Hardy-Rellich inequalities and the critical dimension of fourth order nonlinear elliptic eigenvalue problems

by

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Abstract

This thesis consists of three parts and manuscripts of seven research papers studying improved Hardy and Hardy-Rellich inequalities, nonlinear eigenvalue problems, and simultaneous preconditioning and symmetrization of linear systems.

In the first part that consists of three research papers we study improved Hardy and Hardy-Rellich inequalities. In sections 2 and 3, we give necessary and sufficient conditions on a pair of positive radial functions V and W on a ball B of radius R in R^n, $n \geq 1$, so that the following inequalities hold for all $u \in C_0^\infty(B)$:

$$\int_B V(x)|\nabla u|^2 dx \geq \int_B W(x)u^2 dx,$$

$$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B \Big(\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|}\Big)|\nabla u|^2 dx.$$

This characterization makes a very useful connection between Hardy-type inequalities and the oscillatory behavior of certain ordinary differential equations. This allows us to improve, extend, and unify many results about Hardy and Hardy-Rellich type inequalities. In section 4, with a similar approach, we present various classes of Hardy-Rellich inequalities on $H^2 \cap H_0^1$.

The second part of the thesis studies the regularity of the extremal solution of fourth order semilinear equations. In sections 5 and 6 we study the extremal solution u_{λ^*} of the semilinear biharmonic equation $\Delta^2 u = \frac{\lambda}{(1-u)^2}$, which models a simple Micro-Electromechanical System (MEMS) device on a ball $B \subset R^N$, under Dirichlet or Navier boundary conditions. We show that u^* is regular provided $N \leq 8$ while u_{λ^*} is singular for $N \geq 9$. In section 7, by a rigorous mathematical proof, we show that the extremal solutions of the bilaplacian with exponential nonlinearity is singular in dimensions $N \geq 13$.

In the third part, motivated by the theory of self-duality we propose new templates for solving non-symmetric linear systems. Our approach is efficient when dealing with certain ill-conditioned and highly non-symmetric systems.

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Dedication

This thesis is dedicated to my wonderful parents, Saed and Iran, who have raised me to be the person I am today. It is also dedicated to my sister, Aram, who has supported me in all of my life. You have been with me every step of the way, through good times and bad. Thank you for everything.

Co-authorship Statement

- Chapters 2, 3, and 8 were jointly authored by Nassif Ghoussoub and Amir Moradifam.
- Chapter 5 was jointly authored by Craig Cowan, Pierpaolo Esposito, Nassif Ghoussoub and Amir Moradifam.

In all of the joint papers of this thesis, all authors contributed equally to the identification and design of the research problem, performing the research, data analysis, and manuscript preparation.

Chapter 1

Introduction

The main focus of this thesis is improved Hardy and Hardy-Rellich inequalities, fourth order nonlinear eigenvalue problems, and the preconditioning issue of non-symmetric sparse linear systems. In the first part of the thesis we prove various classes of improved Hardy and Hardy-Rellich inequalities. In the second part we study the regularity of the extremal solutions of nonlinear fourth order eigenvalue problems. Although the first two parts do not seem related at the first glance, we shall see in the second part that our improved Hardy-Rellich inequalities are crucial to show the singular nature of the extremal solutions in large dimensions close to the critical dimension. In the last part, motivated by the theory of self-duality, we propose new templates for solving large non-symmetric linear systems. The method consists of combining a new scheme that simultaneously preconditions and symmetrizes the problem, with various well known iterative methods for solving linear and symmetric problems.

1.1 Optimal improved Hardy and Hardy-Rellich inequalities

Let Ω be smooth bounded domain in R^n and $0 \in \Omega$. Hardy and Hardy-Rellich inequalities assert that

$$\int_\Omega |\nabla u|^2 dx \geq (\tfrac{n-2}{2})^2 \int_\Omega \tfrac{|u|^2}{|x|^2} dx, \tag{1.1}$$

for all $u \in H_0^1(\Omega)$ and $n \geq 3$, and

$$\int_\Omega |\Delta u|^2 dx \geq \tfrac{n^2(n-4)^2}{16} \int_\Omega \tfrac{u^2}{|x|^4} dx, \tag{1.2}$$

for $u \in H_0^2(\Omega)$ and $n \geq 5$, respectively. These inequalities and their various improvements are used in many contexts, such as in the study of the stability of solutions of semi-linear elliptic and parabolic equations, in the analysis of the asymptotic behavior of the heat equation with singular potentials, as well as in the study of the stability of eigenvalues in elliptic problems such as Schrödinger operators. It is well known that the constants appearing in the above inequalities are the best constants and they never achieved. So, one could anticipate improving these inequalities. Indeed, ever since Brézis-Vazquez [4] showed that Hardy's inequality can be improved once restricted to a smooth bounded domain Ω in R^n, there was a flurry of activity about possible improvements of the following type:

$$\text{If } n \geq 3 \quad \text{then} \quad \int_\Omega |\nabla u|^2 dx - (\tfrac{n-2}{2})^2 \int_\Omega \tfrac{|u|^2}{|x|^2} dx \geq \int_\Omega V(x)|u|^2 dx, \tag{1.3}$$

for all $u \in H_0^1(\Omega)$, as well as its fourth order counterpart

$$\text{If } n \geq 5 \quad \text{then} \quad \int_\Omega |\Delta u|^2 dx - \tfrac{n^2(n-4)^2}{16} \int_\Omega \tfrac{u^2}{|x|^4} dx \geq \int_\Omega W(x) u^2 dx \tag{1.4}$$

for $u \in H_0^2(\Omega)$, where V, W are certain explicit radially symmetric potentials of order lower than $\frac{1}{r^2}$ (for V) and $\frac{1}{r^4}$ (for W) (see [1],[3], [4], [8], [12], [19]).

In chapter 2 and 3, we provide an approach that completes, simplifies and improves most related results to-date regarding the Laplacian on Euclidean space as well as its powers. We also establish new inequalities some of which cover critical dimensions such as $n = 2$ for inequality (1.3) and $n = 4$ for (1.4).

We start by giving necessary and sufficient conditions on positive radial functions V and W on a ball B in R^n, so that the following inequality holds for some $c > 0$:

$$\int_B V(x)|\nabla u|^2 dx \geq c \int_B W(x) u^2 dx \text{ for all } u \in C_0^\infty(B). \tag{1.5}$$

Assuming that the ball B has radius R and that $\int_0^R \frac{1}{r^{n-1}V(r)} dr = +\infty$, the condition is simply that the ordinary differential equation

$$(\mathrm{B}_{V,cW}) \qquad y''(r) + (\tfrac{n-1}{r} + \tfrac{V_r(r)}{V(r)}) y'(r) + \tfrac{cW(r)}{V(r)} y(r) = 0$$

has a positive solution on the interval $(0, R)$. We shall call such a couple (V, W) a *Bessel pair on* $(0, R)$. The *weight* of such a pair is then defined as

$$\beta(V, W; R) = \sup \big\{ c;\ (\mathrm{B}_{V,cW}) \text{ has a positive solution on } (0, R) \big\}. \tag{1.6}$$

This characterization makes an important connection between Hardy-type inequalities and the oscillatory behavior of the above equations. For example, by using recent results on ordinary differential equations, we can then infer that an integral condition on V, W of the form

$$\limsup_{r \to 0} r^{2(n-1)} V(r) W(r) \Big(\int_r^R \frac{d\tau}{\tau^{n-1} V(\tau)} \Big)^2 < \frac{1}{4} \tag{1.7}$$

is sufficient (and "almost necessary") for (V, W) to be a Bessel pair on a ball of sufficiently small radius ρ.

Applied in particular, to a pair $(V, \frac{1}{r^2}V)$ where the function $\frac{rV'(r)}{V(r)}$ is assumed to decrease to $-\lambda$ on $(0, R)$, we obtain the following extension of Hardy's inequality: If $\lambda \leq n-2$, then

$$\int_B V(x)|\nabla u|^2 dx \geq (\tfrac{n-\lambda-2}{2})^2 \int_B V(x)\tfrac{u^2}{|x|^2} dx \quad \text{for all } u \in C_0^\infty(B) \tag{1.8}$$

and $(\frac{n-\lambda-2}{2})^2$ is the best constant. The case where $V(x) \equiv 1$ is obviously the classical Hardy inequality and when $V(x) = |x|^{-2a}$ for $-\infty < a < \frac{n-2}{2}$, this is a particular case of the Caffarelli-Kohn-Nirenberg inequality. One can however apply the above criterion to obtain new inequalities such as the following: For $a, b > 0$

- If $\alpha\beta > 0$ and $m \leq \frac{n-2}{2}$, then for all $u \in C_0^\infty(R^n)$

$$\int_{R^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}}|\nabla u|^2 dx \geq (\frac{n-2m-2}{2})^2 \int_{R^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 dx, \tag{1.9}$$

  and $(\frac{n-2m-2}{2})^2$ is the best constant in the inequality.

- If $\alpha\beta < 0$ and $2m - \alpha\beta \leq n-2$, then for all $u \in C_0^\infty(R^n)$

$$\int_{R^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}}|\nabla u|^2 dx \geq (\frac{n-2m+\alpha\beta-2}{2})^2 \int_{R^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 dx, \tag{1.10}$$

  and $(\frac{n-2m+\alpha\beta-2}{2})^2$ is the best constant in the inequality.

We can also extend some of the recent results of Blanchet-Bonforte-Dolbeault-Grillo-Vasquez [3].

- If $\alpha\beta < 0$ and $-\alpha\beta \leq n-2$, then for all $u \in C_0^\infty(R^n)$

$$\int_{R^n} (a+b|x|^\alpha)^\beta |\nabla u|^2 dx \geq b^{\frac{2}{\alpha}} (\frac{n-\alpha\beta-2}{2})^2 \int_{R^n} (a+b|x|^\alpha)^{\beta-\frac{2}{\alpha}} u^2 dx, \tag{1.11}$$

  and $b^{\frac{2}{\alpha}}(\frac{n-\alpha\beta-2}{2})^2$ is the best constant in the inequality.

- If $\alpha\beta > 0$, and $n \geq 2$, then there exists a constant $C > 0$ such that for all $u \in C_0^\infty(R^n)$

$$\int_{R^n} (a+b|x|^\alpha)^\beta |\nabla u|^2 dx \geq C \int_{R^n} (a+b|x|^\alpha)^{\beta-\frac{2}{\alpha}} u^2 dx. \tag{1.12}$$

  Moreover, $b^{\frac{2}{\alpha}}(\frac{n-2}{2})^2 \leq C \leq b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$.

On the other hand, by considering the pair

$$V(x) = |x|^{-2a} \quad \text{and} \quad W_{a,c}(x) = (\tfrac{n-2a-2}{2})^2|x|^{-2a-2} + c|x|^{-2a}W(x)$$

we get the following improvement of the Caffarelli-Kohn-Nirenberg inequalities:

$$\int_B |x|^{-2a}|\nabla u|^2 dx - (\frac{n-2a-2}{2})^2 \int_B |x|^{-2a-2}u^2 dx \geq c\int_B |x|^{-2a}W(x)u^2 dx, \tag{1.13}$$

for all $u \in C_0^\infty(B)$, if and only if the following ODE

$$(\mathrm{B}_{cW}) \qquad y'' + \tfrac{1}{r}y' + cW(r)y = 0$$

has a positive solution on $(0, R)$. Such a function W will be called a *Bessel potential* on $(0, R)$.

More importantly, we establish that Bessel pairs lead to a myriad of optimal Hardy-Rellich inequalities of arbitrary high order, therefore extending and completing a series of new results by Adimurthi et al. [2], Tertikas-Zographopoulos [19] and others. They are mostly based on the following theorem.

Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball centered at zero with radius R in R^n $(n \geq 1)$ such that $\int_0^R \frac{1}{r^{n-1}V(r)}dr = +\infty$ and $\int_0^R r^{n-1}V(r)dr < +\infty$. The following statements are then equivalent:

1. (V, W) is a Bessel pair on $(0, R)$ and $\beta(V, W; R) \geq 1$.
2. $\int_B V(x)|\nabla u|^2 dx \geq \int_B W(x)u^2 dx$ for all $u \in C_0^\infty(B)$.
3. If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < n-2$, then the above are equivalent to

   $$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|})|\nabla u|^2 dx,$$

   for all radial $u \in C_{0,r}^\infty(B)$.
4. If in addition, $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ on $(0, R)$, then the above are equivalent to

   $$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|})|\nabla u|^2 dx,$$

   for all $u \in C_0^\infty(B)$.

In other words, one can obtain as many Hardy and Hardy-Rellich type inequalities as one can construct Bessel pairs on $(0, R)$. The relevance of the above result stems from the fact that there are plenty of such pairs that are easily identifiable. Indeed, even the class of *Bessel potentials* – equivalently those W such that $\big(1, (\frac{n-2}{2})^2|x|^{-2} + cW(x)\big)$ is a Bessel pair– is quite rich and contains several important potentials. Here are some of the most relevant properties of the class of C^1 Bessel potentials W on $(0, R)$, that we shall denote by $\mathcal{B}(0, R)$.

First, the class is a closed convex *solid* subset of $C^1(0, R)$, that is if $W \in \mathcal{B}(0, R)$ and $0 \leq V \leq W$, then $V \in \mathcal{B}(0, R)$. The "weight" of each $W \in \mathcal{B}(R)$, that is

$$\beta(W; R) = \sup\big\{c > 0;\ (B_{cW}) \text{ has a positive solution on } (0, R)\big\}, \tag{1.14}$$

will be an important ingredient for computing the best constants in corresponding functional inequalities. Here are some basic examples of Bessel potentials and their corresponding weights.

- $W \equiv 0$ is a Bessel potential on $(0, R)$ for any $R > 0$.
- $W \equiv 1$ is a Bessel potential on $(0, R)$ for any $R > 0$, and $\beta(1; R) = \frac{z_0^2}{R^2}$ where $z_0 = 2.4048...$ is the first zero of the Bessel function J_0.
- If $a < 2$, then there exists $R_a > 0$ such that $W(r) = r^{-a}$ is a Bessel potential on $(0, R_a)$.
- For $k \geq 1$, $R > 0$ and $\rho = R(e^{e^{e^{\cdot^{\cdot^{e((k-1)-times)}}}}})$, let

  $$W_{k,\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2}\Big(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{r}\Big)^{-2},$$

  where the functions $log^{(i)}$ are defined iteratively as follows: $log^{(1)}(.) = log(.)$ and for $k \geq 2$, $log^{(k)}(.) = log(log^{(k-1)}(.))$. $W_{k,\rho}$ is then a Bessel potential on $(0, R)$ with $\beta(W_{k,\rho}; R) = \frac{1}{4}$.
- For $k \geq 1$, $R > 0$ and $\rho \geq R$, define

  $$\tilde{W}_{k;\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2} X_1^2(\frac{r}{\rho})X_2^2(\frac{r}{\rho})\ldots X_{j-1}^2(\frac{r}{\rho})X_j^2(\frac{r}{\rho}),$$

  where the functions X_i are defined iteratively as follows: $X_1(t) = (1 - \log(t))^{-1}$ and for $k \geq 2$, $X_k(t) = X_1(X_{k-1}(t))$. Then again $\tilde{W}_{k,\rho}$ is a Bessel potential on $(0, R)$ with $\beta(\tilde{W}_{k,\rho}; R) = \frac{1}{4}$.

- More generally, if W is any positive function on R such that

$$\liminf_{r\to 0} \ln(r)\int_0^r sW(s)ds > -\infty,$$

then for every $R>0$, there exists $\alpha := \alpha(R) > 0$ such that $W_\alpha(x) := \alpha^2 W(\alpha x)$ is a Bessel potential on $(0,R)$.

