Large N Gauge Theory and k-Strings

by

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Abstract

We considered the k-antisymmetric representation of U(N) gauge group on two dimensional lattice space and derived the free energy by saddle point approximation in large N limit. k is a large integer comparable with N. Besides Gross-Witten phase transition[1], which happens as the coupling constant changes, we found a new phase transition in the strong coupling system that happens as k changes. The free energy of the weak coupling system is a smooth function of k under continuous limit. We have carefully selected the right saddle point solution among other possible ones. The numerical results match our saddle point calculations.

Table of Contents

List of Figures

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Thank my family for their support.

To My Parents.

Chapter 1

Introduction

Physicists have been trying hard to understand confinement of quarks for about half a century. Although this problem has not been solved with satisfactory, many branches of study in physics have been created and flourished as a result. In 1970s, Kenneth Wilson initiated the method of quantizing the gauge field on space-time lattice in his efforts to demystify quark confinement [2][3]. On the lattice space the minimal interval of space-time is the lattice spacing and there is no need to renormalize the gauge theory. The expectation value of Wilson loop operator is the potential between fermions, and the string tension is the potential energy divided by the spatial length of the loop. Combining his former work on Wilson loop theory, Wilson made the theory of lattice QCD understandable and computable. Another influential method proposed by G. 'tHooft is the large N expansion[6]. Since perturbative expansion fails in the strong interacting systems, 'tHooft expanded the action of SU(N) gauge theory in power of 1/N, where N is large. This new point of view gave encouraging results to explain some phenomena beyond standard model[4]. Large N QCD became an important method in the study of strong interactions of quarks[5]. The U(N) lattice gauge theory is the combination of lattice quantization and large N gauge theory. The theory has been built up and some basic techniques have been developed[7]. In this paper we discussed k-strings, which is the k-representation of the U(N) gauge group on 2 dimensional lattice space. Using the saddle point approximation, we derived the free energy of k-strings. Some numerical results are presented to compare with the saddle point approximation.

Gross-Witten model is a U(N) gauge theory on two dimensional lattice space[1]. In Gross and Witten's paper, they showed that the Wilson loop operator of the fundamental representation can be reduced to the loop operator on one plequatte by the U(N) gauge symmetry. In the large N limit, they used the saddle point method to determine the distribution function of the eigenvalues of the U(N) gauge field, then they derived the string tension of the fundamental representation and found a phase transition as the coupling constant changes. We further studied the k-antisymmetric representation of U(N) gauge group, where k is also large and comparable with N. k is a new parameter in our work. We showed that the action and the Wilson loop of the k-antisymmetric representation can be also reduced to one plequatte on lattice space as the case of the fundamental representation. Using the generating function and the saddle point method We calculate the expectation value of the Wilson loop operator and the free energy Γ, which is a function of both the coupling constant λ and the integer k. In the weak coupling system ($\lambda \leq 2$), We find several branches of free energy. None of them is smooth and none could serve as the free energy in the whole range of $\frac{k}{N} \in [0,1]$. The free energy is not simlpy the minimal solution, because we need to choose a proper contour of the integration of the generating function in complex plane. Further work has been done to decide which one of the two is the real saddle point solution. Under strong coupling, there are two branches of free energy from the saddle point equation. We also need to make the choice of which one is the proper saddle solution in different ranges of $\frac{k}{N} \in [0,1]$. A phase transition appears at $\frac{k}{N} = \frac{1}{2}$. This phase transition is easily determined to be of first order, but it changes to be third order at the boundary between weak and strong coupling, where the coupling constant $\lambda = 2$.

Chapter 2

The U(N) Gauge Theory on The Two Dimensional Lattice Space

In this chapter we fisrt review the U(N) gauge theory on the 2-dimensional lattice space and present some results of the fundamental representation done by D. Gross and E. Witten. They showed that by proper gauge choosing, the action and the Wilson loop operator can be reduced into simple forms, which only involve the integarl over one plaquette in the lattice space. Then we further the study on the k-antisymmetric representation. We are going to derive the Wilson loop operator of k-representation and reduce it into a simple form as Gross and Witten did for the fundamental representaion.

2.1 Gross-Witten Model

U(N) Gauge thoery on two dimensional lattice is somehow trivial, because we could make the transverse dimension trivial by gauge transformation. This leads to the absence of physics gluons and the result that all physical quantities only depend on the integration over the gauge field on a single plaquette.

The dynamical variables are $N \times N$ unitary matrices $U_{\vec{n},\vec{i}}$, which are functions of $\vec{n}$ and $\vec{i}$. $\{\vec{i}, \vec{j}\}$ is an orthonormal basis of the two dimensional lattice space, and $\vec{n} = n_1\vec{i} + n_2\vec{j}$ is a lattice vector. $U_{\vec{n},\vec{i}}$ transprots a matter field from site $\vec{n}$ to $\vec{n}+\vec{i}$ along $\vec{i}$ under fundamental representation of U(N). In this paper, we have nothing to do with the matter fields, the fermions. Since the action of the gauge fields can be reduced into one plaquette, the gluons are not physical particles due to the lack of time dependency. But by studying the non-physical gluons - the U(N) gauge fields we can derive the interaction between fermions and further derive the string tension.

From Wilson loop theory, we have

$$U^{\dagger}_{\vec{n},\vec{i}} = U^{-1}_{\vec{n},\vec{i}} = U_{\vec{n}+\vec{i},-\vec{i}}. \tag{2.1}$$

The Wilson action is defined as

$$S(U) = \sum_{\vec{n}} \frac{1}{g^2} \mathrm{Tr} \left(\prod_{plaquette} U + \text{H.c.} \right), \tag{2.2}$$

summing all plaquettes on the lattice space, and

$$\prod_{plaquette} U = U_{\vec{n},\vec{i}} U_{\vec{n}+\vec{i},\vec{j}} U_{\vec{n}+\vec{i}+\vec{j},-\vec{i}} U_{\vec{n}+\vec{j},-\vec{j}}, \tag{2.3}$$

is a loop orperator over one plaquette. The expectation values of physical observables are

$$< O(U) >= \frac{1}{Z} \int [DU] \exp\{S(U)\} O(U), \tag{2.4}$$

where Z is the vacuum-to-vacuum ampliude

$$Z = \int [DU] \exp\{S(U)\}, \tag{2.5}$$

and DU is the multiple integration over U(N) group on each lattice with each direction:

$$DU = \prod_{\vec{n},\,\vec{k}} dU_{\vec{n},\vec{k}}. \tag{2.6}$$

It also has the property:

$$DU = D(UV) = D(VU), \tag{2.7}$$

and

$$\int dU_{\vec{n},\vec{i}} = \int dU_{\vec{n},\vec{j}} = 1. \tag{2.8}$$

The expectation value of the Wilson loop operator is

$$W_L(g^2N) = \frac{1}{ZN}\int [DU]\exp\{S(U)\}\mathrm{Tr}\left[\prod_L U\right], \tag{2.9}$$

where L is a closed loop on the lattice space. The calculation of Z and W_L can be greatly simplified considering the gauge transformation:

$$U_{\vec{n},\vec{k}} \to V_{\vec{n}} U_{\vec{n},\vec{k}} V^{\dagger}_{\vec{n}+\vec{k}}\ , \quad \vec{k} = \vec{i}\ or\ \vec{j}. \tag{2.10}$$

For arbitrary unitary matrices $V_{\vec{n}}$, the action $S(U)$ is invariant under this transformation. By properly choosing $V_{\vec{n}}$, we can make the paralell transportation alone transverse dirction trivial

$$U'_{\vec{n},\vec{i}} = I, \tag{2.11}$$

for all $\vec{n}$. To give such gauge transformation explicitly, we need to find a class of unitary matrices $\{V_{\vec{n}}\}$ to satisfy

$$U'_{\vec{n},\vec{i}} = V_{\vec{n}} U_{\vec{n},\vec{i}} V^{\dagger}_{\vec{n}+\vec{i}} = I, \tag{2.12}$$

for any lattice vector $\vec{n}$. To do this we first choose an arbitrary unitary matrix $V_{\vec{0}}$, and from Eqn. (2.12) we have

$$U'_{\vec{0},\vec{i}} = V_{\vec{0}} U_{\vec{0},\vec{i}} V^{\dagger}_{\vec{i}} = I. \tag{2.13}$$

Then we get the required unitary matrix on the site of $\vec{i}$:

$$V_{\vec{i}} = V_{\vec{0}} U_{\vec{0},\vec{i}}\ . \tag{2.14}$$

Further, at the site $\vec{i}$ we have

$$U'_{\vec{i},\vec{i}} = V_i U_{\vec{i},\vec{i}} V^{\dagger}_{2\vec{i}} = I, \tag{2.15}$$

and therefore

$$V_{2\vec{i}} = V_{\vec{i}} U_{\vec{i},\vec{i}} \ . \tag{2.16}$$

Following this procedure, we can have

$$V_{m\vec{i}} = V_{(m-1)\vec{i}} U_{(m-1)\vec{i},\vec{i}} \tag{2.17}$$

$$= V_{\vec{0}} \prod_{l=0}^{l=m-1} U_{l\vec{i},\vec{i}}, \tag{2.18}$$

for arbitrary integer m. Along $\vec{j}$ direction, we can't put new restriction at this moment, because it may possibly bring in contradition to Eqn.(2.12). Therefore we leave $V_{n\vec{j}}$, $n \in \mathbb{Z}$ to be arbitrary U(N) matrices. Based on $\{V_{m\vec{i}}, m \in \mathbb{Z}\}$ and $\{V_{n\vec{j}}, n \in \mathbb{Z}\}$, we can construct the whole class of $\{V_{\vec{n}}\}$. From Eqn. (2.12) we have

$$V_{n\vec{j}} U_{n\vec{j},\vec{i}} V^{\dagger}_{n\vec{j}+\vec{i}} = I, \tag{2.19}$$

so

$$V_{n\vec{j}+\vec{i}} = V_{n\vec{j}} U_{n\vec{j},\vec{i}}. \tag{2.20}$$

Similarly,

$$V_{n\vec{j}+2\vec{i}} = V_{n\vec{j}+\vec{i}} U_{n\vec{j}+\vec{i},\vec{i}}. \tag{2.21}$$

Continuing this procedure, we have

$$\begin{aligned} V_{n\vec{j}+m\vec{i}} =& V_{n\vec{j}+(m-1)\vec{i}} U_{n\vec{j}+(m-1)\vec{i}} \\ =& V_{n\vec{j}} \left(\prod_{l=0}^{m-1} U_{n\vec{j}+l\vec{i},\vec{i}} \right). \end{aligned} \tag{2.22}$$

Thererfore, We can determine $V_{\vec{n}}$ on each lattice site $\vec{n}$ leaving the arbitrariness of $V_{m\vec{j}}$, which is the remaining gauge freedom.

