On the computation of Kronecker coefficients

by

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Abstract

A major open problem in algebraic combinatorics is to find a combinatorial rule to compute the Kronecker product of two Schur functions. This is the same as decomposing the inner tensor product of two irreducible characters of the symmetric group as a sum of irreducible characters. Given that there is a combinatorial rule, namely the Littlewood-Richardson rule, which describes a way to compute the outer tensor product of two irreducible characters of the symmetric group, one would expect an algorithm which achieves the same purpose in the case of the inner tensor product. Jeffrey Remmel and Tamsen Whitehead first came up with a description of the Kronecker coefficients occurring in the Kronecker product of two Schur functions, both indexed by partitions of length at most 2. Mercedes Rosas later arrived at the same result using a different approach. The solution of the general problem would have implications in Complexity Theory and Quantum Information Theory.

Our goal in this thesis is to derive formulae for computing the Kronecker product in certain cases where the Schur functions are indexed by partitions which are nearly rectangular. In particular, we study $s_{(n,n-1,1)} * s_{(n,n)}$, $s_{(n-1,n-1,1)} * s_{(n,n-1)}$, $s_{(n-1,n-1,2)} * s_{(n,n)}$, $s_{(n-1,n-1,1,1)} * s_{(n,n)}$ and $s_{(n,n,1)} * s_{(n,n,1)}$. Our approach relies mainly on the fruitful interplay between manipulation of symmetric functions and the representation theory of the symmetric group. As a consequence of these formulae, we also derive an expression enumerating certain standard Young tableaux of bounded height.

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Chapter 1

Introduction and Background

Symmetric functions play a central role in algebraic combinatorics because of the wealth of information they carry about permutations and partitions, objects which are at the heart of the field. To add to that, the theory of symmetric functions also interacts with other branches of mathematics, including group theory, representation theory and algebraic geometry. The Schur functions are a basis of the ring of symmetric functions with a variety of applications. They mirror the underlying algebra and combinatorics in a clear fashion. An outstanding open problem in algebraic combinatorics is to arrive at a combinatorial rule to compute the Kronecker product of two Schur functions. Given partitions λ, μ and ν, the Kronecker coefficients, $g^{\lambda}_{\mu\nu}$, occur as the multiplicities of the irreducible representations of the symmetric group in the tensor product of the representations χ_{λ} and χ_{μ} of the symmetric group. Alternatively, they occur as coefficients in the decomposition of the Kronecker product, $s_{\mu} * s_{\nu}$, of Schur functions in the Schur basis, as shown in the following equation

$$s_\mu * s_\nu = \sum_\lambda g_{\mu\nu}^\lambda s_\lambda . \tag{1.1}$$

The aim of this thesis is to derive explicit formulae for Kronecker coefficients in certain special cases. But prior to that, we will equip the reader with an overview of symmetric functions and the multitude of operations one can carry out on them, so as to render the computation of the Kronecker coefficients transparent in specific cases.

1.1 Symmetric functions

A *symmetric function* is a formal power series $f(\mathbf{x})$ in the variables $\mathbf{x} = \{x_1, x_2, \ldots\}$ with the property that $f(x_{\pi(1)}, x_{\pi(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation π of the positive integers $\mathbb{N}$. Corresponding to a finite multiset of positive integers $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ where the λ_i's are ordered in a weakly decreasing sequence, one can define a monomial x^λ as follows

$$x^\lambda = x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_l}^{\lambda_l} \quad , \tag{1.2}$$

where the i_j for $j = 1, \ldots, l$ are distinct positive integers. The *degree* of the monomial is defined to be $\sum_{i=1}^{l} \lambda_i$. A symmetric function f is said to be *homogeneous* of degree n if all the monomials occuring in f are of degree n. The set of all homogeneous symmetric functions of a given degree n forms a vector space over the rational numbers, denoted by Λ^n. Given $f \in \Lambda^n$ and $g \in \Lambda^m$, the product fg (as a formal power series) is an element of Λ^{m+n}. Now we define the ring of symmetric functions Λ to be the vector space direct sum of the Λ^n, i.e.

$$\Lambda = \bigoplus_{n \geq 0} \Lambda^n \quad , \tag{1.3}$$

and Λ is considered as a graded algebra. A common theme in the theory of symmetric functions is the description of bases for the vector space Λ^n and the relationships amongst themselves.

There are many different $\mathbb{Q}$-bases for symmetric functions and we will now describe several of these bases. We will also introduce the combinatorial object used to describe the Schur functions.

1.1.1 Monomial symmetric functions

The *monomial symmetric functions* are the functions obtained by symmetrizing a monomial. To elaborate on this concept of symmetrization, we need a notion of a partition. A *partition* of a positive integer n is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that the sum $\sum_{i=1}^{l} \lambda_i$ equals n. The integers λ_i are called the parts of λ. We let ϕ denote the empty partition, i.e. the partition of 0. The notation $\lambda \vdash n$ implies that λ is a partition of n. For example, if $n = 12$, then $\lambda = (5,3,3,1,0,0)$ is a partition of 12 and we write $\lambda \vdash 12$. The *length*, $l(\lambda)$ is the largest index i such that $\lambda_i > 0$. At times, the length of the partition might also be called the height of the partition. Again, if $\lambda = (5,3,3,1,0,0)$, then $l(\lambda) = 4$. We will denote the number of distinct non-zero parts of λ by d_λ. Therefore, for the same λ as before, we have $d_\lambda = 3$. Since the zero parts of a given partition do not really matter for our purposes, we will identify partitions which only differ in the number of trailing zeros. So, for example, the partition $\lambda = (5,3,3,1,0,0)$ is the same as the partition $\mu = (5,3,3,1)$.

Definition 1.1.1. *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ *be a partition of some positive integer n. Then the* monomial symmetric function m_λ *corresponding to* λ *is*

$$m_\lambda(\boldsymbol{x}) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_l}^{\lambda_l} \quad , \tag{1.4}$$

where the sum is over all distinct monomials with exponents $\lambda_1, \lambda_2, \ldots, \lambda_l$ *in the variables* $\boldsymbol{x} = \{x_1, x_2, \ldots\}$.

For example,

$$\begin{aligned}
m_\phi &= 1, \\
m_{(1)} &= \sum_{i \geq 1} x_i, \\
m_{(2)} &= \sum_{i \geq 1} x_i^2, \\
m_{(1,1)} &= \sum_{1 \leq i < j} x_i x_j.
\end{aligned}$$

Each symmetric function can be written as a sum of monomial symmetric functions with rational coefficients. This implies that the set $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ^n. Since the monomial symmetric functions are indexed by partitions, the dimension of this vector space is equal to the number of partitions of n, denoted $p(n)$.

1.1.2 Other symmetric function bases

The *elementary symmetric functions*, e_λ can be considered as a product of certain monomial symmetric functions.

Definition 1.1.2.

$$e_0 = m_\phi = 1,$$

$$e_n = m_{(1^n)} = \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \qquad (n \geq 1),$$

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l} \textit{ for } \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l).$$

The set $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ^n. For any partition λ, e_λ can be written as a positive sum of monomial symmetric functions.

One defines the *homogeneous symmetric functions*, h_λ by the formulae

Definition 1.1.3.

$$h_0 = m_\phi = 1,$$

$$h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq i_2 \leq \ldots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \qquad (n \geq 1),$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l} \textit{ for } \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l).$$

The homogeneous symmetric functions can be written as a linear combination of the monomial symmetric functions with positive integer coefficients, and the set $\{h_\lambda : \lambda \vdash n\}$ is also a basis of Λ^n. A third basis given by the *power sum symmetric functions* p_λ is defined by the formulae

Definition 1.1.4.

$$p_0 = m_\phi = 1,$$

$$p_n = m_{(n)} = \sum_{i \geq 1} x_i^n \qquad (n \geq 1),$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l} \text{ for } \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l).$$

Example 1.1.5. *Some examples of the symmetric functions introduced above are the following:*

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + \ldots = \sum_{i<j} x_i x_j = m_{(1,1)}.$$

$$e_{(2,1)} = e_2e_1 = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + \ldots = m_{(2,1)} + 3m_{(1,1,1)}.$$

$$h_2 = m_{(2)} + m_{(1,1)} = \sum_{i \geq 1} x_i^2 + \sum_{1 \leq i \neq j} x_i x_j.$$

$$h_{(2,1)} = h_2h_1 = x_1^3 + x_1^2x_2 + x_1^2x_3 + \cdots + x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + \cdots + x_1^2x_3 + x_1x_3^2 + x_1x_2x_3 + \cdots = 3m_{(1,1,1)} + 2m_{(2,1)} + m_{(3)}.$$

$$p_2 = \sum_{i \geq 1} x_i^2.$$

$$p_{(2,1)} = p_2 p_1 = x_1^3 + x_2^3 + \cdots + x_1^2 x_2 + x_1 x_2^2 + \cdots = m_{(3)} + m_{(2,1)}.$$

Both the elementary and the homogeneous symmetric functions can be written in terms of the power sum symmetric functions. Given a partition λ, let m_i denote the number of i's in λ for $i \geq 1$. Define

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i! \quad . \tag{1.5}$$

The quantity z_λ has a combinatorial interpretation as well. Given a permutation $\pi \in S_n$, define the *cycle type* of π to be the partition obtained when the cycle lengths occurring in π's cycle decomposition are listed in weakly decreasing order. Then, z_λ is the number of permutations in S_n that commute with a fixed permutation of cycle type λ.

Example 1.1.6. *The permutation* $\pi = (135)(26)(48)(7)$ *has cycle type* $(3,2,2,1)$ *and the number of permutations in* S_8 *that commute with* π*,* $z_{(3,2,2,1)}$*, is* $1 \times 2^2 \times 2! \times 3 = 24$.

There also exist the following relations

$$h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}, \tag{1.6}$$

$$e_n = \sum_{\lambda \vdash n} \varepsilon_\lambda \frac{p_\lambda}{z_\lambda}, \tag{1.7}$$

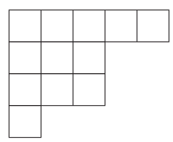
where $\varepsilon_\lambda = (-1)^{n-l(\lambda)}$.

For an exhaustive discussion on symmetric function bases, the reader should refer to [23].

1.2 Semi-standard Young tableaux

Each partition λ of n is associated to a collection of cells (or squares) called a *Ferrers diagram* (or a *Young diagram*), where the i-th row consists of (left justified) λ_i cells. We will be identifying λ with its Ferrers diagram.

Example 1.2.1. *The Ferrers diagram of* $\lambda = (5,3,3,1)$ *is*



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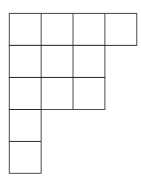
The *conjugate*, λ^t, of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is the shape obtained by transposing the Ferrers diagram of λ. Thus, the conjugate of the partition $\lambda = (5,3,3,1)$ is $\lambda^t = (4,3,3,1,1)$, shown below,



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A *filling* of a Ferrers diagram λ is a map $T : \lambda \to \mathbb{N}$, i.e., T is a map assigning a positive integer to every square/cell of the Ferrers diagram λ. A *semi-standard Young tableau (SSYT)* of shape λ is a filling of λ satisfying the condition that T is weakly increasing along each row and strictly increasing along each column in λ.

A *standard Young tableau (SYT)* of shape $\lambda \vdash n$ is a certain type of SSYT in which the map T is a bijection from λ to $[n] = \{1, 2, \ldots, n\}$. The fact that T

is a bijection implies that the entries in the rows and columns of λ are strictly increasing.

Example 1.2.2. *A SSYT of shape* $\lambda = (5,3,3,1)$ *is*

1	3	3	3	4
2	4	4		
5	5	5		
6				

,

while an example of an SYT of the same shape is

1	3	5	7	9
2	4	8		
6	10	11		
12				

.

The multiset of positive integers that appear in a filling of a shape λ is called the *content* of T. We can also let content(T) be the tuple α where α_i equals the number of times i appears in T. Clearly, the sum of the entries in the tuple α is $|\lambda|$, the size of the partition. For example, the content of the SSYT in Example 1.2.2 is (1,1,3,3,3,1) as a tuple. As a multiset, the content is $\{1,2,3,3,3,4,4,4,5,5,5,6\}$. From this point onwards, we will think of the content as a tuple.

Given a partition $\lambda \vdash n$ and a tuple μ of non-negative integers summing to n, the *Kostka number* $K_{\lambda,\mu}$ is defined as the number of SSYT of shape λ and content μ. Currently, a simple formula to compute $K_{\lambda,\mu}$ is not known. But there is one important case where such a formula is known and the next section covers that particular case.

1.2.1 The hook length formula

The number of SYTs of shape $\lambda \vdash n$ is the Kostka number $K_{\lambda,(1^n)}$. We will denote this number by f_λ, and it can be easily calculated by using the Hook Length Formula of Frame, Robinson and Thrall [11]. Before we can state the formula, we will need the following definitions.

Definition 1.2.3. *Let c be a cell contained in a Ferrers diagram λ. Its* arm length, *$a(c)$, equals the number of cells to the right of c in the same row of λ as c. Its* leg length, *$l(c)$, equals the number of cells below c in the same column of λ as c.*

Definition 1.2.4. *Let c be a cell belonging to a Ferrers diagram λ. The* hook length *of c, denoted by $h(c)$, is one plus the sum of the arm length and leg length of c, i.e., $h(c) = a(c) + l(c) + 1$.*

Example 1.2.5. *Let $\lambda = (5,3,3,1)$. Then the entry in each cell of the Ferrers diagram of λ below gives the respective hook length.*

8	6	5	2	1
5	3	2		
4	2	1		
1				

This brings us to the celebrated hook length formula[11].

Theorem 1.2.6. *Given a partition λ of n,*

$$f_\lambda = \frac{n!}{\prod_{c\in\lambda} h(c)} \quad ,$$

where $h(c)$ denotes the hook length of the cell c.

On using the hooklength formula for Example 1.2.5 we obtain that the number of SYT of shape $(5,3,3,1)$ is

$$f_{(5,3,3,1)} = \frac{12!}{(8\cdot 6\cdot 5\cdot 5\cdot 4\cdot 3\cdot 2\cdot 2\cdot 2\cdot 1\cdot 1\cdot 1)} = 4158.$$

The importance and beauty of this result can be gauged from the fact that it has been proved in a variety of ways, including a probabilistic proof by Greene, Nijenhuis and Wilf [15] and a bijective proof by Novelli, Pak and Stoyanovskii [17].

Some classical results, present in [23], in this context are

$$\sum_{\lambda\vdash n} f_\lambda^2 = n! \quad ,$$

and

$$\sum_{\lambda\vdash n} f_\lambda = \text{ coefficient of } \frac{z^n}{n!} \text{ in } e^{\frac{z+z^2}{2}} \quad .$$

Note that the second of these identities gives us the number of SYTs of a fixed size. Now it is natural to ask how many SYTs are there of fixed size n if one imposes a constraint that the number of parts of $\lambda \vdash n$ is bounded above by some fixed positive integer k. This means we are interested in the sum

$$\tau_k(n) = \sum_{\substack{\lambda\vdash n \\ l(\lambda)\le k}} f_\lambda \quad .$$

This is also a well studied question, as is evident from [3, 4, 13, 19]. In [13], Gessel considered the generating function

$$\tau_k(z) = \sum_{n=0}^{\infty} \tau_k(n) \frac{z^n}{n!} \quad ,$$

and obtained determinantal formulae for $\tau_k(z)$. He also showed [13] that the generating function $\tau_k(z)$ is D-finite. That is to say that they satisfy linear differential equations with polynomial coefficients in z. The expressions for $\tau_k(n)$ are unwieldy when k is large. But for relatively small values of k, these expressions are more succinct than what one would expect from the hooklength formula. Regev [19] worked out the values for $\tau_2(n)$ and $\tau_3(n)$. In particular, he obtained

$$\tau_2(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}, \quad \tau_3(n) = \sum_{i \geq 0} \frac{1}{i+1} \binom{n}{2i} \binom{2i}{i} \quad .$$

Note that $\tau_3(n)$ is actually the Motzkin number M_n which has quite a few interpretations [23]. Gessel [13] and Gouyou-Beauchamps [14] found an expression for $\tau_4(n)$ and Gouyou-Beauchamps in the same paper [14] found an expression for $\tau_5(n)$. The expressions were:

$$\tau_4(n) = C_{\lfloor \frac{n+1}{2} \rfloor} C_{\lceil \frac{n+1}{2} \rceil}, \quad \tau_5(n) = 6 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} C_i \frac{(2i+2)!}{(i+2)!(i+3)!} \quad ,$$

where

$$\begin{aligned} C_i &= \frac{1}{i+1} \binom{2i}{i} \quad , \\ &= \frac{1}{i+1} \frac{(2i)!}{(i)!(i)!} \quad , \end{aligned}$$

is the i-th Catalan number. For the sake of ready reference, we will put the above results as a theorem, the way they are presented in Theorem 15 of [13].