What is remarkable is that the class of Bessel potentials W is also the one that leads to optimal improvements for fourth order inequalities (in dimension $n \geq 3$) of the following type:

$$\int_B |\Delta u|^2 dx - C(n)\int_B \frac{|\nabla u|^2}{|x|^2} dx \geq c(W,R)\int_B W(x)|\nabla u|^2 dx \ , \qquad (1.15)$$

for all $u \in H_0^2(B)$, where $C(3) = \frac{25}{36}$, $C(4) = 3$ and $C(n) = \frac{n^2}{4}$ for $n \geq 5$. The case when $W \equiv \tilde{W}_{k,\rho}$ and $n \geq 5$ was recently established by Tertikas-Zographopoulos [19]. Note that W can be chosen to be any one of the examples of Bessel potentials listed above. Moreover, both $C(n)$ and the weight $\beta(W;R)$ are the best constants in the above inequality.

Appropriate combinations of (1.5) and (1.15) then lead to a myriad of Hardy-Rellich inequalities in dimension $n \geq 4$. For example, if W is a Bessel potential on $(0,R)$ such that the function $r\frac{W_r(r)}{W(r)}$ decreases to $-\lambda$, and if $\lambda \leq n-2$, then we have for all $u \in C_0^\infty(B_R)$

$$\begin{aligned}\int_B |\Delta u|^2 dx \quad &- \quad \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx \qquad (1.16)\\ &\geq \quad \left(\frac{n^2}{4} + \frac{(n-\lambda-2)^2}{4}\right)\beta(W;R)\int_B \frac{W(x)}{|x|^2}u^2 dx.\end{aligned}$$

By applying (1.16) to the various examples of Bessel functions listed above, one improves in many ways the recent results of Adimurthi et al. [2] and those by Tertikas-Zographopoulos [19]. Moreover, besides covering the critical dimension $n=4$, we also establish that the best constant is $(1+\frac{n(n-4)}{8})$ for all the potentials W_k and $\tilde{W}_k$ defined above. For example we have for $n \geq 4$,

$$\begin{aligned}\int_B |\Delta u(x)|^2 dx \quad &\geq \quad \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx \qquad (1.17)\\ &+ \quad \left(1+\frac{n(n-4)}{8}\right)\sum_{j=1}^k \int_B \frac{u^2}{|x|^4}\left(\prod_{i=1}^j log^{(i)}\frac{\rho}{|x|}\right)^{-2} dx.\end{aligned}$$

More generally, we show that for any $m < \frac{n-2}{2}$, and any W Bessel potential on a ball $B_R \subset R^n$ of radius R, the following inequality holds for all $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W;R) \int_{B_R} W(x) \frac{|\nabla u|^2}{|x|^{2m}} dx, \quad (1.18)$$

where $a_{m,n}$ and $\beta(W;R)$ are best constants that we compute in chapter 3 for all m and n and for many Bessel potentials W.

We also establish a more general version of equation (1.16). Assuming again that $\frac{rW'(r)}{W(r)}$ decreases to $-\lambda$ on $(0,R)$, and provided $m \leq \frac{n-4}{2}$ and $\lambda \leq n-2m-2$, we then have for all $u \in C_0^\infty(B_R)$,

$$\begin{aligned} &\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx \geq \beta_{n,m} \int_{B_R} \frac{u^2}{|x|^{2m+4}} dx \quad (1.19)\\ + \quad &\beta(W;R)(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4}) \int_{B_R} \frac{W(x)}{|x|^{2m+2}} u^2 dx, \end{aligned}$$

where again the best constants $\beta_{n,m}$ are computed in chapter 3. This completes the results in Theorem 1.6 of [19], where the inequality is established for $n \geq 5$, $0 \leq m < \frac{n-4}{2}$, and the particular potential $\tilde{W}_{k,\rho}$.

Another inequality that relates the Hessian integral to the Dirichlet energy is the following: Assuming $-1 < m \leq \frac{n-4}{2}$ and W is a Bessel potential on a ball B of radius R in R^n, then for all $u \in C_0^\infty(B)$,

$$\begin{aligned} &\int_B \frac{|\Delta u|^2}{|x|^{2m}} dx - \frac{(n+2m)^2(n-2m-4)^2}{16} \int_B \frac{u^2}{|x|^{2m+4}} dx \geq \quad (1.20)\\ &\beta(W;R) \frac{(n+2m)^2}{4} \int_B \frac{W(x)}{|x|^{2m+2}} u^2 dx + \beta(|x|^{2m};R) ||u||_{H_0^1}. \end{aligned}$$

This improves considerably Theorem A.2. in [2] where it is established – for $m = 0$ and without best constants – with the potential $W_{1,\rho}$ in dimension $n \geq 5$, and the potential $W_{2,\rho}$ when $n = 4$.

Finally, we establish several higher order Rellich inequalities for integrals of the form $\int_{B_R} \frac{|\Delta^m u|^2}{|x|^{2k}} dx$, improving in many ways several recent results in [19].

Hardy-Rellich inequalities on $H^2 \cap H_0^1$ are important in the study of fourth order elliptic equations with Navier boundary condition and systems of second order elliptic equations. In [16], I developed a general approach to prove optimal weighted Hardy-Rellich inequalities on $H^2 \cap H_0^1$ which leads to various new Hardy-Rellich inequalities.

The approach developed in [11], [12], and [18] basically finishes the problem of improving Hardy and Hardy-Rellich inequalities in R^n.

1.2 Fourth order nonlinear eigenvalue problems

Consider the fourth order elliptic problem

$$\beta\Delta^2 u - \tau\Delta u = \lambda f(u) \quad \text{in } \Omega, \ . \qquad (G_\lambda)$$

with either Dirichlet boundary condition $u = \partial_\nu u = 0$ or Navier boundary condition $u = \Delta u = 0$ on $\partial\Omega$. Here $\lambda > 0$ is a parameter, $\tau > 0$, $\beta > 0$ are fixed constants, and $\Omega \subset R^N$ $(N \geq 2)$ is a bounded smooth domain. The case $\beta = 0$, $\tau = 1$, and $f(u) = e^u$ is the well known Gelfand problem. Under some technical assumptions on f one can show that there exists $\lambda^* > 0$ such that for every $0 < \lambda < \lambda^*$, there exists a smooth minimal (smallest) solution of $(S)_{\lambda,f}$, while for $\lambda > \lambda^*$ there is no solution even in a weak sense. Moreover, the branch $\lambda \mapsto u_\lambda(x)$ is increasing for each $x \in \Omega$, and therefore the function $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ can be considered as a generalized solution that corresponds to the pull-in voltage λ^*. Now the issue of the regularity of this extremal solution is an important question for many reasons, not the least of which being the fact that it decides whether the set of solutions stops there, or whether a new branch of solutions emanates from a bifurcation state (u^*, λ^*). One of the main obstacles for this problem is the well-known difficulty of extracting energy estimates for solutions of fourth order problems from their stability properties which means that the methods used to analyze the regularity of the extremal solution of the second order problem could not carry to the corresponding fourth order problem. In the second part of this thesis I study the above problem with various nonlinearities f with both Dirichlet and Navier boundary conditions. Consider the fourth order elliptic problem

$$\begin{cases} \beta\Delta^2 u - \tau\Delta u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ 0 < u \leq 1 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (G_\lambda)$$

where $\lambda > 0$ is a parameter, $\tau > 0$, $\beta > 0$ are fixed constants, and $\Omega \subset R^N$ $(N \geq 2)$ is a bounded smooth domain. This equation is derived in the study of the charged plates in electrostatic actuators.

In chapter 4, we study the regularity of the extremal solution of the above equation.

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } B \\ 0 < u < 1 & \text{in } B \\ u = \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \qquad (P)_\lambda$$

where B is the unit ball in R^N. This problem models a simple electrostatic Micro-Electromechanical Systems (MEMS) device. We proved that the extremal solution u_{λ^*} is regular ($\sup_B u_{\lambda^*} < 1$) provided $N \le 8$ while u_{λ^*} is singular ($\sup_B u_{\lambda^*} = 1$) for $N \ge 9$, in which case $1 - C_0|x|^{4/3} \le u_{\lambda^*}(x) \le 1 - |x|^{4/3}$ on the unit ball, where $C_0 := \left(\frac{\lambda^*}{\bar{\lambda}}\right)^{\frac{1}{3}}$ and $\bar{\lambda} := \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3})$. This completes the results of F.H. Lin and Y.S. Yang [15].

In chapter 5, we study the problem (G_λ) on the unit ball in R^N and showed that the critical dimension for (P_λ) is $N = 9$. Indeed I proved that the extremal solution of (P_λ) is regular ($\sup_B u^* < 1$) for $N \le 8$ and $\beta, \tau > 0$ and it is singular ($\sup_B u^* = 1$) for $N \ge 9$, $\beta > 0$, and $\tau > 0$ with $\frac{\tau}{\beta}$ small.

Our proof for the regularity of the extremal solutions is based on a blow up analysis and certain energy estimates which was recently introduced by Dávila et al. [7]. To show the singularity of the exterimal solutions in dimensions $N \ge 9$ we use various improved Hardy-Rellich inequalities that are consequences of our main results in the first part of the thesis.

Recently Dávila et al. [7] studied the fourth order counter part of the Gelfand problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases} \tag{1.21}$$

They developed a new method to prove the regularity of the extremal solutions in low dimensions and showed that for $N \le 12$, u^* is regular. They used a computer assisted proof to show that the extremal solution is singular in dimensions $13 \le N \le 31$ while they gave a mathematical proof in dimensions $N \ge 32$. In [17], I introduced a unified mathematical approach to deal with this problem and showed that for $N \ge 13$, the extremal solution is singular. The following lemma plays an essential role in our proof.

Lemma 1.22. *Suppose there exist $\lambda' > 0$ and a radial function $u \in H^2(B) \cap W^{4,\infty}_{loc}(B \setminus \{0\})$ such that*

$$\Delta^2 u \le \lambda' e^u \quad \textit{for all} \;\; 0 < r < 1, \tag{1.23}$$

$$u(1) = 0, \qquad \frac{\partial u}{\partial n}(1) = 0, \tag{1.24}$$

$$u \notin L^\infty(B), \tag{1.25}$$

and

$$\beta \int_B e^u \varphi^2 \leq \int_B (\Delta\varphi)^2 \quad \textit{for all} \quad \varphi \in C_0^\infty(B), \tag{1.26}$$

for some $\beta > \lambda'$. Then u^* is singular and

$$\lambda^* \leq \lambda' \tag{1.27}$$

The proof is based on the above lemma combined with certain improved Hardy-Rellich inequalities obtained in chapter 3.

1.3 Preconditioning and symmetrization of non-symmetric linear systems

Many problems in scientific computing lead to systems of linear equations of the form,

$$Ax = b, \tag{1.28}$$

where $A \in R^{n\times n}$ is a nonsingular but sparse matrix, and b is a given vector in R^n, and various iterative methods have been developed for a fast and efficient resolution of such systems. In [13], motivated by the theory of self-duality ([10]) we propose new templates for solving non-symmetric linear systems. Our approach consists of symmetrizing the problem so as to be able to apply CG, MINRES, or SYMMLQ. We argue that for a large class of non-symmetric, ill-conditionned matrices, it is beneficial to replace problem (8.1) by one of the form

$$A^T M A x = A^T M b, \tag{1.29}$$

where M is a symmetric and positive definite matrix that can be chosen properly so as to obtain good convergence behavior for CG when it is applied to the resulting symmetric $A^T MA$. This reformulation should not only be seen as a symmetrization, but also as preconditioning procedure. While it is difficult to obtain general conditions on M that ensure higher efficiency of our approach, we show theoretically and numerically that by choosing M to be either the inverse of the symmetric part of A, or its resolvent, one can get surprisingly good numerical schemes to solve (8.1).

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Part I

Hardy and Hardy-Rellich Inequalities

Chapter 2

On the best possible remaining term in the Hardy inequality [1]

2.1 Introduction

Let Ω be a bounded domain in R^n, $n \geq 3$, with $0 \in \Omega$. The classical Hardy inequality asserts that

$$\int_\Omega |\nabla u|^2 dx \geq (\tfrac{n-2}{2})^2 \int_\Omega \tfrac{|u|^2}{|x|^2} dx \quad \text{for all } u \in H_0^1(\Omega). \tag{2.1}$$

This inequality and its various improvements are used in many contexts, such as in the study of the stability of solutions of semi-linear elliptic and parabolic equations [6, 7, 21], in the analysis of the asymptotic behavior of the heat equation with singular potentials [8, 22], as well as in the study of the stability of eigenvalues in elliptic problems such as Schrödinger operators [10, 12].

Now it is well known that $(\frac{n-2}{2})^2$ is the best constant for inequality (2.1), and that this constant is however not attained in $H_0^1(\Omega)$. So, one could anticipate improving this inequality by adding a non-negative correction term to the right hand side of (2.1) and indeed, several sharpened Hardy inequalities have been established in recent years [4, 5, 11, 12, 22], mostly triggered by the following improvement of Brezis and Vázquez [6].

$$\int_\Omega |\nabla u|^2 dx \geq (\tfrac{n-2}{2})^2 \int_\Omega \tfrac{|u|^2}{|x|^2} dx + \lambda_\Omega \int_\Omega |u|^2 dx \quad \text{for every } u \in H_0^1(\Omega). \tag{2.2}$$

The constant λ_Ω in (2.2) is given by

$$\lambda_\Omega = z_0^2 \omega_n^{2/n} |\Omega|^{-\frac{2}{n}}, \tag{2.3}$$

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where ω_n and $|\Omega|$ denote the volume of the unit ball and Ω respectively, and z_0 is the first zero of the bessel function $J_0(z)$. Moreover, λ_Ω is optimal when Ω is a ball, but is –again– not achieved in $H_0^1(\Omega)$. This led to one of the open problems mentioned in [6] (Problem 2), which is whether the two terms on the RHS of inequality (2.2) (i.e., the coefficients of $|u|^2$) are just the first two terms of an infinite series of correcting terms.

This question was addressed by several authors. In particular, Adimurthi et all [1] proved that for every integer k, there exists a constant c depending on n, k and Ω such that

$$\int_\Omega |\nabla u|^2 dx \geq (\tfrac{n-2}{2})^2 \int_\Omega \tfrac{|u|^2}{|x|^2} dx + c\sum_{j=1}^k \int_\Omega \tfrac{|u|^2}{|x|^2} \big(\prod_{i=1}^j log^{(i)} \tfrac{\rho}{|x|}\big)^{-2} dx, \quad (2.4)$$

for $u \in H_0^1(\Omega)$, where $\rho = (\sup_{x\in\Omega}|x|)(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$. Here we have used the notations $log^{(1)}(.) = log(.)$ and $log^{(k)}(.) = log(log^{(k-1)}(.))$ for $k \geq 2$.

Also motivated by the question of Brezis and Vázquez, Filippas and Tertikas proved in [11] that the inequality can be repeatedly improved by adding to the right hand side specific potentials which lead to an infinite series expansion of Hardy's inequality. More precisely, by defining iteratively the following functions,

$$X_1(t) = (1 - \log(t))^{-1}, \quad X_k(t) = X_1(X_{k-1}(t)) \quad k = 2, 3, ...,$$

they prove that for any $D \geq \sup_{x\in\Omega}|x|$, the following inequality holds for any $u \in H_0^1(\Omega)$:

$$\begin{aligned} \int_\Omega |\nabla u|^2 dx &\geq (\frac{n-2}{2})^2 \int_\Omega \frac{|u|^2}{|x|^2} dx \\ &+ \frac{1}{4}\sum_{i=1}^\infty \int_\Omega \frac{1}{|x|^2} X_1^2(\frac{|x|}{D}) X_2^2(\frac{|x|}{D}) ... X_i^2(\frac{|x|}{D}) |u|^2 dx. \end{aligned} \quad (2.5)$$

Moreover, they proved that the constant $\frac{1}{4}$ is the best constant for the corresponding $k-$improved Hardy inequality which is again not attained in $H_0^1(\Omega)$.