Under this gauge the action is

$$S(U) = \frac{1}{g^2}\sum_{\vec{n}} \mathrm{Tr}(U_{\vec{n},\vec{j}}U^{\dagger}_{\vec{n}+\vec{i},\vec{j}} + \mathrm{H.c.}). \tag{2.23}$$

Originally, the action contains products of gauge fields $U_{\vec{n},\vec{k}}$ on four neighboring lattice sites: $\vec{n}, \vec{n}+\vec{i}, \vec{n}+\vec{i}+\vec{j}$ and $\vec{n}+\vec{j}$ and of two different transporting directions: $\vec{i}$ and $\vec{j}$. After the gauge transformation, it contains products of gauge fields only on two neighboring lattices: $\vec{n}$ and $\vec{n}+\vec{j}$ and just one direction: $\vec{j}$. However, since the $\int[DU]$ is integrating ove all lattice sites, we hope to completely disentangle the products of gauge fields on different lattice sites. This can be done by changing variables:

$$U_{\vec{n}+\vec{i},\vec{j}} = W_{\vec{n}}U_{\vec{n},\vec{j}}. \tag{2.24}$$

Then the action is

$$Z = \int \prod_{\vec{n}} [dW_{\vec{n}}] \exp\left[\sum_{n} \frac{1}{g^2}\mathrm{Tr}(W_n + W^{\dagger}_{\vec{n}})\right]. \tag{2.25}$$

We notice that the gauge fields $W_{\vec{n}}$ in $S(W)$ are disentangled. So $Z = z^{V/a^2}$, where the V is the area of the two dimensional lattice, and a is the spacing. So V/a^2 is the number of plaquttes encircled in the loop, and

$$z = \int dW \exp\left[\frac{1}{g^2}\mathrm{Tr}(W + W^{\dagger})\right], \tag{2.26}$$

which is the action on one plaquette and is independent of the lattice vector $\vec{n}$.

We further make gauge choice $U_{n\vec{j},\vec{j}} = I$. This can also be done explicitly. We kept the arbitrariness of $\{V_{m\vec{j}}\ ,\ m \in \mathbb{Z}\}$ above, but now we need make further restriction to the set of $\{V_{m\vec{j}}\ ,\ m \in \mathbb{Z}\}$ to realize the required gauge

$$U'_{n\vec{j},\vec{j}} = I. \tag{2.27}$$

In general, we have

$$U'_{\vec{n},\vec{j}} = V_{\vec{n}}U_{\vec{n},\vec{j}}V^{\dagger}_{\vec{n}+\vec{j}}\ . \tag{2.28}$$

Still starting from the origin, we have

$$U'_{\vec{0},\vec{j}} = V_{\vec{0}} U_{0,\vec{j}} V^{\dagger}_{\vec{j}} = I. \tag{2.29}$$

So $V_{\vec{j}}$ is no longer arbitrary and

$$V_{\vec{j}} = V_{\vec{0}} U_{0,\vec{j}}. \tag{2.30}$$

At the site $\vec{j}$, we have

$$U'_{\vec{j},\vec{j}} = V_{\vec{j}} U_{\vec{j},\vec{j}} V^{\dagger}_{2\vec{j}} = I, \tag{2.31}$$

and so

$$\begin{aligned} V_{2\vec{j}} &= V_{\vec{j}} U_{\vec{j},\vec{j}} \\ &= V_{\vec{0}} U_{\vec{0},\vec{j}} U_{\vec{j},\vec{j}}. \end{aligned} \tag{2.32}$$

Iterating this procedure, we have $m \in \mathbb{Z}$

$$V_{m\vec{j}} = V_{\vec{0}} \left(\prod_{l=0}^{m-1} U_{l\vec{j},\vec{j}} \right), \tag{2.33}$$

and the required gauge is realized.

Under this gauge, Wilson loop operator turns to be

$$W_L(g^2, N) = \frac{1}{N} < \mathrm{Tr} \prod_{k=0}^{R-1} U_{T\vec{i}+k\vec{j},\vec{j}} >= \frac{1}{N} < \mathrm{Tr} \prod_{k=0}^{R-1} \prod_{l=T-1}^{0} W_{l\vec{i}+k\vec{j}} >, \tag{2.34}$$

where $a^2 T \times R$ is the area inclosed by the loop L. As we have disentangle the integration of the action into to each lattice point, let's first look into the integration on a specific lattice site, say $\int dW_{\vec{n}}$. When we look into the gauge field on site $\vec{n}$, the gauge fields on other sites $W_{\vec{m}}$ do not have any interference. We write

$$\prod_{k=0}^{R-1} \prod_{l=T-1}^{0} W_{l\vec{i}+k\vec{j}} = A W_{\vec{n}} B, \tag{2.35}$$

where A and B stand for the remaining products of the gauge fields on the sites other than $\vec{n}$. For arbitrary unitary matrix V, $\mathrm{Tr}(W_{\vec{n}} + W^{\dagger}_{\vec{n}})$ is invariant under the transformation $W_{\vec{n}} \to VW_{\vec{n}}V^\dagger$. Also $\int dW = \int d(VWV^\dagger)$ from Eqn.(2.7). We have

$$\begin{aligned}\frac{1}{N}\int dW_{\vec{n}} \exp\left[\frac{1}{g^2}\mathrm{Tr}(W_{\vec{n}}+W_{\vec{n}}^\dagger)\right]\mathrm{Tr}(AW_{\vec{n}}B)\\ =\frac{1}{N}\int dW_{\vec{n}} \exp\left[\frac{1}{g^2}\mathrm{Tr}(W_{\vec{n}}+W_{\vec{n}}^\dagger)\right]\mathrm{Tr}(AVW_{\vec{n}}V^\dagger B) \qquad (2.36)\\ =\frac{1}{N}\int dV\int dW_{\vec{n}} \exp\left[\frac{1}{g^2}\mathrm{Tr}(W_{\vec{n}}+W_{\vec{n}}^\dagger)\right]\mathrm{Tr}(AVW_{\vec{n}}V^\dagger B),\end{aligned}$$

where we insert the integration $\int dV$ over the U(N) group. We can interchange the order of integral $\int dV$ and $\int dW_{\vec{n}}$. Further, we have the fact

$$\int dV V_{ij}V_{kl}^\dagger = \frac{1}{N}\delta_{il}\delta_{jk}. \qquad (2.37)$$

Then

$$\begin{aligned}\int dV\,\mathrm{Tr}(AVWV^\dagger B) &= \int dV A_{mi}V_{ij}W_{jk}V_{kl}^\dagger B_{lm} \qquad (2.38)\\ &= \frac{1}{N}A_{ml}B_{lm}W_{jj} = \frac{1}{N}\mathrm{Tr}(AB)\mathrm{Tr}W.\end{aligned}$$

By iterating this procedure, we separate out each $W_{\vec{n}}$, and have

$$W_L(g^2,N) = (w(g^2,N))^{RT}, \qquad (2.39)$$

where

$$w(g^2,N) = \frac{1}{z}\int dW_{\vec{n}}\frac{1}{N}\mathrm{Tr}W_{\vec{n}}\exp\left[\frac{1}{g^2}\mathrm{Tr}(W_{\vec{n}}+W_{\vec{n}}^\dagger)\right]. \qquad (2.40)$$

Now, we reduce the Wilson loop operator of fundamental representaion into one plaquette as well as the action.

2.2 k-Antisymmetric Representation

We are going to reduce the Wilson loop operator ot k-representation in a similar way. Let's begin with 2-antisymmetrical representation first. The fundamental gauge field is still $W_{\vec{n}} \in U(N)$, while the representation of the $U(N)$ gauge group is:

$$W^{(2)}_{ijkl} = \frac{1}{2!} W^{[j}_i W^{l]}_k = \frac{1}{2!}(W^j_i W^l_k - W^l_i W^j_k). \tag{2.41}$$

The 2-antisymmetric representation of U(N) group is $\frac{N(N-1)}{2}$ dimensional. The Wilson loop operator of this representation is:

$$W^{(2)}_L(g^2, N) = \frac{1}{N} \left\langle \mathrm{Tr} \prod_{k=0}^{R-1} \prod_{l=T-1}^{0} W^{(2)}_{l\vec{i}+k\vec{j}} \right\rangle. \tag{2.42}$$

To make the equations look simple, we label the Ws by integers, as W_α. And now $W^{(2)}_L = \frac{1}{N} \left\langle \mathrm{Tr} \prod_{\alpha=1}^{R\times T} W^{(2)}_\alpha \right\rangle$.

$W^{(2)}$ is a map from $\mathbb{R}^n \wedge \mathbb{R}^n$ to $\mathbb{R}^n \wedge \mathbb{R}^n$, which is a subspace of $\mathbb{R}^n \otimes \mathbb{R}^n$. To be explicit:

$$W^{(2)}(\vec{v}, \vec{w}) = (W\vec{v}, W\vec{w}), \tag{2.43}$$

where $(\vec{v}, \vec{w}) \in \mathbb{R}^n \wedge \mathbb{R}^n$. To write out the components:

$$(v_i, w_j) \to (W_{il} v_l, W_{jk} w_k). \tag{2.44}$$

In the Wislon loop operator, we have the product of a series of $W^{(2)}s$. We want to separate the trace of the product of the seroes into a product of traces of each one $W^{(2)}$ as we did above for the fundamental representation Let's first consider a two successive maps: $W^{(2)}$ and $Y^{(2)}$

$$\begin{aligned} Y^{(2)} W^{(2)}(\vec{v}, \vec{w}) &= Y^{(2)}(W\vec{v}, W\vec{w}) \\ &= (YW\vec{v}, YW\vec{w}) \\ &= (YW)^{(2)}(\vec{v}, \vec{w}), \end{aligned} \tag{2.45}$$

for any $\vec{v}, \vec{w} \in \mathbb{R}^n$. Therefore, we have

$$Y^{(2)} W^{(2)} = (YW)^{(2)}. \tag{2.46}$$

By induction, we have

$$\prod_{\alpha=1}^{R\times T} W^{(2)}_\alpha = \left(\prod_{\alpha=1}^{R\times T} W_\alpha \right)^{(2)}. \tag{2.47}$$

Since the gauge transformation $W_1 \to VW_1V^\dagger$, $V \in U(N)$ will not change the Wilson loop operator, we have:

$$W_L^{(2)}(g^2, N) = \frac{1}{N}\left\langle \mathrm{Tr}(VW_1V^\dagger B)^{(2)} \right\rangle \tag{2.48}$$

where B stands for the product of the other $W_\alpha s$. To calculate the trace, let insert the integration $\int dV$ as before,

$$\left\langle \mathrm{Tr}(VW_1V^\dagger B)^{(2)} \right\rangle = \int dV \left\langle \mathrm{Tr}\left[V^{(2)}W_1^{(2)}(V^\dagger)^{(2)}B^{(2)}\right] \right\rangle, \tag{2.49}$$

where we have also used Eqn.(2.47). We can put the integarl of U(N) group $\int dV$ into the expectation $< ... >$. Eqn. (2.37) is based on Peter-Weyl theorem, which also applies to U(N) group, and we have

$$\int dV V_{ij}^R (V^{R'})^\dagger_{kl} = \frac{1}{\dim R}\delta_{RR'}\delta_{il}\delta_{jk}, \tag{2.50}$$

where R stands for certain representation of the group and $V_{ij}^{(R)}$ is a matix element of R-representation. Therefore,

$$\int dV \mathrm{Tr}\left[V^{(2)}W_1^{(2)}(V^\dagger)^{(2)}B^{(2)}\right] = \frac{1}{\dim(\mathbb{R}^n \wedge \mathbb{R}^n)}\mathrm{Tr}W_1^{(2)}\mathrm{Tr}B^{(2)} \tag{2.51}$$

Then we have:

$$\left\langle \mathrm{Tr}\left[W_1^{(2)}B^{(2)}\right] \right\rangle = \left\langle \frac{1}{N(N-1)/2}\mathrm{Tr}W_1^{(2)}\mathrm{Tr}B^{(2)} \right\rangle. \tag{2.52}$$

