Toward a Quantum Theory of Cognition: History, Development, and Perspectives

by

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Abstract

The representation and processing of concepts is considered to be one of the hardest challenges in cognitive science. While computer scientists and engineers have focused on developing advances for particular tasks, philosophers and cognitive scientists have focused on elucidating the structural nature of meaning.

A remarkable bridge between these two limited-success approaches can be found in behavioral research, since, in a variety of tasks, humans process information at a conceptual level in a way that is incompatible with classical probability and fuzzy set theory. Recently, this incompatibility has been shown to occur at a deep structural level, and attempts have been made to use mathematical schemes founded on quantum structures as alternative approaches. For this reason, the application of quantum structures to this type of phenomena has received increasing attention. The quantum approach allows to faithfully model a number of non-classical deviations observed in experimental data. Moreover, it shows that genuine quantum theoretical notions, such as contextuality, superposition, emergence, and entanglement, are powerful epistemic tools to understand and represent cognitive phenomena.

In this thesis, we identify the limitations of classical theories to handle some important cognitive tasks, and introduce the fundamentals of the quantum cognitive approach to concepts. Next, we perform a mathematical analysis of current concept combination models and develop an extension that allows for concrete representations of multiple exemplars simultaneously. Our analysis indicates that a superposition of logical reasoning and a specific form of non-logical reasoning, where non-logical reasoning is dominant, allows to faithfully represent the experimental data. Therefore, the non-logical reasoning introduced by this model represents an important but unexplored form of reasoning in humans.

In addition, we develop novel experimental methodologies to identify quantum conceptual structures for concept combinations in the context of natural language processing and psychological experiments. Namely, we present a methodology to build entangled concepts represented as sets of words with respect to a corpus of text, and present a computational and psychological methodology to discern if a collection of concepts behaves statistically as a collection of quantum or classical particles. Using both methodologies we have identified a significant presence of quantum conceptual structure in the context of natural language processing and psychological experiments.

Preface

In this thesis, I performed a systematic study of the quantum-cognitive approach to concepts. First, I made a comprehensive literature review to motivate the use of the quantum-cognitive approach in cognitive science. Next, I developed a mathematical analysis of the current quantum-cognitive models of concepts, and introduced a novel mathematical tools to produce concrete representations of experimental data. Finally, I introduced new experimental methodologies to identify quantum conceptual structures in the context of natural language processing and psychological experiments.

Some of the material presented in the thesis has been published in the following scientific journals[1]:
Chapter 2:
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I performed the theoretical and data analysis.

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I performed the theoretical analysis.

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I developed the theoretical analysis, the experiment design, and the data analysis.

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Chapter 1

Introduction

A well-established fact in cognitive science is that cognitive phenomena cannot be appropriately modeled using the traditional representational tools [Fod98, Gar90, Daw13]. This fact has serious implications in our understanding of what cognition is and, it is one of the major impediments to the advance of many research areas related to cognitive modeling such as knowledge representation and decision-making [McC].

An alternative approach to cognitive modeling borrows the representational tools of quantum theory to study cases where traditional methods fail. For example, in the field of decision-making, the conjunction fallacy [Fra09] and the Ellsberg paradox [ADS11] are important cases where quantum-inspired models have been used to overcome the limitations of traditional modeling. Quantum-inspired models have recently been developed for phenomena in multiple areas including psychology [BPFT11, BPB13], economics [Khr10], and computer science [MP13, BKL13]. The research field that applies the mathematical formalism of quantum theory to study cognitive phenomena is known as quantum cognition [BBG13].

One area where quantum cognition has found interesting results is the field of concept modeling. Scholars, from a wide range of communities such as philosophy, linguistics, and psychology, agree that concept combinations cannot in most cases be represented using traditional tools such as logic and probability theory. In fact, it has been shown that the conditions for a logical or probabilistic model for concept combinations are usually violated by data collected in psychological experiments [SO81, Ham88a, Ham88b]. However, quantum-inspired models, with genuine quantum features such as state superposition, interference, and entanglement, provide faithful representations for most concept combinations [Aer09, AGS13].

In view of the promising results provided by the quantum-cognitive approach to concept combinations, we propose to carry out a systematic review of this approach to better understand why quantum cognition provides adequate representations of concept combinations, and to propose a framework that enhances the range of applications of concept combination models.

This thesis is divided into three parts:

1. A systematic review of the structural properties of conceptual phenomena, and an introduction to the quantum approach to cognitive modeling.

2. An analysis of the mathematical framework for quantum-cognitive models of concept combinations, and the development of new models that have broader applications.

3. A philosophical argument and some empirical evidence for the application of quantum cognition in artificial intelligence.

In the first part of the thesis, we introduce concept modeling, and identify three structural problems that prevent the development of an adequate theory of concepts. This is done in part by presenting cognitive phenomena that cannot be represented by using traditional mathematical tools. Next, we introduce the quantum approach to cognitive modeling, and demonstrate how quantum-cognitive models can be used to represent those cognitive phenomena.

In the second part, we give a detailed mathematical analysis of the two most important quantum models of concept combination developed in the literature: The Hilbert space and tensor product models for concept conjunctions and disjunctions. We focus on the conditions required by each model to represent experimental data, and identify the minimal dimension that is required by each model to reach maximal modeling power. We then show that the Hilbert and tensor product models entail two fundamentally different ways to reason about concepts, and combine these two models into a more general model: the two-sector Fock space model.

In the two-sector Fock space model, the Hilbert and tensor product models are recovered as extreme cases. Intermediate cases between these two extremes correspond to superposed modes of thought. These superposed modes of thought can represent instances of concept combinations that do not have a representation in the original two models. In addition, we show that the concrete representations provided by the aforementioned models for concept combinations are not consistent with the quantum cognitive principles that inspire the abstract model: conceptual states must be independent of the exemplar, and measurement operators must be exemplar-dependent. We use unitary transformations in the concrete spaces $\mathbb{C}^3$ and $\mathbb{C}^3 \otimes \mathbb{C}^3$ to construct representations of multiple exemplars in accordance with the quantum modeling principles, and extend this representation method to the two-sector Fock space model.

Next, we extend the two-sector Fock space model of conjunctions to the case of conjunctions and negations. We first develop a theoretical analysis that characterizes classical data for the case of conjunctions and negations. Then, we introduce experimental data showing that concept combinations involving negations of concepts do not satisfy the conditions of classical data, elaborate concrete representations in the space $\mathbb{C}^8 \oplus \mathbb{C}^8 \otimes \mathbb{C}^8$, and analyze these representations to show that the extended two-sector Fock space model can faithfully represent the experimental data.

In the last part, we consider the limitations of current artificial intelligence methodologies from the perspective of quantum cognition, and provide examples that justify the development of quantum-inspired models in artificial intelligence. In particular, we explain how the problems of vagueness, contextuality, and non-compositionality are relevant to a sub-area of artificial intelligence, known as natural language processing. We provide experimental evidence of quantum structures in natural language processing by showing that quantum entanglement can be found in the word co-occurrence statistics of a corpus of text, and that Bose-Einstein statistics can be found in psychological experiments and in the retrieval statistics of a search engine.

Chapter 2

Basics of Cognitive Modeling

2.1 From the Mind-Body Problem to a Theory of Concepts

Human beings have the capacities to observe elements of reality, to identify and represent relations among such elements, and to hypothesize and test unobserved relations. These capacities have lead to the emergence of a number of fields of knowledge that have developed to explain physical reality. Among these fields, the basic sciences occupy a privileged place because their methodologies have lead to the development of important technological advances.

All known human cultures have recognized the existence of a second non-physical realm that must be incorporated to the physical realm to complete the picture of the factuality of human existence. This realm, where human manifestations such as ideas, emotions, and self-awareness reside, is known as '*The Mind*' [Sea04]. Whether or not these two realms exist independently of each other is one of the most fundamental questions in western philosophy. This is known as the 'the mind-body problem' [Wig61]. In modern science, the interdisciplinary effort toward the study of the realm of the mind is known as cognitive science [Daw13].

2.1.1 Cognitive Science

Cognitive science is defined as the scientific study of the mind and its processes. It examines what cognition is, what it does and how it works, and, like any other science, it aims to develop technologies and tools to advance our understanding of, in this case, the mind. Such investigation includes considerations of multiple aspects of intelligence and behaviour, and focuses on how information is represented, processed, and transformed. In particular, cognitive science investigates cognitive phenomena such as perception, language, memory, reasoning, and emotion.

The majority of cognitive scientists assume that cognition is the product of neurological processes occurring mostly in the brain [Tho85]. Hence, the dominant attitude regarding the mind-body problem is that the mind is a 'result' of the body. This is better understood by noting that most approaches to study cognition start from a basic 'cognitive architecture.' For example, neuroscience and clinical psychology assume that cognitive phenomena are the output of a nervous system that is controlled by the human brain [Daw13]. For artificial intelligence, the cognitive phenomena are asssumed to be the output of a specific software implemented on a machine [Gar90].

An alternative view, held mainly by a mix of applied mathematicians and cognitive psychologists [Nei76], focuses on understanding the structural aspects of the cognitive phenomena from an abstract, and usually mathematical, perspective. This alternative approach, known as cognitive modeling, is the one we follow here. Therefore, we will identify some fundamental structural properties underlying cognitive phenomena, and attempt to represent them using the language of mathematics.

2.1.2 Cognitive Modeling

A cognitive model is an approximation to a cognitive phenomenon for the purpose of comprehension and prediction. Cognitive models normally focus on a single cognitive phenomenon. For example, we could study how a person directs visual attention to certain images, or how a person decides which links to follow on a webpage. Cognitive modeling can also study how two phenomena interact. For example, we can combine the last two phenomena to study the effect of how we direct our visual attention on the choice of links we make in a webpage.

There are many mathematical approaches to cognitive modeling. They range from basic arithmetic operations to highly abstract representations based on category theory. The most popular mathematical tools applied to cognitive modeling are logic, probability theory, linear algebra, and network theory [RN95]. We will cover some of these approaches in Appendix A. Regardless of the mathematical approach, we can divide the modeling efforts developed within the cognitive modeling community into two main classes.

The first kind consists of 'ad-hoc' cognitive models. Here, the purpose is to model a particular phenomena in a specific domain of application. One example is the model of visual categorization of geometrical shapes based on ontologies presented in [MT08]. The authors introduce a list of features that play an important role in visual categorization and their relations, and an algorithmic procedure, based on Bayesian statistics, to categorize them. Examples of such categorization elements in the ontology are *sphere-like*, *rounded*, *uniform texture*, etc., and an example of a relation is (*rounded*,*uniform texture*)→(*sphere-like*). The algorithms in this model assume three incremental stages: i) knowledge acquisition, ii) learning, and iii) categorization. The model is useful for the task for which it was developed, especially in the case of smooth shapes. However, its design is not meant to represent anything else other than visual categorization, nor is it compatible with other models.

The second kind includes the so-called concept theories, which are general representation frameworks for cognitive phenomena. Here, concepts are envisaged as the units that underlie cognitive phenomena. Since understanding the nature of these units leads to a first-principles basis for a theory of cognition [RMG+76, SBZ01, Gär00], the aim of the theories of concepts is to reveal the formal structure of concepts.

In this work, we focus on the second kind of approach. Namely, we are interested in the structural aspects that the notion of concept needs in order to be properly applied to produce cognitive models. We aim at a characterization that, on the one hand, identify the fundamental structural aspects of concepts, and on the other, can be represented within a mathematical theory.

2.1.3 Theories of Concepts

Traditional models of concepts concentrate on categories possessing concrete or imaginary instances, such as 'horse,' or 'dragon' [Bea64, Ros73, Mac09a]. Modern approaches, however, extend to include abstract instances such as topics of discussion [SG07, BL06], music genres [AP03], and images [BHAT05]. In cognitive science, there are three main proposals for a theory of concepts that are mathematically sound: the classical theory [Med89], the prototype theory [RMG+76], and the exemplar theory [Nos86].

The classical theory follows the tradition of classical logic, and assumes that concepts are determined by a fixed set of attributes. Hence, any instance that holds these attributes is a member of the concept. Classical logic or some of its extensions are applied for inferential tasks, and for concept combinations. This theory of concepts thus assigns membership truth values: an instance is or is not associated with a particular concept (Appendix A).

The prototype theory proposes that concepts are not defined by a fixed set of attributes, but instead by one or multiple prototypes that incorporate the most relevant properties. Each exemplar has a degree of membership and, if the membership is positive, a degree of typicality. The prototype has the maximum degree of typicality. Prototype theory, formulated in the language of fuzzy sets (Appendix A.2), is more general than the classical theory of concepts in that it introduces a graded structure for the membership in terms of such things as typicality, similarity, and representativeness [GA02].

The exemplar theory assumes that a concept is defined by a list of stored entities that represent the current understanding of a certain agent concerning the concept in a given context. One can assess similarity estimations among the instances, and apply logical techniques to infer the similarity to new instances, as well as to combine concepts. Thus, the notion of prototype is recovered in this theory, for one can refer to some instances as more typical. However, the mathematical framework of this theory requires a number of parameters that grows with the number of exemplars, and these parameters do not have a clear interpretation [Nos86].

Throughout this thesis, we will denote concepts with single quotations in italic style with the first letter capitalized on each word, and by capital caligraphic letters when denoted in abstract form. For example, let $\mathcal{A}$ denote the concept '*Animal*.' Conceptual instances, also called exemplars, will be denoted between quotations without italics, and by lowercase letters when denoted in abstract form. For example, we say $p =$'dog' is an exemplar of concept $\mathcal{A}$. Properties, also called attributes or features, apply to both concepts and instances. We will denote properties in italics without quotations: *has four legs* is a property of the exemplar 'dog' of the concept '*Animal*.'

2.2 Challenges for a Theory of Concepts

There is a particular set of phenomena in concept research that highlights the problems of current cognitive models. These problematic phenomena challenge not only the accuracy of traditional models, but also the philosophical principles these models are built upon. We take a closer look at these phenomena to better understand the traditional models, and to identify their weaknesses. We can identify three issues that are problematic in cognitive modeling: vagueness, contextuality, and non-compositionality. In this chapter, we introduce these three issues, and present cognitive phenomena that characterize problems associated with them.

2.2.1 Vagueness

Concepts we reason with in our daily life are not sharply defined, neither in their boundaries nor in their implications [Wit58, Zad65]. Cognitive psychologists, mostly during the seventies and eighties, investigated the imprecise use of concepts in reasoning. They carried out a large number of experiments to characterize how people understand the meaning of concepts we use in daily life, and concluded that the way people estimate the meaning of concepts cannot be modeled using binary systems ('yes'/'no'), but requires instead graded relations that reflect their structural vagueness [Ros73, RMG+76, SM81, SL97].

This is illustrated in studying how people estimate the membership of different exemplars with respect to a concept. For example, consider the concept '*Pet*,' and suppose we want to estimate the membership of the exemplars 'dog,' 'snake,' and 'robot.' Clearly, we can be more certain about the first instance being a member of '*Pet*' than about the second instance, and in turn, we can be more certain about the second instance being a member of '*Pet*' than about the third instance. This suggests that membership should be quantified and that a graded structure is required.

Several cognitive scientists believe that the membership of an exemplar with respect to a concept depends on how much the exemplar resembles the prototype for the concept. Here, the prototype represents the most typical exemplar of a concept. Hence, membership of an exemplar with respect to a concept is measured by the similarity between the prototype and the exemplar [RMG+76]. This idea is one of the milestones of the prototype theory of concepts. In particular, prototypes of concepts can be experimentally obtained by requesting participants to estimate the typicality of a list of exemplars with respect to a concept [RMG+76, Ros99]. Using similar methods, we can also measure experimentally the extent to which an exemplar resembles the prototype of a concept. Experiments confirm that the more similar an exemplar is to the prototype of a concept, the larger its degree of membership is [Ham07].

But the prototype-based approach to membership becomes unclear when a concept has more than one prototype because, after assessment of the similarity of an exemplar to one of the prototypes, there are now different ways to assign membership to the concept as a function of prototypes. Although several similarity measures have been proposed in the literature [Gol94], none of them gives a satisfactory answer to the relation between membership and similarity to prototypes [Tve77, TG82, Ham07]. Furthermore, since prototypes are highly specific, it is difficult to determine a priori which prototypes are required to characterize a concept. Consider for example the concept '*Pet*' with two prototypes 'cat' and 'dog.' Neither of these two prototypes is similar enough to the exemplar 'goldfish' to provide a membership assessment. Hence, 'goldfish' should also be considered a prototype. By applying a similar reasoning to other possible pets such as 'spider,' 'robot,' and 'rabbit,' it becomes clear that similarity-based approaches are inadequate to assess the membership of exemplars.

An alternative way to measure the degree of membership of an exemplar with respect to a concept is to consider the most representative properties of the concept. In fact, the prototypes of a concept can be recovered from the set of most representative properties [Nos87, Bal04]. In the literature, the notions of typicality and similarity have also been assessed using the idea of representative properties of a concept, or of its prototypes [Tve77]. However, the relation between the representative properties and the membership of exemplars to a concept is unclear for at least two reasons. First, since there can be many properties for a given concept, selecting the set of representative properties of a concept is subjective [GA09]. Second, the selection from these properties can mislead our membership estimations. For example, *able to fly* is generally a representative property of the concept '*Bird*,' but a 'penguin' is a '*Bird*' that is not *able to fly* [AG05a].

In conclusion, although researchers have put forward several alternative methods to assess degrees of membership, experimental evidence does not provide conclusive results concerning the relation between these methods and the membership of a concept. Therefore, the way in which typicality, similarity, and representativeness relate to the notion of membership is vague [Fod98]. Throughout the thesis, we will refer to membership, typicality, similarity, and representativeness using the generic term semantic estimation, and refer to the fact that the relation among these forms of semantic estimation is unclear as the "vagueness problem" of concepts.

2.2.2 Context Dependence

Because the meaning of concepts people use in daily life is generally contextual, we can achieve significant improvements by incorporating the notion of context in the study of concepts. People do not think about concepts in isolation, but rather, in an environment that involves both internal and external circumstances. From now on, we refer to this problem as the "contextuality problem" of conceptual structures.

Paradoxically, the notion of context seems harder to define than the notion of concept itself. Depending on the area of application, different perceptions of what constitutes a context become the focus of the definition. While context is roughly understood as 'the circumstances in which something occurs' [Mei12], a total of more than 150 definitions have been proposed in different areas such as linguistics, cognitive science, psychology, and philosophy [BB05].

For concept theories, context entails all the priors at the moment of eliciting a concept. Cognitive psychologists have performed multiple experiments to observe how these priors affect the semantic estimations of concepts. These semantic estimations involve exemplar membership [Ros99, RMG$^+$76, Ham07], typicality [GA02, AG05a, VGEA11], property relevance [MS88, AG05a], and similarity [Nos87, Nos88] among others. For a detailed review of contextual effects on different semantic estimations see [PH14]. In all cases, they conclude that context radically affects the meaning of a concept.

In an experiment reported in [VGEA11], ninety-eight University of British Columbia undergraduates who were taking a first-year psychology course estimated the typicality of different exemplars of a concept given different contexts. The concept chosen was '*Hat*,' and the chosen exemplars p_1 ='cowboy hat,' p_2 ='baseball cap,' p_3 ='helmet,' p_4 ='top hat,' p_5 ='coonskincap,'

p_6 =‘toque,’ and p_7 =‘Medicine Hat[2].’ Properties of ‘*Hat*’ were used to create a context. We denote the context by the name of the property in italic letters and capitalize the first letter to differentiate it from a property. The contexts chosen for the experiment were e_1 = *Is a hat*, e_2 = *Is worn to be funny*, e_3 = *Is worn for protection*, e_4 = *Is worn in the South*, and e_5 = *Is not worn by a person*. Typicality estimations were made using a Likert-scale ranging from 0 to 7. The average and normalized typicality estimates for each context are shown in Table 2.1. The normalized typicality corresponds to the ratio between the average exemplar typicality and the sum of the exemplar typicalities for a given context.

The data shows that the typicality of exemplars for the concept ‘*Hat*’ is strongly affected by the contexts under consideration. Context e_1 was specifically introduced to minimize the contextual influence, so that the conventional typicality of the exemplars with respect to the concept can be determined. The other contexts were chosen to influence the meaning of the concept. Particularly, e_5 was chosen to induce a context that is counterin-

[2]Medicine Hat is a city in Canada.

Table 2.1: Data table of context-dependence typicality experiment for concepts. For each pair of numbers, the first number indicates the average typicality of the exemplar, and the second number indicates the normalized typicality. The total typicality of each context is shown in the last row. Contexts e_1 =*Is a hat*, and e_5 =*Is not worn by a person* are shaded according to their normalized typicality (the larger the number, the darker the shade).

Exp. Data	e_1	e_2	e_3	e_4	e_5
cowboy hat	(5.44;0.18)	(3.57;0.14)	(3.06;0.13)	(6.24;0.28)	(0.69;0.05)
baseball cap	(6.32;0.21)	(1.67;0.06)	(3.16;0.13)	(4.83;0.21)	(0.64;0.04)
helmet	(3.45;0.11)	(2.19;0.08)	(6.85;0.28)	(2.85;0.13)	(0.86;0.06)
top hat	(5.12;0.17)	(4.52;0.17)	(2.00;0.08)	(2.81;0.12)	(0.92;0.06)
coonskincap	(3.55;0.11)	(5.10;0.19)	(2.57;0.10)	(2.70;0.12)	(1.38;0.1)
toque	(4.96;0.16)	(2.31;0.09)	(4.11;0.17)	(1.52;0.07)	(0.77;0.05)
pylon	(0.56;0.02)	(5.46;0.21)	(1.36;0.05)	(0.68;0.03)	(3.95;0.29)
Medicine Hat	(0.86;0.02)	(1.14;0.04)	(0.67;0.03)	(0.56;0.02)	(4.25;0.31)
$N(e)$	30.30	25.98	23.80	22.22	13.51

tuitive to the meaning of '*Hat*.' Exemplars were chosen to cover the wide range of uses of the concept '*Hat*.' For example, the exemplars 'pylon' and 'Medicine Hat' are hardly members of '*Hat*.' This can be verified by the extremely small typicality they receive in the context e_1. It is interesting to note that 'pylon' and 'Medicine Hat' become the most typical exemplars under context e_5. Moreover, the correlation coefficient for the typicality estimations between contexts e_1 and e_5 is $p = -0.93$. This strong anti-correlation is evidence for the possibility that, when contexts of a concept have opposite meanings, then the corresponding typicality estimations of the concept are anticorrelated. This suggests that structural comparisons between the typicality estimations obtained for different contexts could be used to characterize semantic relations between contexts and between exemplars [VGEA11].

From a mathematical point of view, there are cognitive experiments showing that the way context influences concepts is incompatible with the assumptions of probability theory (Appendix A.3). In what follows, we present experiments that reveal the incompatibility of probabilistic approaches in three cognitive situations: direct probability estimation, order-effects in psychological surveys, and decision-making experiments involving successive bets.

Direct Probability Estimation: The Conjunction Fallacy

In the course of their extremely influential research program on decision making, Amos Tversky and the Nobel laureate Daniel Kahnemann [3] introduced for the first time the conjunction fallacy [TK83]. This phenomenon states that people generally estimate the occurrence of conjunctions of events to be more likely than the occurrence of the former events alone. Thus, it contradicts probabilistic rules about conjunction. For example, let E_1 and E_2 be two events, and the probability of their conjunction be given by

$$P(E_1 \, and \; E_2) = P(E_1 \cap E_2). \tag{2.1}$$

Then, the following inequality should hold:

$$P(E_1 \cap E_2) \leq \min(P(E_1), P(E_2)). \tag{2.2}$$

[3] As an historical note, Kahneman recognized in his Nobel prize acceptance speech that he should have shared the award with Tversky, who died six years before.

Note that in a standard logical setting such as classical and fuzzy logic, the membership for the conjunction of two categories is smaller or equal than the minimum of the memberships of the former categories (Appendix A.2). However, experimental data shows that people's estimations usually violate Eq. (2.1). The example used by Tversky and Kahneman in [TK83] is presented in Fig. 2.1.

Advocates of Boolean or fuzzy logical approaches to natural language have proposed multiple participants' misunderstandings to explain this effect. Namely, participants might misunderstand the meaning of the words 'and' [BHN93] and 'probable' [Gig96], or participants might tend to believe that a) implies that 'Linda is not a feminist' [TK83, BTO04]. Measures were taken in subsequent experiments to mitigate these and other possible misunderstandings. They included either the training of participants, or explicitly stating the logical consequences of the possible choices in explanatory text. Although in most cases the percentage of participants committing the fallacy is reduced, the fallacy remains significant (above 30%) in all reasonable experimental settings. For a detailed review of the experiments where the paradox has been tested, see [Mor09].

The conjunction fallacy has been confirmed in several studies that included hypothetical situations as well as real life situations like diagno-

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Which is more probable?
a) Linda is a bank teller
b) Linda is active in the feminist movement
c) Linda is a bank teller and is active in the feminist movement

Figure 2.1: Tversky and Kahneman experiment on conjunction probability estimation. The original experiment contained five other alternatives. We present only three for the sake of simplicity.

sis and prognosis in clinical settings [DH91, Rao09], forecasts of sports results [NA10], effects of government policies [BTO04], and political outcomes [LGS09]. Moreover, the fact that the same result is confirmed in different experimental settings, ranging from those choosing children [Agn91] to those using statistics experts [TK81] as participants, and considering different methodologies like choice [TBO04], ranking [SOSS03], and frequency [TC12] among others [TML96, BTO04, WM08], provides powerful empirical evidence for the conjunction fallacy.

Survey Answering: Order Effects

Researchers in psychology know that the order in which questions are presented influences the statistics of the responses. This is because earlier questions can provide context for the questions that follow, and hence can produce non-commutative effects. For example, when people are asked, '*What is the most important problem facing the nation?*,' the answer participants give becomes the object of focus for their answer to a subsequent question: '*Do you approve or disapprove of the way the president is handling his job?*' Indeed, most people will tend to judge the president's performance primarily on the issue they selected in the first question [KK90].

But even though these non-commutative effects are understood, most decision-making models in psychology are based on classical probability, where the probability of joint events commute by definition.

In a classical probabilistic setting, answers 'yes' to two questions F and H are represented by sets $F_y,\ H_y \subseteq \Sigma$, where Σ is the space of events (Appendix A.3). The event corresponding to answer 'yes' to F and H is defined by

$$F_y \textit{ and } H_y = F_y \cap H_y, \tag{2.3}$$

which is commutative. In Bayesian probability, the likelihood that a subject answers 'yes' to the question H given that the answer to F is 'yes' is represented by the conditional probability $P(H_y|F_y)$. Analogously, $P(F_y|H_y)$ represents conditional probabilities for the reverse order. The two probabilities are related by Bayes rule:

$$P(F_y)P(H_y|F_y) = P(F_y \cap H_y) = P(H_y)P(F_y|H_y). \tag{2.4}$$

Experimental evidence confirms that Eq. (2.4) does not hold in general [BW07, WB13, TB11]. Bayesian models involving more elaborated forms of conditioning can account for order effects. However, these models involve ad-hoc assumptions that can be accomodated only *a posteriori*, and thus have no predictive capacity [WSSB14]. Similarly, Markov models that can account for order effects have been constructed, but they also require the introduction of ad-hoc elements to accommodate the different kinds of deviations reported in the literature [WB13]. For extensive reviews of the kinds of experiments and deviations measuring order effects, we refer to [SB74, SP96, HE92, TRR00].

Decision Making: Ellsberg and Machina Paradoxes

In economics, the predominant model of decision making is given by the Expected Utility Theory [VNM07]. A fundamental principle, the so-called Savage's Sure-Thing Principle, ensures that if the possible outcomes of a variable x do not change the utility of a decision situation S, then the variable x can be neglected in the decision analysis. In [Sav72], the principle is introduced with the story shown in Fig. 2.2:

Let the events D and R represent the two disjoint possible outcomes of the presidential election, and let B represent the businessman buys the property. We have

> *A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant. So, to clarify the matter to himself, he asks whether he would buy if he knew that the Democratic candidate were going to win, and decides that he would. Similarly, he considers whether he would buy if he knew that the Republican candidate were going to win, and again finds that he would. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say.*

Figure 2.2: An example of the Sure-Thing principle.

$$\begin{aligned} P(D) + P(R) &= 1, \\ P(D \cap R) &= 0. \end{aligned} \tag{2.5}$$

Moreover, the fact that in both possible outcomes of the presidential election the businessman prefers to buy the property implies that

$$\begin{aligned} P(B|D) &\geq 0.5, \\ P(B|R) &\geq 0.5. \end{aligned} \tag{2.6}$$

As $P(B|R) + P(B|D) = P(B) \leq 1$, we conclude that $P(B) = 1$. Therefore, because $P(B)$ is equal to one, buying the property is deterministic with respect to the presidential election.

In a well-known study [Ell61], Daniel Ellsberg demonstrated that the Savage's Sure-Thing principle is inconsistent with the reality of human decision-making. The experiment performed by Ellsberg describes a situation such as the one in Fig. 2.3:

Participants in the experiment are confronted by the following 4 options: (I) bet on red, (II) bet on black, (III) bet on red or yellow, (IV) bet on black or yellow. Subjects must decide between options (I) and (II), and then decide between options (III) and (IV).

The experimental results presented in [Ell61] show that a high proportion of participants prefer (I) over (II), and (IV) over (III). But, this violates the Sure-Thing Principle, which requires that (I) preferred over (II) would mean (III) is preferred over (IV). A possible explanation for this violation could be that people make a mistake in their choice, and that the paradox is caused by an error of reasoning, or by aversion to ambiguity [FT95]. A number of

> *Consider an urn with 30 red balls and 60 balls that are either black or yellow in an unknown proportion. A bet regarding the color of a ball drawn from the urn is proposed under the following rules: When bet on c, a prize is given if the ball drawn is c, otherwise no prize is given. c can be a color or a disjunction of two colors.*

Figure 2.3: An Ellsberg paradox situation.

models have tried to account for these possible errors in judgement. Most notable among them, Choquet expected utility [Gil87], max-min expected utility [GS89], variational preferences [MMR06], and second-order probabilities [KMM05]. All of these models are generalizations of the expected utility model based on either Bayesian inference schema, or on a framework that generalizes some specific aspect of a σ-algebra classical probabilistic setting [AST12]. Recently, a new decision situation, the so-called Machina paradox [Mac09b], was shown to be incompatible with all the above models [BLP11, Mac14].

The Machina paradox considers an urn with four kinds of balls, each allocated a number between 1 and 4. The number of balls with the number 1 together with the number of balls with the number 2 is fifty, and the number of balls with the number 3 or 4 is fifty-one. The event E_j indicates that a ball with a number j has been drawn from the urn. In a first stage of the experiment, participants are explained that the choices $f_i,\ i = 1, ..., 4$, have payoffs defined by Table 2.2. Next, participants are asked to decide between betting on f_1 or f_2.

If a participant is sufficiently ambiguity averse, he will prefer f_1 over f_2, because although f_2 presents a slight Bayesian advantage, f_1 has no ambiguity in its payoffs. The person is then asked to bet on f_3 or f_4. In this case, both choices present ambiguity in their payoffs. Thus, a decision maker who values unambiguous information would be indifferent between f_3 and f_4. On the other hand, f_4 benefits from the 51 balls, hence in this case the Bayesian advantage implies that f_4 should be preferred over f_3 because of the different payoffs for cases E_2 and E_3. However, most participants preferring f_1 over f_2, later prefer f_4 over f_3.

Table 2.2: Payoff table of Machina paradox. In the above, E_1, f_1 pays 202, in E_2, f_1 pays 201, and so on.

Act	E_1	E_2	E_3	E_4
f_1	202	202	101	101
f_2	202	101	202	101
f_3	303	202	101	0
f_4	303	101	202	0

The paradox appears because none of the existing models can represent the participants' contextual behaviour [AST12]. Namely, in a first stage, participants are in a context where there is enough information to discern which act has a more ambiguous payoff, and choose f_1 over f_2, exposing their aversion towards ambiguity. In the second stage, however, participants are in a context where there is not enough information to discern which act has a more ambiguous payoff, and choose f_4 over f_3, recognizing the Bayesian advantage.

2.2.3 Non-Compositionality

The vagueness and contextuality problems, mentioned in § 2.2.1 and § 2.2.2 respectively, occur for individual concepts. In a general setting, a cognitive situation might include multiple concepts forming aggregated structures [Rip95]. For example, the concepts '*Fruit*' and '*Vegetable*' can be combined to form a new concept '*Fruit or Vegetable*' [Ham88a]. This concept combination is built with the connective '*or*,' which is also an operation mathematically defined in logic and probability. The question becomes, is it possible to apply the mathematical definition of the connective '*or*' to build the structure of '*Fruit or Vegetable*' from the structures of '*Fruit*' and '*Vegetable*'?

Traditional approaches to the study of cognitive phenomena assume that this question has a positive answer. This assumption, known as the principle of compositionality [Pel94], was first introduced to formalize logical inference, and later applied to linguistics [Gra90] and concept theory [RMG$^+$76]. But modern cognitive psychologists still don't agree on whether or not concepts are compositional [FP88, Fod98]. They have performed several experiments measuring various semantic estimations for concept combinations built with connectives used in logic such as '*Pet and Fish*,' and '*Not Sport*' [Ham88b, Ham88a, Ham97a], and adjective-noun compounds such as '*Red Apple*' [MS88, KP95, Med89] among others [Ham97b]. The evidence collected during two decades of research reveals that concept combinations are not compositional in general, at least in the sense suggested by fuzzy logic and probability theory. From now on, we refer to this problem as the "non-compositionality" problem of conceptual structures.

One of the most illustrative phenomena, called borderline contradiction, considers the gradedness structure of a concept in conjunction with its negation. Namely, a borderline contradiction case is a logical sentence of the type $p(x)$ *and Not* $p(x)$ that is estimated to be 'true,' for a certain predicate p and a borderline exemplar x. For example, in [AP11], participants estimate the truth value of the predicate $p(x) =$'*x Is Tall*' for an instantiation $x =$'John,' whose height assumes the values in Table 2.3.

To fit the evidence found in borderline contradiction, pioneering investigations [BOVW99] assumed the ignorance of participants, and proposed weakening of logical rules for truth estimations. Other models assumed truth gaps based on pragmatic logic [AP11], slight relaxations of probability theory [Rip11], paraconsistent logic [Rip13], and models inspired by fuzzy logic [Sau11]. None of these approaches has provided faithful modeling of empirical data with a coherent explanation of how to model concept combinations [BPB13, Soz14].

And for combinations involving any two concepts combined by conjunction or disjunction, the gradedness structure exhibits features that are even less obvious than what has been found in borderline contradiction research. Aerts, in [Aer09], formally states the conditions that characterize the existence of a classical probability model for concept conjunction and disjunction:

Definition 2.1. Let $\mu_x(A), \mu_x(B)$, and $\mu_x(A \textit{ and } B)$ be the membership weights of an item x with respect to a pair of concepts $\mathcal{A}$ and $\mathcal{B}$ and their conjunction $\mathcal{A}$ *and* $\mathcal{B}$. We say that these membership weights are classical conjunction data if there exists a Kolmogorovian probability space $(\Omega, \sigma(\Omega), P)$, and events $E_A, E_B \in \sigma(\Omega)$ such that

$$\begin{aligned} &P(E_A) = \mu_x(A), \\ &P(E_B) = \mu_x(B), \\ &P(E_A \cap E_B) = \mu_x(A \textit{ and } B). \end{aligned} \tag{2.7}$$

Table 2.3: Experimental data in borderline contradiction for $x =$'John.'

height	$5'4''$	$5'7''$	$5'11''$	$6'2''$	$6'6''$
% $p(x) =$'true'	14.5	21.1	44.6	28.9	5.3

Theorem 2.2. *The membership weights $\mu_x(A), \mu_x(B)$, and $\mu_x(A \text{ and } B)$ of an item x with respect to concepts $\mathcal{A}$, $\mathcal{B}$ and their conjunction $\mathcal{A}$ and $\mathcal{B}$ are classical conjunction data if and only if*

$$
\begin{aligned}
0 \leq \mu_x(A \text{ and } B) \leq \mu_x(A) \leq 1, \\
0 \leq \mu_x(A \text{ and } B) \leq \mu_x(B) \leq 1, \\
\mu_x(A) + \mu_x(B) - \mu_x(A \text{ and } B) \leq 1.
\end{aligned}
\tag{2.8}
$$

Definition 2.3. Let $\mu_x(A), \mu_x(B)$, and $\mu_x(A \text{ or } B)$ be the membership weights of an item x with respect to a pair of concepts $\mathcal{A}$ and $\mathcal{B}$ and their disjunction $\mathcal{A}$ *or* $\mathcal{B}$. We say that these membership weights are classical disjunction data if there exists a Kolmogorovian probability space $(\Omega, \sigma(\Omega), P)$, and events $E_A, E_B \in \sigma(\Omega)$ such that

$$
\begin{aligned}
&P(E_A) = \mu_x(A), \\
&P(E_B) = \mu_x(B), \\
&P(E_A \cup E_B) = \mu_x(A \text{ or } B).
\end{aligned}
\tag{2.9}
$$

Theorem 2.4. *The membership weights $\mu_x(A), \mu_x(B)$, and $\mu_x(A \text{ or } B)$ of an item x with respect to concepts A, B and their conjunction A or B are classical disjunction data if and only if*

$$
\begin{aligned}
0 \leq &\mu_x(A) \leq \mu_x(A \text{ or } B) \leq 1, \\
0 \leq &\mu_x(B) \leq \mu_x(A \text{ or } B) \leq 1, \\
0 \leq &\mu_x(A) + \mu_x(B) - \mu_x(A \text{ or } B).
\end{aligned}
\tag{2.10}
$$

A large body of experimental evidence indicates that the membership weights of exemplars with respect to concept combinations do not form classical conjunction or classical disjunction data. In particular, for the case of conjunction, the membership weight with respect to the conjunction of concepts is generally larger than the membership weight of at least one of the former concepts. This phenomenon is called "single overextension." When it is larger than both of the former membership weights it is called "double overextension."

In Table 2.4, we show two cases reported in [Ham88b]. Here the membership weights $\mu_{x_1}(A_1)$, $\mu_{x_1}(B_1)$, and $\mu_{x_1}(A_1 B_1)$ of the item x_1 ='coffee table' with respect to concepts $\mathcal{A}_1$ ='*Furniture*,' $\mathcal{B}_1$ ='*Household Appliances*,' and their conjunction $\mathcal{A}_1\mathcal{B}_1$ show single overextension, and the membership weights $\mu_{x_2}(A_2)$, $\mu_{x_2}(B_2)$, and $\mu_{x_2}(A_2B_2)$ of the item $x_2 =$'tree house' with respect to concepts $\mathcal{A}_2 =$'*Building*,' $\mathcal{B}_2 =$'*Dwelling*,' and their conjunction $\mathcal{A}_2\mathcal{B}_2$ show double overextension.

The phenomenon of overextension has also been demonstrated not only for membership weights, but also in typicality estimations. A famous example, known as the guppy effect, states that the typicality of 'guppy' with respect to '*Pet and Fish*,' is larger than the typicalities of '*Pet*,' and of '*Fish*' [SO81, SDBVMR98, Ham96]. Estimations of the applicability of relevant properties of concepts and their conjunctions exhibit the same effect. For example, *talk* is not a relevant property for either '*Pet*' or '*Bird*,' but it is for '*Pet and Bird*' [Ham97a, Ham97c, FL96, AG05a, AG05b]. Overextensions have also been observed in experiments considering negated concepts. For example, 'chess' is overextended with respect to the concepts '*Game*' and '*not Sport*' and their conjunction [ASV14b].

For the disjunction of two concepts, the analogous underextension effect also occurs. That is, semantic estimations of an exemplar with respect to the disjunction of two concepts can be smaller than the semantic estimation of exemplar of the individual concepts [Ham88a, Ham97b, Ham07].

Evidence supports the idea that overextension of conjunction and underextension of disjunction are common traits of conceptual combinations rather than mere cognitive effects. A case study in [ABGV12] reports that all 16 exemplars studied were overextended (Fig. 2.4).

Furthermore, the average seems to be a better estimator for the typicality of the conjunction than the fuzzy minimum rule. In particular, the difference between the normalized data typicality estimations of the con-

Table 2.4: Experimental membership weights for exemplars $x_1 =$'coffee table,' and $x_2 =$'tree house.'

	$X = A_1$	$X = B_1$	$X = A_1B_1$	$X = A_2$	$X = B_2$	$X = A_2B_2$
$\mu_{x_1}(X)$	1	0.18	0.35	-	-	-
$\mu_{x_2}(X)$	-	-	-	0.5	0.9	0.95

junction and the fuzzy minimum formula is 0.026 on average, while it is 0.011 with the average of the former concepts' typicalities. Moreover, the correlation between the normalized data and the minimum is 0.795, while it is 0.899 for the average of the former concepts' typicalities.

From a structural point of view however, the average formula is still not a solid estimator because of the existence of double overextended exemplars. In Fig. 2.4, the 4th and 14th value on the x-axis are double overextended. These exemplars correspond to 'hifi' and 'desk lamp' respectively. Formally speaking, double overextended exemplars cannot be described in terms of t-norms that entail all possible convex combinations of the former concepts' typicalities (Appendix A.2).

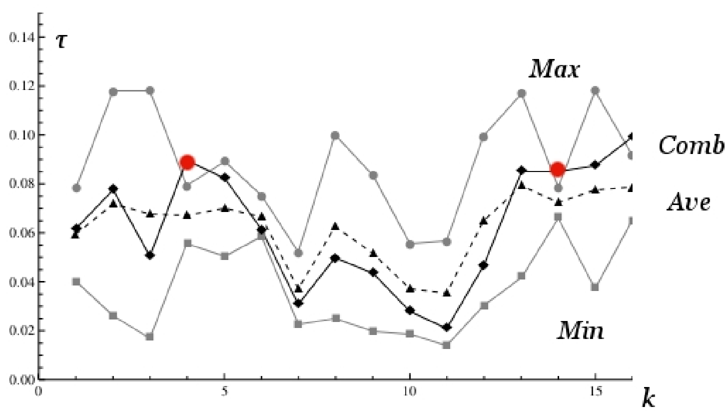

Figure 2.4: Normalized typicality estimations of the concepts '*Furniture*,' '*Household Appliance*,' and their conjunction with respect to 16 exemplars (on x-axis). The minimum and maximum of the former concepts in the combination are shown in grey lines, the typicality of the conjunction is the black line, and the average formula is the black-dashed line. Double overextended exemplars are marked by red points.

Chapter 3

The Quantum Approach to Cognitive Modeling

3.1 Quantum Physics and Quantum Structures

Quantum theory emerged at the beginning of the 20th century as a theory for microscopic phenomena that could not be explained by the current classical theories. These include the radiation profile of black bodies at different temperature levels and the measurement of electric currents in materials exposed to light. These two phenomena are known as the black-body problem and the photoelectric effect, respectively. Quantum theory was able to explain and incorporate these challenging phenomena into a unified representation of the microscopic realm that proposed an entirely different way of thinking.

This new perspective captured the attention of not only physicists, but also of philosophers and mathematicians. Whereas philosophers were concerned with the ontological nature of quantum entities, mathematicians focused on the development of suitable mathematical tools to describe quantum theory and its relation to other theories such as classical, relativity, and information theory. The area of research that lies in the intersection of the physical, philosophical, and mathematical aspects of quantum systems is named *quantum structures*.

From a philosophical standpoint, the differences between classical and quantum theories are the following: in the classical theory, the outcomes we observe when performing experiments exist as concrete states in the system prior to measurement, and are deterministically obtained from measurements; in the quantum theory, the outcomes we observe when performing experiments exist in potential states prior to measurement, and the measurement acts as a context that co-determines the observed outcome in a non-deterministic manner.

These differences are best illustrated by taking a closer look at the notions used to represent physical systems. While in classical physics systems are described by particles, and are not influenced by measurements, in quantum physics systems are described by waves modeled by complex valued functions, and measurements influence the system. Specifically, a quantum system is a superposition of waves of different wavelengths. Each wavelength represents a possible energy level for the system. Therefore, the state of superposition of a quantum system does not represent a real physical system, but embodies the potentiality of encountering different physical energy-level configurations. When a measurement is performed, a probabilistic change occurs to the superposition state. This change consists of the collapse of the superposition of waves into only one wave. Thus a measurement acts as a context that destroys the evolution of potentialities of the quantum system, and collapses the quantum system into a realistic physical configuration in which its properties can be observed.

An interesting phenomenon related to the potentiality of quantum systems is quantum interference. The states that make up the superposed state of a quantum system interact prior to measurement. This interaction changes the probabilistic structure of the wave and, unlike classical interference, quantum interference does not require the existence of an observable flow of particles. So, quantum interference is a phenomenon that occurs for one entity, prior to measurements, and is related to the potentiality of quantum systems.

