mathfrak{Y}$ of $I\mathfrak{X}$ which is an $I^{z,0}\mathfrak{X}$-torsor over $\mathfrak{X}$. By Proposition 7.6, the motivic class $[\mathfrak{Y}]$ is a linear combination of $[\mathfrak{X}]$ and the intermediate covers $\overline{\mathfrak{X}}/\operatorname{Stab}_\Gamma(I)$ for a minimal splitting cover $\overline{X} \to X$ of the maximal torus $T \to X$ of the central band of $\mathfrak{X}$. We note that each of these stacks is a clear gerbe over the intermediate cover $\overline{X}/\operatorname{Stab} I$ and the maximal torus of $\mathfrak{X}$ pulls back to the maximal torus $T|_{\overline{X}/\operatorname{Stab} I}$ of the central band of $\overline{\mathfrak{X}}/\operatorname{Stab}_\Gamma(I)$. Since $\Gamma' = \operatorname{Stab} I$ is a proper subgroup of the Galois group $\Gamma = \pi_1(\overline{X}/X)$, the orbit space $\underline{r}/\Gamma'$ has strictly more elements than $\underline{r}/\Gamma$. Therefore the twist type of $\overline{\mathfrak{X}}/\operatorname{Stab}_\Gamma(I)$ is strictly bigger than that of $\mathfrak{X}$ in the well-ordering defined above.□

We can now finish the proof of our main result.

Theorem 7.10. *The operator I is diagonalizable on $K(\mathsf{QS})(q) = \mathbb{Q}(q) \otimes_{\mathbb{Q}[q]} K(\mathsf{QS})$ as a linear endomorphism of a $\mathbb{Q}(q)$-vector space. The eigenvalue spectrum of it is the set of all polynomials of the form*

$$nq^u \prod_{i=1}^{k} (q^{r_i} - 1).$$

Proof. In Lemma 7.9 we showed that I is triangularizable and by Proposition 7.6 the action of it on each graded piece $K_{\geq(r,n,\gamma)}/K_{>(r,n,\gamma)}$ is multiplication by the polynomial $nq^{r-t}\mathcal{Q}_{\gamma\vdash t}(q)$. This polynomial uniquely determines the triply (r,n,λ) resulting distinct eigenvalues associated to each graded piece. □

Remark 7.11. In fact I is diagonalizable as an endomorphism of the $\mathbb{Z}[q]$-module

$$K(\mathsf{QS})[q^{-1}, \{(\mathcal{Q}_\lambda - \mathcal{Q}_\mu)^{-1} : \forall \lambda \vdash t, \mu \vdash s\}).$$

7.5 Spectrum of the semisimple inertia of quasi-split stacks

We can now prove a semisimple version of Proposition 6.9 in terms of the reductive ranks of the quasi-split clear gerbes rather than their central ranks. Note that the finite group scheme $I^z/I^{z,0}$ is semisimple and product of semisimple commuting elements is semisimple. Therefore there are $\nu(\mathfrak{X})$ connected components of $I^{z,ss}/I^{z,ss,0}$ that map isomorphically to $\mathfrak{X}$. [1]

Proposition 7.12. *Let $\mathfrak{Y}$ be a stratum of $I^{ss}\mathfrak{X}$ not completely contained in $I^{ss,z}\mathfrak{X}$. Then $\rho(\mathfrak{Y}) \geq \rho(\mathfrak{X})$ and if $\rho(\mathfrak{Y}) = \rho(\mathfrak{X})$ then $\nu(\mathfrak{Y}) > \nu(\mathfrak{X})$.*

Proof. Let $\mathfrak{Y} \subseteq I^{ss}\mathfrak{X}$ be a strata of the semisimple inertia, in particular a locally closed substack of $I\mathfrak{X}$. Diagram 5.2 has a semisimple verison. The downward arrows π_1, π_2 and π_3 are all structure morphisms of relative commutative group schemes. Since

[1] In fact, the morphism $Z^{ss}/Z^{ss,0} \to Z/Z^0$ is an isomorphism of finite X-group schemes.

unipotency is preserved under the group homomorphisms we may divide each of these commutative group schemes with their unipotent radical.

$$\begin{array}{ccccc} I^{ss,z}\mathfrak{Y} & \overset{j}{\hookleftarrow} & \mathfrak{Y}\times_{\mathfrak{X}} I^{ss,z}\mathfrak{X} & \longrightarrow & I^{ss,z}\mathfrak{X} \\ & \searrow^{\pi_3} & \downarrow \pi_2 & \square & \downarrow \pi_1 \\ & & \mathfrak{Y} & \longrightarrow & \mathfrak{X} \end{array} \tag{7.3}$$

From previous diagram it is obvious that $\rho(\mathfrak{Y}) \geq \rho(\mathfrak{X})$. Now suppose $\mathrm{rank}^{z}(\mathfrak{Y}) = \mathrm{rank}^{z}(\mathfrak{X})$. Similar to the case of Proposition 6.9 we now pass to the quotients by connected components of unity to get a commutative diagram

$$\begin{array}{ccccc} I^{ss,z/ss,z,0}\mathfrak{Y} & \overset{\bar{j}}{\hookleftarrow} & (I^{ss,z/ss,z,0}\mathfrak{X})_{\mathfrak{Y}} & \longrightarrow & I^{ss,z/ss,z,0}\mathfrak{X} \\ & \searrow^{\pi_3} & \downarrow \pi_2 & & \downarrow \pi_1 \\ & & \mathfrak{Y} & \longrightarrow & \mathfrak{X} \end{array} \tag{7.4}$$

in level of stacks. Same analysis as in case of Proposition 6.9 shows that π_3 has strictly more sections that π_1. □

When $\mathfrak{X}$ is a quasi-split clear gerbe, the connectedness components of the semisimple central inertia $I^{ss,z}\mathfrak{X} \to \mathfrak{X}$ are also quasi-split clear gerbes; this yields a canonical stratification of $I^{ss,z}\mathfrak{X}$. The analogue of Propositions 6.10 and 6.11 is stated below:

Proposition 7.13. *Let $\mathfrak{Y}$ be a connectedness component of $I^{z,ss}\mathfrak{X} \to \mathfrak{X}$. Then $\rho(\mathfrak{Y}) = \rho(\mathfrak{X})$. We always have $\nu(\mathfrak{Y}) \geq \nu(\mathfrak{X})$, with equality happening if and only if the image of $\mathfrak{Y}$ in $I^{ss,z/ss,z,0}\mathfrak{X}$ maps down isomorphically to $\mathfrak{X}$.*

Proof. The proof is similar to that of Propositions 6.10 and 6.11 by considering the commutative diagram

$$\begin{array}{ccccc} I^{ss,z}(I^{ss,z}\mathfrak{X}) & \longrightarrow & I^{ss,z}\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & \square & \downarrow & & \downarrow \\ Z^{ss}\times_X Z^{ss} & \longrightarrow & Z^{ss} & \longrightarrow & X \end{array}$$

for the first claim where by restricting to $\mathfrak{Y}$ we get

$$\begin{array}{ccc} I^{ss,z}(\mathfrak{Y}) & \longrightarrow & \mathfrak{Y} \\ \downarrow & \square & \downarrow \\ Z^{ss}\times_X Y & \longrightarrow & Y. \end{array}$$

Note that $I^{z,ss}\mathfrak{X} \longrightarrow I^{ss,z/ss,z,0}\mathfrak{X}$ is a principal bundle for the connected group scheme $I^{ss,z,0}\mathfrak{X}$ over $\mathfrak{X}$, and therefore there is a bijection between connectedness components of the source and the target of this morphism. So $\mathfrak{Y}$ sits over a connectedness component $\mathfrak{Y}' \subseteq I^{z/z,0}\mathfrak{X}$. Similar to the case in Proposition 6.11 we now can form the following cartesian diagram

$$\begin{array}{ccc} I^{ss,z/ss,z,0}\mathfrak{Y} & \longrightarrow & I^{ss,z/ss,z,0}\mathfrak{X} \\ \downarrow{\scriptstyle \overline{\pi}_2} & \square & \downarrow{\scriptstyle \overline{\pi}_1} \\ \mathfrak{Y} & \longrightarrow & \mathfrak{X} \end{array}$$

and the rest of the proof is now similar to Proposition 6.11. □

The proofs of local finiteness and diagonalization of I^{ss} now follow similar to the case of the full inertia operator. We skip the details and only carry the computation of the spectrum as follows. When $\mathfrak{X}$ is a quasi-split clear gerbe,

$$\begin{aligned}[]
[I^{ss}\mathfrak{X}] = [I^{ss,z}\mathfrak{X}] \\
&+ \text{terms with larger reductive rank} \\
&+ \text{terms with same reductive rank but larger discrete-split central number.}
\end{aligned}$$

For central strata of the semisimple inertia has a simpler description than the case of the full inertia. The relative group space $I^{ss,z,0}\mathfrak{X} \to \mathfrak{X}$ is the base change of a quasi-split torus

$$\begin{array}{ccc} I^{ss,z,0}\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \end{array}$$

where X is the coarse moduli space of $\mathfrak{X}$ and T is the maximal torus of the central band of $\mathfrak{X}$. Computation of Proposition 7.6 therefore shows

$$\begin{aligned}[]
[I^{z,ss}\mathfrak{X}] &= \nu(\mathfrak{X})[I^{ss,z,0}\mathfrak{X}] + \text{terms of lower co-untwistedness} \\
&= \nu(\mathfrak{X})\mathfrak{Q}_{\lambda\vDash r}(q)[\mathfrak{X}] + \text{terms with finer twist types } \gamma > \lambda.
\end{aligned}$$

We may now consider a simpler filtration, $K_{\geq(\cdot,\cdot,\cdot)}$ than before: given integers $r, n \geq 0$, and a partition $\lambda \vdash r$ the submodule $K_{\geq(r,n,\lambda)}$ is generated by those quasi-split clear gerbes that have reductive rank at least r and if this rank is exactly r then the untwistedness of the central group scheme is at least n and if this quantity is exactly n, then the twist type is at least λ. This filtration is preserved by the semisimple inertia endomorphism and leads to

Theorem 7.14. *The endomorphism $I^{ss}: K(\mathsf{QS}) \to K(\mathsf{QS})$ is locally finite and triangularizable and the eigenvalue spectrum of it is the set of all polynomials of the form*

$$n\prod_{i=1}^{k}(q^{r_i}-1).$$

Moreover I^{ss} is a diagonalizable $\mathbb{Q}(q)$-linear endomorphism of the vector space $K(\mathsf{QS})(q)$.

Chapter 8

Examples

Example 5. A first simple example is the case of $[\mathrm{BGL}_2]$. Here, and in following examples we are suppressing the notation [.] for quotient stacks; thus unless mentioned otherwise, all quotients (of schemes) are stack quotients. Note that we have

$$I\,\mathrm{BGL}_2 = \mathrm{GL}_2\,/\,\mathrm{GL}_2 = (\mathrm{GL}_2)^{\mathrm{ss,\,eq}}/\,\mathrm{GL}_2 \sqcup (\mathrm{GL}_2)^{\mathrm{dist}}/\,\mathrm{GL}_2 \sqcup (\mathrm{GL}_2)^{\mathrm{ns}}/\,\mathrm{GL}_2\,.$$

The first stratum contains diagonalizable matrices with one eigenvalue, the second stratum diagonalizable matrices with distinct eigenvalues, and the third stratum the non-semisimple matrices. We study these three strata and their inertia:

First stratum: Consider the mapping $\mathbb{G}_m \to \mathrm{GL}_2$ via $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$. This is equivariant with respect to the natural GL_2-action, so we get an induced morphism of stacks

$$\mathbb{G}_m \times \mathrm{BGL}_2 \to \mathrm{GL}_2\,/\,\mathrm{GL}_2$$

which is easily seen to be an isomorphism onto the first stratum.

Second stratum: Let T be the standard maximal torus of GL_2. Let Δ be the centre of GL_2, which is the diagonal subtorus of T. Let N be the normalizer of T. We have a short exact sequence

$$0 \longrightarrow T \longrightarrow N \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

where $\mathbb{Z}_2$ is the Weyl group of GL_2. Note that $N = \mathbb{G}_m^2 \rtimes \mathbb{Z}_2$ is in fact a semi-direct product, by taking $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as the nontrivial element of $\mathbb{Z}_2 \subset N$. The induced action of the Weyl group $\mathbb{Z}_2$ on T is by swapping the two entries. The natural inclusion map $T \setminus \Delta \to \mathrm{GL}_2$ is equivariant for the inclusion of groups $N \subset \mathrm{GL}_2$, so we get an induced morphism of stacks

$$(T \setminus \Delta)/N \longrightarrow \mathrm{GL}_2\,/\,\mathrm{GL}_2$$

which is an isomorphism onto the second stratum. We will abbreviate this as

$$\mathfrak{X} = (T \setminus \Delta)/N.$$

Third stratum: Let H be the (commutative) subgroup of all matrices of the form $\left(\begin{smallmatrix} \lambda & \mu \\ 0 & \lambda \end{smallmatrix}\right)$. Note that H is the centralizer of every matrix of the form $\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$, with $a \neq 0$. Thus we see that the third stratum is isomorphic to $\mathbb{G}_m \times \mathrm{B}H$.

We conclude that in the level of motivic classes the inertia of the class $[\mathrm{BGL}_2]$ is given by

$$I[\mathrm{BGL}_2] = (q-1)[\mathrm{BGL}_2] + [\mathfrak{X}] + (q-1)[\mathrm{B}H].$$

Since H is commutative, we also have $I[\mathrm{B}H] = q(q-1)[\mathrm{B}H]$. We will now find the inertia of the second stratum $\mathfrak{X}$. Note that that the coarse moduli space of $\mathfrak{X}$ is the smooth variety $X = T \setminus \Delta/\mathbb{Z}_2$. Note also that $I\mathfrak{X} = I^z\mathfrak{X}$ as the stabilizer of any point in $T \setminus \Delta$ is commutative. We will write $\widetilde{X} = T \setminus \Delta$ to emphasize the fact that $T \setminus \Delta$ is a degree 2 cover of X.

Associated to the $\mathbb{Z}_2$-action on the group T, there exists a commutative X-group scheme

$$T' = \widetilde{X} \times_{\mathbb{Z}_2} T,$$

with fibre T. By Lemma 8.1, $\mathfrak{X}$ is the neutral gerbe, $\mathfrak{X} = B_X T'$. And $I^z\mathfrak{X} = I\mathfrak{X}$ fits in the cartesian diagram

$$\begin{array}{ccc} I\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ T' & \longrightarrow & X. \end{array}$$

The representation of $\mathbb{Z}_2$ on $\mathbb{A}^2$ given by swapping entries, yields a canonical closed embedding $T' \subset V$, into a rank 2 vector bundle over X. As in Proposition 7.1, this leads to

$$I[\mathfrak{X}] = (q^2-1)[\mathfrak{X}] - (q-1)^2(q-2)[\mathrm{B}\mathbb{G}_m^2]$$

Thus, the 4-dimensional $K(\mathrm{Var})$-module, L, of motives generated by the 4 motives

$$[\mathrm{BGL}_2], [\mathrm{B}H], [\mathfrak{X}], \text{ and } [\mathrm{B}\mathbb{G}_m^2]$$

is preserved by the inertia endomorphism I. The first element in this set is of central rank 1 and the other three are of central rank 2. $\mathrm{B}H$ has reductive rank 1, $\mathfrak{X}$ has reductive rank 2 with the nontrivial partition of 2 associated to it, and $\mathrm{B}\mathbb{G}_m^2$ has a rank 2 torus with the trivial partition of 2 associated to it. The eigenvalue spectrum is hence $\{q-1, q(q-1), q^2-1, (q-1)^2\}$. Inertia endomorphism is lower triangularizable on L and we have

$$I = \begin{pmatrix} q-1 & 0 & 0 & 0 \\ q-1 & q(q-1) & 0 & 0 \\ 1 & 0 & q^2-1 & 0 \\ 0 & 0 & -(q-1)^2(q-2) & (q-1)^2 \end{pmatrix}$$

with a set of eigenvectors

Eigenvalues	Eigenvectors
$q-1$	$-q(q-1)[\mathrm{BGL}_2] + q[\mathrm{B}H] + [\mathfrak{X}] + (q-1)[\mathrm{B}\mathbb{G}_m^2]$
$q(q-1)$	$[\mathrm{B}H]$
q^2-1	$[\mathfrak{X}] - \frac{(q-1)(q-2)}{2}[\mathrm{B}\mathbb{G}_m^2]$
$(q-1)^2$	$[\mathrm{B}\mathbb{G}_m^2]$

Table 8.1: Spectrum of the inertia endomorphism on a 4-dimensional $K(\mathrm{Var})$-submodule of $K(\mathsf{St})$ containing $[\mathrm{BGL}_2]$

Also I is diagonalizable on $L[q^{-1},(q-1)^{-1}]$ and the eigenprojections of $[\mathrm{BGL}_2]$ are

Eigenvalues	Eigenvectors
Π_{q-1}	$[\mathrm{BGL}_2] - \frac{q}{q-1}[\mathrm{B}H] - \frac{1}{q(q-1)}[\mathfrak{X}] - \frac{1}{q}[\mathrm{B}\mathbb{G}_m^2]$
$\Pi_{q(q-1)}$	$\frac{q}{q-1}[\mathrm{B}H]$
Π_{q^2-1}	$\frac{1}{q(q-1)}[\mathfrak{X}] - \frac{(q-2)}{2q}[\mathrm{B}\mathbb{G}_m^2]$
$\Pi_{(q-1)^2}$	$\frac{1}{2}[\mathrm{B}\mathbb{G}_m^2]$

Table 8.2: Eigenprojections of $[\mathrm{BGL}_2]$

Lemma 8.1. *Let Γ be a group acting on the group T by automorphisms, $\Gamma \to \mathrm{Aut}(T)$, and let $G = N \rtimes H$ be the associated semi-direct product of groups. Let X be a variety, $\widetilde{X} \to X$ a principal Γ-bundle, and $T' \to X$ is the associated form of T over X. The $B_X T' = \widetilde{X}/G$.*

Proof. Consider the diagram

$$\begin{array}{ccc} \widetilde{X} \times T & \longrightarrow & \widetilde{X} \times T \\ \downarrow & & \downarrow \\ \widetilde{X} & \longrightarrow & \widetilde{X} \end{array}$$

where Γ acts on the first column and G on the second column in the obvious way. Then the horizontal arrows are a morphism of T-bundles which is $\Gamma \to G$ equivariant. Thus we get an induced cartesian diagram of stacks

$$\begin{array}{ccc} (\widetilde{X} \times T)/\Gamma & \longrightarrow & (\widetilde{X} \times T)/G \\ \downarrow & & \downarrow \\ \widetilde{X}/H & \longrightarrow & \widetilde{X}/G \end{array} \quad \text{which we may rewrite as} \quad \begin{array}{ccc} T' & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & \widetilde{X}/G. \end{array}$$

Then the latter diagram induces a morphism $\mathfrak{X} \to \widetilde{X}/G$, which is then obviously an isomorphism. □

Example 6. In the previous example the central group schemes were always connected. We will now present an example that demonstrates how non-cennected central group schemes contribute to non-monic eigenvalues. Let $N = \mathbb{G}_m^2 \rtimes \mathbb{Z}_2$ be the group scheme introduced in previous example. In this example we study $\mathrm{B}N$. The inertia of $\mathrm{B}N$ has two obvious connectedness components:

$$I\,\mathrm{B}N = N/N = \mathbb{G}_m^2/N \sqcup \mathbb{G}_m^2 \times \{\sigma\}/N$$

where σ is the nontrivial element of $\mathbb{Z}_2$.