By repeating this procedure, we can separate the trace of the product of $W^{(2)}s$ into product of traces:

$$\left\langle \mathrm{Tr}\prod_{\alpha=1}^{RT} W_\alpha^{(2)} \right\rangle = \left\langle \prod_{\alpha=1}^{RT} \mathrm{Tr}W_\alpha^{(2)} \right\rangle \cdot \left(\frac{1}{N(N-1)/2}\right)^{RT} \tag{2.53}$$

Therefore

$$W_L^{(2)}(g^2, N) = \left[w_L^{(2)}(g^2, N)\right]^{RT} \tag{2.54}$$

where

$$w_L^{(2)}(g^2, N) = \frac{1}{z}\int dW \frac{1}{N(N-1)/2}\mathrm{Tr}W^{(2)} \exp\left[\frac{1}{g^2}\mathrm{Tr}(W + W^\dagger)\right]. \tag{2.55}$$

Next we want to derive that for k-antisymmetric representation the expectation value of the Wilson loop operator can be reduced into a single plaquette as well. The k-antisymmetric representaion has its components:

$$W^{[j_1 j_2 ... j_k]}_{i_1 i_2 ... i_k} = W^{[j_1}_{i_1} W^{j_2}_{i_2} ... W^{j_k]}_{i_k}, \tag{2.56}$$

where $W \in U(N)$. First, it's easy to verify the formular:

$$\frac{1}{k!} Y^{[i_1 i_2 ... i_k]}_{r_1 r_2 ... r_k} \frac{1}{k!} W^{[j_1 j_2 ... j_k]}_{i_1 i_2 ... i_k} = \frac{1}{k!} (YW)^{[j_1 j_2 ... j_k]}_{r_1 r_2 ... r_k} . \tag{2.57}$$

The formular above means

$$Y^{(k)} W^{(k)} = (YW)^{(k)}. \tag{2.58}$$

By induction, we have

$$\prod_\alpha W^{(k)}_\alpha = \left(\prod_\alpha W_\alpha \right)^{(k)} . \tag{2.59}$$

So,

$$\begin{aligned} W^{(k)}_L(g^2, N) &= \left\langle \mathrm{Tr} \prod_{\alpha=1}^{RT} W^{(k)}_\alpha \right\rangle \\ &= \left\langle \mathrm{Tr} \left(\prod_{\alpha=1}^{RT} W_\alpha \right)^{(k)} \right\rangle \\ &= \left\langle \mathrm{Tr} (W_1 B)^{(k)} \right\rangle , \end{aligned} \tag{2.60}$$

where B is the product of the other $W_\alpha s$. As before, from the invariant gauge transformation $W_1 \to V W_1 V^\dagger$ we have

$$\begin{aligned} \left\langle \mathrm{Tr}(W_1 B)^{(k)} \right\rangle &= \left\langle \mathrm{Tr}(V W_1 V^\dagger B)^{(k)} \right\rangle \\ &= \left\langle \mathrm{Tr} \left(V^{(k)} W^{(k)}_1 (V^\dagger)^{(k)} B^{(k)} \right) \right\rangle . \end{aligned} \tag{2.61}$$

We use the similar trick: inserting the integration $\int dV$ over the U(N) group,

$$\int dV \mathrm{Tr}\left(V^{(k)}W_1^{(k)}(V^\dagger)^{(k)}B^{(k)}\right) = \frac{1}{\dim(\mathbb{R}^n \wedge \mathbb{R}^n \wedge ... \wedge \mathbb{R}^n)}\mathrm{Tr}W_1^{(k)}\mathrm{Tr}B^{(k)}, \tag{2.62}$$

where we also have

$$\dim(\underbrace{\mathbb{R}^n \wedge \mathbb{R}^n \wedge ... \wedge \mathbb{R}^n}_{\textbf{k times}}) = \frac{N!}{k!(N-k)!}. \tag{2.63}$$

Continue this procedure, we have

$$W_L^{(k)}(g^2, N) = \left[w_L^{(k)}(g^2, N)\right]^{RT}, \tag{2.64}$$

where

$$w_L^{(k)}(g^2, N) = \frac{1}{z}\int dW \frac{1}{N!/k!/(N-k)!}\mathrm{Tr}W^{(k)} \exp\left\{\frac{1}{g^2}\mathrm{Tr}(W + W^\dagger)\right\}. \tag{2.65}$$

Therefore, for any $k \in \mathbb{Z}$, the Wilson loop operator of k-antisymmetric representation is reduced in the same way as in the fundamental representation.

Chapter 3

The Distribution of The Eigenvalues of The U(N) Gauge Group

In this chapter we present the distribution of eigenvalues of U(N) gauge field derived by Gross and Witten[1], and the results following the distribution function. We will also present the Gross-Witten phase transition in fundamental representaion.

Following from the arguement in the last chapter, to determine the free energy of the k-antisymmetric representation

$$\Gamma_{A_k}(g^2, N) = -\frac{1}{N} \ln W_L^{(k)}(g^2, N), \tag{3.1}$$

we only need to consider the action and the Wilson loop operator of the gauge field on one plaquette. The integration $\int dW$ is over the U(N) group. Since the integrands only depend on the eigenvalues of U(N), we can reduce $\int W$ into the integral over the eigenvalues of the U(N) group, and diagnolize W by gauge tranformation: $VWV^\dagger = \mathrm{diag}\{e^{i\phi_1}, e^{i\phi_2}, ..., e^{i\phi_N}\}, V \in U(N)$. As U(N) is a compact group, its Haar measure can be reduced to a measure on a subspace of $\mathbb{R}^n$ multiplied by a modular function. We have[8],

$$\int dW = \int_0^{2\pi} \prod_{i=1}^{N} d\phi_i \Delta^2(\phi_i), \tag{3.2}$$

where

$$\Delta(\phi) = \prod_{i<j}(e^{i\phi_j} - e^{i\phi_i}) \tag{3.3}$$

is the Vandermonde deterninant. Then we have the action

$$z(g^2, N) = \mathrm{c} \int_0^{2\pi} \prod_{i=1}^{N} d\phi_i \prod_{i<j} \sin^2 \left| \frac{\phi_i - \phi_j}{2} \right| \exp\left[\frac{2}{g^2} \sum_{i=1}^{N} \cos(\phi_i) \right], \tag{3.4}$$

where the constant c is to normalize such that $z(\infty, N) = 1$. We are going to use the steepest-descent method to evaluate the action, so we write the integrand in the form of $e^{f(\phi)}$:

$$z(g^2, N) = \mathrm{c} \int_0^{2\pi} \prod_{i=1}^{N} d\phi_i \mathrm{exp} \left\{ \frac{2}{g^2} \sum_{i=1}^{N} \cos(\phi_i) + \sum_{i \neq j} \ln \left| \sin \frac{\phi_i - \phi_j}{2} \right| \right\}. \tag{3.5}$$

We use the steepest-descent method, and the approximated energy is

$$\begin{aligned} E(\lambda) &= \lim_{N\to\infty} \frac{\ln z(g^2, N)}{N^2} \\ &= \lim_{N\to\infty} \frac{1}{N^2} \left\{ \frac{2}{g^2} \sum_{i=1}^{N} \cos(\phi_i) + \sum_{i \neq j} \ln \left| \sin \frac{\phi_i - \phi_j}{2} \right| \right\} + const. \end{aligned} \tag{3.6}$$

The eigrnvalues ϕ_i can be approximately determined by the stationarity condition:

$$\partial_{\phi_i} \left\{ \frac{2}{g^2} \sum_{i=1}^{N} \cos(\phi_i) + \sum_{i \neq j} \ln \left| \sin \frac{\phi_i - \phi_j}{2} \right| \right\} = 0\,, \tag{3.7}$$

which is simplified as

$$\frac{2}{\lambda} \mathrm{sin}\phi_i = \frac{1}{N} \sum_{i \neq j} \cot \left| \frac{\phi_i - \phi_j}{2} \right| \,, \tag{3.8}$$

where $\lambda = g^2 N$. As N goes to infinity, we treat ϕ_i as a value assumed by a continuous function ϕ at $\frac{i}{N}$:

$$\phi_i = \phi(i/N). \tag{3.9}$$

Then the stationary condition becomes:

$$\frac{2}{\lambda}\sin\phi(x) = P\int_0^1 dy\cot\frac{\phi(x)-\phi(y)}{2}, \tag{3.10}$$

where P denotes the principal value of the integration. To find the function $\phi(x)$, we need to determine the density of eignevalues $\rho(\phi) = \frac{dx}{d\phi}$, which satisfies $\int_{-\phi_c}^{\phi_c}\rho(\phi)d\phi = \int_0^1 dx = 1$. By solving the equation above, Gross and Witten gave the result in their paper[1]:

$$\rho(\phi) = \begin{cases} \frac{2}{\pi\lambda}\cos\frac{\phi}{2}(\frac{\lambda}{2}-\sin^2\frac{\phi}{2})^{1/2}, & \lambda \le 2 \quad \phi_c = 2\sin^{-1}(\frac{\lambda}{2})^{1/2} \\ \\ \frac{1}{2\pi}\left(1+\frac{2}{\lambda}\cos\phi\right), & \lambda \ge 2 \quad \phi_c = \pi. \end{cases} \tag{3.11}$$

In the first case, the density function is gapped, which means it doesn't cover the whole interval of $[0, 2\pi]$. In the second case, the density function is ungapped, and all the values between 0 and 2π can be assumed by $\phi(x)$.

As the density function has been sovled, the energy can be calculated by replacing the summation $\sum_{i=1}^{N}$ by the integral $\int d\phi\rho(\phi)$. We have

$$-E(\lambda) = \begin{cases} \frac{1}{\lambda^2}, & \lambda \ge 2 \\ \\ \frac{2}{\lambda}+\frac{1}{2}\ln\frac{\lambda}{2}-\frac{3}{4}, & \lambda \le 2. \end{cases} \tag{3.12}$$

The expectation value of the Wilson loop operator is

$$w_L(\lambda) = -\frac{\lambda^2}{2N^2}\frac{\partial \ln z}{\partial\lambda} = \lim_{N\to\infty} w_L(g^2,N) = \begin{cases} \frac{1}{\lambda}, & \lambda \ge 2 \\ \\ 1-\frac{\lambda}{4}, & \lambda \le 2, \end{cases} \tag{3.13}$$

and the string tesion

$$\sigma(\lambda) = -\frac{1}{a^2}\ln w_L = \lim_{N\to\infty}\sigma(g^2,N) = \begin{cases} \frac{1}{a^2}\ln\lambda, & \lambda \ge 2 \\ \\ \frac{1}{a^2}\ln\frac{4}{4-\lambda}, & \lambda \le 2. \end{cases} \tag{3.14}$$

Gross-Witten phase transition happens at $\lambda = 2$. By checking the derivatives of the energy Eqn.(3.12), we can see that this phase transition is of third order.

Chapter 4

The k-Antisymmetric Representation

As we have reduced the Wilson loop operator of k-antisymmetric representation into one plaquette and have had the distribution function $\rho(\phi)$ of the U(N) eigenvalues, we are ready to derive the free energy of k-antisymmetric representation.