Theorem 1.2.7. ([13, Theorem 15]) *Let* $\tau_k(n)$ *be the number of standard tableaux with n entries and at most k rows, and let* C_n *be the Catalan number* $\frac{1}{n+1}\binom{2n}{n}$. *Then*

$$\begin{aligned}
\tau_2(n) &= \binom{n}{\lfloor \frac{n}{2} \rfloor}, \\
\tau_3(n) &= \sum_{i \geq 0} \frac{1}{i+1} \binom{n}{2i} \binom{2i}{i}, \\
\tau_4(n) &= C_{\lfloor \frac{n+1}{2} \rfloor} C_{\lceil \frac{n+1}{2} \rceil}, \\
\tau_5(n) &= 6 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} C_i \frac{(2i+2)!}{(i+2)!(i+3)!}.
\end{aligned} \tag{1.8}$$

For over sixty interpretations of the Catalan numbers, the interested reader is referred to [23]. The one that concerns us is that C_n is equal to the number of SYTs of shape (n,n), i.e.,

$$f_{(n,n)} = C_n. \tag{1.9}$$

This can be easily seen to be true by using the hooklength formula. Another byproduct of the hooklength formula is the fact that

$$f_{(n,n-1)} = C_n. \tag{1.10}$$

We will also be needing expressions for $f_{(n,n-1,1)}$ and $f_{(n-1,n-1,1)}$ for our purposes.

Proposition 1.2.8. *Given $n \geq 2$, we have*

$$f_{(n,n-1,1)} = \left(\frac{(n-1)(n+1)}{2n+1}\right) C_{n+1} \quad , \tag{1.11}$$

$$f_{(n-1,n-1,1)} = \left(\frac{n-1}{2}\right) C_n \quad . \tag{1.12}$$

Proof. For $n = 2$ and 3, the proof that follows doesn't hold, but a manual check can quickly reveal that the identities hold. Henceforth, $n \geq 4$.

The hooklengths for each of the cells in the first row of the Ferrers diagram of the partition $(n, n-1, 1)$ from left to right are $(n+2, n, n-1, \ldots, 3, 1)$. The hooklengths for each of the cells in the second row are $(n, n-2, n-3, \ldots, 2, 1)$. The third row has just one cell and the associated hooklength is 1. The hooklength formula implies

$$\begin{aligned} f_{(n,n-1,1)} &= \frac{(2n)!}{(n+2)\cdot(n)\cdot(n-1)\cdots 3\cdot 1\cdot(n)\cdot(n-2)\cdots 2\cdot 1\cdot 1} \\ &= \frac{2(n-1)\times(2n)!}{(n+2)\times n!\times n!} \\ &= \frac{2(n-1)(n+1)^2(2n+1)\times(2n)!}{(2n+1)\times(n+2)!\times(n+1)!} \\ &= \frac{(n-1)(n+1)\times(2n+2)!}{(2n+1)\times(n+2)!\times(n+1)!} \\ &= \frac{(n-1)(n+1)}{(2n+1)} C_{n+1}. \end{aligned} \tag{1.13}$$

We will approach the calculation of $f_{(n-1,n-1,1)}$ similarly. In this case, the hooklengths of the cells in the first row are $(n+1, n-1, n-2, \ldots, 3, 2)$ and the hooklengths of the cells in the second row are $(n, n-2, n-3, \ldots, 2, 1)$. The hooklength formula then gives us

$$
\begin{aligned}
f_{(n-1,n-1,1)} &= \frac{(2n-1)!}{(n+1)\cdot(n-1)\cdot(n-2)\cdots3\cdot2\cdot(n)\cdot(n-2)\cdots2\cdot1\cdot1} \\
&= \frac{2n(n-1)\times(2n-1)!}{2\times(n+1)!\times n!} \\
&= \frac{(n-1)\times(2n)!}{2\times(n+1)!\times n!} \\
&= \frac{(n-1)}{2}C_n. \qquad (1.14)
\end{aligned}
$$

□

1.3 Schur functions

Given that the Schur functions are so protean, it is natural to expect that they can be defined in many ways. We will stick to the approach of defining the Schur functions combinatorially by appealing to semi-standard Young tableaux. This approach, and an alternate formulation that is algebraic and expresses the Schur functions as a quotient of determinants can be found in [23].

1.3.1 Combinatorial definition

Like the other symmetric functions we have encountered, the Schur functions are indexed by partitions. We will proceed to describe how to associate a Schur function s_λ given a partition λ.

Let $SSYT(\lambda)$ be the set of all semi-standard Young tableaux of shape λ. We may associate a monomial x^T to every semi-standard Young tableau $T \in SSYT(\lambda)$ in the following manner. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be the content of T. Note that we are considering the content here as the tuple α where α_i counts the number of times

i appears in T (and not as a multiset). Then the monomial weight is associated as follows:

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots \quad .$$

Example 1.3.1. *Let* $\lambda = (5,3,3,1)$ *and* $T \in SSYT(\lambda)$ *be*

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 \\ \hline 3 & 4 & 4 \\ \cline{1-3} 5 & 5 & 7 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array} \quad .$$

Then content$(T) = (2,1,3,2,2,1,1)$ *and* $x^T = x_1^2 x_2 x_3^3 x_4^2 x_5^2 x_6 x_7$.

There is a generalization of SSYT's of shape λ that fits naturally in the theory of symmetric functions. If λ and μ are partitions such that $\mu \subseteq \lambda$, i.e., $l(\mu) \leq l(\lambda)$ and $\mu_i \leq \lambda_i$ for all $i = 1,2,\ldots,l(\mu)$, then the *skew shape* λ/μ is determined by removing the first μ_i cells from the i-th row of the Ferrers diagram of λ. A semi-standard Young tableau of shape λ/μ is a skew shape λ/μ that has cells filled with positive integers so that the rows are weakly increasing and the columns are strictly increasing. The content and the monomial weight associated with a semi-standard Young tableau of skew shape are defined in the same way as before.

Example 1.3.2. *If* $\lambda = (5,3,3,1)$ *and* $\mu = (3,2,1)$*, then an SSYT* T *of skew shape* $(5,3,3,1)/(3,2,1)$ *is*

$$\begin{array}{ccccc} \cline{4-5} & & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} \\ \cline{3-5} & & \multicolumn{1}{|c|}{2} & & \\ \cline{2-3} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & & \\ \cline{1-3} \multicolumn{1}{|c|}{4} & & & & \\ \cline{1-1} \end{array} \quad ,$$

and $x^T = x_1^3 x_2 x_3 x_4$.

With the definition of a SSYT of skew shape in hand, we are ready to define skew Schur functions.

Definition 1.3.3. *Given a skew shape* λ/μ*, the* skew Schur function $s_{\lambda/\mu}$ *of shape* λ/μ *is the formal power series*

$$s_{\lambda/\mu}(\boldsymbol{x}) = \sum_T x^T \quad ,$$

where the sum is over all SSYTs T *of shape* λ/μ*. If* $\mu = \phi$*, then* $\lambda/\mu = \lambda$*, and we call* s_λ *the* Schur function *of shape* λ*.*

Though by no means obvious, it turns out that the skew Schur functions are symmetric and the elements of the set $\{s_\lambda : \lambda \vdash n\}$ form a basis for Λ^n. It is clear that $s_\lambda(\mathbf{x})$ is a formal power series in infinitely many variables $\{x_1, x_2, \ldots\}$, and one could potentially restrict it to the set of variables $\{x_1, x_2, \ldots, x_n\}$ by setting $x_i = 0$ for $i > n$. This is equivalent to considering only SSYTs with entries in the set $[n] = \{1, 2, \ldots, n\}$ as the weight of any SSYT with an entry greater than n will be 0, and hence its contribution to the above formal power series defining $s_\lambda(\mathbf{x})$ will be 0. In the example given below, we are limiting ourselves to three variables x_1, x_2 and x_3.

Example 1.3.4. *Let* $\lambda = (2,1)$*. Then SSYT*(λ) *with maximum entry being* 3 *is the collection*

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \;,$$

and therefore

$$s_{(2,1)}(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \quad .$$

The coefficient of a monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ in s_λ is the Kostka number $K_{\lambda,\alpha}$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ corresponds to the content and λ is the shape. This equals the number of SSYTs of shape λ and content α. In particular, if $\lambda \vdash n$ then the coefficient of $x_1x_2\cdots x_n$ is the number of standard Young tableaux of shape λ, f_λ. More generally, one can define the skew Kostka number $K_{\lambda/\mu,\alpha}$ to be the number of SSYTs of shape λ/μ and content α.

Now that we are done defining bases for the space of symmetric functions Λ^n, we will equip this space with an inner product $\langle,\rangle_{\Lambda^n}$. This is called the *Hall inner product* and is defined by setting $\langle s_\lambda, s_\mu\rangle_{\Lambda^n} = \delta_{\lambda\mu}$ and then defining the inner product for any $f, g \in \Lambda^n$ by linear extension. One can clearly extend this to an inner product on Λ, in which case we will refer to it as $\langle,\rangle_\Lambda$. The Hall inner product has the following properties:

$$\langle p_\lambda, p_\mu\rangle_{\Lambda^n} = \delta_{\lambda\mu}z_\lambda \quad ,$$

$$\langle h_\lambda, m_\mu\rangle_{\Lambda^n} = \delta_{\lambda\mu} \quad ,$$

where δ denotes the Kronecker delta. The way we have defined the inner product, we ensure that the Schur functions are an orthonormal basis of Λ^n. Equipped with the Hall inner product and the definition of skew Schur functions, one can establish a very fundamental property of skew Schur functions.

Theorem 1.3.5. *For any $f \in \Lambda$, we have*

$$\langle f s_{\mu,}, s_\lambda \rangle_\Lambda = \langle f, s_{\lambda/\mu} \rangle_\Lambda.$$

An alternate description could be as follows. Given a fixed partition μ, one can think of multiplication by s_μ as a linear operator on Λ. If one defines the linear transformations $s_\mu : \Lambda \to \Lambda$ and $s_\mu^\perp : \Lambda \to \Lambda$ defined by $s_\mu(f) = s_\mu f$ and $s_\mu^\perp(s_\lambda) = s_{\lambda/\mu}$, then the operators s_μ and $s_\mu^\perp$ are adjoint with respect to the inner product $\langle , \rangle_\Lambda$. In particular,

$$\langle s_\nu s_\mu, s_\lambda \rangle_\Lambda = \langle s_\nu, s_{\lambda/\mu} \rangle_\Lambda \quad .$$

For further such details about Schur functions, the reader is referred to [23].

Now we will describe a combinatorial procedure for multiplying two Schur functions. This also corresponds to the outer tensor product of characters of the symmetric group. We will need a few more definitions before we can outline the procedure for multiplication.

The *skew reading word* corresponding to an SSYT of skew shape is the word obtained by reading the entries bottom to top, left to right. Thus, the skew reading word of the tableau in Example 1.3.2 is 413211.

Definition 1.3.6. *A word w is called a* reverse lattice word *if w^r, that is the word w read backwards, has the property that any prefix of w^r contains at least as many instances of a positive integer i as it does of $i+1$.*

For instance, $w = 413211$ is a reverse lattice word while $w = 1233221$ is not.

The *Littlewood-Richardson coefficient*, $c^\lambda_{\mu\nu}$, is equal to the number of skew tableaux of shape λ/μ and content ν such that the skew reading word of the tableau is a reverse lattice word. This given, the following identity [23] describes the product of two Schur functions

$$s_\mu(\mathbf{x})s_\nu(\mathbf{x}) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(\mathbf{x}),$$

where the sum is over all λ such that $\mu \subseteq \lambda$ and $|\mu| + |\nu| = |\lambda|$. In terms of the inner product on Λ, this identity is equivalent to $\langle s_\mu s_\nu, s_\lambda \rangle_\Lambda = c^\lambda_{\mu\nu}$. Thus, we obtain a combinatorial rule to multiply two Schur functions by counting SSYTs of skew shape satisfying certain contraints (commonly called the Littlewood-Richardson rule).

Example 1.3.7. *Consider the computation of the product* $s_{(2,1)}s_{(2,2)}$. *To accomplish this, we are required to list all skew tableaux of shape* $\lambda/(2,1)$ *with content* $(2,2)$ *such that the skew reading word is a reverse lattice word.*

$$\begin{array}{cccc} \bullet & \bullet & 1 & 1 \\ \bullet & 2 & 2 & \end{array}\ ,\quad \begin{array}{cccc} \bullet & \bullet & 1 & 1 \\ \bullet & 2 & & \\ 2 & & & \end{array}\ ,\quad \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 1 & 2 \\ 2 & & \end{array}\ ,\quad \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 1 & \\ 2 & 2 & \end{array}\ ,\quad \begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 2 & \\ 1 & & \\ 2 & & \end{array}\ ,\quad \begin{array}{cc} \bullet & \bullet \\ \bullet & 1 \\ 1 & 2 \\ 2 & \end{array}\ .$$

Thus, for example, we have $c^{(3,2,2)}_{(2,1)(2,2)} = 1$, *and*

$$s_{(2,1)}s_{(2,2)} = s_{(4,3)} + s_{(4,2,1)} + s_{(3,3,1)} + s_{(3,2,2)} + s_{(3,2,1,1)} + s_{(2,2,2,1)}.$$

We will now look at a special case of the Littlewood-Richardson rule. It describes the multiplication of a Schur functions with a Schur function indexed by a row shape or a column shape. This amounts to multiplying a Schur function

$s_\lambda(\mathbf{x})$ with a complete homogeneous symmetric function $h_n(\mathbf{x})$ or an elementary symmetric function $e_n(\mathbf{x})$. It is called the *Pieri rule* but before we state the rule we need to describe certain skew shapes. A skew shape λ/μ is called a *horizontal strip* if it does not contain squares in the same column, and is called a *vertical strip* if it does not contain squares in the same row.

Theorem 1.3.8. *If μ is a partition, then*

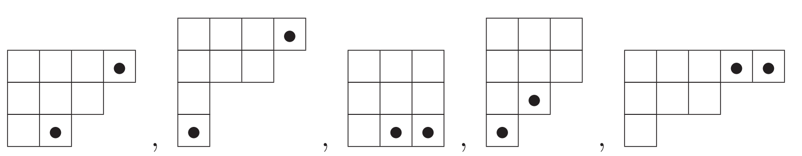
$$s_\mu s_{(n)} = s_\mu h_n = \sum_{\substack{\nu \vdash |\mu|+n \\ \nu/\mu = \text{horizontal strip of size } n}} s_\nu \ ,$$

and

$$s_\mu s_{(1^n)} = s_\mu e_n = \sum_{\substack{\nu \vdash |\mu|+n \\ \nu/\mu = \text{vertical strip of size } n}} s_\nu \ .$$

For a demonstration of how this rule aids calculations, consider the following example.