Another important difference, which comes as a result of this shift in perspective, involves the mathematical representation of a system as a collection of sub-systems. A system formed by two sub-systems A and B with state spaces S_A and S_B respectively, is represented in classical physics by an element of the Cartesian product $S_A \times S_B$. In a Cartesian product, each subsystem is separately described within the joint system. In quantum theory, however, the state of the sub-systems are unit vectors in the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, and the joint system is a unit vector of the tensor product denoted by $\mathcal{H}_A \otimes \mathcal{H}_B$. In a tensor product, it is not always the case that subsystems are described separately. Specifically, a joint system in quantum theory can exist in a state that is not the composition of the states of the two separate systems, but rather, is an entangled state (see Definition 3.8). Thus joint systems in quantum theory are in general non-compositional.

A consequence of non-compositionality in quantum theory is the phenomenon of "quantum entanglement." Namely, when two or more particles interact, their superposed states may become entangled, and thus evolve as a single emergent entity even if they become spatially separated. In particular, measuring one of the two particles will result in a collapsed state for both regardless of the distance separating them. This was first conjectured in the celebrated Einstein Podolsky Rosen (EPR) paradox [EPR35], then mathematically formalized by the Bell-inequality formulation [B$^+$64], and finally tested experimentally by Alain Aspect's Bell-test experiment [ADR82].

In summary, quantum theory is a formal theory that is fundamentally different from its classical counterpart. The differences involve what is known as quantum structures. In particular, the central features of quantum structures are: i) they exist in states of potentiality, ii) they acquire concrete features through contextual processes, and iii) they evolve as emergent systems when combined.

3.2 Conditions of Possible Experience and Non-classical Statistics

Non-deterministic cognitive phenomena must be studied using a probabilistic model that describes a system by quantifying its tendency to behave in one way or another (Appendix A.3). The model is generally verified by observing the phenomena a large number of times through some experimental procedure. The behavioral tendencies of the system are reflected in the observed relative frequencies, also called statistics, of the experimental outcomes.

For example, consider an urn containing a large number of balls, and let us define the following measurements: E_1 ='the ball is red,' and E_2 ='the ball is wooden.' Note that other measurements such as

$$\begin{aligned} \text{not } E_1 &= \text{`the ball is not red,'} \\ \text{not } E_2 &= \text{`the ball is not wooden,'} \\ E_1 \cap E_2 &= \text{`the ball is red and wooden,'} \\ E_1 \cup \text{not } E_2 &= \text{`the ball is red or not wooden,'} \end{aligned} \tag{3.1}$$

can be defined using σ-algebraic constructions.

Draw a ball from the urn, and record the result of one or more measurements. A probabilistic model for this experiment should give a consistent description of the outcomes of these measurements. Consider the probabilities $P(E_1^{\text{yes}})$, $P(E_2^{\text{yes}})$ and, $P(E_1^{\text{yes}} \cap E_2^{\text{yes}})$ to obtain the outcome 'yes' for the measurements E_1, E_2, and $E_1 \cap E_2$. Then, the following consistency conditions must be satisfied:

$$P(E_1^{\text{yes}} \cap \ E_2^{\text{yes}}) \leq \min(P(E_1^{\text{yes}}), P(E_2^{\text{yes}})), \tag{3.2}$$
$$P(E_1^{\text{yes}}) + P(E_2^{\text{yes}}) - P(E_1^{\text{yes}} \cap \ E_2^{\text{yes}}) \leq 1. \tag{3.3}$$

Eq. (3.3) is equivalent to requiring that $P(E_1^{\text{yes}} \cup \ E_2^{\text{yes}})$ be well defined. The consistency conditions given in Eqs. (3.2)-(3.3) are some of the conditions of possible experience derived by George Boole to restrict the possible statistics of an experimental situation to plausible results [Boo54].

Suppose we extract 100 balls and obtain 60 balls are red, 75 balls are wooden, and 32 are both red and wooden. Then, the estimated probabilities are $P(E_1^{\text{yes}}) = 0.6$, $P(E_2^{\text{yes}}) = 0.75$, and $P(E_1^{\text{yes}} \cap \ E_1^{\text{yes}}) = 0.32$. Note that Eq. (3.3) is not satisfied since

$$P(E_1^{\text{yes}}) + P(E_2^{\text{yes}}) - P(E_1^{\text{yes}} \cap \ E_2^{\text{yes}}) = 1.03. \tag{3.4}$$

Clearly, this example cannot occur for any real urn since these proportions of the balls pose a logical contradiction [Pit94].

If all properties are measurable within a single sample, then the conditions of possible experience cannot be violated, and there exists a classical probabilistic representation [Pit89]. However, not all systems allow all properties to be measured in a single sample. For example, because most measurements in quantum systems will involve non-deterministic disturbances, they cannot allow for multiple measurements in a single sample. In these cases measurements are called incompatible. It may still be possible to build a probabilistic representation of the system from a subset of the complete σ-algebra of measurements using the notions of marginal and joint probability (Appendix A.3). But there are statistical situations where a detailed analysis reveals non-trivial violations even though the marginal probabilities appear to satisfy the conditions of possible experience.

The first example of this type was put forward by the mathematician Vorob'ev [Vor62]. He considered an abstract system with three experiments E_1, E_2, and E_3, each one having two outcomes E_i^j, for $i = 1, 2, 3$, and $j = 1, 2$. For the experiment to involve incompatible measurements, he assumed that only two out of the three experiments can be performed on each sample. We now show that the system violates the conditions of possible experience.

Consider the following marginal probabilities:

$$\mathbf{P}(E_1^1, E_2^1) = \mathbf{P}(E_1^2, E_2^2) = 1/2, \tag{3.5}$$
$$\mathbf{P}(E_1^1, E_3^1) = \mathbf{P}(E_1^2, E_3^2) = 1/2, \tag{3.6}$$
$$\mathbf{P}(E_2^1, E_3^2) = \mathbf{P}(E_2^2, E_3^1) = 1/2. \tag{3.7}$$

Note that Eq. (3.5) implies that

$$\mathbf{P}(E_1^2, E_2^2, E_3^1) + \mathbf{P}(E_1^2, E_2^2, E_3^2) = 1/2. \tag{3.8}$$

Analogously, we can use Eq. (3.6) and (3.7) to obtain

$$\mathbf{P}(E_1^1, E_2^1, E_3^1) + \mathbf{P}(E_1^1, E_2^2, E_3^1) = 1/2, \tag{3.9}$$
$$\mathbf{P}(E_1^2, E_2^1, E_3^2) + \mathbf{P}(E_1^1, E_2^1, E_3^2) = 1/2, \tag{3.10}$$
$$\mathbf{P}(E_1^1, E_2^2, E_3^1) + \mathbf{P}(E_1^2, E_2^2, E_3^1) = 1/2. \tag{3.11}$$

If we subtract Eq. (3.9) and Eq. (3.11), we have

$$\mathbf{P}(E_1^1, E_2^1, E_3^1) = \mathbf{P}(E_1^2, E_2^2, E_3^1). \tag{3.12}$$

Hence, replacing the right-hand side of Eq. (3.12) in Eq. (3.8) yields

$$\mathbf{P}(E_1^1, E_2^1, E_3^1) + \mathbf{P}(E_1^2, E_2^2, E_3^2) = 1/2. \tag{3.13}$$

Now, to have a well-defined probability for these events, we also require that

$$\mathbf{P}(E_1^1, E_2^1, E_3^1) + \mathbf{P}(E_1^1, E_2^1, E_3^2) + \mathbf{P}(E_1^1, E_2^2, E_3^1) + \mathbf{P}(E_1^1, E_2^2, E_3^2) + \mathbf{P}(E_1^2, E_2^1, E_3^1) + \mathbf{P}(E_1^2, E_2^1, E_3^2) + \mathbf{P}(E_1^2, E_2^2, E_3^1) + \mathbf{P}(E_1^2, E_2^2, E_3^2) = 1. \tag{3.14}$$

Substituting Eqs. (3.13), (3.10), and (3.11) in Eq. (3.14) gives

$$\mathbf{P}(E_1^2, E_2^1, E_3^1) + \mathbf{P}(E_1^1, E_2^2, E_3^2) = -1/2. \tag{3.15}$$

Since probabilities cannot be negative, this shows that the conditions of possible experience are violated. Therefore, this system cannot be represented by a classical probabilistic model.

3.3 The Birth of Quantum Cognition

Some scientists and philosophers, and remarkably, among them the founding fathers of quantum theory such as Bohr [Boh63] and Heisenberg [Sch92], have recognized that the joint measurement of properties is a relevant issue for cognitive phenomena. However, if properties measured in cognitive phenomena cannot be jointly measured, then non-classical probabilistic models, and particularly quantum-probabilistic modeling, might yield more accurate results [BK99, Bor10, Ama93, Khr10, Smi03]. In these cases, it is possible that cognitive phenomena could exhibit quantum-probabilistic features.

The first example of a cognitive phenomenon exhibiting non-classical probabilistic features was put forward by Aerts in [AA97]. The example consists of an opinion poll that contains three questions, each question having only two possible answers: 'yes' or 'no.' What brings the non-classicality to this situation is the fact that some participants do not have a predefined answer to the questions; but rather their answer must be formed at the moment the question is posed.

To draw an analogy between this cognitive situation and the urn example in the previous section, a participant 'forming his answer at the moment the question is posed' would correspond to 'a ball acquiring its colour when the ball is extracted from the urn.' Clearly, balls do not acquire their color when they are extracted from an urn, but our thought process can be influenced by a question. Similarly, quantum systems do acquire their properties when observed. This is exactly what the collapse of the wave function embodies: an enquiry itself influencing an outcome. And it is why the formalism of quantum theory provides a reasonable approach to model cognitive phenomena.

The questions for the opinion poll are given in Table 3.1.

Consider the case where for each question, 50% of the participants answer 'yes,' and only 30% of the total participants are certain of their answers before the question is posed, with 15% 'yes' and 15% 'no.' This means that 70% of the participants form their answer when the question is posed.

A probabilistic model, known as the ϵ-model, was constructed to assign probabilities for the various outcomes (Fig. 3.1). The ϵ-model has also been applied to study the relation between classical and quantum probabilities [Aer98, Aer96, AA97]. We can use this model to compute the probabilities for this experiment for question U as follows. Assume that the points on the perimeter of the circle represent all the possible states a participant can be in before the question is asked, and let the points u and $-u$ represent the

Table 3.1: Cognitive experiment revealing non-classical statistics.

U:	*Are you a proponent of the use of nuclear energy?*
V:	*Do you think it would be a good idea to legalize soft-drugs?*
W:	*Do you think it is better for people to live in a capitalistic system?*

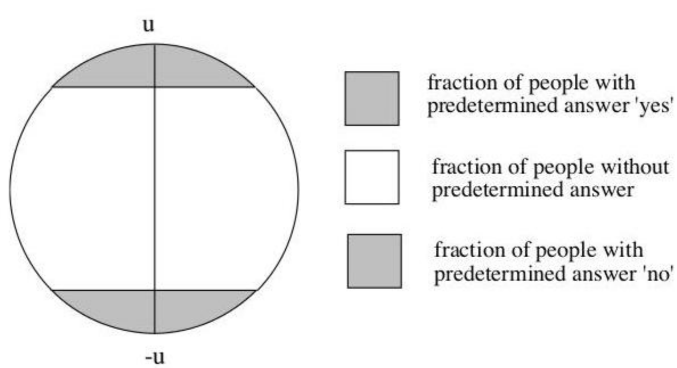

Figure 3.1: Graphical description of the proportion of participants with or without predetermined answers for question U [AA97].

states where answers ‘yes’ and ‘no’ are completely deterministic. Imagine an elastic band joining these two points, and assume that this elastic band can break at any point along the unshaded portion. When the question is asked, the points fall sideways into the line determined by the elastic and the elastic breaks. We say that the participant answered ’yes,’ if it is on the portion of the elastic that fell toward u, and ‘no’ if it is on the portion that fell toward $-u$. This process is shown in Fig. 3.2.

Note that, because the elastic can only break in the unshaded portion, all points in the shaded region are deterministic. They correspond to participants with predetermined answers. Furthermore, because the elastic can break at any point in the unshaded region, the answer for the participants without a predetermined answer is obtained in a probabilistic manner. Moreover, the closer the point representing a participant is to one of the certainty regions, the more likely the process will lead to the answer the region represents. The case in which 50% of the participants answer ‘yes’ to each question is modeled by assuming that the probability that the elastic breaks at each point is given by a symmetric distribution with respect to the midpoint. The calculation of the probability of a certain outcome corresponds to the expected value of getting the outcome for all the states and all the elastic breaking points. We refer to [AA97] for a detailed description of how to compute the probabilities in the $\epsilon-$model.

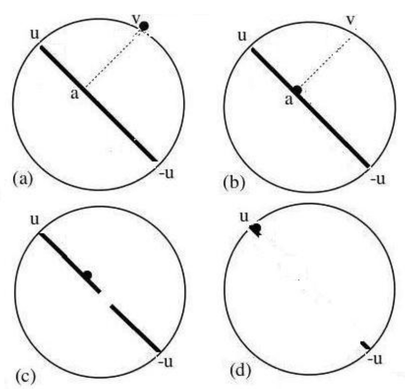

Figure 3.2: Measurement process in the $\epsilon-$model [AA97]. In a) the state of the participant prior to that question is on the circle, in b) the point falls into the elastic, in c) the elastic breaks, and in d) the point is attached to one of the extremes revealing the outcome [AA97].

Now consider the full experiment as described in Fig. 3.3. We can compute the relative proportions of participants with or without predetermined answers with respect to the three questions using the $\epsilon-$model graphical description. Region 1 in the figure corresponds to the participants whose answer is 'yes' for question U prior consultation, but who do not have a predetermined answer for questions V and W. Region 2 corresponds to participants with a predetermined answer 'yes' for questions U and V, but no predetermined answer for W, and so on.

The conditional probabilities of getting an answer, given that we know the answer to another question, can now be computed. The conditional probability of obtaining the answer $U =$'yes' given that $V =$'yes,' denoted by $P(U = \text{'yes'}|V = \text{'yes'})$, is obtained by estimating how likely it is that participants in the regions 2, 3, and 4 are attached to point u after the measurement process. Since they are deterministic, all points in region 2 will be attached to u. We refer to [AA97] for the trigonometric calculations required to compute the likelihood that the points in regions 3 and 4 lead to u after the measurement process. The conditional probabilities of all other outcomes can be calculated from the respective deterministic and non-deterministic regions as shown in Fig. 3.3. In particular, we have that

$$P(U = \text{'yes'}|V = \text{'yes'}) \quad = 0.78, \tag{3.16}$$

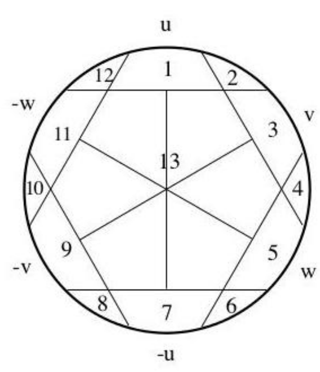

Figure 3.3: Graphical description of the proportion of participants with or without predetermined answers to the three questions U, V, and W [AA97].

$$P(U = \text{`yes'}|W = \text{`yes'}) \quad = 0.22, \tag{3.17}$$
$$P(V = \text{`yes'}|W = \text{`yes'}) \quad = 0.78. \tag{3.18}$$

The conditional probabilities given in Eqs. (3.16)-(3.18) cannot be represented by a classical statistical model [AA97]. To prove this, denote the outcomes U ='yes,' V ='yes,' W ='yes,' by U_+, V_+, and W_+ respectively. Analogously, denote U ='no,' V ='no,' W ='no,' by U_-, V_-, and W_- respectively. Since the probability of having a 'yes' outcome for each question is equal to 0.5, the following marginal probabilities are

$$P(U_+) = P(V_+) = P(W_+) = 0.5. \tag{3.19}$$

Bayes rule (Eq. (2.4)) gives

$$P(U_+|W_+) = \frac{P(U_+ \cap W_+)}{P(U_+)}. \tag{3.20}$$

We use this fact to obtain

$$P(U_+ \cap W_+) = P(U_+ \cap V_- \cap W_+) + P(U_+ \cap V_+ \cap W_+) = 0.11. \tag{3.21}$$

A similar procedure yields

$$P(U_- \cap V_+ \cap W_+) + P(U_+ \cap V_+ \cap W_+) = 0.39. \tag{3.22}$$

Subtracting Eqs. (3.22) and (3.21), we obtain

$$P(U_- \cap V_+ \cap W_+) = 0.28 + P(U_+ \cap V_- \cap W_+). \tag{3.23}$$

On the other hand, Eq. (3.16) implies

$$P(U_-|V_+) = 1 - P(U_+|V_+) = 0.22. \tag{3.24}$$

Using Bayes rule, and repeating the previous comparison, yields

$$P(U_- \cap V_+ \cap W_+) = 0.11 - P(U_- \cap V_+ \cap W_-). \tag{3.25}$$

Finally, we combine Eqs. (3.23) and (3.25) to obtain

$$P(U_+ \cap V_- \cap W_+) + P(U_- \cap V_+ \cap W_-) = -0.17. \tag{3.26}$$

Since all the probabilities must be positive, we have shown that this system cannot be represented by a classical probabilistic model. It can however, be represented by the non-classical ϵ-model.

The discovery of non-classical statistical results in this cognitive phenomenon inspired the development of quantum models for multiple cognitive phenomena including decision making [BB12], psychology of categorization [Aer09], human memory [BKNM09], and finances [HK13], among others [Khr10, PB13]. The use of quantum probability, and of quantum-inspired models for cognitive systems, is an emergent area of research known as Quantum Cognition [BBG13].

3.4 Fundamentals of Quantum Modeling in Cognition

This section introduces some mathematical elements of standard quantum theory and shows how they can be applied to cognition.

In quantum theory, the state of a quantum entity is described by a complex-valued vector of unit length. Vectors are denoted using the bra-ket notation introduced by Paul Dirac [Dir39]. In Dirac notation, there are two kinds of vectors: 'bra' vectors denoted by $\langle A|$, and 'ket' vectors denoted by $|A\rangle$. By convention, the state of a quantum entity is described by a 'ket' vector.

Definition 3.1. Let $\alpha, \beta \in \mathbb{C}$. Consider the vectors $\langle A|$ and $|B\rangle$. The operation bra-ket defined by the inner product $\langle A|B\rangle$ is

1. linear in the ket: $\langle A|(\alpha|B\rangle + \beta|C\rangle) = \alpha\langle A|B\rangle + \beta\langle A|C\rangle$, and
2. anti-linear in the bra: $\langle C|(\alpha|A\rangle + \beta|B\rangle) = \alpha^*\langle A|C\rangle + \beta^*\langle B|C\rangle$.

We say that $|A\rangle$ and $|B\rangle$ are orthogonal if and only if $\langle A|B\rangle = 0$. We denote it by $|A\rangle \perp |B\rangle$. Additionally, we say that $\langle A|B\rangle$ is the complex conjugate of $\langle B|A\rangle$. Therefore

$$\langle A|B\rangle = \langle B|A\rangle^*. \tag{3.27}$$

Definition 3.2. The bra-ket operation induces the norm $||\cdot|| = \sqrt{\langle\cdot|\cdot\rangle}$.

The space of complex-valued vectors representing the possible states of a quantum entity, equipped with the bra-ket operation and its induced norm, is called a Hilbert space, denoted by $\mathcal{H}$. The formalism of quantum theory is built upon the mathematics of Hilbert spaces [RS80].

Measurable quantities of a system, known as observables in quantum theory, are represented by self-adjoint operators on the Hilbert space. We focus on a special kind of self-adjoint operators, known as orthogonal projectors, used to represent quantum measurements representing questions whose possible outcomes are 'yes' and 'no.'

Definition 3.3. Let $|A\rangle$, and $|B\rangle \in \mathcal{H}$, and let $\mathbf{M} : \mathcal{H} \to \mathcal{H}$, be an operator defined by

$$|A\rangle \to \mathbf{M}|A\rangle.$$

$\mathbf{M}$ is an orthogonal projector if and only if it is

1. Linear: for $\alpha, \beta \in \mathbb{C}$ we have $\mathbf{M}(\alpha|A\rangle + \beta|B\rangle) = \alpha\mathbf{M}|A\rangle + \beta\mathbf{M}|B\rangle$,
2. self-adjoint: $\langle A|\mathbf{M}|B\rangle = \langle B|\mathbf{M}|A\rangle^*$, and
3. Idempotent: $\mathbf{M} \cdot \mathbf{M} = \mathbf{M}$.

Orthogonal projectors induce a subspace of $\mathcal{H}$ representing the states whose outcome is 'yes.' This space is given by

$$\mathcal{H}_{\mathbf{M}} = \{\mathbf{M}|A\rangle, |A\rangle \in \mathcal{H}\}. \tag{3.28}$$

The probability to obtain an outcome 'yes' for a measurement is given by the extent to which the state $|A\rangle$ belongs to $\mathcal{H}_{\mathbf{M}}$. This is formalized by the Born rule of probability [ST85].

Definition 3.4. Let $|A\rangle$ be the state of an entity $\mathcal{A}$, and $\mathbf{M}_x$ be an orthogonal projector associated to a question x. The probability of an answer 'yes' to the question x is given by

$$\mu_x(A) = \langle A|\mathbf{M}_x|A\rangle. \tag{3.29}$$

For the probabilistic structure to remain valid after a measurement, the state vector must be renormalized so it is still a unit vector.

Definition 3.5. Let $|A\rangle$ be the state of an entity $\mathcal{A}$, and $\mathbf{M}$ be an orthogonal projector. The state vector after measurement is given by

$$|A_{\mathbf{M}}\rangle = \frac{\mathbf{M}|A\rangle}{\sqrt{\langle A|\mathbf{M}|A\rangle}}. \tag{3.30}$$

Eq.(3.30), known as the projection postulate[4], ensures that when a physical quantity is measured two consecutive times, the result encountered in the first measurement is obtained with probability 1 in the second measurement.

The probability of an outcome 'no' is measured in quantum theory by the orthogonal operator $\mathbf{M}^{\perp}$ defined by

$$\mathbf{M}^{\perp} = \mathbb{1} - \mathbf{M}. \tag{3.31}$$

Hence, the probability to obtain the outcome 'no' is given by

$$\mu(\text{Not } A) = \langle A|\mathbf{M}^{\perp}|A\rangle = 1 - \langle A|\mathbf{M}|A\rangle. \tag{3.32}$$

Because $\mathbf{M}$ is idempotent and $\mathbf{M}^{\perp} = \mathbb{1} - \mathbf{M}$, we also have

$$\mathbf{M}\mathbf{M}^{\perp} = \mathbf{M}(\mathbb{1} - \mathbf{M}) = \mathbf{M} - \mathbf{M}^2 = 0. \tag{3.33}$$

When we consider two different measurements, the operator representing the successive application of these two measurements is not necessarilly a measurement, since the order in which they are measured might lead to different results. In terms of operators, this represents non-commutativity.

Definition 3.6. Given two operators $\mathbf{M}_1$ and $\mathbf{M}_2$. We say that $\mathbf{M}_1$ and $\mathbf{M}_2$ represent compatible measurements if and only if the commutator operator

$$[\mathbf{M}_1, \mathbf{M}_2] = \mathbf{M}_1\mathbf{M}_2 - \mathbf{M}_2\mathbf{M}_1 = 0. \tag{3.34}$$

Otherwise, the operators represent incompatible measurements.

So far, we have not considered the internal structure of states. In quantum theory, the set of states is linearily closed. This means that every linear combination of two states that corresponds to a unit vector is also a state. For example, if a quantum system can exist in two different states, $|A\rangle$ and $|B\rangle$, then it can also exist in the superposed state

$$|AB\rangle = z_1|A\rangle + z_2|B\rangle, \tag{3.35}$$

with $z_1, z_2 \in \mathbb{C}$, and $|z_1|^2 + |z_2|^2 = 1$. When a measurement $\mathbf{M}_x$ is applied to a superposed state, the probability $\mu_x(AB)$ of obtaining an outcome 'yes' is given by

[4]Also known in the literature as the collapse postulate.

$$\begin{aligned}\mu_x(AB) &= (z_1\langle A| + z_2\langle B|)\mathbf{M}_x(z_1|A\rangle + z_2|B\rangle),\\ &= |z_1|^2\langle A|\mathbf{M}_x|A\rangle + |z_2|^2\langle B|\mathbf{M}_x|B\rangle + 2\Re(z_1 z_2^*)\langle A|\mathbf{M}_x|B\rangle,\\ &= |z_1|^2\mu_x(A) + |z_2|^2\mu_x(B) + 2\Re(z_1 z_2^*)\langle A|\mathbf{M}_x|B\rangle,\end{aligned} \tag{3.36}$$

where $\Re(z)$ denotes the real part of z. So, the probability of an outcome 'yes' for the measurement represented by $\mathbf{M}_x$ is the weighted sum of the probabilities of the former events, 'yes' on state $|A\rangle$ and 'yes' on state $|B\rangle$, together with an interference term.

To understand why this term is called an "interference," we rewrite the last equation using the polar notation of complex numbers: $z_i = r_i e^{\mathbb{i}\theta_i}$, for $i = 1, 2$. Eq. (3.36) becomes

$$\begin{aligned}\mu_x(AB) &= (r_1 e^{\mathbb{i}\theta_1}\langle A| + r_2 e^{\mathbb{i}\theta_2}\langle B|)\mathbf{M}_x(r_1 e^{\mathbb{i}\theta_1}|A\rangle + r_2 e^{\mathbb{i}\theta_2}|B\rangle)\\ &= r_1^2 e^{\mathbb{i}(\theta_1-\theta_1)}\mu_x(A) + r_2^2 e^{\mathbb{i}(\theta_2-\theta_2)}\mu_x(B) + 2r_1 r_2\cos(\theta_1-\theta_2)\langle A|\mathbf{M}_x|B\rangle\\ &= r_1^2\mu_x(A) + r_2^2\mu_x(B) + 2r_1 r_2\cos(\theta_1-\theta_2)\langle A|\mathbf{M}_x|B\rangle.\end{aligned} \tag{3.37}$$

In this representation, the interference term corresponds to the product of the weights r_1 and r_2, the cosine of the phase difference $\theta_1 - \theta_2$ between the states, and the inner product of $|A\rangle$ and $|B\rangle$ restricted to $\mathcal{H}_{\mathbf{M}_x}$. If only one state is under consideration, the phase angle plays no role in the probability of a given measurement. However, when measurements are performed on superposed states, the interplay between the relative phases induce either positive or negative interferences. Extreme positive or negative interference is reached when $\theta_2 - \theta_1 = 0$, or $\theta_2 - \theta_1 = \pi$ respectively.

In quantum cognition, cognitive tasks are modeled by representing semantic estimations as probabilistic events [AGS13]. Thus, a quantum cognitive model considers a concept $\mathcal{A}$, whose state is represented by a unit vector $|A\rangle$, and semantic estimations are modeled by orthogonal projections on a Hilbert space. Let $\mathbf{M}_x$ be a semantic estimation: $\mathcal{H}_{\mathbf{M}_x}$ represents the space of states of the concept whose measurement outcome is 'yes.' Therefore, the probability of having an answer 'yes' to a certain semantic estimation is obtained by the Born rule of probability:

$$\mu_x(A) = \langle A|\mathbf{M}_x|A\rangle. \tag{3.38}$$

These outcome probabilities are generally called "weights" in the cognitive-science literature. For example, if $\mathbf{M}_x$ represents a membership estimation for a certain exemplar x, then $\mu_x(A)$ represents the membership weight of x with respect to the concept $\mathcal{A}$ being in the state $|A\rangle$ [Aer09].

Philosophers and cognitive scientists have on many occasions proposed that non-logical processes such as intuitive or unconscious thinking could be understood in terms of superposition, interference, or incompatibility, but have not given a formal account on how these notions operate [EF09, Tha97, Kih87, Smi03]. Quantum cognition is the first approach to incorporate these ideas into mathematical models. In particular, superposed states can be used to represent uncertainty [AS11a]; interference is a mechanism for non-logical cognitive coherence [Aer09]; and incompatible measurements correspond to two consecutive cognitive actions where the first action serves as a context for the second action [WB13, BW07].

3.5 Quantum Cognitive Models and Cognitive Challenges

A quantum model for a cognitive phenomena requires the specification of the entities at play, their state spaces, and the operators used to represent measurements. Once this is determined, the mathematical framework of quantum theory is used to compute the probabilities of the measurement outcomes. In this section, we show how a quantum cognitive framework is successful at modeling phenomena that cannot be explained within traditional approaches. In particular, we outline the quantum models developed to resolve the challenges presented in Chapter 2.

3.5.1 The Conjunction Fallacy as Incompatibility

To model the conjunction fallacy, we use a Hilbert space $\mathcal{H}$, and a state vector $|A\rangle \in \mathcal{H}$ that represents the belief state after reading Linda's story [Fra09, BPFT11]. Next, the event 'yes' to questions a), b), and c) is represented by the subspace corresponding to the projectors $\mathbf{M}_1$, $\mathbf{M}_2$, and the operator $\mathbf{M}_1\mathbf{M}_2$ respectively. The key assumption is that $\mathbf{M}_1$ does not commute with $\mathbf{M}_2$.

First, we expand the term representing the probability of event b) so we can compare it with the probability of event c):

$$\begin{aligned}\langle A|\mathbf{M}_2|A\rangle =&\langle A(\mathbf{M}_1+\mathbf{M}_1^{\perp})|\mathbf{M}_2|(\mathbf{M}_1+\mathbf{M}_1^{\perp})A\rangle \\ =&\langle A\mathbf{M}_1|\mathbf{M}_2|\mathbf{M}_1A\rangle + \langle A\mathbf{M}_1^{\perp}|\mathbf{M}_2|\mathbf{M}_1A\rangle + \\ &\langle A\mathbf{M}_1|\mathbf{M}_2|\mathbf{M}_1^{\perp}A\rangle + \langle A\mathbf{M}_1^{\perp}|\mathbf{M}_2|\mathbf{M}_1^{\perp}A\rangle.\end{aligned} \tag{3.39}$$

If $\mathbf{M}_1$ and $\mathbf{M}_2$ commute, and since $\mathbf{M}_1\mathbf{M}_1^{\perp} = 0$, we have

$$\langle A\mathbf{M}_1^{\perp}|\mathbf{M}_2|\mathbf{M}_1A\rangle = \langle A\mathbf{M}_1|\mathbf{M}_2|\mathbf{M}_1^{\perp}A\rangle = 0. \tag{3.40}$$

Therefore,

$$\langle A|\mathbf{M}_2|A\rangle = \langle A\mathbf{M}_1|\mathbf{M}_2|\mathbf{M}_1A\rangle + \langle A\mathbf{M}_1^{\perp}|\mathbf{M}_2|\mathbf{M}_1^{\perp}A\rangle. \tag{3.41}$$

Otherwise, set

$$\delta = \langle A\mathbf{M}_1|\mathbf{M}_2|\mathbf{M}_1^{\perp}A\rangle + \langle A\mathbf{M}_1^{\perp}|\mathbf{M}_2|\mathbf{M}_1A\rangle. \tag{3.42}$$

The incompatibility term[5] δ accounts for the conjunction fallacy as follows: Note that the story does not suggest that Linda is a bank teller; in fact, it is somewhat likely that she is not a bank teller [BPFT11]. Hence, we can assume

$$\langle A\mathbf{M}_1^{\perp}|\mathbf{M}_2|\mathbf{M}_1^{\perp}A\rangle \geq 0. \tag{3.43}$$

Choose δ to be negative, and such that

$$\delta + \langle A\mathbf{M}_1^{\perp}|\mathbf{M}_2|\mathbf{M}_1^{\perp}A\rangle < 0. \tag{3.44}$$

Then, Eq. (3.39) becomes

$$\langle A|\mathbf{M}_2|A\rangle = \langle A\mathbf{M}_1|\mathbf{M}_2|\mathbf{M}_1A\rangle + \delta + \langle A\mathbf{M}_1^{\perp}|\mathbf{M}_2|\mathbf{M}_1^{\perp}A\rangle < \langle A\mathbf{M}_1|\mathbf{M}_2|\mathbf{M}_1A\rangle. \tag{3.45}$$

This shows that by incorporating a non-commutative term, it is possible to model a situation where the probability for the conjunction of two events is greater than one of the events. A concrete example for the state $|A\rangle$, and the operators $\mathbf{M}_1$ and $\mathbf{M}_2$, has been constructed on a complex Hilbert space of dimension 3 to model empirical data collected on the conjunction

[5] In [Fra09], this term is called interference but since it is related to non-commutativity of measurements rather than superposition of states, we refer to it here as an incompatibility term.

fallacy [Fra09].

The same incompatibility term can account for order effects [WB13]. In particular, the quantum model can be used to derive a property about order effects that has no counterpart in the classical theories.

Let $\mathbf{M}_F$ and $\mathbf{M}_H$ be projectors representing two questions F and H whose outcomes 'yes' and 'no' are represented by F_y, F_n, H_y, and H_n respectively. Let μ be a function that measures the probability for the outcomes of these questions. For example, $\mu(F_n)$ is the probability of obtaining the outcome 'no' to question F, and $\mu(F_y H_n)$ is the probability of obtaining 'yes' to question F, and then 'no' to question H.

For each question, the order effect reflects how having the other question as a prior influences the statistics of the outcomes. For example, the order effects for F and H with respect to the answer 'yes' are respectively measured by

$$\begin{aligned} I_F &= \mu(H_y F_y) + \mu(H_n F_y) - \mu(F_y), \\ I_H &= \mu(F_y H_y) + \mu(F_n H_y) - \mu(H_y). \end{aligned} \tag{3.46}$$

To obtain the probability of consecutive measurements in the quantum model, we compute the probability of the outcome 'yes' for the first question, renormalize using Eq. (3.30) to obtain the post-measurement state, and then reapply the Born rule to the post-measurement state to compute the probability of obtaining the outcome 'yes' for the second question. For example, $\mu(H_y F_y)$ is given by

$$\begin{aligned} \mu(H_y F_y) &= \langle A|\mathbf{M}_F|A\rangle \langle A_{\mathbf{M}_F}|\mathbf{M}_H|A_{\mathbf{M}_F}\rangle \\ &= \langle A|\mathbf{M}_F|A\rangle \left(\frac{1}{\langle A|\mathbf{M}_F|A\rangle} \langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F A\rangle \right) \\ &= \langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F A\rangle. \end{aligned} \tag{3.47}$$

Similarly,

$$\begin{aligned} \mu(H_y F_n) &= \langle A\mathbf{M}_F^{\perp}|\mathbf{M}_H|\mathbf{M}_F^{\perp} A\rangle, \\ \mu(F_y H_y) &= \langle A\mathbf{M}_H|\mathbf{M}_F|\mathbf{M}_H A\rangle, \\ \mu(F_y H_n) &= \langle A\mathbf{M}_H^{\perp}|\mathbf{M}_F|\mathbf{M}_H^{\perp} A\rangle. \end{aligned} \tag{3.48}$$

Substituting these in Eq. (3.46) gives the order effect in term of measurement operators:

$$\begin{aligned} I_1 &= \langle A\mathbf{M}_H|\mathbf{M}_F|\mathbf{M}_H A\rangle + \langle A\mathbf{M}_H^{\perp}|\mathbf{M}_F|\mathbf{M}_H^{\perp} A\rangle - \langle A|\mathbf{M}_F|A\rangle, \quad (3.49)\\ I_2 &= \langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F A\rangle + \langle A\mathbf{M}_F^{\perp}|\mathbf{M}_H|\mathbf{M}_F^{\perp} A\rangle - \langle A|\mathbf{M}_H|A\rangle. \quad (3.50)\end{aligned}$$

Note that the probability of the outcome 'yes' to a question is experimentally obtained from the statistics of the experiment when the question is first asked. Hence

$$\begin{aligned} \langle A|\mathbf{M}_F|A\rangle &= \mu(H_yF_y) + \mu(H_nF_y),\\ \langle A|\mathbf{M}_H|A\rangle &= \mu(F_yH_y) + \mu(F_nH_y). \end{aligned} \quad (3.51)$$

Since $\mathbf{M}_i^{\perp} = \mathbb{1} - \mathbf{M}_i$, for $i = 1, 2$, we can expand the second term in Eqs. (3.49) and (3.50), to obtain

$$\begin{aligned} \langle A\mathbf{M}_F^{\perp}|\mathbf{M}_H|\mathbf{M}_F A\rangle &= \langle A|\mathbf{M}_H\mathbf{M}_F|A\rangle - \langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F A\rangle, \quad (3.52)\\ \langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F^{\perp} A\rangle &= \langle A|\mathbf{M}_F\mathbf{M}_H|A\rangle - \langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F A\rangle. \quad (3.53)\end{aligned}$$

We add Eqs. (3.52) and (3.53), and use the facts that $\mathbf{M}_F\mathbf{M}_H = (\mathbf{M}_H^*\mathbf{M}_F^*)^*$, and $\mathbf{M}_F$ and $\mathbf{M}_H$ are self-adjoint, to write

$$\langle A\mathbf{M}_F^{\perp}|\mathbf{M}_H|\mathbf{M}_F A\rangle + \langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F^{\perp} A\rangle = 2\Re(\langle A|\mathbf{M}_H\mathbf{M}_F|A\rangle) - 2\langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F A\rangle. \quad (3.54)$$

A similar procedure yields

$$\langle A\mathbf{M}_H^{\perp}|\mathbf{M}_F|\mathbf{M}_H A\rangle + \langle A\mathbf{M}_H|\mathbf{M}_F|\mathbf{M}_H^{\perp} A\rangle = 2\Re(\langle A|\mathbf{M}_H\mathbf{M}_F|A\rangle) - 2\langle A\mathbf{M}_H|\mathbf{M}_F|\mathbf{M}_H A\rangle. \quad (3.55)$$

Since Eqs. (3.54) and Eq. (3.55) share the term $2\Re(\langle A|\mathbf{M}_H\mathbf{M}_F|A\rangle)$, their combination yields

$$\begin{aligned} &\langle A\mathbf{M}_F^{\perp}|\mathbf{M}_H|\mathbf{M}_F A\rangle + \langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F^{\perp} A\rangle + 2\langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F A\rangle\\ &= \langle A\mathbf{M}_H^{\perp}|\mathbf{M}_F|\mathbf{M}_H A\rangle + \langle A\mathbf{M}_H|\mathbf{M}_F|\mathbf{M}_H^{\perp} A\rangle + 2\langle A\mathbf{M}_H|\mathbf{M}_F|\mathbf{M}_H A\rangle. \end{aligned} \quad (3.56)$$

From the properties of the quantum model, Eq. (3.56) can be used to derive the following identity:

$$\mu(F_y H_n) + \mu(F_n H_y) = \mu(H_y F_n) + \mu(H_n F_y). \tag{3.57}$$

Eq. (3.57) is known as the Quantum Question equality (QQ-equality). This probabilistic identity cannot be obtained from a classical probability theory because it requires the manipulation of the incompatibility terms $\langle A\mathbf{M}_F|\mathbf{M}_H|\mathbf{M}_F^{\perp}A\rangle$ and $\langle A\mathbf{M}_H|\mathbf{M}_F|\mathbf{M}_H^{\perp}\rangle$. Notably, the QQ-equality has been confirmed in a statistical analysis containing more than seventy surveys, each survey having between three hundred and two thousands participants [WSSB14]. Therefore, the QQ-equality presents evidence that quantum models are necessary to accurately represent non-classical aspects of cognitive phenomena.

3.5.2 Overextension and Underextension as Interference

Collected data on concept conjunction and disjunction that is not classical (Defs. 2.1–2.3) can often be explained in terms of state superposition and interference. Consider the concepts $\mathcal{A}$, $\mathcal{B}$, and a concept combination $\mathcal{AB}$, which can be either the conjunction $\mathcal{A}$ *and* $\mathcal{B}$ or the disjunction $\mathcal{A}$ *or* $\mathcal{B}$. These are represented by states $|A\rangle, |B\rangle$, and $|AB\rangle$ respectively. Let x be an exemplar, and let $\mathbf{M}$ represent the semantic estimation that measures the membership of x with respect to the concepts $\mathcal{A}$, $\mathcal{B}$, and their combination $\mathcal{AB}$. Now, assume $|A\rangle \perp |B\rangle$, and choose the following state for the combined concept:

$$|AB\rangle = \frac{1}{\sqrt{2}}(|A\rangle + |B\rangle). \tag{3.58}$$

With this choice, the membership weight of exemplar x with respect to the conjunction or disjunction of concepts $\mathcal{A}$ and $\mathcal{B}$ is given by

$$\mu(AB) = \frac{1}{2}\langle A + B|\mathbf{M}|A + B\rangle = \frac{(\mu(A) + \mu(B))}{2} + \Re\langle A|\mathbf{M}|B\rangle. \tag{3.59}$$

The membership weight $\mu(AB)$ corresponds to the sum of the average of $\mu(A)$ and $\mu(B)$, and an interference term that depends on the way vectors $|A\rangle$ and $|B\rangle$ project onto $\mathcal{H}_{\mathbf{M}}$.

The quantum probability formula in Eq. (3.59) has been applied to model overextension and underextension of semantic estimations for concept conjunction and disjunction reported in [Ham88b, Ham96, Ham88a, Ham97b]. It is interesting to note that deviations from classical models found for both connectives can be explained in terms of the same model. Indeed, in the absence of interference, that is when $\langle A|\mathbf{M}|B\rangle = 0$, the probability formula is reduced to the average of the former probabilities. Therefore, if the membership weights are not equal, Eq. (3.59) is singly overextended and singly underextended even in the absence of interference [ABGV12]. However, different phase angles have to be chosen for each connective in order to give a precise account of the experimental data. In general, positive interference is needed to account for overextension, and negative interference is needed to account for underextension [Aer09].

3.5.3 Ellsberg and Machina Paradoxes

Both Ellsberg and Machina paradoxes can be modeled using similar methods [AS11b, AS11a, AST12]. The idea is to model the subject's uncertainty about the number of balls of each type in the urn by a superposition of possible urn states. For the Ellsberg paradox, let $|r\rangle, |y\rangle$, and $|b\rangle$ be three orthogonal vectors representing the existence of red, yellow, and black balls respectively, and let the projectors $\mathbf{M}_r, \mathbf{M}_y$, and $\mathbf{M}_b$ represent the event of extracting a red, yellow, or black ball from the urn. Consider a pure Ellsberg state reflecting certain belief about the numbers of balls in the urn, represented by

$$|p\rangle = \frac{1}{\sqrt{3}} e^{\mathbb{i}\alpha}|r\rangle + \rho_y e^{\mathbb{i}\beta}|y\rangle + \rho_b e^{\mathbb{i}\gamma}|r\rangle, \tag{3.60}$$

where $\rho_y^2 + \rho_b^2 = \frac{2}{3}$. Note that the probability that a red ball is extracted given this pure Ellsberg state is

$$\langle p|\mathbf{M}_r|p\rangle = \frac{1}{3}. \tag{3.61}$$

Similarly, the probability of extracting either a yellow or black ball is $\frac{2}{3}$.

To represent the uncertainty about the number of yellow and black balls, we introduce an ambiguous Ellsberg state modeled by the superposition of pure states as follows:

$$|s\rangle = \sum_{i=1}^{n} a_i e^{\mathbb{i}\theta_i}|p_i\rangle, \tag{3.62}$$

where $\sum_{i=1}^{n} a_i^2 = 1$. This superposition of state produces an interference term in the probability formula that accounts for the Ellsberg paradox [AST12]. In particular, the simplest superposition choice, involving the superposition of only yellow versus only black balls, is sufficient. Set

$$|s\rangle = a_1 e^{\mathbb{i}\theta_1}|p_1\rangle + a_2 e^{\mathbb{i}\theta_2}|p_2\rangle, \text{ where} \tag{3.63}$$
$$a_1^2 + a_2^2 = 1, \tag{3.64}$$

with $|p_1\rangle$ and $|p_2\rangle$ given by

$$\begin{aligned} |p_1\rangle &= \frac{1}{\sqrt{3}}(e^{\mathbb{i}\alpha_1}|r\rangle + \sqrt{2}e^{\mathbb{i}\beta_1}|y\rangle), \\ |p_2\rangle &= \frac{1}{\sqrt{3}}(e^{\mathbb{i}\alpha_2}|r\rangle + \sqrt{2}e^{\mathbb{i}\beta_2}|b\rangle), \end{aligned} \tag{3.65}$$

for

$$\cos(\alpha_1 - \alpha_2 + \theta_1 - \theta_2) = 0. \tag{3.66}$$

Eq. (3.64) is set to ensure $|s\rangle$ is a unit vector, and Eq. (3.66) ensures that the probability to extract a red ball is $\frac{1}{3}$. Indeed,

$$\begin{aligned} \langle s|\mathbf{M}_r|s\rangle &= a_1 e^{\mathbb{i}\theta_1}\langle p_1| + a_2 e^{\mathbb{i}\theta_2}\langle p_2|\mathbf{M}_r|a_1 e^{\mathbb{i}\theta_1}|p_1\rangle + a_2 e^{\mathbb{i}\theta_2}|p_2\rangle \\ &= \frac{1}{3}(a_1^2 + a_2^2) + \frac{2}{3}a_1 a_2 \cos(\alpha_1 - \alpha_2 + \theta_1 - \theta_2) \\ &= \frac{1}{3}(a_1^2 + a_2^2) = \frac{1}{3}. \end{aligned} \tag{3.67}$$

Different choices of a_1, a_2, and the phase angles θ_i, α_i, and β_i, for $i = 1, 2$, lead to different kinds of reasoning about the Ellsberg paradox. Therefore, interference effects induced by the superposed state explain the deviations from Savage's sure thing principle [AST12].