In this paper, we show that all the above results –and more– follow from a specific characterization of those potentials V that yield an improved Hardy inequality. Here is our main result.

Let V be a radial function on a smooth bounded radial domain Ω in ^{n} with radius R, in such a way that $V(x) = v(|x|)$ for some non-negative function v on $(0, R)$. The following properties are then equivalent:

1. The ordinary differential equation

   (D$_V$) $$y''(r) + \frac{y'(r)}{r} + v(r)y(r) = 0$$

   has a positive solution on the interval $(0, R)$.

2. The following improved Hardy inequality holds

   (H$_V$) $$\int_\Omega |\nabla u|^2 dx - (\frac{n-2}{2})^2 \int_\Omega \frac{|u|^2}{|x|^2} dx \geq \int_\Omega V(|x|)|u|^2 dx,$$

   for $u \in H_0^1(\Omega)$.

Moreover, the best constant $c(V) := \sup\{c;\ (\mathrm{H}_{cV}) \text{ holds}\}$ is the largest c so that $y''(r) + \frac{y'(r)}{r} + cv(r)y(r) = 0$ has a positive solution on the interval $(0, R)$.

We note that the implication 1) implies 2) holds for any smooth bounded domain Ω in ^{n} containing 0, provided $v(r) + (\frac{n-2}{2})^2 \frac{1}{r^2}$ is non-increasing on $(0, \sup_{x\in\Omega} |x|)$ and R is the radius of the ball which has the same volume as Ω (i.e. $R = (\frac{|\Omega|}{\omega_n})^{\frac{1}{n}}$).

It is therefore clear from the above discussion that in order to find what potentials are candidates for an improved Hardy inequality, one needs to investigate the ordinary differential equation $y'' + \frac{y'}{r} + v(r)y(r) = 0$. We shall see that the results of Brezis-Vázquez, Adimurthi et al, and Filippas-Tertikas mentioned above can be easily deduced by simply checking that the potentials V they consider, correspond to equations (D$_V$) where an explicit positive solution can be found.

Our approach turned out to be also useful for determining the best constants in the above mentioned improvements. Indeed, the case when $V \equiv 1$ will follow immediately from Theorem 2.1. A slightly more involved reasoning – but also based of the above characterization – will allow us to find the best constant in the improvement of Adimurthi et al, and to recover the best one established by Filippas-Tertikas.

Since the existence of positive solutions for ODEs of the form (D$_V$) is closely related to the oscillatory properties of second order equations of the form $z''(s)+a(s)z(s) = 0$, Theorem 2.1 also allows for the use of the extensive literature on the oscillatory properties of such equations to deduce various interesting results such as the following corollary.

Let V be a positive radial function on a smooth bounded radial domain Ω in R^n.

1. If $\liminf_{r\to 0} \ln(r) \int_0^r sV(s)ds > -\infty$, then there exists $\alpha := \alpha(\Omega) > 0$ such that an improved Hardy inequality (H$_{V_\alpha}$) holds for the scaled potential $V_\alpha(x) := \alpha^2 V(\alpha x)$.

2. If $\lim_{r\to 0}\ln(r)\int_0^r sV(s)ds = -\infty$, then there are no $\beta, c > 0$, for which $(\mathrm{H}_{V_{\beta,c}})$ holds with $V_{\beta,c} = cV(\beta x)$.

The following is a consequence of the two results above.

For any $\alpha < 2$, inequality (H_{cV}) holds on a bounded domain Ω for $V_\alpha(x) = \frac{1}{|x|^\alpha}$ and some $c > 0$. Moreover, the best constant $c(\frac{1}{|x|^\alpha})$ is equal to the largest c such that the equation

$$y''(r) + \frac{1}{r}y'(r) + c\frac{1}{|x|^\alpha} = 0,$$

has a positive solution on $(0, R)$, where R is the radius of the ball wich has the same volume as Ω. Moreover, if $\alpha \geq 2$ inequality (H_V) does not hold for $V_{\alpha,c}(x) = c\frac{1}{|x|^\alpha}$ for any $c > 0$. Note that the above corollary gives another proof of the fact that $(\frac{n-2}{2})^2$ is the best constant for the classical Hardy inequality.

Define now the class

$$\begin{aligned} A_\Omega = \{v : R \to R^+; \text{v is non-increasing on}(0, \sup_{x\in\Omega}|x|),\\ (D_v) \quad \text{has a positive solution on}(0, (\frac{|\Omega|}{\omega_n})^{\frac{1}{n}})\}. \end{aligned}$$

An immediate application of Theorem 2.1 coupled with Hölder's inequality gives the following duality statement, which should be compared to inequalities dual to those of Sobolev, recently obtained via the theory of mass transport [2, 9].

Suppose that Ω is a smooth bounded domain in R^n containing 0. Then for any $0 < p \leq 2$, we have

$$\begin{aligned} &\inf\left\{\int_\Omega |\nabla u|^2 dx - (\frac{n-2}{2})^2\int_\Omega \frac{|u|^2}{|x|^2}dx;\ u \in H_0^1(\Omega), ||u||_p = 1\right\}\\ \geq\ &\sup\left\{\frac{1}{||V^{-1}(|x|)||_{L^{\frac{p}{p-2}}(\Omega)}.};\ V \in A_\Omega\right\}. \end{aligned} \tag{2.6}$$

Finally, consider the following classes of radial potentials:

$$X = \{V : \Omega \to^+;\ V \in L^\infty_{\text{loc}}(\Omega\setminus\{0\}),\ \ \liminf_{r\to 0}\ln(r)\int_0^r sV(s)ds > -\infty\}, \tag{2.7}$$

and

$$Y = \{V : \Omega \to^{+}; \; V \in L^{\infty}_{\text{loc}}(\Omega \setminus \{0\}), \;\; \lim_{r\to 0} \ln(r) \int_0^r sV(s)ds = -\infty\}. \quad (2.8)$$

For any $0 < \mu < \mu_n := \frac{(n-2)}{2})^2$ we consider the following weighted eigenvalue problem,

$$(E_{V,\mu}) \qquad \begin{cases} -\Delta u - \frac{\mu}{|x|^2}u &= \lambda V u \;\; \text{in} \;\; \Omega, \\ u &= 0 \quad \text{on} \;\; \Omega. \end{cases} \quad (2.9)$$

Our results above combine with standard arguments to yield the following. For any $0 < \mu < \mu_n$, and $V : \Omega \to^{+}$ with $V \in L^{\infty}_{\text{loc}}(\Omega \setminus \{0\})$ and $\lim_{|x|\to 0} |x|^2 V(x) = 0$, the problem $(E_{V,\mu})$ admits a positive weak solution $u_\mu \in H^1_0(\Omega)$ corresponding to the first eigenvalue $\lambda = \lambda^1_\mu(V)$. Moreover, by letting $\lambda_1(V) = \lim_{\mu\uparrow\mu_n} \lambda^1_\mu(V)$, we have

- If $V \in X$, then there exists $c > o$ such that $\lambda_1(V_c) > 0$.
- If $V \in Y$, then $\lambda_1(V_c) = 0$ for all $c > 0$,

where $V_c(x) := V(cx)$.

2.2 Two dimensional inequalities

In this section, we start by establishing the following improvements of "two-dimensional" Poincaré and Poincaré-Wirtinger inequalities.

Let $a < b$, $k > 0$ is a differentiable function on (a, b), and φ be a strictly positive real valued differentiable function on (a, b). Then, every $h \in C^1([a, b])$ with

$$-\infty < \lim_{r\to a} k(r)|h(r)|^2 \frac{\varphi'(r)}{\varphi(r)} = \lim_{r\to b} k(r)|h(r)|^2 \frac{\varphi'(r)}{\varphi(r)} < \infty, \quad (2.10)$$

satisfies the following inequality:

$$\int_a^b |h'(r)|^2 k(r)dr \geq \int_a^b -|h(r)|^2 (\frac{k'(r)\varphi'(r) + k(r)\varphi''(r)}{\varphi(r)})dr. \quad (2.11)$$

Moreover, assuming (2.10), the equality holds if and only if $h(r) = \varphi(r)$ for all $r \in (a, b)$. **Proof.** Define $\psi(r) = h(r)/\varphi(r)$, $r \in [a, b]$. Then

$$\begin{aligned}\int_a^b |h'(r)|^2 k(r)dr &= \int_a^b |\psi(r)|^2 |\varphi'(r)|^2 k(r)dr + \int_a^b 2\varphi(r)\varphi'(r)\psi(r)\psi'(r)k(r)dr \\ &\quad + \int_a^b |\varphi(r)|^2 |\psi'(r)|^2 k(r)dr \\ &= \int_a^b |\psi(r)|^2 |\varphi'(r)|^2 k(r)dr - \int_a^b |\psi(r)|^2 (k\varphi\varphi')'(r)dr \\ &\quad + \int_a^b |\varphi(r)|^2 |\psi'(r)|^2 k(r)dr \\ &= \int_a^b |\psi(r)|^2 (|\varphi'(r)|^2 k(r) - (k\varphi\varphi')'(r)dr \\ &\quad + \int_a^b |\varphi(r)|^2 |\psi'(r)|^2 k(r)dr.\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_a^b |h'(r)|^2 k(r)dr &= \int_a^b -|h(r)|^2 (\frac{k'(r)\varphi'(r) + k(r)\varphi''(r)}{\varphi})dr \\ &\quad + \int_a^b |\varphi(r)|^2 |\psi'(r)|^2 k(r)dr \\ &\geq \int_a^b -|h(r)|^2 (\frac{k'(r)\varphi'(r) + k(r)\varphi''(r)}{\varphi(r)})dr.\end{aligned}$$

Hence (2.11) holds. Note that the last inequality is obviously an idendity if and only if $h(r) = \varphi(r)$ for all $r \in (a, b)$. The proof is complete. $\square$

By applying Theorem 2.2 to the weight $k(r) = r$, we obtain the following generalization of the 2-dimensional Poincaré inequality. (Generalized 2-dimensional Poincaré inequality) Let $0 \leq a < b$ and φ be a strictly positive real valued differentiable function on (a, b). Then every $h \in C^1([a, b])$ with

$$-\infty < \lim_{r \to a} r|h(r)|^2 \frac{\varphi'(r)}{\varphi(r)} = \lim_{r \to b} r|h(r)|^2 \frac{\varphi'(r)}{\varphi(r)} < \infty, \tag{2.12}$$

satisfies the following inequality:

$$\int_a^b |h'(r)|^2 r dr \geq \int_a^b -|h(r)|^2 (\frac{\varphi'(r) + r\varphi''(r)}{\varphi(r)})dr. \tag{2.13}$$

Moreover, under the assumption (2.12) the equality holds if and only if $h(r) = \varphi(r)$ for all $r \in (a,b)$. By applying Theorem 2.2 to the weight $k(r) = 1$, we obtain the following generalization of the 2-dimensional Poincaré-Wirtinger inequality.

(Generalized Poincaré-Wirtinger inequality) Let $a < b$ and φ be a strictly positive real valued differentiable function on (a,b). Then, every $h \in C^1([a,b])$ with

$$-\infty < \lim_{r\to a} |h(r)|^2 \frac{\varphi'(r)}{\varphi(r)} = \lim_{r\to b} |h(r)|^2 \frac{\varphi'(r)}{\varphi(r)} < \infty, \tag{2.14}$$

satisfies the following inequality:

$$\int_a^b |h'(r)|^2 dr \geq \int_a^b -|h(r)|^2 \frac{\varphi''(r)}{\varphi(r)} dr. \tag{2.15}$$

Moreover, under assumption (2.14), the equality holds if and only if $h(r) = \varphi(r)$ for all $r \in (a,b)$.

Remark 2.2.1. *Note that all of inequalities presented in the above theorems hold when we replace the condtions (2.10), (2.12), and (2.14) with the following weaker conditions*

$$\begin{aligned}
\liminf_{r\to b} k(r)|h(r)|^2 \frac{\varphi'(r)}{\varphi(r)} &\geq \limsup_{r\to a} k(r)|h(r)|^2 \frac{\varphi'(r)}{\varphi(r)}, \\
\liminf_{r\to b} r|h(r)|^2 \frac{\varphi'(r)}{\varphi(r)} &\geq \limsup_{r\to a} r|h(r)|^2 \frac{\varphi'(r)}{\varphi(r)}, \\
\liminf_{r\to b} |h(r)|^2 \frac{\varphi'(r)}{\varphi(r)} &\geq \limsup_{r\to a} |h(r)|^2 \frac{\varphi'(r)}{\varphi(r)},
\end{aligned}$$

respectively, provided both sides in the above inequalities are not equal to $-\infty$ or $+\infty$.

2.3 Proof of the main theorem

We start with the sufficient condition of Theorem 2.1 by establishing the following.

Proposition 2.16. *(Improved Hardy Inequality) Let Ω be a bounded smooth domain in R^n with $0 \in \Omega$, and set $R = (|\Omega|/\omega_n)^{1/n}$. Suppose V is a radially symmetric function on Ω and φ is a C^2-function on $(0,R)$ such that*

$$0 \leq V(|x|) \leq -\tfrac{\varphi'(|x|)+r\varphi''(|x|)}{|x|\varphi(|x|)} \text{ for all } x \in \Omega,\ 0 < |x| < R, \tag{2.17}$$

$$\liminf_{r\to 0} r\frac{\varphi'(r)}{\varphi(r)} \geq 0 \quad \textit{and} \quad \limsup_{r\to R} \frac{\varphi'(r)}{\varphi(r)} < \infty, \tag{2.18}$$

$$(\tfrac{n-2}{2})^2\tfrac{1}{|x|^2} + V(|x|) \textit{ is a decreasing function of } |x|. \tag{2.19}$$

Then for any $u \in H_0^1(\Omega)$*, we have*

$$\int_\Omega |\nabla u|^2 dx \geq (\frac{n-2}{2})^2 \int_\Omega \frac{|u|^2}{|x|^2}dx + \int_\Omega V(|x|)|u|^2 dx. \tag{2.20}$$

Moreover, if $\lim_{r\to 0} r\varphi(r)\varphi'(r) = \lim_{r\to R}\varphi(r)\varphi'(r) = 0$*, then equality holds if and only if* u *is a radial function on* Ω *such that* $u(x) = \varphi(|x|)$ *for all* $x \in \Omega$*.*

Proof: We first prove the inequality for smooth radial positive functions on the ball $\Omega = B_R$. For such $u \in C_0^2(B_R)$, we define

$$v(r) = u(r)r^{(n-2)/2}, \quad r = |x|.$$

In view of Corollary 2.2, we can write

$$\begin{aligned}
\int_\Omega |\nabla u(x)|^2 dx \quad &- \quad (\frac{n-2}{2})^2 \int_\Omega \frac{u^2(x)}{|x|^2}dx \\
&= \quad \omega_n \int_0^R |\frac{n-2}{2}r^{-n/2}v(r) - r^{1-n/2}v'(r)|^2 r^{n-1}dr \\
&- \quad (\frac{n-2}{2})^2 \omega_n \int_0^R \frac{v^2(r)}{r}dr \\
&= \quad \omega_n(\frac{n-2}{2})^2 \int_0^R v^2((1-\frac{2v'(r)r}{(n-2)v(r)})^2 - 1)\frac{dr}{r} \\
&= \quad \omega_n \int_0^R (v'(r))^2 r - \omega_n(\frac{n-2}{2})\int_0^R v(r)v'(r)dr \\
&= \quad \omega_n \int_0^R (v'(r))^2 r \\
&\geq \quad \omega_n \int_0^R -v^2(r)(\frac{\varphi'(r)+r\varphi''(r)}{\varphi(r)})dr \\
&= \quad \omega_n \int_0^R -u^2(r)(\frac{\varphi'(r)+r\varphi''(r)}{\varphi(r)})r^{n-2}dr \\
&= \quad -\int_\Omega u^2(x)(\frac{\varphi'(|x|)+|x|\varphi''(|x|)}{|x|\varphi(|x|)})dx.
\end{aligned}$$

Hence, the inequality (2.20) holds for radial smooth positive functions. By density arguments, inequality (2.20) is valid for any $u \in H_0^1$, $u \geq 0$. For

$u \in H_0^1$ which is not positve and general domain Ω, we use symmetrization arguments. Let B_R be a ball having the same volume as Ω with $R = (|\Omega|/\omega_n)^{1/n}$ and let $|u|^*$ be the symmetric decreasing rearrangement of the function $|u|$. Now note that for any $u \in H_0^1(\Omega)$, $|u|^* \in H_0^1(B_R)$ and $|u|^* > 0$. It is well known that the symmetrization does not change the L^p-norm, and that it decreases the Dirichlet energy, while increasing the integrals $\int_\Omega ((\frac{n-2}{2})^2 \frac{1}{|x|^2} + V(|x|)|u|^2 dx$, since the weight $(\frac{n-2}{2})^2 \frac{1}{|x|^2} + V(|x|)$ is a decreasing function of $|x|$. Hence, (2.20) holds for any $u \in H_0^1(\Omega)$.