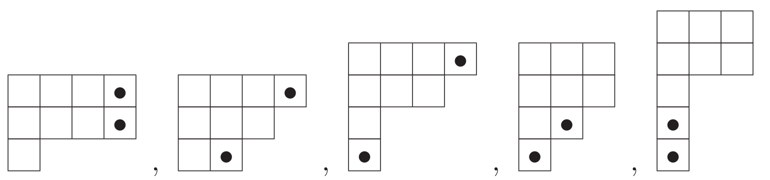
Example 1.3.9. *We will compute $s_{(3,3,1)}s_{(2)}$ using the Pieri rule. This rule implies we should list partitions $\lambda \vdash 9$ such that the skew shape $\lambda/(3,3,1)$ is a horizontal strip of size 2. Thus the possible λ are*



Thus we obtain

$$s_{(3,3,1)}s_{(2)} = s_{(4,3,2)} + s_{(4,3,1,1)} + s_{(3,3,3)} + s_{(3,3,2,1)} + s_{(5,3,1)}.$$

We can compute $s_{(3,3,1)}s_{(1,1)}$ *in a similar fashion. The Pieri rule implies we should list partitions* $\lambda \vdash 9$ *such that the skew shape* $\lambda/(3,3,1)$ *is a vertical strip of size 2. Thus, the possible* λ *are*



and therefore

$$s_{(3,3,1)}s_{(1,1)} = s_{(4,4,1)} + s_{(4,3,2)} + s_{(4,3,1,1)} + s_{(3,3,2,1)} + s_{(3,3,1,1,1)}.$$

1.4 Basic representation theory of S_n

A *representation* of a group G is a homomorphism $\rho : G \to GL(V)$ for some finite dimensional vector space V over $\mathbb{C}$. Here $GL(V)$ denotes the general linear group of V, i.e. the group of all automorphisms of V. After fixing a basis for V, we can think of the representation ρ as a mapping of a group element to an invertible matrix, and we willl call this a *matrix representation*. We might abuse notation and make no explicit reference to ρ by referring to V as the representation, and letting the $[g]$ denote the matrix $\rho(g)$ for an element $g \in G$.

A *subrepresentation* of a representation V is a subspace $W \subseteq V$ that is invariant under the action of G. A representation V is called *irreducible* if the only subrepresentations are $\{0\}$ and V itself.

It is well known that every finite dimensional representation of a finite group is isomorphic to the direct sum of a finite number of irreducible representations. Furthermore, the number of these irreducible representations is the number of conjugacy classes of the group. For details, the reader can refer to [22, Chapter 1].

Our only concern here are the representations of the symmetric group S_n. The irreducible representations of S_n are indexed by partitions, and there is a well known construction [18] of the irreducible representation V^λ for any $\lambda \vdash n$, called the *Specht module* corresponding to λ.

The *character* of a representation of a finite group G is a map $G \to \mathbb{C}$ defined by $g \to trace([g])$. Since the trace of a matrix is conjugation invariant, the trace is a *class function* i.e. it is constant on conjugacy classes. In the case of the symmetric group, the conjugacy classes are also indexed by the partitions as all permutations with the same cycle type are conjugates. We will denote the character of the irreducible Specht module V^λ by χ_λ. The fact that forms the bedrock of most results in finite group representation theory is the following [22].

Proposition 1.4.1. *Every representation of a finite group is determined (upto isomorphism) by its character.*

Let CF^n denote the space of class functions from S_n to $\mathbb{Q}$ (we are using $\mathbb{Q}$ since the characters of S_n can be realized over $\mathbb{Q}$). CF^n has a natural inner product $\langle , \rangle_{CF^n}$ defined by

$$\langle f, g \rangle_{CF^n} = \frac{1}{n!} \sum_{\pi \in S_n} f(\pi) g(\pi).$$

Our main tool that reduces character computations to symmetric functions manipulation is a linear transformation $ch : CF^n \to \Lambda^n$ called the *characteristic map* or the *Frobenius map*. If $f \in CF^n$, then define

$$\begin{aligned} ch(f) &= \frac{1}{n!}\sum_{\pi\in S_n} f(\pi)p_\pi \quad , \\ &= \sum_{\mu\vdash n}\frac{1}{z_\mu} f(\mu)p_\mu \quad , \end{aligned}$$

wherein we have used the fact that f is a class function which basically means that $f(\pi)$ is decided completely by π's cycle type. The p_π above also refers to the power sum symmetric function indexed by the partition that is the cycle type of π. A very important property of the Frobenius map is the following [22, 23].

Proposition 1.4.2. *The linear tranformation ch is an isometry, i.e.,*

$$\langle f,g\rangle_{CF^n} = \langle ch(f),ch(g)\rangle_{\Lambda^n} \quad .$$

It can be shown that $ch(\chi_\lambda) = s_\lambda$, which in particular implies that $\langle s_\lambda, p_\mu\rangle_{\Lambda^n} = \chi_\lambda(\mu)$ where $\mu \vdash |\lambda|$. We have managed to cover those aspects of the representation theory of the symmetric group that concern us most. A detailed account of the same can be found in [23].

1.5 The Kronecker product of Schur functions

We start by describing the Kronecker coefficients at the level of the characters of the symmetric group. Let λ, μ and ν be partitions of n. The *Kronecker coefficients*

$g^{\lambda}_{\mu\nu}$ are defined by

$$g^{\lambda}_{\mu\nu} = \langle \chi_\lambda, \chi_\mu \chi_\nu \rangle_{CF^n} = \frac{1}{n!} \sum_{\pi \in S_n} \chi_\lambda(\pi)\chi_\mu(\pi)\chi_\nu(\pi) = \sum_{\gamma \vdash n} \frac{1}{z_\gamma} \chi_\lambda(\gamma)\chi_\mu(\gamma)\chi_\nu(\gamma) \quad . \tag{1.15}$$

The fact that the Kronecker coefficients are symmetric in λ, μ and ν is clearly highlighted in the above formulation. The relevance of the Kronecker coefficients comes from the fact that follows. Recall that V^μ is the irreducible representation corresponding to the character χ_μ. Then the pointwise product $\chi_\mu \chi_\nu$ is the character of $V^\mu \otimes V^\nu$, the representation obtained by taking the tensor product of V^μ and V^ν. Moreover, $g^{\lambda}_{\mu\nu}$ is the multiplicity of V^λ in $V^\mu \otimes V^\nu$, i.e., it is the number of times a module isomorphic to V^λ occurs in the direct sum decomposition of $V^\mu \otimes V^\nu$ into irreducible representations.

To obtain an interpretation in terms of symmetric functions, let $f, g \in \Lambda^n$. The *Kronecker product*, $f * g$, is defined by

$$f * g = ch(uv) \quad ,$$

where u and v are class function belonging to CF^n such that $ch(u) = f$, $ch(v) = g$ and $uv(\pi) = u(\pi)v(\pi)$. If we set $f = s_\mu, g = s_\nu$ where both μ, ν are partitions of n

then by the property of the Frobenius map we have $u = \chi_\mu, v = \chi_\nu$. Therefore

$$\begin{aligned}
\langle s_\mu * s_\nu, s_\lambda \rangle_{\Lambda^n} &= \langle ch(\chi_\mu \chi_\nu), s_\lambda \rangle_{\Lambda^n} \\
&= \langle \sum_{\gamma \vdash n} \frac{1}{z_\gamma} \chi_\mu \chi_\nu(\gamma) p_\gamma, s_\lambda \rangle_{\Lambda^n} \\
&= \langle \sum_{\gamma \vdash n} \frac{1}{z_\gamma} \chi_\mu(\gamma) \chi_\nu(\gamma) p_\gamma, s_\lambda \rangle_{\Lambda^n} \\
&= \sum_{\gamma \vdash n} \frac{1}{z_\gamma} \chi_\lambda(\gamma) \chi_\mu(\gamma) \chi_\nu(\gamma) \\
&= g_{\mu\nu}^{\lambda} \quad . \qquad (1.16)
\end{aligned}$$

One can also prove easily that

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda\mu} \frac{p_\lambda}{z_\lambda} \quad .$$

Notice that the Kronecker product has the following symmetries:

$$s_\mu * s_\nu = s_\nu * s_\mu \quad ,$$

$$s_\mu * s_\nu = s_{\nu^t} * s_{\mu^t} \quad ,$$

where μ^t denotes the conjugate of the partition μ. Moreover, if $\mu, \nu \vdash n$ then

$$g_{\mu\nu}^{(n)} = g_{\mu\nu^t}^{(1^n)} = \delta_{\mu\nu} \quad .$$

Given that the Kronecker coefficients are positive by the above interpretation as a multiplicity of a character of S_n in a tensor product of two characters, one expects a combinatorial rule for computing these Kronecker coefficients. To date, there is no satisfying positive combinatorial or algebraic formula for the Kronecker product of two Schur functions. Attempts have been made to understand different aspects of these coefficients, for example, special cases [5–7, 20, 21], asymptotics [1, 2], stability [24], the complexity of computing them and conditions which guarantee that the Kronecker coefficients are non-zero [8]. In the next section, we review some of the prior work done on computing Kronecker coefficients by stating the main results that we will be using.

1.6 A brief survey of relevant results

The motivation for the results in this thesis has been the recent results on Kronecker product of two Schur functions, both indexed by two row partitions with one of them being rectangular. Although the description of the Kronecker product of two Schur functions indexed by general two row partitions had already been obtained by [20, 21], an alternative characterisation was sought so as to attack the general problem and for the sake of developing computational tools. Before we recall the relevant results, we will establish some notation that we will stick to throughout. Given a non-negative integer n, let

$$P_n = \{\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) : \lambda \vdash 2n, \lambda_i \geq 0 \text{ and } \lambda_i \text{ all even or all odd})\} \quad ,$$

$$Q_n = \{\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) : \lambda \vdash 2n, \lambda_i \geq 0 \text{ and exactly two of } \lambda_i \text{ are odd})\} \quad .$$

This given, let

$$P = \bigcup_{n \geq 0} P_n \quad ,$$

$$Q = \bigcup_{n \geq 0} Q_n \quad ,$$

and it is amply clear that $P \cup Q$ is the set of all partitions of even size and length at most 4. To reiterate a point we made earlier, when we consider partitions in P or Q, we identify partitions which differ only in the number of trailing zeroes. For instance, if $\lambda = (4,4,2)$, then the statement 'λ belongs to P' is true even though we have not written λ as a partition with 4 parts.

We will also be needing the Knuth bracket for giving truth values to statements. We say

$$((S)) = \begin{cases} 1 & \text{if the statement S is true} \\ 0 & \text{otherwise.} \end{cases} \tag{1.17}$$

Given partitions λ and μ, one can define their difference to be the sequence composed of pointwise differences, i.e., the i-th term of the sequence is $\lambda_i - \mu_i$. We let μP denote the set of partitions λ such that $\lambda - \mu \in P$. For instance, if $\mu = (2,1,1)$ and $\lambda = (7,6,6,3)$ then $\lambda - \mu = (5,5,5,3)$ and thus $\lambda \in \mu P$.

Now we are in a position to state the results of interest. To start it all, the computation of $s_{(n,n)} * s_{(n,n)}$ in the form given below appeared in [12]. This case was also dealt with in [7]. The results stated the following.

Theorem 1.6.1. *Given a positive integer n,*

$$s_{(n,n)} * s_{(n,n)} = \sum_{\substack{\lambda \in P \\ \lambda \vdash 2n}} s_\lambda \quad . \tag{1.18}$$

These results were different from earlier characterisations as they explicitly state which partitions have non-zero coefficients and further establish that the coefficients are all either 0 or 1 without giving a combinatorial rule. This computation originally arose out of solving a mathematical physics problem related to resolving the interference of 4 qubits [25].

Example 1.6.2. *Consider the computation of $s_{(4,4)} * s_{(4,4)}$ in view of the above theorem. Then,*

$$s_{(4,4)} * s_{(4,4)} = s_{(8)} + s_{(6,2)} + s_{(5,1,1,1)} + s_{(4,4)} + s_{(4,2,2)} + s_{(3,3,1,1)} + s_{(2,2,2,2)}.$$

Using the result of [12] as their inspiration, a characterisation of the Kronecker product of $s_{(n,n)} * s_{(n+k,n-k)}$ for $k \geq 0$ was obtained in [7]. Their work indicated that the Schur functions expansion of $s_{(n,n)} * s_{(n+k,n-k)}$ has the pattern of a boolean lattice of subsets, in the sense that it can be written as a certain number intersecting sums of Schur functions each with coefficient 1. The main result in [7] stated the following.

Theorem 1.6.3. *Let λ be a partition of $2n$. Then*

$$\begin{aligned} \langle s_{(n,n)} * s_{(n+k,n-k)}, s_\lambda \rangle = \quad & \sum_{i=0}^{k} ((\lambda \in (k+i,k,i)P)) \\ & + \sum_{i=1}^{k} ((\lambda \in (k+i+1,k+1,i)P)) \quad . \end{aligned} \tag{1.19}$$

As a demonstration of the result, we will consider an example.

Example 1.6.4. *Consider the Kronecker product $s_{(4,4)} * s_{(6,2)}$, i.e., we have n=4 and k=2. Consider $\lambda = (5,2,1)$. Let us see what the above theorem implies for*

$\langle s_{(4,4)} * s_{(6,2)}, s_\lambda \rangle$. We have

$$\langle s_{(4,4)} * s_{(6,2)}, s_\lambda \rangle = \sum_{i=0}^{2}((\lambda \in (2+i,2,i)P)) + \sum_{i=1}^{2}((\lambda \in (3+i,3,i)P)) \ .$$

It is clear that $(5,2,1)$ can not belong to $(3+i,3,i)P$ for $i = 1,2$. Also, $(5,2,1)$ does not belong to $(4,2,2)P$. Thus, we obtain

$$\langle s_{(4,4)} * s_{(6,2)}, s_\lambda \rangle = ((\lambda \in (2,2,0)P)) + ((\lambda \in (3,2,1)P)) \ .$$

Now, since $(5,2,1)-(2,2,0) = (3,0,1) \notin P$ and $(5,2,1)-(3,2,1) = (2,0,0)$ which does happen to be in P, we obtain $\langle s_{(4,4)} * s_{(6,2)}, s_{(5,2,1)} \rangle = 1$.

We have already seen what this theorem implies in the case $k = 0$. For $k = 1$, we obtain the following corollary [7].

Corollary 1.6.5. *Given a positive integer n,*

$$s_{(n,n)} * s_{(n+1,n-1)} = \sum_{\substack{\lambda \in Q \\ \lambda \vdash 2n}} s_\lambda \ . \tag{1.20}$$

Example 1.6.6. *Let us expand $s_{(4,4)} * s_{(5,3)}$ in the Schur function basis. Then the corollary above tells us that*

$$s_{(4,4)} * s_{(5,3)} = s_{(7,1)} + s_{(6,1,1)} + s_{(5,2,1)} + s_{(5,3)} + s_{(4,3,1)} + s_{(4,2,1,1)} + s_{(3,3,2)} + s_{(3,2,2,1)} \ .$$

A result of Littlewood that we will use a lot, and one that aids the computation of Kronecker coefficients is the following [16].

Theorem 1.6.7. *Let α, β and γ be partitions such that $|\alpha| + |\beta| = |\gamma|$. Then,*

$$(s_\alpha s_\beta) * s_\gamma = \sum_{\delta \vdash |\beta|} \sum_{\eta \vdash |\alpha|} c^{\gamma}_{\eta,\delta} (s_\eta * s_\alpha)(s_\delta * s_\beta) \tag{1.21}$$

where $c^{\gamma}_{\eta,\delta}$ are the Littlewood-Richardson coefficients.

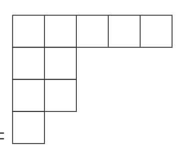
Using this identity of Littlewood and the characterisation of $s_{(n,n)} * s_{(n,n)}$, one can prove the following corollary, present in the following form in [7].