To model the Machina paradox we introduce orthogonal states $|i\rangle$, for $i = 1, ..., 4$, representing the existence of balls of each kind, and orthogonal projectors on each dimension $\mathbf{M}_i$, $i = 1, ..., 4$, representing the bet on a certain kind of ball. Next, we introduce a pure Machina state

$$|p\rangle = \sum_{i=1}^{4} \rho_i e^{\mathbb{i}\alpha_i}|i\rangle \tag{3.68}$$

to represent a distribution of balls, and impose the following consistency constraints:

$$\rho_1^2 + \rho_2^2 = 50, \text{ and } \rho_3^2 + \rho_4^2 = 51. \tag{3.69}$$

Finally, the Machina state

$$|s\rangle = \sum_{k=1}^{n} a_i e^{\mathbb{i}\theta_i} |p_k\rangle \tag{3.70}$$

represents the ambiguity in terms of state superposition. The interference produced in the probability formula from a simple superposition that considers only extreme distributions, accounts for the results of Machina paradox. Thus, the quantum model is the only approach that provides a unified view of the Ellsberg and Machina paradoxes [AST12].

3.6 Entanglement of Conceptual Combinations

We have shown in § 3.5 that the basic structures of the quantum framework can successfully be applied to cognition. The interesting question is the extent to which the tools developed for quantum theory can be applied to cognition. In particular, can we identify other characteristics of quantum theory in the field of cognition? In this section, we provide evidence for a positive answer. Namely, we show that the phenomenon of entanglement arises in the modeling of concept combinations.

3.6.1 Quantum Entanglement

Quantum formalism assumes that quantum systems may exist in superposed states. If a quantum system is formed by the composition of sub-systems, then each sub-system exists in its own superposed state, but the behavior for the emerging system may correspond to that of a non-decomposable entity. In particular, when we perform measurements on the sub-systems of a composite quantum system, the results reveal that the sub-systems may not behave independently, even if the sub-systems are separated by a large distance. In fact, when the system is analyzed, it is possible to encounter states that exhibit non-trivial correlations in the outcomes of their measurements. These states, called entangled states, have inspired some of the most important applications of quantum theory [HHHH09].

Consider for example a composite quantum systems $\mathcal{C}$ obtained by composing two separate quantum systems $\mathcal{C}_1$ and $\mathcal{C}_2$. Formally, the composition of quantum entities corresponds to an element in the tensor product space[6] $\mathcal{H} \otimes \mathcal{H}$.

Definition 3.7. Let $\{|A_i\rangle\}$ form a basis for $\mathcal{H}$. Then, a composite vector $|C\rangle \in \mathcal{H} \otimes \mathcal{H}$ is given by

$$|C\rangle = \sum_{i,j}^{n} c_{ij} |A_i\rangle \otimes |A_j\rangle. \tag{3.71}$$

Definition 3.8. Let $|C\rangle \in \mathcal{H} \otimes \mathcal{H}$. If $|C\rangle$ can be factorized as $|C\rangle = |C_1\rangle \otimes |C_2\rangle$, where $|C_1\rangle \in \mathcal{H}$, and $|C_2\rangle \in \mathcal{H}$, we say that $|C\rangle$ is a separable tensor, also known as product vector. Otherwise, $|C\rangle$ is a non-separable tensor, representing an entangled state.

Because separable tensors can be represented as ordered pairs, when a measurement is performed on one of the sub-systems, the collapse of the wave function induced by the measurement occurs only at the measured sub-system. The other sub-system remains in its original state. When a measurement is performed on a non-separable tensor, the collapse of the wave function induced by the measurement will affect both sub-systems.

An example of an entangled state is given by the famous Einstein-Podolsky-Rosen state [EPR35]. Let $\mathcal{A}$ be an entity whose possible states are $|A_1\rangle$ and $|A_2\rangle$, and let $\mathcal{B}$ be another entity whose possible states are $|B_1\rangle$ and $|B_2\rangle$, where $|A_1\rangle \perp |A_2\rangle$ and $|B_1\rangle \perp |B_2\rangle$. Consider the composite entity $\mathcal{C}$ represented by

$$|C\rangle = \frac{1}{\sqrt{2}} \left(|A_1\rangle \otimes |B_1\rangle + |A_2\rangle \otimes |B_2\rangle \right). \tag{3.72}$$

The state $|C\rangle$ is non-separable because it cannot be decomposed as the product of two state vectors $|C_1\rangle, |C_2\rangle \in \mathcal{H}$.

From a probabilistic perspective, the correlations obtained when measuring entangled states are incompatible with classical probabilistic models that assume independent measurements. We can test whether a statistical situation can be described by a classical probabilistic model using Bell-like inequalities [B$^+$64]. These inequalities are analogous to the conditions of possible experience presented in § 3.2, except that they are based on aggregate probabilistic indicators such as correlations and expected values rather

[6] We assume here a particular form of joint quantum system that is useful for our purposes.

than joint probabilities.

It has been proposed that the quantum description of joint entities is a suitable framework to describe concept combinations. In particular, entangled states can be used to model non-trivial semantic correlations between the concepts that form the combination in the combination itself [AS11c, DCGL$^+$10, HS09, VAZ11, WBAP13, SMR13]. We now present an experimental verification of these non-trivial semantic correlations in concept combinations [AGS13].

3.6.2 Psychological Evidence of Conceptual Entanglement

An abstract formulation to test quantum entanglement in concept combinations was presented in [AABG00]. Consider two entities $\mathcal{A}$ and $\mathcal{B}$ with two measurement; each measurement having two possible outcomes. We denote these measurements and their outcomes by $\mathbf{M}_A = \{A_1, A_2\}$ and $\mathbf{M}_{A'} = \{A'_1, A'_2\}$ for entity $\mathcal{A}$, and $\mathbf{M}_B = \{B_1, B_2\}$ and $\mathbf{M}_{B'} = \{B'_1, B'_2\}$ for entity $\mathcal{B}$. Next, we define the composed operator $XY \in \{\mathbf{M}_{AB}, \mathbf{M}_{A'B}, \mathbf{M}_{AB'}, \mathbf{M}_{A'B'}\}$ and associate the value 1 to the outcomes X_1Y_1 and X_2Y_2, and the value -1 to the outcomes X_1Y_2 and X_2Y_1.

If we perform the experiment XY a large number of times, we can estimate the expected value $E(\mathbf{M}_{XY})$ of each composed experiment. A Bell-like inequality, named the Clauser-Horn-Shimony-Holt (CHSH) inequality, can be used to test the statistics. The CHSH inequality states that if

$$-2 \leq E(\mathbf{M}_{A'B'}) + E(\mathbf{M}_{A'B}) + E(\mathbf{M}_{AB'}) - E(\mathbf{M}_{AB}) \leq 2 \tag{3.73}$$

is violated, then no classical probability model exists for the considered joint experiments [AF82]. Additionally, if the marginal law of probability is satisfied (see Appendix A.3), then the entities are entangled [DK14].

A cognitive experiment in [AS11c, AS14] confirmed that semantic dependencies of concept combinations can violate the CHSH inequality. For example, let the entities $\mathcal{A}$ and $\mathcal{B}$ refer to the concepts *Animal* and *Acts*, respectively. Let $\mathbf{M}_A$, and $\mathbf{M}_{A'}$ be two measurements for concept $\mathcal{A}$, and $\mathbf{M}_B$ and $\mathbf{M}_{B'}$ be two measurements for concept $\mathcal{B}$. The outcomes of these measurements are given by:

$$
\begin{aligned}
\mathbf{M}_A &= \{A_1 = \text{‘horse’},\ A_2 = \text{‘bear’}\}, \\
\mathbf{M}_{A'} &= \{A'_1 = \text{‘tiger’},\ A'_2 = \text{‘cat’}\}, \\
\mathbf{M}_B &= \{B_1 = \text{‘growls’},\ B_2 = \text{‘whinnies’}\}, \\
\mathbf{M}_{B'} &= \{B'_1 = \text{‘snorts’},\ B'_2 = \text{‘meows’}\}.
\end{aligned}
\tag{3.74}
$$

A psychological experiment where 81 participants were asked to choose the combination that best represents the concepts $\mathcal{A}$, $\mathcal{B}$, and the conceptual combination ‘*The Animal Acts*,’ with respect to outcomes of measurements $\mathbf{M}_{AB}$, $\mathbf{M}_{AB'}$, $\mathbf{M}_{A'B}$, and $\mathbf{M}_{A'B'}$ was performed, and the expected values of these joint measurements were calculated (see Table 3.2). From the data we have

$$
\begin{aligned}
E(\mathbf{M}_{AB}) &= P(A_1, B_1) + P(A_2, B_2) - P(A_1, B_2) - P(A_2, B_1) = -0.778, \\
E(\mathbf{M}_{AB'}) &= P(A_1, B'_1) + P(A_2, B'_2) - P(A_1, B'_2) - P(A_2, B'_1) = 0.3580, \\
E(\mathbf{M}_{A'B}) &= P(A'_1, B_1) + P(A'_2, B_2) - P(A'_1, B_2) - P(A'_2, B_1) = 0.6543, \\
E(\mathbf{M}_{A'B'}) &= P(A'_1, B'_1) + P(A'_2, B'_2) - P(A'_1, B'_2) - P(A'_2, B'_1) = 0.6296.
\end{aligned}
\tag{3.75}
$$

Since

$$
E(A'B') + E(A'B) + E(AB') - E(AB) = 2.4197, \tag{3.76}
$$

inequality (3.73) is violated, and so no classical probability model exists for the considered joint experiments on $\mathcal{A}$ and $\mathcal{B}$. Quantum models assuming

Table 3.2: Data table of conceptual entanglement experiment in [AS14].

‘*Animal*’ $\mathbf{M}_X,\ X = A, A'$	$A_1 =$‘horse’ $P(A_1)$=0.5309	$A_2 =$‘bear’ $P(A_2)$=0.4691	$A'_1 =$‘tiger’ $P(A'_1)$=0.7284	$A'_2 =$‘cat’ $P(A'_2)$=0.2716
‘*Acts*’ $\mathbf{M}_Y,\ Y = B, B'$	$B_1 =$‘growls’ $P(B_1)$=0.4815	B_2=‘whinnies’ $P(B_2)$=0.5815	$B'_1 =$‘snorts’ $P(B'_1)$=0.321	$B'_2 =$‘meows’ $P(B'_2)$=0.679
‘*Animal Acts*’ $\mathbf{M}_{AB}$	‘horse growls’ $P(A_1, B_1) = 0.049$	‘horse whinnies’ $P(A_1, B_2) = 0.630$	‘bear growls’ $P(A_2, B_1) = 0.259$	‘bear whinnies’ $P(A_2, B_2) = 0.062$
‘*Animal Acts*’ $\mathbf{M}_{A,B'}$	‘horse snorts’ $P(A_1, B'_1) = 0.593$	‘horse meows’ $P(A_1, B'_2) = 0.025$	‘bear snorts’ $P(A_2, B'_1) = 0.296$	‘bear meows’ $P(A_2, B'_1) = 0.086$
‘*Animal Acts*’ $\mathbf{M}_{A',B}$	‘tiger growls’ $P(A'_1, B_1) = 0.778$	‘tiger whinnies’ $P(A'_1, B_2) = 0.086$	‘cat growls’ $P(A'_2, B_1) = 0.086$	‘cat whinnies’ $P(A'_2, B_1) = 0.049$
‘*Animal Acts*’ $\mathbf{M}_{A',B'}$	‘tiger snorts’ $P(A'_1, B'_1) = 0.148$	‘tiger meows’ $P(A'_1, B'_2) = 0.086$	‘cat snorts’ $P(A'_2, B'_1) = 0.099$	‘cat meows’ $P(A'_2, B'_2) = 0.667$

that $\mathcal{A}$ and $\mathcal{B}$ are entangled concepts can represent this data [AS11c, AS14] even though the marginal law is not satisfied.

Other tests have been carried in psychological experiments confirming entanglement in conceptual combinations [BKR$^+$12, BKRS13, KRBS10, AG05b]. The conclusion is that concept combinations entail semantic correlations that might not be obtained in a classical probabilistic framework. Since quantum entanglement can handle these semantic correlations, the quantum description of joint entities becomes a suitable mathematical framework to represent concept combinations.

Chapter 4

Two Quantum Models for the Conjunction and Disjunction of Concepts

Most prominent authors in the study of concept combinations believe that the problem of non-compositionality of concepts (§2.2.3) is the consequence of non-trivial semantic interactions between the combined concepts [Ham88a]. This interaction can be explained in terms of salient concept's properties [SO81, SO82, Rip95], specialized [CM84, Mur88] or constrained schematas [CK00, CK01], composite prototypes [Ham88b, Ham07], or some combinations of these [WL98, Wis96, Gag00]. Therefore, although they differ in their mathematical approaches, most models look for a coherence mechanism that explains the meaning of concepts combinations [Tha97]. The experiments carried out by James Hampton [Ham88b, Ham88a] and others [SO81, SO82, SDBVMR98] show that such coherence mechanism cannot be accounted for using classical or fuzzy logic.

In §3.5.2, we used a simple quantum model to represents some of the cases of overextension and underextension found in experiments with conceptual combinations. This model includes an interference term that depends on the phase angles used to represent the concepts in combination. Phase angles can therefore be interpreted as a mathematical realization of the coherence mechanism sought by cognitive psychologists.

This chapter further explores quantum modeling of concept combinations. In particular, in §4.1 we analyze the Hilbert space model of concept combinations discussed in §3.5.2, and in §4.2 we introduce another type of modeling for concept combinations based on tensor products of Hilbert spaces[7]. The exploration presented here is theoretical; we focus on the mod-

[7]Tensor products were introduced in §3.6 to model the concept combination '*Animal Acts*' expressed by a noun-verb combination. The model presented in §4.2 is used for

eling power of the introduced frameworks.

4.1 Modeling on a Hilbert Space

The Hilbert space model introduced in §3.5.2 is useful to represent most cases of overextension of conjunctions and underextension of disjunctions found in experimental data. However, some cases of extreme overextension and underextension, as well as some combinations that correspond to classical probabilistic data (see Defs. 2.1 and 2.3) cannot be modeled by the Hilbert space model. In this section, we will determine what conditions are required to find a representation for concept combinations in the Hilbert space model. Rather than focus on a particular combination of concepts, we assume the existence of two concepts $\mathcal{A}$ and $\mathcal{B}$, and of a combined concept $\mathcal{AB}$ that can represent either conjunction or disjunction.

4.1.1 Scope and Dimensionality of a Hilbert Space Model

To explore the type of conceptual combinations that can be represented in the Hilbert space model, we focus on the dimension n of the Hilbert space $\mathbb{C}^n$ equipped with the standard inner product. First, recall that the Hilbert space model of concept combinations requires vectors $|A\rangle, |B\rangle \in \mathcal{H}$, and an orthogonal projector $\mathbf{M} : \mathcal{H} \to \mathcal{H}$, such that the following conditions are satisfied[8]:

$$\begin{aligned} \langle A|A\rangle &= \langle B|B\rangle = 1, && (4.1)\\ \langle A|B\rangle &= 0, && (4.2)\\ \langle A|\mathbf{M}|A\rangle &= \mu(A), \langle B|\mathbf{M}|B\rangle = \mu(B), && (4.3)\\ \mu(AB) &= \frac{1}{2}(\mu(A)+\mu(B)) + \Re(\langle A|\mathbf{M}|B\rangle). && (4.4) \end{aligned}$$

Next, we determine the type of membership data that is compatible with conditions (4.1)–(4.4). We look at the particular cases $\mathcal{H} = \mathbb{C}^2$ and $\mathbb{C}^3$ separately, then show that the general case, $\mathbb{C}^n$ for $n > 3$, is equivalent to the case $\mathbb{C}^3$.

conjunctions and disjunctions of concepts referred by nouns, so it is different.

[8]In this chapter we are concerned with the representation of exemplars individually. For this reason, we will simplify the notation denoting the operator $\mathbf{M}_x$ by $\mathbf{M}$ and the membership weights $\mu_x(\cdot)$ by $\mu(\cdot)$.

Theorem 4.1. *Let $\mu(A), \mu(B)$, and $\mu(AB)$ denote the membership weights of an exemplar with respect to concepts $\mathcal{A}$, $\mathcal{B}$, and a combination of these concepts denoted by $\mathcal{AB}$. The membership weights are compatible with a complex Hilbert space model $\mathcal{H} = \mathbb{C}^2$ if and only if one of the following cases is satisfied*

1. $\mu(A) = \mu(B) = \mu(AB) = 0$,
2. $\mu(A) = \mu(B) = \mu(AB) = 1$,
3. $\mu(A)+\mu(B) = 1$, *and* $\mu(AB) \in [\frac{1}{2}-\sqrt{\mu(A)(1-\mu(A))}, \frac{1}{2}+\sqrt{\mu(A)(1-\mu(A))}]$.

Proof. We use conditions (4.1)–(4.4) to derive the cases stated in the theorem.

$\Leftarrow$: Note that *1* and *2* are trivially satisfied by choosing $\mathbf{M}$ to be a zero- and two-dimensional projector respectively. Then, conditions (4.1)–(4.4) are satisfied by choosing $|A\rangle$ and $|B\rangle$ to be any two unit vectors that are orthogonal.

$\Rightarrow$: Let $\mathbf{M}$ be a one-dimensional projector. Without loss of generality, we set $\mathbf{M}(x, y) \to (x, 0)$, and

$$\begin{aligned} |A\rangle &= (e^{\mathbb{i}\alpha_1} a_1, e^{\mathbb{i}\alpha_2} a_2), \\ |B\rangle &= (e^{\mathbb{i}\beta_1} b_1, e^{\mathbb{i}\beta_2} b_2). \end{aligned} \tag{4.5}$$

Applying condition (4.3), we obtain

$$a_1 = \sqrt{\mu(A)}, \text{ and } b_1 = \sqrt{\mu(B)}. \tag{4.6}$$

Next, we use condition (4.1) to obtain

$$a_2 = \sqrt{1-\mu(A)}, \text{ and } b_2 = \sqrt{1-\mu(B)}. \tag{4.7}$$

Hence, condition (4.4) becomes

$$\begin{aligned} \mu(AB) &= \frac{1}{2}\left(\mu(A) + e^{\mathbb{i}(\alpha_1-\beta_1)}\sqrt{\mu(A)\mu(B)} + e^{\mathbb{i}(\beta_1-\alpha_1)}\sqrt{\mu(A)\mu(B)} + \mu(B)\right) \\ &= \frac{1}{2}(\mu(A)+\mu(B)) + \sqrt{\mu(A)\mu(B)}\cos(\beta_1-\alpha_1). \end{aligned} \tag{4.8}$$

Since $|\cos(\beta_1 - \alpha_1)| \leq 1$,

$$\mu(AB) \in \left[\frac{1}{2}(\mu(A) + \mu(B)) - \sqrt{\mu(A)\mu(B)}, \frac{1}{2}(\mu(A) + \mu(B)) + \sqrt{\mu(A)\mu(B)}\right]. \tag{4.9}$$

We have considered the conditions given by Eqs. (4.1), (4.3), and (4.4). Next, we apply condition (4.2) to obtain

$$\sqrt{\mu(A)\mu(B)}\cos(\beta_1 - \alpha_1) = \sqrt{1-\mu(A)}\sqrt{1-\mu(B)}\cos(\beta_2 - \alpha_2), \tag{4.10}$$

$$\sqrt{\mu(A)\mu(B)}\sin(\beta_1 - \alpha_1) = \sqrt{1-\mu(A)}\sqrt{1-\mu(B)}\sin(\beta_2 - \alpha_2). \tag{4.11}$$

We square both sides and add Eqs. (4.10) and (4.11) to obtain

$$\mu(A)\mu(B) = (1-\mu(A))(1-\mu(B)), \tag{4.12}$$

which is equivalent to

$$\mu(A) + \mu(B) = 1. \tag{4.13}$$

Sustituting Eq. (4.13) in Eq. (4.8) yields

$$\mu(AB) = \frac{1}{2} + \sqrt{\mu(A)(1-\mu(A))}\cos(\beta_1 - \alpha_1). \tag{4.14}$$

Therefore, if $|A\rangle$, $|B\rangle$, and $\mathbf{M}$ satisfy conditions (4.1)–(4.4), then

$$\begin{aligned} &\mu(A) + \mu(B) = 1, \text{ and} \\ &\mu(AB) \in [\frac{1}{2} - \sqrt{\mu(A)(1-\mu(A))}, \frac{1}{2} + \sqrt{\mu(A)(1-\mu(A))}]. \end{aligned} \tag{4.15}$$

We obtain the other side of the implication by choosing $|A\rangle$ and $|B\rangle$ to satisfy Eqs. (4.6) and (4.7), and α_1 and β_1 such that condition (4.4) is satisfied. □

Because Theorem 4.1 requires that $\mu(A) + \mu(B) = 1$, the Hilbert space model with $\mathcal{H} = \mathbb{C}^2$ is strongly constrained. Nonetheless, this simple model can be used to demonstrate how a Hilbert space model with interference extends classical models of concept combinations.

Consider the case $\mu(A) + \mu(B) = 1$. From a classical perspective, this is equivalent to

$$\mathcal{A} = \textit{Not } \mathcal{B}. \tag{4.16}$$

Therefore, in a classical probabilistic model

$$\mu(A \text{ and } B) = \mu(\text{Not } B \text{ and } B) = 0, \tag{4.17}$$

$$\mu(A \text{ or } B) = \mu(\text{Not } B \text{ or } B) = 1. \tag{4.18}$$

The Hilbert space model is more flexible because $\mu(AB)$ can be either overextended or underextended. For example, if $\mu(A) = \mu(B) = \frac{1}{2}$, then for all $x \in [0,1]$ there are angles α_1 and β_1 such that $\mu(A \text{ and } B) = x$.

We use Fig. 4.1 to describe the modeling scope of the $\mathbb{C}^2$ model. The membership weight, $\mu(A)$, is given on the x-axis, and can also be represented by the identity function plotted on the diagonal red line. The membership weight, $\mu(B) = 1 - \mu(A)$, is represented by the antidiagonal red line. The blue curve surrounding the shaded area corresponds to the maximal and minimal membership weight, $\mu(AB)$, that the concept combination $\mathcal{AB}$ can assume in the $\mathbb{C}^2$ model. Therefore, the shaded area corresponds to the region of overextension and underextension that this model can represent. In particular, 1 denotes the single underextended/overextended region, 2 denotes the double underextended region, and 3 denotes the double overextended region.

Note that not all underextended, or overextended, cases admit a representation in this model. Also, since the shaded area does not contain the two

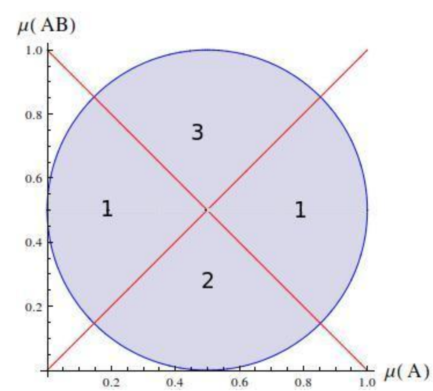

Figure 4.1: Hilbert space model in $\mathbb{C}^2$ for concept combination with $\mu(A) + \mu(B) = 1$.

red curves entirely, this model cannot represent all possible classical cases. For example, when $\mu(A) = 0.9$, $\mu(B) = 0.1$, and $\mu(AB) = 0.1$ there is no representation in the $\mathbb{C}^2$ model.

We now analyze the Hilbert space model in $\mathbb{C}^3$. We introduce the following notation to facilitate the presentation of the mathematical results:

$$\begin{aligned} \mathrm{ave}(AB) &= \tfrac{\mu(A)+\mu(B)}{2}, && (4.19)\\ \mathrm{dev}(AB) &= \sqrt{\min(\mu(A)\mu(B), (1-\mu(A))(1-\mu(B)))}. && (4.20) \end{aligned}$$

Theorem 4.2. *Let $\mu(A)$, $\mu(B)$, and $\mu(AB)$ denote the membership weights of an exemplar with respect to concepts $\mathcal{A}$, $\mathcal{B}$, and a combination of these concepts denoted by $\mathcal{AB}$. The membership weights are compatible with a complex Hilbert space model $\mathcal{H} = \mathbb{C}^3$ if and only if*

$$\mu(AB) \in [\mathrm{ave}(AB) - \mathrm{dev}(AB), \mathrm{ave}(AB) + \mathrm{dev}(AB)]. \tag{4.21}$$

Proof. We will show how conditions (4.1)–(4.4) are used to derive Eq. (4.21).

First, if $\mathbf{M}$ is a zero- or three-dimensional projector, then

$$\begin{aligned} &\mu(A) = \mu(B) = \mu(AB) = 0, \text{ and}\\ &\mu(A) = \mu(B) = \mu(AB) = 1, \end{aligned} \tag{4.22}$$

respectively. Thus Eq. (4.21) is trivially satisfied. Therefore, conditions (4.1)–(4.4) are satisfied by choosing $|A\rangle$ and $|B\rangle$ to be any two unit orthogonal vectors.

The remaining cases are $\mathbf{M}$ is a one- or a two-dimensional projector. We apply conditions (4.1)–(4.4) to vectors $|A\rangle$ and $|B\rangle$ in these two cases separately, and then combine the two analyses to derive Eq. (4.21).

If $\mathbf{M}$ is a one-dimensional projector, then, without loss of generality, set $\mathbf{M}(x, y, z) \to (x, 0, 0)$, and

$$\begin{aligned} |A\rangle &= (e^{\mathbb{i}\alpha_1} a_1, e^{\mathbb{i}\alpha_2} a_2, e^{\mathbb{i}\alpha_3} a_3),\\ |B\rangle &= (e^{\mathbb{i}\beta_1} b_1, e^{\mathbb{i}\beta_2} b_2, e^{\mathbb{i}\beta_3} b_3). \end{aligned} \tag{4.23}$$

Note that conditions (4.1) and (4.3) are satisfied by choosing the coefficients in $|A\rangle$ and $|B\rangle$ as follows:

$$\begin{aligned} a_1 &= \sqrt{\mu(A)},\ a_2 = \sqrt{\lambda}\sqrt{1-\mu(A)}\ , a_3 = \sqrt{1-\lambda}\sqrt{1-\mu(A)}, \\ b_1 &= \sqrt{\mu(B)},\ b_2 = \sqrt{\kappa}\sqrt{1-\mu(B)}\ , b_3 = \sqrt{1-\kappa}\sqrt{1-\mu(B)}, \end{aligned} \tag{4.24}$$

with $0 \leq \lambda \leq 1$, and $0 \leq \kappa \leq 1$. Moreover, condition (4.4) implies that $\mu(AB)$ is given by

$$\mu(AB) = \frac{1}{2}(\mu(A) + \mu(B)) + \sqrt{\mu(A)\mu(B)}\cos(\alpha_1 - \beta_1). \tag{4.25}$$

We apply condition (4.2) to obtain

$$-\sqrt{\mu(A)\mu(B)}\cos(\gamma_1) = \sqrt{(1-\mu(A))(1-\mu(B))}F(\lambda,\kappa,\cos(\gamma_2),\cos(\gamma_3)), \tag{4.26}$$
$$-\sqrt{\mu(A)\mu(B)}\sin(\gamma_1) = \sqrt{(1-\mu(A))(1-\mu(B))}F(\lambda,\kappa,\sin(\gamma_2),\sin(\gamma_3)), \tag{4.27}$$

where

$$F(\lambda,\kappa,f(x),f(y)) = \left(\sqrt{\lambda\kappa}f(x) + \sqrt{(1-\lambda)(1-\kappa)}f(y)\right). \tag{4.28}$$

Note that $F(\lambda,\kappa,\cos(\gamma_2),\cos(\gamma_3))$, and $F(\lambda,\kappa,\sin(\gamma_2),\sin(\gamma_3))$ are convex combinations of $\sqrt{\lambda\kappa}$ and $\sqrt{(1-\lambda)(1-\kappa)}$. Therefore,

$$\begin{aligned} |F(\lambda,\kappa,\cos(\gamma_2),\cos(\gamma_3))| &\leq |\sqrt{\lambda\kappa}| + |\sqrt{(1-\lambda)(1-\kappa)}|, \\ |F(\lambda,\kappa,\sin(\gamma_2),\sin(\gamma_3))| &\leq |\sqrt{\lambda\kappa}| + |\sqrt{(1-\lambda)(1-\kappa)}|. \end{aligned} \tag{4.29}$$

Set

$$\sqrt{\lambda} = \cos(\theta_1),\ \sqrt{\kappa} = \cos(\theta_2), \tag{4.30}$$

for θ_1, θ_2 in $[0, \frac{\pi}{2}]$. Then

$$\begin{aligned} \sqrt{1-\lambda} &= \sin(\theta_1), \\ \sqrt{1-\kappa} &= \sin(\theta_2). \end{aligned} \tag{4.31}$$

Substituting Eqs. (4.30) and (4.31) in Eq. (4.29) we obtain

$$\begin{aligned} |F(\lambda,\kappa,\cos(\gamma_2),\cos(\gamma_3))| &\leq |\cos(\theta_1-\theta_2)| \leq 1, \\ |F(\lambda,\kappa,\sin(\gamma_2),\sin(\gamma_3))| &\leq |\sin(\theta_1-\theta_2)| \leq 1. \end{aligned} \tag{4.32}$$

Since $|F(\lambda, \kappa, \cos(\gamma_2), \cos(\gamma_3))| \leq 1$, Eq. (4.26) implies that

$$|\sqrt{\mu(A)\mu(B)}\cos(\gamma_1)| \leq \sqrt{(1-\mu(A))(1-\mu(B))}. \tag{4.33}$$

Therefore, the interference term is bounded as follows

$$\begin{aligned}|\sqrt{\mu(A)\mu(B)}\cos(\gamma_1)| &\leq \min\left(\sqrt{\mu(A)\mu(B)}, \sqrt{(1-\mu(A))(1-\mu(B))}\right)\\ &= \operatorname{dev}(AB).\end{aligned} \tag{4.34}$$

Next, combining Eqs. (4.26) and (4.27). We obtain

$$\mu(A)\mu(B) = (1-\mu(A))(1-\mu(B))\hat{F}(\lambda, \kappa, \gamma_2, \gamma_3), \tag{4.35}$$

where

$$\hat{F}(\lambda, \kappa, \gamma_2, \gamma_3) = F(\lambda, \kappa, \cos(\gamma_2), \cos(\gamma_3))^2 + F(\lambda, \kappa, \sin(\gamma_2), \sin(\gamma_3))^2. \tag{4.36}$$

Thus,

$$\mu(A) + \mu(B) = 1 + \mu(A)\mu(B)\left(1 - \frac{1}{\hat{F}(\lambda, \kappa, \gamma_2, \gamma_3)}\right). \tag{4.37}$$

Using the parametrization for λ and κ given by Eq. (4.30), and applying Eq. (4.32) to Eq. (4.36), we obtain

$$0 \leq \hat{F}(\lambda, \kappa, \gamma_2, \gamma_3) \leq \cos(\theta_1 - \theta_2)^2 + \sin(\theta_1 - \theta_2)^2 = 1. \tag{4.38}$$

Next, applying Eq. (4.38) to Eq. (4.37) yields

$$\mu(A) + \mu(B) \leq 1. \tag{4.39}$$

Therefore, when $\mathbf{M}$ is a one-dimensional projector, conditions (4.1)–(4.4) imply

$$\begin{aligned}&\mu(AB) \in [\operatorname{ave}(AB) - \operatorname{dev}(AB), \operatorname{ave}(AB) + \operatorname{dev}(AB)], \text{ and}\\ &\mu(A) + \mu(B) \leq 1.\end{aligned} \tag{4.40}$$

Next, consider the case $\mathbf{M}$ is a two-dimensional projector. Without loss of generality we can assume $\mathbf{M}(x, y, z) \to (x, y, 0)$. In this case, we satisfy conditions (4.1) and (4.3) by choosing the coefficients in $|A\rangle$ and $|B\rangle$ as follows

$$\begin{aligned} &a_1 = \sqrt{\lambda}\sqrt{\mu(A)},\ a_2 = \sqrt{1-\lambda}\sqrt{\mu(A)}\ , a_3 = \sqrt{1-\mu(A)},\\ &b_1 = \sqrt{\kappa}\sqrt{\mu(B)},\ b_2 = \sqrt{1-\kappa}\sqrt{\mu(B)}\ , b_3 = \sqrt{1-\mu(B)}, \end{aligned} \tag{4.41}$$

with $0 \le \lambda \le 1$, and $0 \le \kappa \le 1$. Moreover, Eq. (4.4) implies that $\mu(AB)$ is given by

$$\mu(AB) = \frac{1}{2}(\mu(A)+\mu(B)) + \sqrt{\mu(A)\mu(B)}F(\lambda,\kappa,\cos(\gamma_1),\cos(\gamma_2)). \tag{4.42}$$

We apply condition (4.2) to obtain

$$\sqrt{\mu(A)\mu(B)}F(\lambda,\kappa,\cos(\gamma_1),\cos(\gamma_2)) = -\sqrt{(1-\mu(A))(1-\mu(B))}\cos(\gamma_3). \tag{4.43}$$

Since $|F(\lambda,\kappa,\cos(\gamma_1),\cos(\gamma_2))| \le 1$, Eq. (4.43) implies that

$$\begin{aligned} |\sqrt{\mu(A)\mu(B)}F(\lambda,\kappa,\cos(\gamma_1),\cos(\gamma_2))| &\le \min(\sqrt{\mu(A)\mu(B)}, \sqrt{(1-\mu(A))(1-\mu(B))})\\ &= \mathrm{dev}(AB). \end{aligned} \tag{4.44}$$

We repeat the procedure used in the one-dimensional case to deduce

$$\mu(A)\mu(B)\hat{F}(\lambda,\kappa,\gamma_1,\gamma_2) = (1-\mu(A))(1-\mu(B)). \tag{4.45}$$

Since $0 \le \hat{F}(\lambda,\kappa,\gamma_1,\gamma_2) \le 1$, Eq. (4.45) yields

$$1 \le \mu(A)+\mu(B). \tag{4.46}$$

Therefore, when $\mathbf{M}$ is a two-dimensional projector, conditions (4.1)–(4.4) imply

$$\begin{aligned} &\mu(AB) \in [\mathrm{ave}(AB)-\mathrm{dev}(AB), \mathrm{ave}(AB)+\mathrm{dev}(AB)],\ \text{and}\\ &1 \le \mu(A)+\mu(B). \end{aligned} \tag{4.47}$$

Thus, merging conditions (4.40) and (4.47) completes the proof. □

Theorem 4.2 shows that some of the restrictions of the model in $\mathbb{C}^2$ are removed in the $\mathbb{C}^3$ model. Moreover, the cases with $\mu(A)+\mu(B) \leq 1$ are represented by a one-dimensional projector, and the cases with $1 \leq \mu(A)+\mu(B)$ are represented by a two-dimensional projector.

Because extending the Hilbert space model from dimension two to dimension three leads to a reduction of constraints in the model, we might suspect that adding dimensions would lead to further reductions. However, we now show that the constraints of the $\mathbb{C}^3$ model cannot be relaxed in C^n with $n > 3$. To do so, we will prove that the case $\mathbb{C}^n$, for $n > 3$, yields the same constraints on $\mu(AB)$.

Theorem 4.3. *Let $\mu(A), \mu(B)$, and $\mu(AB)$ denote the membership weights of an exemplar with respect to concepts $\mathcal{A}$, $\mathcal{B}$, and a combination of these concepts denoted by $\mathcal{AB}$. If the membership weights are compatible with a complex Hilbert space model in $\mathbb{C}^n$, for $n > 3$, then*

$$\mu(AB) \in [\text{ave}(AB) - \text{dev}(AB), \text{ave}(AB) + \text{dev}(AB)]. \tag{4.48}$$

Proof. We show how conditions (4.1)–(4.4) can be used to derive Eq. (4.48). First note that Eq. (4.48) is trivially satisfied when $\mathbf{M}$ is a zero- or n-dimensional projector. Therefore, conditions (4.1)–(4.4) are satisfied when we choose $|A\rangle$ and $|B\rangle$ to be any two unit orthogonal vectors.

Let $\mathbf{M}$ be a k-dimensional projector. Without loss of generality, we set $\mathbf{M}(x_1, ..., x_n) \to (x_1, ..., x_k, 0, ..., 0)$ with $0 < k < n$, and

$$\begin{aligned} |A\rangle &= (e^{\mathbb{i}\alpha_1} a_1, e^{\mathbb{i}\alpha_2} a_2, ..., e^{\mathbb{i}\alpha_n} a_n), \\ |B\rangle &= (e^{\mathbb{i}\beta_1} b_1, e^{\mathbb{i}\beta_2} b_2, ..., e^{\mathbb{i}\beta_n} b_n). \end{aligned} \tag{4.49}$$

Conditions (4.1) and (4.3) are satisfied by defining the coefficients in $|A\rangle$ and $|B\rangle$ as follows:

$$a_i = \lambda_i \sqrt{\mu(A)}, \quad \text{and } b_i = \kappa_i \sqrt{\mu(B)}, \quad \text{for } i = 1, ..., k, \tag{4.50}$$

$$a_i = \lambda_i \sqrt{1 - \mu(A)}, \quad \text{and } b_i = \kappa_i \sqrt{1 - \mu(B)}, \quad \text{for } i = k+1, ..., n. \tag{4.51}$$

Then,

$$
\begin{aligned}
|A\rangle &= (e^{\mathbb{i}\alpha_1}\lambda_1\sqrt{\mu(A)}, ..., e^{\mathbb{i}\alpha_k}\lambda_k\sqrt{\mu(A)}, e^{\mathbb{i}\alpha_{k+1}}\lambda_{k+1}\sqrt{1-\mu(A)}, ..., e^{\mathbb{i}\alpha_n}\lambda_n\sqrt{1-\mu(A)}),\\
|B\rangle &= (e^{\mathbb{i}\beta_1}\kappa_1\sqrt{\mu(B)}, ..., e^{\mathbb{i}\beta_k}\kappa_k\sqrt{\mu(B)}, e^{\mathbb{i}\beta_{k+1}}\kappa_{k+1}\sqrt{1-\mu(B)}, ..., e^{\mathbb{i}\beta_n}\kappa_n\sqrt{1-\mu(B)}),
\end{aligned}
\tag{4.52}
$$

and condition (4.1) implies

$$
\sum_{i=1}^{k}\lambda_i^2 = \sum_{i=k+1}^{n}\lambda_i^2 = \sum_{i=1}^{k}\kappa_i^2 = \sum_{i=k+1}^{n}\kappa_i^2 = 1. \tag{4.53}
$$

Therefore, condition (4.4) becomes

$$
\langle A|M|B\rangle = \sqrt{\mu(A)\mu(B)}\left(\sum_{i=1}^{k}\lambda_i\kappa_i\cos(\gamma_i)\right), \tag{4.54}
$$

where $\gamma_i = \beta_i - \alpha_i$. Note that Eq. (4.53) allow us to apply the Cauchy-Schwarz inequality in Eq. (4.54) to obtain

$$
-1 \leq \sum_{i=1}^{k}\lambda_i\kappa_i\cos(\gamma_i) \leq 1. \tag{4.55}
$$

Hence, condition (4.4) gives

$$
\mu(AB) \in [\mathrm{ave}(AB) - \sqrt{\mu(A)\mu(B)}, \mathrm{ave}(AB) + \sqrt{\mu(A)\mu(B)}]. \tag{4.56}
$$

Next, condition (4.2) implies that

$$
\begin{aligned}
&\sqrt{\mu(A)\mu(B)}\left(\sum_{i=1}^{k}\lambda_i\kappa_i\cos(\gamma_i)\right)\\
&\qquad = -\sqrt{(1-\mu(A))(1-\mu(B))}\left(\sum_{i=k+1}^{n}\lambda_i\kappa_i\cos(\gamma_i)\right),
\end{aligned}
\tag{4.57}
$$

and

$$
\begin{aligned}
&\sqrt{\mu(A)\mu(B)}\left(\sum_{i=1}^{k}\lambda_i\kappa_i\sin(\gamma_i)\right)\\
&\qquad = -\sqrt{(1-\mu(A))(1-\mu(B))}\left(\sum_{i=k+1}^{n}\lambda_i\kappa_i\sin(\gamma_i)\right).
\end{aligned}
\tag{4.58}
$$

Applying condition (4.4) in Eq. (4.57), yields

$$\mu(AB) - \mathrm{ave}(AB) + \sqrt{(1-\mu(A))(1-\mu(B))}\left(\sum_{i=k+1}^{n} \lambda_i \kappa_i \cos(\gamma_i)\right) = 0. \tag{4.59}$$

We apply Cauchy-Schwarz inequality to Eq. (4.59) to obtain

$$-1 \leq \sum_{i=k+1}^{n} \lambda_i \kappa_i \cos(\beta_i - \alpha_i) \leq 1. \tag{4.60}$$

Therefore, Eqs. (4.59) and (4.60) yield

$$\mu(AB) \in [\mathrm{ave}(AB) - \sqrt{(1-\mu(A))(1-\mu(B))}, \mathrm{ave}(AB) + \sqrt{(1-\mu(A))(1-\mu(B))}]. \tag{4.61}$$

Combining the constraints in Eqs. (4.56) and (4.61) completes the proof. □

Theorem 4.2 establishes a limit to the Hilbert space modeling approach by limiting the values $\mu(A)$, $\mu(B)$, and $\mu(AB)$ can assume. Moreover, Theorem 4.3 confirms that a complex Hilbert space of dimension 3 is sufficient to reach the full modeling power of this model.

4.2 Modeling in the Tensor Product of Hilbert Spaces

The idea of applying the tensor product to model concept conjunctions and disjunctions was first proposed in [AG05b], and has since been applied to other types of combinations (see §3.6). In order to introduce the notation and probabilistic structure of the tensor product model, we present a simplified version of this model in §4.2.1, and then introduce the general model in §4.2.2.

4.2.1 A Simple Tensor Product Model

We can build a simple tensor product model for the membership weight of an exemplar with respect to concepts $\mathcal{A}$, $\mathcal{B}$, and their combination $\mathcal{AB}$ by using the tensor product model introduced in §4.1. Namely, we use unit vectors $|A\rangle$ and $|B\rangle$ to represent the state of concepts $\mathcal{A}$ and $\mathcal{B}$, and a projector $\mathbf{M} : \mathcal{H} \to \mathcal{H}$ to measure the membership weights. Hence,

$$\begin{aligned}\mu(A) &= \langle A|\mathbf{M}|A\rangle, \\ \mu(B) &= \langle B|\mathbf{M}|B\rangle.\end{aligned} \tag{4.62}$$

The state $|C\rangle$, representing the state of the combined concept $\mathcal{AB}$, is given by the tensor product of $|A\rangle$ and $|B\rangle$:

$$|C\rangle = |A\rangle \otimes |B\rangle. \tag{4.63}$$

Note that the membership operator $\mathbf{M}$ can be extended to the tensor product $\mathcal{H}\otimes\mathcal{H}$ by the operators $\mathbf{M}^A = \mathbf{M}\otimes\mathbb{1}$ and $\mathbf{M}^B = \mathbb{1}\otimes\mathbf{M}$ respectively. Indeed,

$$\begin{aligned}\langle C|\mathbf{M}^A|C\rangle &= (\langle A| \otimes \langle B|)\mathbf{M} \otimes \mathbb{1}(|A\rangle \otimes |B\rangle) = \langle A|\mathbf{M}|A\rangle \otimes \langle B|\mathbb{1}|B\rangle = \mu(A), \\ \langle C|\mathbf{M}^B|C\rangle &= (\langle A| \otimes \langle B|)\mathbb{1} \otimes \mathbf{M}(|A\rangle \otimes |B\rangle) = \langle A|\mathbb{1}|A\rangle \otimes \langle B|\mathbf{M}|B\rangle = \mu(B).\end{aligned} \tag{4.64}$$

If we want to measure the membership weight of an exemplar with respect to the conjunction of concepts $\mathcal{A}$ *and* $\mathcal{B}$, we must determine whether the exemplar is a member of both concepts simultaneously. In this case, the membership operator for the conjunction of two concepts is given by

$$\mathbf{M}^{\wedge} = \mathbf{M} \otimes \mathbf{M}. \tag{4.65}$$

The membership weight of an exemplar with respect to the conjunction of concepts $\mathcal{A}$ *and* $\mathcal{B}$ is given by

$$\begin{aligned}\mu(A \textit{ and } B) &= \langle C|\mathbf{M}^{\wedge}|C\rangle = (\langle A| \otimes \langle B|)\mathbf{M} \otimes \mathbf{M}(|A\rangle \otimes |B\rangle) \\ &= \langle A|\mathbf{M}|A\rangle \otimes \langle B|\mathbf{M}|B\rangle = \mu(A)\mu(B).\end{aligned} \tag{4.66}$$

Similarly, if we want to measure the membership weight of the exemplar with respect to the disjunction of concepts $\mathcal{A}$ *or* $\mathcal{B}$, we introduce the operator

$$\begin{aligned}\mathbf{M}^{\vee} &= \mathbf{M} \otimes \mathbf{M} + \mathbf{M} \otimes (\mathbb{1} - \mathbf{M}) + (\mathbb{1} - \mathbf{M}) \otimes \mathbf{M} \\ &= \mathbb{1} \otimes \mathbb{1} - (\mathbb{1} - \mathbf{M}) \otimes (\mathbb{1} - \mathbf{M}).\end{aligned} \tag{4.67}$$

Hence, the membership weight of the exemplar with respect to the disjunction of the concepts $\mathcal{A}$ *or* $\mathcal{B}$ is given by

$$
\begin{aligned}
&\mu(A \ or \ B) = \langle C|\mathbf{M}^{\vee}|C\rangle \\
&= (\langle A| \otimes \langle B|)\mathbf{M} \otimes \mathbf{M} + \mathbf{M} \otimes (\mathbb{1} - \mathbf{M}) + (\mathbb{1} - \mathbf{M}) \otimes \mathbf{M}(|A\rangle \otimes |B\rangle) \\
&= \langle A|\mathbf{M}|A\rangle\langle B|\mathbf{M}|B\rangle + \langle A|\mathbf{M}|A\rangle\langle B|\mathbb{1} - \mathbf{M}|B\rangle + \langle A|\mathbb{1} - \mathbf{M}|A\rangle\langle B|\mathbf{M}|B\rangle \\
&= \mu(A)\mu(B) + \mu(A)(1 - \mu(B)) + (1 - \mu(A))\mu(B) \\
&= \mu(A) + \mu(B) - \mu(A)\mu(B).
\end{aligned}
\tag{4.68}
$$

Note that the formulas for the membership weight of the conjunction and disjunction of two concepts, given by Eqs. (4.66) and (4.68) respectively, are equivalent to the classical probability formulas where the membership estimation for concepts $\mathcal{A}$ and $\mathcal{B}$ are independent events.