We shall need the following lemmas.

Lemma 2.21. *Let $x(r)$ be a function in $C^1(0, R]$ that is a solution of*

$$rx'(r) + x^2(r) \leq -F(r), \qquad 0 < r \leq R, \tag{2.22}$$

where F is a nonnegative continuous function. Then

$$\lim_{r \downarrow 0} x(r) = 0. \tag{2.23}$$

Proof: Divide equation (2.22) by r and integrate once. Then we have

$$x(r) \geq \int_r^R \frac{|x(s)|^2}{s} ds + x(1) + \int_r^R \frac{F(s)}{s} ds. \tag{2.24}$$

It follows that $\lim_{r \downarrow 0} x(r)$ exists. In order to prove that this limit is zero, we claim that

$$\int_r^R \frac{x^2(s)}{s} ds < \infty. \tag{2.25}$$

Indeed, otherwise we have $G(r) := \int_r^R \frac{x^2(s)}{s} ds \to \infty$ as $r \to 0$. From (2.22) we have

$$(-rG'(r))^{\frac{1}{2}} \geq G(r) + x(1) + \int_r^R \frac{F(s)}{s} ds.$$

Note that $F \geq 0$, and G goes to infinity as r goes to zero. Thus, for r sufficiently small we have $-rG'(r) \geq \frac{1}{2} G^2(r)$ hence, $(\frac{1}{G(r)})' \geq \frac{1}{2} \ln(r)$, which contradicts the fact that $G(r)$ goes to infinity as r tends to zero. Thus, our claim is true and the limit in (2.23) is indeed zero. □

Lemma 2.26. *If the equation $\phi'' + \frac{\phi'}{r} + v(r)\phi = 0$ has a positive solution on some interval $(0, R)$, then we have necessarily,*

$$\liminf_{r \to 0} r \frac{\varphi'(r)}{\varphi(r)} \geq 0 \quad \text{and} \quad \limsup_{r \to R} \frac{\varphi'(r)}{\varphi(r)} < \infty. \tag{2.27}$$

Proof: Since $\varphi(\delta) \geq 0$ and $\varphi(r) > 0$ for $0 < r < \delta$, it is obvious that φ satisfies the second condition. To obtain the first condition, set $x(r) = r\frac{\varphi'(r)}{\varphi(r)}$. one may verify that $x(r)$ satisfies the ODE:

$$rx'(r) + x^2(r) = -F(r), \quad \text{for } \ 0 < r \leq \delta,$$

where $F(r) = r^2 v(r) \geq 0$. By Lemma 2.21 we conclude that $\lim_{r\downarrow 0} r\frac{\varphi'(r)}{\varphi(r)} = \lim_{r\downarrow 0} x(t) = 0$. □

Lemma 2.28. *Let V be positive radial potential on the ball Ω of radius R in R^n $(n \geq 3)$. Assume that*

$$\int_\Omega \left(|\nabla u|^2 - (\tfrac{n-2}{2})^2 \tfrac{|u|^2}{|x|^2} - V(|x|)|u|^2\right) dx \geq 0 \text{ for all } u \in H_0^1(\Omega).$$

Then there exists a C^2-supersolution to the equation

$$-\Delta u - \left(\frac{n-2}{2}\right)^2 \frac{u}{|x|^2} - V(|x|)u = 0, \quad \text{in } \Omega, \tag{2.29}$$

$$u > 0 \quad \text{in } \Omega \setminus \{0\}, \tag{2.30}$$

$$u = 0 \quad \text{in } \partial\Omega. \tag{2.31}$$

Proof: Define

$$\lambda_1(V) := \inf\{\frac{\int_\Omega |\nabla\psi|^2 - (\frac{n-2}{2})^2|\psi|^2 - V|\psi|^2}{\int_\Omega |\psi|^2}; \quad \psi \in C_0^\infty(\Omega \setminus \{0\})\}.$$

By our assumption $\lambda(V) \geq 0$. Let (ϕ_n, λ_1^n) be the first eigenpair for the problem

$$\begin{aligned} (L - \lambda_1(V) - \lambda_1^r)\phi_r &= 0 \ \ \textit{on} \ \ \Omega \setminus B_{\frac{R}{n}} \\ \phi(r) &= 0 \ \ \textit{on} \ \ \partial(\Omega \setminus B_{\frac{R}{n}}), \end{aligned}$$

where $L = -\Delta - (\frac{n-2}{2})^2 \frac{1}{|x|^2} - V$, and $B_{\frac{R}{n}}$ is a ball of radius $\frac{R}{n}$, $n \geq 2$. The eigenfunctions can be chosen in such a way that $\phi_n > 0$ on $\Omega \setminus B_{\frac{R}{n}}$ and $\varphi_n(b) = 1$, for some $b \in \Omega$ with $\frac{R}{2} < |b| < R$.

Note that $\lambda_1^n \downarrow 0$ as $n \to \infty$. Harnak's inequality yields that for any compact subset K, $\frac{max_K \phi_n}{max_K \phi_n} \leq C(K)$ with the later constant being independant of ϕ_n. Also standard elliptic estimates also yields that the family (ϕ_n) have also uniformly bounded derivatives on compact sets $\Omega - B_{\frac{R}{n}}$.
Therefore, there exists a subsequence $(\varphi_{n_{l_2}})_{l_2}$ of $(\varphi_n)_n$ such that $(\varphi_{n_{l_2}})_{l_2}$

converges to some $\varphi_2 \in C^2(\Omega \setminus B(\frac{R}{2}))$. Now consider $(\varphi_{n_{l_2}})_{l_2}$ on $\Omega \setminus B(\frac{R}{3})$. Again there exists a subsequence $(\varphi_{n_{l_3}})_{l_3}$ of $(\varphi_{n_{l_2}})_{l_2}$ which converges to $\varphi_3 \in C^2(\Omega \setminus B(\frac{R}{3}))$, and $\varphi_3(x) = \varphi_2(x)$ for all $x \in \Omega \setminus B(\frac{R}{2})$. By repeating this argument we get a supersolution $\varphi \in C^2(\Omega \setminus \{0\})$ i.e. $L\varphi \geq 0$, such that $\varphi > 0$ on $\Omega \setminus \{0\}$. □

Lemma 2.32. *Let a be a locally integrable function on , then the following statements are equivalent.*

1. $z''(s) + a(s)z(s) = 0$, *has a strictly positive solution on* (b, ∞).
2. *There exists a function* $\psi \in C^1(b, \infty)$ *such that* $\psi'(r) + \psi^2(r) + a(t) \leq 0, \quad for \quad r > b$.

Consequently, the equation $y'' + \frac{1}{r}y' + v(r)y = 0$ *has a positive supersolution on* $(0, \delta)$ *if and only if it has a positive solution on* $(0, \delta)$.

Proof: That 1) and 2) are equivalent follows from the work of Wintner [23, 24], a proof of which may be found in [14].
To prove the rest, we note that the change of variable $z(s) = y(e^{-s})$ maps the equation $y'' + \frac{1}{r}y' + v(r)y = 0$ into $z'' + e^{-2s}v(e^{-s})z(s) = 0$. On the other hand, the change of variables $\psi(t) = \frac{-e^{-t}y'(e^{-t})}{y(e^{-t})}$ maps $y'' + \frac{1}{r}y' + v(r)y$ into $\psi'(t) + \psi^2(t) + e^{-2t}v(e^{-t})$. This proves the lemma. □

Proof of Theorem 2.1: The implication 1) implies 2) follows immediately from Proposition 2.16 and Lemma 2.26. It is valid for any smooth bounded domain provided v is assumed to be non-decreasing on $(0, R)$. this condition is not needed if the domain is a ball of radius R.

To show that 2) implies 1), we assume that inequality (H_V) holds on a ball Ω of radius R, and then apply Lemma (2.28) to obtain a C^2-supersolution for the equation (2.29). Now take the surface average of u, that is

$$w(r) = \frac{1}{n\omega_w r^{n-1}} \int_{\partial B_r} u(x)dS = \frac{1}{n\omega_n} \int_{|\omega|=1} u(r\omega)d\omega > 0, \tag{2.33}$$

where ω_n denotes the volume of the unit ball in R^n. We may assume that the unit ball is contained in Ω (otherwise we just use a smaller ball). By a standard calculation we get

$$w''(r) + \frac{n-1}{r}w'(r) \leq \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} \Delta u(x)dS. \tag{2.34}$$

Since $u(x)$ is a supersolution of (2.29), w satisfies the inequality:

$$w''(r)+\frac{n-1}{r}w'(r)+(\frac{n-2}{2})^2\frac{w(r)}{r^2}\leq -v(r)w(r), \quad for \ \ 0<r<R. \tag{2.35}$$

Now define

$$\varphi(r)=r^{\frac{n-2}{2}}w(r), \quad in \ \ 0<r<R. \tag{2.36}$$

Using (2.35), a straightforward calculation shows that φ satisfies the following inequality

$$\varphi''(r)+\frac{\varphi'(r)}{r}\leq -\varphi(r)v(r), \quad for \ \ 0<r<R. \tag{2.37}$$

By Lemma 2.32 we may conclude that the equation $y''(r)+\frac{1}{r}y'+v(r)y=0$ has actually a positive solution ϕ on $(0,R)$.

It is clear that by the sufficient condition $c(V)\geq c$ whenever $y''(r)+\frac{1}{r}y'+cv(r)y=0$ has a positive solution on $(0,R)$. On the other hand, the necessary condition yields that $y''(r)+\frac{1}{r}y'+c(V)v(r)y=0$ has a positive solution on $(0,R)$. The proof is now complete. □

2.4 Applications

In this section we start by applying Theorem 2.1 to recover in a relatively simple and unified way, all previously known improvements of Hardy's inequality. For that we need to investigate whether the ordinary differential equation

$$y''+\frac{y'}{r}+v(r)y(r)=0, \tag{2.38}$$

corresponding to a potential v has a positive solution ϕ on $(0,\delta)$ for some $\delta>0$. In this case, $\psi(r)=\phi(\frac{\delta r}{R})$ is a solution for $y''(r)+\frac{1}{r}y'+\frac{\delta^2}{R^2}v(\frac{\delta}{R}r)y=0$ on $(0,R)$, which means that the scaled potential $V_\delta(x)=\frac{\delta^2}{R^2}V(\frac{\delta}{R}x)$ yields an improved Hardy formula (H_{V_δ}) on a ball of radius R, with constant larger than one. Here is an immediate application of this criterium.

1) The Brezis-Vázquez improvement [6]: Here we need to show that we can have an improved inequality with a constant potential. In this case, the best constant for which the equation

$$y''+\frac{y'}{r}+cy(r)=0, \tag{2.39}$$

has a positive solution on $(0,R)$, with $R=(|\Omega|/\omega_n)^{\frac{1}{n}}$ is $z_0^2\omega_n^{2/n}|\Omega|^{-2/n}$. Indeed, if z_0 is the first root of the solution of the Bessel equation $y''+\frac{y'}{r}+y(r)=0$, then the solution of (2.39) in this case is the Bessel function $\varphi(r)=J_0(\frac{rz_0}{R})$. This readily gives the result of Brezis-Vázquez mentioned in the introduction.

2) The Adimurthi et al. improvement [1]: In this case, one easily sees that the functions $\varphi_j(r)=(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{r})^{\frac{1}{2}}$ is a solution of the equation

$$-\frac{\varphi_j'(r)+r\varphi_j''(r)}{r\varphi_j(r)}=\frac{1}{4r^2}(\prod_{i=1}^{n}\log^{(i)}\frac{\rho}{r})^{-2},$$

on $(0,R)$, which means that the inequality (H_V) holds for the potential $V(x)=\frac{1}{4|x|^2}(\prod_{i=1}^{n}\log^{(i)}\frac{\rho}{|x|})^{-2}$ which yields the result of Adimurthi et al. In the following, we use our characterization to show that the constant appearing in the above improvement is indeed the best constant in the following sense:

$$\frac{1}{4}=\inf_{u\in H_0^1(\Omega)\backslash\{0\}}\frac{\int_\Omega|\nabla u|^2dx-(\frac{n-2}{2})^2\int_\Omega\frac{|u|^2}{|x|^2}dx-\frac{1}{4}\sum_{j=1}^{m-1}\int_\Omega\frac{|u|^2}{|x|^2}\big(\prod_{i=1}^{j}log^{(i)}\frac{\rho}{|x|}\big)^{-2}}{\int_\Omega\frac{|u|^2}{|x|^2}\big(\prod_{i=1}^{m}log^{(i)}\frac{R}{|x|}\big)^{-2}},$$

for all $1\le m\le k$. We proceed by contradiction, and assume that

$$\frac{1}{4}+\lambda=\inf_{u\in H_0^1(\Omega)\backslash\{0\}}\frac{\int_\Omega|\nabla u|^2dx-(\frac{n-2}{2})^2\int_\Omega\frac{|u|^2}{|x|^2}dx-\frac{1}{4}\sum_{j=1}^{m-1}\int_\Omega\frac{|u|^2}{|x|^2}\big(\prod_{i=1}^{j}log^{(i)}\frac{\rho}{|x|}\big)^{-2}}{\int_\Omega\frac{|u|^2}{|x|^2}\big(\prod_{i=1}^{m}log^{(i)}\frac{\rho}{|x|}\big)^{-2}},$$

and $\lambda>0$. From Theorem 2.1 we deduce that there exists a positive function φ such that

$$-\frac{\varphi'(r)+r\varphi''(r)}{\varphi(r)}=\frac{1}{4}\sum_{j=1}^{m-1}\frac{1}{r}(\prod_{i=1}^{j}log^{(i)}\frac{\rho}{r})^{-2}+(\frac{1}{4}+\lambda)\frac{1}{r}(\prod_{i=1}^{m}log^{(i)}\frac{\rho}{r})^{-2}.$$

Now define $f(r)=\frac{\varphi(r)}{\varphi_m(r)}>0$, and calculate,

$$\frac{\varphi'(r)+r\varphi''(r)}{\varphi(r)}=\frac{\varphi_m'(r)+r\varphi_m''(r)}{\varphi_m(r)}+\frac{f'(r)+rf''(r)}{f(r)}-\frac{f'(r)}{f(r)}\sum_{i=1}^{m}\frac{1}{\prod_{j=1}^{i}\log^j(\frac{\rho}{r})}.$$

Thus,

$$\frac{f'(r)+rf''(r)}{f(r)} - \frac{f'(r)}{f(r)}\sum_{i=1}^{m}\frac{1}{\prod_{j=1}^{i}\log^{j}(\frac{\rho}{r})} = -\lambda\frac{1}{r}(\prod_{i=1}^{m} log^{(i)}\frac{\rho}{r})^{-2}. \tag{2.40}$$

If now $f'(\alpha_n)=0$ for some sequence $\{\alpha_n\}_{n=1}^{\infty}$ that converges to zero, then there exists a sequence $\{\beta_n\}_{n=1}^{\infty}$ that also converges to zero, such that $f''(\beta_n)=0$, and $f'(\beta_n)>0$. But this contradicts (2.40), which means that f is eventually monotone for r small enough. We consider the two cases according to whether f is increasing or decreasing:

Case I: Assume $f'(r)>0$ for $r>0$ sufficiently small. Then we will have

$$\frac{(rf'(r))'}{rf'(r)} \le \sum_{i=1}^{m}\frac{1}{r\prod_{j=1}^{i}\log^{j}(\frac{\rho}{r})}.$$

Integrating once we get

$$f'(r) \ge \frac{c}{r\prod_{j=1}^{m}\log^{j}(\frac{\rho}{r})},$$

for some $c>0$. Hence, $\lim_{r\to 0} f(r) = -\infty$ which is a contradiction.