Corollary 1.6.8. *Given a positive integer n,*

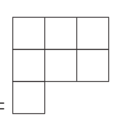
$$s_{(n,n-1)} * s_{(n,n-1)} = \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_\lambda \quad . \tag{1.22}$$

Now to set ourselves up completely for what follows, we just need one final result that, given partitions μ and ν, describes certain partitions λ for which $g^{\lambda}_{\mu\nu} = 0$. Below, $\mu \cap \nu$ denotes the partition obtained by intersecting the corresponding Ferrers diagrams. To illustrate this point, we will give an example.

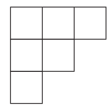
Example 1.6.9. *Let*

$\nu =$  ,

and

$\mu =$  .

Then $\mu \cap \nu$ is represented by the following Ferrers diagram



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Dvir [10] and Clausen and Meier [9] proved the following theorem.

Theorem 1.6.10. *Let μ, ν be partitions of n. Then*

$$\max \{\lambda_1 : g^{\lambda}_{\mu\nu} \neq 0 \text{ for some } \lambda = (\lambda_1, \lambda_2, \ldots)\} = |\mu \cap \nu|,$$

$$\max \{m : g^{\lambda}_{\mu\nu} \neq 0 \text{ for some } \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0)\} = |\mu \cap \nu^t|.$$

The import of this theorem can be gauged by the fact that it already implies that if μ and ν are partitions each with at most two rows, then $g^{\lambda}_{\mu\nu}$ is 0 for all λ such that $l(\lambda) \geq 5$.

1.7 Summary of results present in this thesis

In this thesis, we provide explicit formulae for the Kronecker coefficients occurring in the expansion of $s_{(n,n-1,1)} * s_{(n,n)}$, $s_{(n-1,n-1,1)} * s_{(n,n-1)}$, $s_{(n-1,n-1,2)} * s_{(n,n)}$, $s_{(n-1,n-1,1,1)} * s_{(n,n)}$ and $s_{(n,n,1)} * s_{(n,n,1)}$ on page 39 in Section 2.1, page 41 in Section 2.2, page 49 in Section 2.3, page 56 in Section 2.4 and page 67 in Section 2.5 respectively. One of the main tools we use is the Pieri rule, which gives an easy way to compute products of the form $s_{(1)}s_{\lambda}$, $s_{(2)}s_{\lambda}$ and $s_{(1,1)}s_{\lambda}$ and also provides a succinct description of the skewing operators $s^{\perp}_{(1)}$, $s^{\perp}_{(2)}$ and $s^{\perp}_{(1,1)}$. Another crucial result that we use is an identity of Littlewood, which is Theorem 1.6.7, allowing us to compute the Kronecker coefficients in terms of certain Littlewood-Richardson coefficients and previously obtained characterizations of the Kronecker product of Schur functions indexed by two-rowed partitions.

In arriving at Theorem 2.6.3, we use the knowledge of the Kronecker coefficients occurring in $s_{(n,n-1,1)} * s_{(n,n)}$ and $s_{(n,n,1)} * s_{(n+1,n)}$ to obtain an enumeration of standard Young tableaux of height 5 and smallest part equalling 1. This is done by translating the relevant Kronecker product decomposition to the language of characters and equating dimensions. In doing so, note that what we enumerate is actually the number of lattice words of length n which consist of letters 1,2,3,4 or 5 with exactly one occurrence of 5.

Chapter 2

Description of Kronecker Coefficients in Certain Cases

We start by establishing some notation which the reader has not encountered in the introduction. Let

- λ' = the partition $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, given a partition λ. In case, λ has length less than 4, then we supplement it with an appropriate number of zeroes to get λ'. Thus, for example, if $\lambda = (5,3,3,2,1,1)$, then $\lambda' = (5,3,3,2)$ and if $\lambda = (5,3,2)$ then $\lambda' = (5,3,2,0)$.
- $O_\lambda =$ the number of odd parts in λ' .
- $E_\lambda =$ the number of even non-zero parts in λ'.
- $O'_\lambda =$ the number of distinct odd parts in λ'.
- $E'_\lambda =$ the number of distinct even non-zero parts in λ'.

- $R_\lambda =$ the number of distinct non-zero parts of λ which occur at least twice. For illustration's sake, consider the case $\lambda = (6,5,3,3,3,2,2)$. Then $R(\lambda) = 2$. If the parts in λ are all distinct, then $R_\lambda = 0$.

- $S_{\lambda/\mu} =$ the set of partitions $\theta \vdash (|\lambda| - |\mu|)$ such that $\langle s_\theta, s_{\lambda/\mu} \rangle \neq 0$, given partitions λ and μ. Equivalently, the Littlewood-Richardson coefficient $c^\lambda_{\theta,\mu}$ is non-zero.

- $d_\lambda =$ the number of distinct non-zero parts in λ.

- $d_{\lambda,2} =$ the number of partitions γ such that there exists an index i so that $\gamma_i + 2 = \lambda_i$ and $\gamma_j = \lambda_j$ for all $j \neq i$. Again, if we consider the case where $\lambda = (6,5,3,3,3,2,2)$. Then $d_{\lambda,2} = 2$ as the possible γ that satisfy the condition mentioned above are $(6,3,3,3,3,2,2)$ and $(6,5,3,3,3,2,0)$.

In the sections that follow, the statement '$\lambda \in P$' is considered to be equivalent to '$\lambda' \in P$', and an analogous statement holds for a statement like '$\lambda \in Q$'. For example, consider $\lambda = (5,3,3,1,1)$. Then, even though λ has 5 parts, we say $((\lambda \in P))$ evaluates to 1 because $\lambda' = (5,3,3,1)$ has all 4 parts odd, and thus belongs to P.

2.1 $s_{(n,n-1,1)} * s_{(n,n)}$ $(n \geq 2)$

We will now give an explicit characterization of the Kronecker product of $s_{(n,n-1,1)}$ and $s_{(n,n)}$.

Observe that the Pieri rule (Theorem 1.3.8) implies

$$s_{(n,n-1,1)} = s_{(n,n-1)}s_{(1)} - s_{(n,n)} - s_{(n+1,n-1)}. \tag{2.1}$$

Since we are looking for the coefficient of s_θ, where $\theta \vdash 2n$, in the expansion of $s_{(n,n-1,1)} * s_{(n,n)}$ as a sum of Schur functions, we should compute $\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle$. Using (2.1), we obtain

$$\begin{aligned} \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle &= \langle (s_{(n,n-1)}s_{(1)}) * s_{(n,n)}, s_\theta \rangle \\ &\quad - \langle (s_{(n,n)} + s_{(n+1,n-1)}) * s_{(n,n)}, s_\theta \rangle. \end{aligned} \tag{2.2}$$

We will look to evaluate the inner products appearing on the right hand side of (2.2) individually. The use of Theorem 1.6.7 implies

$$\begin{aligned} (s_{(n,n-1)}s_{(1)}) * s_{(n,n)} &= \sum_{\delta \vdash 1} \sum_{\eta \vdash 2n-1} c^{(n,n)}_{\eta,\delta} (s_\eta * s_{(n,n-1)})(s_\delta * s_{(1)}) \\ &= \sum_{\eta \vdash 2n-1} c^{(n,n)}_{\eta,(1)} (s_\eta * s_{(n,n-1)})(s_{(1)} * s_{(1)}). \end{aligned} \tag{2.3}$$

The Pieri rule yields that $c^{(n,n)}_{\eta,(1)} \neq 0$ if and only if $\eta = (n, n-1)$, in which case $c^{(n,n)}_{(n,n-1),(1)} = 1$. Since $s_{(1)} * s_{(1)} = s_{(1)}$, we conclude that

$$(s_{(n,n-1)}s_{(1)}) * s_{(n,n)} = s_{(1)}(s_{(n,n-1)} * s_{(n,n-1)}). \tag{2.4}$$

This reduces (2.2) to

$$\begin{aligned}
\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle &= \langle s_{(1)}(s_{(n,n-1)} * s_{(n,n-1)}), s_\theta \rangle \\
&\quad - \langle (s_{(n,n)} + s_{(n+1,n-1)}) * s_{(n,n)}, s_\theta \rangle \\
&= \langle s_{(n,n-1)} * s_{(n,n-1)}, s_{(1)}^{\perp} s_\theta \rangle \\
&\quad - \langle (s_{(n,n)} + s_{(n+1,n-1)}) * s_{(n,n)}, s_\theta \rangle \\
&= \langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \le 4}} s_\lambda, s_{\theta/(1)} \rangle - \langle \sum_{\substack{\lambda \vdash 2n \\ l(\lambda) \le 4}} s_\lambda, s_\theta \rangle. \qquad (2.5)
\end{aligned}$$

In arriving at the last step in the above sequence (2.5), we've made use of the identities (1.22),(1.20) and (1.18). Notice $\langle \sum_{\substack{\lambda \vdash 2n \\ l(\lambda) \le 4}} s_\lambda, s_\theta \rangle$ is 1 if $l(\theta) \le 4$ and 0 otherwise. So we will focus on evaluating $\langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \le 4}} s_\lambda, s_{\theta/(1)} \rangle$. We appeal to the Pieri rule again to see how $s_{\theta/(1)}$ decomposes in the Schur basis. It implies

$$s_{\theta/(1)} = \sum_{\theta^- \prec \theta} s_{\theta^-}. \qquad (2.6)$$

In the equality above, $\theta^- \prec \theta$ means that θ covers θ^- in the Young's lattice of partitions. Alternatively put, $s_{\theta/(1)}$ is the sum of all terms of the form s_{θ^-}, where θ^- is obtained by removing an inner corner of the partition θ. Note that the number of terms appearing on the right hand side of (2.6) is equal to the number of distinct parts in the partition θ, i.e. d_θ.

If $\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle \neq 0$, then we must have that $l(\theta) \le 5$ by Theorem 1.6.10, as $|(n,n-1,1) \cap (n,n)^t| \le 5$. We will carry out the rest of the computation in cases depending on the length of the partition θ.

2.1.1 Case I: $l(\theta) = 5$

If $l(\theta) = 5$, but $\theta_5 \geq 2$, then $s_{\theta/(1)}$ is sum of terms of the form s_γ with $l(\gamma) = 5$. The right hand side of (2.5) clearly implies that the coefficient of s_θ in $s_{(n,n-1,1)} * s_{(n,n)}$ is 0 in this instance. If $\theta_5 = 1$, then $s_{\theta/(1)} = s_{\theta'} +$ sum of terms of the form s_γ where $l(\gamma) = 5$. This in turn means that $\langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_\lambda, s_{\theta/(1)} \rangle = 1$. Thus, if $l(\theta) = 5$,

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

2.1.2 Case II: $l(\theta) \leq 4$

We know that if $l(\theta) \leq 4$, then $\langle (s_{(n,n)} + s_{(n+1,n-1)}) * s_{(n,n)}, s_\theta \rangle = 1$, as θ either belongs to P or it belongs to Q. The following computation helps us in finishing this case.

$$\begin{aligned} \langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_\lambda, s_{\theta/(1)} \rangle &= \langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_\lambda, \sum_{\theta^- \prec \theta} s_{\theta^-} \rangle \\ &= d_\theta. \end{aligned} \tag{2.7}$$

Therefore

$$\langle s_{(n,n-1)} * s_{(n,n-1)}, s_{(1)}^{\perp} s_\theta \rangle = d_\theta. \tag{2.8}$$

Thus for $l(\theta) \leq 4$,

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = d_\theta - 1.$$

2.1.3 Summary

On collecting the results of the two cases together, we obtain the following description

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & l(\theta) = 5,\ \theta_5 = 1 \\ d_\theta - 1 & l(\theta) \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

As a demonstration, we give an example.

Example 2.1.1. *Consider the computation of* $s_{(4,3,1)} * s_{(4,4)}$. *Then*

$$\begin{aligned} s_{(4,3,1)} * s_{(4,4)} = \quad & s_{(2,2,2,1,1)} + s_{(3,2,1,1,1)} + 2s_{(3,2,2,1)} + s_{(3,3,1,1)} + s_{(3,3,2)} \\ & + s_{(4,1,1,1,1)} + 2s_{(4,2,1,1)} + s_{(4,2,2)} + 2s_{(4,3,1)} + s_{(5,1,1,1)} \\ & + 2s_{(5,2,1)} + s_{(5,3)} + s_{(6,1,1)} + s_{(6,2)} + s_{(7,1)}. \end{aligned}$$

2.2 $s_{(n-1,n-1,1)} * s_{(n,n-1)}$ $\quad (n \geq 2)$

In the same vein as the previous case, we can try finding the Kronecker product of $s_{(n-1,n-1,1)}$ and $s_{(n,n-1)}$. Again, the Pieri rule implies that

$$s_{(n-1,n-1,1)} = s_{(1)}s_{(n-1,n-1)} - s_{(n,n-1)}. \tag{2.9}$$

An application of Theorem 1.6.7 gives

$$\begin{aligned}
\big(s_{(1)}s_{(n-1,n-1)}\big) * s_{(n,n-1)} &= \sum_{\delta\vdash 1}\sum_{\eta\vdash 2n-2} c^{(n,n-1)}_{\eta,\delta}\big(s_\eta * s_{(n-1,n-1)}\big)\big(s_\delta * s_{(1)}\big) \\
&= \sum_{\eta\vdash 2n-2} c^{(n,n-1)}_{\eta,(1)}\big(s_\eta * s_{(n-1,n-1)}\big)\big(s_{(1)} * s_{(1)}\big).
\end{aligned} \tag{2.10}$$

The Pieri rule dictates that the only cases where $c^{(n,n-1)}_{\eta,(1)} \neq 0$ are when $\eta = (n,n-2)$ or $\eta = (n-1,n-1)$ and in both cases $c^{(n,n-1)}_{\eta,(1)} = 1$. Thus

$$\begin{aligned}
\big(s_{(1)}s_{(n-1,n-1)}\big) * s_{(n,n-1)} &= s_{(1)}\big(s_{(n,n-2)} * s_{(n-1,n-1)}\big) \\
&\quad + s_{(1)}\big(s_{(n-1,n-1)} * s_{(n-1,n-1)}\big).
\end{aligned} \tag{2.11}$$

If $\theta \vdash 2n-1$, (2.9) and (2.11) together bring us to

$$\begin{aligned}
\langle s_{(n-1,n-1,1)} * s_{(n,n-1)}, s_\theta \rangle &= \langle s_{(1)}((s_{(n,n-2)} + s_{(n-1,n-1)}) * s_{(n-1,n-1)}), s_\theta \rangle \\
&\quad - \langle s_{(n,n-1)} * s_{(n,n-1)}, s_\theta \rangle \\
&= \langle (s_{(n,n-2)} + s_{(n-1,n-1)}) * s_{(n-1,n-1)}, s_{(1)}^{\perp} s_\theta \rangle \\
&\quad - \langle s_{(n,n-1)} * s_{(n,n-1)}, s_\theta \rangle \\
&= \langle \sum_{\substack{\lambda \vdash 2n-2 \\ l(\lambda) \le 4}} s_\lambda, s_{\theta/(1)} \rangle - \langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \le 4}} s_\lambda, s_\theta \rangle. \qquad (2.12)
\end{aligned}$$

Now one can easily check that this gives the same characterization as the one obtained from $s_{(n,n-1,1)} * s_{(n,n)}$ and the argument is essentially the same, except that we use (2.12) instead of (2.5). We obtain

$$\langle s_{(n-1,n-1,1)} * s_{(n,n-1)}, s_\theta \rangle = \begin{cases} 1 & l(\theta) = 5,\ \theta_5 = 1 \\ d_\theta - 1 & l(\theta) \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

To see the above characterization in practice, we will consider an example.