4.2.2 Generalizing the States in the Tensor Product Model

The probabilistic independence of the model presented in §4.2.1 is a consequence of the choice of the state vector representing the concept combination and of the membership operator. Specifically, the state of the combined concept $|AB\rangle$ is given by the tensor product $|C\rangle = |A\rangle \otimes |B\rangle$ of the states of the two former concepts $|A\rangle$ and $|B\rangle$, and the operators $\mathbf{M}^A$, $\mathbf{M}^B$, $\mathbf{M}^{\wedge}$, and $\mathbf{M}^{\vee}$ are built from an operator $\mathbf{M}$ that acts on the two sides of the tensor product space separately. This choice for the state and operators is a simplified application of the tensor product model because the state and the operators are separable (see Appendix 6.3.2); it means that we can identify the first part of the tensor space with the concept $\mathcal{A}$, and the second part with the concept $\mathcal{B}$. We now assume a general state $|C\rangle$ that is not necessarily separable. This means that we do not know what part of $|C\rangle$ is inherited from the state of the concept $\mathcal{A}$ or the state of the concept $\mathcal{B}$.

To obtain the membership weights for the single concepts $\mathcal{A}$ and $\mathcal{B}$ from the state $|C\rangle$, we require two projection operators $\mathbf{M}^A, \mathbf{M}^B : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ that recover the membership weights $\mu(A)$ and $\mu(B)$ when applied to the vector $|C\rangle$. Therefore

$$
\begin{aligned}
\langle C|\mathbf{M}^A|C\rangle &= \mu(A), \\
\langle C|\mathbf{M}^B|C\rangle &= \mu(B).
\end{aligned}
\tag{4.69}
$$

We also require two membership operators $\mathbf{M}^{\wedge}, \mathbf{M}^{\vee} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

representing the measurement with respect to concept conjunction and disjunction:

$$\langle C|\mathbf{M}^{\wedge}|C\rangle \quad = \mu(A \text{ } and \text{ } B), \tag{4.70}$$

$$\langle C|\mathbf{M}^{\vee}|C\rangle \quad = \mu(A \text{ } or \text{ } B). \tag{4.71}$$

Therefore, the general tensor product model for concept combinations is given by a four-tuple $(|C\rangle, \mathbf{M}^A, \mathbf{M}^B, \mathbf{M}^{\wedge})$ satisfying conditions (4.69) and (4.70) for conjunction, and by a four-tuple $(|C\rangle, \mathbf{M}^A, \mathbf{M}^B, \mathbf{M}^{\vee})$ satisfying conditions (4.69) and (4.71) for disjunction. For simplicity, we will assume that the membership operators $\mathbf{M}^A, \mathbf{M}^B, \mathbf{M}^{\wedge}$, and $\mathbf{M}^{\vee}$ are built from a measurement operator $\mathbf{M} : \mathcal{H} \to \mathcal{H}$ as in Eqs. (4.62), (4.65), and (4.67).

We now build a concrete representation of this model in a complex tensor space. Let $|C\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ and $\{|i\rangle\}_{i=1}^n$ be the canonical basis of $\mathbb{C}^n$. Without loss of generality, let $\mathbf{M}$ be the orthogonal projector on the subspace of $\mathbb{C}^n$ spanned by the basis elements $|1\rangle, ..., |r\rangle$, for $r < n$:

$$\mathbf{M}(x_1, ..., x_n) \to (x_1, ..., x_r, 0, ..., 0).$$

Next, let $|C\rangle$ be a unit vector in $\mathbb{C}^n \otimes \mathbb{C}^n$. That is,

$$|C\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}e^{\mathbb{i}\gamma_{ij}}|i\rangle \otimes |j\rangle, \tag{4.72}$$

and

$$\begin{aligned}
\langle C|C\rangle &= \sum_{i,j=1}^{n} c_{ij}e^{\mathbb{i}\gamma_{ij}}\langle i| \otimes \langle j| \sum_{k,l=1}^{n} c_{kl}e^{\mathbb{i}\gamma_{kl}}|k\rangle \otimes |l\rangle \\
&= \sum_{i,j,k,l=1}^{n} c_{ij}c_{kl}e^{\mathbb{i}(-\gamma_{ij}+\gamma_{kl})}\langle i|k\rangle\langle j|l\rangle \\
&= \sum_{i,j=1}^{n} c_{ij}^2 = 1.
\end{aligned} \tag{4.73}$$

We can now apply condition (4.69) to the vector $|C\rangle$:

$$\begin{aligned}\langle C|\mathbf{M}^A|C\rangle = \langle C|\mathbf{M}\otimes \mathbb{1}|C\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}e^{-\mathbb{i}\gamma_{ij}}\langle i|\otimes\langle j|\mathbf{M}\otimes\mathbb{1}|\sum_{k=1}^{n}\sum_{l=1}^{n} c_{kl}e^{\mathbb{i}\gamma_{kl}}|k\rangle\otimes|l\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}e^{-\mathbb{i}\gamma_{ij}}\langle i|\otimes\langle j|\sum_{k=1}^{r}\sum_{l=1}^{n} c_{kl}e^{\mathbb{i}\gamma_{kl}}|k\rangle\otimes|l\rangle \\ &= \sum_{i,j=1}^{r}\sum_{k,l=1}^{n} c_{ij}c_{kl}e^{\mathbb{i}(-\gamma_{ij}+\gamma_{kl})}\langle i|k\rangle\langle j|l\rangle \\ &= \sum_{i=1}^{r}\sum_{j=1}^{n} c_{ij}^2 = \mu(A),\end{aligned} \tag{4.74}$$

and

$$\begin{aligned}\langle C|\mathbf{M}^B|C\rangle = \langle C|\mathbb{1}\otimes\mathbf{M}|C\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}e^{-\mathbb{i}\gamma_{ij}}\langle i|\otimes\langle j|\mathbb{1}\otimes\mathbf{M}|\sum_{k=1}^{n}\sum_{l=1}^{n} c_{kl}e^{\mathbb{i}\gamma_{kl}}|k\rangle\otimes|l\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}e^{-\mathbb{i}\gamma_{ij}}\langle i|\otimes\langle j|\sum_{k=1}^{n}\sum_{l=1}^{r} c_{kl}e^{\mathbb{i}\gamma_{kl}}|k\rangle\otimes|l\rangle \\ &= \sum_{i,k=1}^{n}\sum_{j,l=1}^{r} c_{ij}c_{kl}e^{\mathbb{i}(-\gamma_{ij}+\gamma_{kl})}\langle i|k\rangle\langle j|l\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{r} c_{ij}^2 = \mu(B).\end{aligned} \tag{4.75}$$

For the case of conjunction, we apply condition (4.70):

$$\begin{aligned}\langle C|\mathbf{M}^\wedge|C\rangle &= \langle C|\mathbf{M}\otimes\mathbf{M}|C\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}e^{-\mathbb{i}\gamma_{ij}}\langle i|\otimes\langle j|\mathbf{M}\otimes\mathbf{M}|\sum_{k=1}^{n}\sum_{l=1}^{n} c_{kl}e^{\mathbb{i}\gamma_{kl}}|k\rangle\otimes|l\rangle \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} c_{ij}^2 = \mu(A\ and\ B).\end{aligned} \tag{4.76}$$

Finally, we apply condition (4.71) for the case of disjunction:

$$\begin{aligned}
\langle C|\mathbf{M}^{\vee}|C\rangle &= \langle C|\mathbf{M}^{A} + \mathbf{M}^{B} - \mathbf{M}^{\wedge}|C\rangle \\
&= \langle C|(\mathbf{M}\otimes\mathbf{M} + \mathbf{M}\otimes(\mathbb{1}-\mathbf{M}) + (\mathbb{1}-\mathbf{M})\otimes\mathbf{M}|C\rangle, \\
&= \sum_{i=1}^{r}\sum_{j=1}^{n} c_{ij}^2 + \sum_{i=1}^{n}\sum_{j=1}^{r} c_{ij}^2 - \sum_{i=1}^{r}\sum_{j=1}^{r} c_{ij}^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^2 - \sum_{i=r+1}^{n}\sum_{j=r+1}^{n} c_{ij}^2 \\
&= 1 - \sum_{i=r+1}^{n}\sum_{j=r+1}^{n} c_{ij}^2 = \mu(A \textit{ or } B).
\end{aligned} \tag{4.77}$$

With these results, we can prove that the constraints of the tensor product model are exactly those of the classical probabilistic model.

Definition 4.4. Let $\mu = \{\mu(A), \mu(B), \mu(A \textit{ and } B)\}$ be a triplet denoting the membership weights of concepts $\mathcal{A}$, $\mathcal{B}$, and their conjunction $\mathcal{A}$ *and* B. We say that the triplet μ admits a representation in the tensor product space $\mathbb{C}^n \otimes \mathbb{C}^n$ if there exists a unit vector $|C\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$, and an operator $\mathbf{M} : \mathbb{C}^n \to \mathbb{C}^n$, such that conditions (4.73)–(4.76) are satisfied.

Theorem 4.5. *Let $\mu = \{\mu(A), \mu(B), \mu(A \textit{ and } B)\}$ be a triplet denoting the membership weights of concepts $\mathcal{A}$, $\mathcal{B}$, and their conjunction $\mathcal{A}$ and B. The triplet μ is classical conjunction data if and only if it admits a representation in a tensor product space $\mathbb{C}^n \otimes \mathbb{C}^n$ with $n = 2$.*

Proof. If μ admits a representation in the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2$, there exists $|C\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and an operator $\mathbf{M}$ such that (4.73)–(4.76) are satisfied. If $\mu(A) = \mu(B) = \mu(A \textit{ and } B) = 0$ or 1, we can choose $|C\rangle$ to be any unit vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbf{M}$ to be a zero- or two-dimensional projector respectively. Otherwise, let $\{|1\rangle, |2\rangle\}$ be the canonical base of $\mathbb{C}^2$. Without loss of generality, we set $|C\rangle$ to be

$$|C\rangle = c_{11}e^{\mathbb{i}\gamma_{11}}|1\rangle\otimes|1\rangle + c_{12}e^{\mathbb{i}\gamma_{12}}|1\rangle\otimes|2\rangle + c_{21}e^{\mathbb{i}\gamma_{21}}|2\rangle\otimes|1\rangle + c_{22}e^{\mathbb{i}\gamma_{22}}|2\rangle\otimes|2\rangle, \tag{4.78}$$

and let $\mathbf{M}$ be a one-dimensional projector into the subspace determined by $|1\rangle$. With this choice,

$$\begin{aligned} \mu(A) &= \langle C|\mathbf{M}\otimes \mathbb{1}|C\rangle = c_{11}^2 + c_{12}^2, \\ \mu(B) &= \langle C|\mathbb{1}\otimes \mathbf{M}|C\rangle = c_{11}^2 + c_{21}^2, \\ \mu(A \textit{ and } B) &= \langle C|\mathbf{M}\otimes \mathbf{M}|C\rangle = c_{11}^2. \end{aligned} \tag{4.79}$$

Then, clearly

$$\mu(A \textit{ and } B) \leq \mu(A), \tag{4.80}$$

$$\mu(A \textit{ and } B) \leq \mu(B), \text{ and} \tag{4.81}$$

$$\mu(A) + \mu(B) - \mu(A \textit{ and } B) \quad = c_{11}^2 + c_{12}^2 + c_{21}^2 \leq 1. \tag{4.82}$$

Thus, μ is classical conjunction data. The other implication is proven by taking $\mathbf{M}$ to be the same one-dimensional projector, $|C\rangle$ such that

$$\begin{aligned} c_{11} &= \sqrt{\mu(A \textit{ and } B)}, \\ c_{12} &= \sqrt{\mu(A) - \mu(A \textit{ and } B)}, \\ c_{21} &= \sqrt{\mu(B) - \mu(A \textit{ and } B)}, \\ c_{22} &= \sqrt{1 - \mu(A) - \mu(B) + \mu(A \textit{ and } B)}, \end{aligned} \tag{4.83}$$

and $\gamma_{ij} = 0$ for $i, j = 1, 2$. □

Definition 4.6. Let $\mu = \{\mu(A), \mu(B), \mu(A \textit{ or } B)\}$ be a triplet denoting the membership weights of concepts $\mathcal{A}$, $\mathcal{B}$, and their disjunction $\mathcal{A}$ *or* $\mathcal{B}$. We say that the triplet μ admits a representation in the tensor product space $\mathbb{C}^n \otimes \mathbb{C}^n$ if there exists a unit vector $|C\rangle \in \mathbb{C}^n$ and an operator $\mathbf{M} : \mathbb{C}^n \to \mathbb{C}^n$ such that conditions (4.73)–(4.75) and (4.77) are satisfied.

Theorem 4.7. *Let $\mu = \{\mu(A), \mu(B), \mu(A \textit{ or } B)\}$ be a triplet denoting the membership weights of concepts $\mathcal{A}$, $\mathcal{B}$, and their disjunction $\mathcal{A}$ or $\mathcal{B}$. The triplet μ is classical disjunction data if and only if it admits a representation in a tensor product space $\mathbb{C}^n \otimes \mathbb{C}^n$ with $n = 2$.*

Proof. If μ admits a representation in the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2$, there exists $|C\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and an operator $\mathbf{M}$ such that conditions (4.73)–(4.75) and (4.77) are satisfied. If $\mu(A) = \mu(B) = \mu(A \textit{ or } B) = 0$ or 1, we can choose $|C\rangle$ to be any unit vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbf{M}$ to be a zero- or two-dimensional projector respectively. Otherwise, let $\{|1\rangle, |2\rangle\}$ be the canonical basis of $\mathbb{C}^2$. Without loss of generality set $|C\rangle$ to be

$$|C\rangle = c_{11}e^{\mathbb{i}\gamma_{11}}|1\rangle \otimes |1\rangle + c_{12}e^{\mathbb{i}\gamma_{12}}|1\rangle \otimes |2\rangle + c_{21}e^{\mathbb{i}\gamma_{21}}|2\rangle \otimes |1\rangle + c_{22}e^{\mathbb{i}\gamma_{22}}|2\rangle \otimes |2\rangle, \tag{4.84}$$

and let $\mathbf{M}$ be a one-dimensional projector into the subspace determined by $|1\rangle$. With this choice,

$$\begin{aligned} \mu(A) &= \langle C|\mathbf{M} \otimes \mathbb{1}|C\rangle = c_{11}^2 + c_{12}^2, \\ \mu(B) &= \langle C|\mathbb{1} \otimes \mathbf{M}|C\rangle = c_{11}^2 + c_{21}^2, \\ \mu(A \ or \ B) &= \langle C|\mathbf{M} \otimes \mathbf{M} + \mathbf{M} \otimes (\mathbb{1} - \mathbf{M}) + (\mathbb{1} - \mathbf{M}) \otimes \mathbf{M}|C\rangle = c_{11}^2 + c_{12}^2 + c_{21}^2. \end{aligned} \tag{4.85}$$

Then, clearly

$$\begin{aligned} &\mu(A) \leq \mu(A \ or \ B), \\ &\mu(B) \leq \mu(A \ or \ B), \text{ and} \\ &\mu(A) + \mu(B) - \mu(A \ or \ B) = c_{11}^2 \geq 0. \end{aligned} \tag{4.86}$$

Hence, μ is classical disjunction data. The other implication is proven by taking $\mathbf{M}$ to be the same one-dimensional projector, $|C\rangle$ such that

$$\begin{aligned} c_{11} &= \sqrt{\mu(A) + \mu(B) - \mu(A \ or \ B)}, \\ c_{12} &= \sqrt{\mu(A \ or \ B) - \mu(B)}, \\ c_{21} &= \sqrt{\mu(A \ or \ B) - \mu(A)}, \\ c_{22} &= \sqrt{1 - \mu(A \ or \ B)}, \end{aligned} \tag{4.87}$$

and $\gamma_{ij} = 0$ for $i, j = 1, 2$. □

Theorems 4.5 and 4.7 give the strict equivalence between classical conjunction and disjunction data and the models of conjunctions and disjunctions built in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

4.3 Examples and Comparisons

In this section, we compare the scope of the two models developed in §4.1 and §4.2. In particular, we use the experimental data presented in

Hampton [Ham88b, Ham88a] to show examples of the two types of representations, and compute the number of conceptual combinations that each model can represent.

There are four different cases. The first case applies when estimations can be represented by both the Hilbert space and tensor product models, the next two cases when only one of the models can represent the data, and the last case, when none of the models can represent the data. For simplicity we show only explicit examples for data on conjunction.

The first example consists of concepts $\mathcal{A}$ =‘*Machine*’ and $\mathcal{B}$ =‘*Vehicle*,’ and the exemplar p_5 =‘sailboat.’ In this case, we have

$$\begin{aligned}
&\mu_5(A) = 0.56,\ \ \mu_5(B) = 0.8,\ \text{ and } \mu_5(A \textit{ and } B) = 0.42,\\
&\text{ave}_5(AB) = 0.68,\\
&\text{dev}_5(AB) = 0.297.
\end{aligned} \tag{4.88}$$

By Theorem 4.2, since

$$\text{ave}_5(AB) - \text{dev}_5(AB) \leq \mu_5(A \textit{ and } B) \leq \text{ave}_5(AB) + \text{dev}_5(AB), \tag{4.89}$$

the membership estimations can be modeled in the Hilbert space model. Because $\mu_5(A)+\mu_5(B) > 1$, $\mathbf{M}$ is a two-dimensional projector. We represent this case by choosing

$$\begin{aligned}
&|A\rangle = (-0.43 + 0.3\mathbb{i}, 0.02 - 0.53\mathbb{i}, 0.58 + 0.32\mathbb{i}),\ \text{ and}\\
&|B\rangle = (0.63, 0.63, 0.45).
\end{aligned} \tag{4.90}$$

In addition, since

$$\begin{aligned}
\mu_5(A) - \mu_5(A \textit{ and } B) &= 0.14,\\
\mu_5(B) - \mu_5(A \textit{ and } B) &= 0.38, \quad \text{and}\\
\mu_5(A) + \mu_5(B) - \mu_5(A \textit{ and } B) &= 0.06,
\end{aligned} \tag{4.91}$$

by Theorem 4.5 we can also construct a representation in the tensor space model. Here we take $\mathbf{M}$ to be a one-dimensional projector, and

$$|C\rangle = 0.64|1\rangle \otimes |1\rangle + 0.37|1\rangle \otimes |2\rangle + 0.62|2\rangle \otimes |1\rangle + 0.24|2\rangle \otimes |2\rangle. \tag{4.92}$$

In the second example, the data can be represented only in a Hilbert space model. Consider the concepts $\mathcal{A}$ ='*Machine*' and $\mathcal{B}$ ='*Vehicle*,' and the exemplar p_{12} ='skateboard.' We have

$$\begin{aligned}&\mu_{12}(A) = 0.28,\ \mu_{12}(B) = 0.84,\ \text{and}\ \mu_{12}(A\ \mathit{and}\ B) = 0.34,\\&\mathrm{ave}_{12}(AB) = 0.56,\\&\mathrm{dev}_{12}(AB) = 0.339.\end{aligned} \tag{4.93}$$

By Theorem 4.2, since

$$\mathrm{ave}_{12}(AB)-\mathrm{dev}_{12}(AB) \leq \mu_{12}(A\ \mathit{and}\ B) \leq \mathrm{ave}_{12}(AB)+\mathrm{dev}_{12}(AB), \tag{4.94}$$

the membership estimations can be modeled in the Hilbert space model:

$$\begin{aligned}&|A\rangle = (0.034 - 0.37\mathbb{i}, -0.37 - 0.026\mathbb{i}, 0.55 + 0.65\mathbb{i}),\ \text{and}\\&|B\rangle = (0.65, 0.65, 0.4).\end{aligned}$$

However, since $\mu_{12}(A\ \mathit{and}\ B) > \mu_{12}(A)$, this case cannot be modeled in the tensor product space.

In the third example, the data can only be represented in the tensor product model. Consider the concepts $\mathcal{A}$ ='*Bird*' and $\mathcal{B}$ ='*Pet*,' and the exemplar p_{14} ='goldfish.' We have

$$\begin{aligned}&\mu_{14}(A) = 0,\ \mu_{14}(B) = 1,\ \text{and}\ \mu_{14}(A\ \mathit{and}\ B) = 0,\\&\mathrm{ave}_{14}(AB) = 0.5,\\&\mathrm{dev}_{14}(AB) = 0.\end{aligned} \tag{4.95}$$

In this case, we cannot provide a Hilbert space representation of the data because

$$0 = \mu_{14}(A\ \mathit{and}\ B) < \mathrm{ave}_{14}(AB) - \mathrm{dev}_{14}(AB) = 0.5. \tag{4.96}$$

However, the data is compatible with the tensor product model since

$$\begin{aligned}\mu_{14}(A) - \mu_{14}(A\ \mathit{and}\ B) &= 0,\\\mu_{14}(B) - \mu_{14}(A\ \mathit{and}\ B) &= 1,\ \text{and}\\\mu_{14}(A) + \mu_{14}(B) - \mu_{14}(A\ \mathit{and}\ B) &= 1.\end{aligned} \tag{4.97}$$

We obtain a representation by setting $\mathbf{M}$ and $|C\rangle$ as follows:

$$\begin{aligned} &\mathbf{M}(x,y) \to (x,0), \\ &|C\rangle = |2\rangle \otimes |1\rangle. \end{aligned} \tag{4.98}$$

Finally, for data that cannot be modeled by either of the two models, consider again the concepts $\mathcal{A}$ =‘*Bird*’ and $\mathcal{B}$ =‘*Pet*,’ and the exemplar p_6 =‘heron.’ We have

$$\begin{aligned} &\mu_6(A) = 0.94,\ \mu_6(B) = 0.15,\ \text{and}\ \mu_6(A\ \mathit{and}\ B) = 0.26, \\ &\mathrm{ave}_6(AB) = 0.545, \\ &\mathrm{dev}_6(AB) = 0.225. \end{aligned} \tag{4.99}$$

By Theorem 4.2, since

$$0.26 = \mu(A\ \mathit{and}\ B) < \mathrm{ave}_6(AB) - \mathrm{dev}_6(AB) = 0.32, \tag{4.100}$$

we cannot provide a Hilbert space representation. Moreover, since

$$\mu_6(A\ \mathit{and}\ B) > \mu_6(A),$$

we cannot represent the data in the tensor product model.

We now compare the performance of the two models by counting the number of membership estimations that allow a representation in both a Hilbert space model and a tensor product model, in only one of the models, or in neither models, for all the concepts conjunctions and disjunctions tested by Hampton in [Ham88b, Ham88a]. Fig. 4.2 shows the relative frequency of membership estimations for each of these cases. The histogram on the left gives the relative frequency for the conjunction data, while the histogram on the right gives the relative frequencies for the disjunction data.

We observe that for both conjunctions and disjunctions approximately half of the cases cannot be modeled by either of the two models (52.4% and 42% respectively). Considering the cases that can be modeled for conjunctions, the Hilbert space model performs better since 41.6% of cases can be modeled by the Hilbert space model, and 19% of cases can be modeled only by the tensor product. Moreover, 12% of the cases can be modeled by both the tensor and Hilbert space models.

For the case of disjunctions, the Hilbert space model provides a representation in 42.7% of cases, and the tensor product model can represent 37% of the cases.

Although in almost half of the cases neither model is capable of providing a representation of the data, overall the Hilbert space model seems to perform better than the tensor product model. Moreover, since the equivalence established in Theorems 4.5 and 4.7 establishes the equivalence of the tensor product models for conjunctions and disjunctions and their classical probabilistic counterparts, we can conclude that the Hilbert space model is better suited for this type of data than the classical probabilistic models. It is important to note however that the tensor product model can represent some cases that cannot be represented by the Hilbert space model. This indicates that there is a need for a general model that incorporates both the tensor product and the Hilbert space models.

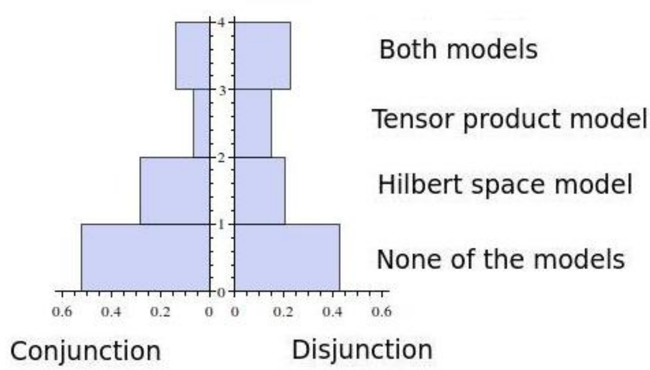

Figure 4.2: Relative frequency of experimental data that can be represented in the Hilbert space or tensor space models.

Chapter 5

Fock Space Modeling of Conjunctions and Disjunctions of Concepts

The Fock space formalism was developed in quantum mechanics to represent systems composed of a varying or unknown number of entities. In quantum theory, the state of a quantum entity is represented as a vector in a Hilbert space $\mathcal{H}$, and the state of a collection of k quantum entities is represented in the tensor product space $\otimes^k \mathcal{H}$. A Fock space $\mathcal{F}^*$ consists of a direct sum of these tensor products for all possible values of k:

$$\mathcal{F}^* = \oplus_{k=1}^{\infty} \otimes^k \mathcal{H}. \tag{5.1}$$

We use the Fock space structure to develop a model that brings together the two models of Chapter 4 from both a mathematical and a cognitive perspective.

5.1 The Two-sector Fock Space Model

In quantum cognition, the Fock space is used to represent different modes of reasoning in the modeling of concepts combinations. We will show a two-sector Fock space model that is a generalization of the Hilbert space and tensor product models developed in Chapter 4. In fact, both models are obtained as two extreme cases of the two-sector Fock space model, each representing a specific mode of reasoning. But before presenting the mathematical formulation of the two-sector Fock space model of concepts, we first take a closer look at the cognitive interpretation of the two models developed in Chapter 4.

5.1.1 Concept Combination in the Hilbert and Tensor Product Models: One or Two Instances in Mind?

Consider the experimental situation of a participant estimating the membership weight of an exemplar with respect to two concepts and their combination. For example, take the concepts '*Fruit*,' and '*Vegetable*,' and their conjunction '*Fruit and Vegetable*,' and suppose we would like to estimate the membership weight of the exemplar 'apple' with respect to the conjunction. When participants estimate the membership weight of 'apple' with respect to the concept conjunction '*Fruit and Vegetable*' two kinds of reasoning can be identified:

1. 'apple' being an exemplar of the concept '*Fruit and Vegetable*,'
2. 'apple' being an exemplar of the concept '*Fruit*,' and the concept '*Vegetable*,' separately.

In the first case, the membership weight of 'apple' is estimated with respect to the meaning of a single concept '*Fruit and Vegetable*.' Thus, a single instance of 'apple' is taken into consideration. In the second case, two instances of 'apple' are taken into consideration, one for each estimation. Namely, the first instance is estimated with respect to the meaning of '*Fruit*,' and the second instance with respect to the meaning of '*Vegetable*.'

To clarify the conceptual distinction between these two kinds of reasoning, note that the first case considers a concept that cannot be decomposed as a combination of '*Fruit*' and '*Vegetable*,' but is a single emergent concept. In the second case, the combined concept is decomposed into two concepts, and the membership weights for the two concepts are analyzed separately. Therefore, the second case corresponds to the traditional compositional understanding of the conjunction where logical rules of concept combinations operate.

Likewise, if we consider 'apple' with respect to '*Fruit*,' *Vegetable*,' and their disjunction '*Fruit or Vegetable*,' the same kinds of reasoning can be used to estimate membership weights. Therefore, we can identify two fundamentally different manners of reasoning about the membership of concept combinations: the first considers a concept combination as an emergent entity that cannot be logically decomposed, and the second considers a concept combination as a logically decomposable entity.

It is interesting that the Hilbert space and the tensor product models presented in Chapter 4 can be identified with these two kinds of reasoning. The first type of reasoning, modeled by the Hilbert space model, creates a concept combination state from the states of the former concepts, and the membership weight of the combined concept is related to the average of the membership weights of the two concepts plus an interference term. This model can represent non-logical effects such as overextension of conjunction and underextension of disjunction. Although it is sometimes compatible with classical data, there are multiple cases of classical data that cannot be represented for by this model. The second type of reasoning, modeled by the tensor product model, is of a logical nature. In fact, Theorems 4.5 and 4.7 show that the classical probabilistic model and the tensor product model are equivalent.

5.1.2 Introduction to Fock Space Modeling

Definition 5.1. Let $\mathcal{H}$ be a Hilbert space, and k be an integer. We define the k^{th} sector, $\mathcal{F}_k$, of a Fock space by

$$\mathcal{F}_k = \otimes_{i=1}^{k} \mathcal{H}. \tag{5.2}$$

Since we have identified two modes of reasoning, we use the first two sectors of the Fock space to model concept combinations. Namely, the first sector, $\mathcal{F}_1 = \mathcal{H}$, represents the emergent mode of reasoning previously modeled by the Hilbert space model, and the second sector, $\mathcal{F}_2 = \mathcal{H} \otimes \mathcal{H}$, represents the logical mode of reasoning previously modeled by the tensor product model. Hence, our model represents the concept combination in the space

$$\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}). \tag{5.3}$$

Let $|A\rangle$, and $|B\rangle$ be the states of the concepts $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{F}_1$, and let $|C\rangle$ be the state of the combination of concepts $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{F}_2$. Also, let $\mathbf{M} : \mathcal{H} \to \mathcal{H}$ be the membership operator associated with an exemplar x, and let $\mathbf{M}^A, \mathbf{M}^B, \mathbf{M}^\wedge$, and $\mathbf{M}^\vee$ be given by Eqs. (4.64)–(4.67). We have

$$\begin{aligned} \mu(A) &= \langle A|\mathbf{M}|\mathcal{A}\rangle = \langle C|\mathbf{M}^A|C\rangle \\ \mu(B) &= \langle B|\mathbf{M}|B\rangle = \langle C|\mathbf{M}^B|C\rangle. \end{aligned} \tag{5.4}$$

The state $|\psi_{AB}\rangle$ representing the concept combination is obtained as a superposition of modes of thought:

$$|\psi_{AB}\rangle = \frac{n_{AB}}{\sqrt{2}}(|A\rangle + |B\rangle) \oplus \sqrt{1 - n_{AB}^2}|C\rangle. \tag{5.5}$$

The first term of $|\psi_{AB}\rangle$ represents the probabilistic structure of the concept combination in the emergent mode of reasoning, which is the contribution from the first sector, $\mathcal{F}_1 = \mathcal{H}$. The second term of $|\psi_{AB}\rangle$ represents the probabilistic structure of the concept combination in the logical mode of reasoning, which is the contribution from the second sector, $\mathcal{F}_2 = \mathcal{H} \otimes \mathcal{H}$.

To estimate the membership weight of a concept combination, we construct membership operators in the two-sector Fock space:

$$\begin{aligned} \mathbf{M}^{\mathcal{F}_\wedge} &= \mathbf{M} \oplus \mathbf{M}^\wedge, \\ \mathbf{M}^{\mathcal{F}_\vee} &= \mathbf{M} \oplus \mathbf{M}^\vee. \end{aligned} \tag{5.6}$$

The membership weights for conjunctions and disjunctions of concepts in the two-sector Fock space model must satisfy the following conditions:

$$\mu(A \ and \ B) = \langle \psi_{AB}|\mathbf{M}^{\mathcal{F}_\wedge}|\psi_{AB}\rangle, \tag{5.7}$$

$$\mu(A \ or \ B) = \langle \psi_{AB}|\mathbf{M}^{\mathcal{F}_\vee}|\psi_{AB}\rangle. \tag{5.8}$$

Therefore, the formula for the membership weight of an exemplar with respect to the concept conjunction is obtained by applying conditions (4.4) and (4.76) to (5.7) as follows:

$$\begin{aligned} &\mu(\mathcal{A} \ and \ \mathcal{B}) = \langle \psi_{AB}|\mathbf{M}^{\mathcal{F}_\wedge}|\psi_{AB}\rangle \\ &= \langle \psi_{AB}|\mathbf{M} \oplus \mathbf{M} \otimes \mathbf{M}|\psi_{AB}\rangle \\ &= \frac{n_{AB}}{2}(((\langle A + \langle B)|\mathbf{M}(|A\rangle + |B\rangle)) + (1 - n_{AB}^2)(\langle C|)\mathbf{M} \otimes \mathbf{M}(|C\rangle) \\ &= n_{AB}\tilde{\mu}(A \ and \ B) + (1 - n_{AB}^2)\breve{\mu}(A \ and \ B). \end{aligned} \tag{5.9}$$

Similarly, we can measure the disjunction of the two concepts as follows:

$$\begin{aligned} &\mu(\mathcal{A} \ or \ \mathcal{B}) = \langle \psi_{AB}|\mathbf{M}^{\mathcal{F}_\vee}|\psi_{AB}\rangle \\ &= \frac{n_{AB}^2}{2}(((\langle A + \langle B)|\mathbf{M}(|A\rangle + |B\rangle)) + (1 - n_{AB}^2)(\langle C|)\mathbf{M}^A + \mathbf{M}^B - \mathbf{M}^\wedge(|C\rangle) \\ &= n_{AB}^2\tilde{\mu}(A \ or \ B) + (1 - n_{AB}^2)\breve{\mu}(A \ or \ B). \end{aligned} \tag{5.10}$$

Eqs. (5.9) and (5.10) show that $\mu(A\ and\ B)$ and $\mu(A\ or\ B)$ are given by a convex combination of the membership weight formulas for concept combination corresponding to each mode of reasoning. Since each sector must respect the constraints of its mode of reasoning, the contribution from the first sector is constrained as in Theorem 4.2:

$$\begin{aligned}\tilde{\mu}(A\ and\ B) &\in [\mathrm{ave}(AB) - \mathrm{dev}(AB), \mathrm{ave}(AB) + \mathrm{dev}(AB)],\\ \tilde{\mu}(A\ or\ B) &\in [\mathrm{ave}(AB) - \mathrm{dev}(AB), \mathrm{ave}(AB) + \mathrm{dev}(AB)].\end{aligned} \tag{5.11}$$

The membership weight contribution from the second sector is constrained by Theorems (4.5) and (4.7) as follows:

$$\breve{\mu}(A\ and\ B) \quad \in [\max(0, 1 - \mu(A) - \mu(B)), \min(\mu(A), \mu(B))], \tag{5.12}$$
$$\breve{\mu}(A\ or\ B) \quad \in [\max(\mu(A), \mu(B)), \min(1, \mu(A) + \mu(B))]. \tag{5.13}$$

The membership formulas in the two-sector Fock space model are convex combinations of membership formulas for two modes of thought. Thus, the two-sector Fock space model not only generalizes the Hilbert space and tensor product models, but can also represent cases that cannot be represented by either model.

For example, consider the concepts $\mathcal{A}$ =‘*Pet*,’ $\mathcal{B}$ =‘Bird,’ and the exemplar p_6 =‘heron.’ Recall that this case was given in § 4.3 as an example that could not be represented by either of the two models. In this case, we have $\mu(A) = 0.94$, $\mu(B) = 0.15$, and $\mu(A\ and\ B) = 0.26$. For simplicity, suppose that conditions (4.3) and (4.69) are satisfied. Then, by applying Theorems 4.2 and 4.5, the possible values $\tilde{\mu}(A\ and\ B)$ and $\breve{\mu}(A\ and\ B)$ are bounded by the intervals I_1 and I_2 respectively:

$$\begin{aligned}I_1 &= [\mathrm{ave}(AB) - \mathrm{dev}(AB), \mathrm{ave}(AB) + \mathrm{dev}(AB)] = [0.32, 0.77],\\ I_2 &= [\max(0, 1 - \mu(A) - \mu(B)), \min(\mu(A), \mu(B))] = [0.09, 0.15].\end{aligned} \tag{5.14}$$

Therefore, neither sector can represent this exemplar. However, this exemplar can be represented in the two-sector Fock space model because $\mu(A\ and\ B)$ belongs to the convex combination of I_1 and I_2:

$$[\min(I_1 \cup I_2), \max(I_1 \cup I_2)]. \tag{5.15}$$

In particular, Eq. (5.9) with the choices $\Re(\langle A|\mathbf{M}|B\rangle) = 0$ and $n_{AB} = 0.294554$ recovers the membership weight $\mu(A \textit{ and } B)$.

The two-sector Fock space model provides the extra parameter n_{AB}. We will show that this parameter becomes crucial in the construction of representations of multiple exemplars in concrete two-sector Fock spaces.

5.2 Data Representation of Multiple Exemplars

The two-sector Fock space model developed in § 5.1.2 provides an abstract model to represent concept combinations. This model outperforms the modeling scope of the Hilbert space and tensor product models. In the current literature, the concrete instantiations of this abstract model have always provided representations that are exemplar-dependent. This means that a different state is given for each exemplar, and a single membership operator represent the semantic estimations of all exemplars [Aer07a, Aer07b, Aer09]. Such concrete representations are useful to explain the way that state vectors, operators, and modes of reasoning operate in the two-sector Fock space model. However, they do not model concepts in accordance with the cognitive principles that have inspired the abstract model.

From a cognitive perspective, a concept is an entity in a state that is independent of the exemplar to be measured. In addition, since semantic estimations are used to compare concepts and exemplars, these estimations should depend on the exemplar being measured. Therefore, a concrete representation in the two-sector Fock space model must satisfy the following two modeling principles of quantum cognition:

1. Concepts are represented by a state that is independent of the exemplar to be measured,
2. Semantic estimations are represented by a measurement operator that depends on the exemplar to be measured.

The following examples show that the concrete representations provided in the literature disagree with these modeling principles. Consider the exemplars ‘filing cabinet’ and ‘heated waterbed’ with respect to the concepts $\mathcal{A}$ =‘*Furniture*,’ $\mathcal{B}$ =‘*Household Appliances*,’ and their conjunction $\mathcal{AB}$ =‘*Furniture and Household Appliances*’ [Ham88b]. For the first exemplar, we have $\mu(A) = 0.97$, $\mu(B) = 0.31$, and $\mu(A \textit{ and } B) = 0.53$. Applying Theorem 4.2, we represent this case in the space $\mathbb{C}^3$ by the vectors

$$\begin{aligned} &|A\rangle = (-0.57 + 0.40\mathbb{i}, 0.29 - 0.63\mathbb{i}, 0.13 + 0.11\mathbb{i}), \text{ and} \\ &|B\rangle = (0.39, 0.39, 0.83). \end{aligned} \tag{5.16}$$

For the second exemplar, we have $\mu(A) = 1$, $\mu(B) = 0.49$, and $\mu(A \textit{ and } B) = 0.78$. Applying Theorem 4.2 yields

$$\begin{aligned} &|A\rangle = (0.71, 0.71, 0), \text{ and} \\ &|B\rangle = (0.49, 0.49, 0.71). \end{aligned} \tag{5.17}$$

Because the vector states $|A\rangle$ and $|B\rangle$ are different for each exemplar, this concrete representation is exemplar-dependent. Moreover, because $\mu(A) + \mu(B) > 1$ for both cases, $\mathbf{M}$ is the operator that projects onto the first two dimensions of these vectors, so the same operator is used for two different exemplars.

The same situation occurs for the the tensor product model. For example, consider the exemplars 'sailboat' and 'roadroller' with respect to the concepts $\mathcal{A} =$'*Machine*,' $\mathcal{B} =$'*Vehicle*,' and their conjunction $\mathcal{AB} =$'*Machine and Vehicle*.' For the first exemplar, we have $\mu(A) = 0.56$, $\mu(B) = 0.8$, and $\mu(A \textit{ and } B) = 0.42$. Applying Theorem 4.5, we represent this case in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ by the tensor

$$|C\rangle = 0.64|1\rangle \otimes |1\rangle + 0.37|1\rangle \otimes |2\rangle + 0.62|2\rangle \otimes |1\rangle + 0.24|2\rangle \otimes |2\rangle. \tag{5.18}$$

For the second exemplar, we have $\mu(A) = 0.94$, $\mu(B) = 0.91$, and $\mu(A \textit{ and } B) = 0.91$. Applying Theorem 4.5 yields

$$|C\rangle = 0.95|1\rangle \otimes |1\rangle + 0.17|1\rangle \otimes |2\rangle + 0.24|2\rangle \otimes |2\rangle. \tag{5.19}$$

Moreover, $\mathbf{M}$ is a one-dimensional projector in both cases. Thus, the construction of concrete representations for concepts in the tensor product model is also exemplar-dependent. It is, however, possible to develop concrete representations for multiple exemplars that are consistent with the quantum modeling principles. To do so, we will exploit the linear structure of the Hilbert space and tensor space models and use special linear operators known as unitary transformations.

Unitary transformations embody the notion of isometry. This means that they do not affect the value of the inner product. Formally, given a unitary operator $\mathbf{U}$ and two vectors $|x\rangle$ and $|y\rangle$, we have

$$\langle \mathbf{U}x|\mathbf{U}y\rangle = \langle x|y\rangle. \tag{5.20}$$

An important consequence of Eq. (5.20) is that, when a unitary operator is applied to transform the state of a concept and a membership operator, the membership weight is preserved.

Unitary operators in a Hilbert space are a generalization of rotation matrices in linear algebra; they correspond to a change of basis for the vector states and for the operators in a Hilbert space. We will apply unitary transformations to the representations obtained from Theorems 4.2, 4.5, and 4.7 to obtain new representations where all the exemplars are represented in the same basis. Next, we will combine these representations to provide a representation in the two-sector Fock space model where the concept is identified with one single state, and the measurement operators are different for each exemplar.

Since the representational problems of conjunctions and disjunctions are similar, we focus on the case of conjunctions of concepts. In what follows, we denote the set of data $\{\mu_i(A), \mu_i(B), \mu_i(A \textit{ and } B)\}_{i=1}^k$ by $\mu_{i=1}^k$. A particular triplet $(\mu_i(A), \mu_i(B), \mu_i(A \textit{ and } B))$ will be denoted by μ_i, and the conjunction $\mu_i(A \textit{ and } B)$ will be denoted by $\mu_i(AB)$.

5.2.1 Hilbert Space Representation

We now show how to concretely represent multiple exemplars in the Hilbert space model using the space $\mathbb{C}^3$.

Definition 5.2.

Theorem 5.3. *The set of data $\mu_{i=1}^k$ has a representation in $\mathbb{C}^3$ if and only if for all $i = 1, ..., k$,*

$$\mu_i(AB) \in [\mathrm{ave}_i(AB) - \mathrm{dev}_i(AB), \mathrm{ave}_i(AB) + \mathrm{dev}_i(AB)]. \tag{5.21}$$

Proof. Let $|A\rangle = |1\rangle$, $|B\rangle = |2\rangle$, and $|C\rangle = |3\rangle$ form the canonical basis of $\mathbb{C}^3$. We prove that if (5.21) is satisfied for each $i = 1, ..., k$ then there exists an orthogonal projector $\mathbf{M}_i$ such that conditions (4.1)–(4.4) are satisfied for

$|A\rangle$, $|B\rangle$, and $\mathbf{M}_i$.

Let $i \in \{1, ..., k\}$. Since $\mu_i(A), \mu_i(B)$, and $\mu_i(AB)$ satisfy (5.21), by Theorem 4.2 there exist two vectors

$$\begin{aligned} |A_i\rangle &= (e^{\mathbb{i}\alpha_1}a_1, e^{\mathbb{i}\alpha_2}a_2, e^{\mathbb{i}\alpha_3}a_3), \text{ and} \\ |B_i\rangle &= (e^{\mathbb{i}\beta_1}b_1, e^{\mathbb{i}\beta_2}b_2, e^{\mathbb{i}\beta_3}b_3), \end{aligned} \tag{5.22}$$

and an orthogonal projector $\tilde{\mathbf{M}}_i$ such that (4.1)–(4.4) are satisfied.