Case II: Assume $f'(r)<0$ for $r>0$ sufficiently small. Then

$$\frac{(rf'(r))'}{rf'(r)} \ge \sum_{i=1}^{m}\frac{1}{r\prod_{j=1}^{i}\log^{j}(\frac{\rho}{r})}.$$

Thus,

$$f'(r) \ge -\frac{c}{r\prod_{j=1}^{m}\log^{j}(\frac{\rho}{r})}, \tag{2.41}$$

for some $c>0$ and $r>0$ sufficiently small. On the other hand

$$\frac{f'(r)+rf''(r)}{f(r)} \le -\lambda\sum_{j=1}^{m}\frac{1}{r}(\prod_{i=1}^{j} log^{(i)}\frac{R}{r})^{-2} \le -\lambda(\frac{1}{\prod_{j=1}^{m}\log^{j}(\frac{\rho}{r})})'.$$

Since $f'(r)<0$, there exists l such that $f(r)>l>0$ for $r>0$ sufficiently small. From the above inequality we then have

$$bf'(b) - af'(a) < -\lambda l(\frac{1}{\prod_{j=1}^{m}\log^{j}(\frac{\rho}{b})} - \frac{1}{\prod_{j=1}^{m}\log^{j}(\frac{\rho}{a})}).$$

From (2.41) we have $\lim_{a\to 0} af'(a) = 0$. Hence,

$$bf'(b) < -\frac{\lambda l}{\prod_{j=1}^{m}\log^{j}(\frac{\rho}{b})},$$

for every $b > 0$, and

$$f'(r) < -\frac{\lambda l}{r\prod_{j=1}^{m}\log^j(\frac{\rho}{r})},$$

for $r > 0$ sufficiently small. Therefore,

$$\lim_{r\to 0} f(r) = +\infty,$$

and by choosing l large enouph (e.g., $l > \frac{c}{\lambda}$) we get to contradict (2.41) and the proof is now complete.

3) The Filippas and Tertikas improvement [11]: Let $D \geq \sup_{x\in\Omega}|x|$, and define

$$\varphi_k(r) = (X_1(\frac{r}{D})X_2(\frac{r}{D})\dots X_{i-1}(\frac{r}{D})X_i(\frac{r}{D}))^{-\frac{1}{2}}, \quad i = 1, 2, \dots.$$

Using the fact that $X_k'(r) = \frac{1}{r}X_1(r)X_2(r)\dots X_{k-1}(r)X_k^2(r)$ for $k = 1, 2, \dots$, we get

$$-\frac{\varphi_k'(r) + r\varphi_k''(r)}{\varphi_k(r)} = \frac{1}{4r}X_1^2(\frac{r}{D})X_2^2(\frac{r}{D})\dots X_{k-1}^2(\frac{r}{D})X_k^2(\frac{r}{D}).$$

This means that the inequality (H_V) holds for the potential

$$V(x) = \frac{1}{4|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})\dots X_{k-1}^2(\frac{|x|}{D})X_k^2(\frac{|x|}{D}),$$

which yields the result of Filippas and Tertikas [11]. We now identify the best constant by showing that:

$$\frac{1}{4} = \inf_{u\in H_0^1(\Omega)\setminus\{0\}}$$
$$\frac{\int_\Omega|\nabla u|^2dx - (\frac{n-2}{2})^2\int_\Omega\frac{|u|^2}{|x|^2}dx - \frac{1}{4}\sum_{j=1}^{m-1}\int_\Omega\frac{|u|^2}{|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})\dots X_{j-1}^2(\frac{|x|}{D})X_j^2(\frac{|x|}{D})}{\int_\Omega\frac{|u|^2}{|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})\dots X_{m-1}^2(\frac{|x|}{D})X_m^2(\frac{|x|}{D})},$$

for all $1 \leq m \leq k$. We proceed again by contradiction and in a way very similar to the above case. Indeed, assuming that

$$\frac{1}{4} + \lambda = \inf_{u\in H_0^1(\Omega)\setminus\{0\}}$$
$$\frac{\int_\Omega|\nabla u|^2dx - (\frac{n-2}{2})^2\int_\Omega\frac{|u|^2}{|x|^2}dx - \frac{1}{4}\sum_{j=1}^{m-1}\int_\Omega\frac{|u|^2}{|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})\dots X_{j-1}^2(\frac{|x|}{D})X_j^2(\frac{|x|}{D})}{\int_\Omega\frac{|u|^2}{|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})\dots X_{m-1}^2(\frac{|x|}{D})X_m^2(\frac{|x|}{D})},$$

and $\lambda > 0$, we use again Theorem 2.1 to find a positive function φ such that

$$\begin{aligned} -\frac{\varphi'(r) + r\varphi''(r)}{\varphi(r)} &= \frac{1}{4}\sum_{j=1}^{m-1}\frac{1}{r}X_1^2(\frac{r}{D})X_2^2(\frac{r}{D})\dots X_{j-1}^2(\frac{r}{D})X_j^2(\frac{r}{D}) \\ &+ (\frac{1}{4}+\lambda)\frac{1}{r}X_1^2(\frac{r}{D})X_2^2(\frac{r}{D})\dots X_{m-1}^2(\frac{r}{D})X_m^2(\frac{r}{D}). \end{aligned}$$

Setting $f(r) = \frac{\varphi(r)}{\varphi_m(r)} > 0$, we have

$$\frac{\varphi'(r) + r\varphi''(r)}{\varphi(r)} = \frac{\varphi'_m(r) + r\varphi''_m(r)}{\varphi_m(r)} + \frac{f'(r) + rf''(r)}{f(r)} - \frac{f'(r)}{f(r)} \sum_{i=1}^{m} \prod_{j=1}^{i} X_j(\frac{r}{D}).$$

Thus,

$$\frac{f'(r) + rf''(r)}{f(r)} - \frac{f'(r)}{f(r)} \sum_{i=1}^{m} \prod_{j=1}^{i} X_j(\frac{r}{D}) = -\lambda \frac{1}{r} \prod_{j=1}^{m} X_j^2(\frac{r}{D}). \tag{2.42}$$

Arguing as before, we deduce that f is eventually monotone for r small enough, and we consider two cases:

Case I: If $f'(r) > 0$ for $r > 0$ sufficiently small, then we will have

$$\frac{(rf'(r))'}{rf'(r)} \leq \sum_{i=1}^{m} \frac{1}{r} \prod_{j=1}^{i} X_j(\frac{r}{D}).$$

Integrating once we get

$$f'(r) \geq \frac{c}{r} \prod_{j=1}^{m} X_j(\frac{r}{D}),$$

for some $c > 0$, and therefore $\lim_{r \to 0} f(r) = -\infty$ which is a contradiction.

Case II: Assume $f'(r) < 0$ for $r > 0$ sufficiently small. Then

$$\frac{(rf'(r))'}{rf'(r)} \geq \sum_{i=1}^{m} \frac{1}{r} \prod_{j=1}^{i} X_j(\frac{r}{D})$$

Thus,

$$f'(r) \geq -\frac{c}{r} \prod_{j=1}^{m} X_j(\frac{r}{D}), \tag{2.43}$$

for some $c > 0$ and $r > 0$ sufficiently small. On the other hand

$$\frac{f'(r) + rf''(r)}{f(r)} \leq -\lambda \sum_{j=1}^{m} \frac{1}{r} \prod_{i=1}^{j} X_j^2 \leq -\lambda (\prod_{j=1}^{m} X_j(\frac{r}{D}))'.$$

Since $f'(r) < 0$, we may assume $f(r) > l > 0$ for $r > 0$ sufficiently small, and from the above inequality we have

$$bf'(b) - af'(a) < -\lambda l (\prod_{j=1}^{m} X_j(\frac{b}{D}) - \prod_{j=1}^{m} X_j(\frac{a}{D})).$$

From (2.43) we have $\lim_{a\to 0} af'(a) = 0$. Hence,

$$f'(r)) < -\frac{\lambda l}{r}\prod_{j=1}^{m} X_j(\frac{r}{D}),$$

for $r > 0$ sufficiently small. Therefore,

$$\lim_{r\to 0} f(r) = +\infty,$$

and by choosing l large enouph (i.e. $l > \frac{c}{\lambda}$) we contradict (2.43) and the proof is complete. □

We shall now make the connection between improved Hardy inequalities and the existence of non-oscillatory solutions (i.e., those $z(s)$ such that $z(s) > 0$ for $s > 0$ sufficiently large) for the second order linear differential equations

$$z''(s) + a(s)z(s) = 0. \tag{2.44}$$

Interesting results in this direction were established by many authors (see [14, 15, 23, 24, 25]). Here is a typical criterium about the oscillatory properties of equation (2.44):

1. If $\limsup_{t\to\infty} t\int_t^\infty a(s)ds < \frac{1}{4}$, then Eq. (2.44) is non-oscillatory.
2. If $\liminf_{t\to\infty} t\int_t^\infty a(s)ds > \frac{1}{4}$, then Eq. (2.44) is oscillatory.

This result combined with Theorem 2.1 and Lemma 2.32 clearly yields Corollary 2.1.

Proof of Corollary 2.1: It follows from Hölder's inequality that

$$(\int_\Omega V(|x|)u^2(x)dx)^{\frac{1}{s}} \geq \frac{\int_\Omega u^{\frac{2}{s}}(x)dx}{(\int_\Omega V^{-\frac{r}{s}}(|x|)dx)^{\frac{1}{r}}},$$

where $s \geq 1$ and $\frac{1}{s} + \frac{1}{r} = 1$. Letting $p = \frac{2}{s}$, we get

$$\int_\Omega V(|x|)u^2(x)dx \geq (\int_\Omega u^p(x)dx)^{\frac{2}{p}} \frac{1}{||V^{-1}(|x|)||_{L^{\frac{p}{2-p}}(\Omega)}}.$$

Inequality (2.1) now follows from Theorem 2.1. □

Proof of Corollary 2.1: Define the functional

$$F_\mu(u) = \int_\Omega |\nabla u(x)|^2 dx - \mu \int_\Omega \frac{u^2(x)}{|x|^2}dx, \tag{2.45}$$

which is continuous, Gateaux differentiable and coercive on $H_0^1(\Omega)$. Let $u_\mu > 0$ be a minimizer of F_μ over the manifold $M = \{u \in H_0^1(\Omega)| \int_\Omega u^2(x)V(x) = 1\}$ and assume λ_μ^1 is the infimum. It is clear that $\lambda_\mu^1 > 0$. By standard arguments we can conclude that u_μ is a weak solution of $(E_{V,\mu})$. The rest of the proof follows from Corollary 2.1 and the fact that

$$\lambda_1(V) = \lim_{\mu \to \mu_n} \lambda_\mu^1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 - \mu_n \frac{u^2(x)}{|x|^2})dx}{\int_\Omega |u(x)|^2 V(x)dx}.$$

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Chapter 3

Bessel pairs and optimal Hardy and Hardy-Rellich inequalities [2]

3.1 Introduction

Ever since Brézis-Vazquez [10] showed that Hardy's inequality can be improved once restricted to a smooth bounded domain Ω in R^n, there was a flurry of activity about possible improvements of the following type:

$$\text{If } n \geq 3 \quad \text{then} \quad \int_\Omega |\nabla u|^2 dx - (\tfrac{n-2}{2})^2 \int_\Omega \tfrac{|u|^2}{|x|^2} dx \geq \int_\Omega V(x)|u|^2 dx, \tag{3.1}$$

for all $u \in H_0^1(\Omega)$, as well as its fourth order counterpart

$$\text{If } n \geq 5 \quad \text{then} \quad \int_\Omega |\Delta u|^2 dx - \tfrac{n^2(n-4)^2}{16} \int_\Omega \tfrac{u^2}{|x|^4} dx \geq \int_\Omega W(x) u^2 dx \tag{3.2}$$

for $u \in H_0^2(\Omega)$, where V, W are certain explicit radially symmetric potentials of order lower than $\frac{1}{r^2}$ (for V) and $\frac{1}{r^4}$ (for W) (see [1], [3], [6], [4], [5], [8], [9], [10], [15], [16], [17], [19], [31]).

In this section, we provide an approach that completes, simplifies and improves most related results to-date regarding the Laplacian on Euclidean space as well as its powers. We also establish new inequalities some of which cover critical dimensions such as $n = 2$ for inequality (3.1) and $n = 4$ for (3.2).

We start by giving necessary and sufficient conditions on positive radial functions V and W on a ball B in R^n, so that the following inequality holds for some $c > 0$:

$$\int_B V(x)|\nabla u|^2 dx \geq c \int_B W(x) u^2 dx \text{ for all } u \in C_0^\infty(B). \tag{3.3}$$

Assuming that the ball B has radius R and that $\int_0^R \frac{1}{r^{n-1}V(r)} dr = +\infty$, the condition is simply that the ordinary differential equation

$$(\mathrm{B}_{V,cW}) \qquad y''(r) + (\tfrac{n-1}{r} + \tfrac{V_r(r)}{V(r)}) y'(r) + \tfrac{cW(r)}{V(r)} y(r) = 0$$

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has a positive solution on the interval $(0,R)$. We shall call such a couple (V,W) a *Bessel pair on* $(0,R)$. The *weight* of such a pair is then defined as

$$\beta(V,W;R) = \sup\big\{c;\ (\mathrm{B}_{V,cW}) \text{ has a positive solution on } (0,R)\big\}. \tag{3.4}$$

This characterization makes an important connection between Hardy-type inequalities and the oscillatory behavior of the above equations. For example, by using recent results on ordinary differential equations, we can then infer that an integral condition on V,W of the form

$$\limsup_{r\to 0} r^{2(n-1)}V(r)W(r)\Big(\int_r^R \frac{d\tau}{\tau^{n-1}V(\tau)}\Big)^2 < \frac{1}{4} \tag{3.5}$$

is sufficient (and "almost necessary") for (V,W) to be a Bessel pair on a ball of sufficiently small radius ρ.

Applied in particular, to a pair $(V, \frac{1}{r^2}V)$ where the function $\frac{rV'(r)}{V(r)}$ is assumed to decrease to $-\lambda$ on $(0,R)$, we obtain the following extension of Hardy's inequality: If $\lambda \le n-2$, then

$$\int_B V(x)|\nabla u|^2 dx \ge (\tfrac{n-\lambda-2}{2})^2 \int_B V(x)\tfrac{u^2}{|x|^2}dx \quad \text{for all } u\in C_0^\infty(B) \tag{3.6}$$

and $(\frac{n-\lambda-2}{2})^2$ is the best constant. The case where $V(x)\equiv 1$ is obviously the classical Hardy inequality and when $V(x)=|x|^{-2a}$ for $-\infty < a < \frac{n-2}{2}$, this is a particular case of the Caffarelli-Kohn-Nirenberg inequality. One can however apply the above criterium to obtain new inequalities such as the following: For $a,b>0$

- If $\alpha\beta>0$ and $m\le\frac{n-2}{2}$, then for all $u\in C_0^\infty(R^n)$

$$\int_{R^n}\frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}}|\nabla u|^2dx \ge (\frac{n-2m-2}{2})^2\int_{R^n}\frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}}u^2dx, \tag{3.7}$$

  and $(\frac{n-2m-2}{2})^2$ is the best constant in the inequality.