In this chapter we first present the generating function of Wilson loop operator of k-antisymmetric representation. Then we are going to use the saddle point method to approximate the free energy. From the saddle point equations, we pluge in the distribution function for the weak coupling system. After laborious derivations, we get the paramter $\frac{k}{N}$ as a function of t and the approximated free energy as a function of t. Further work has been done to combine them and we have the approximated free energy as a function of $\frac{k}{N}$. The calculation for the strong coupling is more straightforward.

4.1 The Generating Function

For the gauge field $W \in U(N)$ we can diagnolize it by gauge transformation. Therefore, without loss gnerality, we can let the gauge field be

$$W = \text{diag}\{e^{i\phi_1}, e^{i\phi_2}...e^{i\phi_N}\}. \tag{4.1}$$

The k-antisymmetric representation of the U(N) group has its components:

$$W_{i_1}^{[\,j_1}W_{i_2}^{j_2}...W_{i_k}^{j_k\,]}, \tag{4.2}$$

and we have the trace of $W^{(k)}$ as

$$\mathrm{Tr}_{A_k}W = \frac{1}{k!}W_{i_1}^{[\,i_1}W_{i_2}^{i_2}...W_{i_k}^{i_k\,]}$$

$$= \sum_{i_1<i_2<...<i_N} e^{i\phi_1}e^{i\phi_2}...e^{i\phi_N}, \tag{4.3}$$

Now, we are going to introduce the generating function. First we have the expansion

$$\ln(1+tW) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(tW)^n, \tag{4.4}$$

and we have

$$\mathrm{Tr}\ln(1+tW) = \sum_{i=1}^{N}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(te^{i\phi_i})^n$$

$$= \sum_{i=1}^{N}\ln(1+te^{i\phi_i}). \tag{4.5}$$

The generating function is

$$\oint dt\frac{e^{\mathrm{Tr}\ln(1+tW)}}{2\pi i t^{k+1}} = \oint dt\frac{\prod_{i=1}^{N}(1+te^{i\phi_i})}{2\pi i t^{k+1}}$$

$$= \sum_{i_1<i_2<...<i_k} e^{i\phi_1}e^{i\phi_2}...e^{i\phi_k}. \tag{4.6}$$

Then we have the trace evaluated by the generating function

$$\mathrm{Tr}_{A_k}W = \oint dt\frac{e^{\mathrm{Tr}\ln(1+tW)}}{2\pi i t^{k+1}}$$

$$= \oint dt\frac{e^{\sum_{i=1}^{N}\ln(1+te^{i\phi_i})}}{2\pi i t^{k+1}}, \tag{4.7}$$

and the expectation value of the Wilson loop operator is

$$\langle \mathrm{Tr}_{A_k} W \rangle = \oint \frac{dt}{2\pi i t^{k+1}} \int \prod_{i=1}^{N} d\phi_i \\ \exp\left\{ \frac{2N}{\lambda} \sum_{i=1}^{N} \cos\phi_i + \sum_{i<j} \ln\cot^2 \frac{\phi_i - \phi_j}{2} + \sum_{i=1}^{N} \ln(1 + t e^{i\phi_i}) \right\}. \tag{4.8}$$

4.2 Saddle Point Approximation

We use the saddle point approximation to evaluate the multiple integration. To do this, we need to put all the integral variables to the exponent, such as $t^{k+1} = e^{(k+1)\ln t}$. Then the saddle point conditions are:

$$\partial_{\phi_i} \left\{ \frac{2N}{\lambda} \sum_{i=1}^{N} \cos\phi_i + \sum_{i<j} \ln\cot^2 \frac{\phi_i - \phi_j}{2} \right. \\ \left. + \sum_{i=1}^{N} \ln(1 + t e^{i\phi_i}) - (k+1)\ln t \right\} = 0, \tag{4.9}$$

and

$$\partial_{t} \left\{ \frac{2N}{\lambda} \sum_{i=1}^{N} \cos\phi_i + \sum_{i<j} \ln\cot^2 \frac{\phi_i - \phi_j}{2} \right. \\ \left. + \sum_{i=1}^{N} \ln(1 + t e^{i\phi_i}) - (k+1)\ln t \right\} = 0, \tag{4.10}$$

From the saddle point conditions we have:

$$-\frac{2N}{\lambda}\sin(\phi_i) + \sum_{j\neq i}^{N} \cot\frac{\phi_i - \phi_j}{2} + \frac{i t e^{i\phi_i}}{1 + t e^{i\phi_i}} = 0, \tag{4.11}$$

$$\sum_{i=1}^{N} \frac{e^{i\phi_i}}{1 + t e^{i\phi_i}} = \frac{k+1}{t}. \tag{4.12}$$

The two equations serve to determine the distribution function of the U(N) eigenvalues for the k-representation and the saddle point t in the complex plane. Since the last term in the first equation is supressed by $\frac{1}{N}$, it will not influence the distribution function much. So we can take the distribution function $\rho(\phi)$ the same as the one given for the fundamental representation.

As N goes to infinity, we can take the continuous limit and replace $\sum_{i=1}^{N}$ by $N\int_{-\pi}^{\pi} d\phi\rho(\phi)$, then the second saddle point equation (4.12) becomes:

$$R_{A_k}(t) = \int_{-\pi}^{\pi} d\phi\rho(\phi)\frac{te^{i\phi}}{1+te^{i\phi}}, \tag{4.13}$$

where $R_{A_k} = \frac{k+1}{N}$. And we have

$$\begin{aligned}
\langle \mathrm{Tr}_{A_k} W\rangle &= \left\langle \oint dt \frac{e^{\sum_{i=1}^{N}\ln(1+te^{i\phi_i})}}{2\pi i t^{k+1}} \right\rangle \\
&\approx \oint \frac{dt}{2\pi i} e^{N\int_{-\pi}^{\pi} d\phi\rho(\phi)\ln(1+te^{i\phi})-(k+1)\ln t} \\
&\approx \exp\left\{ N\int_{-\pi}^{\pi} d\phi\rho(\phi)\ln(1+te^{i\phi}) - (k+1)\ln\hat{t}\right\}, \qquad (4.14)
\end{aligned}$$

where $\rho(\phi)$ and $\hat{t}$ satisfy the saddle point equations (4.11) and (4.12). Since $e^{-N\Gamma_{A_k}} =< \mathrm{Tr}_{A_k}W >$, we have the un-normalized free energy:

$$\Gamma_{A_k} = -\int d\phi\rho\ln(1+te^{i\phi}) + \frac{k+1}{N}\ln(t). \tag{4.15}$$

We are going to combine Eqn.(4.13) and Eqn.(4.15) to determine the free energy Γ_{A_k} as a function of $\frac{k}{N}$. To do this, we need to finish the integration in saddle point equations for specific density function $\rho(\phi)$. We'll work on this in the following.

4.3 The Weak Coupling

Since the density function is symmetric about the origin:

$$\rho(\phi) = \rho(-\phi) \tag{4.16}$$

and $t = e^{2\theta}$, from Eqn.(4.13) we have

$$R_{A_k}(t) - \frac{1}{2} = \int_{-\pi}^{\pi} d\phi\rho(\phi)\frac{1}{2}\left[\left(\frac{e^{2\theta}e^{i\phi}}{1+e^{2\theta}e^{i\phi}} - \frac{1}{2}\right) + c.c.\right]. \tag{4.17}$$

The integrand is

$$\frac{1}{2}\left[\left(\frac{e^{2\theta}e^{i\phi}}{1+e^{2\theta}e^{i\phi}} - \frac{1}{2}\right) + c.c.\right] = \frac{1}{2}\left(\frac{e^{4\theta}-1}{1+e^{4\theta}+2e^{2\theta}\cos(\phi)}\right). \tag{4.18}$$

Let $x = \sin(\frac{\phi}{2})$, and we have:

$$\begin{aligned}\frac{1}{2}\left[\left(\frac{e^{2\theta}e^{i\phi}}{1+e^{2\theta}e^{i\phi}} - \frac{1}{2}\right) + c.c.\right] &= \frac{1}{2}\left(\frac{e^{4\theta}-1}{1+e^{4\theta}+2e^{2\theta}-4e^{2\theta}x^2}\right) \\ &= \frac{\sinh 2\theta}{4\cosh^2\theta}\left[\frac{1}{1-\frac{x^2}{\cosh^2\theta}}\right].\end{aligned} \tag{4.19}$$

From the expasion:

$$\frac{1}{1-\frac{x^2}{\cosh^2\theta}} = \sum_{m=0}^{\infty}\left(\frac{x^2}{\cosh^2\theta}\right)^m, \tag{4.20}$$

we have

$$R_{A_k} - \frac{1}{2} = \int_{-\pi}^{\pi} d\phi\rho(\phi)\frac{\sinh 2\theta}{4\cosh^2\theta}\sum_{m=0}^{\infty}\left(\frac{x^2}{\cosh^2\theta}\right)^m. \tag{4.21}$$

In the weak coupling system, we have the distribution function

$$\rho(\phi) = \frac{\cos\frac{\phi}{2}}{\pi(2-2p)}\sqrt{2-2p-\sin^2\frac{\phi}{2}}\ , \quad -\phi_c \le \phi \le \phi_c \tag{4.22}$$

where $\phi_c = 2\arcsin\sqrt{\frac{\lambda}{2}}$ and $p = w_L$, which is the Wislon loop operator under the fundamental representation. Weaking coupling means $\lambda \le 2$, and from Eqn.(3.13) we know $w_L = 1 - \frac{\lambda}{4}$. So we have $2-2p = \frac{\lambda}{2}$. Let $A = 2-2p$ and change variable: $\sin\frac{\phi}{2} = x$, we have

$$\int_{-\pi}^{\pi} d\phi \rho(\phi) = \int_{-\sqrt{A}}^{\sqrt{A}} dx \frac{2\sqrt{A-x^2}}{A\pi}. \tag{4.23}$$

We need to do the integral:

$$\int_{-\sqrt{A}}^{\sqrt{A}} dx \frac{2\sqrt{A-x^2}}{A\pi} x^{2m} = \frac{A^m \Gamma(m-\frac{1}{2})}{\sqrt{\pi}\Gamma(m+2)} = A^m \frac{(m-\frac{1}{2}) \times \cdots \times \frac{1}{2}}{(m+1)!}. \tag{4.24}$$

From Eqn.(4.21) we have

$$R_{A_k}(t) - \frac{1}{2} = \frac{\sinh 2\theta}{4\cosh^2\theta} \sum_{m=0}^{\infty} \left(\frac{A}{\cosh^2\theta}\right)^m \frac{(m-\frac{1}{2}) \times \cdots \times \frac{1}{2}}{(m+1)!}. \tag{4.25}$$

Further let $y = -\frac{A}{\cosh^2\theta}$ we have:

$$R_{A_k}(t) - \frac{1}{2} = \frac{\sinh 2\theta}{4\cosh^2\theta} \frac{2}{y}(\sqrt{1+y} - 1), \tag{4.26}$$

which is

$$R_{A_k}(t) - \frac{1}{2} = \frac{\sinh\theta}{A}\left(\cosh\theta - \sqrt{\cosh^2\theta - A}\right). \tag{4.27}$$

Next we are going to derive the free energy for weak coupling system. Similarly, because the distribution function $\rho(\phi)$ is even, we have

$$\ln(1+te^{i\phi}) = \frac{1}{2}\ln(1+te^{i\phi}) + c.c., \tag{4.28}$$

where $t = e^{2\theta}$. Then

$$\frac{1}{2}\ln(1+te^{i\phi}) + c.c. = \frac{1}{2}\ln(1+e^{4\theta} + 2e^{2\theta}\cos\phi). \tag{4.29}$$