First stratum: This stratum is not already a gerbe (as the stabilizer of points on diagonal $\Delta \subset \mathbb{G}_m^2$ is not isomorphic to the stabilizer of other points). However the following is a stratification of it into clear gerbes:

$$\mathbb{G}_m^2/N = \Delta/N \sqcup \mathfrak{X},$$

where $\mathfrak{X} = \mathbb{G}_m^2 \setminus \Delta/N$ is the same quotient stack that appeared in previous example. The action of N on Δ is trivial so we have

$$[\mathbb{G}_m^2/N] = (q-1)[\mathrm{B}N] + [\mathfrak{X}].$$

Second stratum: This stratum is already a clear gerbe and we will denote it as $\mathfrak{Y}$. Any point $\langle(\mu,\gamma),\sigma\rangle$ of $\mathbb{G}_m^2 \times \{\sigma\}$ is conjugate to $\langle(\mu\gamma,1),\sigma\rangle$ which is canonical for the orbit. Thus the subscheme Y representing the points,

$$\{\langle(x,1),\sigma\rangle\} \subset \mathbb{G}_m^2 \times \{\sigma\},$$

is a coarse moduli space for this gerbe. This is isomorphic to $\mathbb{A}^1 \setminus \{0\}$. The stabilizer of this subscheme is a subgroup scheme $N' \subseteq N$, the fiber of which over a geometric point $\overline{x}$ of Y is the $\kappa(\overline{x})$-algebraic group

$$N'_{\overline{x}} = \{\langle (t,t),1\rangle : t \in \kappa(\overline{s})^{\times}\} \cup \{(xt,t),\sigma\rangle : t \in \kappa(\overline{s})^{\times}\}.$$

We notice that the mapping $Y/N' \to \mathfrak{Y}$ is an isomorphism of stacks and also that N' is a commutative Y-group scheme, acting trivially on Y. Therefore

$$[\mathfrak{Y}] = [Y][\mathrm{B}N'] = (q-1)[\mathrm{B}N']$$
$$I[\mathrm{B}N'] = [N'/N'] = [N'][\mathrm{B}N']$$

Finally $N'/(N')^0 \to Y$ is a degree two covering of X. The image of $(N')^0$ in $N'/(N')^0$ is isomorphic to Y and therefore so is the image of the other connected component. So N' is Zariski locally the union of two $\mathbb{G}_m$-torsors over Y. Pulling back along $\mathfrak{Y} \to Y$ we have

$$I[\mathrm{B}N'] = 2(q-1)[\mathrm{B}N'].$$

We conclude that the $K(\mathrm{Var})$-submodule of $K(\mathsf{St})$ generated by

$$[\mathrm{B}N], [\mathrm{B}N'], [\mathfrak{X}], \text{ and } [\mathrm{B}\mathbb{G}_m^2]$$

is invariant under inertia endomorphism. The first two generators have central rank one, and $[\mathrm{B}N]$ has split central number one whereas $[\mathrm{B}N']$ has split central number two. The spectrum of I restricted to this submodule is the set

$$\{(q-1), 2(q-1), q^2-1, (q-1)^2\}$$

as expected.

Example 7. Another simple example that shows many features of this theory is the stack BGL_3. As before, the inertia stack is isomorphic to the quotient stack $[\mathrm{GL}_3/\mathrm{GL}_3]$ via conjugation action of GL_3 on itself. We first stratify this quotient according to Jordan canonical forms: let J^k_λ be the subscheme of all general linear matrices with k-distinct eigenvalues and $\lambda \vDash 3$ is a partition of 3 indicating format of the Jordan blocks and $R^k_\lambda \rightrightarrows J^k_\lambda$ is the groupoid representation of restriction of $[\mathrm{GL}_3/\mathrm{GL}_3]$ to J^k_λ. Then we have a stratification

$$\begin{aligned}[\mathrm{GL}_3/\mathrm{GL}_3] = [J^1_{(3)}/R^1_{(3)}] \sqcup [J^1_{(2,1)}/R^1_{(2,1)}] \sqcup [J^1_{(1,1,1)}/R^1_{(1,1,1)}] \\ \sqcup [J^2_{(2,1)}/R^2_{(2,1)}] \sqcup [J^2_{(1,1,1)}/R^2_{(1,1,1)}] \\ \sqcup [J^3_{(1,1,1)}/R^3_{(1,1,1)}]\end{aligned}$$

The action of R^k_λ on J^k_λ by conjugation is always trivial unless in presence of Jordan blocks of same dimension with distinct eigenvalues (which can then be permuted). Thus

$$\begin{aligned}[\mathrm{GL}_3 / \mathrm{GL}_3] = J^1_{(3)} \times \mathrm{B}R^1_{(3)} \sqcup J^1_{(2,1)} \times \mathrm{B}R^1_{(2,1)} \sqcup J^1_{(1,1,1)} \times \mathrm{B}R^1_{(1,1,1)} \\ \sqcup J^2_{(2,1)} \times \mathrm{B}R^2_{(2,1)} \sqcup J^2_{(1,1,1)} \times \mathrm{B}R^2_{(1,1,1)} \\ \sqcup J^3_{(1,1,1)} \times \mathrm{B}R^3_{(1,1,1)}\end{aligned}$$

We recall the notation of Example 5 for the subgroup of upper-triangular 2×2 matrices with a single eigenvalue of multiplicity two:

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{G}_m \right\}.$$

This represents a commutative group scheme. Now, easy computations show that all R^k_λ's are subgroup schemes of GL_3 and in fact

Groupoid	Group scheme structure	Commutative?
$R^1_{(3)}$	$\left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : a \in \mathbb{G}_m, b, c \in \mathbb{A}^1 \right\}$	Yes
$R^1_{(2,1)}$	$\left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix} : a, e \in \mathbb{G}_m, b, c, d \in \mathbb{A}^1 \right\}$	No
$R^1_{(1,1,1)}$	GL_3	No
$R^2_{(2,1)}$	$H \times \mathbb{G}_m$	Yes
$R^2_{(1,1,1)}$	$\mathrm{GL}_2 \times \mathbb{G}_m$	No
$R^3_{(1,1,1)}$	$\mathbb{G}_m^3 \rtimes S_3$	Yes

Table 8.3: Stratification of GL_3

$$\begin{aligned}[\mathrm{GL}_3 / \mathrm{GL}_3] = (q-1)[\mathrm{B}R^1_{(3)}] + (q-1)[\mathrm{B}R^1_{(2,1)}] + (q-1)[\mathrm{BGL}_3] \\ + (q-1)(q-2)[\mathrm{B}H][\mathrm{B}\mathbb{G}_m] + (q-1)(q-2)[\mathrm{BGL}_2][\mathrm{B}\mathbb{G}_m] \\ + [\mathbb{G}_m^3 / \mathbb{G}_m^3 \rtimes S_3]\end{aligned}$$

Since inertia respects the commutative algebra structure of $K(\mathsf{St})$ we may use the previous example to compute the effect of inertia on terms of the second line above. Since $R^1_{(3)}$ is commutative we also have

$$I[\mathrm{B}R^1_{(3)}] = [R^1_{(3)}][\mathrm{B}R_{(3)^1}] = q^2(q-1)[\mathrm{B}R^1_{(3)}].$$

Strata	Canonical form for an orbit	Centralizer of the canonical form
$\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix} : a \neq e$	$\begin{pmatrix} a & b+cd/(a-e) & 0 \\ 0 & a & 0 \\ 0 & 0 & e \end{pmatrix}$	$G_1 = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{pmatrix} : x, z \neq 0 \right\}$
$\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & a \end{pmatrix} : c, d \neq 0$	$\begin{pmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & cd & a \end{pmatrix}$	$G_2 = \left\{ \begin{pmatrix} x & y & z \\ 0 & x & 0 \\ 0 & w & x \end{pmatrix} : x \neq 0 \right\}$
$\begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & d & a \end{pmatrix} : d \neq 0$	$\begin{pmatrix} a & b+d & 0 \\ 0 & a & 0 \\ 0 & d & a \end{pmatrix}$	$G_3 = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & z & x \end{pmatrix} : x \neq 0 \right\}$
$\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : c \neq 0$	$\begin{pmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$	$G_4 = \left\{ \begin{pmatrix} x & y & z \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} : x \neq 0 \right\}$
$\begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$	itself	G

Table 8.4: Stratification of $R^1_{(2,1)}$

The case of $\mathfrak{Y} = [\mathbb{G}_m^3/\mathbb{G}_m^3 \rtimes S_3]$ is similar to that of $[\mathbb{G}_m^2/\mathbb{G}_m^2 \rtimes \mathbb{Z}_2]$. It remains to analyze the action of $G = R^1_{(2,1)}$ on itself. We need to stratify G/G to several substacks which is carried out in Table 8.4. It follows that

$$I[\mathrm{B}G] = q(q-1)[\mathrm{B}G] + q^3(q-1)(q-2)[\mathrm{B}G_1] \\ + q(q-1)^3[\mathrm{B}G_2] + q(q-1)^2[\mathrm{B}G_3] + q(q-1)^2[\mathrm{B}G_4].$$

We conclude that $[\mathrm{BGL}_3]$ is contained in a 9-dimensional $K(\mathrm{Var})$-submodule of $K(\mathsf{St})$ which is diagonalizable (Table 8.5).

Central rank	Reductive rank	Twist type	Pivot elements	Eigenvalue
1	1	(1)	$[\mathrm{BGL}_3]$	$q-1$
2	2	(2,0)	$[\mathrm{BGL}_2][\mathrm{B}\mathbb{G}_m]$	$(q-1)^2$
	1	(1)	$[\mathrm{B}G]$	$q(q-1)$
3	3	(0,0,1)	$[\mathfrak{Y}]$	q^3-1
		(1,1,0)	$[\mathfrak{X}][\mathrm{B}\mathbb{G}_m]$	$(q^2-1)(q-1)$
		(3,0,0)	$[\mathrm{B}\mathbb{G}_m^3]$	$(q-1)^3$
	2	(2,0)	$[\mathrm{B}H][\mathrm{B}\mathbb{G}_m], [\mathrm{B}G_1]$	$q(q-1)^2$
	1	(1)	$[\mathrm{B}R^1_{(3)}], [\mathrm{B}G_3], [\mathrm{B}G_4]$	$q^2(q-1)$
4	1	(1)	$[\mathrm{B}G_2]$	$q^3(q-1)$

Table 8.5: Spectrum of the inertia endomorphism of a 9-dimensional $K(\mathrm{Var})$-submodule of $K(\mathsf{St})$ containing $[\mathrm{BGL}_3]$

Part III

Algebroids and their Hall algebras

Chapter 9

Linear Stacks

9.1 Algebraic stacks

Let us briefly summarize our conventions about algebraic stacks.

We choose a noetherian base ring R, and we fix our base category $\mathscr{S}$ to be the category of R-schemes, endowed with the étale topology.Over $\mathscr{S}$ we have a canonical sheaf of R-algebras $\mathcal{O}_{\mathscr{S}}$, it is represented by $\mathbb{A}^1 = \mathbb{A}^1_{\operatorname{Spec} R}$, and called the *structure sheaf*.

We will assume our algebraic stacks to be locally of finite type. Thus, an *algebraic stack*, is a stack over the site $\mathscr{S}$, which admits a presentation by a smooth groupoid $X_1 \rightrightarrows X_0$, where X_0 and X_1 are algebraic spaces, *locally of finite type*, the source and target morphism $s, t : X_1 \to X_0$ are smooth, and the diagonal $X_1 \to X_0 \times X_0$ is of finite type.

If G is an algebraic group acting on the algebraic space X, we will denote the quotients stack by X/G, because we fear the more common notation $[X/G]$ would lead to confusion with the notation for elements of various K-groups of schemes and stacks.

Coherent sheaves

In particular, an algebraic stack X is a fibered category $X \to \mathscr{S}$. The category X inherits a topology from $\mathscr{S}$, called the étale topology, and X endowed with this topology is the *big étale site* of X. Sheaves over X are by definition sheaves on this big étale site. For example, $\mathcal{O}_{\mathscr{S}}$ induces a sheaf of R-algebras on X, which is denoted by $\mathcal{O}_X$, and called the *structure sheaf* of X. It is represented by $\mathbb{A}^1_X$.

A sheaf $\mathscr{F}$ over X induces for every object x of X lying over the object U of $\mathscr{S}$ a sheaf on the usual (small) étale site of the scheme U, denoted $\mathscr{F}_U$. Moreover, for every morphispm $\alpha : y \to x$ lying over $f : V \to U$, we obtain a morphism of sheaves $\alpha^* : \mathscr{F}_U \to f_* \mathscr{F}_V$. (The α^* satisfy an obvious cocycle condition.) The data of the sheaves

$\mathscr{F}_U$, together with the compatibility morphisms α^*, is equivalent to the data defining $\mathscr{F}$. For example, the structure sheaf $\mathscr{O}_X$ induces the structure sheaf on U, for every such x/U.

A sheaf of $\mathscr{O}_X$-modules is *coherent*, if for all x/U the sheaf $\mathscr{F}_U$ is a coherent sheaf of $\mathscr{O}_U$-modules, and all compatibility morphisms $\alpha^* : f^*\mathscr{F}_U \to \mathscr{F}_V$ are *isomorphisms* of sheaves of $\mathscr{O}_V$-modules.

For example, a groupoid presentation $X_1 \rightrightarrows X_0$ of X, and a coherent sheaf $\mathscr{F}_0$ on X_0, together with an isomorphism $s^*\mathscr{F}_0 \to t^*\mathscr{F}_0$, satisfying the usual cocycle condition on $X_2 = X_1 \times_{X_0} X_1$, give rise to a coherent sheaf on X.

Representable coherent sheaves

A sheaf $\mathscr{F}$ over X is *representable*, if there exists an algebraic stack $Y \to X$, with representable structure morphism $Y \to X$, such that $\mathscr{F}$ is isomorphic to the sheaf of sections of $Y \to X$.

A coherent sheaf $\mathscr{F}$ over the algebraic stack X is representable if and only if for every x/U, the coherent $\mathscr{O}_U$-module $\mathscr{F}_U$ is reflexive, i.e., isomorphic to $\mathscr{F}_U^{\vee\vee}$. So $\mathscr{F}$ is representable if and only if $\mathscr{F}$ is isomorphic to $\mathscr{F}^{\vee\vee}$ as an $\mathscr{O}_X$-module. If $\mathscr{F}$ is representable, it is represented by the finite type affine X-scheme $Y = \operatorname{Spec}_X(\operatorname{Sym}_{\mathscr{O}_X}\mathscr{F}^\vee)$.

9.2 Linear algebraic stacks

We will review the definition of linear algebraic stacks, and some basic constructions. For definitions and basic properties of fibered categories we refer the reader to [20, Exposé VI].

Suppose $\mathfrak{X} \to \mathscr{S}$ is a category over $\mathscr{S}$. We write $\mathfrak{X}(S)$ for the fiber of $\mathfrak{X}$ over the object S of $\mathscr{S}$. If $f : S' \to S$ is a morphism in $\mathscr{S}$, and $x' \in \mathfrak{X}(S')$ and $x \in \mathfrak{X}(S)$ are $\mathfrak{X}$-objects lying over S' and S, respectivly, we write $\operatorname{Hom}_f(x', x)$ for the set of morphisms from x' to x in $\mathfrak{X}$, lying over f. For $S' = S$ and $f = \mathrm{id}_S$, we write $\operatorname{Hom}_S(x', x)$.

Recall that a morphism $\alpha : x' \to x$ lying over $f : S' \to S$ is *cartesian*, if for every object x'' of $\mathfrak{X}(S)$, composition with α induces a bijection $\operatorname{Hom}_S(x'', x') \xrightarrow{\simeq} \operatorname{Hom}_f(x'', x)$. Recall further that $\mathfrak{X} \to \mathscr{S}$ is a *fibered category*, if every composition of cartesian morphisms is cartesian, and if for every $f : S' \to S$ in $\mathscr{S}$, and every x over S, there exists a cartesian morphism over f with target x. A *cartesian functor* between categories over $\mathscr{S}$ is one that preserves cartesian morphisms.

If $\mathfrak{X}$ is a fibered category over $\mathscr{S}$, the subcategory of $\mathfrak{X}$, consisting of the same objects and all cartesian morphisms is a category fibered in groupoids over $\mathscr{S}$. We denote it by $\mathfrak{X}_{\mathrm{cfg}}$, and call it the *underlying category fibered in groupoids*.