- If $\alpha\beta<0$ and $2m-\alpha\beta\le n-2$, then for all $u\in C_0^\infty(R^n)$

$$\int_{R^n}\frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}}|\nabla u|^2dx \ge (\frac{n-2m+\alpha\beta-2}{2})^2\int_{R^n}\frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}}u^2dx, \tag{3.8}$$

  and $(\frac{n-2m+\alpha\beta-2}{2})^2$ is the best constant in the inequality.

We can also extend some of the recent results of Blanchet-Bonforte-Dolbeault-Grillo-Vasquez [4].

- If $\alpha\beta<0$ and $-\alpha\beta\le n-2$, then for all $u\in C_0^\infty(R^n)$

$$\int_{R^n}(a+b|x|^\alpha)^\beta|\nabla u|^2dx \ge b^{\frac{2}{\alpha}}(\frac{n-\alpha\beta-2}{2})^2\int_{R^n}(a+b|x|^\alpha)^{\beta-\frac{2}{\alpha}}u^2dx, \tag{3.9}$$

  and $b^{\frac{2}{\alpha}}(\frac{n-\alpha\beta-2}{2})^2$ is the best constant in the inequality.

- If $\alpha\beta > 0$, and $n \geq 2$, then there exists a constant $C > 0$ such that for all $u \in C_0^\infty(R^n)$

$$\int_{R^n} (a + b|x|^\alpha)^\beta |\nabla u|^2 dx \geq C \int_{R^n} (a + b|x|^\alpha)^{\beta - \frac{2}{\alpha}} u^2 dx. \tag{3.10}$$

  Moreover, $b^{\frac{2}{\alpha}}(\frac{n-2}{2})^2 \leq C \leq b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$.

On the other hand, by considering the pair

$$V(x) = |x|^{-2a} \quad \text{and} \quad W_{a,c}(x) = (\tfrac{n-2a-2}{2})^2|x|^{-2a-2} + c|x|^{-2a}W(x)$$

we get the following improvement of the Caffarelli-Kohn-Nirenberg inequalities:

$$\int_B |x|^{-2a}|\nabla u|^2 dx - (\frac{n-2a-2}{2})^2 \int_B |x|^{-2a-2}u^2 dx \geq c \int_B |x|^{-2a}W(x)u^2 dx, \tag{3.11}$$

for all $u \in C_0^\infty(B)$, if and only if the following ODE

$$(\mathrm{B}_{cW}) \qquad y'' + \tfrac{1}{r}y' + cW(r)y = 0$$

has a positive solution on $(0, R)$. Such a function W will be called a *Bessel potential* on $(0, R)$. This type of characterization was established recently by the authors [19] in the case where $a = 0$, yielding in particular the recent improvements of Hardy's inequalities (on bounded domains) established by Brezis-Vázquez [10], Adimurthi et al. [1], and Filippas-Tertikas [17]. Our results here include in addition those proved by Wang-Willem [34] in the case where $a < \frac{n-2}{2}$ and $W(r) = \frac{1}{r^2(\ln\frac{R}{r})^2}$, but also cover the previously unknown limiting case corresponding to $a = \frac{n-2}{2}$ as well as the critical dimension $n = 2$.

More importantly, we establish here that Bessel pairs lead to a myriad of optimal Hardy-Rellich inequalities of arbitrary high order, therefore extending and completing a series of new results by Adimurthi et al. [2], Tertikas-Zographopoulos [31] and others. They are mostly based on the following theorem which summarizes the main thrust of this chapter.

Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball centered at zero with radius R in R^n ($n \geq 1$) such that $\int_0^R \frac{1}{r^{n-1}V(r)}dr = +\infty$ and $\int_0^R r^{n-1}V(r)dr < +\infty$. The following statements are then equivalent:

1. (V, W) is a Bessel pair on $(0, R)$ and $\beta(V, W; R) \geq 1$.

2. $\int_B V(x)|\nabla u|^2 dx \geq \int_B W(x)u^2 dx$ for all $u \in C_0^\infty(B)$.

3. If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < n - 2$, then the above are equivalent to

   $$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|})|\nabla u|^2 dx,$$

   for all radial $u \in C_{0,r}^\infty(B)$.

4. If in addition, $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ on $(0, R)$, then the above are equivalent to

$$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B \left(\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|}\right)|\nabla u|^2 dx,$$

for all $u \in C_0^\infty(B)$.

In other words, one can obtain as many Hardy and Hardy-Rellich type inequalities as one can construct Bessel pairs on $(0, R)$. The relevance of the above result stems from the fact that there are plenty of such pairs that are easily identifiable. Indeed, even the class of *Bessel potentials* –equivalently those W such that $\big(1, (\frac{n-2}{2})^2|x|^{-2} + cW(x)\big)$ is a Bessel pair– is quite rich and contains several important potentials. Here are some of the most relevant properties of the class of C^1 Bessel potentials W on $(0, R)$, that we shall denote by $\mathcal{B}(0, R)$.

First, the class is a closed convex *solid* subset of $C^1(0, R)$, that is if $W \in \mathcal{B}(0, R)$ and $0 \leq V \leq W$, then $V \in \mathcal{B}(0, R)$. The "weight" of each $W \in \mathcal{B}(R)$, that is

$$\beta(W; R) = \sup\big\{c > 0;\ (B_{cW}) \text{ has a positive solution on } (0, R)\big\}, \tag{3.12}$$

will be an important ingredient for computing the best constants in corresponding functional inequalities. Here are some basic examples of Bessel potentials and their corresponding weights.

- $W \equiv 0$ is a Bessel potential on $(0, R)$ for any $R > 0$.
- $W \equiv 1$ is a Bessel potential on $(0, R)$ for any $R > 0$, and $\beta(1; R) = \frac{z_0^2}{R^2}$ where $z_0 = 2.4048...$ is the first zero of the Bessel function J_0.
- If $a < 2$, then there exists $R_a > 0$ such that $W(r) = r^{-a}$ is a Bessel potential on $(0, R_a)$.
- For $k \geq 1$, $R > 0$ and $\rho = R(e^{e^{e^{\cdot^{\cdot^{e((k-1)-times)}}}}})$, let

  $$W_{k,\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2}\Big(\prod_{i=1}^j log^{(i)}\frac{\rho}{r}\Big)^{-2},$$

  where the functions $log^{(i)}$ are defined iteratively as follows: $log^{(1)}(.) = log(.)$ and for $k \geq 2$, $log^{(k)}(.) = log(log^{(k-1)}(.))$. $W_{k,\rho}$ is then a Bessel potential on $(0, R)$ with $\beta(W_{k,\rho}; R) = \frac{1}{4}$.
- For $k \geq 1$, $R > 0$ and $\rho \geq R$, define

  $$\tilde{W}_{k;\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2} X_1^2(\frac{r}{\rho})X_2^2(\frac{r}{\rho})\dots X_{j-1}^2(\frac{r}{\rho})X_j^2(\frac{r}{\rho}),$$

  where the functions X_i are defined iteratively as follows: $X_1(t) = (1 - \log(t))^{-1}$ and for $k \geq 2$, $X_k(t) = X_1(X_{k-1}(t))$. Then again $\tilde{W}_{k,\rho}$ is a Bessel potential on $(0, R)$ with $\beta(\tilde{W}_{k,\rho}; R) = \frac{1}{4}$.

- More generally, if W is any positive function on R such that
$$\liminf_{r\to 0} \ln(r) \int_0^r sW(s)ds > -\infty,$$
then for every $R > 0$, there exists $\alpha := \alpha(R) > 0$ such that $W_\alpha(x) := \alpha^2 W(\alpha x)$ is a Bessel potential on $(0, R)$.

What is remarkable is that the class of Bessel potentials W is also the one that leads to optimal improvements for fourth order inequalities (in dimension $n \geq 3$) of the following type:

$$\int_B |\Delta u|^2 dx - C(n) \int_B \frac{|\nabla u|^2}{|x|^2} dx \geq c(W, R) \int_B W(x)|\nabla u|^2 dx \ , \tag{3.13}$$

for all $u \in H_0^2(B)$, where $C(3) = \frac{25}{36}$, $C(4) = 3$ and $C(n) = \frac{n^2}{4}$ for $n \geq 5$. The case when $W \equiv \tilde{W}_{k,\rho}$ and $n \geq 5$ was recently established by Tertikas-Zographopoulos [31]. Note that W can be chosen to be any one of the examples of Bessel potentials listed above. Moreover, both $C(n)$ and the weight $\beta(W; R)$ are the best constants in the above inequality.

Appropriate combinations of (3.3) and (3.13) then lead to a myriad of Hardy-Rellich inequalities in dimension $n \geq 4$. The For example, if W is a Bessel potential on $(0, R)$ such that the function $r\frac{W_r(r)}{W(r)}$ decreases to $-\lambda$, and if $\lambda \leq n-2$, then we have for all $u \in C_0^\infty(B_R)$

$$\begin{aligned}\int_B |\Delta u|^2 dx \quad &- \quad \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx \\ &\geq \quad (\frac{n^2}{4} + \frac{(n-\lambda-2)^2}{4})\beta(W; R) \int_B \frac{W(x)}{|x|^2} u^2 dx.\end{aligned} \tag{3.14}$$

By applying (3.14) to the various examples of Bessel functions listed above, one improves in many ways the recent results of Adimurthi et al. [2] and those by Tertikas-Zographopoulos [31]. Moreover, besides covering the critical dimension $n = 4$, we also establish that the best constant is $(1 + \frac{n(n-4)}{8})$ for all the potentials W_k and $\tilde{W}_k$ defined above. For example we have for $n \geq 4$,

$$\begin{aligned}\int_B |\Delta u(x)|^2 dx \quad &\geq \quad \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx \\ &+ \quad (1 + \frac{n(n-4)}{8}) \sum_{j=1}^k \int_B \frac{u^2}{|x|^4} (\prod_{i=1}^j log^{(i)} \frac{\rho}{|x|})^{-2} dx.\end{aligned} \tag{3.15}$$

More generally, we show that for any $m < \frac{n-2}{2}$, and any W Bessel potential on a ball $B_R \subset R^n$ of radius R, the following inequality holds for all $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W; R) \int_{B_R} W(x) \frac{|\nabla u|^2}{|x|^{2m}} dx, \tag{3.16}$$

where $a_{m,n}$ and $\beta(W;R)$ are best constants that we compute in the appendices for all m and n and for many Bessel potentials W. Worth noting is Corollary 3.3 where we show that inequality (3.16) restricted to radial functions in $C_0^\infty(B_R)$ holds with a best constant equal to $(\frac{n+2m}{2})^2$, but that $a_{n,m}$ can however be strictly smaller than $(\frac{n+2m}{2})^2$ in the non-radial case. These results improve considerably Theorem 1.7, Theorem 1.8, and Theorem 6.4 in [31].

We also establish a more general version of equation (3.14). Assuming again that $\frac{rW'(r)}{W(r)}$ decreases to $-\lambda$ on $(0,R)$, and provided $m \leq \frac{n-4}{2}$ and $\lambda \leq n-2m-2$, we then have for all $u \in C_0^\infty(B_R)$,

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx \geq \beta_{n,m} \int_{B_R} \frac{u^2}{|x|^{2m+4}} dx \tag{3.17}$$
$$+ \quad \beta(W;R)(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4}) \int_{B_R} \frac{W(x)}{|x|^{2m+2}} u^2 dx,$$

where again the best constants $\beta_{n,m}$ are computed in section 3. This completes the results in Theorem 1.6 of [31], where the inequality is established for $n \geq 5$, $0 \leq m < \frac{n-4}{2}$, and the particular potential $\tilde{W}_{k,\rho}$.

Another inequality that relates the Hessian integral to the Dirichlet energy is the following: Assuming $-1 < m \leq \frac{n-4}{2}$ and W is a Bessel potential on a ball B of radius R in R^n, then for all $u \in C_0^\infty(B)$,

$$\int_B \frac{|\Delta u|^2}{|x|^{2m}} dx - \frac{(n+2m)^2(n-2m-4)^2}{16} \int_B \frac{u^2}{|x|^{2m+4}} dx \geq \tag{3.18}$$
$$\beta(W;R)\frac{(n+2m)^2}{4} \int_B \frac{W(x)}{|x|^{2m+2}} u^2 dx + \beta(|x|^{2m};R)||u||_{H_0^1}.$$

This improves considerably Theorem A.2. in [2] where it is established – for $m = 0$ and without best constants – with the potential $W_{1,\rho}$ in dimension $n \geq 5$, and the potential $W_{2,\rho}$ when $n = 4$.

Finally, we establish several higher order Rellich inequalities for integrals of the form $\int_{B_R} \frac{|\Delta^m u|^2}{|x|^{2k}} dx$, improving in many ways several recent results in [31].

Theorem 3.1 also leads to certain improved Hardy-Rellich inequalities that are crucial to show the singular nature of the external solutions in various fourth order nonlinear eigenvalue problems [12, 23, 24].

3.2 General Hardy Inequalities

Here is the main result of this section. Let V and W be positive radial C^1-functions on $B_R \backslash \{0\}$, where B_R is a ball centered at zero with radius R $(0 < T \leq +\infty)$ in R^n $(n \geq 1)$. Assume that $\int_0^a \frac{1}{r^{n-1}V(r)} dr = +\infty$ and $\int_0^a r^{n-1}V(r) dr < \infty$ for some $0 < a < R$. Then the following two statements are equivalent:

1. The ordinary differential equation

$$(\mathrm{B}_{V,W}) \qquad y''(r) + (\tfrac{n-1}{r} + \tfrac{V_r(r)}{V(r)})y'(r) + \tfrac{W(r)}{V(r)} y(r) = 0$$

has a positive solution on the interval $(0, R]$ (possibly with $\varphi(R) = 0$).

2. For all $u \in C_0^\infty(B_R)$

$$(\mathrm{H}_{V,W}) \qquad \int_{B_R} V(x)|\nabla u(x)|^2 dx \geq \int_{B_R} W(x)u^2 dx.$$

Before proceeding with the proofs, we note the following immediate but useful corollary.

Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball with radius R in R^n ($n \geq 1$) and centered at zero, such that $\int_0^R \frac{1}{r^{n-1}V(r)} dr = +\infty$ and $\int_0^R r^{n-1}V(r)dr < \infty$. Then (V, W) is a Bessel pair on $(0, R)$ if and only if for all $u \in C_0^\infty(B_R)$, we have

$$\int_{B_R} V(x)|\nabla u|^2 dx \geq \beta(V, W; R) \int_{B_R} W(x)u^2 dx,$$

with $\beta(V, W; R)$ being the best constant.

For the proof of Theorem 3.2, we shall need the following lemmas.