Let $x = \sin\frac{\phi}{2}$, we have

$$\frac{1}{2}\ln(1+te^{i\phi}) + c.c. = \ln(1+e^{2\theta}) + \frac{1}{2}\ln\left(1 - \frac{x^2}{\cosh^2\theta}\right). \tag{4.30}$$

Only the second term depends on x, and we have the expansion:

$$\ln\left(1 - \frac{x^2}{\cosh^2\theta}\right) = -\sum_{n=1}^{\infty} \frac{1}{n} \frac{x^{2n}}{\cosh^{2n}\theta}. \tag{4.31}$$

Then we need to do a similar integration as above, and from Eqn.(4.24) we have:

$$\int_{-\sqrt{A}}^{\sqrt{A}} dx \frac{2\sqrt{A-x^2}}{A\pi} \ln\left(1 - \frac{x^2}{\cosh^2\theta}\right) =$$

$$-\sum_{n=1}^{\infty} \frac{1}{n} \frac{A^n}{\cosh^{2n}\theta} \frac{(n-\frac{1}{2}) \times \cdots \times \frac{1}{2}}{(n+1)!}. \tag{4.32}$$

Further let $y = \frac{A}{\cosh^2\theta}$, and

$$I(y) = -\sum_{n=1}^{\infty} \frac{y^n}{n} \frac{(n-\frac{1}{2}) \times \cdots \times \frac{1}{2}}{(n+1)!}, \tag{4.33}$$

then we have

$$\Gamma_{A_k} = -\ln(1+e^{2\theta}) - \frac{1}{2}I(y) + \frac{k}{N}\ln(e^{2\theta}). \tag{4.34}$$

Now we need to simplify $I(y)$. The first derivation of $I(y)$ is:

$$I'(y) = -\sum_{n=1}^{\infty} y^{n-1} \frac{(n-\frac{1}{2}) \times \cdots \times \frac{1}{2}}{(n+1)!}. \tag{4.35}$$

By adding and multiplying, we get a power series of y which sums up to a simple function,

$$(-y)I'(y) + 1 = \sum_{n=0}^{\infty} y^n \frac{(n-\frac{1}{2}) \times \cdots \times \frac{1}{2}}{(n+1)!}$$

$$= -\frac{2}{y}(\sqrt{1-y} - 1). \tag{4.36}$$

So

$$I'(y) = \frac{2\sqrt{1-y}}{y^2} - \frac{2}{y^2} + \frac{1}{y}. \tag{4.37}$$

Integrating the equation above over y, we have

$$I(y) = -\frac{2\sqrt{1-y}}{y} + 2\ln(\sqrt{1-y}+1) + \frac{2}{y} + C, \tag{4.38}$$

where C is a constant. From (4.33) we know that

$$\lim_{y\to 0} I(y) = 0, \tag{4.39}$$

and

$$\lim_{y\to 0}\left\{-\frac{2\sqrt{1-y}}{y} + 2\ln(\sqrt{1-y}+1) + \frac{2}{y} + C\right\} = 1 + 2\ln 2, \tag{4.40}$$

then we have $C = -1 - 2\ln 2$. From(4.34) we have

$$\begin{aligned}\Gamma_{A_k} = &(2\theta\sinh\theta - \cosh\theta)\left(\frac{\cosh\theta - \sqrt{\cosh^2\theta - A}}{A}\right)\\ &- \ln\left(\cosh\theta + \sqrt{\cosh^2\theta - A}\right) + \frac{1}{2}.\end{aligned} \tag{4.41}$$

4.4 The Strong Coupling

In the strong coupling, the density function of the U(N) eigenvalues is:

$$\rho(\phi) = \frac{1}{2\pi}(1 + 2p\cos(\phi)), \tag{4.42}$$

where now $\lambda \geq 2$ and $p = w_L = \frac{1}{\lambda}$. The expectation value of Wilson loop operator is

$$e^{-N\Gamma_{A_k}} = \oint \frac{dt}{2\pi i}\frac{e^{N\int_{-\pi}^{\pi} d\phi\rho(\phi)\ln(1+te^{i\phi})}}{t^{k+1}}. \tag{4.43}$$

Consider the integral:

$$\int_{-\pi}^{\pi} d\phi\rho(\phi)\ln(1 + te^{i\phi}), \tag{4.44}$$

when $||t|| < 1$, we can expand the logarithm term

$$\ln(1 + te^{i\phi}) = -\sum_{n=1}^{\infty}\frac{(-1)^n(te^{i\phi})^n}{n}. \tag{4.45}$$

Only the first term in the summation contributes to the integral and gives

$$\int_{-\pi}^{\pi} d\phi\frac{1}{2\pi}(1 + 2p\cos(\phi))(te^{i\phi}) = pt. \tag{4.46}$$

Therefore,

$$e^{-N\Gamma_{A_k}} = \oint \frac{dt}{2\pi i}\frac{e^{Npt}}{t^{k+1}}, \qquad ||t|| < 1. \tag{4.47}$$

Expand $e^{Npt} = \sum_{n=0}^{\infty} \frac{(Npt)^n}{n!}$, and only the term $\frac{(Npt)^k}{k!}$ contributes to the contour integration. We have

$$e^{-N\Gamma_{A_k}} = \oint \frac{dt}{2\pi i}\frac{(Npt)^k}{k!t^{k+1}} = \frac{(Np)^k}{k!}. \tag{4.48}$$

Then the un-normalized free energy is

$$\Gamma_{A_k} = -\frac{1}{N}\left(k\ln(Np) - \ln(k!)\right). \tag{4.49}$$

When k is also large, we can further use Stirling's formula:

$$\ln(k!) = k\ln k - k + o(k). \tag{4.50}$$

Then, up to the leading order, we have

$$\Gamma_{A_k} = \frac{k}{N}\ln(\frac{k}{Npe}). \tag{4.51}$$

When $||t|| > 1$, we have

$$\ln(1+te^{i\phi}) = \ln(te^{i\phi}) + \ln(1+\frac{1}{t}e^{-i\phi}), \tag{4.52}$$

and expand $\ln(1+\frac{1}{t}e^{-i\phi})$ in the same way as above. We have

$$\int_{-\pi}^{\pi} d\phi\rho(\phi)\ln(1+te^{i\phi}) = \ln(t) + \frac{p}{t}, \qquad ||t|| > 1 \tag{4.53}$$

and

$$e^{-N\Gamma_{A_k}} = \oint \frac{dt}{2\pi i}\frac{t^N}{t^{k+1}}e^{\frac{Np}{t}} = \frac{(Np)^{N-k}}{(N-k)!}, \qquad ||t|| > 1. \tag{4.54}$$

This branch of the un-normalized free energy is

$$\Gamma_{A_k} = -\frac{1}{N}\left[(N-k)\ln(Np) - \ln((N-k)!)\right]. \tag{4.55}$$

Use Stirling's formula again, we have

$$\Gamma_{A_k} = (1-\frac{k}{N})\ln\left(\frac{1-\frac{k}{N}}{pe}\right). \tag{4.56}$$

When $p > \frac{1}{2}$, it is weak coupling $(\lambda < 2)$. When $p < \frac{1}{2}$ the system is strong coupling $(\lambda > 2)$. We will verify that the two free energies for strong and weak coupling systems match each other at $p = \frac{1}{2}, i.e. A = 1(\lambda = 2)$.

4.5 Comparison at The Boundry

Starting from weak coupling system when $A = 1$, from Eqn.(4.27)we have

$$\frac{k}{N} = \sinh\theta\left(\cosh\theta - \sqrt{\cosh^2\theta - 1}\right) + \frac{1}{2}. \tag{4.57}$$

Consider θ as a complex variable, so $\frac{k}{N}$ is a two-valued function of θ:

$$\frac{k}{N} = \begin{cases} \sinh\theta(\cosh\theta - \sinh\theta) + \frac{1}{2}, & \sqrt{\sinh^2\theta} = \sinh\theta \\ \\ \sinh\theta(\cosh\theta + \sinh\theta) + \frac{1}{2}. & \sqrt{\sinh^2\theta} = -\sinh\theta \end{cases} \tag{4.58}$$

With $t = e^{2\theta}$, we have

$$\frac{k}{N} = \begin{cases} 1 - \frac{1}{2t}, & \sqrt{\sinh^2\theta} = \sinh\theta \\ \\ \frac{t}{2}. & \sqrt{\sinh^2\theta} = -\sinh\theta \end{cases} \tag{4.59}$$

Then let's check the free energy. When $A = 1$, from Eqn.(4.41) we have:

$$\begin{aligned} \Gamma = & (2\theta\sinh\theta - \cosh\theta)\left(\cosh\theta - \sqrt{\sinh^2\theta}\right) \\ & - \ln\left(\cosh\theta + \sqrt{\sinh^2\theta}\right) + \frac{1}{2}. \end{aligned} \tag{4.60}$$

We also have two choices of the sign of $\sqrt{\sinh^2\theta}$:

$$\Gamma = \begin{cases} -\theta e^{-2\theta} - \frac{1}{2}e^{-2\theta}, & \sqrt{\sinh^2\theta} = \sinh\theta \\ \\ \Gamma = \theta e^{2\theta} - \frac{1}{2}e^{2\theta}. & \sqrt{\sinh^2\theta} = -\sinh\theta. \end{cases} \tag{4.61}$$

Pluge in (4.59), we have

$$\Gamma = \begin{cases} \left(1 - \frac{k}{N}\right)\ln\left(\frac{2(1-\frac{k}{N})}{e}\right), & \sqrt{\sinh^2\theta} = \sinh\theta \\ \\ \Gamma = \frac{k}{N}\ln\left(\frac{2k}{eN}\right). & \sqrt{\sinh^2\theta} = -\sinh\theta \end{cases} \tag{4.62}$$

They are the same as the two branches of free energy that we get in the strong coupling system at $p = \frac{1}{2}$.