Definition 9.1. A category $\mathfrak{X}$ over $\mathscr{S}$ is an $\mathscr{O}$-**linear category over** $\mathscr{S}$, if for every $f : S' \to S$ in $\mathscr{S}$ and all $x' \in \mathfrak{X}(S')$, $x \in \mathfrak{X}(S)$, the set $\operatorname{Hom}_f(x', x)$ is endowed with the structure of an $\mathscr{O}(S')$-module, in such a way that for every pair of morphisms $g : S'' \to S'$, $f : S' \to S$, and every triple of objects $x'' \in \mathfrak{X}(S'')$, $x' \in \mathfrak{X}(S')$, $x \in \mathfrak{X}(S)$, the composition

$$\mathrm{Hom}_f(x', x) \times \mathrm{Hom}_g(x'', x') \longrightarrow \mathrm{Hom}_{f \circ g}(x'', x)$$

is $\mathscr{O}(S')$-bilinear. Here the $\mathscr{O}(S'')$-modules, $\mathrm{Hom}_g(x'', x')$ and $\mathrm{Hom}_{f\circ g}(x'', x)$ inherit the structure of $\mathscr{O}(S')$-modules via pullback along g.

An $\mathscr{O}$-**linear functor** $F : \mathfrak{X} \to \mathfrak{Y}$ between $\mathscr{O}$-linear categories is a functor of categories over $\mathscr{S}$, such that for every $f : S' \to S$, and all $x' \in \mathfrak{X}(S')$, $x \in \mathfrak{X}(S)$ the map $\mathrm{Hom}_f(x', x) \to \mathrm{Hom}_f\big(F(x'), F(x)\big)$ is $\mathscr{O}(S')$-linear.

Let $\mathfrak{X}$ be an $\mathscr{O}$-linear fibered category over $\mathscr{S}$. Pullback in $\mathfrak{X}$ is $\mathscr{O}$-linear, i.e., if $f : S' \to S$ is a morphism in $\mathscr{S}$, and $x, y \in \mathfrak{X}(S)$ are objects with pullbacks $x', y' \in \mathfrak{X}(S')$, the pullback map $f^* : \mathrm{Hom}_S(x, y) \to \mathrm{Hom}_{S'}(x', y')$ is $\mathscr{O}(S)$-linear. So if we fix objects $x, y \in \mathfrak{X}(S)$, the presheaf $\underline{\mathrm{Hom}}_S(x, y)$ over the usual (small) étale site of S, defined by $\underline{\mathrm{Hom}}_S(x, y)(T) = \mathrm{Hom}_T(x|_T, y|_T)$, for every étale $T \to S$, is a presheaf of $\mathscr{O}_S$-modules. Moreover, for *any* morphism $f : S' \to S$ in $\mathscr{S}$, we have a natural homomorphism of sheaves of $\mathscr{O}_{S'}$-modules $f^* \underline{\mathrm{Hom}}_S(x, y) \to \underline{\mathrm{Hom}}_{S'}(x', y')$.

Definition 9.2. A **linear algebraic stack** is an $\mathscr{O}$-linear fibered category $\mathfrak{X}$ over $\mathscr{S}$, such that

(i) for every object $S \in \mathscr{S}$, and every pair $x, y \in \mathfrak{X}(S)$, the presheaf of $\mathscr{O}_S$-modules $\underline{\mathrm{Hom}}_S(x, y)$ is a coherent sheaf over S, which is representable by a finite type affine S-scheme;

(ii) for every morphism $f : S' \to S$ the pullback homomorphism $f^* \underline{\mathrm{Hom}}_S(x, y) \to \underline{\mathrm{Hom}}_{S'}(x', y')$ is an isomorphism of coherent $\mathscr{O}_{S'}$-modules; and,

(iii) the underlying category fibered in groupoids $\mathfrak{X}_{\mathrm{cfg}} \to \mathscr{S}$ is an algebraic stack over R (locally of finite type).

A **morphism** of linear algebraic stacks is an $\mathscr{O}$-linear cartesian functor over $\mathscr{S}$.

Remark 9.3. If $\mathfrak{X}$ is a linear algebraic stack, with underling algebraic stack $X = \mathfrak{X}_{\mathrm{cfg}}$, there exists a representable coherent sheaf $\mathscr{H}$ over $X \times X$, which represents the sheaf over $X \times X$, whose set of sections over the pair $x, y \in X(S)$ is the $\mathscr{O}(S)$-module $\mathrm{Hom}_S(x, y)$. The sheaf $\mathscr{H}$ is the *universal sheaf of homomorphisms.* The subsheaf $\mathscr{I} \subset \mathscr{H}$ representing isomorphisms is naturally identified with X, and the projection to $X \times X$ with the diagonal.

Pulling back $\mathscr{H}$ via the diagonal to X, we obtain the *universal sheaf of endomorphisms*, which represents the sheaf whose set of sections over $x \in X(S)$ is the $\mathscr{O}(S)$-algebra $\mathrm{End}_S(x)$.

The linear algebraic stack $\mathfrak{X}$ can be reconstructed from its underlying algebraic stack X, and the sheaf of $\mathcal{O}_{X\times X}$-algebras $\mathcal{H}$. We leave it to the reader to write down axioms for the pair $(X,\mathcal{H})$, which assure that $(X,\mathcal{H})$ comes from a linear algebraic stack.

Examples

Example 8. Let X be a projective R-scheme. The linear stack $\mathfrak{Coh}_X$ has as objects lying over the R-scheme S, the coherent sheaves on $X \times S$, which are flat over S. For a morphism of R-schemes $f : S' \to S$, and $\mathcal{F}' \in \mathfrak{Coh}_X(S')$, and $\mathcal{F} \in \mathfrak{Coh}_X(S)$, we set $\operatorname{Hom}_f(\mathcal{F}',\mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_{X\times S'}}(\mathcal{F}', f^*\mathcal{F})$. A morphism $\mathcal{F}' \to \mathcal{F}$ in $\mathfrak{Coh}_X$ over f in $\mathcal{S}$ is cartesian, if it induces an isomorphism $\mathcal{F}' \cong f^*\mathcal{F}$.

The linear stack $\mathfrak{Coh}_X$ is algebraic. Let $\pi : X \times S \to S$ be the projection on the second component. The fact that $\underline{\operatorname{Hom}}_S(\mathcal{F},\mathcal{G})$ is represented by a finite type affine S-scheme (whose formation commutes with base change) follows from the fact that $\pi_* \underline{\operatorname{Hom}}_S(\mathcal{F},\mathcal{G})$ is reflexive, and equal to the dual of a coherent $\mathcal{O}_S$-module, whose formation commutes with base change (see [21, EGA III 7.7.8, 7.7.9]).

The fact that $(\mathfrak{Coh}_X)_{\mathrm{cfg}}$ is algebraic and locally of finite type is proved in [34, 4.6.2.1].

Example 9. As a special case of the previous example, consider the case $X = \operatorname{Spec} R$. Then the linear algebraic stack $\mathfrak{Coh}_{\operatorname{Spec} R}$ is the *linear stack of vector bundles*, notation $\mathfrak{Vect}$. The underlying algebraic stack $\mathfrak{Vect}_{\mathrm{cfg}}$ is the disjoint union $\bigsqcup_{n\geq 0} BGL_n$. The sheaf $\mathcal{H}$ over

$$\bigsqcup_{n\geq 0} BGL_n \times \bigsqcup_{n\geq 0} BGL_n = \bigsqcup_{n,m\geq 0} B(GL_n \times GL_m)$$

is given by the natural representation $M(m\times n)$ of $GL_n \times GL_m$ over the component $B(GL_n \times GL_m)$.

Example 10. A generalization of $\mathfrak{Vect}$ in a different direction is given by quiver representations.

Let Q be a quiver. The stack of representations of Q, notation $\mathfrak{Rep}_Q$, has as $\mathfrak{Rep}_Q(S)$ the set of diagrams $(\mathcal{F})$ in the shape of Q of locally free finite rank $\mathcal{O}_S$-modules. For a morphism $f : S' \to S$ of R-schemes we have $\operatorname{Hom}_f(\mathcal{F}',\mathcal{F})$ is the $\mathcal{O}(S')$-module of homomorphisms $\mathcal{F}' \to f^*\mathcal{F}$ of diagrams of locally free $\mathcal{O}_{S'}$-modules.

Example 11. As a toy example, let A be an R-algebra scheme of finite type, with group scheme of units $A^\times$, also of finite type. Then we define the linear stack of $A^\times$-torsors to have as objects over the R-scheme S the right $A^{\times}$-torsors over S, and for $f: S' \to S$ and $A^{\times}$-torsors P' over S' and P over S, we set $\mathrm{Hom}_f(P',P) = \mathrm{Hom}_{S'}(P', f^*P) = P' \times_{A^{\times}} A \times_{A^{\times}} f^*P$. In this example, the underlying algebraic stack is $BA^{\times}$ and we have $\mathscr{H} = A^{\times} \backslash A / A^{\times}$.

The case $A = 0$ is not excluded. The associated linear stack is $\mathrm{id}: \mathscr{S} \to \mathscr{S}$. All $\underline{\mathrm{Hom}}_f(x,y)$ are singletons, endowed with their unique module structure. This stack is represented by $\mathrm{Spec}\, R$. It can also be thought of as the stack of zero-dimensional vector bundles.

Substacks

Let $\mathfrak{X}$ be a linear algebraic stack with underlying algebraic stack $X = \mathfrak{X}_{\mathrm{cfg}}$. If $Y \subset X$ is a locally closed algebraic substack, there is a canonical linear algebraic stack $\mathfrak{Y}$, with underlying algebraic stack $\mathfrak{Y}_{\mathrm{cfg}} = Y$. In fact, we can define $\mathfrak{Y}$ to be the full subcategory of $\mathfrak{X}$ consisting of objects which are in X.

Fibered products

Let $F: \mathfrak{X} \to \mathfrak{Z}$ and $G: \mathfrak{Y} \to \mathfrak{Z}$ be cartesian morphisms of $\mathscr{O}$-linear fibered categories. We define a new $\mathscr{O}$-linear fibered category $\mathfrak{W}$ as follows: objects of $\mathfrak{W}$ over the object T of $\mathscr{S}$ are triples (x,α,y), where x is an $\mathfrak{X}$-object over T, y is a $\mathfrak{Y}$-object over T, and α is an isomorphism $\alpha: F(x) \to G(y)$, over T. A morphism from (x',α',y') to (x,α,y) over $T' \to T$ is a pair of morphisms $f: x' \to x$ over $T' \to T$ and $g: y' \to y$ over $T' \to T$, such that $\alpha \circ F(f) = G(g) \circ \alpha'$.

In other words, we can write the set of morphisms from (x',α',y') to (x,α,y) over $\varphi: T' \to T$ as the fibered product

$$\mathrm{Hom}_{\varphi}(x',x) \times_{\mathrm{Hom}_{\varphi}(F(x'),G(y))} \mathrm{Hom}_{\varphi}(y',y)\,,$$

and as each of the sets in this fibered product is an $\mathscr{O}(T')$-module, and the maps are linear, this fibered product is also an $\mathscr{O}(T')$-module. We leave it to the reader to verify that composition is bilinear.

Let us verify that $\mathfrak{W}$ is a fibered category. Suppose that (x,α,y) is a triple over T, and $\varphi: T' \to T$ a morphism in $\mathscr{S}$. We construct a triple (x',α',y') over T' by taking as x' a pullback of x via φ, and for y' a pullback of y via φ. Then, as G is cartesian, $G(y')$ is a pullback of $G(y)$ via φ. Hence there exists a unique morphism $\alpha': F(x') \to G(y')$ covering T', such that $\alpha \circ F(x' \to x) = G(y' \to y) \circ \alpha'$. Then α' is cartesian, because cartesian morphisms satisfy the necessary two out of three property. Then α' is invertible, because cartesion morphisms covering an identity are invertible. The triple (x',α',y') comes with a given morphism to (x,α,y) which covers φ. It is easily verified that this morphism is cartesian.

Therefore, $\mathfrak{W}$ is an $\mathcal{O}$-linear fibered category. By construction, the two projections $\mathfrak{W} \to \mathfrak{X}$ and $\mathfrak{W} \to \mathfrak{Y}$ are cartesian. We call $\mathfrak{W}$ the *fibered product* of $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathfrak{Z}$.

Suppose $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ are linear algebraic stacks, with underlying algebraic stacks X, Y and Z, respectively. For triples (x', α', y') and (x, α, y) over S, the presheaf $\underline{\mathrm{Hom}}_S\big((x',\alpha',y'),(x,\alpha,y)\big)$ is equal to the fibered product

$$\underline{\mathrm{Hom}}_S(x',x) \times_{\underline{\mathrm{Hom}}_S(Fx',Gy)} \underline{\mathrm{Hom}}_S(y',y)\,,$$

and is therefore a representable coherent sheaf of $\mathcal{O}_S$-modules. We see that $\mathfrak{W}$ is again a linear algebraic stack. Morover, the underlying algebraic stack of W is the fibered product $X \times_Z Y$.

Lack of locality

Remark 9.4. Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are linear algebraic stacks, with underlying algebraic stacks X and Y. We can construct a disjoint union linear algebraic stack $\mathfrak{X} \sqcup \mathfrak{Y}$ whose underlying algebraic stack is $X \sqcup Y$, by declaring all homomorphisms between objects of $\mathfrak{X}$ and objects of $\mathfrak{Y}$ to be zero. This concept of disjoint union is not useful for our purposes. For the linear algebraic stacks we are interested in, the underlying algebraic stack often decomposes into a disjoint union, even though the linear algebraic stack does not. An example is given by the linear stack of vector bundles $\mathfrak{Vect}$, Example 9.

Thus linear algebraic stacks exhibit less local behaviour than algebraic stacks, and are therefore less geometrical. This is one of the reasons we prefer to work with *algebroids*, rather than linear algebraic stacks.

Chapter 10

Algebroids

10.1 Finite type algebras

Definition 10.1. Let X be an algebraic stack. By an **algebra** over X, we mean a sheaf of $\mathcal{O}_X$-algebras over X. If the algebra A over the algebraic stack X is an algebraic stack itself, i.e., if the structure morphism $A \to X$ is a representable morphism of stacks, then we say that A is **representable**. If, in addition, the underlying sheaf of $\mathcal{O}$-modules of A coherent, we call A a **finite type algebra** over X.

Inertia representation

Whenever $A \to X$ is an algebra over the algebraic stack X, we have a tautological morphism of sheaves of groups over X

$$I_X \longrightarrow \underline{\mathrm{Aut}}_X(A)\,. \tag{10.1}$$

Here I_X is the inertia stack of X, i.e., the stack of pairs (x, φ), where x is an object of X, and φ an automorphism of x, and $\underline{\mathrm{Aut}}_X(A)$ is the sheaf of automorphisms of the sheaf of algebras A over X. To construct (10.1), consider the stack of sheaves of algebras $\mathfrak{Alg}$ over $\mathcal{S}$, which has as objects over the scheme S, the sheaves of $\mathcal{O}_S$-algebras on the usual (small) étale site of S. A morphism from the sheaf of $\mathcal{O}_{S'}$-algebras A' over S', covering the morphism of schemes $f : S' \to S$, to the sheaf of $\mathcal{O}_S$-algebras A over S, is, by definition, an isomorphism of sheaves of $\mathcal{O}_{S'}$-algebras $A' \to f^*A$. The sheaf of algebras $A \to X$ gives rise to a morphism of $\mathcal{S}$-stacks $a : X \to \mathfrak{Alg}$. We get an induced morphism on inertia stacks $I_X \to I_{\mathfrak{Alg}}$, and notice that $a^*I_{\mathfrak{Alg}} = \underline{\mathrm{Aut}}_X(A)$.

With this definition, an automorphism φ of the object x of the stack X is mapped to the *inverse* of the restriction morphism $\varphi^* : A(x) \to A(x)$.

Lemma 10.2. *Suppose X is a gerbe over the algebraic space S, and $A \to X$ is an algebra. Then there exists a sheaf of $\mathcal{O}_Y$-algebras B, and an isomorphism $A \cong B|_X$ if and only if the inertia representation $I_X \to \underline{\mathrm{Aut}}_X(A)$ is trivial.*

If this is the case, then A is representable or of finite type if and only if B is. □

We can pull back the sheaf of algebras A over X, via the structure morphism $I_X \to X$, to obtain the sheaf of algebras $A|_{I_X}$. This sheaf of algebras is endowed with a tautological automorphism, induced from (10.1). In fact, the morphism 10.1 induces over each object $x \in X$ an action of $\mathrm{Aut}(x)$ on A_x. Therefore the objects of A_{I_X} over an object x of X, are triples (x, φ, a) where φ is an automorphism of x, and $a \in A_x$ is an object of A lying over x. The tautological automorphism of A_{I_X} maps (x, φ, a) to $(x, \varphi, \varphi(a))$. The algebra of invariants for this automorphism consists of objects (φ, a) such that $\varphi(a) = a$. We shall denote this algebra by $A^{\mathrm{fix}}_{I_X}$.

The following statement is somewhat tautological, and holds more generally than for algebras.

Proposition 10.3. *Suppose that A is a representable algebra over the algebraic stack X. Then the inertia stack of A is naturally identified with $A^{\mathrm{fix}}_{I_X}$. In particular, I_A is a representable algebra over I_X.*

Proof. We have a commutative diagram of algebraic stacks

$$\begin{array}{ccc} I_A & \longrightarrow & A \\ \downarrow & & \downarrow \\ I_X & \longrightarrow & X \end{array}$$

which identifies I_A with a substack of $A|_{I_X}$. In fact, the factorization $I_A \to A|_{I_X}$ is a monomorphism if and only if $A \to X$ is representable [46, Tag 04YY]. The algebra $A|_{I_X}$ is the stack of triples (x, φ, a), where x is an object of X, φ is an automorphism of x, and $a \in A(x)$ is an object of A lying over x. Such a triple is in I_A, if and only if $\varphi \in \mathrm{Aut}(x)$ is in the subgroup $\mathrm{Aut}(a) \subset \mathrm{Aut}(x)$. This is equivalent to φ fixing a under the action of $\mathrm{Aut}(x)$ on $A(x)$. This is the claim. □

In fact, the fibre of I_A over the objects x of X is equal to

$$I_A(x) = \{(\varphi, a) \in \mathrm{Aut}(x) \times A(x) \mid \varphi^*(a) = a\}.$$

The fibre of $I_A(x)$ over $\varphi \in \mathrm{Aut}(x)$ is the subalgebra $A(x)^\varphi \subset A(x)$, and the fibre of $I_A(x)$ over $a \in A(x)$ is the subgroup $\mathrm{Stab}_{\mathrm{Aut}(x)}(a) \subset \mathrm{Aut}(x)$.