Lemma 3.19. *Let Ω be a smooth bounded domain in R^n with $n \geq 1$ and let $\varphi \in C^1(0, R := \sup_{x\in\partial\Omega} |x|)$ be a positive solution of the ordinary differential equation*

$$y'' + (\frac{n-1}{r} + \frac{V_r(r)}{V(r)})y' + \frac{W(r)}{V(r)}y = 0, \tag{3.20}$$

on $(0, R)$ for some $V(r), W(r) \geq 0$ where $\int_0^R \frac{1}{r^{n-1}V(r)} dr = +\infty$ and $\int_0^R r^{n-1}V(r)dr < \infty$. Setting $\psi(x) = \frac{u(x)}{\varphi(|x|)}$ for any $u \in C_0^\infty(\Omega)$, we then have the following properties:

1. $\int_0^R r^{n-1}V(r)(\frac{\varphi'(r)}{\varphi(r)})^2 dr < \infty$ *and* $\lim_{r\to 0} r^{n-1}V(r)\frac{\varphi'(r)}{\varphi(r)} = 0$.
2. $\int_\Omega V(|x|)(\varphi'(|x|))^2\psi^2(x)dx < \infty$.
3. $\int_\Omega V(|x|)\varphi^2(|x|)|\nabla\psi|^2(x)dx < \infty$.
4. $|\int_\Omega V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi(x)dx| < \infty$.
5. $\lim_{r\to 0} |\int_{\partial B_r} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)ds| = 0$, *where $B_r \subset \Omega$ is a ball of radius r centered at 0.*

Proof: 1) Setting $x(r) = r^{n-1}V(r)\frac{\varphi'(r)}{\varphi(r)}$, we have

$$\begin{aligned} r^{n-1}V(r)x'(r) + x^2(r) &= \frac{r^{2(n-1)}V^2(r)}{\varphi}(\varphi''(r) + (\frac{n-1}{r} + \frac{V_r(r)}{V(r)})\varphi'(r)) \\ &= -\frac{r^{2(n-1)}V(r)W(r)}{\varphi(r)} \leq 0, \quad 0 < r < R. \end{aligned}$$

Dividing by $r^{n-1}V(r)$ and integrating once, we obtain

$$x(r) \geq \int_r^R \frac{|x(s)|^2}{s^{n-1}V(s)} ds + x(R). \tag{3.21}$$

To prove that $\lim_{r\to 0} G(r) < \infty$, where $G(r) := \int_r^R \frac{x^2(s)}{s^{n-1}V(s)} ds$, we assume the contrary and use (3.21) to write that

$$(-r^{n-1}V(r))G'(r))^{\frac{1}{2}} \geq G(r) + x(R).$$

Thus, for r sufficiently small we have $-r^{n-1}V(r)G'(r) \geq \frac{1}{2}G^2(r)$ and hence, $(\frac{1}{G(r)})' \geq \frac{1}{2r^{n-1}V(r)}$, which contradicts the fact that $G(r)$ goes to infinity as r tends to zero.

Also in view of (3.21), we have that $x_0 := \lim_{r\to 0} x(r)$ exists, and since $\lim_{r\to 0} G(r) < \infty$, we necessarily have $x_0 = 0$ and 1) is proved.

For assertion 2), we use 1) to see that

$$\int_\Omega V(|x|)(\varphi'(|x|))^2\psi^2(x)dx \leq ||u||_\infty^2 \int_\Omega V(|x|)\frac{(\varphi'(|x|))^2}{\varphi^2(|x|)} dx < \infty.$$

3) Note that

$$|\nabla\psi(x)| \leq \tfrac{|\nabla u(x)|}{\varphi(|x|)} + |u(x)|\tfrac{|\varphi'(|x|)|}{\varphi^2(|x|)} \leq \tfrac{C_1}{\varphi(|x|)} + C_2\tfrac{|\varphi'(|x|)|}{\varphi^2(|x|)}, \quad \text{for all } x \in \Omega,$$

where $C_1 = \max_{x\in\Omega} |\nabla u|$ and $C_2 = \max_{x\in\Omega} |u|$. Hence we have

$$\begin{aligned}
&\int_\Omega V(|x|)\varphi^2(|x|)|\nabla\psi|^2(x)dx \leq \int_\omega V(|x|)\frac{(C_1\varphi(|x|) + C_2\varphi'(|x|))^2}{\varphi^2(|x|)} dx \\
= \;& \int_\Omega C_1^2 V(|x|)dx + \int_\Omega 2C_1C_2\frac{|\varphi'(|x|)|}{\varphi(|x|)}V(|x|)dx + \int_\Omega C_2^2(\frac{\varphi'(|x|)}{\varphi(|x|)})^2 V(|x|)dx \\
\leq \;& L_1 + 2C_1C_2\Big(\int_\Omega V(|x|)(\frac{\varphi'(|x|)}{\varphi(|x|)})^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega V(|x|)dx\Big)^{\frac{1}{2}} + L_2 \\
< \;& \infty,
\end{aligned}$$

which proves 3).

4) now follows from 2) and 3) since

$$\begin{aligned}
V(|x|)|\nabla u|^2 \;&=\; V(|x|)(\varphi'(|x|))^2\psi^2(x) \\
&+\; 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi(x) + V(|x|)\varphi^2(|x|)|\nabla\psi|^2.
\end{aligned}$$

Finally, 5) follows from 1) since

$$\begin{aligned}
|\int_{\partial B_r} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)ds| \;&<\; ||u||_\infty^2 |\int_{\partial B_r} V(|x|)\frac{\varphi'(|x|)}{\varphi(|x|)} ds \\
&=\; ||u||_\infty^2 V(r)\frac{|\varphi'(r)|}{\varphi(r)}\int_{\partial B_r} 1ds \\
&=\; n\omega_n ||u||_\infty^2 r^{n-1}V(r)\frac{|\varphi'(r)|}{\varphi(r)}.
\end{aligned}$$

Lemma 3.22. *Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in R^n $(n \geq 1)$ and centered at zero. Assuming*

$$\int_B \left(V(x)|\nabla u|^2 - W(x)|u|^2\right) dx \geq 0 \text{ for all } u \in C_0^\infty(B),$$

then there exists a C^2-supersolution to the following linear elliptic equation

$$\begin{aligned} -\mathrm{div}(V(x)\nabla u) - W(x)u &= 0, \quad \text{in } B, &(3.23)\\ u &> 0 \quad \text{in } B\setminus\{0\}, &(3.24)\\ u &= 0 \quad \text{in } \partial B. &(3.25)\\ & &(3.26)\end{aligned}$$

Proof: Define

$$\lambda_1(V) := \inf\{\frac{\int_B V(x)|\nabla\psi|^2 - W(x)|\psi|^2}{\int_B |\psi|^2}; \quad \psi \in C_0^\infty(B\setminus\{0\})\}.$$

By our assumption $\lambda_1(V) \geq 0$. Let (ϕ_n, λ_1^n) be the first eigenpair for the problem

$$\begin{aligned}(L - \lambda_1(V) - \lambda_1^n)\phi_n &= 0 \;\; on \;\; B\setminus B_{\frac{R}{n}}\\ \phi_n &= 0 \;\; on \;\; \partial(B\setminus B_{\frac{R}{n}}),\end{aligned}$$

where $Lu = -\mathrm{div}(V(x)\nabla u) - W(x)u$, and $B_{\frac{R}{n}}$ is a ball of radius $\frac{R}{n}$, $n \geq 2$. The eigenfunctions can be chosen in such a way that $\phi_n > 0$ on $B\setminus B_{\frac{R}{n}}$ and $\varphi_n(b) = 1$, for some $b \in B$ with $\frac{R}{2} < |b| < R$.

Note that $\lambda_1^n \downarrow 0$ as $n \to \infty$. Harnak's inequality yields that for any compact subset K, $\frac{\max_K \phi_n}{\min_K \phi_n} \leq C(K)$ with the later constant being independant of ϕ_n. Also standard elliptic estimates also yields that the family (ϕ_n) have also uniformly bounded derivatives on the compact sets $B - B_{\frac{R}{n}}$.

Therefore, there exists a subsequence $(\varphi_{n_{l_2}})_{l_2}$ of $(\varphi_n)_n$ such that $(\varphi_{n_{l_2}})_{l_2}$ converges to some $\varphi_2 \in C^2(B\setminus B(\frac{R}{2}))$. Now consider $(\varphi_{n_{l_2}})_{l_2}$ on $B\setminus B(\frac{R}{3})$. Again there exists a subsequence $(\varphi_{n_{l_3}})_{l_3}$ of $(\varphi_{n_{l_2}})_{l_2}$ which converges to $\varphi_3 \in C^2(B\setminus B(\frac{R}{3}))$, and $\varphi_3(x) = \varphi_2(x)$ for all $x \in B\setminus B(\frac{R}{2})$. By repeating this argument we get a supersolution $\varphi \in C^2(B\setminus\{0\})$ i.e. $L\varphi \geq 0$, such that $\varphi > 0$ on $B\setminus\{0\}$. □

Proof of Theorem 3.2: First we prove that 1) implies 2). Let $\phi \in C^1(0, R]$ be a solution of $(B_{V,W})$ such that $\phi(x) > 0$ for all $x \in (0, R)$. Define $\frac{u(x)}{\varphi(|x|)} = \psi(x)$. Then

$$|\nabla u|^2 = (\varphi'(|x|))^2\psi^2(x) + 2\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi + \varphi^2(|x|)|\nabla\psi|^2.$$

Hence,

$$V(|x|)|\nabla u|^2 \geq V(|x|)(\varphi'(|x|))^2\psi^2(x) + 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi(x).$$

Thus, we have

$$\begin{aligned}\int_B V(|x|)|\nabla u|^2dx &\geq \int_B V(|x|)(\varphi'(|x|))^2\psi^2(x)dx \\ &+ \int_B 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx.\end{aligned}$$

Let B_ϵ be a ball of radius ϵ centered at the origin. Integrate by parts to get

$$\begin{aligned}&\int_B V(|x|)|\nabla u|^2dx \\ \geq &\int_B V(|x|)(\varphi'(|x|))^2\psi^2(x)dx + \int_{B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx \\ + &\int_{B\backslash B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx \\ = &\int_{B_\epsilon} V(|x|)(\varphi'(|x|))^2\psi^2(x)dx + \int_{B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx \\ - &\int_{B\backslash B_\epsilon} (V(|x|)\varphi''(|x|)\varphi(|x|) + (\frac{(n-1)V(|x|)}{r} + V_r(|x|))\varphi'(|x|)\varphi(|x|))\psi^2(x)dx \\ + &\int_{\partial(B\backslash B_\epsilon)} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)ds\end{aligned}$$

Let $\epsilon \to 0$ and use Lemma 3.19 and the fact that ϕ is a solution of $(D_{v,w})$ to get

$$\begin{aligned}\int_B V(|x|)|\nabla u|^2dx &\geq -\int_B [V(|x|)\varphi''(|x|) \\ &+ (\frac{(n-1)V(|x|)}{r} + V_r(|x|))\varphi'(|x|)]\frac{u^2(x)}{\varphi(|x|)}dx \\ &= \int_B W(|x|)u^2(x)dx.\end{aligned}$$

To show that 2) implies 1), we assume that inequality $(\mathrm{H}_{V,W})$ holds on a ball B of radius R, and then apply Lemma 3.22 to obtain a C^2-supersolution for the equation (3.23). Now take the surface average of u, that is

$$y(r) = \frac{1}{n\omega_w r^{n-1}}\int_{\partial B_r} u(x)dS = \frac{1}{n\omega_n}\int_{|\omega|=1} u(r\omega)d\omega > 0, \tag{3.27}$$

where ω_n denotes the volume of the unit ball in R^n. We may assume that the unit ball is contained in B (otherwise we just use a smaller ball). We clearly have

$$y''(r) + \frac{n-1}{r}y'(r) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r} \Delta u(x)dS. \tag{3.28}$$

Since $u(x)$ is a supersolution of (3.23), we have

$$\int_{\partial B_r} div(V(|x|)\nabla u)ds - \int_{\partial B} W(|x|)udx \geq 0,$$

and therefore,

$$V(r)\int_{\partial B_r}\Delta u dS - V_r(r)\int_{\partial B_r}\nabla u.x ds - W(r)\int_{\partial B_r} u(x)ds \geq 0.$$

It follows that

$$V(r)\int_{\partial B_r}\Delta u dS - V_r(r)y'(r) - W(r)y(r) \geq 0, \tag{3.29}$$

and in view of (3.27), we see that y satisfies the inequality

$$V(r)y''(r) + (\frac{(n-1)V(r)}{r} + V_r(r))y'(r) \leq -W(r)y(r), \quad \textit{for} \quad 0 < r < R, \tag{3.30}$$

that is it is a positive supersolution for $(B_{V,W})$.

Standard results in ODE now allow us to conclude that $(B_{V,W})$ has actually a positive solution on $(0, R)$, and the proof of theorem 3.2 is now complete.

3.2.1 Integral criteria for Bessel pairs

In order to obtain criteria on V and W so that inequality $(\mathrm{H}_{V,W})$ holds, we clearly need to investigate whether the ordinary differential equation $(B_{V,W})$ has positive solutions. For that, we rewrite $(B_{V,W})$ as

$$(r^{n-1}V(r)y')' + r^{n-1}W(r)y = 0,$$

and then by setting $s = \frac{1}{r}$ and $x(s) = y(r)$, we see that y is a solution of $(B_{V,W})$ on an interval $(0, \delta)$ if and only if x is a positive solution for the equation

$$(s^{-(n-3)}V(\tfrac{1}{s})x'(s))' + s^{-(n+1)}W(\tfrac{1}{s})x(s) = 0 \quad \text{on} \quad (\tfrac{1}{\delta}, \infty). \tag{3.31}$$

Now recall that a solution $x(s)$ of the equation (3.31) is said to be *oscillatory* if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \to +\infty$ and $x(a_n) = 0$. Otherwise we call the solution *non-oscillatory*. It follows from Sturm comparison theorem that all solutions of (3.31) are either all oscillatory or all non-oscillatory. Hence, the fact that (V, W) is a Bessel pair or not is closely related to the oscillatory behavior of the equation (3.31). The following theorem is therefore a consequence of Theorem 3.2, combined with a relatively recent result of Sugie et al. in [29] about the oscillatory behavior of the equation (3.31).

Let V and W be positive radial C^1-functions on $B_R\backslash\{0\}$, where B_R is a ball centered at 0 with radius R in R^n $(n \geq 1)$. Assume $\int_0^R \frac{1}{\tau^{n-1}V(\tau)}d\tau = +\infty$ and $\int_0^R r^{n-1}v(r)dr < \infty$.

- Assume

$$\limsup_{r\to 0} r^{2(n-1)}V(r)W(r)\Big(\int_r^R \frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2 < \frac{1}{4} \tag{3.32}$$

  then (V, W) is a Bessel pair on $(0, \rho)$ for some $\rho > 0$ and consequently, inequality $(\mathrm{H}_{V,W})$ holds for all $u \in C_0^{\infty}(B_\rho)$, where B_ρ is a ball of radius ρ.

- On the other hand, if

$$\liminf_{r\to 0} r^{2(n-1)}V(r)W(r)\Big(\int_r^R \frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2 > \frac{1}{4} \tag{3.33}$$

  then there is no interval $(0,\rho)$ on which (V,W) is a Bessel pair and consequently, there is no smooth domain Ω on which inequality $(\mathrm{H}_{V,W})$ holds.

A typical Bessel pair is $(|x|^{-\lambda}, |x|^{-\lambda-2})$ for $\lambda \le n-2$. It is also easy to see by a simple change of variables in the corresponding ODEs that

$$W \text{ is a Bessel potential if and only if } \big(|x|^{-\lambda}, |x|^{-\lambda}(|x|^{-2}+W(|x|)\big) \text{ is a Bessel pair.} \tag{3.34}$$

More generally, the above integral criterium allows to show the following.

Let V be an strictly positive C^1-function on $(0,R)$ such that for some $\lambda \in R$

$$\tfrac{rV_r(r)}{V(r)} + \lambda \ge 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0} \tfrac{rV_r(r)}{V(r)} + \lambda = 0. \tag{3.35}$$

If $\lambda \le n-2$, then for any Bessel potential W on $(0,R)$, and any $c \le \beta(W;R)$, the couple $(V, W_{\lambda,c})$ is a Bessel pair, where

$$W_{\lambda,c}(r) = V(r)((\frac{n-\lambda-2}{2})^2 r^{-2} + cW(r)). \tag{3.36}$$

Moreover, $\beta\big(V, W_{\lambda,c}; R\big) = 1$ for all $c \le \beta(W;R)$.