Chapter 5

Phase Transition

5.1 The Free Energy of The Weak Coupling System

In the weak coupling system, we already have the free energy Eqn.(4.41) associated with Eqn.(4.27):

$$R(\theta) = \frac{k}{N} = \frac{\sinh\theta}{A}\left(\cosh\theta - \sqrt{\cosh^2\theta - A}\right) + \frac{1}{2}. \tag{5.1}$$

To solve the free energy as a function of $R = \frac{k}{N}$, we must combine the two equations. From the second equation, we could find the inverse function: $\theta(R)$ and then pluge it into the first one $\Gamma(\theta)$. Let $u = e^{\theta}$ we have:

$$\sqrt{\frac{(u+\frac{1}{u})^2}{4} - A} = \frac{u+\frac{1}{u}}{2} - \frac{2A(\frac{k}{2N})}{u - \frac{1}{u}} \tag{5.2}$$

Squaring both sides, it turns

$$\left[4A - 8A\left(\frac{k}{N} - \frac{1}{2}\right)\right]u^4 + \left[16A^2\left(\frac{k}{N} - \frac{1}{2}\right)^2 - 8A\right]u^2 + 8A\frac{k}{N} = 0. \tag{5.3}$$

Solve the quadratic equation of u^2, we have

$$\begin{aligned} u_{\pm}^2(R) =& \frac{1}{2(1-R)}\left(1 - 2A\left(R - \frac{1}{2}\right)\right)^2 \\ &\pm \sqrt{\left(1 - 2A\left(R - \frac{1}{2}\right)^2\right)^2 - 4\left(1-R\right)R}. \end{aligned} \tag{5.4}$$

Then we have

$$\theta_\pm(R) = \frac{1}{2}\ln u_\pm(R). \tag{5.5}$$

We regard θ as a complex variable, then the free energy $\Gamma(\theta)$ is a two-valued complex function due to $\sqrt{\cosh^2\theta - A}$. So we have two choices of the free energy:

$$\begin{aligned}\Gamma_\pm(\theta) =& \frac{1}{A}\left(2\theta\sinh\theta - \cosh\theta\right)\left(\cosh\theta \mp \sqrt{\cosh^2\theta - A}\right)\\ &- \ln\left(\cosh\theta \pm \sqrt{\cosh^2\theta - A}\right) + \frac{1}{2}.\end{aligned} \tag{5.6}$$

Now we have four branches of free energy as a function of $R = \frac{k}{N}$: $\Gamma_+(\theta_+(R))$, $\Gamma_+(\theta_-(R))$, $\Gamma_-(\theta_+(R))$ and $\Gamma_-(\theta_-(R))$, see Fig.(5.1) and Fig.(5.2). At $\frac{k}{N} = \frac{1}{2}$, we have $u_\pm^2(\frac{1}{2}) = 1$, and $\theta_\pm = 0$. So we have

$$\Gamma_+(\theta_+(R = \frac{1}{2})) = \Gamma_+(\theta_-(R = \frac{1}{2})), \tag{5.7}$$

and

$$\Gamma_-(\theta_+(R = \frac{1}{2})) = \Gamma_-(\theta_-(R = \frac{1}{2})). \tag{5.8}$$

We are going to take the bottom curves as free energy. We have two bottome curves as the moment: one is patched by $\Gamma_+(\theta_\pm(R))$, and the other one is patched by $\Gamma_-(\theta_\pm(R))$,

$$\Gamma_+(R) = \begin{cases}\Gamma_+(\theta_-(R)), & R \leq \frac{1}{2}\\ \\ \Gamma_+(\theta_+(R)), & R \geq \frac{1}{2},\end{cases} \tag{5.9}$$

and

$$\Gamma_-(R) = \begin{cases}\Gamma_-(\theta_+(R)), & R \leq \frac{1}{2}\\ \\ \Gamma_-(\theta_-(R)), & R \geq \frac{1}{2},\end{cases} \tag{5.10}$$

see Fig.(5.3) and Fig.(5.4). By checking derivatives, we found that none of these four functions $\Gamma_\pm(\theta_\pm(R))$ are smooth. see Fig.(5.5), Fig.(5.6),

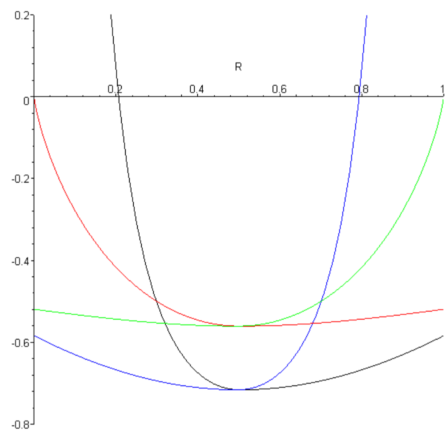

Figure 5.1: A=0.8 (p=0.6), $\Gamma_+(\theta_+(R))$: Green; $\Gamma_+(\theta_-(R))$: Red; $\Gamma_-(\theta_+(R))$: Blue; $\Gamma_-(\theta_-(R))$: Black.

Fig.(5.7) and Fig.(5.8). However the patched are $\Gamma_+(R)$ and $\Gamma_-(R)$ are smooth curves, see Fig.(5.9) and Fig.(5.10).

Now we need to decide which one of the two is the solution from saddle point approximation. However, the issue cannot be simplily solved by choosing the smaller one. We need to look into the problem and know how each solution actually comes out. First, we note that the free energy Γ_{A_k} are multi-valued complex function on t-plane. This is due to the logarithm and the square root terms. The logarithm term contributes a constant to the free energy. Up to a constant, the free energy is 2-valued function, and there are two branch points determined from $\cosh^2\theta = A$

$$t_\pm = 2A - 1 \pm 2\sqrt{A(1-A)}i. \tag{5.11}$$

When $A < 1$, the two branch points are complex numbers and conjugate to each other. To make the free energy single valued, we need draw a branch cut connecting $t_\pm$. Dr. Karczmarek pointed out that the value of free energy depends on the way we draw the branch cut. The reason is as followed. Let's first look into some basic facts about the contour choosing and the saddle

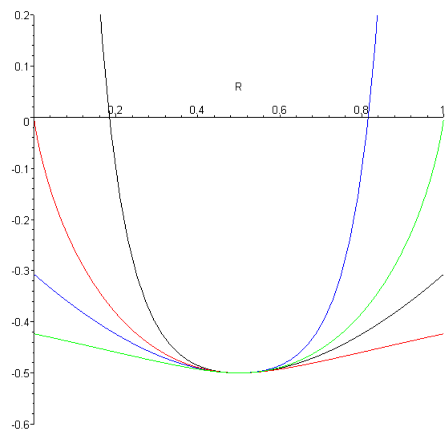

Figure 5.2: A=1 (p=0.3), $\Gamma_+(\theta_+(R))$: Green; $\Gamma_+(\theta_-(R))$: Red; $\Gamma_-(\theta_+(R))$: Blue; $\Gamma_-(\theta_-(R))$: Black.

point. First, the integrating contour must encircle the origin based on the generating function method. Secondly, the saddle point should be on the contour of course, and moreover be the summit on the chosen contour. Also, numerical simulation indictes that the saddle point locates on $\mathbb{R}^+$ axis and the origin is a highest point of the integrand. Based on the 3 facts, we can continue to discuss the possible ways to draw the branch cut and how to choose contours on the t-plane.

To connect the branch points $t_\pm$, which are conjugate to each other, the branch cut necessarily crosses the $\mathbb{R}$ axis. (The branch cuts could also connect the branch points to infinity, but it crosses the real axis at infinity as well.) Let $x_0 \in \mathbb{R}^+$ be the saddle point and t_0 be the crossing point of real axis and the branch cut. There are two cases to be considered. One is that $t_0 \in [0, x_0]$. In this case, t_0 separates the origin and the saddle point on $\mathbb{R}$ asix. The contour must encirle the branch cut, otherwise it is not qualified to be used in saddle point method. The other case is $t_0 \notin [0, x_0]$ and t_0 is far away from the origin such that we could choose a valid contour without encircling the branch cut. Dr Karczmarek shows that the first kind of branch

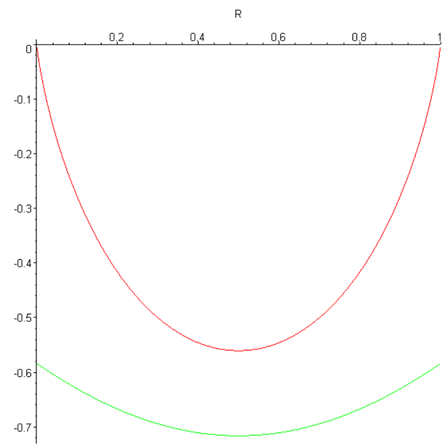

Figure 5.3: A=0.8 (p=0.6), $\Gamma_+(R)$: Red; $\Gamma_-(R)$:Green

cut gives the free energy $\Gamma_-(R)$, and the second gives $\Gamma_+(R)$. Despite the fact that $\Gamma_-(R)$ is lower than $\Gamma_+(R)$ as showed in Fig.(5.3) and Fig.(5.4), we take $\Gamma_+(R)$ as the valid saddle point solution, because the corresponding contour can shrink to the origin without intersecting the branch cut. When $0 < A < 1$, $\Gamma_+(R)$ is a smooth curve, see Fig.(5.9).

5.2 The Free Energy of The Strong Coupling System

In the strong coupling system, we are still facing the problem to choose the right free energy among the two branches, see Fig.(5.11) and Fig.(5.12)

$$\Gamma_1(R) = R\ln(\frac{R}{pe}), \tag{5.12}$$

and

$$\Gamma_2(R) = (1-R)\ln(\frac{1-R}{pe}), \tag{5.13}$$

where $R = \frac{k}{N}$.

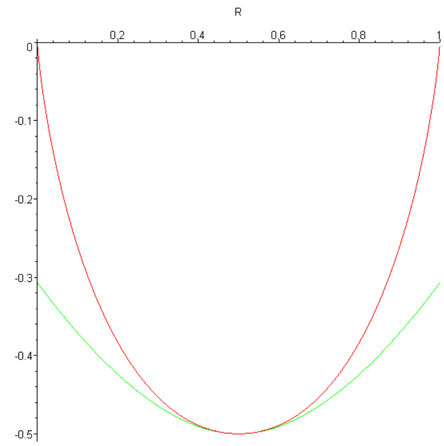

Figure 5.4: A=1 (p=0.5), $\Gamma_+(R)$: Red; $\Gamma_-(R)$:Green

The numerical result indicates the free energy for strong coupling system to be

$$\Gamma_{s.c.}(R) = \begin{cases} R\ln(\frac{R}{pe}) & 0 \leq R \leq \frac{1}{2} \\ \\ (1-R)\ln(\frac{1-R}{pe}). & \frac{1}{2} \leq R \leq 1 \end{cases} \tag{5.14}$$

$\Gamma_{s.c.}(R)$ is the bottom curve only when $0 < p < \frac{1}{e}$, see Fig.(5.11). However, when $\frac{1}{e} < p < \frac{1}{2}$, $\Gamma_{s.c.}(R)$ is not the minimum one in the whole range of $R \in [0,1]$. The bottom curve is patched by 4 parts, as showed in Fig.(5.13). But the numerical simulation favors $\Gamma_{s.c.}(R)$ in the whole range. The reason is unknown yet. This indicates that the continuous limit $\sum_{i=1}^{N}\phi_i \to N\int d\phi\rho(\phi)$ brings in some unknown outgrowth.