Example 2.2.1. *We will compute* $s_{(3,3,1)} * s_{(4,3)}$.

$$\begin{aligned}
s_{(3,3,1)} * s_{(4,3)} = \quad & s_{(2,2,1,1,1)} + s_{(2,2,2,1)} + s_{(3,1,1,1,1)} + 2s_{(3,2,1,1)} + s_{(3,2,2)} + s_{(3,3,1)} \\
& + s_{(4,1,1,1)} + 2s_{(4,2,1)} + s_{(4,3)} + s_{(5,1,1)} + s_{(5,2)} + s_{(6,1)}.
\end{aligned}$$

2.3 $s_{(n-1,n-1,2)} * s_{(n,n)} \qquad (n \geq 3)$

In this particular case, an application of the Pieri rule yields

$$s_{(n-1,n-1,2)} = s_{(2)}s_{(n-1,n-1)} - s_{(n,n-1,1)} - s_{(n+1,n-1)}. \tag{2.13}$$

The Kronecker product of $s_{(n-1,n-1,2)}$ and $s_{(n,n)}$ would require the evaluation of $(s_{(2)}s_{(n-1,n-1)}) * s_{(n,n)}$. Using Theorem 1.6.7, one can see that

$$\begin{aligned}(s_{(2)}s_{(n-1,n-1)}) * s_{(n,n)} &= \sum_{\delta\vdash 2}\sum_{\eta\vdash 2n-2} c^{(n,n)}_{\eta,\delta}(s_\eta * s_{(n-1,n-1)})(s_\delta * s_{(2)}) \\ &= \sum_{\eta\vdash 2n-2} c^{(n,n)}_{\eta,(1,1)}(s_\eta * s_{(n-1,n-1)})(s_{(1,1)} * s_{(2)}) \\ &\quad + \sum_{\eta\vdash 2n-2} c^{(n,n)}_{\eta,(2)}(s_\eta * s_{(n-1,n-1)})(s_{(2)} * s_{(2)}).\end{aligned} \tag{2.14}$$

Notice that for $c^{(n,n)}_{\eta,(2)} \neq 0$ to hold, we must have $\eta = (n, n-2)$ whereas $c^{(n,n)}_{\eta,(1,1)} \neq 0$ implies that $\eta = (n-1, n-1)$. In both cases, $c^{(n,n)}_{\eta,\delta} = 1$ where $\delta = (1,1)$, $\eta = (n-1, n-1)$ or $\delta = (2)$, $\eta = (n, n-2)$. This allows us to rewrite (2.14) as

$$\begin{aligned}(s_{(2)}s_{(n-1,n-1)}) * s_{(n,n)} &= s_{(1,1)}(s_{(n-1,n-1)} * s_{(n-1,n-1)}) \\ &\quad + s_{(2)}(s_{(n,n-2)} * s_{(n-1,n-1)}).\end{aligned} \tag{2.15}$$

Now, let $\theta \vdash 2n$. From the equality above, it follows that

$$
\begin{aligned}
\langle s_{(n-1,n-1,2)} * s_{(n,n)}, s_\theta \rangle = & \quad \langle (s_{(2)} s_{(n-1,n-1)}) * s_{(n,n)}, s_\theta \rangle \\
- & \quad \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle \\
- & \quad \langle s_{(n+1,n-1)} * s_{(n,n)}, s_\theta \rangle \\
= & \quad \langle s_{(1,1)} (s_{(n-1,n-1)} * s_{(n-1,n-1)}), s_\theta \rangle \\
+ & \quad \langle s_{(2)} (s_{(n,n-2)} * s_{(n-1,n-1)}), s_\theta \rangle \\
- & \quad \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle \\
- & \quad \langle s_{(n+1,n-1)} * s_{(n,n)}, s_\theta \rangle \\
= & \quad \langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^\perp_{(1,1)} s_\theta \rangle \\
+ & \quad \langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^\perp_{(2)} s_\theta \rangle \\
- & \quad \langle s_{(n+1,n-1)} * s_{(n,n)}, s_\theta \rangle \\
- & \quad \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle. \qquad (2.16)
\end{aligned}
$$

Since we already have a description of the Kronecker products $s_{(n,n-1,1)} * s_{(n,n)}$ and $s_{(n+1,n-1)} * s_{(n,n)}$, we can focus on evaluating the other two terms on the right hand side of (2.16) individually. But before that, note the fact that if $\langle s_{(n-1,n-1,2)} * s_{(n,n)}, s_\theta \rangle \neq 0$, then $l(\theta) \leq 6$ necessarily, by an application of Theorem 1.6.10. So we will restrict ourselves to the partitions $\theta \vdash 2n$ satisfying $l(\theta) \leq 6$. We will deal with cases based on the length of the partition θ.

2.3.1 Case I: $l(\theta) = 6$

In this case, we already know that both the terms $\langle s_{(n+1,n-1)} * s_{(n,n)}, s_\theta \rangle$ and $\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle$ are 0.

Now we deal with $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle$. To this end, notice that $s^{\perp}_{(2)} s_\theta$ is a sum of terms of the form s_γ where either γ is a partition obtained by subtracting 2 from one of the parts of θ, or γ is a partition obtained by subtracting 1 from 2 distinct parts of the partition θ. This is a consequence of the Pieri rule again. This implies, if $l(\theta) = 6$

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle = 0. \tag{2.17}$$

Next, we will look at $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)} s_\theta \rangle$. As a consequence of the Pieri rule, notice that $s^{\perp}_{(1,1)} s_\theta$ is a sum of terms of the form s_γ, where $\gamma \vdash 2n-2$ is a partition obtained by subtracting 1 each from two different (but not necessarily distinct) parts of θ. We know that if $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_\gamma \rangle \neq 0$ then $l(\gamma) \leq 4$. Thus one can expect $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)} s_\theta \rangle$ to be non-zero if the two smallest parts, θ_5 and θ_6, are both equal to 1. In other instances, $s^{\perp}_{(1,1)} s_\theta$ is a sum of terms of the form s_γ where $l(\gamma) \geq 5$ which, as noted earlier, can not occur in $s_{(n-1,n-1)} * s_{(n-1,n-1)}$. Thus, one obtains the following

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)} s_\theta \rangle = \begin{cases} 1 & \theta' \in P,\ \theta_5 = \theta_6 = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Thus, in the present case, we obtain

$$\langle s_{(n-1,n-1,2)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = \theta_6 = 1,\ \theta \in P \\ 0 & \text{otherwise.} \end{cases}$$

2.3.2 Case II: $l(\theta) = 5$

In this case, we already know that $\langle s_{(n+1,n-1)} * s_{(n,n)}, s_\theta \rangle = 0$. The characterization of $s_{(n,n-1,1)} * s_{(n,n)}$ we obtained earlier implies

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now, consider the computation of $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{(2)}^{\perp} s_\theta \rangle$. One can notice that if $\theta_5 \geq 3$, then $s_{(2)}^{\perp} s_\theta$ is a sum of terms of the form s_γ where $l(\gamma) = 5$. This implies $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{(2)}^{\perp} s_\theta \rangle = 0$. If $\theta_5 = 2$, then $s_{(2)}^{\perp} s_\theta = s_{\theta'} +$ sum of terms of the form s_γ where $l(\gamma) = 5$. Thus, $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{(2)}^{\perp} s_\theta \rangle = 1$ if $\theta^{'} \in Q$. The case where $\theta_5 = 1$ is slightly intricate, mainly because if θ_4 is also 1, then $s_{(2)}^{\perp} s_\theta$ will not have a term with both θ_4 and θ_5 removed. The only terms in $s_{(2)}^{\perp} s_\theta$ which might have a non-zero coefficient in $s_{(n,n-2)} * s_{(n-1,n-1)}$ are of the form s_γ where $\gamma \in S_{\theta'/(1)}$ with the clause that, we neglect $\gamma \in S_{\theta'/(1)}$ which satisfy $l(\gamma) = 3$. This takes care of the case where $\theta_4 = \theta_5 = 1$. Notice that $\theta^{'}$ has exactly 3 odd parts or exactly 3 even parts, as $\theta^{'} \vdash 2n-1$. Note that $s_{(n,n-2)} * s_{(n-1,n-1)}$ has terms of the form s_γ where $\gamma \in Q$ only, by (1.20). If $E_{\theta'} = 1$ (i.e. $\theta^{'}$ has exactly 3 odd parts), then we can subtract 1 from any of the distinct odd parts to obtain a partition which belongs to Q, and on subtracting 1 from the even part we obtain a partition which belongs to P. On the other hand, if $O_{\theta'} = 1$ (i.e. $\theta^{'}$ has exactly 3 even parts), then we can subtract 1 from any of the distinct even parts to obtain a partition which belongs to Q, and on subtracting 1 from the odd part we obtain a partition which belongs to P. This line of reasoning allows us to conclude that

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle = \quad ((E_{\theta'}=1))[O^{'}_{\theta'} - ((\theta^{'}_4 = 1))] + ((O_{\theta'}=1))[E^{'}_{\theta'}]. \tag{2.19}$$

Put succinctly, one has

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle = \begin{cases} 0 & \theta_5 \geq 3 \\ 1 & \theta_5 = 2,\ \theta^{'} \in Q \\ O^{'}_{\theta'} - ((\theta_4 = 1)) & \theta_5 = 1,\ E_{\theta'} = 1 \\ E^{'}_{\theta'} & \theta_5 = 1,\ O_{\theta'} = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

Shifting our focus to computing $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)} s_\theta \rangle$, observe that if $\theta_5 \geq 2$, then $s^{\perp}_{(1,1)} s_\theta$ is a sum of terms of the form s_γ with $l(\gamma) = 5$. Thus, these terms do not appear in $s_{(n-1,n-1)} * s_{(n-1,n-1)}$. This allows us to narrow our consideration to the case $\theta_5 = 1$. In this case $s^{\perp}_{(1,1)} s_\theta$ is a sum of terms of the form s_γ, where either $l(\gamma) = 5$ (i.e. θ_5 is not removed), in which case they can not occur in $s_{(n-1,n-1)} * s_{(n-1,n-1)}$, or $l(\gamma) \leq 4$ and $\gamma \in S_{\theta'/(1)}$ (i.e θ_5 is removed and 1 is subtracted from one of θ_1, θ_2, θ_3 or θ_4 and hence the claim that $\gamma \in S_{\theta'/(1)}$). Since $\theta^{'}$ is a partition of $2n-1$, it has either exactly three parts even, or exactly 3 parts odd. In any case, there is exactly one partition in $S_{\theta'/(1)}$ which belongs to P. This implies the fact that, if $l(\theta)=5$ then

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)} s_\theta \rangle = \begin{cases} 1 & \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

Collecting the results of computing individual terms on the right hand side of (2.16) in the case $l(\theta)=5$ gives us

$$\langle s_{(n-1,n-1,2)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = 2,\ \theta \in Q \\ O^{'}_{\theta} - ((\theta_4 = 1)) & E_\theta = 1,\ \theta_5 = 1 \\ E^{'}_{\theta} & O_\theta = 1,\ \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that that we have used the fact that, by definition $O^{'}_{\theta'} = O^{'}_{\theta}$ and $E^{'}_{\theta'} = E^{'}_{\theta}$.

2.3.3 Case III: $l(\theta) \le 4$

We know that, if $l(\theta) \le 4$, then

$$\langle s_{(n+1,n-1)} * s_{(n,n)}, s_\theta \rangle = ((\theta \in Q)) \quad ,$$

and

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = d_\theta - 1.$$

To help complete this case, we will compute $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle$. Let $\theta \in P$. The partition obtained by subtracting 2 from any part of θ is still in P, and there are no terms of the form s_γ with $\gamma \in P$ in $s_{(n,n-2)} * s_{(n-1,n-1)}$. Thus the only contribution is from terms of the form s_γ where γ is obtained by subtracting 1 from two distinct parts of θ, as such a partition clearly belongs to Q. All such terms appear in $s^\perp_{(2)}s_\theta$ with coefficient 1. If $\theta \in Q$, then all partitions obtained by subtracting 2 from a part of θ are still in Q. To obtain other partitions γ such that γ is in Q and s_γ appears in $s^\perp_{(2)}s_\theta$, we subtract 1 from one of the odd parts and a 1 from one of the even parts. This leads us to

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^\perp_{(2)}s_\theta\rangle = \begin{cases} \binom{d_\theta}{2} & \theta \in P \\ d_{\theta,2} + O'_\theta E'_\theta & \theta \in Q. \end{cases} \tag{2.22}$$

Let us consider $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^\perp_{(1,1)}s_\theta\rangle$ now. Assume $\theta \in P$. Then $s^\perp_{(1,1)}s_\theta$ is a sum of terms of the form s_γ, where $\gamma \in Q$ as the parity of exactly two of the parts of θ is flipped, allowing us to conclude that, in this case $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^\perp_{(1,1)}s_\theta\rangle = 0$. Now $\theta \in Q$ is the remaining case. There are exactly 2 odd parts in θ. Subtracting 1 from each of these parts will give us a partition of $2n-2$, which lies in P. If there are 2 even non-zero parts in θ, then subtracting 1 from each of these will also give us a partition of $2n-2$ lying in P. Both these partitions appear with a coefficient 1 in $s^\perp_{(1,1)}s_\theta$. Thus, if $l(\theta) \le 4$,

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^\perp_{(1,1)}s_\theta\rangle = \begin{cases} 0 & \theta \in P \\ 1 + ((E_\theta = 2)) & \theta \in Q. \end{cases} \tag{2.23}$$

On collecting the results obtained in the case $l(\theta) \le 4$, we obtain

$$\langle s_{(n-1,n-1,2)} * s_{(n,n)}, s_\theta\rangle = \begin{cases} 1 - d_\theta + \binom{d_\theta}{2} & \theta \in P \\ 1 - d_\theta + d_{\theta,2} + O'_\theta E'_\theta + ((E_\theta = 2)) & \theta \in Q. \end{cases}$$

2.3.4 Summary

Just for the sake of convenience, we will collect the results of the three cases together. This gives us the following characterization

$$\langle s_{(n-1,n-1,2)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} ((\theta \in P)) & l(\theta) = 6 \text{ and} \\ & \theta_5 = \theta_6 = 1 \\ ((\theta \in Q)) & l(\theta) = 5,\ \theta_5 = 2 \\ O'_\theta - ((\theta_4 = 1)) & l(\theta) = 5 \text{ and} \\ & E_\theta = 1,\ \theta_5 = 1 \\ E'_\theta & l(\theta) = 5 \text{ and} \\ & O_\theta = 1,\ \theta_5 = 1 \\ 1 - d_\theta + \binom{d_\theta}{2} & l(\theta) \leq 4,\ \theta \in P \\ 1 - d_\theta + d_{\theta,2} + O'_\theta E'_\theta + ((E_\theta = 2)) & l(\theta) \leq 4,\ \theta \in Q \\ 0 & \text{otherwise.} \end{cases}$$

We will now consider an example.

Example 2.3.1. *Consider the product $s_{(7,7,2)} * s_{(8,8)}$, and three partitions $\alpha = (5,5,3,1,1,1)$, $\beta = (6,4,3,2,1)$ and $\gamma = (7,5,2,2)$.*

Note that $l(\alpha) = 6$ and $\alpha_5 = \alpha_6 = 1$ as well. Since $\alpha' = (5,5,3,1)$ belongs to P, we obtain $\langle s_{(7,7,2)} * s_{(8,8)}, s_{(5,5,3,1,1,1)} \rangle = 1$.

Consider the case of β now. We have $l(\beta) = 5$ and $\beta_5 = 1$. Since β' has exactly 1 odd part, we have $O_\beta = 1$. Thus, the above characterization allows one to obtain $\langle s_{(7,7,2)} * s_{(8,8)}, s_{(6,4,3,2,1)} \rangle = E'_{(6,4,3,2,1)}$. Since β' has exactly 3 distinct even non-zero parts, we have $\langle s_{(7,7,2)} * s_{(8,8)}, s_{(6,4,3,2,1)} \rangle = 3$.