Algebra bundles

Definition 10.4. We call a finite type algebra $A \to X$ an **algebra bundle**, if the underlying $\mathcal{O}_X$-module is locally free (necessarily of finite rank).

Lemma 10.5. *Let $A \to X$ be a finite type algebra over the algebraic stack X. There is a non-empty open substack $U \subset X$, such that $A|_U$ is an algebra bundle.*

PROOF. The claim is true for the scheme case, so by considering a smooth presentation of X, we can show that there exists a non-empty scheme V, together with a smooth morphism $V \to X$, such that $A|_V$ is an algebra bundle. The image U of V in X is an open substack, and $A|_U$ is an algebra bundle, because local freeness is local in the smooth topology. □

Remark 10.6. By considering the representation of A on itself by left multiplication, we see that every algebra bundle is a sheaf of subalgebras of the algebra $\mathrm{End}(V)$ of endomorphisms of a vector bundle V over the stack X.

Central idempotents

Lemma 10.7. *The centre of a finite type algebra is a finite type algebra.*

PROOF. By

$$\mathcal{H}om(\mathcal{E}^\vee, \mathcal{H}om(\mathcal{E}, \mathcal{O})) = \mathcal{H}om(\mathcal{E}^\vee \otimes \mathcal{E}, \mathcal{O}),$$

the endomorphism sheaf of a reflexive sheaf is reflexive, and therefore the algebra of $\mathcal{O}$-linear endomorphisms of a finite type algebra is of finite type.

The centre of A is the kernel of the $\mathcal{O}$-linear homomorphism of representable coherent sheaves $A \to \underline{\mathrm{End}}_{\mathcal{O}}(A)$, given by $a \mapsto [a, \cdot\,]$. As such, it is a representable coherent sheaf itself. □

For a representable algebra A over X, we denote the closed substack of idempotents in A by $E(A)$.

Lemma 10.8. *Suppose $A \to X$ is a commutative finite type algebra. The stack $E(A) \to X$ of idempotents in A is unramified over X.*

PROOF. The claim is local in the smooth topology, so we may assume that X is a scheme. It is then sufficient to prove the claim over a stratification of X, so we may assume that A is an algebra bundle, and hence a subalgebra of $GL(V)$, for a vector bundle V over X, which we may as well assume is trivial, of rank n.

We use the formal criterion. So let $T \subset T'$ be a square zero extension of affine schemes, with ideal I, and e, f two $n \times n$ matrices with entries in $\mathscr{O}(T')$, which agree on T. Hence the entries of $e - f$ are in I, which implies that $(e-f)^2 = 0$. Therefore, we have

$$0 = e(e-f)^2 = e(e - 2ef + f) = e - 2ef + ef = e - ef\,.$$

So we have $e = ef$, and by symmetry also $f = fe$, and as e and f commute, we have $e = f$. □

Corollary 10.9. *If $A \to X$ is a commutative finite type algebra, and X is reduced, there is a non-empty open substack U of X, such that $E(A|_U) \to U$ if finite étale.*

Proof. First we use generic flatness, and the fact that a flat and unramified morphisms is necessarily étale, to prove that $E(A) \to X$ is generically étale. Then we use Zariski's main theorem to prove that $E(A) \to X$ is generically finite. □

By this corollary, when studying the centre of finite type algebras over the finite type stack X, we may, after passing to a locally closed stratificationof X, assume that the stack of central idempotents is finite étale over X.

Also note that a finite étale morphism to the stack X is locally trivial in the smooth topology.

Recall that a non-zero idempotent e is called *primitive*, if $e = e_1 + e_2$, for orthogonal idempotents e_1, e_2, implies that $e_1 = 0$ or $e_2 = 0$.

In a commutative algebra, the following is true

(i) every idempotent is in a unique way (up to order of the summands) a sum of primitive idempotents, this is the *primitive decomposition*,

(ii) orthogonal idempotents have disjoint primitive decompositions,

(iii) distinct primitive idempotents are orthogonal to each other,

(iv) the primitive idempotents add up to 1.

Definition 10.10. Assume that the commutative finite type algebra A has finite étale stack of idempotents $E(A) \to X$. An idempotent local section of $A \to X$ is **primitive**, if it is a primitive idempotent locally in the smooth topology.

Lemma 10.11. *Let $A \to X$ be a commutative finite type algebra, with finite étale stack of idempotents $E(A) \to X$. There is an open and closed substack $PE(A) \subset E(A)$, such that an idempotent factors through $PE(A)$ if and only if it is primitive.*

Proof. We may assume that $I(A) \to X$ is constant. Then the multiplication operation and the partially defined addition operation on $E(A)$ are also constant. The claim follows. □

Definition 10.12. Let $A \to X$ be a finite type algebra, with centre $Z \to X$. Let $ZE(A)$ be the stack of idempotents in Z, in other words the stack of central idempotents in A. Assume that $ZE(A) \to X$ is finite étale. The substack of primitive idempotents in $ZE(A)$ is denoted by $PZE(A)$, and called the stack of **primitive central idempotents** of A. It is finite étale over X. The degree of $PZE(A) \to X$ is called the **central rank** of A.

If X is smooth and connected, the number of connected components of $PZE(A)$ is the **split central rank** of A. More precisely, the partition of the central rank given by the connected components of $PZE(A)$ is called the **central type** of A. (So the split central rank is the length of the type.)

Remark 10.13. Let X be smooth and connected, and let $A \to X$ be a commutative finite type algebra, with finite étale stack of idempotents $E(A) \to X$. Then there is a one-to-one correspondence between the connected components of $PE(A)$ and the primitive idempotents in the algebra of global sections $\Gamma(X, A)$.

The semisimple centre

For a finite type algebra over a field, we have

(i) the primitive idempotents are linearly independent,

(ii) an element is semisimple if and only if it is a linear combination of primitive idempotents.

We need a version of this statement for finite type algebras over stacks.

Let $A \to X$ be a commutative finite type algebra whose stack of idempotents $E(A)$ is finite étale. Consider the finite étale cover of primitive idempotents $\pi : PE(A) \to X$. We have a tautological global section e of $A|_{PE(A)}$, and $a \mapsto ae$ defines a homomorphism of $\mathcal{O}_{PE(A)}$-modules $\mathcal{O}_{PE(A)} \to A|_{PE(A)}$. Pushing forward with π and composing with the trace map $\pi_*(A|_{PE(A)}) \to A$ defines the morphism of algebras over X

$$\pi_* \mathcal{O}_{PE(A)} \longrightarrow A. \tag{10.2}$$

In fact, (10.2) is an isomorphism onto A^{ss}, the subalgebra of semisimple elements in A.

If we drop the assumption that A is commutative, we get a canonical embedding of algebras

$$\pi_* \mathcal{O}_{PZE(A)} \longrightarrow A, \tag{10.3}$$

whose image is the semisimple centre $Z(A)^{ss}$.

Permanence of rank and split rank

Proposition 10.14. *Let $A \hookrightarrow A'$ be a monomorphism of commutative finite type algebras with finite étale stacks of idemtpotents over the smooth and connected stack X. Denote the ranks of A and A' by n and n', and the split ranks by k and k', repsectively. Then $n \le n'$ and $k \le k'$. Morover,*

(i) if A' admits a semisimple global section, which is not in A, then $n < n'$,

(ii) if A' admits an idempotent global section, which is not in A, then $k < k'$.

PROOF. The embedding $A \hookrightarrow A'$ induces an embedding of finite étale X-stacks $E(A) \hookrightarrow E(A')$. Every idempotent e in A can be decomposed uniquely into a sum of orthogonal primitive idempotents in A'. Let us call this the *primitive decomposition* of e in A'. Consider the correspondence $Q \subset PE(A) \times_X PE(A')$ defined by

$$(e, e') \in Q \quad \Longleftrightarrow \quad e' \text{ partakes in the primitive decomposition of } e \text{ in } A'.$$

One shows that Q is a finite étale cover of X locally in the étale topology of X, reducing to the case where both $E(A)$ and $E(A')$ are trivial covers. By properties of the primitive decomposition, the projection $Q \to PE(A)$ is surjective, and the projection $Q \to PE(A')$ is injective. Thus we have

$$n = \deg PE(A) \le \deg Q \le \deg PE(A') = n.$$

If $n = n'$, then both $Q \to PE(A)$ and $Q \to PE(A')$ are isomorphisms, showing that $PE(A) = PE(A')$, and hence $A^{ss} = (A')^{ss}$. This proves (i).

We can repeat the argument for the algebras of global sections $\Gamma(X, A) \hookrightarrow \Gamma(X, A')$. We deduce that $k \le k'$, and if $k = k'$, every primitive idempotent in $\Gamma(X, A)$ remains primitive in $\Gamma(X, A')$, and every primitive idempotent of $\Gamma(X, A')$ is in $\Gamma(X, A)$. We deduce that $\Gamma(X, A)$ and $\Gamma(X, A')$ have the same idempotents, which proves (ii). □

Families of idempotents

Definition 10.15. For a finite type algebra $A \to X$, we denote by $E_n(A) \to X$ the stack of n-tuples of non-zero idempotents in A, which are pairwise orthogonal, and add up to unity. We call sections of $E_n(A)$ also *complete sets of orthogonal idempotents.*

Note that the family members of sections of $E_n(A)$ need not be central.

The stack $E_n(A)$ is algebraic, and of finite type over X. For $n = 0$, the stack $E_0(A)$ is empty, unless $A = 0$, in which case it is identified with X. For $n = 1$, the stack $E_1(A)$ contains exactly the unit in A (so is identified with X), unless $A = 0$, in which case $E_1(A)$ is empty.

10.2 Algebroids

Definition 10.16. An **algebroid** is a triple (X, A, ι), where X is an algebraic stack, A is a finite type algebra over X, and $\iota : I_X \to A^\times$ is an isomorphism of sheaves of groups over X, such that the natural diagram

$$\begin{array}{ccc} I_X & \xrightarrow{\iota} & A^\times \\ & \searrow & \downarrow \\ & & \underline{\mathrm{Aut}}_X(A) \end{array} \tag{10.4}$$

of sheaves of groups over X commutes.The vertical map $A^\times \to \underline{\mathrm{Aut}}_X(A)$ associates to a unit u of A the inner automorphism $x \mapsto uxu^{-1}$. The diagonal map $I_X \to \underline{\mathrm{Aut}}_X(A)$ is the inertia representation (10.1). We sometimes refer to the commutativity of (10.4) as the **algebroid condition**.

Thus, if (X, A) is an algebroid, the inertia representation acts through inner automorphisms on A.

We will usually abbreviate the triple (X, A, ι) to X, and write A_X for A, if we need to specify the algbra.

Example 12. Let $\mathfrak{X}$ be a linear algebraic stack with underlying algebraic stack X, and let $A \to X$ be the universal sheaf of endomorphisms of Remark 9.3. Then automorphisms are invertible endomorphisms, so we use for $\iota : I_X \to A^\times$ the tautological embedding. The inertia representation being the inverse of the pullback action, it is, indeed, given by (left) inner automorphisms.

We call (X, A) the *algebroid underlying* the linear algebraic stack $\mathfrak{X}$.

Example 13. As toy example, consider an R-algebra scheme of finite type A, with group scheme of units $A^\times$, as in Example 11. Let $A^\times$ act on A from the left by inner automorphisms. Then $A^\times \backslash A$ is an a finite type algebra over the stack of right $A^\times$-torsors $BA^\times$. The inertia stack of $BA^\times$ is the quotient stack $A^\times \backslash A^\times$, where $A^\times$ acts on $A^\times$ from the left by inner automorphisms. The morphism $\iota : A^\times \backslash A^\times \to A^\times \backslash A$ is the inclusion map.

Example 14. Every scheme Z is an algebroid via the definition $A_Z = 0_Z$. There is no natural way to enhance the algebroid $(Z, 0_Z)$ to a linear algebraic stack. This exhibits one way in which algebroids are more flexible than linear algebraic stacks.

Example 15. Consider the linear stack of vectorbundles $\mathfrak{Vect}$, Example 9. The underlying algebroid consists of the disjoint union of the quotient stacks $\mathrm{GL}_n \backslash M(n\times n)$, given by the adjoint representations $M(n\times n)$, for $n \geq 0$. Thus, in passing from the linear stack to the underlying algebroid, we discard all $M(m\times n)$, for $m \neq n$, and for $m = n$, restrict the left-right bi-action of GL_n on $M(n\times n)$ to the (left only) adjoint action. Thus we remove exactly the information which we consider non-local, see Remark 9.4.

Example 16. If $X \to \operatorname{Spec} R$ is a gerbe, any algebroid over X can be promoted to a linear algebraic stack, whose underlying algebraic stack is X.

Definition 10.17. A **morphism** of algebroids $X \to Y$ is a pair (f,φ), where $f: X \to Y$ is a morphism of algebraic stacks, and $\varphi: A_X \to A_Y$ is a morphism of algebras over f, such that the diagram

$$\begin{array}{ccc} I_X & \xrightarrow{\iota} & A_X^\times \\ {\scriptstyle I_f}\downarrow & & \downarrow{\scriptstyle \varphi} \\ I_Y & \xrightarrow{\iota} & A_Y^\times \end{array}$$

commutes.

Definition 10.18. A morphism (f,φ) of algebroids is **representable**, if $f: X \to Y$ is a representable morphism of algebraic stacks, and $\varphi: A_X \to f^* A_Y$ is a monomorphism of sheaves of algebras over X.

There is a natural notion of 2-morphism of algebroid, which makes algebroids into a 2-category. Representable algebroids over a fixed algebroid form a 1-category.

Remark 10.19. Suppose (X,A) is an algebroid, $f: Y \to X$ is a representable morphism of algebraic stacks, $B \to Y$ is a finite type algebra over Y, and $\varphi: B \to A$ is a morphism of algebras, covering f, such that $B \to f^*A$ is a monomorphism. If the image of the composition $I_Y \to I_X|_Y \to A^\times|_Y$ is equal to the image of $B^\times \to A^\times|_Y$, then the induced isomorphism $I_Y \to B^\times$ makes (Y,B) into an algebroid, and (f,φ) into a representable morphism of algebroids. (The algebroid condition on (X,B) is automatic.)

Commutative algebroids

Proposition 10.20. *If X is a gerbe over the algebraic space S, and $A \to X$ is a commutative algebroid over X, then the algebra A descends to a finite type algebra over S.*

Proof. This follows from the fact that the inertia representation is trivial, being given by inner automorphisms, and Lemma 10.2. □

Fibered products

Let $(X,A_X)\to(Z,A_Z)$ and $(Y,A_Y)\to(Z,A_Z)$ be morphisms of algebroids. The fibered product of (X,A_X) and (Y,A_Y) over (Z,A_Z) has as underlying algebraic stack the fibered product $W=X\times_Z Y$. The algebra A_W is given by the fibered product of algebras over W

$$A_W = A_X|_W \times_{A_Z|_W} A_Y|_W\,.$$

This fibered product (W,A_W) has a natural universal mapping property. If (X,A_X), (Y,A_Y) and (Z,A_Z) are the algebroids underlying linear algebraic stacks $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$, respectively, then the algebroid (W,A_W) is the algebroid underlying the linear algebraic stack $\mathfrak{W}=\mathfrak{X}\times_{\mathfrak{Z}}\mathfrak{Y}$.

Pullback algebroid

A morphism of algebraic stacks $f:Y\to X$ is called *inertia preserving* if $I_Y=I_X|_Y$. Examples of inertia preserving morphispms include monomorphisms, and projections $Z\times X\to X$, for algebraic spaces Z, as well as pullback of gerbes via morphisms to the coarse moduli space. Inertia preserving morphisms are necessarily representable.

If (X,A_X) is an algebroid, and $f:Y\to X$ is inertia preserving, then (Y,f^*A_X) is an algebroid, called the *pullback* of the algebroid A_X via f. The pullback (Y,f^*A_X) comes wtih a representable structure morphism to (X,A_X).

Definition 10.21. A **closed immersion** of algebroids $(Y,A_Y)\to(X,A_X)$ is a morphism where $Y\to X$ is a closed immersion of algebraic stacks, such that $A_Y=A_X|_Y$.

10.2.1 Algebroid inertia

A key observation is that if (X,A) is an algebroid, then (I_X,I_A) is another algebroid. In fact, by Proposition 10.3, $I_A\to I_X$ makes I_A into a finite type algebra over I_X, and the algebra $I_A(x,\varphi)$ lying over the object (x,φ) of I_X is equal to the fixed algebra $A(x)^\varphi$. The fibre of $I(I_X)\to I_X$ over (x,φ) is the centralizer in $\mathrm{Aut}(x)$ of φ, which we shall denote by $\mathrm{Aut}(x)^\varphi$. The isomorphism $\iota(x):\mathrm{Aut}(x)\to A(x)^\times$ restricts to an isomorphism

$$\mathrm{Aut}(x)^\varphi \longrightarrow \big(A(x)^\times\big)^\varphi = \big(A(x)^\varphi\big)^\times,$$

and therefore we get an induced isomorphism $I(I_X)\to(I_A)^\times$ of groups over I_X. One checks that (I_X,I_A) inherits the commutativity of (10.4) from (X,A). The algebroid (I_X,I_A) comes with a natural representable morphism to (X,A).

There is also a semisimple version.It is given by (I_X^{ss},I_A^{ss}), and it is isomorphic to the pullback of (I_X,I_A) via the inertia preserving morphism $I_X^{ss}\to I_X$. The fibre of I_A^{ss} over the object x of X is

$$I_A^{ss}(x)=\{(\varphi,a)\in\mathrm{Aut}(x)\times A(x)\mid \varphi^*(a)=a \text{ and } \varphi \text{ is semisimple}\}\,.$$

Definition 10.22. We call (I_X, I_A) the **inertia algebroid** of the algebroid (X,A), notation $I_{(X,A)}$. Also, (I_X^{ss}, I_A^{ss}) is called the **semisimple inertia** of the algebroid (X,A), notation $I_{(X,A)}^{ss}$.