5.3 The Phase Transition

$\Gamma_{s.c.}(R)$ has a phase transition at $R = \frac{1}{2}$, which is obvious from Fig.(5.11), Fig.(5.12) and Fig.(5.13) but not so obvious from Fig.(5.14). The order of the Phase transition can be determined by checking the derivatives of Γ_1

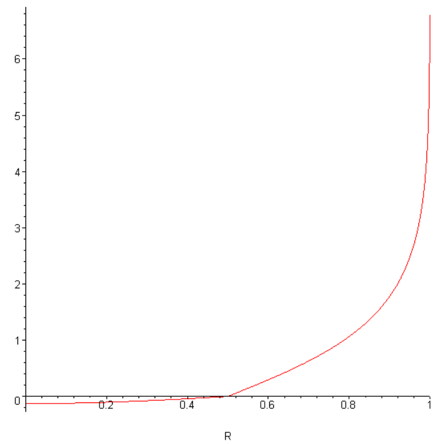

Figure 5.5: A=0.8 (p=0.6), $\frac{d}{dR}\Gamma_+(\theta_+(R))$

and Γ_2 at $R = \frac{1}{2}$, which are the left and right derivatives of $\Gamma_{s.c.}$ at $R = \frac{1}{2}$.

$$\left.\frac{d}{dR}\right|_{R=1/2} \Gamma_1(R) = \ln \left.\frac{R}{p}\right|_{R=\frac{1}{2}} = \ln \frac{1}{2p}, \tag{5.15}$$

$$\left.\frac{d}{dR}\right|_{R=1/2} \Gamma_2(R) = -\ln \left.\frac{1-R}{p}\right|_{R=\frac{1}{2}} = -\ln \frac{1}{2p}. \tag{5.16}$$

Their derivatives at $R = \frac{1}{2}$ are distinct in general, but both equal zero when $p = \frac{1}{2}$. So for the case of $p = \frac{1}{2}$, we need to consider higher order derivatives.

$$\left.\frac{d^2}{dR^2}\right|_{R=\frac{1}{2}} \Gamma_1(R) = \left.\frac{1}{R}\right|_{R=\frac{1}{2}} = 2, \tag{5.17}$$

$$\left.\frac{d^2}{dR^2}\right|_{R=\frac{1}{2}} \Gamma_2(R) = \left.\frac{1}{1-R}\right|_{R=\frac{1}{2}} = 2. \tag{5.18}$$

The second derivatives match each other at $R = \frac{1}{2}$. Then we need to further consider the third derivatives:

$$\left.\frac{d^3}{dR^3}\right|_{R=\frac{1}{2}} \Gamma_1(R) = -\left.\frac{1}{R^2}\right|_{R=\frac{1}{2}} = -4 \tag{5.19}$$

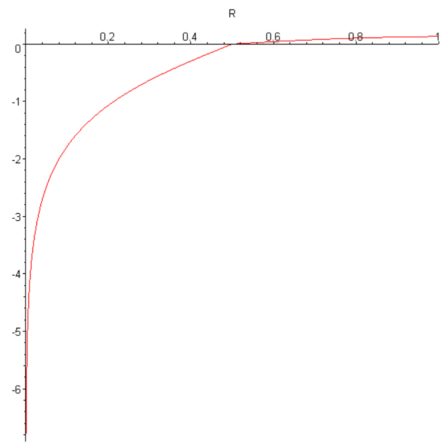

Figure 5.6: A=0.8 (p=0.6), $\frac{d}{dR}\Gamma_+(\theta_-(R))$

$$\left.\frac{d^3}{dR^3}\right|_{R=\frac{1}{2}} \Gamma_2(R) = \left.\frac{1}{(1-R)^2}\right|_{R=\frac{1}{2}} = 4 \tag{5.20}$$

We see that the third left and right derivatives of $\Gamma_{s.c.}$ at $R = \frac{1}{2}$ are not equal, and conclude that it's a third order phase transition at $p = \frac{1}{2}$.

Therefore, when $p < \frac{1}{2}$, the phase transition is 1st order, when $p = \frac{1}{2}$, it is 3rd order. When $p > \frac{1}{2}$, the system is weak coupling, and the free energy is a smooth function of $\frac{k}{N}$.

5.4 Gross-Witten Phase Transition for k-Representation

We have discussed the phase transition that happens when $\frac{k}{N}$ changes. The coupling constant is also a parameter of the free energy. In the fundamental representation, Gross-Witten phase transition happens when the coupling constant changes from weak to strong. In the k-antisymmetric representation we have already check that the free energy is continuous at the boundary between weak and strong coupling in Section 4.5. We are going to check

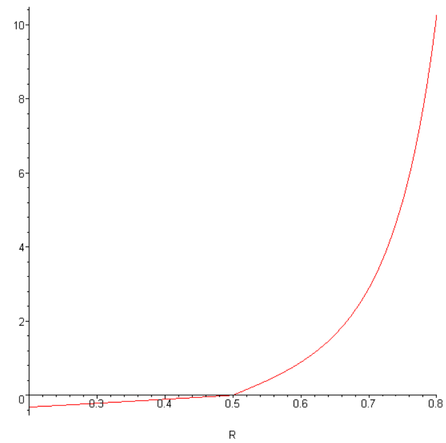

Figure 5.7: A=0.8 (p=0.6), $\frac{d}{dR}\Gamma_-(\theta_+(R))$

the derivatives of the free energy from both sides to see if there is a phase transition.

The free energy is also a function of the coupling constant λ. From Eqn.(5.4) and Eqn.(5.5) we know that $u_\pm(R, A)$ and $\theta_\pm(R, A)$ are functions of R and A. Since $A = 2 - 2p$ and

$$p = w_L(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq 2 \quad \text{strong coupling} \\ \\ 1 - \frac{\lambda}{4}. & \lambda \leq 2 \quad \text{weak coupling} \end{cases} \tag{5.21}$$

Therefore, A or p is related to the coupling constant λ. The free energy is function of both R and λ, $\Gamma(R, \lambda)$. From Eqn.(5.9) the free energy in the weak coupling system $\lambda \leq 2$) is

$$\Gamma_{w.c.}(R, \lambda) = \begin{cases} \Gamma_+(\theta_-(R, \lambda), \lambda), & R \leq \frac{1}{2} \\ \\ \Gamma_+(\theta_+(R, \lambda), \lambda), & R \geq \frac{1}{2}, \end{cases} \tag{5.22}$$

and from Eqn.(5.14) the free energy for the strong coupling system ($\lambda \geq 2$)

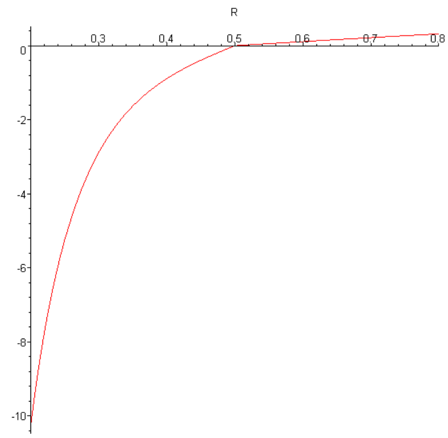

Figure 5.8: A=0.8 (p=0.6), $\frac{d}{dR}\Gamma_{-}(\theta_{-}(R))$

is

$$\Gamma_{s.c.}(R,\lambda) = \begin{cases} R\ln\left(\frac{\lambda R}{e}\right) & 0 \leq R \leq \frac{1}{2} \\ \\ (1-R)\ln\left(\frac{\lambda(1-R)}{e}\right). & \frac{1}{2} \leq R \leq 1 \end{cases} \tag{5.23}$$

Then we'll compare the derivatives of $\Gamma_{w.c.}(R,\lambda)$ and $\Gamma_{s.c.}(R,\lambda)$. Let's first consider the case at $\frac{k}{N} = \frac{1}{2}$. From Eqn.(5.4) and Eqn.(5.5) we have

$$\theta_{\pm}(R = 0.5) = 0, \tag{5.24}$$

and from Eqn.(5.6) we know

$$\Gamma_{w.c.}(\theta = 0, A) = -\frac{1}{A}(1-\sqrt{1-A}) - \ln(1+\sqrt{1-A}) + \frac{1}{2}. \tag{5.25}$$

Since $A = \frac{\lambda}{2}$ in the weak coupling system, we have

$$\Gamma_{w.c.}(R = 0.5, \lambda) = -\frac{2}{\lambda}\left(1-\sqrt{1-\frac{\lambda}{2}}\right) - \ln\left(1+\sqrt{1-\frac{\lambda}{2}}\right) + \frac{1}{2}. \tag{5.26}$$

Then we have

$$\frac{\partial\Gamma_{w.c.}}{\partial\lambda}(\lambda) = \frac{2}{\lambda^2}\left(1-\sqrt{1-\frac{\lambda}{2}}\right) - \frac{1}{2\lambda}, \tag{5.27}$$

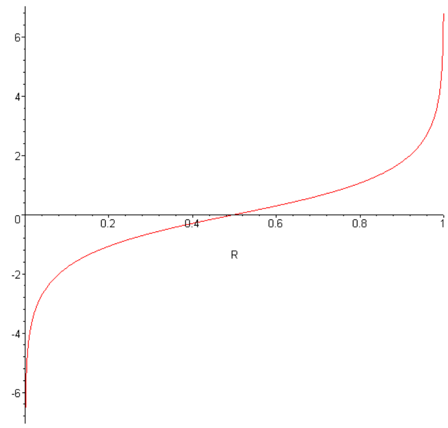

Figure 5.9: A=0.8 (p=0.6), $\Gamma'_+(R)$

and

$$\frac{\partial \Gamma_{s.c.}}{\partial \lambda}(\lambda) = \frac{1}{2\lambda}. \tag{5.28}$$

Then we find

$$\left.\frac{\partial \Gamma_{w.c.}}{\partial \lambda}\right|_{\lambda=2} = \left.\frac{\partial \Gamma_{s.c.}}{\partial \lambda}\right|_{\lambda=2} = \frac{1}{4}. \tag{5.29}$$

So we need to consider the second derivatives:

$$\frac{\partial^2 \Gamma_{w.c.}}{\partial \lambda^2}(\lambda) = -\frac{4}{\lambda^3}\left(1 - \sqrt{1 - \frac{\lambda}{2}}\right) + \frac{1}{2\lambda\sqrt{1-\frac{\lambda}{2}}} + \frac{1}{2\lambda^2}, \tag{5.30}$$

and

$$\frac{\partial^2 \Gamma_{s.c.}}{\partial \lambda^2}(\lambda) = -\frac{1}{2\lambda^2}. \tag{5.31}$$

We see that

$$\lim_{\lambda \to 2} \frac{\partial^2 \Gamma_{w.c.}}{\partial \lambda^2}(\lambda) \to \infty. \tag{5.32}$$

So we conclude that the phase transition is of second order at $\lambda = 2$ and $\frac{k}{N} = \frac{1}{2}$. In the case $\frac{k}{N} \in (0, 0.5) \cup (0.5, 1)$, $\left.\frac{\partial^2 \Gamma_{w.c.}}{\partial \lambda^2}\right|_{\lambda=2}$ does not diverge, but we still have

$$\left.\frac{\partial \Gamma_{w.c.}}{\partial \lambda}\right|_{\lambda=2} = \left.\frac{\partial \Gamma_{s.c.}}{\partial \lambda}\right|_{\lambda=2}, \tag{5.33}$$

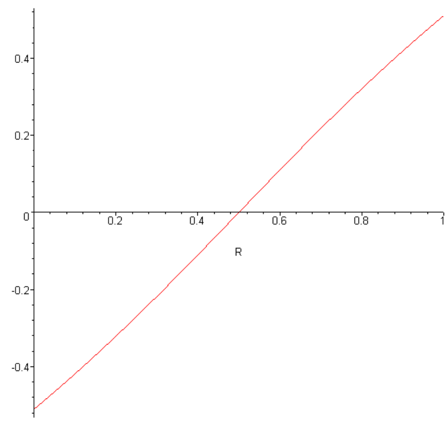

Figure 5.10: A=0.8 (p=0.6), $\Gamma'_{-}(R)$

and

$$\left.\frac{\partial^2 \Gamma_{w.c.}}{\partial \lambda^2}\right|_{\lambda=2} \neq \left.\frac{\partial^2 \Gamma_{s.c.}}{\partial \lambda^2}\right|_{\lambda=2}, \tag{5.34}$$

see Fig.(5.15). Therefore, the phase transition is of second order at $\lambda = 2$, $\frac{k}{N} \in (0, 1)$. Now, we have the whole phase diagram Fig.5.16.