Let

$$\begin{aligned} |C_i\rangle =& |A_i\rangle \times |B_i\rangle \\ =& (a_2 b_3 e^{-\mathbb{i}(\beta_3+\alpha_2)} - a_3 b_2 e^{-\mathbb{i}(\beta_2+\alpha_3)}, \\ & a_1 b_3 e^{-\mathbb{i}(\beta_3+\alpha_1)} - a_3 b_1 e^{-\mathbb{i}(\beta_1+\alpha_3)}, \\ & a_1 b_2 e^{-\mathbb{i}(\beta_2+\alpha_1)} - a_2 b_1 e^{-\mathbb{i}(\beta_1+\alpha_2)}). \end{aligned} \tag{5.23}$$

The vector $|C_i\rangle$ is chosen to complete an orthonormal basis for $\mathbb{C}^3$ from $|A_i\rangle$ and $|B_i\rangle$. This ensures that $|C_i\rangle \perp |A_i\rangle$, $|C_i\rangle \perp |B_i\rangle$, and $\||C_i\rangle\| = 1$. Now we define the operator $\mathbf{U}_i$ by

$$\mathbf{U}_i = \begin{pmatrix} \langle A_i|A\rangle & \langle A_i|B\rangle & \langle A_i|C\rangle \\ \langle B_i|A\rangle & \langle B_i|B\rangle & \langle B_i|C\rangle \\ \langle C_i|A\rangle & \langle C_i|B\rangle & \langle C_i|C\rangle \end{pmatrix}. \tag{5.24}$$

$\mathbf{U}_i$ is a unitary operator whose action induces a change from the basis $(|A_i\rangle, |B_i\rangle, |C_i\rangle)$ to the basis $(|A\rangle, |B\rangle, |C\rangle)$. In fact,

$$\mathbf{U}_i|A_i\rangle = |A\rangle,\ \mathbf{U}_i|B_i\rangle = |B\rangle, \text{ and } \mathbf{U}_i|C_i\rangle = |C\rangle.$$

We apply the operator $\mathbf{U}_i$ to represent $\tilde{\mathbf{M}}_i$ in the orthogonal basis $\{|A_i\rangle, |B_i\rangle, |C_i\rangle\}$. Set

$$\mathbf{M}_i = \mathbf{U}_i \tilde{\mathbf{M}}_i \mathbf{U}_i^{-1}. \tag{5.25}$$

$\mathbf{M}_i$ is the operator $\tilde{\mathbf{M}}_i$ represented in the basis $(|A\rangle, |B\rangle, |C\rangle)$. Since

$$\mathbb{1} = \mathbf{U}_i^{-1}\mathbf{U}_i = \mathbf{U}_i\mathbf{U}_i^{-1}, \tag{5.26}$$

we obtain

$$\begin{aligned}\mu_i(A) &= \langle A_i|\tilde{\mathbf{M}}_i|A_i\rangle = \langle A_i\mathbf{U}_i^{-1}|\mathbf{U}_i\tilde{\mathbf{M}}_i\mathbf{U}_i^{-1}|\mathbf{U}_iA_i\rangle = \langle A|\mathbf{M}_i|A\rangle,\\ \mu_i(B) &= \langle B_i|\tilde{\mathbf{M}}_i|B_i\rangle = \langle B_i\mathbf{U}_i^{-1}|\mathbf{U}_i\tilde{\mathbf{M}}_i\mathbf{U}_i^{-1}|\mathbf{U}_iB_i\rangle = \langle B|\mathbf{M}_i|B\rangle,\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}\mu_i(AB) &= \frac{1}{2}(\mu_i(A)+\mu_i(B)) + \Re(\langle A_i|\tilde{\mathbf{M}}_i|B_i\rangle)\\ &= \frac{1}{2}(\mu_i(A)+\mu_i(B)) + \Re(\langle A_i\mathbf{U}_i^{-1}|\mathbf{U}_i\tilde{\mathbf{M}}_i\mathbf{U}_i^{-1}|\mathbf{U}_iA_i\rangle)\\ &= \frac{1}{2}(\mu_i(A)+\mu_i(B)) + \Re(\langle A|\mathbf{M}_i|B\rangle).\end{aligned} \tag{5.28}$$

The other side of the implication is a direct consequence of Definition 5.2. □

Theorem 5.3 provides a data representation in terms of a single pair of vectors $|A\rangle$ and $|B\rangle$, and a set of projectors $\mathbf{M}_i,\ i = 1, ..., k$, corresponding to the membership operator for each exemplar.

Recall that the exemplars p ='filing cabinet,' and q ='heated waterbed' were represented by different state vectors and the same measurement operator in § 5.1.2. We can now apply Theorem 5.3 to obtain a representation consistent with the modeling principles of quantum cognition using the state vectors

$$\begin{aligned}|A\rangle &= (1,0,0) = |1\rangle,\\ |B\rangle &= (0,1,0) = |2\rangle,\end{aligned} \tag{5.29}$$

and two measurement operators corresponding to the exemplars p and q:

$$\begin{aligned}\mathbf{M}_p &= \begin{pmatrix} 0.97 & -0.11+0.09\mathbb{i} & 0.09+0.01\mathbb{i}\\ -0.11-0.09\mathbb{i} & 0.31 & 0.28+0.34\mathbb{i}\\ 0.09-0.01\mathbb{i} & 0.28-0.34\mathbb{i} & 0.72\end{pmatrix},\\ \mathbf{M}_q &= \begin{pmatrix} 1 & 0 & 0\\ 0 & 0.49 & 0.499\\ 0 & 0.499 & 0.51\end{pmatrix}.\end{aligned} \tag{5.30}$$

The construction introduced in the proof of Theorem 5.3 is independent of the choice of the vectors $|A\rangle$ and $|B\rangle$.

Corollary 5.4. *Let* $(|A\rangle, |B\rangle, \{\mathbf{M}_i)\}_{i=1}^k)$ *be a representation of* $\mu_{i=1}^k$ *in* $\mathbb{C}^3$*, and let* $|A'\rangle, |B'\rangle \in \mathbb{C}^3$ *be two orthogonal unit vectors. Then, there exists a unitary transformation* $\mathbf{U}$ *such that* $(|A'\rangle, |B'\rangle, \{\mathbf{U}^{-1}\mathbf{M}_i\mathbf{U})\}_{i=1}^k)$ *is a representation of* $\mu_{i=1}^k$*.*

Corollary 5.4 shows that all representations in $\mathbb{C}^3$ are equivalent up to a unitary transformation.

5.2.2 Tensor Product Model Representation

We now apply unitary transformations in the concrete representations of the tensor product model in $\mathbb{C}^n \otimes \mathbb{C}^n$. We first define different types of representations for multiple exemplars, and then provide explicit representation theorems for the cases $n = 2$ and 3. These will be useful to study the performance of the two-sector Fock space model.

Definition 5.5. A zero-type representation of $\mu_{i=1}^k$ on the tensor product space $\mathbb{C}^n \otimes \mathbb{C}^n$ is a unit vector $|C\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$, and a collection of orthogonal projectors $\{\mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k$ from $\mathbb{C}^n \otimes \mathbb{C}^n$ to $\mathbb{C}^n \otimes \mathbb{C}^n$, such that conditions (4.73)–(4.76) are satisfied with $\mathbf{M}_i^\wedge = \mathbf{M}_i^A\mathbf{M}_i^B$, for $i = 1, ..., k$. We say $(|C\rangle, \{\mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k)$ is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^n \otimes \mathbb{C}^n$.

The zero-type representation is, mathematically speaking, the most general representation in the tensor product model that is consistent with the modeling principles of quantum cognition because it assumes a single concept state $|C\rangle$, and a collection of measurements that represent the membership weight estimations. However, this representation cannot be appropriately interpreted because $\mathbf{M}_A$ and $\mathbf{M}_B$ can be entangled measurements[9].

A more reasonable representation of data assumes that the measurements $\mathbf{M}_A$ and $\mathbf{M}_B$ act on different sides of $\mathbb{C}^n \otimes \mathbb{C}^n$ so they are not entangled.

Definition 5.6. A first-type representation of $\mu_{i=1}^k$ on the tensor product space $\mathbb{C}^n \otimes \mathbb{C}^n$ is a unit vector $|C\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$, and a collection of orthogonal projectors $\mathbf{M}_i$ from $\mathbb{C}^n$ to $\mathbb{C}^n$, for $i = 1, ..., k$, such that $(|C\rangle, \{\mathbf{M}_i \otimes \mathbb{1}, \mathbb{1} \otimes \mathbf{M}_i\}_{i=1}^k)$ is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^n \otimes \mathbb{C}^n$.

The first-type representation is a direct extension of the representation of individual exemplars in Definition 4.4, and thus it is interpreted according to such representation: The state $|C\rangle$ describes the situation having two

[9]Entangled measurements appear in non-trivial analysis of entanglement in physics. A possible interpretation of entangled measurements in this model is left for future work.

concepts and their combination, and $\mathbf{M}_i$ represents the semantic estimation of exemplar p_i, $i = 1, ..., k$.

We now introduce another representation that is mathematically simpler, and thus will facilitate the data analysis.

Definition 5.7. A second-type representation of $\mu_{i=1}^k$ on the tensor product space $\mathbb{C}^n \otimes \mathbb{C}^n$ is a pair of unit vectors $|A\rangle$, and $B\rangle \in \mathbb{C}^n$, and a collection of orthogonal projectors $\mathbf{M}_i$ from $\mathbb{C}^n$ to $\mathbb{C}^n$, for $i = 1, ..., k$, such that $(|A\rangle \otimes |B\rangle, \{\mathbf{M}_i \otimes \mathbb{1}, \mathbb{1} \otimes \mathbf{M}_i\}_{i=1}^k)$ is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^n \otimes \mathbb{C}^n$.

The zero-, first-, and second-type representations require different conditions to represent a collection of exemplars for a pair of concepts and their conjunction. While the first-type corresponds to the natural way to represent a pair of systems in quantum physics, and thus is the natural way to define a representation in the tensor product model for concepts, the zero-type provides a general way to build concrete representations because it does not impose a product structure on the concept state or the membership operators for the exemplars. The second-type is a mathematical simplification of the first-type representation that assumes $|C\rangle$ to be a product state. In fact, it is trivial to deduce that a second-type representation is also a first-type representation, and a first-type representation is also a zero-type representation from Definitions 5.5–5.7 .

The following theorem characterizes the cases when a set of data has a zero-type representation in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Theorem 5.8. *The set of data $\mu_{i=1}^k$ has a zero-type representation in $\mathbb{C}^2 \otimes \mathbb{C}^2$ if and only if μ_i is classical conjunction data for $i = 1, ..., k$.*

Proof. For each $i = 1, ..., k$, we use the construction in the proof of Theorem 4.5 to obtain a tensor $|\tilde{C}_i\rangle$ and a one-dimensional projector $\tilde{\mathbf{M}}$ such that $\tilde{\mathbf{M}}_i^A = \tilde{\mathbf{M}} \otimes \mathbb{1}$, $\tilde{\mathbf{M}}_i^B = \mathbb{1} \otimes \tilde{\mathbf{M}}$, and $\tilde{\mathbf{M}}_i^\wedge = \tilde{\mathbf{M}} \otimes \tilde{\mathbf{M}}$. This gives the tensor product representation for μ_i. Next, we use unitary transformations to change this representation so that $|\tilde{C}_i\rangle$ is a vector in the canonical basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$. To facilitate the notation, we will make use of the isomorphism $\mathcal{I}$ between $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^4$. Let

$$\begin{aligned} (1,0,0,0) &= |e_1\rangle, \\ (0,1,0,0) &= |e_2\rangle, \\ (0,0,1,0) &= |e_3\rangle, \\ (0,0,0,1) &= |e_4\rangle. \end{aligned} \tag{5.31}$$

We define

$$\begin{aligned}
\mathcal{I}(|1\rangle \otimes |1\rangle) &= |e_1\rangle, \\
\mathcal{I}(|1\rangle \otimes |2\rangle) &= |e_2\rangle, \\
\mathcal{I}(|2\rangle \otimes |1\rangle) &= |e_3\rangle, \\
\mathcal{I}(|2\rangle \otimes |2\rangle) &= |e_4\rangle.
\end{aligned} \tag{5.32}$$

The isomorphism $\mathcal{I}$ allows us to represent $|\tilde{C}_i\rangle$ by a vector $|C_i\rangle$ in $\mathbb{C}^4$.

We can prove the theorem by building a unitary transformation that takes $|C_i\rangle$ to one of the canonical basis vectors of $\mathbb{C}^4$, and use this transformation to represent the operators $\tilde{\mathbf{M}}_i^A$, $\tilde{\mathbf{M}}_i^B$, and $\tilde{\mathbf{M}}_i^\wedge$ by the operators $\mathbf{M}_i^A$, $\mathbf{M}_i^B$, and $\mathbf{M}_i^\wedge$ in $\mathbb{C}^4$. Next, we apply the the inverse isomorphism $\mathcal{I}^{-1}$ to map these new representations to $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Let $|D_i\rangle, |E_i\rangle, |F_i\rangle$ be three vectors in $\mathbb{C}^4$ such that

$$\begin{aligned}
\langle D_i|D_i\rangle = \langle E_i|E_i\rangle = \langle F_i|F_i\rangle = 1, \\
\langle C_i|D_i\rangle = \langle C_i|E_i\rangle = \langle C_i|F_i\rangle = 0, \\
\langle D_i|E_i\rangle = \langle D_i|F_i\rangle = \langle E_i|F_i\rangle = 0.
\end{aligned} \tag{5.33}$$

The vectors $|C_i\rangle, |D_i\rangle, |E_i\rangle$, and $|F_i\rangle$ form an orthonormal basis for $\mathbb{C}^4$. Set

$$\mathbf{U}_i = \begin{pmatrix} \langle C_i|e_1\rangle & \langle C_i|e_2\rangle & \langle C_i|e_3\rangle & \langle C_i|e_4\rangle \\ \langle D_i|e_1\rangle & \langle D_i|e_2\rangle & \langle D_i|e_3\rangle & \langle D_i|e_4\rangle \\ \langle E_i|e_1\rangle & \langle E_i|e_2\rangle & \langle E_i|e_3\rangle & \langle E_i|e_4\rangle \\ \langle F_i|e_1\rangle & \langle F_i|e_2\rangle & \langle F_i|e_3\rangle & \langle F_i|e_4\rangle \end{pmatrix}. \tag{5.34}$$

Note that $\mathbf{U}_i$ is a unitary matrix whose action induces a change from the basis $\{|C_i\rangle, |D_i\rangle, |E_i\rangle, |F_i\rangle\}$ to the basis $\{|e_j\rangle\}_{j=1}^4$. In fact,

$$\mathbf{U}_i|C_i\rangle = |e_1\rangle,\ \mathbf{U}_i|D_i\rangle = |e_2\rangle\ \mathbf{U}_i|E_i\rangle = |e_3\rangle, \text{ and } \mathbf{U}_i|F_i\rangle = |e_4\rangle.$$

The operator $\mathbf{U}_i$ can now be used to change the basis in which $\mathbf{M}_i^A$, $\mathbf{M}_i^B$, and $\mathbf{M}_i^\wedge$ are represented to the basis $\{|e_j\rangle\}_{j=1}^4$:

$$\begin{aligned}
\bar{\mathbf{M}}_i^A &= \mathbf{U}_i \mathbf{M}_i^A \mathbf{U}_i^{-1}, \\
\bar{\mathbf{M}}_i^B &= \mathbf{U}_i \mathbf{M}_i^B \mathbf{U}_i^{-1}, \\
\bar{\mathbf{M}}_i^\wedge &= \mathbf{U}_i \mathbf{M}_i^\wedge \mathbf{U}_i^{-1}.
\end{aligned} \tag{5.35}$$

Since $\mathbb{1} = \mathbf{U}_i^{-1}\mathbf{U}_i = \mathbf{U}_i\mathbf{U}_i^{-1}$, we obtain

$$\begin{aligned}
\mu_i(A) &= \langle C_i|\mathbf{M}_i^A|C_i\rangle = \langle C_i\mathbf{U}_i^{-1}|\mathbf{U}_i\mathbf{M}_i^A\mathbf{U}_i^{-1}|\mathbf{U}_iC_i\rangle = \langle e_1|\bar{\mathbf{M}}_i^A|e_1\rangle,\\
\mu_i(B) &= \langle C_i|\mathbf{M}_i^B|C_i\rangle = \langle C_i\mathbf{U}_i^{-1}|\mathbf{U}_i\mathbf{M}_i^B\mathbf{U}_i^{-1}|\mathbf{U}_iC_i\rangle = \langle e_1|\bar{\mathbf{M}}_i^B|e_1\rangle,\\
\mu_i(AB) &= \langle C_i|\mathbf{M}_i^\wedge|C_i\rangle = \langle C_i\mathbf{U}_i^{-1}|\mathbf{U}_i\mathbf{M}_i^\wedge\mathbf{U}_i^{-1}|\mathbf{U}_iC_i\rangle = \langle e_1|\bar{\mathbf{M}}_i^\wedge|e_1\rangle.
\end{aligned} \tag{5.36}$$

We then use the inverse isomorphism $\mathcal{I}^{-1}$ to obtain a zero-type representation in $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\begin{aligned}
|C\rangle &= \mathcal{I}^-1(|e_1\rangle) = |1\rangle \otimes |1\rangle,\\
\tilde{\mathbf{M}}_i^A &= \mathcal{I}^{-1}\bar{\mathbf{M}}_i^A\mathcal{I},\\
\tilde{\mathbf{M}}_i^B &= \mathcal{I}^{-1}\bar{\mathbf{M}}_i^B\mathcal{I},\\
\tilde{\mathbf{M}}_i^\wedge &= \mathcal{I}^{-1}\bar{\mathbf{M}}_i^\wedge\mathcal{I}.
\end{aligned} \tag{5.37}$$

□

We have constructed a zero-type representation $(|1\rangle\otimes|1\rangle, \{\mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k)$ from a collection of representations $(|C_i\rangle, \mathbf{M})$ for the exemplars p_i with $\mathbf{M}(x, y) \to (x, 0)$ obtained from Theorem 4.5.

In the construction of Theorem 5.8, note that when Eq. (5.37) entails operators $\mathbf{M}_i^A$ and $\mathbf{M}_i^B$ that are of the form $\mathbf{M}_A^i = \check{\mathbf{M}}_i \otimes \mathbb{1}$ and $\mathbf{M}_B^i = \mathbb{1} \otimes \check{\mathbf{M}}_i$, then the representation is also of the first-type. Stating the necessary and sufficient conditions required for a set of data to have first-type representation is out of the scope of this thesis. However, since second-type are also first-type representations, we can obtain sufficient conditions for the existence of a first-type representation by characterizing the conditions required for the data to have a second-type representation:

Lemma 5.9. *The set of data $\mu_{i=1}^k$ has a second-type representation in $\mathbb{C}^2 \otimes \mathbb{C}^2$ if and only if for each $i = 1, ..., k$, there exist $|A_i\rangle$, $|B_i\rangle$, $\check{\mathbf{M}}_A^i$, and $\check{\mathbf{M}}_B^i$ such that Eqs.* (4.62)–(4.66) *are satisfied.*

Proof. Let $\mathbf{U}_i(A)$ and $\mathbf{U}_i(B)$ be the unitary transformations that map $|A_i\rangle$ to $|1\rangle$ and $|B_i\rangle$ to $|1\rangle$ for $i = 1, ..., k$. Then $(|1\rangle \otimes |1\rangle, \{\mathbf{M}_i^A \otimes \mathbb{1}, \mathbb{1} \otimes \mathbf{M}_i^B\}_{i=1}^k)$ is a tensor space zero-type representation of $\mu_{i=1}^k$ with

$$\begin{aligned}
\mathbf{M}_i^A &= \mathbf{U}_i(A)^{-1}\check{\mathbf{M}}_i^A\mathbf{U}_i(A),\\
\mathbf{M}_i^B &= \mathbf{U}_i(B)^{-1}\check{\mathbf{M}}_i^B\mathbf{U}_i(B).
\end{aligned} \tag{5.38}$$

□

Note that Theorem 5.8 and Lemma 5.9 characterize the sets of data that have a zero- and second-type representations. Since the first-type representation is less general than the zero-type representation, but more general than the second-type representation, Theorem 5.8 and Lemma 5.9 can be applied to obtain an upper and lower bound on the number of exemplars that have a first-type representation.

We need to extend Theorem 5.8 to $\mathbb{C}^3 \otimes \mathbb{C}^3$ so the zero-, first-, and second-type representations become compatible with the representation developed in § 5.2.1 for a Hilbert space model in $\mathbb{C}^3$. The next corollary extends the proof of Theorem 5.8 to the space $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Corollary 5.10. *If the set of data $\mu_{i=1}^k$ has a zero-type representation in $\mathbb{C}^2 \otimes \mathbb{C}^2$, then $\mu_{i=1}^k$ has a zero-type representation in $\mathbb{C}^3 \otimes \mathbb{C}^3$.*

Proof. Let $(|C\rangle, \{\mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k)$ be a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$. We can create a vector

$$|C^*\rangle = \sum_{i,j=1}^{3} c_{ij}^* |i\rangle \otimes |j\rangle \tag{5.39}$$

such that it is the trivial embedding of

$$|C\rangle = \sum_{i,j=1}^{2} c_{ij} |i\rangle \otimes |j\rangle \tag{5.40}$$

in $\mathbb{C}^3 \otimes \mathbb{C}^3$ by choosing

$$c_{ij}^* = \begin{cases} c_{ij} & i, j \in \{1, 2\}, \\ 0 & \text{else.} \end{cases} \tag{5.41}$$

Similarly, we can also create operators $\mathbf{M}_i^{A^*}$ and $\mathbf{M}_i^{B^*}$ by using the trivial embedding so that the actions of the operators $\mathbf{M}_i^A$ and $\mathbf{M}_i^B$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ are preserved. This completes the proof. □

Since second-type representations are also first- and zero-type representations, we can apply Corollary 5.10 to obtain a first- and second-type representation in $\mathbb{C}^3 \otimes \mathbb{C}^3$.

5.2.3 Two-sector Fock Space Representation

We now combine the representations of multiple exemplars developed in § 5.2.1 and § 5.2.2 to represent sets of data in the two-sector Fock space model in a way that is consistent with the modeling principles of quantum cognition in the concrete space $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$.

Definition 5.11. A zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$ consists of the vectors $|A\rangle, |B\rangle \in \mathbb{C}^3$ and $|C\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$, a collection of operators $\{\mathbf{M}_i, \mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k$ from $\mathbb{C}^3 \otimes \mathbb{C}^3$ to $\mathbb{C}^3 \otimes \mathbb{C}^3$, and a coefficient $n_{AB} \in [0,1]$ such that for all $i = 1,..,,k$, condition (5.4) is satisfied, and the vector $|\psi_{AB}\rangle$, defined in Eq. (5.5), and the operator $\mathbf{M}_i^{\mathcal{F}\wedge}$, defined in Eq. (5.6), satisfy condition (5.7). We say that $(n_{AB}, |A\rangle, |B\rangle, |C\rangle, \{\mathbf{M}_i, \mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k)$ is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$.

Definition 5.12. A first-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$ consists of a tensor $|C\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$, a collection of operators $\{\mathbf{M}_i\}_{i=1}^k$ from $\mathbb{C}^3$ to $\mathbb{C}^3$, and a coefficient $n_{AB} \in [0,1]$ such that $(n_{AB}, |A\rangle, |B\rangle, |C\rangle, \{\mathbf{M}_i, \mathbf{M}_i \otimes \mathbb{1}, \mathbb{1} \otimes \mathbf{M}_i\}_{i=1}^k)$ is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$. We say that $(n_{AB}, |A\rangle, |B\rangle, \{\mathbf{M}_i\}_{i=1}^k)$ is a first-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$.

Definition 5.13. A second-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$ consists of the vectors $|A\rangle, |B\rangle \in \mathbb{C}^3$, a collection of operators $\{\mathbf{M}_i\}_{i=1}^k$ from $\mathbb{C}^3$ to $\mathbb{C}^3$, and a coefficient $n_{AB} \in [0,1]$ such that $(n_{AB}, |A\rangle, |B\rangle, |A\rangle \otimes |B\rangle, \{\mathbf{M}_i, \mathbf{M}_i \otimes \mathbb{1}, \mathbb{1} \otimes \mathbf{M}_i\}_{i=1}^k)$ is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$. We say that $(n_{AB}, |A\rangle, |B\rangle, \{\mathbf{M}_i\}_{i=1}^k)$ is a second-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$.

We now identify the conditions for a zero- and second-type representation of a set of data in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$.

Theorem 5.14. *The set of data $\mu_{i=1}^k$ admits a zero-type representation in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$ if and only if there exists $n_{AB} \in [0,1]$ such that for all $i = 1, ..., k$*

$$\mu_i(AB) = n_{AB}\tilde{\mu}_i(AB) + \sqrt{1 - n_{AB}^2}\breve{\mu}_i(AB), \tag{5.42}$$

with $\tilde{\mu}_{i=1}^k$ satisfying conditions (4.1)–(4.4)*, and $\breve{\mu}_{i=1}^k$ is classical conjunction data.*

Proof. Since μ_i satisfies conditions (4.1)–(4.4) for $i = 1, ..., k$, we apply Theorem 5.3, and Corollary 5.4 to obtain the representation $(|A\rangle, |B\rangle, \{\mathbf{M}_i\}_{i=1}^k)$ of $\tilde{\mu}_{i=1}^k$ in $\mathbb{C}^3$ with $|A\rangle = |1\rangle$ and $|B\rangle = |2\rangle$. Similarly, since μ_i is classical data for $i = 1, ..., k$, we apply Theorem 5.8 and Corollary 5.10 to obtain a zero-type representation $(|C\rangle, \{\mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k)$ of $\breve{\mu}_{i=1}^k$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Next, let $i = 1, ..., k$. We apply Eq. (5.44), to show that the state $|\psi_{AB}\rangle$ satisfies

$$\begin{aligned}\langle\psi_{AB}|\mathbf{M}_i^{\mathcal{F}\wedge}|\psi_{AB}\rangle &= \frac{n_{AB}^2}{2}\left(\langle A| + \langle B|\right)\mathbf{M}_i(|A\rangle + |B\rangle) + (1 - n_{AB}^2)\langle C|\mathbf{M}_i^{\wedge}|C\rangle \\ &= n_{AB}^2\tilde{\mu}_i(AB) + (1 - n_{AB}^2)\breve{\mu}_i(AB) = \mu(AB).\end{aligned} \tag{5.43}$$

This completes the proof. □

This result can be similarly obtained for the second-type representation.

Corollary 5.15. *The set of data $\mu_{i=1}^k$ admits a second-type representation in $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$ if and only if there exists $n_{AB} \in [0, 1]$ such that for all $i = 1, ..., k$*

$$\mu_i(AB) = n_{AB}\tilde{\mu}_i(AB) + \sqrt{1 - n_{AB}^2}\breve{\mu}_i(AB), \tag{5.44}$$

with $\tilde{\mu}_i$ satisfying conditions (4.1)–(4.4), and $\breve{\mu}_i(AB) = \mu_i(A)\mu_i(B)$.

Proof. The result follows from the proof of Theorem 5.14 replacing $\breve{\mu}_i(AB)$ by $\mu_i(A)\mu_i(B)$. □

Theorem 5.14 shows that a zero-type representation of a set of data that respects the modeling principles of quantum cognition requires the existence of a value for n_{AB} such that the convex combination of the membership weights $\tilde{\mu}(AB)$, representing the contribution given by $\mathbb{C}^3$, and $\breve{\mu}(AB)$, representing the contribution given by $\mathbb{C}^3 \otimes \mathbb{C}^3$, are equal to the membership weight $\mu_i(AB)$, for $i = 1, ..., k$.

The second-type representation additionally imposes the conditions that the membership weight operators for concepts $\mathcal{A}$ and $\mathcal{B}$ act separately on the two sides of $\mathbb{C}^3 \otimes \mathbb{C}^3$, and that $|C\rangle = |A\rangle \otimes |B\rangle$.

As an example of how to identify whether or not a collection of exemplars can be represented, consider the exemplars p_1 ='filing cabinet' and

p_{11} ='painting' for the concepts $\mathcal{A}$ ='*Furniture*,' $\mathcal{B}$ ='*Household Appliances*,' and their conjunction $\mathcal{A}$ *and* $\mathcal{B}$ ='*Furniture and Household Appliances*' [Ham88b]. The membership values are

$$\begin{aligned} &\mu_1(A) = 0.97,\ \mu_1(B) = 0.31,\ \mu_1(AB) = 0.53,\ \text{and} \\ &\mu_2(A) = 0.62,\ \mu_2(B) = 0.05,\ \mu_2(AB) = 0.11. \end{aligned} \tag{5.45}$$

A simple calculation shows that we can obtain a separate zero-type representation of μ_1 and μ_2 by choosing $n_{AB} \in [0.3215, 1]$, and $n_{AB} \in [0.119, 0.692]$, respectively. Therefore, a zero-type representation of $\mu_{i=1}^2$ requires $n_{AB} \in [0.3215, 0.692]$.

5.3 Data Representation Analysis

In this section, we provide an analysis of Hampton's data on conjunction to compare the two-sector Fock space model to both the Hilbert space and tensor product models presented in Chapter 4. We identify how many exemplars can be represented by the zero- and second-type of representations in the Fock space using Theorem 5.14 and Corollary 5.15 respectively. This give us a upper and lower bound on the number of exemplars that can be simultaneously represented by the first-type representation.

For the representation of individual exemplars, the zero- and second-type of representations in the two-sector Fock space can model 78.1% and 77% of the exemplars in the data set respectively. This is an improvement over the performance of the Hilbert space model (41%) and of the tensor product model (20%).

For the representation of multiple exemplars, Figure 5.1 shows the fraction of exemplars that can be simultaneously represented for each value of n_{AB} for the set of concept conjunctions tested by Hampton.

We now elaborate on six general statements, inferred from these graphs, that explain how the two-sector Fock space model developed in § 5.1.2 is an improvement of the Hilbert space and tensor product models.

The first statement is that at the extreme values, $n_{AB} = 0$ or 1, corresponding to the tensor product and Hilbert space models respectively, the model recovers the previous performances, 19% for the tensor product model and 41.6% for the Hilbert space model, obtained in the analysis of Chapter 4.

The second statement is that, as we expected, the zero-type representation performs better than the second-type representation. However, the difference is small. Since we know that the first-type representation is more constrained than the zero-type representation, but less constrained than the second-type representation, we conclude that the performance of the first-type representation should be similar to the performance of the second-type representation.

The third statement is that there is a small decrease in performance for n_{AB} between 0 and 0.3. This implies that the logical-based representation performs better than the superposition of logic and emergent thought, when logical thought is dominant.

The fourth statement is that there a is steady improvement for n_{AB} be-

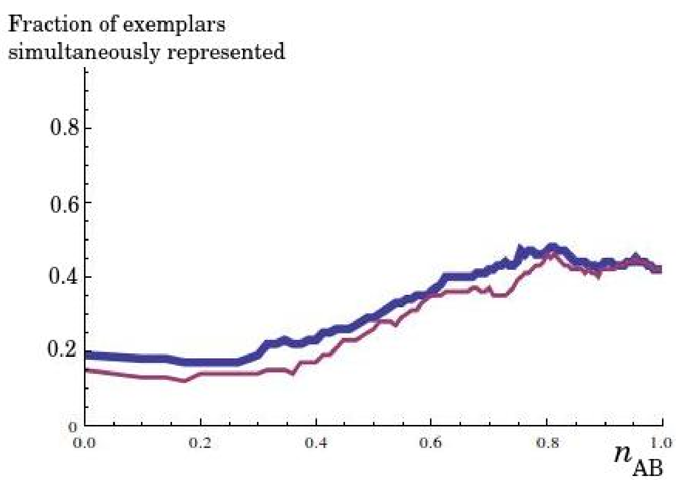

Figure 5.1: Fraction of Hampton's experimental data that can be simultaneously modeled in the two-sector Fock space model for different values of n_{AB}. The blue and red curves correspond to the fraction of exemplars that can be simultaneously modeled using the zero-type and second-type representations respectively.

tween 0.3 and 0.8. This implies that in this range, the stronger the influence of the emergent mode of thought, the better the performance.

The fifth statement is that there is a slight decrease in the performance for n_{AB} between 0.8 and 0.9, and that the performance remain stable for $n_{AB} > 0.9$. This implies that the maximal performance of the two-sector Fock space model is reached at a value of n_{AB} close to 0.8.

The sixth statement is the two-sector Fock space model outperforms the tensor product model for $0.3 \leq n_{AB} \leq 1$, and outperforms the Hilbert space model for $0.7 \leq n_{AB} \leq 0.9$.

We conclude that the two-sector Fock space model gives a better performance for the representation of individual exemplar and multiple exemplars simultaneously, and that the model reaches its best performance when the first sector is dominant at $n_{AB} \sim 0.8$.

Chapter 6

Fock Space Modeling of Negations and Conjunctions of Concepts

The Fock space model, introduced in Chapter 5, uses the idea of a superposition of modes of thought to model conjunctions and disjunctions. Since conjunction, disjunction, and negation are the primary operations in logic, we now consider how the negations of concepts are represented in the Fock space model. We restrict our analysis to conjunctions and negations because this is the only experimental data available to contrast theory with data.

The notation for this chapter is as follows: Let $\mathcal{A}$ and $\mathcal{B}$ be two concepts and let p_i be an exemplar. We denote the negation of concept $\mathcal{A}$ by $\bar{\mathcal{A}} =$ Not $\mathcal{A}$, and the conjunctions '$\mathcal{A}$ *and* $\mathcal{B}$,' '$\bar{\mathcal{A}}$ *and* $\mathcal{B}$,' '$\mathcal{A}$ *and* $\bar{\mathcal{B}}$,' and '$\bar{\mathcal{A}}$ *and* $\bar{\mathcal{B}}$,' by $\mathcal{AB}$, $\bar{\mathcal{A}}\mathcal{B}$, $\mathcal{A}\bar{\mathcal{B}}$, and $\bar{\mathcal{A}}\bar{\mathcal{B}}$, respectively. We denote the set of data for the membership weights of $\mathcal{A}$, $\mathcal{B}$, $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$, and their conjunctions by

$$\mu_i = \{\mu_i(A), \mu_i(B), \mu_i(\bar{A}), \mu_i(\bar{B}), \mu_i(AB), \mu_(\bar{A}B), \mu_i(A\bar{B}), \mu_i(\bar{A}\bar{B})\}, \quad (6.1)$$

and the set of data for the exemplars p_i, for $i = 1, ..., k$ by $\mu_{i=1}^k$.

First, we develop a theoretical analysis that characterizes classical data for the case of conjunctions and negations. Next, we introduce experimental data showing that concept combinations involving negations of concepts do not satisfy the conditions of classical data. Finally, we develop an extension of the model presented in Chapter 5 to represent conjunctions and negations of concepts, and we give some examples of data representation.

6.1 Conditions for a Classical Model

We now introduce the conditions for data representation within a classical Kolmogorovian probability model, and use this information to characterize classical data for the case of concept combinations involving conjunctions and negations.

Definition 6.1. The set of data μ_i is a classical data set, or classical data, if and only if there exists a Kolmogorovian probability space $(\Omega, \sigma(\Omega), P)$ and events $E_A, E_B \in \sigma(\Omega)$ such that

$$P(E_A) = \mu_i(A), \tag{6.2}$$
$$P(E_B) = \mu_i(B), \tag{6.3}$$
$$P(\Omega \setminus E_A) = \mu_i(\bar{A}), \tag{6.4}$$
$$P(\Omega \setminus E_B) = \mu_i(\bar{B}), \tag{6.5}$$
$$P(E_A \cap E_B) = \mu_i(AB), \tag{6.6}$$
$$P(E_A \cap (\Omega \setminus E_B)) = \mu_i(A\bar{B}), \tag{6.7}$$
$$P((\Omega \setminus E_A) \cap E_B) = \mu_i(\bar{A}B), \tag{6.8}$$
$$P((\Omega \setminus E_A) \cap (\Omega \setminus E_B)) = \mu_i(\bar{A}\bar{B}). \tag{6.9}$$

Note that the conditions for classical data for conjunctions and negations contain the conditions for classical data for conjunctions given by Theorem 2.2. Indeed, because

$$E_A = (E_A \cap E_B) \cup (E_A \cap (\Omega / E_B)), \tag{6.10}$$

combining Eqs. (6.6) and (6.7) yields

$$\mu_i(A) = \mu_i(AB) + \mu_i(A\bar{B}). \tag{6.11}$$

From this $\mu_i(AB) \leq \mu_i(A)$. The other two conditions for classical data for conjunctions can be obtained similarly.

Moreover, note that Eqs. (6.2)–(6.9) imply that a concept and its negation entail 'opposite' membership evaluations. For example, from Eqs. (6.2) and (6.4), we have

$$\mu_i(A) + \mu_i(\bar{A}) = P(E_A) + P(\Omega/E_A) = P(E_A \cup (\Omega/E_A)) = P(\Omega) = 1. \tag{6.12}$$

In addition, Eqs. (6.11) and (6.12) provide examples of the marginal probability law (see Eq. (A.4) in Appendix A.3).

We now identify a set of conditions for μ_i to be classical data.

Theorem 6.2. *The set of data μ_i is classical data if and only if*

$$\mu_i(A) + \mu_i(\bar{A}) = 1, \tag{6.13}$$
$$\mu_i(B) + \mu_i(\bar{B}) = 1, \tag{6.14}$$
$$\mu_i(A) = \mu_i(AB) + \mu_i(A\bar{B}), \tag{6.15}$$
$$\mu_i(B) = \mu_i(AB) + \mu_i(\bar{A}B), \tag{6.16}$$
$$\mu_i(\bar{A}) = \mu_i(\bar{A}\bar{B}) + \mu_i(\bar{A}B), \tag{6.17}$$
$$\mu_i(\bar{B}) = \mu_i(\bar{A}\bar{B}) + \mu_i(A\bar{B}). \tag{6.18}$$

Proof. Since μ_i is classical data, Eqs. (6.2)–(6.9) are satisfied. Therefore, the marginal probability formulas, given by conditions (6.15)–(6.18), are directly satisfied. Moreover, since $P(\Omega) = 1$, we add (6.2) and (6.4) to obtain condition (6.13), and add conditions (6.3) and (6.5) to obtain condition (6.14).

Now suppose that μ_i satisfies Eqs. (6.13)–(6.18). We need to prove that there exists a probability space, $(\Omega, \sigma(\Omega), P)$, that satisfies (6.2)–(6.9). Consider the set $\Omega = \{1, 2, 3, 4\}$, and let $\sigma(\Omega) = \mathcal{P}(\Omega)$ be the set of all subsets of Ω. Set

$$P(\{1\}) \;= \mu_i(AB), \tag{6.19}$$
$$P(\{2\}) \;= \mu_i(A\bar{B}), \tag{6.20}$$
$$P(\{3\}) \;= \mu_i(\bar{A}B), \tag{6.21}$$
$$P(\{4\}) \;= \mu_i(\bar{A}\bar{B}), \tag{6.22}$$

and for any arbitrary subset $S \subseteq \{1, 2, 3, 4\}$, define

$$P(S) = \sum_{a \in S} P(\{a\}). \tag{6.23}$$

From Eqs. (6.15)–(6.18), we obtain

$$\begin{aligned} E_A &= \{1, 2\}, \\ E_B &= \{1, 3\}, \\ E_{\bar{A}} &= \{3, 4\}, \\ E_{\bar{B}} &= \{2, 4\}. \end{aligned} \tag{6.24}$$

It is easy to verify that, given these choices, Eqs. (6.2)–(6.9) are satisfied. Since a Kolmogorovian probability space additionally requires that $P(\Omega) = 1$, we apply Eq. (6.13) to obtain

$$P(\Omega) = P(\{1, 2, 3, 4\}) = 1. \tag{6.25}$$

This completes the proof. □

Theorem 6.2 characterizes classical data using the marginal probability law for estimations of membership weights for individual and combined concepts. We now introduce an alternative form that will be useful to measure the deviations in the experimental data.

Definition 6.3. Let

$$\Lambda_A = 1 - \mu_i(A) - \mu_i(\bar{A}), \tag{6.26}$$

$$\Lambda_B = 1 - \mu_i(B) - \mu_i(\bar{B}), \tag{6.27}$$

$$I_A = \mu_i(A) - \mu_i(AB) - \mu_i(A\bar{B}), \tag{6.28}$$

$$I_B = \mu_i(B) - \mu_i(AB) - \mu_i(\bar{A}B), \tag{6.29}$$

$$I_{\bar{A}} = \mu_i(\bar{A}) - \mu_i(\bar{A}\bar{B}) - \mu_i(\bar{A}B), \tag{6.30}$$

$$I_{\bar{B}} = \mu_i(\bar{B}) - \mu_i(\bar{A}\bar{B}) - \mu_i(A\bar{B}), \tag{6.31}$$

$$I_{AB\bar{A}\bar{B}} = 1 - \mu_i(AB) - \mu_i(A\bar{B}) - \mu_i(\bar{A}B) - \mu_i(\bar{A}\bar{B}). \tag{6.32}$$

The following result summarizes the conditions for classical data using the parameters defined in Eqs. (6.28)–(6.32).

Corollary 6.4. *The set of data μ_i is classical conjunction data if and only if*

$$I_{AB\bar{A}\bar{B}} = I_A = I_B = I_{\bar{A}} = I_{\bar{B}} = 0. \tag{6.33}$$

Proof. We obtain Eq. (6.13) by combining $I_{AB\bar{A}\bar{B}} = 0$ with Eqs. (6.28) and (6.30). Similarly, we obtain Eq. (6.14) by combining $I_{AB\bar{A}\bar{B}} = 0$ with Eqs. (6.29) and (6.31). Next, $I_A = 0$ implies Eq. (6.15). Similarly, $I_B = I_{\bar{A}} = I_{\bar{B}} = 0$ implies Eqs. (6.15)–(6.18). □

6.2 Experiment on Conjunctions and Negations of Concepts

We use an experimental data set μ_i obtained for four pairs of concepts[10]. In this experiment, participants fill a questionnaire in which they have to estimate the membership of different exemplars with respect to concepts and concept combinations involving conjunctions and negations. The pairs of concepts considered in the experiment are

$$\begin{aligned}
&(\mathcal{A}_1, \mathcal{B}_1) = (\text{‘}\textit{Home Furnishing}\text{’},\text{‘}\textit{Furniture}\text{’}),\\
&(\mathcal{A}_2, \mathcal{B}_2) = (\text{‘}\textit{Spices}\text{’},\text{‘}\textit{Herbs}\text{’}),\\
&(\mathcal{A}_3, \mathcal{B}_3) = (\text{‘}\textit{Pets}\text{’},\text{‘}\textit{Farmyard Animals}\text{’}), \text{ and}\\
&(\mathcal{A}_4, \mathcal{B}_4) = (\text{‘}\textit{Fruits}\text{’},\text{‘}\textit{Vegetables}\text{’}).
\end{aligned} \tag{6.34}$$

The membership of 24 exemplars was tested for each pair of concepts. The data is shown in Appendix B. The choice of exemplars and concepts was inspired by Hampton's experiments for concept disjunctions [Ham88b].

The methodology for the experiment is that of the "within-subjects" design. That is, all participants were exposed to the same conditions. The set of 24 exemplars was assigned to all participants. In the experiment, participants were requested to estimate the membership of the 24 exemplars in the following order: i) $\mathcal{A}_j, \mathcal{B}_j$, and $\mathcal{A}_j\mathcal{B}_j$, ii) $\mathcal{A}_j, \bar{\mathcal{B}}_j$, and $\mathcal{A}_j\bar{\mathcal{B}}_j$, iii) $\bar{\mathcal{A}}_j, \mathcal{B}_j$, and $\bar{\mathcal{A}}_j\mathcal{B}_j$, and iv) $\bar{\mathcal{A}}_j, \bar{\mathcal{B}}_j$, and $\bar{\mathcal{A}}_j\bar{\mathcal{B}}_j$, for $j = 1, ..., 4$.

The membership weight of exemplars was estimated using the scale

$$\{-3, -2, -1, 0, +1, +2, +3\}, \tag{6.35}$$

where the extreme values -3 and $+3$ indicate strong non-membership and strong membership respectively, and zero, the inability to decide.

The analysis presented in this chapter uses only the data for membership vs non-membership in the interval $[0, 1]$. Therefore, we average the membership estimations asssuming a value equal to 0 with each negative response, 0.5 with each response equal to zero, and $+1$ with each positive response.

[10] The experiment was tested on 40 participants, and was carried out by a collaborator, Sandro Sozzo.

6.2.1 Results

A statistical analysis for the data suggests two strong tendencies:

1. membership estimations satisfy conditions (6.13) and (6.14) for classical data, and

2. membership estimations violate conditions (6.15)–(6.18), and the value of the deviation is approximately constant.