Proposition 10.23. *Let (X,A) be an algebroid. Then $E_n(A)$ is the base of a canonical algebroid, mapping to X with a representable morphism of algebroids.*

PROOF. The pullback of A via the structure morphism $E_n(A) \to X$ is an algebra endowed with an n-tuple of idempotents $e_1, \ldots, e_n$. We define A^{fix} to be the subalgebra of $A|_{E_n(A)}$ commuting with $e_1, \ldots, e_n$. The inertia stack of $E_n(A)$ is the subgroup stack of $I_X|_{E_n(A)}$ consisting of automorphisms which stabilize each of the e_i, under the inertia representation on A. If we map it via ι into $A^\times|_{E_n(A)}$, we obtain exactly $(A^\times)^{\mathrm{fix}}$, by the algebroid property of A, applied to the idempotents $e_1, \ldots, e_n$. By Remark 10.19, this is enough to assure that $(E_n(A), A^{\mathrm{fix}})$ is an algebroid. □

10.2.2 Clear algebroids

Suppose that (X,A) is an algebroid, and that X is a gerbe over the scheme $\overline{X}$. Then the centre $Z(A)$ descends to a commutative finite type algebra over $\overline{X}$, by Proposition 10.20. Note that this is analogue to Proposition 3.1 in the case of bands of gerbes.

Definition 10.24. We call an algebroid (X,A) **clear**, if

(i) X is a gerbe with coarse moduli space $\overline{X}$,

(ii) the space $\overline{X}$ is a non-singular variety,

(iii) A and $Z(A)$ are algebra bundles over X,

(iv) $ZE(A) \to X$ is finite étale.

For a clear algebroid, $ZE(A)$ and $PZE(A)$ descend to a finite étale $\overline{X}$-schemes. The definition of *central rank*, *split central rank*, and *central type* apply to clear algebroids.

For every algebroid (X,A), over a finite type algebraic stack X, there exists a stratification of X, such that the restricted algebroids over the pieces of the stratification are all clear. This follows from Lemma 10.5 and Corollary 10.9.

Chapter 11

K-algebra of stack functions

11.1 Stack functions

Let $\mathfrak{M}$ be a linear stack, and $\mathfrak{A} \to \mathfrak{M}$ its universal endomorphism algebra. In Example 12 we showed that $(\mathfrak{M}, \mathfrak{A})$ is an algebroid. A main example to have in mind is the linear stack $\mathfrak{Coh}_X$ of Example 8.

Definition 11.1. A **stack function** is a representable morphism of algebroids $(X, A) \to (\mathfrak{M}, \mathfrak{A})$, such that X is of finite type.

Recall Remark 10.19, which says that to prove that a given $(X, A) \to (\mathfrak{M}, \mathfrak{A})$ is a stack function, we do not need to check the algebroid condition on (X, A).

The **K-algebra** of $\mathfrak{M}$, notation $K(\mathfrak{M})$, is the free $\mathbb{Q}$-vector space on (isomorphism classes of) stack functions, modulo the scissor relations relative $\mathfrak{M}$. The class in $K(\mathfrak{M})$ defined by a stack function $X \to \mathfrak{M}$ will be denoted $[X \to \mathfrak{M}]$. A **scissor relation relative** $\mathfrak{M}$ is

$$[X \to \mathfrak{M}] = [Z \to X \to \mathfrak{M}] + [X \setminus Z \to X \to \mathfrak{M}],$$

for any closed immersion of algebroids $Z \hookrightarrow X$, and any stack function $X \to \mathfrak{M}$. The substacks Z and $X \setminus Z$ are endowed with their respective pullback algebroids.

Example 17. Consider the linear stack $\mathfrak{Vect}$. Stack functions are triples (X, V, A), where X is a finite type algebraic stack, V is a vector bundle over X, and $A \subset \underline{\mathrm{End}}_X(V)$ is a subalgebra, such that the inertia representation $I_X \to \underline{\mathrm{Aut}}_X(V)$ identifies I_X with $A^\times \subset \underline{\mathrm{Aut}}_X(V)$.

Example 18. As a trivial example, consider the case where $\mathfrak{M}$ is the stack of zero dimensional vector bundles. In this case the underlying algebroid of $\mathfrak{M}$ is $\mathrm{Spec}\, R$, endowed with the 0-algebra. Stack functions are the same as algebraic spaces over R. We denote the corresponding K-algebra by $K(\mathrm{Var})$. Fibered product over $\operatorname{Spec} R$ makes $K(\mathrm{Var})$ into a $\mathbb{Q}$-agebra.

The vector space $K(\mathfrak{M})$ is a commutative ring with the multiplication

$$[X \to \mathfrak{M}] \cdot [Y \to \mathfrak{M}] = [X \times Y \to \mathfrak{M} \times \mathfrak{M} \xrightarrow{\oplus} \mathfrak{M}].$$

Also a $K(\mathrm{Var})$-module structure on $K(\mathfrak{M})$ is given by $[Z] \cdot [X \to \mathfrak{M}] = [Z \times X \to X \to \mathfrak{M}]$. This turns $K(\mathfrak{M})$ into a $K(\mathrm{Var})$-algebra. The additive zero in $K(\mathfrak{M})$ is given by the empty algebroid $0 = [\emptyset \to \mathfrak{M}]$ and the multiplicative unit is $1 = [\operatorname{Spec} R \xrightarrow{0} \mathfrak{M}]$.

11.2 The filtration by split central rank

Definition 11.2. We introduce the **filtration by split central rank** $K^{\geq k}(\mathfrak{M})$ on $K(\mathfrak{M})$, by declaring $K^{\geq k}(\mathfrak{M})$ to be generated as a $\mathbb{Q}$-vector space by stack functions $[X \to \mathfrak{M}]$, where X is a clear algebroid (Definition 10.24) such that A_X admits k orthogonal central non-zero idempotents (globally).

Alternatively, $K^{\geq k}(\mathfrak{M})$ is generated by $[X \to \mathfrak{M}]$, where X is a clear algebraoid such that $PZE(A_X)$ has at least k components.

Remark 11.3. Trying to define a direct sum decomposition of $K(\mathfrak{M})$ by split central rank would not work, because a clear algebroid X of split central rank k may very well admit a closed substack $Z \subset X$ whose restricted algebroid is again clear, but of split central rank larger than k.

The 0-ring has no non-zero central idempotents, but any non-zero ring has at least one. Therefore, $K(\mathrm{Var}) \subset K(\mathfrak{M})$ is a complement for $K^{>0}(\mathfrak{M})$ in $K(\mathfrak{M}) = K^{\geq 0}(\mathfrak{M})$, i.e., $K(\mathfrak{M}) = K(\mathrm{Var}) \oplus K^{>0}(\mathfrak{M})$. In particular, we have

$$K^{\geq 0}(\mathfrak{M})/K^{>0}(\mathfrak{M}) = K(\mathrm{Var}).$$

11.3 The idempotent operators E_r

Let E_r denote the operator on $K(\mathfrak{M})$ which maps a stack function $[X \to \mathfrak{M}]$ to $[E_r(X) \to X \to \mathfrak{M}]$, where $E_r(X) = E_r(A_X)$ is the stack of r-tuples of non-zero orthogonal idempotents adding to unity in A_X, see Definition 10.15. The algebroid structure on $E_r(X)$ is described in Proposition 10.23.

This definition applies also to $r = 0$. The stack $E_0(X)$ is empty if $A_X \neq 0$, and $E_0(X) = X$, if X is a scheme. Hence E_0 is diagonalizable, and has eigenvalues 0 and 1.

The kernel (0-eigenspaces) is $K^{>0}(\mathfrak{M}) \subset K(\mathfrak{M})$, the image (1-eigenspaces) is $K(\mathrm{Var}) \subset K(\mathfrak{M})$.

For $r = 1$, the operator E_1 vanishes on stack functions $[X \to \mathfrak{M}]$, where X is an algebraic space, and acts as identity on stack functions for which $A_X \neq 0$. Hence, E_1 is also diagonalizable with eigenvalues 0, and 1. The kernel of E_1 is $K(\mathrm{Var})$, and the image is $K^{>0}(\mathfrak{M})$. Hence E_0 and E_1 are complementary idempotent operators on $K(\mathfrak{M})$, i.e., they are orthogonal to each other and add up to the identity.

Recall the Stirling number of the second kind, $S(k,r)$, which is defined in such a way that $r!S(k,r)$ is the number of surjections from $\underline{k}$ to $\underline{r}$.

Theorem 11.4. *The operators E_r, for all $r \geq 0$, preserve the filtration $K^{\geq k}(\mathfrak{M})$ by split central rank. On the quotient $K^{\geq k}(\mathfrak{M})/K^{>k}(\mathfrak{M})$, the operator E_r acts as multiplication by $r!S(k,r)$.*

PROOF. Consider a clear algebroid (X,A) with a morphism $X \to \mathfrak{M}$ defining the stack function $[X \to \mathfrak{M}]$ in $K(\mathfrak{M})$. Let n be the central rank of X, and k the split central rank of X. The filtered piece $K^{\geq k}(\mathfrak{M})$ is generated by such $[X \to \mathfrak{M}]$.

Denote by $X \to \overline{X}$ the coarse space of X, which is a non-singular variety, by assumption.

Let $\widetilde{X} \to X$ be a connected Galois cover with Galois group Γ, which trivializes $PZE(A) \to X$. As $PZE(A)$ descends to $\overline{X}$, this Galois cover can be constsructed as a pullback from the non-singular variety $\overline{X}$. Therefore, the morphism $\widetilde{X} \to X$ is inertia preserving and hence $\widetilde{X}$ inherits, via pullback, the structure of an algebroid, and hence $[\widetilde{X} \to X \to \mathfrak{M}]$ is a stack function.

Recall that the degree of the cover $PZE(A) \to X$ is n, and the number of components of $PZE(A)$ is k.

By labelling the components of the pullback of $PZE(A)$ to $\widetilde{X}$, we obtain an action of Γ on the set $\underline{n} = \{1,\dots,n\}$ and an isomorphism of finite étale covers of X

$$\begin{aligned} \widetilde{X} \times_\Gamma \underline{n} &\xrightarrow{\simeq} PZE(A) \\ [x,v] &\longmapsto e_{[x,v]}\,. \end{aligned}$$

Both source and target of this isomorphism support natural algebroids and the isomorphism preserves them.The number of orbits of Γ on $\underline{n}$ is k.

Then we also have an isomorphism

$$\begin{aligned} \widetilde{X} \times_\Gamma \mathrm{Epi}(\underline{n},\underline{r}) &\xrightarrow{\simeq} ZE_r(A) \\ [x,\varphi] &\longmapsto \Big(\sum_{\varphi(v)=\rho} e_{[x,v]}\Big)_{\rho=1,\dots,r}, \end{aligned}$$

where ZE_r denotes the stack of labelled complete sets of r orthogonal central idempotents. Again, both stacks involved are in fact algebroids, and this isomorphism is an isomorphism of algebroids.

Hence, we may calculate as follows (all stacks involved are endowed with their natural algebroid structures):

$$\begin{aligned}
ZE_r[X \to \mathfrak{M}] &= [\widetilde{X} \times_\Gamma \operatorname{Epi}(\underline{n},\underline{r}) \to \mathfrak{M}] \\
&= [\widetilde{X} \times_\Gamma \sqcup_{\varphi \in \operatorname{Epi}(\underline{n},\underline{r})/\Gamma} \Gamma / \operatorname{Stab}_\Gamma \varphi \to \mathfrak{M}] \\
&= \sum_{\varphi \in \operatorname{Epi}(\underline{n},\underline{r})/\Gamma} [\widetilde{X} / \operatorname{Stab}_\Gamma \varphi \to \mathfrak{M}] \\
&= \sum_{\varphi \in \operatorname{Epi}(\underline{n},\underline{r})^\Gamma} [X \to \mathfrak{M}] + \sum_{\substack{\varphi \in \operatorname{Epi}(\underline{n},\underline{r})/\Gamma \\ \operatorname{Stab}_\Gamma \varphi \neq \Gamma}} [\widetilde{X} / \operatorname{Stab}_\Gamma \varphi \to \mathfrak{M}] .
\end{aligned}$$

Now, we have $\operatorname{Epi}(\underline{n},\underline{r})^\Gamma = \operatorname{Epi}(\underline{n}/\Gamma,\underline{r})$, and hence

$$\# \operatorname{Epi}(\underline{n},\underline{r})^\Gamma = r! S(|\lambda|, r) = r! S(k,r) .$$

Thus, we conclude,

$$ZE_r[X \to \mathfrak{M}] = r! S(k,r) [X \to \mathfrak{M}] + \sum_{\substack{\varphi \in \operatorname{Epi}(\underline{n},\underline{r})/\Gamma \\ \operatorname{Stab}_\Gamma \varphi \neq \Gamma}} [\widetilde{X} / \operatorname{Stab}_\Gamma \varphi \to \mathfrak{M}] .$$

For any proper subgroup $\Gamma' \subset \Gamma$, the quotient $X' = \widetilde{X}/\Gamma'$ is an intermediate cover $\widetilde{X} \to X' \to X$, such that $X' \neq X$. The pullback of $PZE(A)$ to X' has more than k components, because the number of orbits of Γ' on $\underline{n}$ is larger than k. Thus we have proved the theorem for ZE_r, instead of E_r.

Now observe that $ZE_r(A) \subset E_r(A)$ is a closed substack, because $ZE_r(A) \to X$ is proper and $E_r(A) \to X$ is separated. So we can write

$$E_r[X \to \mathfrak{M}] = ZE_r[X \to \mathfrak{M}] + [NZE_r(A) \to X \to \mathfrak{M}] ,$$

where $NZE_r(A)$ is the complement of $ZE_r(A)$ in $E_r(A)$. To prove that $[NZE_r(A) \to \mathfrak{M}] \in K^{>k}(\mathfrak{M})$, let $Y \hookrightarrow NZE_r(A)$ be a locally closed embedding, such that the algebroid $(E_r(A), A^{\text{fix}})|_Y$ is clear.

Consider the embedding of algebras $A^{\text{fix}}|_Y \hookrightarrow A|_Y$. It induces an embedding of commutative algebras $Z(A|_Y) \hookrightarrow Z(A^{\text{fix}}|_Y)$, because $Z(A|_Y) \subset A^{\text{fix}}|_Y$. The algebra $A|_Y$ comes with r tautological idempotent sections, all of which are contained in $Z(A^{\text{fix}}|_Y)$, but at least one of which is not contained in $Z(A|_Y)$. So by Proposition 10.14 (ii), the split central rank of $A^{\text{fix}}|_Y$ is strictly larger than the split central rank of $A|_Y$. The latter is at least as big as k, the split central rank of A, because the split central rank cannot decrease under base extension. This shows that $[Y \to \mathfrak{M}] \in K^{>k}(\mathfrak{M})$ and finishes the proof. □

Corollary 11.5. *The operators E_r, for $r \geq 0$ are simultaneously diagonalizable. The common eigenspaces form a family $K^k(\mathfrak{M})$ of subspaces of $K(\mathfrak{M})$ indexed by non-negative integers $k \geq 0$, and*

$$K(\mathfrak{M}) = \bigoplus_{k\geq 0} K^k(\mathfrak{M}) . \tag{11.1}$$

Let π_k denote the projection onto $K^k(\mathfrak{M})$. We have

$$E_r \pi_k = r! S(k,r) \pi_k ,$$

for all $r \geq 0$, $k \geq 0$.

Proof. First remark that for given r, the numbers $r!S(k,r)$ form a monotone increasing sequence of integers.

Then note that the operators E_r pairwise commute: the composition $E_r \circ E_{r'}$ associates to an algebroid (X,A) the stack of pairs (e,e'), where both e and e' are complete families of non-zero orthogonal idempotents in A, the length of e being r and the length of e' being r', and the members of e commuting with the members of e'.

Finally, let us prove that, for every k and every r, the $\mathbb{Q}$-vector space $K^{\geq k}(\mathfrak{M})$ is a union of finite-dimensional subspaces invariant by E_r.

For this, define $K(\mathfrak{M})_{\leq N}$ to be generated as $\mathbb{Q}$-vector space by stack functions $[X \to \mathfrak{M}]$, where X is a clear algebroid, such that the rank of the vector bundle underlying the algebra $A_X \to X$ is bounded above by N. This is an ascending filtration of $K(\mathfrak{M})$, which is preserved by E_r. Set

$$K^{\geq k}(\mathfrak{M}) \cap K(\mathfrak{M})_{\leq N} = K^{\geq k}(\mathfrak{M})_{\leq N} .$$

Suppose $x = [X \to \mathfrak{M}]$ is a stack function with X a clear algebroid of split central rank k, and let N be the rank of the vector bundle underlying A_X. Note that $k \leq N$, because for a commutative algebra, the number of primitive idempotents is bounded by the rank of the underlying vector bundle. We deduce that for $k > N$, we have $K^{\geq k}(\mathfrak{M})_{\leq N} = 0$.

On the other hand, Theorem 11.4 implies by induction that

$$E_r^i(x) \in \mathbb{Q}x + \mathbb{Q}E_r(x) + \ldots + \mathbb{Q}E_r^{i-1}(x) + K^{\geq k+i}(\mathfrak{M}) .$$

Applying this for $i = N - k + 1$, we see that

$$E_r(E_r^{N-k}(x)) \in \mathbb{Q}x + \mathbb{Q}E_r(x) + \ldots + \mathbb{Q}E_r^{N-k}(x) ,$$

and hence that $\mathbb{Q}x + \mathbb{Q}E_r(x) + \ldots + \mathbb{Q}E_r^{N-k}(x)$ is invariant under E_r.

This proves that any $x \in K^{\geq k}(\mathfrak{M})$ is contained in a finite-dimensional subspace invariant under E_r. Standard techniques from finite-dimensional linear algebra now imply the result. □

Remark 11.6. The proof of Theorem 11.4 and its corollary show that the central versions ZE_r of the E_r are also diagonalizable. On the other hand, the ZE_r do not commute with each other, and so are less useful.

Corollary 11.7. *For $r \geq 1$, we have*

$$\ker E_r = \bigoplus_{k<r} K^k(\mathfrak{M}) .$$

In particular, for any element x, we have $E^r x = 0$, for $r \gg 0$.