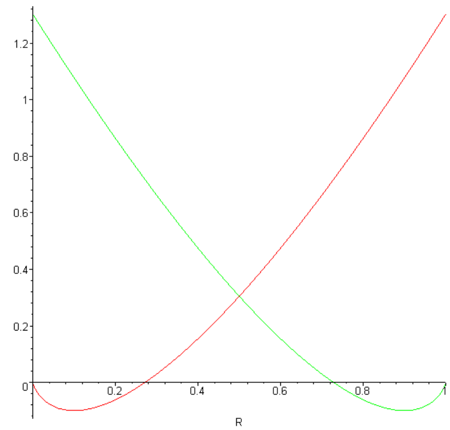

Figure 5.11: p=0.1, $\Gamma_1(R)$: Red; $\Gamma_2(R)$:Green

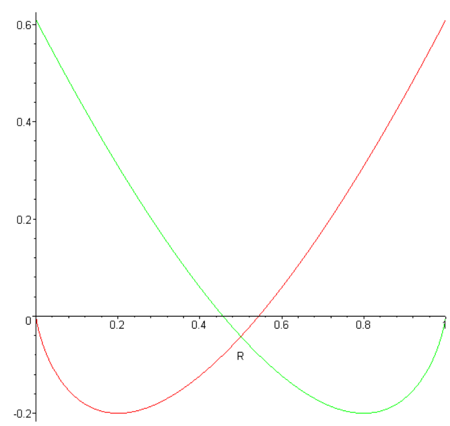

Figure 5.12: p=0.2, $\Gamma_1(R)$: Red; $\Gamma_2(R)$:Green

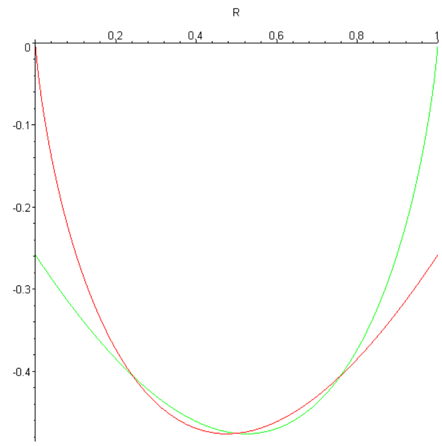

Figure 5.13: $p = \frac{1}{2.1}$, $\Gamma_1(R)$: Red; $\Gamma_2(R)$:Green

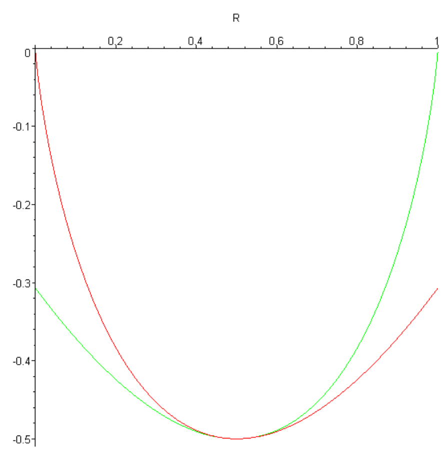

Figure 5.14: p=0.5, $\Gamma_1(R)$: Red; $\Gamma_2(R)$:Green

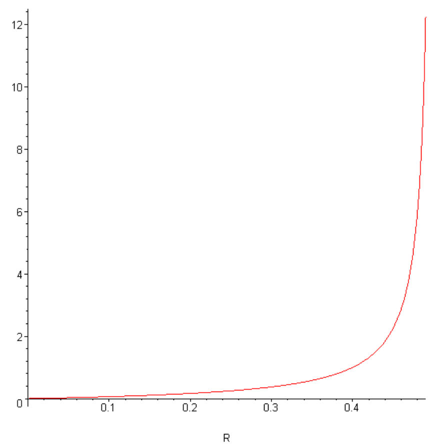

Figure 5.15: $\left.\frac{\partial^2\Gamma_{w.c}}{\partial\lambda^2}\right|_{\lambda=2} - \left.\frac{\partial^2\Gamma_{s.c}}{\partial\lambda^2}\right|_{\lambda=2}$.

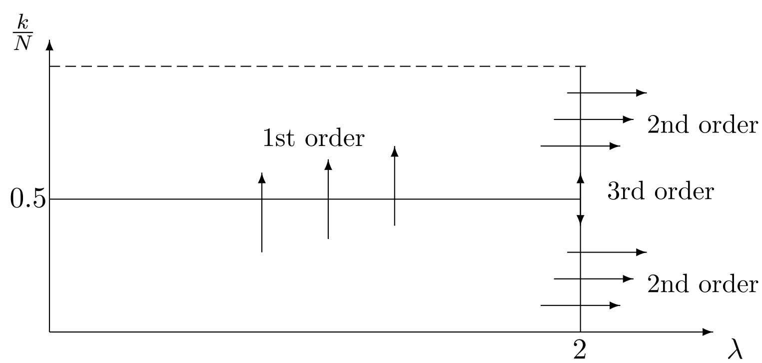

Figure 5.16: Phase Diagram

Chapter 6

Remarks and Conclusions

6.1 Discussion About The Distribution Function

We stated the the last term of Eqn.(4.11) $\frac{te^{i\phi}}{1+te^{i\phi}}$ is supressed by $\frac{1}{N}$, but this is true only if t is not closed to $e^{i(-\phi+\pi)}$. However, this is not necessarily true from the saddle point equations Eqn.(4.11) and Eqn.(4.12). Let's first assume that the distribution function is the same as in fundamental representation, then find the saddle point $\hat{t}$. If there is no contradiction, i.e. $\hat{t}$ is not close to $e^{i(-\phi+\pi)}$ for any $\phi \in [-\phi_c, \phi_c]$, we can say that our assumption is reasonable.

Let's start with the weak coupling system. The distribution function of fundamental representation is gapped, i.e. $\phi_c < \pi$. Based on the assumed distribution function, the numerical simulation tells us that the saddle point $\hat{t}$ is positive and real. So the distance between $\hat{t}$ and $\{e^{i(-\phi+\pi)}|\phi \in [-\phi_c, \phi_c]\}$ is finite. Therefore, in the large N limit, $\frac{\hat{t}e^{i\phi}}{1+\hat{t}e^{i\phi}}$ is finite and negligible.

Next, let's consider the case in the strong coupling system. The distribution function is ungapped, which means $\phi_c = \pi$. All the values on the unit circle in complex plane is assumed by the distribution function. From Eqn.(4.47) and Eqn.(4.54), we get the saddle point

$$\hat{t} = \begin{cases} \frac{k+1}{Np}, & ||\hat{t}|| < 1 \\ \\ \frac{p}{\frac{k+1}{N}-1}, & ||\hat{t}|| > 1 \end{cases} \tag{6.1}$$

where $0 \leq p \leq \frac{1}{2}$. We can see that, when $k+1 = Np$ or $\frac{k+1}{N} - 1 = p$ the saddle point $\hat{t} = 1$ and $\frac{\hat{t}e^{i\phi}}{1+\hat{t}e^{i\phi}}$ diverges at $\phi = \pi$. This suggests that near $\phi = \pi$ the distribution function $\rho(\phi)$ is different from the one of the fundamental representation. The revision of the distribution function is left as an open question.

6.2 The k-String Tension

As we have had the expectation value of the Wilson loop operator for k-antisymmetric representation of U(N), we can further calculate the k-string tension:

$$\sigma_k = -\frac{1}{a^2} \ln w_L^{(k)}(g^2, N). \tag{6.2}$$

With the normalization factor, the expecation value of Wilson loop operator on one plaquette under k-representation is

$$w_L^{(k)} = \frac{1}{N!/k!/(N-k)!} < \mathrm{Tr} W^{(k)} > . \tag{6.3}$$

And the k-string tension is

$$\sigma_k = -\frac{1}{a^2} \left(- \ln \left(\frac{N!}{k!(N-k)!} \right) + \ln < \mathrm{Tr} W^{(k)} > \right). \tag{6.4}$$

Let's start with the strong coupling system first, because we have the exact solution under strong coulping and things are more straightforward. From (equation 4.*) we have under strong coupling

$$w_L^{(k)} = \frac{k!(N-k)!}{N!} \frac{(Np)^k}{k!} = \frac{(N-k)!(Np)^k}{N!}. \tag{6.5}$$

Noting that when $k = 1$, $w^{(1)} = p$, which matches (equation 3. *). When k is relatively small, say a finite integer,

$$\lim_{N \to \infty} \frac{(N-k)! N^k}{N!} = 1, \tag{6.6}$$

so

$$w_L^{(k)} \approx p^k \qquad \text{for finite k.} \tag{6.7}$$

Then we have the ratio of k-string tension σ_k to the fundamental string tension σ:

$$\frac{\sigma_k}{\sigma} = \frac{\ln w_L^{(k)}}{\ln w_L^{(1)}} \approx k \qquad \text{for finite k.} \tag{6.8}$$

Both the sine-formular

$$\frac{\sigma_k}{\sigma} = \frac{\sin(k\pi/N)}{\sin(\pi/N)} \tag{6.9}$$

and the Casimir scaling law

$$\frac{\sigma_k}{\sigma} = \frac{k(N-k)}{N-1} \tag{6.10}$$

are satisfied when k is finite and N goes to infinity.

We are more concerned about the case when k is comparable with N. From

$$\ln w_L^{(k)} = \ln((N-k)!) - \ln(N!) + k\ln(Np), \tag{6.11}$$

ans using Stirling's formular, we have

$$\ln w_L^{(k)} = (N-k)\ln\left(\frac{N-k}{N}\right) + k(1+\ln p). \tag{6.12}$$

Now the ration of k-string tension to the fundamental one is

$$\frac{\sigma_k}{\sigma} = k + \frac{(N-k)\ln\left(\frac{N-k}{N}\right) + k}{\ln p}. \tag{6.13}$$

The ratio also depends on the coupling constant p. This is not similar as sine-formular nor the Casimir scaling law.

Next consider the weak coupling system:

$$\sigma_k = \frac{1}{a^2}\left(-N\Gamma(k/N,p) + \ln\left(\frac{k!(N-k)!}{N!}\right)\right), \qquad 0.5 < p \leq 1. \tag{6.14}$$

Using Stirling's formular again, we have the k-string tension ration:

$$\frac{\sigma_k}{\sigma} = \frac{1}{-\ln p}\left(-N\Gamma(k/N,p) + N\ln\frac{N-k}{N} - k\ln\frac{N-k}{k}\right). \tag{6.15}$$

Bibliography

[1] D. J. Gross and E. Witten, "Possible Third Order Phase Transition In The Large N Lattice Gauge Theory," Phys. Rev. D **21**, 446 (1980).

[2] K. G. Wilson, "Quark Confinement,"

[3] J. B. Kogut and L. Susskind, "Hamiltonian Formulation Of Wilson's Lattice Gauge Theories," Phys. Rev. D **11**, 395 (1975).

[4] E. Witten, "Current Algebra Theorems For The U(1) Goldstone Boson," Nucl. Phys. B **156**, 269 (1979).

[5] A. V. Manohar, "Large N QCD," arXiv:hep-ph/9802419.

[6] Sidney Coleman, "1/N," Presented at The 1979 International School of Subnuclear Physics: Pointlike Structures Inside and Outside Hadron

[7] I. Bars and F. Green, "Complete Integration Of U (N) Lattice Gauge Theory In A Large N Limit," Phys. Rev. D **20**, 3311 (1979).

[8] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, "2-D Gravity and random matrices," Phys. Rept. **254**, 1 (1995) [arXiv:hep-th/9306153].