Turning our attention to γ, we see that $l(\gamma) = 4$ and $\gamma \in Q$. We have $d_\gamma =$

3, $d_{\gamma,2} = 3$, $O'_{\gamma} = 2$, $E'_{\gamma} = 1$ *and* $E_{\gamma} = 2$. *Thus, we get the fact that* $\langle s_{(7,7,2)} * s_{(8,8)}, s_{(7,5,2,2)} \rangle = 4$.

2.4 $s_{(n-1,n-1,1,1)} * s_{(n,n)}$ $(n \geq 2)$

In essence, this is pretty similar to the previous case. On applying Pieri's rule, we obtain

$$s_{(n-1,n-1,1,1)} = s_{(1,1)} s_{(n-1,n-1)} - s_{(n,n)} - s_{(n,n-1,1)}. \tag{2.24}$$

Using Theorem 1.6.7, we deduce that

$$\begin{aligned}
(s_{(1,1)} s_{(n-1,n-1)}) * s_{(n,n)} &= \sum_{\delta \vdash 2} \sum_{\eta \vdash 2n-2} c^{(n,n)}_{\eta,\delta} (s_\eta * s_{(n-1,n-1)})(s_\delta * s_{(1,1)}) \\
&= \sum_{\eta \vdash 2n-2} c^{(n,n)}_{\eta,(1,1)} (s_\eta * s_{(n-1,n-1)})(s_{(1,1)} * s_{(1,1)}) \\
&\quad + \sum_{\eta \vdash 2n-2} c^{(n,n)}_{\eta,(2)} (s_\eta * s_{(n-1,n-1)})(s_{(2)} * s_{(1,1)}).
\end{aligned} \tag{2.25}$$

We've already worked out which η satisfies $c^{(n,n)}_{\eta,\delta} \neq 0$ for $\delta \vdash 2$ in the previous section. Since $s_{(1,1)} * s_{(1,1)} = s_{(2)}$, we obtain

$$\begin{aligned}
(s_{(1,1)} s_{(n-1,n-1)}) * s_{(n,n)} &= s_{(2)} (s_{(n-1,n-1)} * s_{(n-1,n-1)}) \\
&\quad + s_{(1,1)} (s_{(n,n-2)} * s_{(n-1,n-1)}).
\end{aligned} \tag{2.26}$$

Given $\theta \vdash 2n$, this means

$$
\begin{aligned}
\langle s_{(n-1,n-1,1,1)} * s_{(n,n)}, s_\theta\rangle = & \quad \langle (s_{(1,1)} s_{(n-1,n-1)}) * s_{(n,n)}, s_\theta\rangle \\
& - \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta\rangle \\
& - \langle s_{(n,n)} * s_{(n,n)}, s_\theta\rangle \\
= & \quad \langle s_{(2)} (s_{(n-1,n-1)} * s_{(n-1,n-1)}), s_\theta\rangle \\
& + \langle s_{(1,1)} (s_{(n,n-2)} * s_{(n-1,n-1)}), s_\theta\rangle \\
& - \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta\rangle \\
& - \langle s_{(n,n)} * s_{(n,n)}, s_\theta\rangle \qquad (2.27)
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
\langle s_{(n-1,n-1,1,1)} * s_{(n,n)}, s_\theta\rangle = & \quad \langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^\perp_{(2)} s_\theta\rangle \\
& + \langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^\perp_{(1,1)} s_\theta\rangle \\
& - \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta\rangle \\
& - \langle s_{(n,n)} * s_{(n,n)}, s_\theta\rangle. \qquad (2.28)
\end{aligned}
$$

As we did in the previous case, we proceed to evaluate individual terms on the right hand side of (2.28). We only need to work on $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^\perp_{(2)} s_\theta\rangle$ and $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^\perp_{(1,1)} s_\theta\rangle$.

2.4.1 Case I: $l(\theta) = 6$

From existing characterizations, we know that both $\langle s_{(n,n)} * s_{(n,n)}, s_\theta \rangle$ and $\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle$ are 0 in the present case.

Let us look at $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle$. This is clearly 0 if $l(\theta) \geq 6$ because in these cases, $s^{\perp}_{(2)} s_\theta$ is a sum of terms of the form s_γ with $l(\gamma) \geq 5$.

Now consider $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)} s_\theta \rangle$. If $l(\theta) = 6$, the only possibility for a non-zero coefficient is when $\theta_5 = \theta_6 = 1$. By (1.20), this readily gives

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)} s_\theta \rangle = \begin{cases} 1 & \theta_5 = \theta_6 = 1,\ \theta' \in Q \\ 0 & \text{otherwise.} \end{cases} \tag{2.29}$$

Thus, in the current case, using (2.28) we obtain

$$\langle s_{(n-1,n-1,1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = \theta_6 = 1,\ \theta \in Q \\ 0 & \text{otherwise.} \end{cases}$$

2.4.2 Case II: $l(\theta) = 5$

Firstly, recall that, in this particular case we have $\langle s_{(n,n)} * s_{(n,n)}, s_\theta \rangle = 0$ and

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.30}$$

Consider $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle$ now. If $l(\theta) = 5$ and $\theta_5 \geq 3$, this is 0 because $s^{\perp}_{(2)} s_\theta$ is a sum of terms of the form s_γ with $l(\gamma) = 5$.

If $\theta_5 = 2$, then the only term in $s^{\perp}_{(2)} s_\theta$ which is of the form s_γ with $l(\gamma) \leq 4$ is

$s_{\theta'}$, and this occurs with coefficient 1 in it. Thus, if $\theta_5 = 2$, one obtains

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle = \begin{cases} 1 & \theta' \in P \\ 0 & \text{otherwise.} \end{cases} \tag{2.31}$$

If $\theta_5 = 1$, the only terms in $s^{\perp}_{(2)} s_\theta$ with length ≤ 4 occur by removing θ_5 and subtracting 1 from one of the other parts provided that it is not equal to θ_5. θ' has either exactly 3 odd parts or exactly 3 even parts. The expansion of $s_{(n-1,n-1)} * s_{(n-1,n-1)}$ has terms of the form s_γ where $\gamma \in P$. Thus, after removing θ_5, the other 1 should be removed from the single even part if there are 3 odd parts, or from the odd part if there are exactly 3 even parts. Thus

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle = \begin{cases} 1 & E_\theta = 1 \\ 1 - ((\theta_4 = 1)) & O_\theta = 1. \end{cases} \tag{2.32}$$

The above facts put together, imply

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle = \begin{cases} 1 & \theta_5 = 2,\ \theta' \in P \\ 1 & \theta_5 = 1,\ E_\theta = 1 \\ 1 - ((\theta_4 = 1)) & \theta_5 = 1,\ O_\theta = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.33}$$

Now we analyse $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)} s_\theta \rangle$. One sees that if $\theta_5 \geq 2$, the coefficient of $s^{\perp}_{(1,1)} s_\theta$ in $s_{(n,n-2)} * s_{(n-1,n-1)}$ is 0. If $\theta_5 = 1$, then the only way to get a term of the form s_γ with $l(\gamma) \leq 4$ in $s^{\perp}_{(1,1)} s_\theta$ is to remove θ_5 and subtract 1 from one of the other parts in θ. We need γ to belong to Q if it is to have a non-zero coefficient. The way to achieve this is to subtract 1 from one of the odd parts if there are exactly 3 odd parts in $\theta^{'}$, or subtract 1 from one of the even parts if there are exactly 3 even parts. All things considered, we find that, if $l(\theta) = 5$, then

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)} s_\theta \rangle = \begin{cases} E^{'}_{\theta'} & \theta_5 = 1,\ E_{\theta'} = 3 \\ O^{'}_{\theta'} & \theta_5 = 1,\ O_{\theta'} = 3 \\ 0 & \text{otherwise.} \end{cases} \tag{2.34}$$

Using (2.28), if $l(\theta) = 5$, we obtain the following

$$\langle s_{(n-1,n-1,1,1)} * s_{(n,n)}, s_\theta \rangle \quad = \quad \begin{cases} 1 & \theta_5 = 2,\ \theta \in P \\ O^{'}_{\theta} & \theta_5 = 1,\ E_\theta = 1 \\ E^{'}_{\theta} - ((\theta_4 = 1)) & \theta_5 = 1,\ O_\theta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

2.4.3 Case III: $l(\theta) \leq 4$

In this case, we have

$$\langle s_{(n,n)} * s_{(n,n)}, s_\theta \rangle = ((\theta \in P)) \ , \tag{2.35}$$

and

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = d_\theta - 1. \tag{2.36}$$

Consider $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(2)} s_\theta \rangle$ now. If $\theta \in P$, then the only terms in $s^{\perp}_{(2)} s_\theta$ which can appear in $s_{(n-1,n-1)} * s_{(n-1,n-1)}$ are of the form s_γ where γ has been obtained by subtracting 2 from a part of θ. Subtracting 1 from 2 different parts of θ gives a partition which lies in Q. These can't appear in $s_{(n-1,n-1)}s * s_{(n-1,n-1)}$. If, on the other hand, $\theta \in Q$, then to get a term of the form s_γ in $s^{\perp}_{(2)}s_\theta$ such that γ lies in P, the only possibility is to either subtract 1 each from two distinct non-zero even parts of θ, or subtract 1 each from two distinct odd parts of θ. These arguments imply

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s^{\perp}_{(2)}s_\theta\rangle = \begin{cases} d_{\theta,2} & \theta \in P \\ ((E'_\theta = 2)) + ((O'_\theta = 2)) & \theta \in Q. \end{cases} \tag{2.37}$$

Now for $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)}s_\theta\rangle$. If $\theta \in P$, then subtracting 1 each from any two parts in θ will give a partition in Q, if what is obtained after the subtraction is indeed a partition. If $\theta \in Q$, then subtracting 1 from one of the even parts and subtracting 1 from one of the odd parts gives a partition in Q. Thus, we conclude that

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s^{\perp}_{(1,1)}s_\theta\rangle = \begin{cases} \binom{d_\theta}{2} + R_\theta & \theta \in P \\ O'_\theta E'_\theta & \theta \in Q. \end{cases} \tag{2.38}$$

Now, using (2.28) we collect the different parts together in the case $l(\theta) \leq 4$ to obtain

$$\langle s_{(n-1,n-1,1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} d_{\theta,2} - d_\theta + \binom{d_\theta}{2} + R_\theta & \theta \in P \\ 1 - d_\theta + O'_\theta E'_\theta & \\ +((E'_\theta = 2)) + ((O'_\theta = 2)) & \theta \in Q. \end{cases}$$

2.4.4 Summary

On gleaning the relevant information from the three cases above, we obtain

$$\langle s_{(n-1,n-1,1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} ((\theta \in Q)) & l(\theta) = 6 \text{ and} \\ & \theta_5 = \theta_6 = 1 \\ 1 & l(\theta) = 5 \text{ and} \\ & \theta_5 = 2,\ \theta \in P \\ O'_\theta & l(\theta) = 5 \text{ and} \\ & E_\theta = 1,\ \theta_5 = 1 \\ E'_\theta - ((\theta_4 = 1)) & l(\theta) = 5 \text{ and} \\ & O_\theta = 1,\ \theta_5 = 1 \\ d_{\theta,2} - d_\theta & \\ + \binom{d_\theta}{2} + R_\theta & l(\theta) \leq 4,\ \theta \in P \\ 1 - d_\theta + O'_\theta E'_\theta & \\ +((E'_\theta = 2)) + ((O'_\theta = 2)) & l(\theta) \leq 4,\ \theta \in Q \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider an example now.

Example 2.4.1. *Consider the Kronecker product* $s_{(7,7,1,1)} * s_{(8,8)}$, *and three parti-*

tions $\alpha = (5,5,3,1,1,1)$, $\beta = (6,4,3,2,1)$ *and* $\gamma = (7,5,2,2)$.

Notice that even though $\alpha_5 = \alpha_6 = 1$, *the partition* $\alpha' = (5,5,3,1)$ *does not belong to Q. Thus* $\langle s_{(7,7,1,1)} * s_{(8,8)}, s_{(5,5,3,1,1,1)} \rangle = 0$.

Consider the case of β *now. We have* $l(\beta) = 5$ *and* $\beta_5 = 1$. *Since* $\beta' = (6,4,3,2)$, *we have* $O_\beta = 1$. *The characterization above gives us* $\langle s_{(7,7,1,1)} * s_{(8,8)}, s_{(6,4,3,2,1)} \rangle = E'_{(6,4,3,2,1)} - ((\beta_4 = 1))$. *Since* β' *has exactly 3 distinct even non-zero parts and* $\beta_4 \neq 1$, *we obtain* $\langle s_{(7,7,1,1)} * s_{(8,8)}, s_{(6,4,3,2,1)} \rangle = 3$.

As far as γ *is concerned, we have* $l(\gamma) = 4$ *and* $\gamma \in Q$. *We have* $d_\gamma = 3$, $O'_\gamma = 2$ *and* $E'_\gamma = 1$. *Thus* $\langle s_{(7,7,1,1)} * s_{(8,8)}, s_{(7,5,2,2)} \rangle = 1 - 3 + 2 + 1 = 1$.

2.5 $s_{(n,n,1)} * s_{(n,n,1)} \qquad (n \geq 2)$

In this section we will answer the question of giving a characterisation of the Kronecker product $s_{(n,n,1)} * s_{(n,n,1)}$. Before we begin our calculations, we need to introduce certain statistics on partitions, the purpose of which, will become clear soon enough. We need to figure out the relation between the number of distinct parts in a partition θ, and that in the partition θ^- obtained by removing an inner corner of θ.

2.5.1 Relating d_θ and d_{θ^-}

Fix an alphabet $X = \{0,1,2\}$. We will associate a string σ of length $l(\theta)+1$ to a partition θ. For $1 \leq i \leq l(\theta)$, define

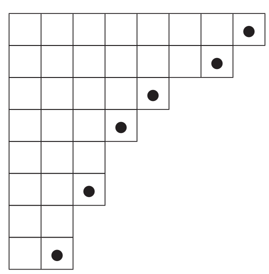
$$\sigma_i = \begin{cases} 0 & (\theta_i = \theta_{i+1}) \\ 1 & (\theta_i - \theta_{i+1} = 1) \\ 2 & (\theta_i - \theta_{i+1} \geq 2). \end{cases} \tag{2.39}$$

Here we are assuming that when $i = l(\theta)$, then $\theta_{i+1} = 0$. For the sake of convenience, define $\sigma_0 = \sigma_1$. Once σ has been found, define the following sets

$$\begin{aligned}
A_{\theta,1} &= \{i : 1 \leq i \leq l(\theta),\ \sigma_i = 1 \text{ and } \sigma_{i-1} = 0\} \\
A_{\theta,2} &= \{i : 1 \leq i \leq l(\theta),\ \sigma_i = 2 \text{ and } \sigma_{i-1} = 0\} \\
B_{\theta,1} &= \{i : 1 \leq i \leq l(\theta),\ \sigma_i = 1 \text{ and } \sigma_{i-1} \neq 0\} \\
B_{\theta,2} &= \{i : 1 \leq i \leq l(\theta),\ \sigma_i = 2 \text{ and } \sigma_{i-1} \neq 0\}
\end{aligned} \tag{2.40}$$

Before we describe the relation between d_θ and d_{θ^-}, we will consider an example to make the above mentioned notions clear.