Corollary 11.8. *For every $k \geq 0$, we have*

$$\pi_k = \sum_{r=k}^{\infty} \frac{s(r,k)}{r!} E_r ,$$

where the $s(n,k)$ are the Stirling numbers of the first kind. In particular, $\pi_0 = E_0$, and

$$\pi_1 = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} E_r .$$

Proof. We have

$$\mathrm{id} = \sum_{\ell \geq 0} \pi_\ell ,$$

and hence

$$E_r = \sum_{\ell \geq 0} E_r \pi_\ell = \sum_{\ell \geq 0} r! S(\ell, r) \pi_\ell ,$$

and therefore

$$\sum_{r \geq 0} \frac{s(r,k)}{r!} E_r = \sum_{r \geq 0} \frac{s(r,k)}{r!} \sum_{\ell \geq 0} r! S(\ell,r) \pi_\ell$$
$$= \sum_{\ell \geq 0} \Big(\sum_{r \geq 0} S(\ell,r) s(r,k) \Big) \pi_\ell = \sum_{\ell \geq 0} \delta_{\ell,k} \pi_\ell = \pi_k ,$$

by the inverse relationship between the Stirling numbers of the first and second kind. □

Remark 11.9. The Stirling numbers of the first kind appear in the Taylor expansions of the powers of the logarithm:

$$\sum_{r=k}^{\infty} \frac{s(r,k)}{r!} t^r = \log(1+t)^k .$$

Example 19. The universal rank 2 vector bundle $GL_2 \backslash \mathbb{A}^2 \to BGL_2$, and its classifying morphism to $\mathfrak{Vect}$ define a Hall algebra element $[BGL_2 \to \mathfrak{Vect}] \in K(\mathfrak{Vect})$, which we will abbreviate to $[BGL_2]$. To decompose $[BGL_2]$ into its pieces according to (11.1), we consider the action of E_2, as we have $E_r[BGL_2] = 0$, for all $r > 2$. In fact,

$$E_2[BGL_2] = [BT], \qquad \text{and} \qquad E_2[BT] = 2[BT],$$

where T is a maximal torus in GL_2. Thus $\mathbb{Q}[BGL_2] + \mathbb{Q}[BT]$ is a subspace of $K(\mathfrak{Vect})$ invariant under E_2, and the matrix of E_2 acting on this subspace is

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} . \tag{11.2}$$

This matrix is lower triangular, with different numbers on the diagonal, hence diagonalizable over $\mathbb{Q}$. In fact, the diagonal entries are $2S(1,2) = 0$ and $2S(2,2) = 2$. Diagonalizing (11.2) gives the eigenvectors

(i) $v_1 = [BGL_2] - \frac{1}{2}[BT]$ with eigenvalue 0,

(ii) $v_2 = \frac{1}{2}[BT]$ with eigenvalue 2.

Therefore, we have $v_1 \in K^1(\mathfrak{Vect})$ and $v_2 \in K^2(\mathfrak{Vect})$, and since $[BGL_2] = v_1 + v_2$, we have found the required decomposition of $[BGL_2]$.

11.4 The spectrum of semisimple inertia

The semisimple inertia operator on $K(\mathfrak{M})$ is the $\mathbb{Q}$-linear endomorphism

$$\begin{aligned} I^{ss} : K(\mathfrak{M}) &\longrightarrow K(\mathfrak{M}) \\ [X \to \mathfrak{M}] &\longmapsto [I_X^{ss} \to X \to \mathfrak{M}] . \end{aligned}$$

Here I_X^{ss} denotes the semisimple algebroid inertia of Definition 10.22. In fact, I^{ss} is linear over $K(\mathrm{Var})$. We will use as scalars the subring $\mathbb{Q}[q] \subset K(\mathrm{Var})$, and extend scalars to the quotient field $\mathbb{Q}(q)$. Thus, we will consider I^{ss} as a $\mathbb{Q}(q)$-linear operator

$$I^{ss} : K(\mathfrak{M})(q) \longrightarrow K(\mathfrak{M})(q) ,$$

where $K(\mathfrak{M})(q) = K(\mathfrak{M}) \otimes_{\mathbb{Q}[q]} \mathbb{Q}(q)$.

Recall the definition of $\mathcal{Q}_\lambda$ for a partition $\lambda \vdash n$ as in §7.1. This is a polynomial in q, of degree n, which vanishes to order $|\lambda|$ at $q = 1$. Recall the well-ordering on the set of all partitions of all integers introduced in §7.4.

Theorem 11.10. *The operator $I^{ss} : K(\mathfrak{M})(q) \to K(\mathfrak{M})(q)$ is diagonalizable. Its eigenvalue spectrum consists of the $\mathcal{Q}_\lambda \in \mathbb{Q}(q)$, for all paritions λ. Denote the eigenspace corresponding to the eigenvalue $\mathcal{Q}_\lambda$ by $K^\lambda(\mathfrak{M})(q)$. We have*

$$K^k(\mathfrak{M})(q) = \bigoplus_{|\lambda|=k} K^\lambda(\mathfrak{M})(q),$$

where $K^k(\mathfrak{M})(q) = K^k(\mathfrak{M}) \otimes_{\mathbb{Q}[q]} \mathbb{Q}(q)$. Thus, the decomposition of $K(\mathfrak{M})(q)$ according to eigenspaces of I^{ss} refines the decomposition according to eigenspaces of the family of operators (E_r).

Proof. We define a *clear* stack function to be a stack function $X \to \mathfrak{M}$ for which X is a clear algebroid.

Then define a decreasing filtration $K^{\geq\lambda}(\mathfrak{M})(q)$ indexed by partitions, by declaring $K^{\geq\lambda}(\mathfrak{M})(q)$ to be generated by clear stack functions whose central type is $\geq \lambda$. We will prove

(i) the operator I^{ss} preserves the filtration by partitions,

(ii) on the quotient $K^{\geq\lambda}(\mathfrak{M})(q)/K^{>\lambda}(\mathfrak{M})(q)$, the operator I^{ss} acts as multiplication by $\mathcal{Q}_\lambda$.

(iii) the operator I^{ss} is locally finite.

This will prove the claims concerning diagonalizability of I^{ss}.

Consider a clear stack function $X \to \mathfrak{M}$, of central rank n, central type $\lambda \vdash n$, and split central rank $k = |\lambda|$. As in the proof of Theorem 11.4, let $\widetilde{X} \to X$ be a connected Galois cover with Galois group Γ, acting on the set $\underline{n}$, such that

$$\widetilde{X} \times_\Gamma \underline{n} \xrightarrow{\simeq} PZE(A_X).$$

We get an induced isomorphism

$$\widetilde{X} \times_\Gamma \mathbb{A}^n \xrightarrow{\simeq} Z(A_X)^{ss}$$

onto the semisimple centre of A_X, by reformulating (10.3). It follows that we have an isomorphism

$$\widetilde{X} \times_\Gamma \mathbb{G}_m^n \xrightarrow{\simeq} ZI^{ss}(X)$$

onto the semisimple central inertia of X. A computation similar to that of §7.1 would now show that

$$ZI^{ss}[X \to \mathfrak{M}] = \mathfrak{Q}(\lambda)[X \to \mathfrak{M}] + \sum_{\substack{I_\bullet \in F(\underline{n})/\Gamma \\ \mathrm{Stab}_\Gamma(I) \subsetneq \Gamma}} (-1)^{\ell(I_\bullet)} q^{|I_{\max}|} [\widetilde{X}/\,\mathrm{Stab}_\Gamma(I) \to \mathfrak{M}].$$

Recall that $F(\underline{n})$ is the set of all flags $I_\bullet = I_k \subsetneq \ldots \subsetneq I_1 \subsetneq I_0 = \underline{n}$ in $\underline{n}$. The length of a flag $I_\bullet$ is denoted by $k = \ell(I_\bullet)$, and the maximal index is $k = \max$.

Note that the cover $\widetilde{X} \to X$, as well as all intermediate covers $\widetilde{X} \to \widetilde{X}/\Gamma' \to X$, for any subgroup $\Gamma' \subset \Gamma$, come via base extension from covers of the variety $\overline{X}$, and are therefore endowed with canonical structures of algebroids over $\mathfrak{M}$, and define elements of $K(\mathfrak{M})(q)$, as in the proof of Theorem 11.4.

As in the proof of Theorem 11.4, all stack functions $[\widetilde{X}/\Gamma' \to \mathfrak{M}]$, for $\Gamma' \subsetneq \Gamma$ are contained in $K^{>k}(\mathfrak{M})(q)$, hence in $K^{>\lambda}(\mathfrak{M})(q)$ by the first property of our partition ordering.

Let us now consider I^{ss}, instead of ZI^{ss}. We have a closed immersion of algebroids $ZI^{ss}X \hookrightarrow I^{ss}X$. Thus we can write

$$I^{ss}[X \to \mathfrak{M}] = ZI^{ss}[X \to \mathfrak{M}] + [NZI^{ss}(X) \to X \to \mathfrak{M}].$$

Now consider a locally closed embedding $Y \hookrightarrow NZI^{ss}(X)$, such that Y is a clear algebroid. Over Y, we have the inclusion of algebras $A^{\mathrm{fix}}|_Y \hookrightarrow A|_Y$, inducing an embedding of the centres in the opposite direction $Z(A)|_Y \hookrightarrow Z(A^{\mathrm{fix}})|_Y$. There is one semisimple section, namely the tautological one, which is in $A(A^{\mathrm{fix}})|_Y$, but not in $Z(A)|_Y$, and so by Proposotion 10.14 (i), we have that the central rank of Y is larger than n. By the second property of our partition ordering, we have therefore $[Y \to \mathfrak{M}] \in K^{>\lambda}(\mathfrak{M})$

This proves the first two claims we made about I^{ss}. To prove local finiteness of I^{ss}, proceed as in the proof of Corollary 11.5. Every time we apply I^{ss} either the central rank or the split central rank will increase, but their sum can be bounded in terms of the dimension.

We have now proved that I^{ss} is diagonalizable. Next, note that the E_r are linear over $K(\mathrm{Var})$, and hence also induce $\mathbb{Q}(q)$-linear endomoprhism of $K(\mathfrak{M})(q)$. Moreover, I^{ss} commutes with E_r, for every r. Both compositions $E_r \circ I^{ss}$ and $I^{ss} \circ E_r$ associate to an algebroid (X, A) the stack of pairs (a, e), where a is a semisimple unit in A, and e a labelled complete set of r orthogonal idempotents, all commuting with a.

Therefore, E_r preserves the eigenspace $K^\lambda(\mathfrak{M})(q)$ of the I^{ss}. As

$$K^{\geq|\lambda|} \supset K^{\geq\lambda} \supset K^{>\lambda} \supset K^{>|\lambda|}$$

it follows from Theorem 11.4, that E_r acts on $K^\lambda(\mathfrak{M})(q)$ by scalar multiplication by

$r!S(|\lambda|,r)$, and hence that $K^{\lambda}(\mathfrak{M})(q) \subset K^{|\lambda|}(\mathfrak{M})(q)$. □

Example 20. Let us continue with Example 19. The stack function $[BGL_2]$ is clear, its central rank is 1. We have

$$\begin{aligned} I^{ss}[BGL_2] &= [\Delta/GL_2] + [T^*/N] \\ &= (q-1)[GL_2] + [T^*/N], \end{aligned}$$

where Δ is the central torus of GL_2, and $T^* = T\setminus\Delta$. Also, N is the normalizer of T in GL_2. Moreover, $[\Delta/GL_2] = \mathbb{G}_m \times [BGL_2]$ is a clear stack function of central rank 1, and $[T^*/N]$ is a clear stack function of central rank 2, and split central rank 1.

Applying I^{ss} to $[T^*/N]$, we get

$$\begin{aligned} I^{ss}[T^*/N] &= (q^2-1)[T^*/N] - (q-1)[T^*/T] \\ &= (q^2-1)[T^*/N] - (q-1)[T^*][BT] \\ &= (q^2-1)[T^*/N] - (q-1)^2(q-2)[BT]. \end{aligned}$$

Finally, $[BT]$ is a clear stack function of central rank 2 and split central rank 2. It is an eigenvector for I^{ss}:

$$I^{ss}[BT] = [T/T] = [T][BT] = (q-1)^2[BT].$$

We see that $\mathbb{Q}(q)[BGL_2] + \mathbb{Q}(q)[T^*/N] + \mathbb{Q}(q)[BT]$ is invariant under I^{ss}, and the matrix of I^{ss} on this subspace is

$$\begin{pmatrix} q-1 & 0 & 0 \\ 1 & q^2-1 & 0 \\ 0 & -(q-1)^2(q-2) & (q-1)^2 \end{pmatrix}$$

This matrix is lower trianguler, with distinct scalars on the diagonal, and is therefore diagonalizable over $\mathbb{Q}(q)$. Diagonalizing, we get the following eigenvectors

(i) $v_{(1)} = [BGL_2] - \frac{1}{q(q-1)}[T^*/N] - \frac{1}{q}[BT]$,

(ii) $v_{(2)} = \frac{1}{q(q-1)}[T^*/N] - \frac{q-2}{2q}[BT]$,

(iii) $v_{(1,1)} = \frac{1}{2}[BT]$,

where $v_{(1)} \in K^{(1)}(\mathfrak{Vect})(q)$, $v_{(2)} \in K^{(2)}(\mathfrak{Vect})(q)$ and $v_{(1,1)} \in K^{(1,1)}(\mathfrak{Vect})(q)$. Moreover, $[BGL_2] = v_{(1)} + v_{(2)} + v_{(1,1)}$, and this is the spectral decomposition of $[BGL_2]$ into eigenvectors of I^{ss}. Of course, $v_{(1)} + v_{(2)} = v_1$ and $v_{(1,1)} = v_2$.

11.5 Graded structure of multiplication

Proposition 11.11. *For $x, y \in K(\mathfrak{M})$, we have*

$$I^{ss}(x \cdot y) = I^{ss}(x) \cdot I^{ss}(y).$$

PROOF. This follows immediately from the fact that, for any two algebroids X, Y, we have $I^{ss}(X \times Y) = I^{ss}(X) \times I^{ss}(Y)$, as algebroids over $X \times Y$. □

Denote the disjoint union of two partitions λ and μ by $\lambda + \mu$.

Corollary 11.12. *We have $K^{\lambda}(\mathfrak{M})(q) \cdot K^{\mu} \subset K^{\lambda+\mu}(\mathfrak{M})(q)$, and hence also $K^{k}(\mathfrak{M})(q) \cdot K^{\ell}(\mathfrak{M})(q) \subset K^{k+\ell}(\mathfrak{M})(q)$.*

Remark 11.13. So the $\mathbb{Q}(q)$-vector space

$$K(\mathfrak{M})(q) = \bigoplus_{k \geq 0} K^{k}(\mathfrak{M})(q)$$

is a graded $\mathbb{Q}(q)$-algebra, with respect to the commutative product on $K(\mathfrak{M})(q)$. One can show that this fact is true for $K(\mathfrak{M})$ itself, without tensoring with $\mathbb{Q}(q)$. In other words if $x \in K^{k}(\mathfrak{M})$ and $y \in K^{\ell}(\mathfrak{M})$, then $x.y \in K^{k+\ell}(\mathfrak{M})$.

Chapter 12

Hall Algebra of algebroids

Let $\mathfrak{M}$ be a linear stack, and $\mathfrak{A} \to \mathfrak{M}$ be its universal endomorphism algebra. Recall that $(\mathfrak{M},\mathfrak{A})$ forms an algebroid (c.f. Example 12). In this chapter, we define the Hall product of the category of stack functions over $\mathfrak{M}$ following a similar treatment as to that of [12]. This restricts the possible choices for $\mathfrak{M}$. For example we need existence of linear algebraic stacks of flags of objects of $\mathfrak{M}$, denoted by $\mathfrak{M}^{(n)}$. We also require the projection $\mathfrak{M}^{(n)} \to \mathfrak{M}$ onto largest element in the flag to be a morphism of linear algebraic stacks of finite type. In particular, $\mathfrak{M}^{(2)}$ is the linear algebraic stack of short exact sequences of objects of $\mathfrak{M}$ and will be used to define a non-commutative but associate Hall product on $K(\mathfrak{M})$.

For sake of simplicity we focus on $\mathfrak{M} = \mathfrak{Coh}_X$ of Examples 8, although some other possible choices are $\mathfrak{Vect}$ and $\mathfrak{Rep}_Q$.

12.1 The Hall algebra of a linear stack

We define the linear category $\mathfrak{M}^{(n)}$ by setting the objects of over every R-scheme S being the flags $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ of objects of $\mathfrak{M}$ over the same scheme S, where each factor M_i/M_{i-1} is also an object of $\mathfrak{M}$ over S. Morphisms over $f : T \to S$ are diagrams

$$\begin{array}{ccccccc} f^*(M_1) & \longrightarrow & f^*(M_2) & - - - & f^*(M_{n-1}) & \longrightarrow & f^*(M_n) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N_1 & \longrightarrow & N_2 & - - - - - & N_{n-1} & \longrightarrow & N_n \end{array} \tag{12.1}$$

where each vertical arrow is a morphism in $\mathfrak{M}$ and each square is a commutative diagram of objects of $\mathfrak{M}$. For all $i = 1,\dots,n$ there are morphisms of linear categories $\alpha_i : \mathfrak{M}^{(n)} \to \mathfrak{M}$, sending the above flag to its i-th factor M_i/M_{i-1}. And there is another morphism $b : \mathfrak{M}^{(n)} \to \mathfrak{M}$ sending the above flag to $A_n = A$.

$$\begin{array}{ccc} \mathfrak{M}^{(n)} & \xrightarrow{b} & \mathfrak{M} \\ \big\downarrow{\scriptstyle a_1 \times \cdots \times a_n} & & \\ \mathfrak{M} \times \cdots \times \mathfrak{M}. & & \end{array}$$

In particular $\mathfrak{M}^{(2)}$ is the linear category of short exact sequences $M' \to M \to M''$ in $\mathfrak{M}$ and morphism between the short exact sequences.