To support the first claim, we give the 95% confidence interval for the deviations Λ_A and Λ_B defined in Eqs. (6.26) and (6.27) respectively for the set of exemplars of each concept combination. In all cases, the deviations fall within a narrow band that is very close to zero. In fact, because $\mu_i(A)$ and $\mu_i(\bar{A})$ have values between $[0,1]$, Λ_A and Λ_B are contained in an interval of length 2. However, we see in Table 6.1 that the experimental data falls within an interval that is of length smaller than 0.05 with 95% certainty. Moreover, the center of the interval is also contained in a narrow region between the values -0.016 (for Λ_B, and $i = 2$) and -0.105 (for Λ_B and $i = 1$). This result confirms that participants' reasoning about the membership of exemplars with respect to individual concepts and their negations obeys the rules of classical logic and probability.

To support the second claim, we give the 95% confidence interval for I_A, I_B, $I_{\bar{A}}$, and $I_{\bar{B}}$ defined in Eqs. (6.28)–(6.31) for the set of exemplars of each concept combination.

In all cases, the deviations fall within a narrow band of similar values. In fact, since all the values of the data set μ_i are contained in the interval $[0,1]$, I_A, I_B, $I_{\bar{A}}$, and $I_{\bar{B}}$ must fall in an interval of length 2. However, the experimental data falls within an interval that is of length smaller than 0.09, and whose center is between -0.471 (I_B, and $j = 4$), and -0.274 ($I_{\bar{B}}$, and $j = 4$) with 95% certainty. This result confirms that participants' reasoning about the membership with respect to concept combinations deviates from

Table 6.1: 95% confidence interval for Λ_A and Λ_B for the data on conjunctions and negations in Tables B.1–B.4, Appendix B.

95% CI	$j = 1$	$j = 2$	$j = 3$	$j = 4$
Λ_A	$(-0.074, -0.032)$	$(-0.064, -0.037)$	$(-0.034, -0.014)$	$(-0.036, 0.000)$
Λ_B	$(-0.125, -0.078)$	$(-0.038, 0.005)$	$(-0.041, -0.012)$	$(-0.047, -0.023)$

the rules of classical logic and probability, and the value of the deviation is approximately constant.

The pattern of deviations from classicality can be further refined by looking separately at I_X for $X = A, B$, and I_Y, for $Y = \bar{A}, \bar{B}$. If we consider the average and variance of the extremes of the intervals, we obtain the interval

$$(-0.460 \pm 0.0006, -0.395 \pm 0.0001), \tag{6.36}$$

for I_X with $X = A, B$, and

$$(-0.371 \pm 0.0029, -0.311 \pm 0.0035), \tag{6.37}$$

for I_Y with $Y = \bar{A}, \bar{B}$.

Note that while the length of both average intervals is approximately 0.06, the center of these intervals is different. In particular, the center for the case $X = A, B$ is 0.428, and 0.341 for the case $Y = \bar{A}, \bar{B}$. Moreover, although the variances are small in both cases, it is one order of magnitude smaller for $X = A, B$. This means that the violations of the conditions for classical data have larger value and are more pronounced in the membership estimations of concepts than in the estimations of the negated concepts.

To visualize these patterns, we show the extreme values of the 95% confidence interval for I_A, $I_{\bar{A}}$, I_B, and $I_{\bar{B}}$ in Fig. 6.1. Blue points denote the interval for concepts and red points for the negated concepts. For example, the interval I_A for $i = 1$ is the blue point with coordinates $(-0.469, -0.406)$.

It is easy to observe that the blue points are more concentrated, and further from the origin than the red points. This visually confirms the anal-

Table 6.2: 95% confidence interval for I_A, $I_B, I_{\bar{A}}$, and $I_{\bar{B}}$, for the data on conjunctions and negations in Tables B.1–B.4, Appendix B.

95% CI	$j = 1$	$j = 2$	$j = 3$	$j = 4$
I_A	$(-0.469, -0.406)$	$(-0.476, -0.390)$	$(-0.426, -0.349)$	$(-0.463, -0.398)$
I_B	$(-0.482, -0.427)$	$(-0.443, -0.376)$	$(-0.429, -0.368)$	$(-0.495, -0.446)$
$I_{\bar{A}}$	$(-0.458, -0.393)$	$(-0.375, -0.326)$	$(-0.332, -0.259)$	$(-0.359, -0.302)$
$I_{\bar{B}}$	$(-0.390, -0.329)$	$(-0.429, -0.387)$	$(-0.323, -0.241)$	$(-0.298, -0.251)$

ysis above: the deviation from classical data is larger for the membership estimations of concepts than for those of negated concepts.

6.3 Fock Space Modeling of Conjunctions and Negations

Since the data for concept combinations involving conjunctions and negations does not satisfy the conditions for classical data, a quantum model is necessary. We extend the model for concept conjunctions presented in §5.1.2 to the negations of concepts. To construct this model extension, we introduce a set of conditions that relate a concept to its negation in the two sectors of the Fock space model, and add them to the conditions obtained for the case of conjunctions.

First, we introduce some notation to facilitate the presentation. Let $X = \mathcal{A}$ or $\bar{\mathcal{A}}$ and $Y = \mathcal{B}$ or $\bar{\mathcal{B}}$ and set

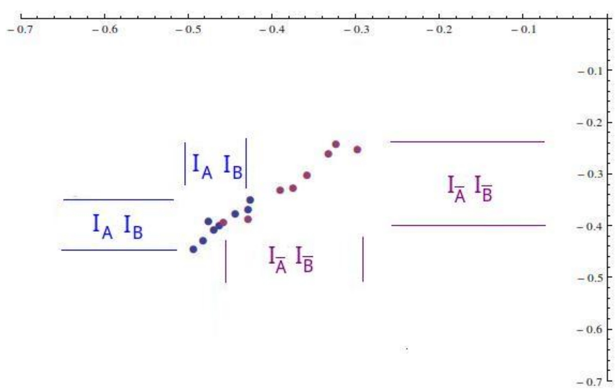

Figure 6.1: Representation of intervals I_A and I_B on blue, and of intervals $I_{\bar{A}}$ and $I_{\bar{B}}$ on red.

$$\begin{aligned} h_{\min}(XY) &= \mathrm{ave}(XY) - \mathrm{dev}(XY), \\ h_{\max}(XY) &= \mathrm{ave}(XY) + \mathrm{dev}(XY), \\ t_{\min}(XY) &= \max(0, 1 - \mu_i(X) - \mu_i(Y)), \\ t_{\max}(XY) &= \min(\mu_i(X), \mu_i(Y)). \end{aligned} \tag{6.38}$$

6.3.1 First Sector Analysis

Recall that the requirements for a model for the concepts $\mathcal{A}$, $\mathcal{B}$, and their conjunction $\mathcal{AB}$ are

$$\langle A|A\rangle = \langle B|B\rangle = 1, \tag{6.39}$$

$$\langle A|B\rangle = 0, \tag{6.40}$$

$$\mu_i(A) = \langle A|\mathbf{M}|A\rangle,\ \ \mu_i(B) = \langle B|\mathbf{M}|B\rangle, \tag{6.41}$$

$$\mu_i(AB) = \frac{1}{2}(\mu_i(A) + \mu_i(B)) + \Re(\langle A|\mathbf{M}|B\rangle). \tag{6.42}$$

To extend the model for negations, we represent the state of the conceptual negations $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ by the vectors $|\bar{A}\rangle$ and $|\bar{B}\rangle$ respectively, and require that the set $\{|A\rangle, |B\rangle, |\bar{A}\rangle, |\bar{B}\rangle\}$ forms an orthonormal set. Moreover, we require that

$$\mu_i(\bar{A}) = \langle \bar{A}|\mathbf{M}|\bar{A}\rangle,\ \text{and} \quad \mu_i(\bar{B}) = \langle \bar{B}|\mathbf{M}|\bar{B}\rangle. \tag{6.43}$$

Next, we build the state for the concept combinations as a superposition of states

$$|XY\rangle = \frac{1}{\sqrt{2}}(|X\rangle + |Y\rangle), \tag{6.44}$$

and extend condition (6.42) to the other concept combinations:

$$\mu_i(XY) = \frac{1}{2}(\mu_i(X) + \mu_i(Y)) + \Re\langle X|\mathbf{M}|Y\rangle. \tag{6.45}$$

To measure negated concepts, we use the standard negated operator $\mathbf{M}^{\perp}$ from quantum theory [BVN75]:

$$\mathbf{M}^{\perp} = \mathbb{1} - \mathbf{M}. \tag{6.46}$$

Because we consider four orthogonal vectors $|A\rangle$, $|\bar{A}\rangle$, $|B\rangle$, and $|\bar{B}\rangle$, and two projection operators $\mathbf{M}$ and $\mathbb{1} - \mathbf{M}$, the maximal number of subspaces we can obtain is eight[11]. Therefore, we set $\mathcal{H} = \mathbb{C}^8$.

Definition 6.5. The set of data μ_i has a representation in the Hilbert space $\mathbb{C}^8$ if and only if there exist vectors $|A\rangle, |B\rangle, |\bar{A}\rangle, |\bar{B}\rangle \in \mathbb{C}^8$, and $\mathbf{M} : \mathbb{C}^8 \to \mathbb{C}^8$ such that Eqs. (6.39)–(6.45) are satisfied.

The following theorem summarizes the type of data that can be represented by this model.

Theorem 6.6. *The set of data μ_i has a representation in the Hilbert space $\mathbb{C}^8$ if and only if*

$$\mu_i(XY) \in [h_{\min}(XY), h_{\max}(XY)], \tag{6.47}$$

for $X = A$ or $\bar{A}$, $Y = B$ or $\bar{B}$.

Proof. Let the set $\{|1\rangle, |2\rangle, ..., |8\rangle\}$ denote the canonical basis of $\mathbb{C}^8$, and define the operators $\mathbf{M}$ and $\mathbb{1} - \mathbf{M}$ by

$$\begin{aligned} \mathbf{M}((x_1, ..., x_8)) &= (0, 0, 0, 0, x_5, x_6, x_7, x_8), \text{ and} \\ \mathbb{1} - \mathbf{M}((x_1, ..., x_8)) &= (x_1, x_2, x_3, x_4, 0, 0, 0, 0). \end{aligned} \tag{6.48}$$

If we set

$$|A\rangle = e^{i\phi_A}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8), \tag{6.49}$$
$$|\bar{A}\rangle = e^{i\phi_{\bar{A}}}(a'_1, a'_2, a'_3, a'_4, a'_5, a'_6, a'_7, a'_8), \tag{6.50}$$
$$|B\rangle = e^{i\phi_B}(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8), \tag{6.51}$$
$$|\bar{B}\rangle = e^{i\phi_{\bar{B}}}(b'_1, b'_2, b'_3, b'_4, b'_5, b'_6, b'_7, b'_8), \tag{6.52}$$

[11]Since we have shown in § 6.2.1 that $\mu_i(A) + \mu_i(\bar{A})$ and $\mu_i(B) + \mu_i(\bar{B})$ are usually very close to one, a good approximation of μ_i could be constructed using less than eight independent subspaces.

then since these vectors must be orthonormal, we have

$$\langle A|\bar{A}\rangle = \textstyle\sum_{i=1}^{8} a_i a'_i = 0, \tag{6.53}$$
$$\langle B|\bar{B}\rangle = \textstyle\sum_{i=1}^{8} b_i b'_i = 0, \tag{6.54}$$
$$\langle A|B\rangle = \textstyle\sum_{i=1}^{8} a_i b_i = 0, \tag{6.55}$$
$$\langle A|\bar{B}\rangle = \textstyle\sum_{i=1}^{8} a_i b'_i = 0, \tag{6.56}$$
$$\langle \bar{A}|B\rangle = \textstyle\sum_{i=1}^{8} a'_i b_i = 0, \tag{6.57}$$
$$\langle \bar{A}|\bar{B}\rangle = \textstyle\sum_{i=1}^{8} a'_i b'_i = 0. \tag{6.58}$$
$$\langle A|A\rangle = \textstyle\sum_{i=1}^{8} a_i a_i = 1, \tag{6.59}$$
$$\langle B|B\rangle = \textstyle\sum_{i=1}^{8} b_i b_i = 1, \tag{6.60}$$
$$\langle \bar{A}|\bar{A}\rangle = \textstyle\sum_{i=1}^{8} a'_i a'_i = 1, \tag{6.61}$$
$$\langle \bar{B}|\bar{B}\rangle = \textstyle\sum_{i=1}^{8} b'_i b'_i = 1. \tag{6.62}$$

We use Eqs (6.41) and (6.43) to compute the membership weights:

$$\mu_i(A) = \langle A|\mathbf{M}|A\rangle = a_5^2 + a_6^2 + a_7^2 + a_8^2, \tag{6.63}$$
$$1 - \mu_i(A) = \langle A|\mathbb{1} - \mathbf{M}|A\rangle = a_1^2 + a_2^2 + a_3^2 + a_4^2, \tag{6.64}$$
$$\mu_i(\bar{A}) = \langle \bar{A}|\mathbf{M}|\bar{A}\rangle = a'^2_5 + a'^2_6 + a'^2_7 + a'^2_8, \tag{6.65}$$
$$1 - \mu_i(\bar{A}) = \langle \bar{A}|\mathbb{1} - \mathbf{M}|\bar{A}\rangle = a'^2_1 + a'^2_2 + a'^2_3 + a'^2_4, \tag{6.66}$$
$$\mu_i(B) = \langle B|\mathbf{M}|B\rangle = b_5^2 + b_6^2 + b_7^2 + b_8^2, \tag{6.67}$$
$$1 - \mu_i(B) = \langle B|\mathbb{1} - \mathbf{M}|B\rangle = b_1^2 + b_2^2 + b_3^2 + b_4^2, \tag{6.68}$$
$$\mu_i(\bar{B}) = \langle \bar{B}|\mathbf{M}|\bar{B}\rangle = b'^2_5 + b'^2_6 + b'^2_7 + b'^2_8, \tag{6.69}$$
$$1 - \mu_i(\bar{B}) = \langle \bar{B}|\mathbb{1} - \mathbf{M}|\bar{B}\rangle = b'^2_1 + b'^2_2 + b'^2_3 + b'^2_4. \tag{6.70}$$

Next, Eqs. (6.42) yields

$$\begin{aligned} \mu_i(AB) &= \frac{1}{2}(\mu_i(A) + \mu_i(B)) + \Re\langle A|\mathbf{M}|B\rangle \\ &= \mathrm{ave}(AB) + \sum_{i=5}^{8} a_i b_i \cos(\phi_B - \phi_A), \end{aligned} \tag{6.71}$$

$$\begin{aligned} \mu_i(A\bar{B}) &= \frac{1}{2}(\mu_i(A) + \mu_i(\bar{B})) + \Re\langle A|\mathbf{M}|\bar{B}\rangle \\ &= \mathrm{ave}(A\bar{B}) + \sum_{i=5}^{8} a_i b'_i \cos(\phi_{\bar{B}} - \phi_A), \end{aligned} \tag{6.72}$$

$$\mu_i(\bar{A}B) = \frac{1}{2}(\mu_i(\bar{A}) + \mu_i(B)) + \Re\langle \bar{A}|\mathbf{M}|B\rangle$$

$$= \text{ave}(\bar{A}B) + \sum_{i=5}^{8} a_i' b_i \cos(\phi_B - \phi_{\bar{A}}), \tag{6.73}$$

$$\mu_i(\bar{A}\bar{B}) = \frac{1}{2}(\mu_i(\bar{A}) + \mu_i(\bar{B})) + \Re\langle \bar{A}|\mathbf{M}|\bar{B}\rangle$$

$$= \text{ave}(\bar{A}\bar{B}) + \sum_{i=5}^{8} a_i' b_i' \cos(\phi_{\bar{B}} - \phi_{\bar{A}}). \tag{6.74}$$

To finish the proof, we need to show that the interference terms for the combinations XY are bounded by $\text{dev}(XY)$. We apply the Cauchy-Schwarz lemma to the interference term in Eq. (6.71) to obtain

$$|\Re(\langle A|\mathbf{M}|B\rangle)| \leq \sqrt{\mu_i(A)\mu_i(B)}. \tag{6.75}$$

Eq. (6.56) implies that

$$0 = \langle A|B\rangle = \langle A|\mathbf{M}|B\rangle + \langle A|\mathbb{1} - \mathbf{M}|B\rangle. \tag{6.76}$$

Since the real and imaginary parts of Eq. (6.76) must be zero, we obtain from the real part

$$\Re(\langle A|\mathbf{M}|B\rangle)^2 = (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)^2 \cos^2(\phi_B - \phi_A), \tag{6.77}$$

and apply Cauchy-Scwharz lemma to obtain

$$\Re(\langle A|\mathbf{M}|B\rangle)^2 \leq (a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2)\cos(\phi_B - \phi_A)^2. \tag{6.78}$$

Thus,

$$|\Re(\langle A|\mathbf{M}|B\rangle)| \leq \sqrt{(1 - \mu_i(A))(1 - \mu_i(B))}. \tag{6.79}$$

Combining Eqs. (6.75) and (6.79) yields

$$|\Re(\langle A|\mathbf{M}|B\rangle)| \leq \sqrt{\min(\mu_i(A)\mu_i(B), (1 - \mu_i(A))(1 - \mu_i(B))}, \tag{6.80}$$

and thus

$$\mu_i(AB) \in [h_{\min}(AB), h_{\max}(AB)]. \tag{6.81}$$

We repeat this procedure with Eqs. (6.72)–(6.74) to obtain Eq. (6.47). The other side of the implication follows directly from the construction. □

Similarly to § 5.2.1, we can obtain a representation in $\mathbb{C}^8$ that is compatible with the quantum modeling principles by applying unitary transformations to the representations of individual exemplars.

Definition 6.7. A Hilbert space representation of $\mu_{i=1}^k$ is a four-tuple of unit vectors $|A\rangle, |B\rangle, |\bar{A}\rangle, |\bar{B}\rangle \in \mathbb{C}^8$, and a collection of orthogonal projectors $\mathbf{M}_i : \mathbb{C}^3 \to \mathbb{C}^3$, for $i = 1, ..., k$, such that conditions (4.1)–(4.4) are satisfied for $i = 1, ..., k$. We say $(|A\rangle, |B\rangle, |\bar{A}\rangle, |\bar{B}\rangle, \{\mathbf{M}_i\}_{i=1}^k)$ is a representation of $\mu_{i=1}^k$ in $\mathbb{C}^8$.

The following is a corollary of Theorem 5.3 for the representation in Definition 6.7.

Corollary 6.8. *The set of data $\mu_{i=1}^k$ has a representation in $\mathbb{C}^8$ if and only if for all $i = 1, ..., k$,*

$$\mu_i(AB) \in [h_{\min}(AB), h_{\max}(AB)]. \tag{6.82}$$

Proof. The proof follows the same construction as in Theorem 5.3 starting from four vectors in $\mathbb{C}^8$ instead of two vectors in $\mathbb{C}^3$. □

6.3.2 Second Sector Analysis

To ensure that the two sectors of the two-sector Fock space model are compatible, we use $\mathbb{C}^8 \otimes \mathbb{C}^8$ for the second sector. In order to represent an exemplar, the vector $|C\rangle$ describes the conceptual situation where concepts and their combinations are jointly represented, and projection operators measure the membership weight for each combination. Let $|C\rangle$ be a unit vector in $\mathbb{C}^8 \otimes \mathbb{C}^8$. That is,

$$|C\rangle = \sum_{i,j=1}^{8} c_{ij} e^{i\gamma_{ij}} |i\rangle \otimes |j\rangle, \tag{6.83}$$

and

$$\begin{aligned}\langle C|C\rangle &= (\sum_{k,l=1}^{8} c_{kl} e^{-i\gamma_{kl}} \langle k| \otimes \langle l|)(\sum_{i,j=1}^{8} c_{ij} e^{i\gamma_{ij}} |i\rangle \otimes |j\rangle) \\ &= \sum_{k,l=1}^{8} \sum_{i,j=1}^{8} c_{kl} c_{ij} e^{i(\gamma_{ij}-\gamma_{kl})} \langle k|i\rangle\langle l|j\rangle \\ &= \sum_{i,j=1}^{8} c_{ij} c_{ij} e^{i(\gamma_{ij}-\gamma_{ij})} = \sum_{i,j=1}^{8} c_{ij}^2 = 1.\end{aligned} \tag{6.84}$$

We extend the membership operator $\mathbf{M}$, defined in Eq. (6.48), to the tensor product using $\mathbf{M}_A = \mathbf{M} \otimes \mathbb{1}$, $\mathbf{M}_B = \mathbb{1} \otimes \mathbf{M}$, and $\mathbf{M}_{AB} = \mathbf{M}_A \mathbf{M}_B$. Likewise, the operators that measure membership for concepts and conjunctions involving negated concepts are defined as follows:

$$\begin{aligned} \mathbf{M}_{\bar{A}} &= (\mathbb{1} - \mathbf{M}) \otimes \mathbb{1}, \\ \mathbf{M}_{\bar{B}} &= \mathbb{1} \otimes (\mathbb{1} - \mathbf{M}), \\ \mathbf{M}_{\bar{A}B} &= \mathbf{M}_{\bar{A}} \mathbf{M}_B, \\ \mathbf{M}_{A\bar{B}} &= \mathbf{M}_A \mathbf{M}_{\bar{B}}, \\ \mathbf{M}_{\bar{A}\bar{B}} &= \mathbf{M}_{\bar{A}} \mathbf{M}_{\bar{B}}. \end{aligned} \tag{6.85}$$

Therefore, the formulas for the membership weight for the concepts $\mathcal{A}$ and $\mathcal{B}$ are

$$\begin{aligned} \mu_i(A) &= \langle C|\mathbf{M}_A|C\rangle = \langle C|\mathbf{M} \otimes \mathbb{1}|C\rangle \\ &= (\sum_{k,l=1}^{8} c_{kl} e^{-i\gamma_{kl}} \langle k| \otimes \langle l|)|\mathbf{M} \otimes \mathbb{1}|(\sum_{i,j=1}^{8} c_{ij} e^{i\gamma_{ij}} |i\rangle \otimes |j\rangle) \\ &= \sum_{k,l=1}^{8} \sum_{i,j=1}^{8} c_{kl} c_{ij} e^{i(\gamma_{ij} - \gamma_{kl})} \langle k|\mathbf{M}|i\rangle \langle l|\mathbb{1}|j\rangle \\ &= \sum_{i,j,k=1}^{8} c_{kj} c_{ij} e^{i(\gamma_{ij} - \gamma_{kj})} \langle k|\mathbf{M}|i\rangle \\ &= \sum_{i=5}^{8} \sum_{j=1}^{8} c_{ij} c_{ij} e^{i(\gamma_{ij} - \gamma_{ij})} = \sum_{i=5}^{8} \sum_{j=1}^{8} c_{ij}^2, \end{aligned} \tag{6.86}$$

and

$$
\begin{aligned}
\mu_i(B) &= \langle C|\mathbf{M}_B|C\rangle = \langle C|\mathbb{1}\otimes\mathbf{M}|C\rangle \\
&= (\sum_{k,l=1}^{8} c_{kl}e^{-i\gamma_{kl}}\langle k|\otimes\langle l|)\mathbb{1}\otimes\mathbf{M}|(\sum_{i,j=1}^{8} c_{ij}e^{i\gamma_{ij}}|i\rangle\otimes|j\rangle) \\
&= \sum_{k,l=1}^{8}\sum_{i,j=1}^{8} c_{kl}c_{ij}e^{i(\gamma_{ij}-\gamma_{kl})}\langle k|\mathbb{1}|i\rangle\langle l|\mathbf{M}|j\rangle \\
&= \sum_{i,j,l=1}^{8} c_{il}c_{ij}e^{i(\gamma_{ij}-\gamma_{il})}\langle l|\mathbf{M}|j\rangle \\
&= \sum_{i=1}^{8}\sum_{j=5}^{8} c_{ij}c_{ij}e^{i(\gamma_{ij}-\gamma_{ij})} = \sum_{i=1}^{8}\sum_{j=5}^{8} c_{ij}^2.
\end{aligned}
\tag{6.87}
$$

The membership weight formulas of the negated concepts $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are

$$
\begin{aligned}
\mu_i(\bar{A}) &= 1-\mu_i(A) = \langle C|\mathbf{M}_{\bar{A}}|C\rangle = \langle C|(\mathbb{1}-\mathbf{M})\otimes\mathbb{1}|C\rangle \\
&= (\sum_{k,l=1}^{8} c_{kl}e^{-i\gamma_{kl}}\langle k|\otimes\langle l|)(\mathbb{1}-\mathbf{M})\otimes\mathbb{1}|(\sum_{i,j=1}^{8} c_{ij}e^{i\gamma_{ij}}|i\rangle\otimes|j\rangle) \\
&= \sum_{k,l=1}^{8}\sum_{i,j=1}^{8} c_{kl}c_{ij}e^{i(\gamma_{ij}-\gamma_{kl})}\langle k|\mathbb{1}-\mathbf{M}|i\rangle\langle l|\mathbb{1}|j\rangle \\
&= \sum_{i,j,k=1}^{8} c_{kj}c_{ij}e^{i(\gamma_{ij}-\gamma_{kj})}\langle k|\mathbb{1}-\mathbf{M}|i\rangle \\
&= \sum_{i=1}^{4}\sum_{j=1}^{8} c_{ij}c_{ij}e^{i(\gamma_{ij}-\gamma_{ij})} = \sum_{i=1}^{4}\sum_{j=1}^{8} c_{ij}^2,
\end{aligned}
\tag{6.88}
$$

and

$$
\begin{aligned}
\mu_i(\bar{B}) &= 1-\mu_i(B) = \langle C|\mathbf{M}_{\bar{A}}|C\rangle = \langle C|\mathbb{1}\otimes(\mathbb{1}-\mathbf{M})|C\rangle \\
&= (\sum_{k,l=1}^{8} c_{kl}e^{-i\gamma_{kl}}\langle k|\otimes\langle l|)\mathbb{1}\otimes(\mathbb{1}-\mathbf{M})|(\sum_{i,j=1}^{8} c_{ij}e^{i\gamma_{ij}}|i\rangle\otimes|j\rangle) \\
&= \sum_{k,l=1}^{8}\sum_{i,j=1}^{8} c_{kl}c_{ij}e^{i(\gamma_{ij}-\gamma_{kl})}\langle k|\mathbb{1}|i\rangle\langle l|\mathbb{1}-\mathbf{M}|j\rangle \\
&= \sum_{i,j,l=1}^{8} c_{il}c_{ij}e^{i(\gamma_{ij}-\gamma_{il})}\langle l|\mathbb{1}-\mathbf{M}|j\rangle
\end{aligned}
$$

$$= \sum_{i=1}^{8}\sum_{j=1}^{4} c_{ij}c_{ij}e^{i(\gamma_{ij}-\gamma_{ij})} = \sum_{i=1}^{8}\sum_{j=1}^{4} c_{ij}^2. \tag{6.89}$$

And the membership weight formulas for concept combinations involving conjunctions and negations are

$$\begin{aligned}
\mu_i(AB) &= \langle C|\mathbf{M}_{AB}|C\rangle = \langle C|\mathbf{M}\otimes\mathbf{M}|C\rangle \\
&= (\sum_{k,l=1}^{8} c_{kl}e^{-i\gamma_{kl}}\langle k|\otimes\langle l|)\mathbf{M}\otimes\mathbf{M}|(\sum_{i,j=1}^{8} c_{ij}e^{i\gamma_{ij}}|i\rangle\otimes|j\rangle) \\
&= \sum_{k,l=1}^{8}\sum_{i,j=1}^{8} c_{kl}c_{ij}e^{i(\gamma_{ij}-\gamma_{kl})}\langle k|\mathbf{M}|i\rangle\langle l|\mathbf{M}|j\rangle \\
&= \sum_{i=5}^{8}\sum_{j=5}^{8} c_{ij}c_{ij}e^{i(\gamma_{ij}-\gamma_{ij})} = \sum_{i=5}^{8}\sum_{j=5}^{8} c_{ij}^2, && (6.90)\\
\mu_i(A\bar{B}) &= \langle C|\mathbf{M}_{A\bar{B}}|C\rangle = \langle C|\mathbf{M}\otimes(\mathbb{1}-\mathbf{M})|C\rangle \\
&= \sum_{i=5}^{8}\sum_{j=1}^{4} c_{ij}^2, && (6.91)\\
\mu_i(\bar{A}B) &= \langle C|\mathbf{M}_{\bar{A}B}|C\rangle\langle C|(\mathbb{1}-\mathbf{M})\otimes\mathbf{M}|C\rangle \\
&= \sum_{i=1}^{4}\sum_{j=5}^{8} c_{ij}^2, && (6.92)\\
\mu_i(\bar{A}\bar{B}) &= \langle C|\mathbf{M}_{\bar{A}\bar{B}}|C\rangle = \langle C|(\mathbb{1}-\mathbf{M})\otimes(\mathbb{1}-\mathbf{M})|C\rangle \\
&= \sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}^2. && (6.93)
\end{aligned}$$

Definition 6.9. A representation of μ_i in the second sector of the Fock space is a pair $(|C\rangle, \mathbf{M})$, where $|C\rangle \in \mathbb{C}^8 \otimes \mathbb{C}^8$ and $\mathbf{M} : \mathbb{C}^8 \to \mathbb{C}^8$ are such that Eqs. (6.13)–(6.18) are satisfied.

The following theorem characterizes the cases when data involving conjunctions and negations can be represented in the second sector.

Theorem 6.10. *The set of data μ_i has a representation in the second sector of the Fock space if and only μ_i is classical data.*

Proof. Assume that we have $|C\rangle$ and $\mathbf{M}$ such that Eqs. (6.86)–(6.93) are satisfied. Then, it is easy to prove that the classicality conditions (6.13)–(6.18) are satisfied. For example, Eq. (6.17) is proven as follows:

$$\begin{aligned}\mu_i(\bar{A}B)+\mu_i(\bar{A}\bar{B}) &= \langle C|(\mathbb{1}-\mathbf{M})\otimes\mathbf{M}|C\rangle+\langle C|(\mathbb{1}-\mathbf{M})\otimes(\mathbb{1}-\mathbf{M})|C\rangle \\ &= \langle C|(\mathbb{1}-\mathbf{M})\otimes\mathbf{M}+(\mathbb{1}-\mathbf{M})|C\rangle \\ &= \langle C|(\mathbb{1}-\mathbf{M})\otimes\mathbb{1}|C\rangle=\mu_i(\bar{A}).\end{aligned} \tag{6.94}$$

We prove the other side of the implication. Suppose that μ_i is classical data and thus satisfies conditions (6.15)–(6.18). If we choose $|C\rangle = \sum_{i,j=1}^{8} c_{ij}$ such that

$$c_{ij}=\begin{cases}\sqrt{\frac{1}{16}\mu_i(AB)} & \text{for } 5\le i\le 8 \text{ and } 5\le j\le 8,\\ \sqrt{\frac{1}{16}\mu_i(A\bar{B})} & \text{for } 5\le i\le 8 \text{ and } 1\le j\le 4,\\ \sqrt{\frac{1}{16}\mu_i(\bar{A}B)} & \text{for } 1\le i\le 4 \text{ and } 5\le j\le 8,\\ \sqrt{\frac{1}{16}\mu_i(\bar{A}\bar{B})} & \text{for } 1\le i\le 4 \text{ and } 1\le j\le 4,\end{cases} \tag{6.95}$$

and $\mathbf{M}$, such that

$$\mathbf{M}(x_1,...,x_8)=(0,0,0,0,x_5,x_6,x_7,x_8), \tag{6.96}$$

then, Eqs. (6.86)–(6.93) are easily satisfied. This completes the proof. □

Similarly to § 5.2.2, we can obtain representations in $\mathbb{C}^8\otimes\mathbb{C}^8$ that are compatible with the modeling principles of quantum cognition by applying unitary transformations to the representations of individual exemplars. We introduce the extensions of the zero-, first-, and second-type representation for this case.

Definition 6.11. A zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8\otimes\mathbb{C}^8$ is a unit vector $|C\rangle\in\mathbb{C}^8\otimes\mathbb{C}^8$, and a collection of orthogonal projectors $\{\mathbf{M}_i^A,\mathbf{M}_i^B\}_{i=1}^k$ from $\mathbb{C}^8\otimes\mathbb{C}^8$ to $\mathbb{C}^8\otimes\mathbb{C}^8$, such that conditions(6.13)–(6.18) are satisfied with $\mathbf{M}_i^\wedge=\mathbf{M}_i^A\mathbf{M}_i^B$, for $i=1,...,k$. We say $(|C\rangle,\{\mathbf{M}_i^A,\mathbf{M}_i^B\}_{i=1}^k)$ is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8\otimes\mathbb{C}^8$.

Definition 6.12. A first-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8\otimes\mathbb{C}^8$ is a unit vector $|C\rangle\in\mathbb{C}^8\otimes\mathbb{C}^8$, and a collection of orthogonal projectors $\{\mathbf{M}_i\}_{i=1}^k$ from $\mathbb{C}^8$ to $\mathbb{C}^8$, such that $(|C\rangle,\{\mathbf{M}_i\otimes\mathbb{1},\mathbb{1}\otimes\mathbf{M}_i\}_{i=1}^k)$ is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8\otimes\mathbb{C}^8$.

Definition 6.13. A second-type representation of $\mu_{i=1}^k$ on the tensor product space $\mathbb{C}^8\otimes\mathbb{C}^8$ is a pair of unit vectors $|A\rangle,B\rangle\in\mathbb{C}^8$, and a collection of orthogonal projectors $\mathbf{M}_i$ from $\mathbb{C}^8$ to $\mathbb{C}^8$, for $i=1,...,k$, such that

$(|A\rangle \otimes |B\rangle, \{\mathbf{M}_i \otimes \mathbb{1}, \mathbb{1} \otimes \mathbf{M}_i\}_{i=1}^k)$ is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8 \otimes \mathbb{C}^8$.

As for the case of conjunction, the first-type representation is in accordance with the modeling principles of quantum cognition, the second-type and zero-type representations are a mathematical simplification and a generalization respectively. These two representations will facilitate the data analysis.

Since Definitions 6.11–6.13 are trivial extensions of Definitions 5.5–5.7, obtaining a zero-, first-, and second-type representation by applying unitary transformations to a collection of representations for individual exemplars follows the same procedure presented in Theorem 5.8.

Corollary 6.14. *The set of data $\mu_{i=1}^k$ has a zero-type representation in $\mathbb{C}^8 \otimes \mathbb{C}^8$ if and only if μ_i is classical data for $i = 1, ..., k$.*

Corollary 6.15. *The set of data $\mu_{i=1}^k$ has a second-type representation in $\mathbb{C}^8 \otimes \mathbb{C}^8$ if and only if for all $i = 1, ..., k$*

$$\mu_i(XY) = \mu_i(X)\mu_i(Y) \tag{6.97}$$

for $X = A, \bar{\mathcal{A}}$, and $Y = \mathcal{B}, \bar{\mathcal{B}}$.

6.3.3 Fock Space Representation of Experimental Data

We now combine the representations of multiple exemplars developed in § 6.3.1 and § 6.3.2 to represent sets of data in the two-sector Fock space model in a way that is consistent with the modeling principles of quantum cognition in the concrete space $\mathbb{C}^8 \oplus \mathbb{C}^8 \otimes \mathbb{C}^8$.

First, we need to introduce the state vectors and the membership formulas. The model requires state vectors that represent the state of the concept combinations. These states correspond to the superposition of the concept combination represented in the first and second sectors. So the state vectors for the concept combinations are given by

$$|\psi_{XY}\rangle = n_{XY}\frac{e^{i\rho}}{\sqrt{2}}(|X\rangle + |Y\rangle) + \sqrt{1 - n_{XY}^2}e^{i\theta}|C\rangle. \tag{6.98}$$

Hence, the membership weights for the concept combinations are given by

$$\begin{aligned}
\mu_i(XY) = &\langle \psi_{XY} | \mathbf{M} \oplus \mathbf{M} \otimes \mathbf{M} | \psi_{XY} \rangle \\
= &\frac{n_{XY}^2}{2} ((\langle X| + \langle Y|) \mathbf{M} (|X\rangle + |Y\rangle) + (1 - n_{XY}^2) \langle C | \mathbf{M} \otimes \mathbf{M} | C \rangle \\
= &\frac{n_{XY}^2}{2} (\langle X|\mathbf{M}|X\rangle + \langle Y|\mathbf{M}|Y\rangle + \langle X|\mathbf{M}|Y\rangle + \langle Y|\mathbf{M}|X\rangle) + (1 - n_{XY}^2) \sum_{i,j=5}^{8} c_{ij}^2 \\
= &\frac{n_{XY}^2}{2} (\mu_i(X) + \mu_i(X)) + \Re \langle X|\mathbf{M}|Y\rangle) + (1 - n_{XY}^2) \sum_{i,j=5}^{8} c_{ij}^2 \\
= &n_{XY}^2 (\frac{\mu_i(X) + \mu_i(Y)}{2} + \left(\sum_{i=5}^{8} a_i b_i \right) \cos(\phi_B - \phi_A)) + (1 - n_{XY}^2) \check{\mu}_i(XY).
\end{aligned} \tag{6.99}$$

Eq. (6.99) expresses the membership weights for conjunctions of concepts $\mathcal{A}$, $\mathcal{B}$, and their negations. We can introduce the representations of multiple exemplars as in §5.2.3.

Definition 6.16. A zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8 \oplus \mathbb{C}^8 \otimes \mathbb{C}^8$ consists of an orthonormal set $\{|A\rangle, |B\rangle, |\bar{A}\rangle, |\bar{B}\rangle\}$ of vectors in $\mathbb{C}^8$, a tensor $|C\rangle \in \mathbb{C}^8 \otimes \mathbb{C}^8$, a collection of operators $\{\mathbf{M}_i, \mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k$ from $\mathbb{C}^8 \otimes \mathbb{C}^8$ to $\mathbb{C}^8 \otimes \mathbb{C}^8$, and coefficients $0 \leq n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}, n_{\bar{A}\bar{B}} \leq 1$ such that for all $i = 1, ..., k$, Eqs. (6.41), (6.43), (6.84), (6.86)–(6.89), and (6.99) are satisfied. We say that

$$(n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}, n_{\bar{A}\bar{B}}, |A\rangle, |B\rangle, |\bar{A}\rangle, |\bar{B}\rangle, |C\rangle, \{\mathbf{M}_i, \mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k)$$

is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8 \oplus \mathbb{C}^8 \otimes \mathbb{C}^8$.

Definition 6.17. A first-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8 \oplus \mathbb{C}^8 \otimes \mathbb{C}^8$ is a zero-type representation

$$(n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}, n_{\bar{A}\bar{B}}, |A\rangle, |B\rangle, |\bar{A}\rangle, |\bar{B}\rangle, |C\rangle, \{\mathbf{M}_i, \mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k)$$

of $\mu_{i=1}^k$ in $\mathbb{C}^8 \oplus \mathbb{C}^8 \otimes \mathbb{C}^8$ such that for all $i = 1, ..., k$, $\mathbf{M}_i^A = \mathbf{M}_i \otimes \mathbb{1}$, and $\mathbf{M}_i^B = \mathbb{1} \otimes \mathbf{M}_i$.

Definition 6.18. A second-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8 \oplus \mathbb{C}^8 \otimes \mathbb{C}^8$ is a zero-type representation

$$(n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}, n_{\bar{A}\bar{B}}, |A\rangle, |B\rangle, |\bar{A}\rangle, |\bar{B}\rangle, |C\rangle, \{\mathbf{M}_i, \mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k)$$

of $\mu_{i=1}^k$ in $\mathbb{C}^8 \oplus \mathbb{C}^8 \otimes \mathbb{C}^8$ such that $|C\rangle = |A\rangle \otimes |B\rangle$.

The following result summarizes the cases where the two-sector Fock space model for conjunctions and negations of concepts can represent the membership weights for a collection of exemplars:

Theorem 6.19. *There exists a zero-type representation of $\mu_{i=1}^k$ in the Fock space model if and only if there exist parameters $0 \leq n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}, n_{\bar{A}\bar{B}} \leq 1$ such that for all $i = 1, ..., k$*

$$\mu_i(XY) = n_{XY}^2 \tilde{\mu}_i(XY) + \sqrt{1 - n_{XY}^2} \breve{\mu}_i(XY), \tag{6.100}$$

with

$$\begin{aligned} \tilde{\mu}_i(XY) &\in [h_{min}(XY), h_{max}(XY)], \\ \breve{\mu}_i(XY) &\in [t_{min}(XY), t_{max}(XY)]. \end{aligned} \tag{6.101}$$

Proof. Eq. (6.101) implies we can build a representation $(|A\rangle, |B\rangle, \{\mathbf{M}_i\})$ of $\tilde{\mu}_{i=1}^k$ in $\mathbb{C}^8$, and a zero-type representation $(|C, \{\mathbf{M}_i^A, \mathbf{M}_i^B\})$ of $\breve{\mu}_{i=1}^k$ in $\mathbb{C}^8 \otimes \mathbb{C}^8$. Next, Eq. (6.100) implies there are parameters $n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}$, and $n_{\bar{A}\bar{B}}$ such that Eq. (6.99) is satisfied by $|A\rangle, |B\rangle, |C\rangle, \mathbf{M}_i, \mathbf{M}_i^A$, and $\mathbf{M}_i^B$ for each $i = 1, ..., k$. Therefore,

$$(n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}, n_{\bar{A}\bar{B}}, |A\rangle, |B\rangle, |C\rangle, \{\mathbf{M}_i, \mathbf{M}_i^A, \mathbf{M}_i^B\}_{i=1}^k)$$

is a zero-type representation of $\mu_{i=1}^k$ in $\mathbb{C}^8 \oplus \mathbb{C}^8 \otimes \mathbb{C}^8$. □

The following corollary characterizes that sets of data that allow for a second-type representation,

Corollary 6.20. *There exists a second-type representation of $\mu_{i=1}^k$ in the Fock space model if and only if there exist parameters $0 \leq n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}, n_{\bar{A}\bar{B}} \leq 1$ such that for all $i = 1, ..., k$*

$$\mu_i(XY) = n_{XY}^2 \tilde{\mu}_i(XY) + \sqrt{1 - n_{XY}^2} \mu_i(X)\mu_i(Y), \tag{6.102}$$

with

$$\tilde{\mu}_i(XY) \in [h_{min}(XY), h_{max}(XY)]. \tag{6.103}$$

Proof. Because we impose the extra constraint $|C\rangle = |A\rangle \otimes |B\rangle$ for the case of second-type representations, we can apply the proof of Theorem 6.19 using $\check{\mu}_i(XY) = \mu_i(X)\mu_i(Y)$. $\square$

We can now build representations of the data collected in the experiment described in § 6.2. Before representing the data, we revisit the results obtained in § 6.2.1 in light of this model. In particular we show that this model is capable of describing the experimental deviations from the classicality parameters in Definition 6.3.

Since we know from § 6.2.1 that individual concepts behave classically, we focus our analysis on concept combinations. Therefore, we calculate the value of the parameters I_A, I_B, $I_{\bar{A}}$, and $I_{\bar{B}}$ in the Fock space model. For simplicity, we assume

$$n_{XY} = n, \tag{6.104}$$

and

$$\begin{aligned} \Re\langle A|\mathbf{M}|B\rangle + \Re\langle A|\mathbf{M}|\bar{B}\rangle &= 0, \\ \Re\langle A|\mathbf{M}|B\rangle + \Re\langle \bar{A}|\mathbf{M}|B\rangle &= 0, \\ \Re\langle \bar{A}|\mathbf{M}|B\rangle + \Re\langle \bar{A}|\mathbf{M}|\bar{B}\rangle &= 0, \\ \Re\langle A|\mathbf{M}|\bar{B}\rangle + \Re\langle \bar{A}|\mathbf{M}|\bar{B}\rangle &= 0. \end{aligned} \tag{6.105}$$

Then, applying Eq. (6.99) to the definition of I_A in Eq. (6.28) yields

$$\begin{aligned} I_A &= \mu_i(A) - \mu_i(AB) - \mu_i(A\bar{B}) \\ &= \mu_i(A) - n^2(\mu_i(A) - \frac{\mu_i(B) + \mu_i(\bar{B})}{2}) - (1-n^2)(\check{\mu}_i(AB) + \check{\mu}_i(A\bar{B})). \end{aligned} \tag{6.106}$$

From our results in § 6.2.1, we can assume that individual concepts satisfy the classicality conditions:

$$\mu_i(B) + \mu_i(\bar{B}) = 1. \tag{6.107}$$

Theorem 6.10 shows that the second sector also satisfies the classical conditions. Therefore

$$\check{\mu}_i(AB) + \check{\mu}_i(A\bar{B}) = \mu_i(A). \tag{6.108}$$

Substituting Eqs. (6.107) and (6.108) in Eq. (6.106) yields

$$I_A = \mu_i(A) - n^2(\mu_i(A) - \frac{1}{2}) - (1-n^2)\mu_i(A) = -\frac{n^2}{2}. \tag{6.109}$$

Eq. (6.109) shows that the classicality condition $I_A = 0$ is violated by a factor that is proportional to n^2. The same result can be obtained for the parameters I_B, $I_{\bar{A}}$, and $I_{\bar{B}}$.