Example 2.5.1. *Consider the partition $\theta = (8,7,5,4,3,3,2,2)$. The Ferrers diagram of θ is*



,

where the bulleted squares represent the inner corner of θ. An inner corner is a square whose removal leaves the Ferrers diagram of a partition. The string σ associated with θ is 112110102. Recall that the indexing of the string starts from zero. Observe that the bulleted squares belong to only those parts θ_i for which $\sigma_i = 1$ or 2. Note further that, on removing a square from θ_2 or θ_6, the *resulting partition has the same number of distinct parts as θ itself. On removing a square from θ_1, θ_3 or θ_4, the number of distinct parts in the resulting partition is one less that d_θ. On the other hand, on removing a square from θ_8 gives a partition with number of distinct parts being one more than d_θ. Note also that $\{2,6\} = A_{\theta,1} \cup B_{\theta,2}$, $\{1,3,4\} = B_{\theta,1}$ and $\{8\} = A_{\theta,2}$. These observations motivate what follows.*

As observed earlier, to obtain θ^- from θ, one can remove a corner only from those parts θ_i such that $\sigma_i = 1$ or 2. If θ^- is obtained by subtracting 1 from θ_i where $i \in A_{\theta,1}$ or $i \in B_{\theta,2}$, then $d_{\theta^-} = d_\theta$. For $i \in A_{\theta,2}$, subtracting 1 from θ_i results in d_{θ^-} being $d_\theta + 1$. Finally, for $i \in B_{\theta,1}$, subtracting 1 from θ_i results in d_{θ^-} being $d_\theta - 1$. We will use $a_{\theta,1}$, $a_{\theta,2}$, $b_{\theta,1}$ and $b_{\theta,2}$ for the cardinalities of the sets $A_{\theta,1}$, $A_{\theta,2}$, $B_{\theta,1}$ and $B_{\theta,2}$ respectively.

2.5.2 Computation

Note that the Pieri rule implies

$$s_{(1)}s_{(n,n)} = s_{(n,n,1)} + s_{(n+1,n)}. \tag{2.41}$$

This means

$$\begin{aligned} s_{(n,n,1)} * s_{(n,n,1)} &= \left(s_{(1)}s_{(n,n)} - s_{(n+1,n)}\right) * s_{(n,n,1)} \\ &= \left(s_{(1)}s_{(n,n)}\right) * s_{(n,n,1)} - s_{(n+1,n)} * s_{(n,n,1)}. \end{aligned} \tag{2.42}$$

As we have done before, we will use the formula of Littlewood (Theorem 1.6.7) which says

$$\begin{aligned}\left(s_{(1)}s_{(n,n)}\right)*s_{(n,n,1)} &= \sum_{\delta\vdash 1}\sum_{\eta\vdash 2n} c^{(n,n,1)}_{\eta,\delta}\left(s_\eta * s_{(n,n)}\right)\left(s_\delta * s_{(1)}\right)\\ &= \sum_{\eta\vdash 2n} c^{(n,n,1)}_{\eta,(1)}\left(s_\eta * s_{(n,n)}\right)\left(s_{(1)} * s_{(1)}\right). \qquad (2.43)\end{aligned}$$

Now we need to figure out which partitions $\eta \vdash 2n$ give a non-zero value for $c^{(n,n,1)}_{\eta,(1)}$. The Pieri rule yields that $c^{(n,n,1)}_{(\eta,(1)} = 0$ for all η except $\eta = (n,n)$ and $\eta = (n,n-1,1)$. It also tells us that $c^{(n,n,1)}_{(n,n),(1)} = c^{(n,n,1)}_{(n,n-1,1),(1)} = 1$. Using the fact that $s_{(1)} * s_{(1)} = s_{(1)}$, (2.43) becomes

$$\left(s_{(1)}s_{(n,n)}\right)*s_{(n,n,1)} = s_{(1)}\left(s_{(n,n)} * s_{(n,n)}\right) + s_{(1)}\left(s_{(n,n)} * s_{(n,n-1,1)}\right), \qquad (2.44)$$

and using this in (2.42) gives us

$$\begin{aligned}s_{(n,n,1)} * s_{(n,n,1)} &= s_{(1)}\left(s_{(n,n)} * s_{(n,n)} + s_{(n,n)} * s_{(n,n-1,1)}\right)\\ &\quad - s_{(n+1,n)} * s_{(n,n,1)}. \qquad (2.45)\end{aligned}$$

Since we are looking to compute the coefficient of s_θ in $s_{(n,n,1)} * s_{(n,n,1)}$ where $\theta \vdash 2n+1$, we must find out what $\langle s_{(n,n,1)} * s_{(n,n,1)}, s_\theta\rangle$ is. To this end, (2.45) gives

$$\begin{aligned}\langle s_{(n,n,1)} * s_{(n,n,1)}, s_\theta\rangle &= \langle s_{(1)}\left(s_{(n,n)} * s_{(n,n)} + s_{(n,n)} * s_{(n,n-1,1)}\right), s_\theta\rangle \\ &\quad -\langle s_{(n+1,n)} * s_{(n,n,1)}, s_\theta\rangle \\ &= \langle s_{(n,n)} * s_{(n,n)}, s_{(1)}^{\perp} s_\theta\rangle + \langle s_{(n,n)} * s_{(n,n-1,1)}, s_{(1)}^{\perp} s_\theta\rangle \\ &\quad -\langle s_{(n+1,n)} * s_{(n,n,1)}, s_\theta\rangle. \qquad (2.46)\end{aligned}$$

Notice that we do have characterizations for $s_{(n,n)} * s_{(n,n)}$, $s_{(n,n)} * s_{(n,n-1,1)}$, and $s_{(n+1,n)} * s_{(n,n,1)}$, terms appearing on the right hand side in (2.46) and hence we just have to bring the results together. It is clear via Theorem 1.6.10 that if s_θ has a non-zero coefficient in $s_{(n,n,1)} * s_{(n,n,1)}$, then $l(\theta) \leq 6$. We will carry out the computation case by case.

2.5.3 Case I: $l(\theta) = 6$

Notice that we already obtained a characterization for the Kronecker coefficients appearing in $s_{(n,n,1)} * s_{(n+1,n)}$ in Section 2.2. That tells us $\langle s_{(n+1,n)} * s_{(n,n,1)}, s_\theta\rangle = 0$. Now, using results in Section 2.1, we know that the Kronecker product $s_{(n,n)} * s_{(n,n-1,1)}$ is a sum of terms of the form s_γ and $l(\gamma) \leq 5$ for each such term. It is also known that if $l(\gamma) = 5$ then s_γ appears with coefficient 1 iff $\gamma_5 = 1$ otherwise the coefficient is 0. This implies that $\langle s_{(n,n)} * s_{(n,n-1,1)}, s_{(1)}^{\perp} s_\theta\rangle = 1$ iff $\theta_6 = \theta_5 = 1$, and 0 otherwise. Since $s_{(n,n)} * s_{(n,n)}$ is a sum of terms of the form s_γ where $l(\gamma) \leq 4$, it is clear that $\langle s_{(n,n)} * s_{(n,n)}, s_{(1)}^{\perp} s_\theta\rangle = 0$. This gives us the following

$$\langle s_{(n,n,1)} * s_{(n,n,1)}, s_\theta \rangle = \begin{cases} 1 & (\theta_6 = \theta_5 = 1) \\ 0 & \text{otherwise.} \end{cases}$$

2.5.4 Case II: $l(\theta) = 5$

In this case, if we look at $\langle s_{(n+1,n)} * s_{(n,n,1)}, s_\theta \rangle$, then we know that this is 1 if $\theta_5 = 1$ and 0 otherwise, because of the results in Section 2.2.

Now we will compute $\langle s_{(n,n-1,1)} * s_{(n,n)}, s^{\perp}_{(1)} s_\theta \rangle$. Consider the case where $\theta_5 \geq 3$. Then $s^{\perp}_{(1)} s_\theta$ is a sum of terms of the form s_γ where $\gamma_5 \geq 2$. We know that such terms do not appear in $s_{(n,n)} * s_{(n,n-1,1)}$. On considering $\theta_5 = 2$, we see that $s^{\perp}_{(1)} s_\theta$ has terms of the form s_γ with $\gamma_5 = 2$, in which case they do not appear in $s_{(n,n)} * s_{(n,n-1,1)}$, and a term with $\gamma_5 = 1$ which will appear in $s_{(n,n)} * s_{(n,n-1,1)}$ with coefficient 1, as we calculated earlier.

The one remaining sub-case for this case is $\theta_5 = 1$. We know that

$$s^{\perp}_{(1)} s_\theta = s_{\theta'} + \sum_{\substack{\gamma \vdash 2n, \gamma_5 = 1 \\ \langle s^{\perp}_{(1)} s_\theta, s_\gamma \rangle \neq 0}} s_\gamma \quad , \tag{2.47}$$

and hence

$$
\begin{aligned}
\langle s_{(n,n)} * s_{(n,n-1,1)}, s^{\perp}_{(1)} s_\theta \rangle &= \langle s_{(n,n)} * s_{(n,n-1,1)}, s_{\theta'} \rangle \\
&\quad + \langle s_{(n,n)} * s_{(n,n-1,1)}, \sum_{\substack{\gamma \vdash 2n, \gamma_5 = 1 \\ \langle s^{\perp}_{(1)} s_\theta, s_\gamma \rangle \neq 0}} s_\gamma \rangle.
\end{aligned}
\tag{2.48}
$$

Now we will look at individual terms of the right hand side of (2.48). We know that

$$
\langle s_{(n,n)} * s_{(n,n-1,1)}, s_{\theta'} \rangle = d_{\theta'} - 1. \tag{2.49}
$$

Notice that $d_{\theta'} - 1$ can be related to the number of distinct parts of θ itself in the following manner

$$
d_{\theta'} - 1 = \begin{cases} d_\theta - 2 & (\theta_4 \geq 2) \\ d_\theta - 1 & (\theta_4 = 1). \end{cases} \tag{2.50}
$$

Also, observe that every term s_γ other than $s_{\theta'}$ in $s^{\perp}_{(1)} s_\theta$ occurs with coefficient 1 and there are $d_\theta - 1$ such terms clearly. This gives us

$$
\langle s_{(n,n)} * s_{(n,n-1,1)}, \sum_{\substack{\gamma \vdash 2n, \gamma_5 = 1 \\ \langle s^{\perp}_{(1)} s_\theta, s_\gamma \rangle \neq 0}} s_\gamma \rangle = d_\theta - 1. \tag{2.51}
$$

Then (2.49), (2.50) and(2.51) together imply

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s^{\perp}_{(1)} s_\theta \rangle = \begin{cases} 2d_\theta - 3 & (\theta_5 = 1,\ \theta_4 \geq 2) \\ 2d_\theta - 2 & (\theta_5 = 1,\ \theta_4 = 1). \end{cases} \tag{2.52}$$

Now consider $\langle s_{(n,n)} * s_{(n,n)}, s^{\perp}_{(1)} s_\theta \rangle$. Since $s_{(n,n)} * s_{(n,n)}$ only consists of terms s_γ where $\gamma \in P$ and $l(\gamma) \leq 4$, we have that if $l(\theta) = 5$, then

$$\langle s_{(n,n)} * s_{(n,n)}, s^{\perp}_{(1)} s_\theta \rangle = \begin{cases} 1 & (\theta_5 = 1,\ \theta \in P) \\ 0 & \text{otherwise.} \end{cases} \tag{2.53}$$

Summarising the case $l(\theta) = 5$ we have

$$\langle s_{(n,n,1)} * s_{(n,n,1)}, s_\theta \rangle = \begin{cases} 2d_\theta - 3 + ((\theta \in P)) & (\theta_5 = \theta_4 = 1) \\ 2d_\theta - 4 + ((\theta \in P)) & (\theta_4 \geq 2, \theta_5 = 1) \\ 1 & (\theta_5 = 2) \\ 0 & \text{otherwise.} \end{cases}$$

2.5.5 Case III: $l(\theta) \leq 4$

Firstly, for this case we know from Section 2.2 that

$$\langle s_{(n+1,n)} * s_{(n,n,1)}, s_\theta \rangle = d_\theta - 1. \tag{2.54}$$

Given a partition $\gamma \vdash 2n$ and $l(\gamma) \leq 4$, from Section 2.1, we also have

$$\langle s_{(n,n)} * s_{(n,n-1,1)}, s_\gamma \rangle = d_\gamma - 1. \tag{2.55}$$

Since $s_{(1)}^\perp s_\theta = \sum_{\theta^- \prec \theta} s_{\theta^-}$, and we are computing $\langle s_{(n,n)} * s_{(n,n-1,1)}, s_{(1)}^\perp s_\theta \rangle$, central to our task is the relation between d_{θ^-} and d_θ when $\theta^- \prec \theta$. θ^- is obtained by removing a corner from θ, hence we need to figure out when and how does removing a corner change the number of distinct parts of a partition θ. Having accomplished this task in Section 2.5.1, the computation is now routine. We obtain

$$\begin{aligned}
\langle s_{(n,n)} * s_{(n,n-1,1)}, s_{(1)}^\perp s_\theta \rangle &= \sum_{\theta^- \prec \theta} (-1 + d_{\theta^-}) \\
&= -d_\theta + \sum_{\theta^- \prec \theta} d_{\theta^-} \\
&= -d_\theta + a_{\theta,2}(d_\theta + 1) + b_{\theta,1}(d_\theta - 1) \\
&\quad + a_{\theta,1} d_\theta + b_{\theta,2} d_\theta.
\end{aligned} \tag{2.56}$$

Notice that $a_{\theta,1} + a_{\theta,2} + b_{\theta,1} + b_{\theta,2} = d_\theta$. That reduces (2.56) to

$$\langle s_{(n,n)} * s_{(n,n-1,1)}, s_{(1)}^\perp s_\theta \rangle = -d_\theta + d_\theta^2 - b_{\theta,1} + a_{\theta,2}. \tag{2.57}$$

Next we consider the task of computing $\langle s_{(n,n)} * s_{(n,n)}, s_{(1)}^\perp s_\theta \rangle$. If $\theta \vdash 2n+1$ and $l(\theta) = 4$, then either θ has 3 parts odd and 1 part even, or it has 3 parts even and 1 odd. Since $s_{(n,n)} * s_{(n,n)}$ has terms of the form s_γ where $\gamma \in P$, it is easily seen that if $l(\theta) = 4$, then $\langle s_{(n,n)} * s_{(n,n)}, s_{(1)}^\perp s_\theta \rangle = 1$. If $l(\theta) = 3$, then θ has either 3 parts odd, or 2 parts even and 1 odd. In the former case, $s^{\perp}_{(1)}s_\theta$ will not have terms of the form s_γ with $\gamma \in P$ whereas in the latter, the only term giving a non-zero coefficient is the term s_γ with γ obtained by removing a corner from the part with odd length in θ. Arguments on very similar lines yield that for $l(\theta) \leq 2$, we have $\langle s_{(n,n)} * s_{(n,n)}, s^{\perp}_{(1)}s_\theta \rangle = 1$. To see this, assume $l(\theta) = 2$. Then θ has one part odd and the other part even. Subtracting one from the odd part gives a partition in P, while subtracting one from the even part gives a partition in Q. If $l(\theta) = 1$, then the lone part has to be odd, and subtracting one from it gives a partition which is in P. Thus, for $l(\theta) \leq 2$, we do get $\langle s_{(n,n)} * s_{(n,n)}, s^{\perp}_{(1)}s_\theta \rangle = 1$. Finally we have

$$\langle s_{(n,n)} * s_{(n,n)}, s^{\perp}_{(1)}s_\theta \rangle = \begin{cases} 1 & (l(\theta) = 4, 2 \text{ or } 1) \\ 1 & (l(\theta) = 3 \text{ and } \theta \text{ has exactly 1 odd part}) \\ 0 & \text{otherwise.} \end{cases} \tag{2.58}$$

Summarising the case $l(\theta) \leq 4$, we have

$$\langle s_{(n,n,1)} * s_{(n,n,1)}, s_\theta \rangle = \begin{cases} (d_\theta - 1)^2 + 1 - b_{\theta,1} + a_{\theta,2} & (l(\theta) = 4, 2 \text{ or } 1) \\ (d_\theta - 1)^2 + 1 - b_{\theta,1} + a_{\theta,2} & (l(\theta) = 3 \text{ and } \theta \\ & \text{has exactly 1 odd part}) \\ (d_\theta - 1)^2 - b_{\theta,1} + a_{\theta,2} & \text{otherwise.} \end{cases}$$

2.5.6 Summary

After simplifying some of the above relations, we have

$$\langle s_{(n,n,1)} * s_{(n,n,1)}, s_\theta \rangle = \begin{cases} 1 & l(\theta) = 6, \theta_6 = \theta_5 = 1 \\ 2d_\theta - 3 + ((\theta \in P)) & l(\theta) = 5, \theta_5 = \theta_4 = 1 \\ 2d_\theta - 4 + ((\theta \in P)) & l(\theta) = 5, \theta_4 \geq 2, \theta_5 = 1 \\ 1 & l(\theta) = 5, \theta_5 = 2 \\ (d_\theta - 1)^2 + 1 - b_{\theta,1} + a_{\theta,2} & l(\theta) = 4 \\ (d_\theta - 1)^2 + 1 - b_{\theta,1} + a_{\theta,2} & l(\theta) = 3 \text{ and } \theta \\ & \text{has exactly 1 odd part} \\ (d_\theta - 1)^2 - b_{\theta,1} + a_{\theta,2} & l(\theta) = 3 \text{ and } \theta \\ & \text{has all parts odd} \\ 2 - b_{\theta,1} + a_{\theta,2} & l(\theta) = 2 \\ 1 & l(\theta) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let us now consider an example.