Proposition 12.1. *For the linear algebraic stacks $\mathfrak{M} = \mathfrak{Coh}_X$, (1) the linear category $\mathfrak{M}^{(n)}$ is a linear algebraic stack; (2) b is a representable morphism of linear algebraic stacks; and (3) $a_1 \times \cdots \times a_n$ is a morphism of linear algebraic stacks of finite type.*

PROOF. The underlying algebraic stacks of $\mathfrak{M}^{(n)}$ are all constructed by restricting the vertical morphisms in (12.1) to be isomorphisms of sheaves. The claims then follow from results of [12, Section 4.1]. □

Recall the space $K(\mathfrak{M})$ of stack functions as defined in §11. We have the following structures on the Hall algebra $K(\mathfrak{M})$.

1. **Module structure**. There is an action of $K(\mathrm{Var})$ on $K(\mathfrak{M})$, given by $[Z] \cdot [X \to \mathfrak{M}] = [Z \times X \to X \to \mathfrak{M}]$. This action turns $K(\mathfrak{M})$ into a $K(\mathrm{Var})$-module.

2. **Multiplication**. We multiply two stack functions $[X \to \mathfrak{M}]$ and $[Y \to \mathfrak{M}]$ by the formula
$$[X \to \mathfrak{M}] \cdot [Y \to \mathfrak{M}] = [X \times Y \to \mathfrak{M} \times \mathfrak{M} \xrightarrow{\oplus} \mathfrak{M}].$$

3. **Hall product**. The Hall product of the stack functions $[X \to \mathfrak{M}]$ and $[Y \to \mathfrak{M}]$ is defined by first constructing the fibered product
$$\begin{array}{ccc} X * Y & \longrightarrow & \mathfrak{M}^{(2)} \\ \big\downarrow & & \big\downarrow{\scriptstyle a_1 \times a_2} \\ X \times Y & \longrightarrow & \mathfrak{M} \times \mathfrak{M} \end{array}$$
and then setting
$$[X \to \mathfrak{M}] * [Y \to \mathfrak{M}] = [X * Y \longrightarrow \mathfrak{M}^{(2)} \xrightarrow{b} \mathfrak{M}].$$

The additive zero in $K(\mathfrak{M})$ is given by the empty algebroid $0 = [\emptyset \to \mathfrak{M}]$. The multiplication is associative and commutative, and the Hall product is associative by [12, Theorem 4.3]. The unit with respect to both multiplications is $1 = [\operatorname{Spec} R \xrightarrow{0} \mathfrak{M}]$, i.e. the 0-object of $\mathfrak{M}$.

12.1.1 Filtered structure of the Hall algebra

Definition 12.2. For $n \geq 0$, we define

$$K^{\leq n}(\mathfrak{M}) = \ker E_{n+1} = \bigoplus_{k \leq n} K^k(\mathfrak{M}) .$$

This is an ascending filtration on $K(\mathfrak{M})$, called the **filtration by the order of vanishing of inertia at $q = 1$**, or simply the **order filtration** of $K(\mathfrak{M})$.

This is a slight abuse of language, because we have to tensor with $\mathbb{Q}(q)$, before we can state that $K^{\leq n}(\mathfrak{M})(q)$ is the direct sum of all eigenspaces of I^{ss} whose coresspond-ing eigenvalue $\mathfrak{Q} \in \mathbb{Q}[q]$ has order of vanishing at $q = 1$ less than or equal to n.

Theorem 12.3. *The Hall product respects the order filtration: if $\xi \in K^{\leq m}(\mathfrak{M})$ and $\eta \in K^{\leq n}(\mathfrak{M})$, then $\xi * \eta \in K^{\leq m+n}(\mathfrak{M})$ and $\xi.\eta \in K^{\leq m+n}(\mathfrak{M})$. On the associated graded the Hall product coincides with the commutative product.*

12.1.2 Proof of Theorem 12.3

Analysis of $E_p(E_n * E_m)$

Suppose $\xi = (X \to \mathfrak{M})$ and $\chi = (Y \to \mathfrak{M})$ are stack functions. The stack function $\xi * \chi$ is defined by the cartesian diagram:

$$\begin{array}{ccccc} X * Y & \longrightarrow & \mathfrak{M}^{(2)} & \longrightarrow & \mathfrak{M} \\ \downarrow & & \downarrow & & \\ X \times Y & \longrightarrow & \mathfrak{M} \times \mathfrak{M} & & \end{array}$$

Explicitely, $X * Y$ is the stack of triples (x, M, y),

$$\begin{array}{ccccc} x & & & & y \\ \downarrow & & & & \downarrow \\ M' & \longrightarrow & M & \longrightarrow & M'' \end{array} \tag{12.2}$$

where x and y are objects of X and Y, respectively, M is an object of $\mathfrak{M}^{(2)}$, i.e., a short exact sequence $M' \to M \to M''$ of objects in $\mathfrak{M}$, and $x \to M'$ and $y \to M''$ are isomorphisms from the images of x and y in $\mathfrak{M}$ to M' and M'', respectively. (We omit these isomorphisms from the triple to simplify the notation.)

The stack function $E_n(\xi) * E_m(\chi)$ is defined by the enlarged diagram:

$$\begin{array}{ccccccc} E_n(X) * E_m(Y) & \longrightarrow & X * Y & \longrightarrow & \mathfrak{M}^{(2)} & \longrightarrow & \mathfrak{M} \\ \downarrow & & \downarrow & & \downarrow & & \\ E_n(X) \times E_m(Y) & \longrightarrow & X \times Y & \longrightarrow & \mathfrak{M} \times \mathfrak{M} & & \end{array}$$

Explicitly, $E_n(X) * E_m(Y)$ is the stack of 5-tuples $\big(x, (e_\nu), M, y, (f_\mu)\big)$, where (x, M, y) represents a diagram (12.2), and $(e_\nu) = (e_1, \ldots, e_n)$ is a complete set of non-zero orthogonal idempotents in $A(x)$, and $(f_\mu) = (f_1, \ldots, f_m)$ is a complete set of non-zero orthogonal idempotents in $A(y)$.

Finally, the stack $E_p\big(E_n(X) * E_m(Y)\big)$ is the stack of objects of $E_n(X) * E_m(Y)$, endowed with a complete set of p non-zero labelled idempotents. Explicitly, it consits of 6-tuples

$$\big(x, (e_{\nu,\rho}), M, (g_\rho), y, (f_{\mu,\rho})\big)\,, \tag{12.3}$$

where (x, M, y) is as in (12.2), and $(g_\rho)_{\rho \in \underline{p}}$ is a complete set of non-zero orthogonal idempotents for the short exact sequence $M' \to M \to M''$. Moreover, $(e_{\rho,\nu})_{\rho \in \underline{p}, \nu \in \underline{n}}$ is a pn-tuple of orthogonal idempotents in $A(x)$, and $(f_{\rho,\mu})_{\rho \in \underline{p}, \mu \in \underline{m}}$ is a pm-tuple of orthogonal idempotents in $A(y)$, such that for every $\rho = 1, \ldots, p$ we have $\sum_{\nu=1}^{n} e_{\rho,\nu} = g_\rho|_{E'}$ and $\sum_{\mu=1}^{m} f_{\rho,\mu} = g_\rho|_{E''}$. Finally, we require for all $\nu = 1, \ldots, n$ that $e_\nu = \sum_{\rho=1}^{p} e_{\rho,\nu} \neq 0$ and for all $\mu = 1, \ldots, m$ that $f_\mu = \sum_{\rho=1}^{p} f_{\rho,\mu} \neq 0$.

Decomposing $E_p(E_n * E_m)$

We use the notation $\sigma \vDash \underline{u}$ for partitions of sets as opposed to $\sigma \vDash_\ell \underline{r}$ for *labelled* set partitions (where the order of blocks matter). For more details in labelled and unlabelled partitions we refer the reader to §A.3. If $b(\sigma) = p$ then for an element $\omega \in \underline{u}$ we say $\sigma(\omega) = \rho$ or $\omega \mapsto \rho$ if $\rho \in \underline{p}$ is the label of the partition ω belongs to. Also if σ and γ are both labelled set partitions of $\underline{u}$, we write $\omega \mapsto (\rho, \mu)$ if ω is in the set with label ρ in σ and in the set with label μ in γ.

Given non-negative integers p, u, v, and labelled set partitions $\gamma \vDash \underline{u} \sqcup \underline{v}$ such that $b(\gamma) = p$, we define a new stack function $(X * Y)_\gamma \to \mathfrak{M}$, denoted $(\xi * \chi)_\gamma$, as follows. Let $(X * Y)_\gamma$ be the algebraic stack of 6-tuples

$$\big(x, (e_\omega), M, (g_\rho), y, (f_\eta)\big)\,, \tag{12.4}$$

where (x, M, y) is as in (12.2), and $(e_\omega)_{\omega \in \underline{u}}$, $(f_\eta)_{\eta \in \underline{v}}$ and $(g_\rho)_{\rho \in \underline{p}}$ are complete sets of non-zero orthogonal idempotents for x, y and the short exact sequence E, respectively. Moreover, we require that for all $\rho = 1, \dots, p$ we have

$$g_\rho|_{M'} = \sum_{\gamma(\omega)=\rho} e_\omega \qquad \text{and} \qquad g_\rho|_{M''} = \sum_{\gamma(\eta)=\rho} f_\eta \,. \tag{12.5}$$

There is a natural algebroid structure on $(X * Y)_\gamma$. The morphism to $\mathfrak{M}$ given by mapping the 6-tuple (12.4) to the middle object $b(M)$ of the short exact sequence M, makes $(X * Y)_\gamma$ into a stack function.

There is a morphism

$$E_u(X) \times E_v(Y) \longrightarrow (X * Y)_\gamma \tag{12.6}$$

which maps a quadruple $(x, (e_\omega), y, (f_\eta))$ to the 6-tuple (12.4) where $M = M' \oplus M''$, with M' denoting the image of x in $\mathfrak{M}$, and M'' the image of y in $\mathfrak{M}$. The family of idempotents (g_ρ) on M is defined by formulas (12.5).

Lemma 12.4. *If for every $\rho = 1, \dots, p$ exactly one of the two preimages $\gamma^{-1}(\rho) \cap \underline{u}$ and $\gamma^{-1}(\rho) \cap \underline{v}$ is empty, the morphism (12.6) is an isomorphism.*

Proof. Given an object (12.4) of $(X * Y)_\gamma$, the short exact sequence M is split into a direct sum of p short exact sequences. Each one of these sequences is canonically split, because either the subobject or the quotient object vanishes, by the assumption on φ and ψ. Therefore the sequence M is split, canonically, too. □

Now suppose given $\varphi \vDash_\ell \underline{u}$ with n labelled blocks and $\psi \vDash_\ell \underline{v}$ with m labelled blocks such that $\varphi \cap \gamma|_{\underline{u}} = 0$ and $\psi \cap \gamma|_{\underline{v}} = 0$. Then we can define a morphism of stacks

$$(X * Y)_\gamma \longrightarrow E_p\big(E_n(X) * E_m(Y)\big) \,, \tag{12.7}$$

by mapping the 6-tuple (12.4) to the 6-tuple (12.3) by defining

$$e_{\rho,\nu} = \sum_{\omega \mapsto (\rho,\nu)} e_\omega \qquad \text{and} \qquad f_{\rho,\mu} = \sum_{\eta \mapsto (\rho,\mu)} f_\eta \,.$$

By our assumptions, these sums are either empty or consist of a single summand, so the $e_{\rho,\nu}$ and the $f_{\rho,\mu}$ are obtained from the e_ω and the f_η essentially by relabelling.

Lemma 12.5. *The morphism (12.7) gives rise to a morphism of stack functions $(\xi * \chi)_\gamma \to E_p\big(E_n(\xi) * E_m(\chi)\big)$, which is both an open and a closed immersion. If we change any of u, v or γ or φ or ψ, we get a morphism with disjoint image. The images of all morphisms (12.7) cover $E_p\big(E_n(X) * E_m(Y)\big)$.*

Proof. This follows from the fact that the source and target of (12.7) only differ in the way the idempotents in A_x and A_y are indexed. □

Corollary 12.6. *In $K(\mathfrak{M})$ we have the equation*

$$E_p(E_n(\xi) * E_m(\chi)) = \sum_{u,v,\gamma} \sum_{\varphi,\psi} (\xi * \eta)_\gamma = \sum_{u,v,\gamma} n!m! \begin{bmatrix} n \\ \gamma|_{\underline{u}} \end{bmatrix} \begin{bmatrix} m \\ \gamma|_{\underline{v}} \end{bmatrix} (\xi * \eta)_\gamma ,$$

where u,v run over all positive integers and γ runs over all labelled partitions of $\underline{u} \sqcup \underline{v}$ with p blocks. The bracket notation is adapted from §A.2.

The Hall algebra is filtered

We can now calculate as follows:

$$\begin{aligned} E_p(\pi_k(\xi) * \pi_\ell(\chi)) &= \sum_{n,m} \frac{s(n,k)}{n!} \frac{s(m,\ell)}{m!} E_p(E_n(\xi) * E_m(\chi)) \\ &= \sum_{n,m} s(n,k)s(m,\ell) \sum_{u,v,\gamma} \begin{bmatrix} n \\ \gamma|_{\underline{u}} \end{bmatrix} \begin{bmatrix} m \\ \gamma|_{\underline{u}} \end{bmatrix} (\xi * \chi)_\gamma \\ &= \sum_{u,v,\gamma} \left(\sum_n s(n,k) \begin{bmatrix} n \\ \gamma|_{\underline{u}} \end{bmatrix} \right) \left(\sum_n s(m,\ell) \begin{bmatrix} m \\ \gamma|_{\underline{u}} \end{bmatrix} \right) (\xi * \chi)_\gamma . \end{aligned}$$

For both brackets to be non-zero, the number of blocks of $\gamma|_{\underline{u}}$ must be at most k, and the number of blocks of $\gamma|_{\underline{v}}$ must be at most ℓ, by Lemma A.3. We conclude that for all $p > k+\ell$ we have $E_p(\pi_k(\xi) * \pi_\ell(\chi)) = 0$, which proves the first part of the theorem.

If $p = k+\ell$ then the only possible case is if the number of blocks of $\gamma|_{\underline{u}}$ is exactly k, and the number of blocks of $\gamma|_{\underline{v}}$ is exactly ℓ, by Lemma A.3. In this case we have

$$E_p(\pi_k(\xi) * \pi_\ell(\chi)) = \sum_{u,v,\gamma} \mu(0_{\underline{u}}, \gamma|_{\underline{u}}) \mu(0_{\underline{v}}, \gamma|_{\underline{v}}) (\xi * \chi)_\gamma ,$$

where γ runs over those partitions of $\underline{u} \sqcup \underline{v}$ such that for every $\rho = 1, \dots, p$ exactly one of the two preimages $\gamma^{-1}(\rho) \cap \underline{u}$ and $\gamma^{-1}(\rho) \cap \underline{v}$ is non-empty. By Lemma 12.4, we have therefore

$$\begin{aligned} E_p(\pi_k(\xi) * \pi_\ell(\chi)) &= \sum_{u,v,\gamma} \mu(0_{\underline{u}}, \gamma|_{\underline{u}}) \mu(0_{\underline{v}}, \gamma|_{\underline{v}}) E_u \xi . E_v \chi \\ &= \sum_{u,v} \frac{(k+\ell)!}{u!v!} E_u \xi . E_v \chi \left(\sum_{\substack{\gamma_1 \vDash \underline{u} \\ b(\gamma_1) = k}} \mu(0, \gamma_1) \right) \left(\sum_{\substack{\gamma_2 \vDash \underline{v} \\ b(\gamma_2) = \ell}} \mu(0, \gamma_2) \right) \\ &= (k+\ell)! \left(\sum_u \frac{s(u,k)}{u!} E_u \xi \right) \left(\sum_v \frac{s(v,\ell)}{v!} E_v \chi \right) \\ &= (k+\ell)! \, \pi_k(\xi) \, \pi_\ell(\chi) . \end{aligned}$$

The associated graded algebra

By what we just proved, we have

$$
\begin{aligned}
\pi_{k+\ell}(\pi_k(\xi) * \pi_\ell(\chi)) &= \sum_p \frac{s(p,k+\ell)}{p!} E_p(\pi_k(\xi) * \pi_\ell(\chi)) \quad \text{only nonzero if } p = k+\ell \\
&= \frac{s(k+\ell,k+\ell)}{(k+\ell)!} E_{k+\ell}(\pi_k(\xi) * \pi_\ell(\chi)) \\
&= \pi_k(\xi).\pi_\ell(\chi)\,.
\end{aligned}
$$

The proof for the commutative product is similar and gives

$$\pi_{k+\ell}(\pi_k(\xi).\pi_\ell(\chi)) = \pi_k(\xi).\pi_\ell(\chi)\,.$$

This finishes the proof of the theorem.

12.2 Epsilon functions

We consider a stack function $\xi = (X \to \mathfrak{M})$, and an idempotent operator E_k. Let us denote by $F_n X$ the fibered product

$$
\begin{array}{ccc}
F_n X & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathfrak{M}^{(n)} & \xrightarrow{\;b\;} & \mathfrak{M}
\end{array}
$$

Then we will consider $E_k(F_n X)$. This is the stack of triples

$$(x, (e_\kappa), F)$$

where x is an object of X, and $F = (F_1 \to \ldots \to F_n)$ is a flag in $\mathfrak{M}$, such $F_n = M$, the image of x in $\mathfrak{M}$. Moreover, $(e_\kappa)_{\kappa=1,\ldots k}$ is a complete set of non-zero orthogonal idempotents for x, such that the induced idempotent operators on M respect the flag F. We get induced idempotents $f_{\kappa,\nu} = e_\kappa|_{F_\nu/F_{\nu-1}}$. They have the properties that $\sum_\kappa f_{\kappa,\nu} = 1$, for all ν, and for every κ, at least one of the $f_{\kappa,\nu}$ does not vanish.

We will decompose $E_k(F_n X)$ according to which of the $f_{\kappa,\nu}$ vanish. For this, consider a non-negative integer p and a labelled partition $\alpha \vDash_\ell \underline{p}$ with k blocks. Then we define $F_\alpha X$ to be the stack of triples

$$(x, (e_\kappa), (F_\kappa))\,.$$

Here x is an object of X, and (e_κ) is a complete set of orthogonal non-zero idempotents for x. If we denote the image of x in $\mathfrak{M}$ by M, then these idempotents define a direct sum decomposition $M = M_1 \oplus \ldots \oplus M_k$. For every κ, we have a filtration F_κ of E_κ indexed by $\alpha^{-1}(\kappa)$.