We can compare experimental values of these parameters with the result of Eq. (6.109). We use the experimental values of the parameters I_A, I_B, $I_{\bar{A}}$, and $I_{\bar{B}}$ (see Table 6.2) to compute n. The result shows that n fluctuates between 0.7 and 1. This is consistent with the data analysis of § 5.3. There, we demonstrated that the best performance for the Fock space model for conjunctions is obtained when n_{AB} is equal to approximately 0.8 (see Figure 5.1).

This confirms that the contribution from the first sector of the Fock space is larger than that of the second sector. And since each sector represent a different mode of thought, this suggests that the contribution from the emergent mode of thought is larger than that of the logical mode of thought.

6.4 Examples and Data Representation Analysis

Before analyzing the performance of the model, we give a concrete representation for an example from the experimental data. Consider the exemplar 'olive' in Table B.4 (Appendix B). The membership weights are

$$\begin{gathered}\mu_i(A) = 0.53,\ \mu_i(B) = 0.63,\ \mu_i(\bar{A}) = 0.47,\ \mu_i(\bar{B}) = 0.44,\\ \mu_i(AB) = 0.65,\ \mu_i(A\bar{B}) = 0.34,\ \mu_i(\bar{A}B) = 0.51, \mu_i(\bar{A}\bar{B}) = 0.36.\end{gathered} \tag{6.110}$$

This exemplar can be represented in the Fock space $\mathbb{C}^8 \oplus (\mathbb{C}^8 \otimes \mathbb{C}^8)$ by making the choices

$$\begin{aligned}
&n_{AB} = n_{A\bar{B}} = n_{\bar{A}B} = n_{\bar{A}\bar{B}} = 1,\\
&|A\rangle = e^{i\phi_A}(0.47, 0.48, 0.26, 0.14, -0.61, -0.20, -0.04, 0.23),\\
&|B\rangle = e^{i\phi_B}(0.05, -0.44, -0.64, 0.14, -0.33, -0.44, -0.22, 0.15),\\
&|\bar{A}\rangle = e^{i\phi_{\bar{A}}}(0.46, -0.039, -0.42, 0.28, -0.034, 0.62, 0.37, -0.04),\\
&|B'\rangle = e^{i\phi_{\bar{B}}}(0.43, 0.1, 0.047, 0.49, 0.59, -0.33, -0.27, -0.18),
\end{aligned} \tag{6.111}$$

with

$$\begin{aligned}
\phi_{AB} &= \phi_B - \phi_A = 102.18°, \\
\phi_{A\bar{B}} &= \phi_{\bar{B}} - \phi_A = 116.27°, \\
\phi_{\bar{A}B} &= \phi_B - \phi_{\bar{A}} = 97.28°, \\
\phi_{\bar{A}\bar{B}} &= \phi_{\bar{B}} - \phi_{\bar{A}} = 107.51°,
\end{aligned} \tag{6.112}$$

and by characterizing the state $|C\rangle$ as follows:

$$\begin{aligned}
&\sum_{i=5}^{8}\sum_{i=5}^{8} c_{ij}^2 = 0.640, \sum_{i=5}^{8}\sum_{i=1}^{4} c_{ij}^2 = 0.347, \\
&\sum_{i=1}^{4}\sum_{j=5}^{8} c_{ij}^2 = 0.469, \sum_{i=1}^{4}\sum_{i=1}^{4} c_{ij}^2 = 0.5.
\end{aligned} \tag{6.113}$$

We estimated the number of exemplars that can individually be represented assuming Eq. (6.104), and have found that both the zero- and second-type representations can model 95% of exemplars in the data set. We also calculated the performance achieved for specific values of $n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}$, and $n_{\bar{A}\bar{B}}$. In Fig. 6.2, we show the number of exemplars for which Eq. (6.99) is satisfied for $0 \leq n_{AB}, n_{\bar{A}B}, n_{A\bar{B}}$, and $n_{\bar{A}\bar{B}} \leq 1$ for $j = 1, .., 4$. The blue, red, yellow, and green curves in the graphs of the first column correspond to the combinations $\mathcal{A}_j\mathcal{B}_j$, $\bar{\mathcal{A}}_j\mathcal{B}_j$, $\mathcal{A}_j\bar{\mathcal{B}}_j$, and $\bar{\mathcal{A}}_j\bar{\mathcal{B}}_j$ respectively. The first column assumes that $\breve{\mu}_i(XY)$ in Eq. (6.99) must satisfy the constraints for classical data, and the second column assumes that $\breve{\mu}_i(XY) = \mu_i(X)\mu_i(Y)$. Therefore, by Theorem (6.19) and Corollary (6.20), the first and second columns indicate the number of exemplars that have a zero- and a second-type representation respectively.

We observed a very similar pattern across all the concept combinations. For the first column, the performance remains low for small values of n_{XY}, the number of exemplars increases steadily reaching the highest performance for n_{XY} between 0.6 and 0.8, and remains stable thereafter. The same pattern appear in the second column, except that the performance remains low up to n_{XY} equal to approximately 0.5, and the highest performance is reached for values between 0.7 and 0.9. Therefore, we conclude that the two-sector Fock space model is able to represent almost all exemplars when the first sector is dominant. Interestingly, this result is consistent with our

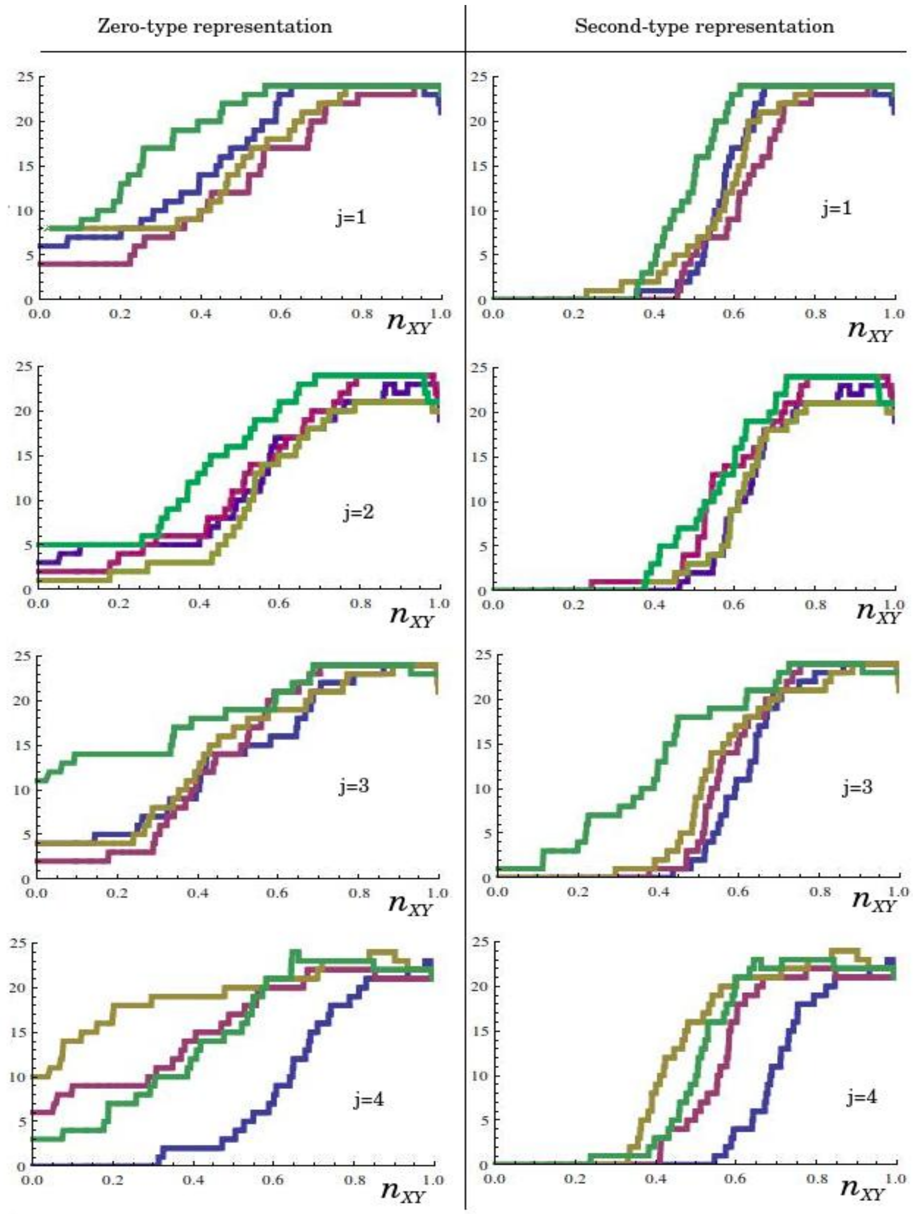

Figure 6.2: Number of exemplars having a zero- and second-type representation for different values of n_{XY}.

analysis for conjunction in Chapter 5.

To provide a simpler presentation of our results, we present in Fig. 6.3 the second-type representation when the four parameters n_{XY} are equal to the same n value. In this restricted case, we show the fraction of exemplars that simultaneously satisfy Eq. 6.99. The blue, red, yellow, and green curves represent the cases $j = 1, 2, 3$, and 4 respectively.

Here we can clearly see the representation pattern explained above. In fact, none of the exemplars can be represented for $n < 0.7$. In addition, the fraction of exemplars that can be modeled increases abruptly from 0 to approximately 0.8 for $0.7 \leq n \leq 0.9$, and remains stable for $n > 0.9$. Moreover, the maximal performance in all cases does not surpass 90%. This implies that there are some exemplars in the data set that cannot be represented for a fixed n since, from our previous analysis, we know that 95.8% of the exemplars can be represented by the model.

From our theoretical and experimental analyses, we conclude that when concepts are combined using conjunctions and negations, the emergent mode

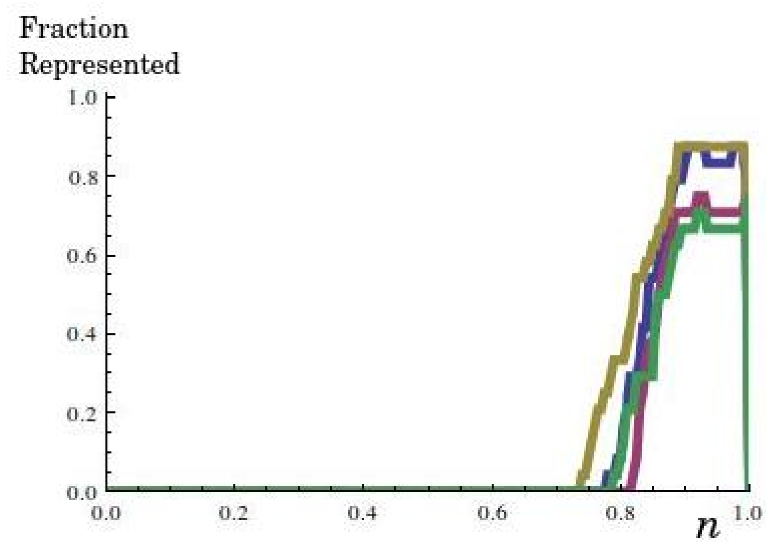

Figure 6.3: Fraction of experimental data that can be simultaneously represented in the Fock space models for specific values of n.

of thought, represented by the first sector of the Fock space, is predominant over the logical mode of thought, represented by the second sector of the Fock space. In particular, the emergent mode of thought contributes approximately 70% or more to the conceptual combination state.

Chapter 7

Quantum Structures in Natural Language Processing

The advent of the internet and the consequent technological revolution have transformed the field of language processing into a major challenge for science. The area of research that focuses on performing non-trivial information tasks such as translating a document to a foreign language, known as machine translation, or identifying relevant information from a collection of documents, known as information retrieval, is called natural language processing (NLP) [JMK$^+$00].

It is well-known in the artificial intelligence community that many information tasks are not easily automated [RN95]. The most famous example of such a task was introduced by one of the founding fathers of the theory of computation, Alan Turing [SCA03]. He proposed the following intelligence test for machines: Suppose we have a chat opened for two computers in different rooms. In one of the rooms there is a human who does not know who he/she is going to talk to, and in the other room there is a machine that has no interaction with or feedback from humans, except for the human on the other side of the chat. The 'Turing test' consists of asking the human, after a few minutes of conversation, whether the conversation he/she had on the chat corresponds to an interaction with a machine or with a human. If participants tend to believe they are speaking with a human, we say the machine has passed the Turing test. When the test was proposed, most researchers in the field thought that the Turing test would be passed after 10 or 20 years of research. Now, after nearly 70 years, the test remains an open problem, and there is no theory that sheds light on how it could be resolved [McC]. The Turing test illustrates the difficulties associated with automated language processing.

7.1 Language, Concepts, and Quantum Structures

A NLP model operates on 'linguistic units' that can correspond to words, sentences, paragraphs, documents, or even collections of documents. The model usually involves a 'syntactical' part that is concerned with the grammatical correctness of the linguistic units, and a 'semantic' part that focuses on the meaning of such units.

In NLP, syntactical models are generally built upon one of the many mathematically well-grounded theories of syntax. Examples of these theories are generative, dependency, and functionalist grammars [Cho02]. For the case of semantics however, there are no widely accepted formal theories. In fact, semantic models in NLP are usually built in an ad-hoc manner [MS99]. For example, semantic approaches based on ontologies are often dependent on the topic of discourse that is being processed [Sow00]. Since semantics is a fundamental part of NLP models, most researchers believe that the lack of a theory for semantics is one of the most important impediments for the achievement of human-level performance, and that fundamentally new ideas are required to achieve significant progress [McC].

Although philosophers of language have proposed that the meaning of words can be represented using a concept-theoretical framework [Sea04], the dominant research methodology in NLP does not follow this approach. In fact, most researchers in NLP do not look for representational frameworks for the meaning of linguistic units, but rather focus on matching the performance of human annotators' 'gold-standards' using ad-hoc models for particular language tasks [Pel06].

From a methodological perspective, concept modeling and NLP are similar because they are generally approached in an ad-hoc manner. Moreover, because both areas are concerned with the study of meaning, they have similar structural problems.

For example, it is a well-known fact in NLP that semantic relations in language are graded [Zad65, Mur03, Tur01], and that the gradedness structure of these relations can be better understood using contextual information [STZ05, BYBM11, Nav09, LC98]. However, it is not clear whether it is better to define semantic relations for words [Fel98], sets of words [BFL98], or grammar structures [Cow98], and whether contexts should be defined in terms of windows of text[12], or grammar structures. As in the case of concept modeling, several theories have been proposed to define the basic units of study and their contexts, but there is no general agreement among researchers [BB05].

There is an open debate in NLP as to whether the meaning of the combination of lexical units can be represented in terms of the meaning of the original linguistic units. While several models introduce methods to combine linguistic units using syntactical structures [Cho02], vector-based representations [Bar13], or heuristic approaches [BP03], it is not clear which method has the best performance [ML10]. In fact, various scholars have proposed that the meaning of word combinations is non-compositional [BZL10, Sve08], and that the study of such meaning will require new representational tools [Gra90].

Quantum cognition has been successful in handling similar structural problems for the modeling of concepts. Note that the meaning of a certain piece of text in a document can be associated to a concept. And, pieces of text, such as words or paragraphs, can be thought of as exemplars for a concept. Therefore, quantum cognition could offer an alternative approach to represent certain NLP tasks. By studying how words and sentences tend to appear in a document or a collection of documents, we can identify the concepts that give meaning to the document, and thus determine whether or not these concepts exhibit a quantum structure.

In the following section, we present two examples using this methodology. In the first case, we study word co-occurrence in a corpus of text to identify entangled concepts in the corpus and, in the second case, we study a property of quantum particles, called indistinguishability, using statistical information obtained from a web search engine.

7.2 Evidence of Quantum Structure in Natural Language Processing

We present the results of experiments conducted to identify quantum structures in the statistical analysis of natural language data.

[12] A window of text is a sequence of words that contain a linguistic unit in a piece of text and is larger than the linguistic unit.

7.2.1 Quantum Entanglement in Text Corpora

Recall that to test whether two abstract entities $\mathcal{A}$ and $\mathcal{B}$ are entangled, we need two measurements for each entity with each measurement having two possible outcomes. Then, the concepts are entangled if the Clauser-Horn-Shimony-Holt (CHSH) inequality, Eq. (3.73), is violated. Since we can associate the exemplars of a concept with words, and estimate the elicitation of concepts in a document by counting the number of times their exemplars appear in the document, we propose to test entanglement of concepts in a corpus of text by conducting the following experiment:

Consider a corpus of text T and two concepts $\mathcal{A}$ and $\mathcal{B}$. Choose eight words $w_1, ..., w_8$, where $w_1, ..., w_4$ are exemplars of the concept $\mathcal{A}$, and $w_5, ..., w_8$ are exemplars of the concept $\mathcal{B}$. Since entanglement is measured using the statistical co-occurrence of the exemplars of a concept, we partition T as a collection of n consecutive windows of text $\{t_1, ..., t_n\}$, and estimate how the words representing the exemplars of $\mathcal{A}$ and $\mathcal{B}$ co-occur in these windows of text.

Let $N \in \mathbb{N}$ and assume that each t_i is a window of N words, for $i = 1, ..., n$, and $n = n(N)$. Next, let $\mathbf{M}^i_A = \{w_1, w_2\}$, $\mathbf{M}^i_{A'} = \{w_3, w_4\}$, $\mathbf{M}^i_B = \{w_5, w_6\}$, and $\mathbf{M}^i_{B'} = \{w_7, w_8\}$ be the measurements whose outcome is $+1$ if the first word is in t_i, -1 if the second word is in t_i, and 0 if neither or both words are in t_i. Finally, let $\mathbf{M}^i_{XY}$ be the joint measurements whose outcome is associated with the product of the outcomes of the former experiments, and denote by $E(\mathbf{M}_{XY})$ the expected value of such measurement for $X = A$ or A', and $Y = B$ or B'.

We say that the concepts $\mathcal{A}$ and $\mathcal{B}$ of the corpus of text cannot be represented using a classical probabilistic model whenever the inequality

$$-2 \leq E(\mathbf{M}_{AB}) + E(\mathbf{M}_{A'B}) + E(\mathbf{M}_{AB'}) - E(\mathbf{M}_{A'B'}) \leq 2 \tag{7.1}$$

is violated.

Let the frequency matrix $\mathbf{M}(A, A', B, B', N)$ be defined by

$$\mathbf{M}(A, A', B, B', N) = \left(\begin{array}{cc|cc} F(A_1B_1) & F(A_1B_2) & F(A_1B'_1) & F(A_1B'_2) \\ F(A_2B_1) & F(A_2B_2) & F(A_2B'_1) & F(A_2B'_2) \\ \hline F(A'_1B_1) & F(A'_1B_2) & F(A'_1B'_1) & F(A'_1B'_2) \\ F(A'_2B_1) & F(A'_2B_2) & F(A'_2B'_1) & F(A'_2B'_2) \end{array}\right) \tag{7.2}$$

where

$$F(XY) = \sum_{i=1}^{n} \mathbf{M}^i_{XY}, \tag{7.3}$$

with $X = A_j$ or A'_j, and $Y = B_j$ or B'_j, for $j = 1, 2$.

Note that each quadrant in the matrix corresponds to the frequency table of one of the joint experiments. Also, since

$$E(\mathbf{M}_{AB}) = \frac{F(A_1B_1) + F(A_2B_2) - F(A_1B_2) - F(A_2B_1)}{F(A_1B_1) + F(A_2B_2) + F(A_1B_2) + F(A_2B_1)}, \tag{7.4}$$

we can estimate Eq. (7.1) from $\mathbf{M}(A, A', B, B', N)$. Because we are concerned with the statistics of joint experiments, it is also important to verify whether or not the marginal probability law holds[13].

The marginal probability law implies that

$$\begin{aligned} p(A_1, B) &= \frac{F(A_1B_1) + F(A_1B_2)}{\sum_{i,j=1}^2 F(A_iB_j)} = \frac{F(A_1B'_1) + F(A_1B'_2)}{\sum_{i,j=1}^2 F(A_iB'_j)} = p(A_1, B'), \\ p(A'_1, B) &= \frac{F(A'_1B_1) + F(A'_1B_2)}{\sum_{i,j=1}^2 F(A'_iB_j)} = \frac{F(A'_1B'_1) + F(A'_1B'_2)}{\sum_{i,j=1}^2 F(A'_iB'_j)} = p(A'_1, B'), \\ p(A_2, B) &= \frac{F(A_2B_1) + F(A_2B_2)}{\sum_{i,j=1}^2 F(A_iB_j)} = \frac{F(A_2B'_1) + F(A_2B'_2)}{\sum_{i,j=1}^2 F(A_iB'_j)} = p(A_2, B'), \\ p(A'_2, B) &= \frac{F(A'_2B_1) + F(A'_2B_2)}{\sum_{i,j=1}^2 F(A'_iB_j)} = \frac{F(A'_2B'_1) + F(A'_2B'_2)}{\sum_{i,j=1}^2 F(A'_iB'_j)} = p(A'_2, B'). \end{aligned} \tag{7.5}$$

Similarly

[13] It has recently been proven that the simultaneous verification of CHSH inequalities and marginal probability law is a sufficient test of entanglement (see [DK14, ASV14a]).

$$\begin{aligned} p(A, B_1) &= p(A', B_1), \\ p(A, B_2) &= p(A', B_2), \\ p(A, B_1') &= p(A', B_1'), \\ p(A, B_2') &= p(A', B_2'). \end{aligned} \tag{7.6}$$

We define the vector

$$\begin{aligned} r =\Big(&p(A_1, B) - p(A_1, B'), p(A_1', B) - p(A_1', B'), p(A_2, B) - p(A_2, B'), \\ &p(A_2', B) - p(A_2', B'), p(B_1, A) - p(B_1, A'), p(B_1', A) - p(B_1', A'), \\ &p(B_2, A) - p(B_2, A'), p(B_2', A) - p(B_2', A')\Big) \end{aligned} \tag{7.7}$$

to record the extent to which the marginal probability law is satisfied by each joint experiment, and quantify the violation of the marginal probability law by

$$\delta = \sup_{i=1,\ldots,8} ||r_i||_\infty, \tag{7.8}$$

where $||\cdot||_\infty$ is the supreme norm.

The first step in the experiment is to identify two sets of four words that are exemplars of the concepts $\mathcal{A}$ and $\mathcal{B}$. Since a concept can be associated with multiple sets of words, and a set of words can be associated with multiple concepts [RS10], this is not a trivial procedure. We propose the following methodology:

1. Select a set of statistically relevant words, W, in the corpus.

2. Determine concepts $\mathcal{A}$ and $\mathcal{B}$ using all the possible combinations of four words in W.

3. Compute the CHSH inequality and the value of r for each choice of $\mathcal{A}$ and $\mathcal{B}$.

We applied this methodology on a collection of corpus called 'TREC collection WSJ8792 Lemur 4.12' [14]. The corpus was pre-processed by removing

[14] Lemur is an open source project that develops search engines and text analysis tools for research and development of information retrieval and text mining softwares.

stop terms and applying the Porter stemmer. 32 topics having 70 or more documents were selected among TREC topics 151-200. For these topics, we segmented the documents into windows of words of lenght $N = 5, 10$, and 20. For each topic, 2 sets of 10 words were chosen using two popular relevance criteria. The first, known as tf score, simply rank words by their frequency in the corpus, and the second, called term frequency-inverse document frequency or tf-idf score, is a numerical statistic that reflects how important a word is to a document in a corpus. The tf-idf score is often used as a weighting factor in information retrieval and text mining [RUUU12]. For each criteria, the first and second sets of 10 words were used to build all possible measurements $\mathbf{M}_A, \mathbf{M}_{A'}$, and $\mathbf{M}_B, \mathbf{M}_{B'}$, respectively.

Our statistical analysis indicates that we can identify entangled concepts from the statistical co-occurrence of words using a corpus of text. Fig. 7.1 shows the proportion, $p_N(T)$, of subsets of words for the 32-topic corpus that violate Eq. (7.1). The black curve corresponds to $N = 20$, the gray dotted curve corresponds to $N = 10$, and the black dashed curve corresponds to $N = 5$. The left plot is based on the word choice using the frequency relevance criteria, and the right plot is based on word choice using the tf-idf relevance criteria.

We observe that the tf-idf score selects more sets of words violating Eq. (7.1) than the tf score. This is consistent with the fact that td-idf is a better word-relevance measure than tf. Although some topics exhibit more violations than others, we identify a strong tendency for violations of Eq. (7.1) for most topics. Moreover, the violation decreases when N increases. This is consistent with the fact that word correlations are noisy for large window sizes [iCS01]. For shorter window sizes, it is more likely that only meaningful correlations between words will be kept, and hence the violation of Eq. (7.1) is observed more frequently.

In Fig. 7.2, each point is associated to a set of words. The x-axis corresponds to the CHSH inequality value obtained using Eq. (7.1), and the y-axis corresponds to the δ value. We plot points whose CHSH value is 1.5 or larger to better visualize the behavior near the violation threshold. Since we are interested in observing sets of words that violate the CHSH inequality and satisfy the marginal probability law, the left plot shows $\delta \in [0, 1]$, the middle plot zooms to $\delta \in [0, 0.4]$, and the right plot zooms to $\delta \in [0, 0.1]$. We can visualize the extent to which the CHSH inequality will be violated for different values of δ by the density of points in each region of the plots.

In most cases where the CHSH inequality is violated, the marginal probability law is not preserved. However, it is possible to identify a region where the CHSH inequality is violated and the violation of the marginal probability law is very small. In fact, since δ is a supreme norm, the right plot shows that there are sets of words that violate the CHSH inequality

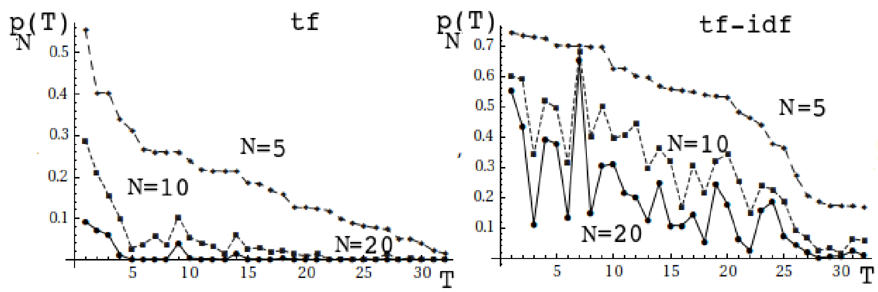

Figure 7.1: Frequency of the violation of Eq. (7.1) for the 20 most relevant terms. The left plot corresponds to the co-occurrence data for relevance associated to term-frequency score, and the right plot corresponds to the co-occurrence data for relevance associated to td-idf score. In both plots, the topics were sorted such that $p_5(T)$ is decreasing so as to avoid that curves crossed each other.

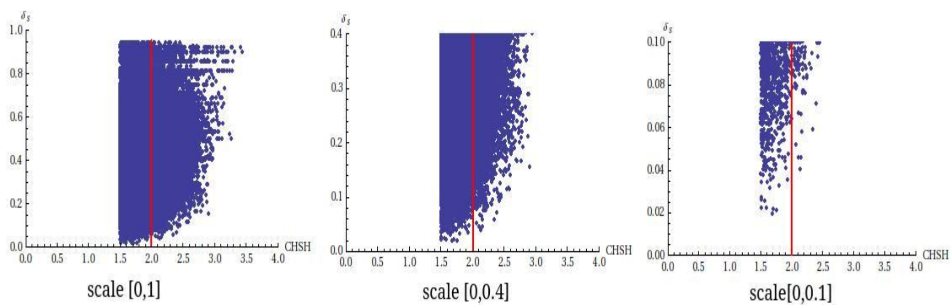

Figure 7.2: Each point corresponds to the choice of particular measurements A, A', B, and B'. The x-axis represents the extent to which equation (7.1) is violated and the y-axis denote the δ value. We consider three scales for the δ value, and one single scale for the the middle term of the CHSH inequality. Points to the right of the red line violate the CHSH inequalty.

with $r < 0.05$. This indicates that there could be sets of words that violate the CHSH inequality and satisfy the marginal probability law.

7.2.2 Indistinguishability of Concepts and Bose-Einstein Statistics

One of the most profound differences between quantum and classical physics is how identical particles behave statistically. While classical particles are distinguishable, and thus governed by the Maxwell-Boltzmann (MB) distribution, quantum particles are indistinguishable. Quantum particles are governed by the Bose-Einstein (BE) distribution in the case of integer-spin particles, and by the Fermi-Dirac (FD) distribution in the case of half-integer spin particles.

Since the statistics of identical particles illustrate a fundamental difference between classical and quantum entities, we propose to study the statistical behaviour of a collection of concepts to determine whether they behave as classical or quantum entities.

Consider for example the linguistic expression 'eleven animals.' This expression can be viewed as the combination of concepts '*Eleven*' and '*Animals*' into '*Eleven Animals*.' The concept '*Eleven Animals*' corresponds to an abstract idea of eleven animals. So the linguistic expression 'eleven animals' elicits the thought of eleven indistinguishable entities. However, the same linguistic expression can also elicit the thought of eleven animals as objects existing in space and time, and thus distinguishable from each other. This intuitive difference between the reasoning about concepts and objects is what motivates the development of a methodology to test what type of elicitation is predominant.

In order to explain the next experiment, we first summarize the statistical differences in the classical and quantum distributions of physical entities. These differences are then analyzed with respect to concepts using experimental data from both a psychological and computational studies.

The Statistics of Indistinguishability

In classical mechanics, the state of an individual particle is represented by a pair $(q, p) \in \Omega$, where q denotes the particle's position and p its momentum. The set Ω is called the phase space of the particle. The particle's evolution is ruled by specific dynamical laws. As the number of particles increases, the dynamical description of the system becomes intractable. In this case, classical statistical mechanics is introduced to describe the properties of the system [Mar12]. For a classical system, the MB distribution estimates the likelihood of finding the system in each of its energy states.

A fundamental assumption in the derivation of MB distribution is that all particles are 'distinguishable.' That is, one can always follow the trajectories of each particle and label them differently. Consider a system of N distinguishable particles, and suppose that each particle can be in one of M possible states. Then the total number of possible system configurations is $W_{MB}(N, M) = M^N$. Hence, the probability that a specific configuration s is realized is

$$P_{MB}(s) = \frac{T_{MB}(s)}{W_{MB}(N, M)}, \tag{7.9}$$

where $T_{MB}(s)$ is the number of ways in which s can be realized. For example, consider a system of $N = 2$ classical particles that can be distributed in $M = 2$ energy states. If one applies MB distribution to this simple situation, then the number of possible arrangements is $W_{MB}(2, 2) = 4$, each one with a probability $\frac{1}{4}$.

The situation is radically different in quantum mechanics where the state of a system is represented by a probability wave-function in a Hilbert space. Since the measurement of a system induces a collapse of the wave function describing the system, we cannot know, post-measurement, which particle collapsed to which state. Indeed, given two identical quantum particles, it is not possible to recognize if an exchange has occurred between the two particles. More concretely, consider a system of two quantum particles q_1 and q_2, and suppose that it is represented by the unit vector $|\Psi(q_1, q_2)\rangle$ in a Hilbert space. Then, indistinguishability implies that

$$|\langle\Psi(q_1, q_2)|\Psi(q_1, q_2)\rangle|^2 = |\langle\Psi(q_2, q_1)|\Psi(q_2, q_1)\rangle|^2. \tag{7.10}$$

Therefore, we have either

$$\begin{aligned} |\Psi(q_2, q_1)\rangle &= |\Psi(q_1, q_2)\rangle, \text{ or} \\ |\Psi(q_2, q_1)\rangle &= -|\Psi(q_1, q_2)\rangle. \end{aligned} \tag{7.11}$$

It follows from the spin-statistics Theorem [KB62] that integer-spin particles, called 'bosons,' and half-integer spin particles, called 'fermions,' correspond to the first and second cases of Eq. (7.11) respectively. Moreover, the spin-statistics theorem implies that fermions are subject to the 'Pauli exclusion principle,' which means that only one fermion can occupy a specific quantum state at a specific time. This follows directly from the anti-symmetry of the wave function. For bosons there is no restriction in occupying the same state.

The above difference between fermions and bosons has a dramatic influence in the way both types of particles behave statistically. Let us consider again the situation of N particles that can be distributed in M single-particle states, and suppose that the particles are identical. For a system of N identical bosons, the number of possible configurations is

$$W_{BE}(N,M) = \frac{(N+M-1)!}{N!(M-1)!}, \tag{7.12}$$

where $N! = N(N-1)(N-2)\dots 1$. There are fewer arrangements available due to indistinguishability. In the case of fermions, the Pauli exclusion principle dictates that two fermions cannot be in the same state, which further reduces the number of possible configurations to

$$W_{FD}(N,M) = \frac{M!}{N!(M-N)!}. \tag{7.13}$$

By considering again the case $N = M = 2$, we have that, for a system of 2 identical bosons, $W_{BE}(2,2) = (2+2-1)!/2!(2-1)! = 3$, and the probability for each realization is $1/3$. For a system of 2 identical fermions, only one realization is possible and it occurs with probability 1.

The above differences between distinguishable and indistinguishable particles are statistically significant and can be used to characterize empirical evidence for the indistinguishability of concept combinations. Suppose we consider two states of '*Animal*,' namely $p =$'cat' and $q =$'dog.' Then, the concept '*Eleven Animals*,' gives rise to twelve possible states. We denote them by $p_{11,0} =$'eleven cats,' $p_{10,1} =$'ten cats and one dog,' and so on. For simplicity, we assume the existence of two probability values $\mu(p)$ and $\mu(q)$ that account for possible bias towards one of the states. Thus, $\mu(p)$ and $\mu(q)$ are independent probabilities such that $\mu(p) + \mu(q) = 1$.

For the MB statistics, the probability of obtaining a state with n cats and $11-n$ dog is given by

$$P_{MB}(p_{n,11-n},\mu(p),\mu(q)) = \frac{11!}{n!(11-n)!}\mu(p)^n\mu(q)^{11-n}. \tag{7.14}$$

For example, if $\mu(q) = \mu(p) = 0.5$, the number of possible arrangements for the state 'eleven cats' and for the state 'eleven dogs' is 1. Hence, the corresponding probability for these configurations is $P_{MB}(p_{0,11}, 0.5, 0.5) = 0.0005$.

Since we consider two states, and the FD distribution is subjected to the Pauli exclusion principle, we cannot apply the FD statistics in this case. For the BE statistics, the probability to obtain a state with n cats and $11-n$ dog is given by

$$P_{BE}(p_{n,11-n},\mu(q),\mu(p)) = \frac{n\mu(p)+(11-n)\mu(q)}{\frac{12\times 11}{2}}. \tag{7.15}$$

Since $\mu(p) = 1-\mu(q)$, then $P_{BE}(p_{n,11-n},\mu(p),\mu(q))$ is linear with respect to n. Moreover, when $\mu(p) = \mu(q) = 0.5$, we have that $P_{BE}(p_{n,11-n}, 0.5, 0.5) = \frac{1}{12}$ is constant.

The above analysis shows that, if one performs experiments on a collection of concepts to estimate the probability of elicitation for each state, then it is possible to determine which type of distribution, MB or BE, is generated.

Psychological Experiment to Test Indistinguishability

The psychological experiment involved 88 participants. We considered a list of concepts $\mathcal{A}^i$, for $i = 1, \ldots, 14$, of different nature, both physical and non-physical, and two possible exemplars, p_1^i and p_2^i, for each concept. Next, we requested participants to choose one exemplar from the combination $N^i\mathcal{A}^i$ of concepts for $N^i \in \mathbb{N}$. The exemplars of these combinations of concepts are the states p^i_{k,N^i-k} describing the conceptual combination 'k exemplars in state p^i and (N^i-k) exemplars in state p^i', where k is an integer such that $k = 0, \ldots, N^i$. For example, the first collection of concepts we considered is $N^1\mathcal{A}^1$ corresponding to the compound conceptual entity '*Eleven Animals*,' with p^1 and q^1 describing the exemplars 'cat' and 'dog' of the concept '*Animal*,' and $N^1 = 11$. The exemplars considered are $p^1_{11,0}$, $p^1_{10,1}$, …, $p^1_{1,10}$, and $p^1_{0,11}$. The collections of concepts used in the experiment and their corresponding exemplars are listed in Table 7.1.

For each $i = 1, ..., 14$, we fitted the experimental data using the distributions $P_{MB}(p_{n,11-n}, \mu(p^i), \mu(q^i))$ and $P_{BE}(p_{n,11-n}, \mu(p^i), \mu(q^i))$ by choosing the values for $\mu(p^i)$ that minimize the R-squared value of the fit[15], for $i = 1, \ldots, 14$.

Next, we used the 'Bayesian Information Criterion' (BIC) [KR95] to estimate which model provides the best fit. Table 7.2 summarizes the statistical analysis. The first column of this table identifies the collection of concepts, the second and third columns show the value of the probability parameter $\mu(p^i)$ and the R^2 value of the best MB statistical fit, the fourth and fifth columns show the value of the probability parameter $\mu(p^i)$ and the R^2 value of the best BE statistical fit. The sixth column shows the Δ_{BIC} criterion to discern between $P_{MB}(p^i n, 11-n, \mu(p^i)\mu(q^i)$ and $P_{BE}(p^i_{n,11-n}, \mu(p^i)\mu(q^i)$, and the seventh column identifies the distribution that best represent the data for concept $\mathcal{A}^i$, $i = 1, \ldots, 14$. Negative Δ_{BIC} values imply that the concept is best fitted by a MB distribution, whereas positive Δ_{BIC} values

[15]Only one of the two values is sufficient since $\mu(p^i) = 1 - \mu(q^i)$.

Table 7.1: List of concepts and their respective exemplars for the psychological experiment on indistinguishability.

i	N^i	$\mathcal{A}^i$	p^i	q^i
1	11	'*Animals*'	'cat'	'dog'
2	9	'*Humans*'	'man'	'woman'
3	8	'*Expressions of Emotion*'	'laugh'	'cry'
4	7	'*Expressions of Affection*'	'kiss'	'hug'
5	11	'*Moods*'	'happy'	'sad'
6	8	'*Parts of Face*'	'nose'	'chin'
7	9	'*Movements*'	'step'	'run'
8	11	'*Animals*'	'whale'	'condor'
9	9	'*Humans*'	'child'	'elder'
10	8	'*Expressions of Emotion*'	'sigh'	'moan'
11	7	'*Expressions of Affection*'	'caress'	'present'
12	11	'*Moods*'	'thoughtful'	'bored'
13	8	'*Parts of Face*'	'eye'	'cheek'
14	9	'*Movements*'	'jump'	'crawl'

imply that the concept $\mathcal{A}^i$ is best fitted by a BE distribution. However, these statements are weak for $|\Delta_{\text{BIC}}| < 2$, moderate for $2 < |\Delta_{\text{BIC}}| < 8$, and strong for $8 < |\Delta_{\text{BIC}}|$ [KR95].

We see that concepts 2 and 9 show a strong Δ_{BIC} value towards MB statistics, and that concepts $1, 3, 5, 7, 11, 12$, and 14 show a strong Δ_{BIC} value towards BE statistics. Complementary to the BIC criterion, the R^2 value indicates of Δ_{BIC} can be confirmed with a good fit of the data. The concepts that have a strong indication towards one type of statistics and an R^2 value larger than 0.78 have their R^2 value in bold text. These cases are confirmed by both statistical indicators. Moreover, in all the cases with strong tendency towards one type of statistics, the R^2 value of the other type of statistics is poor. Interestingly, we can observe that the concepts that exhibit MB behavior are associated to physical entities, and that all the concepts associated to non-physical entities exhibit BE behavior.

We conclude that collections of concepts can behave statistically like quantum entities.

Table 7.2: Results of statistical fit for the psychological experiment. Each column refers to the 14 collections of concepts introduced in Table 7.1.

i	$\mu(p^i)$ MB	R^2_{MB}	$\mu(p^i)$ BE	R^2_{BE}	Δ_{BIC}	Best Model
1	0.55	-0.05	0.16	0.78	19.31	BE strong
2	0.57	**0.78**	0.42	0.44	-9.54	MB strong
3	0.82	0.29	0.96	**0.79**	10.81	BE strong
4	0.71	0.81	0.53	0.77	-1.69	MB weak
5	0.25	**0.79**	0.39	0.93	14.27	BE strong
6	0.62	0.59	0.61	0.57	-0.37	MB weak
7	0.72	0.41	0.64	**0.83**	12.66	BE strong
8	0.63	0.58	0.47	0.73	5.53	BE positive
9	0.45	**0.87**	0.26	0.67	-9.69	MB strong
10	0.59	0.50	0.63	0.77	7.17	BE positive
11	0.86	0.46	1.00	**0.87**	11.4	BE strong
12	0.21	0.77	0.00	0.87	6.68	BE strong
13	0.62	0.54	0.71	0.67	2.97	BE weak
14	0.81	0.20	0.91	**0.90**	20.68	BE strong

Indistinguishability in Concepts on the Web

We have adapted the experimental methodology of § 7.2.2 to study the indistinguishability of concepts on the web. We use a search engine to estimate the number of web pages in which different exemplars of a collection of concepts appear. In this way, we use the relative frequency of the exemplars to verify if the indistinguishability of concepts, identified in our psychological experiments, can also be manifested on the web.

Let $N^i \geq 3$ be an integer number, and consider four pairs of states (p^j, q^j), for each $j = 1, \ldots, 4$. Next, for each number $3 \leq k \leq N^i$ and pair (p^j, q^j) of states, we build a set of sentences r^j_{k,N^i-k} that refer to the state p^j_{k,N^i-k}. The states and numbers chosen for this experiment are shown in Tables 7.3 and 7.4. For example, the states p^1 and q^1 correspond to 'cat' and 'dog,' and the state $p^1_{1,3}$ describing 'three cats and one dog' is referred by the sentences $r^1_{1,3} = \{$'three cats and one dog', 'one dog and three cats'$\}$.

In this experiment, we counted the total number n^i_{k,N^i-k} of web pages where the sentences of r^j_{k,N^i-k} are found using the Bing search API for web developers[16]. Since n^i_{k,N^i-k} estimates the number of references to the state p^j_{k,N^i-k} in the web, we use their relative frequencies to estimate a distribution $P(p^i k, N^i - k, \mu(p^j), \mu(q^j))$ of the exemplars on the web. Thus, we can study if this distributions can be best described using the MB or BE distributions, for different values of N^i. We have built the distribution $P(p^i k, N^i - k, \mu(p^j), \mu(q^j))$ for $k = 3, ..., N^i$, for using $3 \leq N^i \leq 15$.

[16]For more information, see `http://datamarket.azure.com/dataset/bing/search`.

Table 7.3: List of singular/plural reference to states used to perform the web-based experiment.

j	p^j_1	p^j_2
1	"cat" / "cats"	"dog" / "dogs"
2	"man" / "men"	"woman" / "women"
3	"win" / "wins"	"loss" / "losses"
4	"son" / "sons"	"daughter" / "daughters"

Because the sample size is small, this study can only be considered preliminary. Moreover, there are certain technical difficulties, well-known in the field of computational semantics [Pyl84], that affect the result. They include the fact that a state can potentially be referred to by an infinite number of linguistic expressions, or be linked to an infinite number of concepts. Also, due to semantic ambiguities, the linguistic expression might in some cases not refer to the state we assume it to refer to. Even though these are strong limitations, we have found interesting evidence for BE statistics in the data.

We summarize our results in Table 7.5. The first column specifies N^i, the other four columns specify the pair of states used in the experiment. Each entry in the table contains a pair of numbers. The first number is the BIC criteria, Δ_{BIC}, and the second number is the R^2 value of the best fit. As before, negative Δ_{BIC} values imply that the concept is best fitted by a MB distribution, whereas positive Δ_{BIC} values imply that the concept is

Table 7.4: List of references to numbers used to perform web-based experiment.

N^i	List of references
0	"0","no","zero"
1	"1","a","one"
2	"2","two","a couple of"
3	"3","three"
4	"4","four"
5	"5","five"
6	"6","six"
7	"7","seven"
8	"8","eight"
9	"9","nine"
10	"10","ten"
11	"11","eleven"
12	"12","twelve"
13	"13","thirteen"
14	"14","fourteen"
15	"15","fifteen"
16	"16","sixteen"

best fitted by a BE distribution. These statements are weak for $|\Delta_{\text{BIC}}| < 2$, moderate for $2 < |\Delta_{\text{BIC}}| < 8$, and strong for $8 < |\Delta_{\text{BIC}}|$ [KR95]. If $R^2 > 0.65$, we omit the value to emphasize that we do not have a significant fit.

We can identify three trends:

(i) when $3 \leq N^i \leq 8$, the majority of pairs of states exhibit MB statistics,

(ii) when $9 \leq N^i \leq 15$, the majority of pairs of states exhibit BE statistics, and

(iii) for $N^i \in \{11, 13, 14, 15\}$, at least two pairs of states show a poor R^2 fit.