Example 2.5.2. *Consider the computation of* $s_{(8,8,1)} * s_{(8,8,1)}$. *Let* $\alpha = (6,5,3,2,1)$, $\beta = (8,6,2,1)$ *and* $\gamma = (7,5,5)$ *be three partitions.*

Consider first $\langle s_{(8,8,1)} * s_{(8,8,1)}, s_\alpha \rangle$. *We can see that* $l(\alpha) = d_\alpha = 5$, $\alpha_5 = 1$ *and* $\alpha_4 \geq 2$. *Note also that* $\alpha' = (6,5,3,2)$ *does not belong to P. Thus* $\langle s_{(8,8,1)} * s_{(8,8,1)}, s_{(6,5,3,2,1)} \rangle = 2 \times 5 - 4 = 6$.

Next, consider $\langle s_{(8,8,1)} * s_{(8,8,1)}, s_\beta \rangle$. *We have* $l(\beta) = d_\beta = 4$. *The string* σ *associated with* $(8,6,2,1)$ *is* 22211. *This immediately yields* $a_{\beta,2} = 0$ *and* $b_{\beta,1} = 2$.

This implies $\langle s_{(8,8,1)} * s_{(8,8,1)}, s_{(8,6,2,1)} \rangle = (4-1)^2 + 1 + 0 - 2 = 8$.

Finally, consider $\langle s_{(8,8,1)} * s_{(8,8,1)}, s_\gamma \rangle$. *We can see that* $l(\gamma) = 3$, $d_\gamma = 2$ *and the string* σ *associated with* $\gamma = (7,5,5)$ *is* 2202. *Thus, we have* $a_{\gamma,2} = 1$ *and* $b_{\gamma,1} = 0$. *This gives* $\langle s_{(8,8,1)} * s_{(8,8,1)}, s_{(7,5,5)} \rangle = (2-1)^2 + 1 - 0 = 2$.

2.6 Combinatorial implications

We start by recalling relevant notation established prior to this section, and establishing some new notation to be used henceforth. Let

- $f_\lambda =$ number of standard Young tableau of shape given by a partition λ.
- $d_\lambda =$ number of distinct parts of a partition λ.
- $\tau_k(n) =$ the number of standard Young tableau of height $\leq k$ and size n and if the subscript k is omitted, that means we are just counting all standard Young tableau of size n.
- $\sigma_k(n) = \sum_{\substack{\lambda \vdash n \\ l(\lambda) \leq k}} d_\lambda f_\lambda$.
- $C_n =$ the n-th Catalan number and equal to $\dfrac{1}{n+1}\dbinom{2n}{n}$.
- $M_n =$ the n-th Motzkin number given by $\sum_{i \geq 0} \frac{1}{i+1}\binom{n}{2i}\binom{2i}{i}$.

The results obtained above about the Kronecker products $s_{(n,n-1,1)} * s_{(n,n)}$ and $s_{(n-1,n-1,1)} * s_{(n,n-1)}$ in Sections 2.5.1 and 2.5.2 imply the following:

$$\sum_{\substack{\lambda \vdash 2n \\ l(\lambda) \leq 4}} (d_\lambda - 1) s_\lambda + \sum_{\substack{\lambda \vdash 2n \\ l(\lambda) = 5 \\ \lambda_5 = 1}} s_\lambda = s_{(n,n-1,1)} * s_{(n,n)}, \tag{2.59}$$

and

$$\sum_{\substack{\lambda\vdash 2n-1\\ l(\lambda)\leq 4}}(d_\lambda-1)s_\lambda+\sum_{\substack{\lambda\vdash 2n-1\\ l(\lambda)=5\\ \lambda_5=1}}s_\lambda=s_{(n-1,n-1,1)}*s_{(n,n-1)}. \tag{2.60}$$

Looking at (2.60) and (2.59) in the language of characters of the symmetric group, i.e. looking at the inverse image of the identities above under the Frobenius map, we obtain

$$\sum_{\substack{\lambda\vdash 2n\\ l(\lambda)\leq 4}}(d_\lambda-1)\chi_\lambda+\sum_{\substack{\lambda\vdash 2n\\ l(\lambda)=5\\ \lambda_5=1}}\chi_\lambda=\chi_{(n,n-1,1)}\chi_{(n,n)}, \tag{2.61}$$

and

$$\sum_{\substack{\lambda\vdash 2n-1\\ l(\lambda)\leq 4}}(d_\lambda-1)\chi_\lambda+\sum_{\substack{\lambda\vdash 2n-1\\ l(\lambda)=5\\ \lambda_5=1}}\chi_\lambda=\chi_{(n-1,n-1,1)}\chi_{(n,n-1)}. \tag{2.62}$$

If one evaluates (2.61) and (2.62) at the identity element of S_{2n} and S_{2n-1} respectively, then one obtains the following identity

$$\sum_{\substack{\lambda\vdash 2n\\ l(\lambda)\leq 4}}(d_\lambda-1)f_\lambda+\sum_{\substack{\lambda\vdash 2n\\ l(\lambda)=5\\ \lambda_5=1}}f_\lambda=f_{(n,n-1,1)}f_{(n,n)}, \tag{2.63}$$

and

$$\sum_{\substack{\lambda\vdash 2n-1\\ l(\lambda)\leq 4}}(d_\lambda-1)f_\lambda+\sum_{\substack{\lambda\vdash 2n-1\\ l(\lambda)=5\\ \lambda_5=1}}f_\lambda=f_{(n-1,n-1,1)}f_{(n,n-1)}. \tag{2.64}$$

Now, our aim is two-fold. We will find suitable closed form expressions for $\sigma_4(n)$ and $\sum_{\substack{\lambda\vdash n\\ l(\lambda)=5\\ \lambda_5=1}} f_\lambda$. We will start by giving an expression for $\sigma_k(n)$.

Theorem 2.6.1. *Given positive integers m and k, we have*

$$\sigma_k(m) \quad = \quad \tau_k(m+1) - \tau_{k-1}(m). \tag{2.65}$$

Proof.

$$\tau_k(m+1) \quad = \quad \sum_{\substack{\lambda \vdash m+1 \\ l(\lambda) \le k}} f_\lambda. \tag{2.66}$$

Using [22, Lemma 2.8.2], which says $f_\lambda = \sum_{\mu \prec \lambda} f_\mu$, we obtain the following sequence of equalities

$$\begin{aligned}
\sum_{\substack{\lambda \vdash m+1 \\ l(\lambda) \le k}} f_\lambda &= \sum_{\substack{\lambda \vdash m+1 \\ l(\lambda) \le k}} \sum_{\mu \prec \lambda} f_\mu \\
&= \sum_{\substack{\mu \vdash m \\ l(\mu) \le k}} \sum_{\substack{\lambda \succ \mu \\ l(\lambda) \le k}} f_\mu \\
&= \sum_{\substack{\mu \vdash m \\ l(\mu) \le k-1}} (d_\mu + 1) f_\mu + \sum_{\substack{\mu \vdash m \\ l(\mu) = k}} d_\mu f_\mu \\
&= \sum_{\substack{\lambda \vdash m \\ l(\lambda) \le k}} d_\lambda f_\lambda + \tau_{k-1}(m) \\
&= \sigma_k(m) + \tau_{k-1}(m).
\end{aligned} \tag{2.67}$$

Here $\mu \prec \lambda$ means that the partition λ covers the partition μ in the Young's lattice.

□

This gives us the following corollary.

Corollary 2.6.2. *Given a positive integer n, we have*

$$\sigma_4(n) = C_{\lfloor \frac{n}{2} \rfloor + 1} C_{\lceil \frac{n}{2} \rceil + 1} - M_n. \tag{2.68}$$

Proof. Theorem 2.6.1 implies

$$\sigma_4(n) = \tau_4(n+1) - \tau_3(n). \tag{2.69}$$

Theorem 1.2.7 then yields the values for $\tau_4(n+1)$ and $\tau_3(n)$, allowing us to prove the corollary. □

Now we come to our next enumerative result which makes use of (2.63) and (2.64). It relates to a very specific case of counting standard Young tableau with a fixed height.

Theorem 2.6.3. *Given a positive integer $k \geq 3$, we have*

$$\sum_{\substack{\lambda \vdash k \\ l(\lambda)=5 \\ \lambda_5=1}} f_\lambda = \frac{\lfloor \frac{k+1}{2} \rfloor (\lceil \frac{k+1}{2} \rceil + 1)}{k+1} C_{\lfloor \frac{k+1}{2} \rfloor} C_{\lceil \frac{k+1}{2} \rceil} - C_{\lfloor \frac{k}{2} \rfloor + 1} C_{\lceil \frac{k}{2} \rceil + 1} + M_k. \tag{2.70}$$

Proof. We will treat the cases where k is odd and k is even separately. Firstly assume $k = 2n$ for some integer $n \geq 2$. Then (2.63) implies

$$\sum_{\substack{\lambda \vdash 2n \\ l(\lambda)=5 \\ \lambda_5=1}} f_\lambda = f_{(n,n-1,1)} f_{(n,n)} - \sum_{\substack{\lambda \vdash 2n \\ l(\lambda)\le 4}} (d_\lambda - 1) f_\lambda$$
$$= f_{(n,n-1,1)} f_{(n,n)} - \sigma_4(2n) + \tau_4(2n). \qquad (2.71)$$

By (1.9), we have $f_{(n,n)} = C_n$ and by Proposition 1.2.8,

$$f_{(n,n-1,1)} = \left(\frac{(n+1)(n-1)}{2n+1} \right) C_{n+1}. \qquad (2.72)$$

Using Corollary 2.6.2, we obtain

$$\sigma_4(2n) = C_{n+1}^2 - M_{2n}. \qquad (2.73)$$

Theorem 1.2.7 also implies

$$\tau_4(2n) = C_n C_{n+1}. \qquad (2.74)$$

Therefore

$$\sum_{\substack{\lambda \vdash 2n \\ l(\lambda)=5 \\ \lambda_5=1}} f_\lambda = \left(\frac{n^2-1}{2n+1} \right) C_n C_{n+1} - C_{n+1}^2 + M_{2n} + C_n C_{n+1}$$
$$= \left(\frac{n(n+2)}{2n+1} \right) C_n C_{n+1} - C_{n+1}^2 + M_{2n}. \qquad (2.75)$$

Now, assume $k = 2n-1$ for $n \geq 2$. Then (2.64) implies

$$\sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda)=5 \\ \lambda_5=1}} f_\lambda = f_{(n-1,n-1,1)} f_{(n,n-1)} - \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda)\leq 4}} (d_\lambda - 1) f_\lambda$$
$$= f_{(n-1,n-1,1)} f_{(n,n-1)} - \sigma_4(2n-1) + \tau_4(2n-1). \tag{2.76}$$

Now, by (1.10), we have $f_{(n,n-1)} = C_n$ and by Proposition 1.2.8, we have

$$f_{(n-1,n-1,1)} = \binom{n-1}{2} C_n. \tag{2.77}$$

Furthermore, by Corollary 2.6.2, one obtains

$$\sigma_4(2n-1) = C_n C_{n+1} - M_{2n-1}, \tag{2.78}$$

and by Theorem 1.2.7, we get

$$\tau_4(2n-1) = C_n^2. \tag{2.79}$$

Therefore

$$\sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda)=5 \\ \lambda_5=1}} f_\lambda = \binom{n-1}{2} C_n^2 - C_n C_{n+1} + C_n^2 + M_{2n-1}$$
$$= \binom{n+1}{2} C_n^2 - C_n C_{n+1} + M_{2n-1}. \tag{2.80}$$

One can check that the claim is just a unified way of rewriting the formulae obtained in the two cases, $k = 2n$ and $k = 2n - 1$. $\square$

Chapter 3

Conclusion

3.1 Further directions

It is clear from the techniques we have used here that if we know a combinatorial rule for computing the Kronecker product of Schur functions indexed by partitions of height at most k, for some positive integer k, then we can give a description of the Kronecker product of Schur functions indexed by partitions of height $k+1$, where the smallest part is 1 or 2. Our concern here was mainly with nearly rectangular partitions. One can go a step further and try to compute $s_{(n+k,n-k-1,1)} * s_{(n,n)}$ for $n \geq k+2$ and $k \geq 1$. On performing calculations akin to those described in the previous chapter, one obtains the following relation

$$\begin{aligned} \langle s_{(n+k,n-k-1,1)} * s_{(n,n)}, s_\theta \rangle &= \sum_{j=0}^{k} (-1)^{k+j} \langle s_{(n+j,n-j)} * s_{(n,n)}, s_{(1)}^{\perp} s_{(1)} s_\theta \rangle \\ &\quad - \langle s_{(n+k+1,n-k-1)} * s_{(n,n)}, s_\theta \rangle \\ &\quad - \langle s_{(n+k,n-k)} * s_{(n,n)}, s_\theta \rangle. \end{aligned} \tag{3.1}$$

Note that the case where $k = 0$ has already been dealt with in Section 2.1. Coming back to (3.1), there is a combinatorial rule for computing Kronecker products of the form $s_{(n+j,n-j)} * s_{(n,n)}$, as described in [7]. One could use the Pieri rule twice to describe the terms appearing in $s^{\perp}_{(1)} s_{(1)} s_{\theta}$. Thus, in theory, one could compute $\langle s_{(n+k,n-k-1,1)} * s_{(n,n)}, s_{\theta} \rangle$ given (3.1). But that is not satisfying since there is an alternating sum in (3.1), and we already know what we are computing is a positive integer.

Another area worth looking into is counting standard Young tableaux with added constraints as described below. Given $k, i, n \geq 0$, consider the set

$$S(k,i,n) = \{\lambda \vdash n : \lambda_{k+1} = i \text{ and } l(\lambda) \leq k+1\} \quad .$$

Let $\rho_{k,i}(n)$ be defined by

$$\rho_{k,i}(n) = \sum_{\lambda \in S(k,i,n)} f_{\lambda} \quad . \tag{3.2}$$

The first thing to note is that $\rho_{k,0}(n) = \tau_k(n)$, and what we have enumerated in Theorem 2.6.3 is $\rho_{4,1}(n)$. Furthermore, for $k \geq 1$, we have

$$\tau_k(n) = \sum_{i=0}^{n} \rho_{k-1,i}(n) \quad . \tag{3.3}$$

Thus, the numbers $\rho_{k,i}(n)$ provide a refinement of the sequence $\tau_k(n)$. To the best of our knowledge, the numbers $\rho_{k,i}(n)$ have not been studied, and the question of enumerating them could possibly yield new combinatorial identities.

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