Now let us suppose given another labelled partition $\beta \vDash_\ell p$ with n blocks such that $\alpha \cap \beta = 0$. Using β, we define a morphism

$$F_\alpha X \longrightarrow E_k(F_n X)\,, \tag{12.8}$$

by defining the flag F in terms of the k-tuple of flags (F_κ) by

$$F_\nu = \bigoplus_\kappa \sum_{\substack{\alpha(\rho)=\kappa \\ \beta(\rho)\le n}} F_\rho\,.$$

Note that the sum for fixed κ is not really a sum, it is just the largest of the spaces F_ρ, such that $\alpha(\rho) = \kappa$ and $\beta(\rho) \le n$.

Lemma 12.7. *The morphism (12.8) is an isomorphism onto the locus in $E_k(F_n X)$, given by $f_{\kappa,\nu} \neq 0$ if and only if $\alpha^{-1}(\kappa) \cap \beta^{-1}(\nu)$ is nonempty (i.e. is a singleton).*

Corollary 12.8. *We have*

$$E_k(F_n\xi) = \sum_p \sum_{\alpha:b(\alpha)=k} \begin{bmatrix} n \\ \alpha \end{bmatrix} F_\alpha\xi\,.$$

A special case of the following result is the main outcome of theories of generalized Donaldson-Thomas invariants and is used in many applications (cf. for example [11, §6.3]).

Corollary 12.9. *We have*

$$\varepsilon_k(\xi) = \sum_n \frac{s(n,k)}{n!} F_n(\xi) \in K^{\le k}(\mathfrak{M})\,.$$

Proof. Let us do a calculation:

$$\begin{aligned} E_{k+1} \sum_n \frac{s(n,k)}{n!} F_n(\xi) &= \sum_n \frac{s(n,k)}{n!} E_{k+1} F_n(\xi) \\ &= \sum_n s(n,k) \sum_p \sum_{\alpha:b(\alpha)=k+1} \begin{bmatrix} n \\ \alpha \end{bmatrix} F_\alpha \xi \\ &= \sum_p \sum_{\alpha:b(\alpha)=k+1} \left(\sum_n s(n,k) \begin{bmatrix} n \\ \alpha \end{bmatrix} \right) F_\alpha \xi \\ &= 0\,. \end{aligned}$$

The expression in the bracket vanishes, by Corollary A.3.□

In particular, in the case of $k = 1$, we have

$$\varepsilon_1(\xi) = \sum_{n>0} \frac{(-1)^n}{n} F_n(\xi).$$

For $\mathfrak{M} = \mathfrak{Coh}_X$ the moduli stack of coherent sheaves on a projective $\mathbb{C}$-scheme X (cf. Example 8) and τ a stability condition as in §1.1, let $\xi = [\mathrm{SS}^\gamma(\tau) \hookrightarrow \mathfrak{Coh}_X]$ be the inclusion of the substack of semistable sheaves of class γ. Applying the above result to this case produces

$$\varepsilon_1(\xi) = \sum_n \frac{(-1)^n}{n!} \sum_{\substack{0 \neq \gamma_1,\dots,\gamma_n \in \Gamma \\ \tau(\gamma_i)=\tau, \sum \gamma_i = \gamma}} \mathrm{SS}^{\gamma_1}(\tau) * \dots * \mathrm{SS}^{\gamma_n}(\tau) \in K^{\leq 1}(\mathfrak{Coh}_X).$$

This is what is called the no-poles theorem in [29] or absence of poles in [33] and proving them is involved. In contrast, our framework provides a simple derivation of this result.

12.3 The semi-classical Hall algebra

By Theorem 12.3, the submodule

$$\mathscr{K}(\mathfrak{M}) = \bigoplus_{n \geq 0} t^n K^{\leq n}(\mathfrak{M})$$

of $K(\mathfrak{M})[\![t]\!]$ is a $K(\mathrm{Var})[t]$-subalgebra with respect to the Hall product. The algebra $\mathscr{K}(\mathfrak{M})$ is a one-parameter flat family of algebras. The special fibre at $t = 0$ is canonically isomorphic to the graded algebra associated to the filtered algebra $(K(\mathfrak{M}), *)$. The quotient map $\mathscr{K} \to \mathscr{K}/t\mathscr{K}$ is identified with the map $\sum_n x_n t^n \mapsto \sum_n \pi_n(x_n)$.

The graded algebra associated to the filtered algebra $(K(\mathfrak{M}), *)$, is canonically isomorphic to the commutative graded algebra $(K(\mathfrak{M}), \cdot)$, by Theorem 12.3. The special fibre inherits therefore a Poisson bracket, which encodes the Hall product to second order. This Poisson bracket has degree -1, and is given by the formula

$$\{x, y\} = \pi_{k+\ell-1}(x * y - y * x), \qquad \text{for } x \in K^k(\mathfrak{M}),\ y \in K^\ell(\mathfrak{M}). \tag{12.9}$$

Corollary 12.10. *The graded $K(\mathrm{Var})$-algebra $(K(\mathfrak{M}), \cdot)$ is endowed with a Poisson bracket of degree -1, given by (12.9).*

Corollary 12.11. *In particular, $K^1(\mathfrak{M})$ is a Lie algebra with respect to the Poisson bracket (12.9). In fact, for $x, y \in K^1(\mathfrak{M})$, we have that $x * y - y * x \in K^1(\mathfrak{M})$, so in this case, the Poisson bracket is equal to the Lie bracket.*

Corollary 12.12. *For every stack function $\xi = (X \to \mathfrak{M})$, the epsilon element $\varepsilon_1 \xi$ defines a virtual indecomposable.*

Thus, $K^1(\mathfrak{M})$ is a Lie algebra over the ring of scalars $K(\mathrm{Var})$. We call $K^1(\mathfrak{M})$ the Lie algebra of **virtually indecomposable** stack functions. This terminology is used in analogy with that of [29]. It is a work in progress to show that $K^1(\mathfrak{M})$ serves a similar purpose as the virtual indecomposables of [29] in defining generalized Donaldson-Thomas invariants.

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Appendix A

Partitions

A.1 Mobius numbers

We follow the notation of [9]. Let $(S, \leq)$ be a poset. The two elements $0, 1 \in S$ are respectively the smallest and the largest elements in the poset. The *Mobius function* is an integer valued function of two variables on S that associates to a pair (x, y) of elements of S the *Mobius number* of it, defined by $\mu(x, z) = 0$ when $x \nleq z$ and when $x \leq z$ by

$$\sum_{y: x \leq y \leq z} \mu(x, y) = \delta(x, z)\,. \tag{A.1}$$

Here δ is the Kronecker delta function, defined as $\delta(x, z) = 1$ if $x = z$ and is zero otherwise.

The meet of two elements $x, y \in S$ is defined by $x \wedge y$ being the unique greatest lower bound for x and y. A poset with well-defined meet operator $\wedge$ is called a *meet semilattices* [15].

For every two elements $x, y \in S$, the notation $[x, y]$ stands for the subset of all nodes z such that $z \geq x$ and $z \leq y$. If S is a met semi-lattice then so it $[x, y]$. S is called *locally finite* if for every two elements $x, y \in S$, the segment $[x, y]$ has finitely many nodes.

Proposition A.1. *Given three fixed elements x, z, w of a locally finite meet semilattice S*

$$\sum_{y: y \wedge w = x} \mu(y, z) = \begin{cases} \mu(x, z) & w \geq z \\ 0 & w \ngeq z. \end{cases}$$

In the following proof we use the $\mu_w(x, z)$ for this type of summations.

Proof. The case of $w \geq z$ is easy because if $y \wedge w = x$, and $y \leq z \leq w$ then $y = x$. By locally finiteness of S the segment $[x, 1]$ is a finite poset. So we may proceed by induction on x. If $x = 1$ then the claim is easy to check from the above definition. Suppose the claim is proved for every element $\overline{x} > x$. The set of all $y \in [x, z]$ is the disjoint union of the following sets

$$\{y : y \wedge w = \overline{x}\} \quad \forall \overline{x} \in [x, z]$$

which means

$$\mu_w(x, z) = \sum_{\overline{x}: x \le \overline{x} \le z} \mu_w(\overline{x}, z) = \sum_{y: x \le y \le z} \mu(x, y) = 0$$

where the first identity follows from the induction hypothesis and the last identity follows from (A.1) in the definition of μ. □

A.2 Identities involving Stirling numbers

Let u be a positive integer. We use the notation $\underline{u}$ for the set of first u integers $\{1, 2, \ldots, u\}$ and Ω_u for the set of all partitions $\lambda \vDash \underline{u}$. Given two partitions $\lambda, \mu \vDash \underline{u}$ we write $\lambda \le \mu$ if every element of λ is a subset of an element of μ. In this case we say λ is *finer* than μ and μ is *coarser* than λ. This way, $(\Omega_u, \le)$ is a poset. We use the notations $0_{\underline{u}}$ and $1_{\underline{u}}$ respectively for the finest and coarsest partitions of $\underline{u}$. In places where no confusion should arise we suppress the subscript $\underline{u}$ and simply write 0 and 1 respectively for $0_{\underline{u}}$ and $1_{\underline{u}}$.

Given any other partition $\alpha \vDash \underline{u}$ the partition $\alpha \cap \lambda$ is the coarsest partition that is finer than both α and λ. This turns Ω_u into a meet semilattice.

We use the notation $b(\lambda)$ for the number of blocks of a partition. In [47, Example 3.10.4] the k-th *Whitney number of the first kind* associated to $\Omega_{\underline{u}}$ is defined as

$$w_k = \sum_{\substack{\tau \vDash \underline{u} \\ b(\tau) = u - k}} \mu(0_{\underline{u}}, \tau)$$

and is it shown to be equal to a Stirling number of the first kind by $w_k = s(u, u - k)$. We like to rewrite this result as

$$s(b(\alpha), k) = \sum_{\substack{\tau \vDash \underline{u} \\ b(\tau) = k}} \mu(\alpha, \tau) \tag{A.2}$$

by substituting Ω_u with the segment $[\alpha, 1_{\underline{u}}]$. Note that if $\tau \notin [\alpha, 1_{\underline{u}}]$ then $\mu(\alpha, \tau) = 0$ hence the above identity holds without the need to restrict τ to be in $[\alpha, 1_{\underline{u}}]$.

Lemma A.2. *We have*

$$\sum_{\alpha : \alpha \cap \lambda = 0} s(b(\alpha), k) = \begin{cases} 0 & \text{if } k < b(\lambda) \\ \mu(0, \lambda) & \text{if } k = b(\lambda). \end{cases}$$

Here α runs over all partitions of $\underline{u}$.

PROOF. We use the identity (A.2) and have

$$\begin{aligned}\sum_{\alpha:\alpha\cap\lambda=0} s(b(\alpha),k) &= \sum_{\alpha:\alpha\cap\lambda=0}\ \sum_{\tau:b(\tau)=k} \mu(\alpha,\tau) \\ &= \sum_{\tau:b(\tau)=k}\ \sum_{\alpha:\alpha\cap\lambda=0} \mu(\alpha,\tau) \\ &= \sum_{\tau:b(\tau)=k:\tau\leq\lambda} \mu(0,\tau)\end{aligned}$$

where the last identity follows from Proposition A.1. The set of all such τ can only be nonempty if $k \geq b(\lambda)$ proving the lemma. □

Let u be a positive integer and $\lambda \vDash \underline{u}$ a set partition. The notation $\begin{bmatrix} n \\ \lambda \end{bmatrix}$ means the number of partitions $\alpha \vDash \underline{u}$ with n blocks such that $\alpha \cap \lambda = 0$.

Corollary A.3. *We have*

$$\sum_n s(n,k)\begin{bmatrix} n \\ \lambda \end{bmatrix} = \begin{cases} 0 & \textit{if } k < b(\lambda) \\ \mu(0,\lambda) & \textit{if } k = b(\lambda)\,. \end{cases}$$

PROOF. This follows from previous lemma and the following computation

$$\sum_n s(n,k)\begin{bmatrix} n \\ \lambda \end{bmatrix} = \sum_n \sum_{\alpha,b(\alpha)=n,\alpha\cap\lambda=0} s(n,k) = \sum_{\alpha:\alpha\cap\lambda=0} s(b(\alpha),k).$$

□

A.3 Labelled partitions and integer partitions

A *labelled partition* of $\underline{u}$ is a partition $\lambda \vDash \underline{u}$ where the order of elements of λ is important. We denote a labelled partition by $\lambda \vDash_\ell \underline{u}$. If we forget the labelling the resulting partition is called the associated *unlabelled* partition to λ. Note that given a labelled partition λ there are $b(\lambda)!$ other labelled partitions with the same associated unlabelled partition as that of λ. If $\lambda, \mu \vDash_\ell \underline{u}$ are two labelled partitions we adapt the notation $\mu(\lambda,\mu)$ to denote the Mobius number of the associated unlabelled partitions of λ and μ.

Recall that given an integer t, an integer partition is a sequence $(s_i)_{i\geq 1}$ of integers such that $\sum_{i\geq 1} i s_i = t$. To a set partition $\underline{\lambda} \vDash \underline{t}$ we may therefore associate an integer partition $\lambda \vdash t$ defined as $\lambda = (\lambda_1, \lambda_2, \ldots)$ if $\underline{\lambda}$ has λ_i elements of size i. We call λ the *integer partition type* of $\underline{\lambda}$. We define the *length* of λ as $b(\lambda) = \sum_i \lambda_i$. So the length of the integer type of λ is identical to the number of blocks of $\underline{\lambda}$.

Appendix B

Splitting covers of gerbes

Let $\mathfrak{X} \to X$ be a gerbe over a smooth connected scheme X and G an X-group scheme. Recall that triviality of $\mathfrak{X}$ is obstructed [16]. However $\mathfrak{X}$ is by definition étale locally neutral. After passing to a dense open on X, we may assume there exists a finite étale covering $\overline{X} \to X$, such that the pullback of $\mathfrak{X}$ is a neutral gerbes over $\overline{X}$, written as

$$\mathfrak{X}|_{\overline{X}} \cong B_{\overline{X}}G$$

for some X-group scheme G. We say $\overline{X}$ is a *trivializing* cover for $\mathfrak{X}$. We may also assume that $\overline{X} \to X$ is a Galois cover [48, Proposition 5.3.9] with Galois group $\Gamma = \mathrm{Aut}(\overline{X}/X)$. Then by classification of gerbes there exists a homomorphism

$$\varphi : \Gamma \to \mathrm{Aut}(BG) = [\mathrm{Aut}\, G/G]$$

such that $\mathfrak{X} \cong \overline{X} \times_{\Gamma,\varphi} BG$. We may assume φ is injective by passing to an intermediate cover:

$$\overline{X} \xrightarrow{\ker\varphi} \widetilde{X} \xrightarrow{\Gamma/\ker\varphi} X.$$

Note that $\widetilde{X}$ is now the minimal trivializing Galois cover of $\mathfrak{X} \to X$.

Definition B.1. The above covering $\widetilde{X} \to X$ is called a splitting cover of $\mathfrak{X}$ over X.

Lemma B.2. *Given a Galois covering $\overline{X} \to X$ and $\Gamma = \mathrm{Aut}(\overline{X}/X)$, an action of Γ on BG that results $\mathfrak{X}$ as a form of BG on X is unique up to 2-isomorphisms of stacks.*

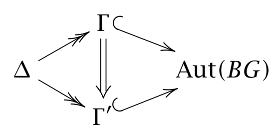
Proof. Let $\varphi : \Gamma \to \mathrm{Aut}(BG)$ and $\varphi' : \Gamma \to \mathrm{Aut}(BG)$ be two such actions. If

$$\widetilde{X} \times_{\Gamma,\varphi} BG \cong \widetilde{X} \times_{\Gamma,\psi} BG$$

as gerbes over X, then from classification of gerbes we can construct a natural transformation between the functors φ and ψ. □

Lemma B.3. *Splitting covers are unique up to isomorphisms of stacks.*

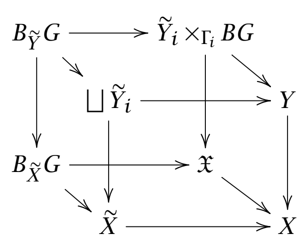
Proof. Let $\overline{X}_1$ and $\overline{X}_2$ be two splitting covers with corresponding $\varphi : \Gamma \to \mathrm{Aut}(BG)$ and $\varphi' : \Gamma' \to \mathrm{Aut}(BG)$. We can create a product trivializing cover $\widetilde{X} \to X$ with $\Delta = \mathrm{Aut}(\widetilde{X}/X)$ a new group of automorphisms over X. Then by the previous lemma there is a natural transformation between the two composition functors



both corresponding to trivializing covers of $\mathfrak{X}$. Therefore there exists a natural transformation between φ and φ'. This proves uniqueness. □

We finally show that under base change, each connected component of the pullback of a splitting cover is a splitting cover.

Corollary B.4. *Let $Y \to X$ be a smooth morphism of schemes. If $\widetilde{X} \to X$ is the splitting cover for $\mathfrak{X}$, then $\widetilde{Y} = Y \times_X \widetilde{X} \to Y$ is a union $\bigsqcup \widetilde{Y}_i$ of splitting covers of $\mathfrak{X}|_Y \to Y$.*



Proof. Let Γ be the Galois group of $\widetilde{X} \to X$ and $\widetilde{Y}_i$ a connected component of $\widetilde{Y}$. Let Γ_i be the stabilizer of $\widetilde{Y}$ as a subgroup of $\mathrm{Aut}(\widetilde{Y}/Y)$. All Γ_i are conjugate to each other and all $\widetilde{Y}_i$ are isomorphic to each other. Note that since $\mathfrak{X}$ is the form of BG twisted by $\varphi : \Gamma \to \mathrm{Aut}(BG)$, then

$$\mathfrak{X}|_Y \cong \widetilde{Y}_i \times_{\Gamma_i, \varphi_i} \mathrm{Aut}(BG).$$

where φ_i is the restriction of the homomorphism φ to the subgroup Γ_i. Since φ is injective, so is φ_i, which implies that $\widetilde{Y}_i$ is the splitting cover for $\mathfrak{X}|_Y$ for any i. □