These results indicate that, for $N^i \leq 8$, the concepts in the combination behave like distinguishable entities, while for $9 \leq N^i$, they become indistinguishable. This suggest that, when numbers are large enough, humans tend to treat collections of concepts as indistinguishable entities. This is consistent with the fact that we cannot generally remember, repeat, or compare collections of more than seven or eight distinguishable entities [CMC07]. However, by not trying to distinguish the entities when elicited in large collections, we make use of language to properly communicate large collections of concepts and reason about them. The third trend shows that for some

Table 7.5: Results of statistical fit of web-based experiment. The numbers in bold correspond to the cases where the BE-distribution provides a best fit according to the Δ_{BIC} and R^2 criteria.

N^i	$j = 1$	$j = 2$	$j = 3$	$j = 4$
3	$-3.9, 0.79$	$-4.4, 0.82$	$-1.5, -$	$-3.9, 0.80$
4	$-8.9, 0.92$	$-7.4, 0.91$	$-4.2, 0.80$	$-9.2, 0.93$
5	$-10.5, 0.94$	$-3.83, 0.81$	$-8.20, 0.90$	$-14.7, 0.97$
6	$-5.0, 0.84$	$-3.6, 0.82$	$-2.6, 0.80$	$-15.5, 0.96$
7	$-2.0, 0.77$	$\mathbf{3.1, 0.72}$	$-1.5, 0.75$	$-4.7, 0.85$
8	$\mathbf{2.0, 0.72}$	$-0.1, 0.74$	$-0.8, 0.77$	$-1.5, 0.79$
9	$\mathbf{5.5, 0.69}$	$\mathbf{7.3, 0.76}$	$\mathbf{6.4, 0.78}$	$-7.4, 0.87$
10	$\mathbf{9.0, 0.70}$	$0.5, -$	$\mathbf{10.5, 0.77}$	$-11.4, 0.89$
11	$2.4, -$	$10.0, -$	$\mathbf{9.3, 0.73}$	$-5.2, 0.80$
12	$\mathbf{10.4, 0.70}$	$\mathbf{7.0, 0.72}$	$\mathbf{11.1, 0.72}$	$-6.4, 0.79$
13	$6.6, -$	$11.1, -$	$\mathbf{12.7, 0.76}$	$9.4, -$
14	$13.6, -$	$\mathbf{17.3, 0.71}$	$\mathbf{10.8, 0.76}$	$-8.9, -$
15	$9.1, -$	$\mathbf{23.0, 0.79}$	$2.3, -$	$-17.6, 0.8$

numbers above ten, the data does not fit BE or MB distributions. This is probably because the data is sparse which leads to strongly irregular distributions.

We conclude that the statistical behavior of collections of concepts can resemble the statistical behavior of quantum particles in both psychological and NLP experimental settings. Moreover, since quantum structures can be observed in different NLP settings, this suggests that quantum cognition tools should be applied in the context of NLP.

Chapter 8

Conclusion

8.1 General Conclusion

In this thesis, we performed a systematic review of the quantum-cognitive approach to concepts (part I), proposed a framework that enhances the range of applications of concept combination models (part II), and presented evidence of quantum conceptual structures in the context of natural language processing (part III).

We have elucidated the mathematical structure of the two-sector Fock space model for concept combinations based on either conjunctions or disjunctions and studied how the dimension of the space $\mathcal{H} = \mathbb{C}^n$ influences the modeling power on each sector of the model, and concluded that $\mathcal{H} = \mathbb{C}^3$ is sufficient for maximal modeling power in both sectors.

Next, we introduced unitary transformations to represent concept combinations for multiple exemplars for each sector separately, and then combined these representations to obtain a representation in the two-sector Fock space $\mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \mathbb{C}^3$. This representation is consistent with the cognitive principles of quantum modeling, it also maximizes the modeling power and permits the representation of multiple exemplars simultanously. Our data analysis shows that, when the first sector is approximately 80% dominant with respect to the second sector, the two-sector Fock space model provides the optimal performance.

We later studied concept combinations built from conjunctions and negations. We first identified the conditions that characterize classical data, and found that this data is regularly violated. Moreover, we performed a statistical analysis to characterize this violation of data, and found that the violation is precisely characterized by a constant value. Next, we extended the representations developed for the two-sector Fock space model for conjunctions to the case of conjunctions and negations. Our data analysis indicates that, when the first sector is approximately 80% dominant with respect to the second sector, the pattern we identified for the violation of classical conditions is duplicated, and moreover, the two-sector Fock space model provides the optimal performance.

The conclusion for the second part of the thesis is that the two-sector Fock space model is not only a powerful tool to represent conceptual combinations, but it also provides a sensible explanation for the fact that humans do not reason logically. In particular, our results indicate that the emergent mode of reasoning modeled by the first sector is 80% dominant with respect to its logical counterpart represented in the second sector.

In the third part of the thesis, we considered the application of quantum cognition in the context of natural language processing. We presented two studies identifying quantum structures in natural language phenomena. In the first, we developed a methodology to identify sets of words that statistically behave as quantum entangled particles in a corpus of text, and showed that in many cases sets of words can behave as entangled entities. In the second study, we have demonstrated that references to exemplars of collections of concepts statistically behave as indistinguishable (quantum) entities using data from psychological and web-based studies. Moreover, we found in this study that there is a tendency for non-physical concepts to follow the statistics of indistinguishable particles, while for physical concepts the tendency is to follow the statistics of classical particles.

The conclusion of this thesis is that quantum cognition proposes a suitable framework for a theory for concepts that can be applied to model cognitive phenomena. In particular, the possibility to model non-classical processes by means of superposition, entanglement, and indistinguishability, entails a fundamental feature that deserves further exploration.

8.2 Future Work

Here, we propose three ideas for future work inspired by our results in Chapters 5, 6, and 7 respectively:

8.2.1 Incompatible Exemplars

In the concrete representations of concepts and their combinations introduced in Chapter 5, all measurements are expressed in the same basis.

This enables us to investigate the relation between the exemplars by analyzing the structure of their measurement operators. For example, given two measurements $\mathbf{M}_1$ and $\mathbf{M}_2$, we have that $\mathbf{M}_1$ and $\mathbf{M}_2$ represent compatible observables if and only the commutator operator

$$[\mathbf{M}_1, \mathbf{M}_2] = \mathbf{M}_1\mathbf{M}_2 - \mathbf{M}_2\mathbf{M}_1 = 0. \tag{8.1}$$

Otherwise, the operators represent incompatible observables.

The existence of incompatible measurements is one of the most prominent examples of how quantum mechanics differs from the classical world. In particular, the famous Heisenberg uncertainty principle is implied by the existence of incompatible measurements [Hei27]. Therefore, an important question in quantum cognition is to elucidate if semantic estimations can be incompatible. Indeed, the existence of incompatible measurements would imply that the application of consecutive semantic estimations could create uncontrollable disturbances.

We have performed a preliminary calculation showing that membership operators in the representations derived in Chapter 5 are, in some cases, incompatible. Consider the concepts $\mathcal{A} =$ '*Machine,*' $\mathcal{B} =$'*Vehicle*,' and the exemplars $p_5 =$'sailboat,' and $p_{12} =$'skateboard.' For the case of conceptual conjunction we have

$$\begin{aligned} \mu_5(A) = 0.56, \mu_5(B) = 0.8, \mu_5(AB) = 0.42, \\ \mu_{12}(A) = 0.28, \mu_{12}(B) = 0.84, \mu_{12}(AB) = 0.34. \end{aligned} \tag{8.2}$$

Note that exemplars p_5 and p_{12} satisfy the conditions of Theorem 5.3. Thus, we obtain a concrete representation $\{|A\rangle, |B\rangle, \{\mathbf{M}_5, \mathbf{M}_{12}\}\}$ of these exemplars in $\mathbb{C}^3$. In this representation, we have that

$$\begin{aligned} \langle A|[\mathbf{M}_5, \mathbf{M}_{12}]|A\rangle = 0.084\mathbb{i} \\ \langle B|\mathbf{M}_5, \mathbf{M}_{12}]|B\rangle = 0.097\mathbb{i} \end{aligned} \tag{8.3}$$

Therefore, exemplars p_5 and p_{12} are incompatible with respect to the states $|A\rangle$ and $|B\rangle$.

Since the data we analyzed in Chapter 5 was collected presenting the exemplars in only one specific order [Ham88b], these computations demonstrate that it may be possible to predict order effects in membership measurements for exemplars that are incompatible. This results is, however, speculative since there is no experimental data where membership weight estimations have been done presenting exemplars in different orders.

One area of reseach would be to generate experimental data to check whether the predictions are accurate. If order effects are predictable, then the canonical representation proposed in this thesis could be used to develop Heisenberg-like uncertainty relations in the context of concept combination models.

8.2.2 Modeling Concept Combinations for Real-World Applications

The models for concepts and concept combinations presented in this thesis are not general enough for real-world applications. One of the reasons for this is that we do not have a model for the conjunction, disjunction, and negation of concepts. In fact, these three connectives are required to build the simplest concept combination structure used in computational applications [RN95]. Therefore, it is important to extend the models presented in thesis to incorporate these three connectives together. In fact, we could achieve a model for this connective structure by incorporating disjunctions to the model of Chapter 6. For the case of two concepts, this involves representations for states and measurements for the case of disjunction of concepts, and of disjunctions and negations.

Real-world applications generally require combinations of more than two concepts. In logic and computer science, the study of conjunctions, disjunctions, and negations of three or more concepts, usually called propositions, is known as the satisfiability problem [AN96]. This is necessary to determine whether or not there is a possible instantiation of a concept combination whose truth value is positive[17].

It would be interesting to study the satisfiability problem from the point of view of quantum cognition. That is, we could test the classical logical satisfiability conditions using psychological experiments and, in those cases where deviations from classical and fuzzy theoretical rules are found, develop a quantum model to handle these deviations.

[17] This means 'true' in the context of propositional logic, and a value above a certain threshold in fuzzy logic.

Because real-world applications usually involve a large number of concepts and a large number of exemplars for each concept, it is necessary to test our models against larger datasets. When we consider a large number of exemplars and concepts, it is very likely that some concepts have exemplars in common. This imposes a new type of constraint that has not been studied. The constraints for the representation of exemplars shared across multiple concepts will probably require representations in spaces of larger dimension.

8.2.3 Indistinguishability and Modes of Reasoning

The identification of concepts with the Maxwell-Boltzmann (MB) and Bose-Einstein (BE) statistics in § 7.2.2 assumes that a collection of concepts can be elicited as an exemplar representing a collection of entities existing in space and time, or a collection of indistinguishable entities of abstract nature. In the former case, the entities corresponding to the exemplar are distinguishable and behave according to the MB statistics, while in the latter case the entities are indistinguishable and behave according to the BE statistics.

In cognitive science, it is well-known that natural categories, usually referred by nouns in language, can be represented in a 'hierarchy' according to their 'level of abstraction' [Ros99]. For example, the concepts 'Puppy,' 'Dog,' and 'Mammal' are concepts ordered from lower to higher level of abstraction. This notion of abstraction is useful to explain that a concept at a lower level of abstraction can be an exemplar of a concept at a higher level. In fact, our analysis in § 7.2.2 reveals that concepts at a lower level of abstraction tend to behave as distinguishable entities, while concepts at a higher level of abstraction tend to behave as indistinguishable entities.

A possible extension to this thesis is to consider the notion of abstraction as a mode of reasoning rather than a property of a concept. In fact, although our results indicate that the more concrete the category is, the more distinguishable our reasoning about the category is, we can always consciously induce on ourselves a mode of reasoning that contradicts this tendency. Therefore, it would be interesting to develop experimental settings where the kind of reasoning applied to elicit concepts is controlled. Note that creating such methodology would allow us to compare concepts at different level of abstraction for a fixed kind of reasoning and, therefore, would generalize the results of § 7.2.2. Moreover, we could investigate different kinds of reasoning ranging from personal experiences that are constrained by our perceptual limitations and the structure of reality, to hypothetical worlds created by pure imagination where ‘the impossible’ can occur.

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Appendices

Appendices A

Traditional Modeling Tools

Different mathematical tools have been used to model cognitive phenomena. In what follows, we give a brief overview of the most relevant mathematical structures used in cognitive modelling to date. We refer to [RN95] for a more comprehensive review.

A.1 Classical logic

Classical logic is the first and most explored mathematical structure used to represent and process meaning. Indeed, classical logic has been formulated first by ancient greeks, in their search for a notion of truth and deductive procedures [CM72]. The idea behind logic is that a reasoning process, starting from certain 'true' basic facts, should allow us to deduce all possible true (or false) facts. The basic elements of any logical approach are i) a set of basic postulates that forms the starting universe of discourse, ii) certain connectives and relations to build new postulates from the basic ones, iii) deductive rules to reason.

Logic can be used, in principle, to formalize any process where some form of analytic reasoning is present. For example, propositional logic (PL) is defined as a system $\mathcal{L} = (P, C, R, A)$, where P is a set of propositional variables, C is a set of connectives and relations that include 'and,' 'or,' 'not,' and 'then' denoted by $\wedge, \vee, \neg$ and $\rightarrow$, R is a set of deduction rules, and A is the set of axioms. A typical element of P is a basic proposition such as $p_1 =$*Today is Monday*, and $p_2 =$*John goes to a restaurant*. R contains the modus ponens such as from p and $p \rightarrow q$, infer q. And axioms such as $p \rightarrow p$ and $\neg\neg p \rightarrow p$ are basic truths in the logical system.

A logical system should satisfy at least two basic conditions. The first, called soundness, requires that deduction rules only prove formulas that are true. The second feature, called completeness, requires that every true proposition must be provable within the logical system. Unfortunately, the latter condition is hard to apply in practice, since it is known that the complexity of the satisfiability of a conjunctive proposition requires at least non-deterministic polynomial time with respect to the proposition's lenght [Pyl84].

The set of connectives C is of particular interest since it may limit how much can be proven in a certain logical system. There are different expressivity levels depending on the types of connectives we use. For example, the connectives of PL are always finitely evaluated. However, connectives such as $\forall$, which means 'for all,' require evaluations of formulas in potentially an infinite number of cases. Consider the simple mathematical inequality $\forall x < 1 \rightarrow x < 2$. This proposition is true for infinitely many numbers (all numbers smaller than 1), so it cannot be stated within a PL system. The use of connectives and relations with different levels of expressivity leads to an extremely fine-grained development of 'modal' logics that reveal a hierarchy of logical systems organized according to their degree of expressivity [Cha97].

Logic is also deeply connected with the notion of computation. Indeed, the basic operations that a computer performs correspond to logical operations, and thus all procedures that a computer performs can be reduced to logical formulas. In particular, an interesting connection between logic and computation is in the area of descriptive complexity [Imm87], where different classes of computational complexity are mapped to different logical languages.

Formal Concept Analysis (FCA) is an example of the application of logic to model concepts. FCA is based on a particular formalization of the notion of concept, inspired by the view of concepts in traditional logic, which assumes that a concept can be described in terms of a set of attributes, where each exemplar corresponds to a propositional-logic combination of some of these attributes [AN96]. In its basic form, FCA analyses input data consisting of objects determined by a set of attributes assumed to be held by the object.

The primary aim in FCA is to extract from the input data all consistent or formal concepts that form a hierarchy, called the concept lattice, and a set of particular attribute dependencies known as attribute implications [GSW05]. A formal concept is defined as a pair consisting of two sets: a set of objects to which the concept applies, the concepts extent, and a set of attributes that characterize the concept, the concepts intent. An attribute implication is an expression $A \rightarrow B$. If an object has all the attributes in A, then it also has all the attributes in B. For example, let $A = \{$drinksalcohol, smokes} and $B = \{\text{heart-problems}\}$.

The concept lattice is used to derive the minimal set of attribute implications. This set can be used to find hidden causal relations that are not evident from the dataset. It has been shown that the attribute implication has a non-redundant minimal base from which all the dependences can be obtained applying standard deduction rules [GSW05].

Applications of FCA can be found in many areas, including engineering, natural and social sciences, and mathematics [Aré03, GSW05]. In cognitive modelling, FCA has been proposed as a possible structure to study property correlations that might foster categorization [Boe97]. However from a structural perspective, FCA is a non-graded mathematical formalism, so it cannot account for the membership and typicality functions of a concept Theory.

A.2 Fuzzy Logic and Fuzzy Set Theory

Fuzzy logic is an extension of classical logic where propositions can have degrees of truth. Fuzzy set theory is the analogous extension of classical set theory. It allows elements of a set to have degrees of membership ranging from null to total membership. The degree of membership is measured on a $[0, 1]$ scale, where 0 means *not a member*, and 1 means *full member*. This is useful for logical settings where it is impossible to assign binary membership values [Zad65]. Fuzzy set theory and fuzzy logic are mathematically related. Here, we elaborate on fuzzy set theory because it has been used as a mathematical framework for the prototype theory of concepts.

Given a set $X \neq \{\emptyset\}$, a fuzzy set is a pair $(\mathbf{X}, f)$ where $f : \mathbf{X} \to [0, 1]$ [Zad65]. The value $f(x)$ is usually interpreted as a degree of membership of x in $\mathbf{X}$. One of the important notions in fuzzy set theory, is the threshold dependent set $\mathbf{X}_\alpha$, the set that contains elements having a degree of membership above α. This emphasizes the elements that have higher membership, and represents a fingerprint of the structure of the set when we view $\mathbf{X}_\alpha$ as a function of α.

Fuzzy set theory is equipped with algebraic operators that are extensions of the logical operators of classical set theory, including union, intersection, and complement operators. However, these operators can be extended in many different ways, and there is no formal criteria to decide which extension is more approriate [BK11]. An interesting class of extensions, called triangular norms (or t-norms), provide the most reasonable extension for the binary set theoretical operators $\wedge$ ('and'), and $\vee$ ('or'.) These operators are built in terms of minimum and maximum membership functions and, in some cases, include parameters that can be used to tailor the norm to better fit experimental data [BK11].

Aggregation operators that depend on the membership of all the elements in the fuzzy set can also be defined. These operators are used to extract information regarding the membership structure of the set. For example, the cardinality of a fuzzy set is defined as the sum of the membership of all its elements. From here, the average membership of the fuzzy set is defined as the ratio between the fuzzy and classical cardinality. Aggregation operators are used to develop fuzzy notions of connectives belonging to high expressivity formal languages found in Modal logic.

Fuzzy set theory has been applied in a broad spectrum of areas related to automatization such as control theory and expert systems [Low96]. In cognitive modelling, fuzzy set theory was first used to frame the prototype theory of concepts [Ros99]. A concept $\mathcal{A}$ is defined by a universal set of exemplars U, a membership function f, and a membership threshold α. f can also be interpreted as a measure of typicality, or of similarity to a specific prototype [Lak73]. The concept is modeled by the fuzzy set $(\mathbf{U}, f)$. To recover truth evaluations, a threshold is imposed for f so the categories can be treated as sets. This approach extends classical approach to concepts, but still lacks a formal procedure to combine categories forming complex categories or sentences.

A.3 Probabilistic Approaches

Probability theory has been applied to almost every possible science-related endeavour [HPS71]. In cognitive modeling, the degree of membership of an exemplar with respect to a concept can be thought of as an estimation of the probability of being a member of the category. There are procedures, developed using standard probability theory, to infer the membership probability of some exemplars from the membership probability of others [TKGG11]. Analogously, the notions of typicality, property relevance, and similarity can also be framed in a probabilistic manner [Nos88].

Before showing how probability theory has been applied to concepts research, we consider the foundational aspects of probability as it relates to concept theories.

A.3.1 Interpretations of Probability

There are three main interpretations of probability. The first assumes that probabilities are the relative frequencies with which the possible outcomes of a situation occur [Khr99]. This interpretation is useful when observations are repeatable. For example, when studying the possible outcomes of throwing a die, the relative frequencies of each outcome tend to $\frac{1}{6}$ as the number of repetitions of the experiment grows. We may assume, therefore, that there is an underlying probability that accounts for the relative frequencies of occurrences describing the phenomena.

There are several probability estimations that do not correspond to such a view [Khr99]. For example, we know from atomic physics that the decaying time of certain radioactive elements corresponds to thousands of years. Clearly, we cannot afford to repeat many experiments, so the decaying time is obtained through a formula whose parameters are established by a series of observations.

A second interpretation assumes that the outcome of a situation is ruled by an intrinsic propensity that would generate the observed relative frequencies if the experiment was repeated a large number of times [Khr99]. This interpretation however does not account for situations such as the likelihood that the current president will be reelected. This kind of subjective probability estimation cannot be interpreted from a relative frequency nor from an intrinsic propensity view.

The third interpretation of probability involves a subjective belief interpretation. That is, it is assumed that estimators have a degree of belief concerning the possible outcomes of a certain situation [Khr99]. Because estimations about concepts must be built upon cognitive mechanisms based on the information available to estimators, this is the interpretation most often used in theories of concepts. It is interesting to note that while philosophers are concerned with the relation between foundations of probability and concepts [Wit58], the modeling community has often ignored this issue [Aer02].

A.3.2 Probability Spaces

Formally, a probabilistic space $\mathcal{P}$ is defined by the triple

$$\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{P}), \tag{A.1}$$

where Ω is a set of *elementary events* $w \in \Omega$, $\mathcal{F}$ is a σ-algebra of the subsets of Ω, and $\mathbf{P}$ is a σ-additive probability measure from $\mathcal{F}$ to $[0, 1]$ such that

$$\begin{aligned} &\mathbf{P}(A) \geq 0, \text{ for } A \in \mathcal{F}, \\ &\mathbf{P}(\Omega) = 1, \\ &\mathbf{P}(\cup_i A_i) = \sum_i \mathbf{P}(A_i), \text{ for disjoints sets } A_i \in \mathcal{F}, \ i \in \mathbb{N}. \end{aligned} \tag{A.2}$$

Here we briefly explain how experiments on a system are modeled in a probabilistic setting.

A partition $\mathcal{E}$ of Ω is a set of subsets of Ω such that for all $E_1, E_2 \in \mathcal{E}$

$$E_1 \cap E_2 = \emptyset, \quad \text{and} \quad \cup_{E \in \mathcal{E}} E = \Omega.$$

Because outcomes of each experiment are exclusive, experiments are partitions of Ω. The events representing the outcomes form disjoint sets. Also, each possible post-experimental situations of the system corresponds to an outcome so the set of outcomes of each experiment is complete and thus the union of the set of outcomes of an experiment is Ω.

Denote the experiments by $\mathcal{E}_i$ for $i = 1, ..., n$, and let $E_i = \{E_i^1, E_i^2, ..., \}$ be the set of outcomes for $\mathcal{E}_i$. E_i can be either finite or infinite. For example, let $\mathcal{E}_1$ be the experiment of measuring whether a particle is on the positive or negative side of a referential axis. In this case, we have two outcomes, E_1^+ and E_1^-, depending on which side the particle is. Let $\mathcal{E}_2$ be the experiment of measuring the exact position of the particle on a referential axis. In this case the set of outcomes is infinite.

We are interested in the joint probability distribution, $\mathbf{P}(E_1, ..., E_n)$, which gives us the probability of all possible outcome configurations of our experimental setting. In particular, a specific outcome configuration

$(E_1^{j_1}, ..., E_n^{j_n})$ has probability

$$\mathbf{P}(\cap_{i=1}^{n} E_i^{j_i}) = \mathbf{P}(E_1^{j_1}, ..., E_n^{j_n}). \tag{A.3}$$

In many cases, rather than performing all possible experiments in our system, we focus on a few experiments. For simplicity let $n = 2$, and assume we would like to know only the probabilities of the outcomes of experiment $\mathcal{E}_2$. The probability distribution that only considers the outcomes of $\mathcal{E}_2$ is known as the marginal probability distribution, $\mathbf{P}(E_2)$, computed by

$$\mathbf{P}(E_2) = \sum_{j=1}^{|E_1|} \mathbf{P}(E_1^j, E_2). \tag{A.4}$$

This formula, known as the marginal probability law, margins out the probability of outcomes for $\mathcal{E}_2$ by summing over all the possible outcomes of $\mathcal{E}_1$. We can also compute the probability of the outcomes for $\mathcal{E}_2$ given that the experiment $\mathcal{E}_1$ has already been performed. This is known as the conditional probability, $\mathbf{P}(E_2|E_1^k)$ given by

$$\mathbf{P}(E_2|E_1^k) = \frac{\mathbf{P}(E_1^k \cap E_2)}{\mathbf{P}(E_1^k)}, \text{ for } \mathbf{P}(E_1^k) > 0 \tag{A.5}$$

Appendices B

Membership of Conjunctions and Negations of Concepts

Table B.1: Representation of the membership weights in the case of the concepts *Home Furnishing* and *Furniture*.

A=Home Furnishing, B=Furniture																	
Exemplar	$\mu(A)$	$\mu(B)$	$\mu(A')$	$\mu(B')$	$\mu(AB)$	$\mu(A\bar{B})$	$\mu(\bar{A}B)$	$\mu(\bar{A}\bar{B})$	Δ_{AB}	$\Delta_{A\bar{B}}$	$\Delta_{\bar{A}'B}$	$\Delta_{\bar{A}\bar{B}}$	$I_{AB\bar{A}\bar{B}}$	I_A	I_B	$I_{\bar{A}}$	$I_{\bar{B}}$
Mantelpiece	0.9	0.61	0.12	0.5	0.71	0.75	0.21	0.21	0.1	0.25	0.09	0.09	-0.89	-0.56	-0.31	-0.31	-0.46
Window Seat	0.5	0.48	0.47	0.55	0.45	0.49	0.39	0.41	-0.03	-0.01	-0.08	-0.06	-0.74	-0.44	-0.36	-0.33	-0.35
Painting	0.8	0.49	0.35	0.64	0.64	0.6	0.33	0.38	0.15	-0.04	-0.03	0.03	-0.94	-0.44	-0.48	-0.35	-0.33
Light Fixture	0.88	0.6	0.16	0.51	0.73	0.63	0.33	0.16	0.13	0.11	0.16	0	-0.84	-0.48	-0.45	-0.33	-0.28
Kitchen Counter	0.67	0.49	0.31	0.62	0.55	0.54	0.38	0.33	0.06	-0.08	0.06	0.01	-0.79	-0.42	-0.44	-0.39	-0.24
Bath Tub	0.73	0.51	0.28	0.46	0.59	0.59	0.36	0.29	0.08	0.13	0.08	0.01	-0.83	-0.45	-0.44	-0.37	-0.41
Deck Chair	0.73	0.9	0.27	0.2	0.74	0.41	0.54	0.18	0.01	0.21	0.27	-0.03	-0.86	-0.42	-0.38	-0.44	-0.39
Shelves	0.85	0.93	0.24	0.13	0.84	0.39	0.53	0.08	-0.01	0.26	0.29	-0.05	-0.83	-0.38	-0.43	-0.36	-0.34
Rug	0.89	0.58	0.18	0.61	0.7	0.68	0.41	0.21	0.13	0.07	0.24	0.04	-1	-0.48	-0.54	-0.45	-0.28
Bed	0.76	0.93	0.26	0.11	0.79	0.36	0.61	0.14	0.03	0.26	0.35	0.03	-0.9	-0.39	-0.48	-0.49	-0.39
Wall-Hangings	0.87	0.46	0.21	0.68	0.55	0.71	0.35	0.24	0.09	0.03	0.14	0.03	-0.85	-0.39	-0.44	-0.38	-0.27
Space Rack	0.38	0.43	0.63	0.62	0.41	0.49	0.43	0.58	0.04	0.11	0	-0.04	-0.9	-0.53	-0.41	-0.37	-0.44
Ashtray	0.74	0.4	0.32	0.64	0.49	0.6	0.36	0.39	0.09	-0.04	0.04	0.07	-0.84	-0.34	-0.45	-0.43	-0.35
Bar	0.72	0.63	0.37	0.51	0.61	0.61	0.4	0.4	-0.01	0.11	0.03	0.03	-1.03	-0.51	-0.39	-0.43	-0.51
Lamp	0.94	0.64	0.15	0.49	0.75	0.7	0.4	0.2	0.11	0.21	0.25	0.05	-1.05	-0.51	-0.51	-0.45	-0.41
Wall Mirror	0.91	0.76	0.13	0.45	0.83	0.66	0.44	0.14	0.07	0.21	0.31	0.01	-1.06	-0.58	-0.51	-0.45	-0.35
Door Bell	0.75	0.33	0.32	0.79	0.5	0.64	0.34	0.51	0.17	-0.11	0.02	0.19	-0.99	-0.39	-0.51	-0.53	-0.36
Hammock	0.62	0.66	0.41	0.41	0.6	0.5	0.56	0.31	-0.02	0.09	0.16	-0.09	-0.98	-0.48	-0.5	-0.47	-0.41
Desk	0.78	0.95	0.31	0.09	0.78	0.33	0.75	0.15	-0.01	0.24	0.44	0.06	-1	-0.32	-0.58	-0.59	-0.39
Refrigerator	0.74	0.73	0.26	0.41	0.66	0.55	0.46	0.25	-0.06	0.14	0.21	-0.01	-0.93	-0.47	-0.4	-0.46	-0.39
Park Bench	0.53	0.66	0.59	0.46	0.55	0.29	0.56	0.39	0.02	-0.17	-0.03	-0.07	-0.79	-0.31	-0.45	-0.36	-0.22
Waste Paper Basket	0.69	0.54	0.36	0.63	0.59	0.41	0.46	0.49	0.04	-0.22	0.1	0.13	-0.95	-0.31	-0.51	-0.59	-0.27
Sculpture	0.83	0.46	0.34	0.66	0.58	0.73	0.46	0.36	0.11	0.07	0.13	0.03	-1.13	-0.48	-0.58	-0.49	-0.43
Sink Unit	0.71	0.57	0.34	0.58	0.6	0.56	0.38	0.38	0.03	-0.01	0.04	0.04	-0.91	-0.46	-0.41	-0.41	-0.36

Table B.2: Representation of the membership weights in the case of the concepts *Spices* and *Herbs*.

A=Spices, B=Herbs																	
Exemplar	$\mu(A)$	$\mu(B)$	$\mu(A')$	$\mu(B')$	$\mu(AB)$	$\mu(A\bar{B})$	$\mu(\bar{A}B)$	$\mu(\bar{A}\bar{B})$	Δ_{AB}	$\Delta_{A\bar{B}}$	$\Delta_{\bar{A}'B}$	$\Delta_{\bar{A}\bar{B}}$	$I_{AB\bar{A}\bar{B}}$	I_A	I_B	$I_{\bar{A}}$	$I_{\bar{B}}$
Molasses	0.36	0.13	0.67	0.84	0.24	0.54	0.25	0.73	0.11	0.18	0.12	0.06	-0.75	-0.41	-0.36	-0.31	-0.43
Salt	0.67	0.04	0.36	0.92	0.24	0.69	0.09	0.6	0.19	0.02	0.04	0.24	-0.61	-0.26	-0.28	-0.33	-0.37
Peppermint	0.67	0.93	0.38	0.1	0.7	0.38	0.55	0.15	0.03	0.28	0.18	0.05	-0.78	-0.41	-0.33	-0.33	-0.43
Curry	0.96	0.28	0.04	0.78	0.54	0.88	0.16	0.21	0.26	0.1	0.13	0.18	-0.79	-0.45	-0.42	-0.34	-0.31
Oregano	0.81	0.86	0.21	0.13	0.79	0.4	0.5	0.08	-0.03	0.28	0.29	-0.05	-0.76	-0.38	-0.43	-0.36	-0.35
MSG	0.44	0.12	0.59	0.85	0.23	0.58	0.24	0.73	0.11	0.13	0.12	0.13	-0.76	-0.36	-0.34	-0.37	-0.45
Chili Pepper	0.98	0.53	0.05	0.56	0.8	0.9	0.28	0.13	0.27	0.34	0.23	0.08	-1.1	-0.73	-0.54	-0.35	-0.46
Mustard	0.65	0.28	0.39	0.71	0.49	0.65	0.23	0.46	0.21	0	-0.05	0.08	-0.83	-0.49	-0.44	-0.3	-0.41
Mint	0.64	0.96	0.43	0.09	0.79	0.31	0.64	0.11	0.14	0.23	0.21	0.03	-0.85	-0.46	-0.47	-0.32	-0.34
Cinnamon	1	0.49	0.02	0.51	0.69	0.79	0.21	0.15	0.19	0.28	0.19	0.13	-0.84	-0.48	-0.41	-0.34	-0.43
Parsley	0.54	0.9	0.54	0.09	0.68	0.26	0.73	0.18	0.14	0.18	0.19	0.09	-0.84	-0.4	-0.5	-0.36	-0.35
Saccarin	0.34	0.14	0.68	0.88	0.24	0.54	0.24	0.8	0.1	0.19	0.1	0.12	-0.81	-0.43	-0.34	-0.36	-0.46
Poppy Seeds	0.82	0.47	0.29	0.54	0.59	0.66	0.31	0.28	0.12	0.13	0.02	-0.02	-0.84	-0.43	-0.43	-0.29	-0.4
Pepper	0.99	0.47	0.1	0.58	0.7	0.9	0.18	0.14	0.23	0.32	0.08	0.04	-0.91	-0.61	-0.41	-0.21	-0.46
Turmeric	0.88	0.53	0.11	0.43	0.74	0.69	0.28	0.21	0.21	0.26	0.16	0.1	-0.91	-0.54	-0.49	-0.38	-0.47
Sugar	0.45	0.34	0.59	0.77	0.35	0.56	0.25	0.65	0.01	0.11	-0.09	0.06	-0.81	-0.46	-0.26	-0.31	-0.44
Vinegar	0.3	0.11	0.76	0.88	0.15	0.41	0.26	0.83	0.04	0.11	0.16	0.07	-0.65	-0.26	-0.31	-0.33	-0.36
Sesame Seeds	0.8	0.49	0.3	0.59	0.59	0.7	0.34	0.29	0.1	0.11	0.04	-0.01	-0.91	-0.49	-0.44	-0.33	-0.4
Lemon Juice	0.28	0.2	0.74	0.81	0.15	0.43	0.39	0.81	-0.05	0.15	0.19	0.07	-0.78	-0.3	-0.34	-0.46	-0.43
Chocolate	0.27	0.21	0.78	0.8	0.2	0.46	0.38	0.78	-0.01	0.19	0.16	-0.01	-0.81	-0.39	-0.36	-0.37	-0.44
Horseradish	0.61	0.67	0.48	0.28	0.61	0.4	0.53	0.33	0	0.12	0.04	0.04	-0.86	-0.4	-0.47	-0.37	-0.44
Vanilla	0.76	0.51	0.3	0.49	0.63	0.61	0.33	0.35	0.11	0.13	0.03	0.05	-0.91	-0.48	-0.44	-0.38	-0.48
Chives	0.66	0.89	0.43	0.26	0.76	0.28	0.64	0.31	0.1	0.02	0.21	0.06	-0.99	-0.38	-0.51	-0.53	-0.33
Root Ginger	0.84	0.56	0.23	0.44	0.69	0.59	0.41	0.23	0.13	0.14	0.18	-0.01	-0.91	-0.43	-0.54	-0.41	-0.37

Table B.3: Representation of the membership weights in the case of the concepts *Pets* and *Farmyard Animals*.

A=Pets, B=Farmyard Animals																	
Exemplar	$\mu(A)$	$\mu(B)$	$\mu(A')$	$\mu(B')$	$\mu(AB)$	$\mu(A\bar{B})$	$\mu(\bar{A}B)$	$\mu(\bar{A}\bar{B})$	Δ_{AB}	$\Delta_{A\bar{B}}$	$\Delta_{\bar{A}'B}$	$\Delta_{\bar{A}\bar{B}}$	$I_{AB\bar{A}\bar{B}}$	I_A	I_B	$I_{\bar{A}}$	$I_{\bar{B}}$
Goldfish	0.93	0.17	0.12	0.81	0.43	0.91	0.18	0.43	0.26	0.1	0.06	0.31	-0.94	-0.41	-0.43	-0.48	-0.53
Robin	0.28	0.36	0.71	0.64	0.31	0.35	0.46	0.46	0.04	0.08	0.1	-0.18	-0.59	-0.39	-0.41	-0.22	-0.18
Blue-tit	0.25	0.31	0.76	0.71	0.18	0.39	0.44	0.56	-0.08	0.14	0.13	-0.15	-0.56	-0.31	-0.3	-0.24	-0.24
Collie Dog	0.95	0.77	0.03	0.35	0.86	0.56	0.25	0.11	0.09	0.21	0.23	0.09	-0.79	-0.48	-0.34	-0.34	-0.33
Camel	0.16	0.26	0.89	0.75	0.2	0.31	0.51	0.68	0.04	0.16	0.26	-0.08	-0.7	-0.36	-0.46	-0.3	-0.24
Squirrel	0.3	0.39	0.74	0.65	0.28	0.26	0.46	0.59	-0.03	-0.04	0.07	-0.06	-0.59	-0.24	-0.34	-0.31	-0.2
Guide Dog for Blind	0.93	0.33	0.13	0.69	0.55	0.73	0.16	0.33	0.23	0.03	0.04	0.2	-0.76	-0.35	-0.39	-0.36	-0.36
Spider	0.31	0.39	0.73	0.63	0.31	0.31	0.44	0.51	0	0	0.05	-0.12	-0.58	-0.31	-0.36	-0.23	-0.19
Homing Pigeon	0.41	0.71	0.61	0.34	0.56	0.25	0.59	0.34	0.16	-0.09	-0.03	0	-0.74	-0.41	-0.44	-0.31	-0.25
Monkey	0.39	0.18	0.65	0.79	0.2	0.49	0.29	0.61	0.03	0.09	0.11	-0.04	-0.59	-0.29	-0.31	-0.25	-0.31
Circus Horse	0.3	0.48	0.74	0.6	0.34	0.35	0.53	0.48	0.04	0.05	0.04	-0.13	-0.69	-0.39	-0.38	-0.26	-0.23
Prize Bull	0.13	0.76	0.88	0.26	0.43	0.28	0.83	0.34	0.29	0.14	0.06	0.08	-0.86	-0.57	-0.49	-0.28	-0.35
Rat	0.2	0.36	0.85	0.68	0.21	0.28	0.54	0.63	0.01	0.08	0.18	-0.05	-0.65	-0.29	-0.39	-0.31	-0.23
Badger	0.16	0.28	0.88	0.73	0.14	0.26	0.44	0.66	-0.03	0.1	0.16	-0.07	-0.5	-0.24	-0.3	-0.23	-0.19
Siamese Cat	0.99	0.5	0.05	0.53	0.74	0.75	0.18	0.24	0.24	0.23	0.13	0.19	-0.9	-0.5	-0.41	-0.36	-0.46
Race Horse	0.29	0.7	0.71	0.39	0.51	0.31	0.65	0.31	0.23	0.03	-0.05	-0.08	-0.79	-0.54	-0.46	-0.26	-0.24
Fox	0.13	0.3	0.86	0.68	0.18	0.29	0.46	0.59	0.04	0.16	0.16	-0.09	-0.51	-0.33	-0.34	-0.19	-0.19
Donkey	0.29	0.9	0.78	0.15	0.56	0.18	0.81	0.23	0.28	0.03	0.04	0.08	-0.78	-0.45	-0.48	-0.26	-0.25
Field Mouse	0.16	0.41	0.83	0.59	0.23	0.24	0.43	0.58	0.06	0.08	0.02	-0.01	-0.46	-0.3	-0.24	-0.18	-0.23
Ginger Tom-cat	0.82	0.51	0.21	0.54	0.59	0.58	0.26	0.29	0.08	0.03	0.05	0.08	-0.71	-0.34	-0.34	-0.34	-0.32
Husky in Slead team	0.64	0.51	0.37	0.53	0.56	0.51	0.44	0.29	0.06	-0.01	0.07	-0.08	-0.8	-0.43	-0.49	-0.36	-0.28
Cart Horse	0.27	0.86	0.76	0.15	0.53	0.2	0.84	0.23	0.26	0.05	0.08	0.08	-0.79	-0.46	-0.5	-0.31	-0.28
Chicken	0.23	0.95	0.8	0.06	0.58	0.11	0.81	0.18	0.34	0.05	0.01	0.11	-0.68	-0.46	-0.44	-0.19	-0.23
Doberman Guard Dog	0.88	0.76	0.14	0.27	0.8	0.55	0.45	0.23	0.04	0.28	0.31	0.09	-1.03	-0.47	-0.49	-0.54	-0.51

Table B.4: Representation of the membership weights in the case of the concepts *Fruits* and *Vegetables*.

A=Fruits, B=Vegetables																	
Exemplar	$\mu(A)$	$\mu(B)$	$\mu(A')$	$\mu(B')$	$\mu(AB)$	$\mu(A\bar{B})$	$\mu(\bar{A}B)$	$\mu(\bar{A}\bar{B})$	Δ_{AB}	$\Delta_{A\bar{B}}$	$\Delta_{\bar{A}'B}$	$\Delta_{\bar{A}\bar{B}}$	$I_{AB\bar{A}\bar{B}}$	I_A	I_B	$I_{\bar{A}}$	$I_{\bar{B}}$
Apple	1	0.23	0	0.82	0.6	0.89	0.13	0.18	0.38	0.07	0.13	0.18	-0.79	-0.49	-0.5	-0.3	-0.24
Parsley	0.02	0.78	0.99	0.25	0.45	0.1	0.84	0.44	0.43	0.08	0.06	0.19	-0.83	-0.53	-0.51	-0.29	-0.29
Olive	0.53	0.63	0.47	0.44	0.65	0.34	0.51	0.36	0.12	-0.11	0.04	-0.08	-0.86	-0.46	-0.53	-0.41	-0.26
Chili Pepper	0.19	0.73	0.83	0.35	0.51	0.2	0.68	0.44	0.33	0.01	-0.06	0.09	-0.83	-0.53	-0.46	-0.29	-0.29
Broccoli	0.09	1	0.94	0.06	0.59	0.09	0.9	0.25	0.49	0.03	-0.04	0.19	-0.83	-0.58	-0.49	-0.21	-0.28
Root Ginger	0.14	0.71	0.81	0.33	0.46	0.14	0.71	0.43	0.33	0	0	0.1	-0.74	-0.46	-0.46	-0.33	-0.24
Pumpkin	0.45	0.78	0.51	0.26	0.66	0.21	0.63	0.18	0.21	-0.05	0.11	-0.09	-0.68	-0.43	-0.51	-0.29	-0.13
Raisin	0.88	0.27	0.13	0.76	0.53	0.75	0.25	0.34	0.26	-0.01	0.12	0.21	-0.86	-0.39	-0.51	-0.46	-0.33
Acorn	0.59	0.4	0.49	0.64	0.46	0.49	0.38	0.51	0.06	-0.1	-0.03	0.02	-0.84	-0.36	-0.44	-0.39	-0.36
Mustard	0.07	0.39	0.87	0.6	0.29	0.23	0.55	0.75	0.22	0.16	0.16	0.15	-0.81	-0.44	-0.45	-0.43	-0.38
Rice	0.12	0.46	0.9	0.52	0.21	0.23	0.59	0.59	0.09	0.11	0.13	0.07	-0.61	-0.32	-0.34	-0.28	-0.29
Tomato	0.34	0.89	0.64	0.19	0.7	0.2	0.74	0.23	0.36	0.01	0.1	0.04	-0.86	-0.56	-0.55	-0.33	-0.24
Coconut	0.93	0.32	0.17	0.7	0.56	0.69	0.2	0.34	0.24	-0.01	0.03	0.17	-0.79	-0.33	-0.44	-0.37	-0.33
Mushroom	0.12	0.66	0.9	0.38	0.33	0.13	0.66	0.5	0.21	0.01	0	0.12	-0.61	-0.33	-0.33	-0.26	-0.24
Wheat	0.17	0.51	0.8	0.52	0.34	0.21	0.61	0.56	0.17	0.04	0.11	0.04	-0.73	-0.38	-0.44	-0.38	-0.26
Green Pepper	0.23	0.61	0.81	0.41	0.49	0.24	0.61	0.43	0.26	0.01	0	0.02	-0.76	-0.5	-0.49	-0.23	-0.26
Watercress	0.14	0.76	0.89	0.25	0.49	0.1	0.79	0.35	0.35	-0.04	0.03	0.1	-0.73	-0.45	-0.51	-0.24	-0.2
Peanut	0.62	0.29	0.48	0.75	0.48	0.55	0.25	0.53	0.18	-0.07	-0.04	0.05	-0.8	-0.41	-0.43	-0.3	-0.33
Black Pepper	0.21	0.41	0.81	0.61	0.38	0.21	0.5	0.63	0.17	0.01	0.09	0.01	-0.71	-0.38	-0.46	-0.31	-0.23
Garlic	0.13	0.79	0.88	0.24	0.53	0.1	0.75	0.45	0.4	-0.03	-0.04	0.21	-0.83	-0.5	-0.49	-0.33	-0.31
Yam	0.38	0.66	0.71	0.43	0.59	0.24	0.65	0.44	0.21	-0.14	-0.01	0.01	-0.91	-0.45	-0.58	-0.38	-0.24
Elderberry	0.51	0.39	0.54	0.61	0.45	0.41	0.46	0.48	0.06	-0.09	0.07	-0.07	-0.8	-0.36	-0.52	-0.39	-0.28
Almond	0.76	0.29	0.28	0.72	0.48	0.61	0.24	0.48	0.18	-0.11	-0.04	0.19	-0.8	-0.33	-0.42	-0.43	-0.37
Lentils	0.11	0.66	0.89	0.38	0.38	0.11	0.7	0.53	0.26	0	0.04	0.15	-0.71	-0.38	-0.41	-0.33	-0.26