We need the following easy lemma.

Lemma 3.37. *Assume the equation*

$$y'' + \frac{a}{r}y' + V(r)y = 0,$$

has a positive solution on $(0,R)$, where $a \ge 1$ and $V(r) > 0$. Then y is strictly decreasing on $(0,R)$.

Proof: First observe that y can not have a local minimum, hence it is either increasing or decreasing on $(0,\delta)$, for δ sufficiently small. Assume y is increasing. Under this assumption if $y'(a) = 0$ for some $a > 0$, then $y''(a) = 0$ which contradicts the fact that y is a positive solution of the above ODE. So we have $\frac{y''}{y'} \le -\frac{a}{r}$, thus,

$$y' \ge \frac{c}{r^a}.$$

Therefore, $x(r) \to -\infty$ as $r \to 0$ which is a contradiction. Since, y can not have a local minimum it should be strictly decreasing on $(0,R)$. □

Proof of Theorem 3.2.1: Write $\frac{V_r(r)}{V(r)} = -\frac{\lambda}{r} + f(r)$ where $f(r) \ge 0$ on $(0,R)$

and $\lim_{r\to 0} rf(r) = 0$. In order to prove that $\big(V(r), V(r)((\frac{n-\lambda-2}{2})^2 r^{-2} + cW(r))\big)$ is a Bessel pair, we need to show that the equation

$$y'' + (\frac{n-\lambda-1}{r} + f(r))y' + ((\frac{n-\lambda-2}{2})^2 r^{-2} + cW(r))y(r) = 0, \tag{3.38}$$

has a positive solution on $(0, R)$. But first we note that the equation

$$x'' + (\frac{n-\lambda-1}{r})x' + ((\frac{n-\lambda-2}{2})^2 r^{-2} + cW(r))x(r) = 0,$$

has a positive solution on $(0, R)$, whenever $c \leq \beta(W; R)$. Since now $f(r) \geq 0$ and since, by the proceeding lemma, $x'(r) \leq 0$, we get that x is a positive subsolution for the equation (3.38) on $(0, R)$, and thus it has a positive solution of $(0, R)$. Note that this means that $\beta(V, W_{\lambda,c}; R) \geq 1$.

For the reverse inequality, we shall use the criterium in Theorem 3.2.1. Indeed apply criteria (3.32) to $V(r)$ and $W_1(r) = C\frac{V(r)}{r^2}$ to get

$$\begin{aligned}
& \lim_{r\to 0} r^{2(n-1)} V(r) W_1(r) \Big(\int_r^R \frac{1}{\tau^{n-1} V(\tau)} d\tau \Big)^2 \\
= \ & C \lim_{r\to 0} r^{2(n-2)} V^2(r) \Big(\int_r^R \frac{1}{\tau^{n-1} V(\tau)} d\tau \Big)^2 \\
= \ & C \Big(\lim_{r\to 0} r^{(n-2)} V(r) \int_r^R \frac{1}{\tau^{n-1} V(\tau)} d\tau \Big)^2 \\
= \ & C \Big(\lim_{r\to 0} \frac{\frac{1}{r^{n-1} V(r)}}{\frac{(n-2) r^{n-3} V(r) + r^{n-2} V_r(r)}{r^{2(n-2)} V^2(r)}} \Big)^2 \\
= \ & C \Big(\lim_{r\to 0} \frac{1}{(n-2) + r\frac{V_r(r)}{V(r)}} \Big)^2 \\
= \ & \frac{C}{(n-\lambda-2)^2}.
\end{aligned}$$

For $\big(V, CV(r^{-2} + cW)\big)$ to be a Bessel pair, it is necessary that $\frac{C}{(n-\lambda-2)^2} \leq \frac{1}{4}$, and the proof for the best constant is complete. □

With a similar argument one can also prove the following. Let V and W be positive radial C^1-functions on $B_R \backslash \{0\}$, where B_R is a ball centered at zero with radius R in R^n $(n \geq 1)$. Assume that

$$\lim_{r\to 0} r\frac{V_r(r)}{V(r)} = -\lambda \text{ and } \lambda \leq n-2. \tag{3.39}$$

- If $\limsup_{r\to 0} r^2 \frac{W(r)}{V(r)} < (\frac{n-\lambda-2}{2})^2$, then (V, W) is a Bessel pair on some interval $(0, \rho)$, and consequently there exists a ball $B_\rho \subset R^n$ such that inequality $(\mathrm{H}_{V,W})$ holds for all $u \in C_0^\infty(B_\rho)$.
- On the other hand, if $\liminf_{r\to 0} r^2 \frac{W(r)}{V(r)} > (\frac{n-\lambda-2}{2})^2$, then there is no smooth domain $\Omega \subset R^n$ such that inequality $(\mathrm{H}_{V,W})$ holds on Ω.

3.2.2 New weighted Hardy inequalities

An immediate application of Theorem 3.2.1 and Theorem 3.2 is the following very general Hardy inequality.

Let $V(x) = V(|x|)$ be a strictly positive radial function on a smooth domain Ω containing 0 such that $R = \sup_{x\in\Omega} |x|$. Assume that for some $\lambda \in R$

$$\frac{rV_r(r)}{V(r)} + \lambda \geq 0 \text{ on } (0, R) \text{ and } \lim_{r\to 0} \frac{rV_r(r)}{V(r)} + \lambda = 0. \tag{3.40}$$

1. If $\lambda \leq n - 2$, then the following inequality holds for any Bessel potential W on $(0, R)$:

$$\int_\Omega V(x)|\nabla u|^2 dx \geq (\frac{n-\lambda-2}{2})^2 \int_\Omega \frac{V(x)}{|x|^2} u^2 dx + \beta(W;R) \int_\Omega V(x)W(x)u^2 dx, \tag{3.41}$$

   for all $u \in C_0^\infty(\Omega)$ and both $(\frac{n-\lambda-2}{2})^2$ and $\beta(W;R)$ are the best constants.

2. In particular, $\beta(V, r^{-2}V;R) = (\frac{n-\lambda-2}{2})^2$ is the best constant in the following inequality

$$\int_\Omega V(x)|\nabla u|^2 dx \geq (\frac{n-\lambda-2}{2})^2 \int_\Omega \frac{V(x)}{|x|^2} u^2 dx \quad \text{for all } u \in C_0^\infty(\Omega). \tag{3.42}$$

Applied to $V_1(r) = r^{-m}W_{k,\rho}(r)$ and $V_2(r) = r^{-m}\tilde{W}_{k,\rho}(r)$ where

$$W_{k,\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2} (\prod_{i=1}^{j} log^{(i)} \frac{\rho}{r})^{-2}$$

and $\tilde{W}_{k;\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2} X_1^2(\frac{r}{\rho}) X_2^2(\frac{r}{\rho}) \ldots X_{j-1}^2(\frac{r}{\rho}) X_j^2(\frac{r}{\rho})$ are the iterated logs introduced in the introduction, and noting that in both cases the corresponding λ is equal to $2m + 2$, we get the following new Hardy inequalities.

Let Ω be a smooth bounded domain in R^n ($n \geq 1$) and $m \leq \frac{n-4}{2}$. Then the following inequalities hold.

$$\int_\Omega \frac{W_{k,\rho}(x)}{|x|^{2m}} |\nabla u|^2 dx \geq (\frac{n-2m-4}{2})^2 \int_\Omega \frac{W_{k,\rho}(x)}{|x|^{2m+2}} u^2 dx \tag{3.43}$$

$$\int_\Omega \frac{\tilde{W}_{k,\rho}(x)}{|x|^{2m}} |\nabla u|^2 dx \geq (\frac{n-2m-4}{2})^2 \int_\Omega \frac{\tilde{W}_{k,\rho}(x)}{|x|^{2m+2}} u^2 dx. \tag{3.44}$$

Moreover, the constant $(\frac{n-2m-4}{2})^2$ is the best constant in both inequalities.

Remark 3.2.1. The two following theorems deal with Hardy-type inequalities on the whole of R^n. Theorem 3.2 already yields that inequality $(H_{V,W})$ holds for all $u \in C_0^\infty(R^n)$ if and only if the ODE $(B_{V,W})$ has a positive solution on $(0, \infty)$. The latter equation is therefore non-oscillatory, which will again be a very useful fact for computing best constants, in view of the following criterium at infinity (Theorem 2.1 in [29]) applied to the equation

$$(a(r)y')' + b(r)y(r) = 0, \tag{3.45}$$

where $a(r)$ and $b(r)$ are positive real valued functions. Assuming that $\int_d^\infty \frac{1}{a(\tau)} d\tau < \infty$ for some $d > 0$, and that the following limit

$$L := \lim_{r\to\infty} a(r)b(r) \left(\int_r^\infty \frac{1}{a(r)} dr \right)^2,$$

exists. Then for the equation (3.45) equation to be non-oscillatory, it is necessary that $L \leq \frac{1}{4}$.

Let $a, b > 0$, and α, β, m be real numbers.

- If $\alpha\beta > 0$, and $m \leq \frac{n-2}{2}$, then for all $u \in C_0^\infty(R^n)$

$$\int_{R^n} \frac{(a + b|x|^\alpha)^\beta}{|x|^{2m}} |\nabla u|^2 dx \geq (\frac{n-2m-2}{2})^2 \int_{R^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 dx, \tag{3.46}$$

  and $(\frac{n-2m-2}{2})^2$ is the best constant in the inequality.

- If $\alpha\beta < 0$, and $2m - \alpha\beta \leq n - 2$, then for all $u \in C_0^\infty(R^n)$

$$\int_{R^n} \frac{(a + b|x|^\alpha)^\beta}{|x|^{2m}} |\nabla u|^2 dx \geq (\frac{n-2m+\alpha\beta-2}{2})^2 \int_{R^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 dx, \tag{3.47}$$

  and $(\frac{n-2m+\alpha\beta-2}{2})^2$ is the best constant in the inequality.

Proof: Letting $V(r) = \frac{(a+br^\alpha)^\beta}{r^{2m}}$, then

$$r\frac{V'(r)}{V(r)} = -2m + \frac{b\alpha\beta r^\alpha}{a + br^\alpha} = -2m + \alpha\beta - \frac{a\alpha\beta}{a + br^\alpha}.$$

Hence, in the case $\alpha, \beta > 0$ and $2m \leq n - 2$, (3.46) follows directly from Theorem 3.2.2. The same holds for (3.47) since it also follows directly from Theorem 3.2.2 in the case where $\alpha < 0$, $\beta > 0$ and $2m - \alpha\beta \leq n - 2$.

For the remaining two other cases, we will use Theorem 3.2. Indeed, in this case the equation $(B_{V,W})$ becomes

$$y'' + (\frac{n-2m-1}{r} + \frac{b\alpha\beta r^{\alpha-1}}{a+br^\alpha})y' + \frac{c}{r^2}y = 0, \tag{3.48}$$

and the best constant in inequalities (3.46) and (3.47) is the largest c such that the above equation has a positive solution on $(0, +\infty)$. Note that by Lemma 3.37, we have that $y' < 0$ on $(0, +\infty)$. Hence, if $\alpha < 0$ and $\beta < 0$, then the positive solution of the equation

$$y'' + \frac{n-2m-1}{r}y' + \frac{(\frac{n-2m-2}{2})^2}{r^2}y = 0$$

is a positive super-solution for (3.48) and therefore the latter ODE has a positive solution on $(0, +\infty)$, from which we conclude that (3.46) holds. To prove now that $(\frac{n-2m-2}{2})^2$ is the best constant in (3.46), we use the fact that if the equation (3.48) has a positive solution on $(0, +\infty)$, then the equation is necessarily non-oscillatory. By rewriting (3.48) as

$$\left(r^{n-2m-1}(a+br^{\alpha})^{\beta}y'\right)' + cr^{n-2m-3}(a+br^{\alpha})^{\beta}y = 0, \tag{3.49}$$

and by noting that

$$\int_d^{\infty} \frac{1}{r^{n-2m-1}(a+br^{\alpha})^{\beta}} < \infty,$$

and

$$\lim_{r\to\infty} cr^{2(n-2m-2)}(a+br^{\alpha})^{2\beta}\left(\int_r^{\infty} \frac{1}{r^{n-2m-1}(a+br^{\alpha})^{\beta}}dr\right)^2 = \frac{c}{(n-2m-2)^2},$$

we can use Theorem 2.1 in [29] to conclude that for equation (3.49) to be non-oscillatory it is necessary that

$$\frac{c}{(n-2m-2)^2} \le \frac{1}{4}.$$

Thus, $\frac{(n-2m-2)^2}{4}$ is the best constant in the inequality (3.46).

A very similar argument applies in the case where $\alpha > 0$, $\beta < 0$, and $2m < n-2$, to obtain that inequality (3.47) holds for all $u \in C_0^{\infty}(R^n)$ and that $(\frac{n-2m+\alpha\beta-2}{2})^2$ is indeed the best constant. □

Note that the above two inequalities can be improved on smooth bounded domains by using Theorem 3.2.2.

We shall now extend the recent results of Blanchet-Bonforte-Dolbeault-Grillo-Vasquez [4] and address some of their questions regarding best constants.

Let $a, b > 0$, and α, β be real numbers.

- If $\alpha\beta < 0$ and $-\alpha\beta \le n-2$, then for all $u \in C_0^{\infty}(R^n)$

$$\int_{R^n}(a+b|x|^{\alpha})^{\beta}|\nabla u|^2 dx \ge b^{\frac{2}{\alpha}}(\frac{n-\alpha\beta-2}{2})^2\int_{R^n}(a+b|x|^{\alpha})^{\beta-\frac{2}{\alpha}}u^2 dx, \tag{3.50}$$

  and $b^{\frac{2}{\alpha}}(\frac{n-\alpha\beta-2}{2})^2$ is the best constant in the inequality.

- If $\alpha\beta > 0$ and $n \ge 2$, then there exists a constant $C > 0$ such that for all $u \in C_0^{\infty}(R^n)$

$$\int_{R^n}(a+b|x|^{\alpha})^{\beta}|\nabla u|^2 dx \ge C\int_{R^n}(a+b|x|^{\alpha})^{\beta-\frac{2}{\alpha}}u^2 dx. \tag{3.51}$$

Moreover, $b^{\frac{2}{\alpha}}(\frac{n-2}{2})^2 \leq C \leq b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$.

Proof: Letting $V(r) = (a+br^\alpha)^\beta$, then we have

$$r\frac{V'(r)}{V(r)} = \frac{b\alpha\beta r^\alpha}{a+br^\alpha} = \alpha\beta - \frac{a\alpha\beta}{a+br^\alpha}.$$

Inequality (3.50) and its best constant in the case when $\alpha < 0$ and $\beta > 0$, then follow immediately from Theorem 3.2.2 with $\lambda = -\alpha\beta$. The proof of the remaining cases will use Theorem 3.2 as well as the integral criteria for the oscillatory behavior of solutions for ODEs of the form $(B_{V,W})$.

Assuming still that $\alpha\beta < 0$, then with an argument similar to that of Theorem 3.2.2 above, one can show that the positive solution of the equation $y'' + (\frac{n+\alpha\beta-1}{r})y' + \frac{(n+\alpha\beta-2)^2}{4r^2}y = 0$ on $(0,+\infty)$ is a positive supersolution for the equation

$$y'' + (\frac{n-1}{r} + \frac{V'(r)}{V(r)})y' + \frac{b^{\frac{2}{\alpha}}(n+\alpha\beta-2)^2}{4(a+br^\alpha)^{\frac{2}{\alpha}}}y = 0.$$

Theorem 3.2 then yields that the inequality (3.50) holds for all $u \in C_0^\infty(R^n)$. To prove now that $b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$ is the best constant in (3.50) it is enough to show that if the following equation

$$\left(r^{n-1}(a+br^\alpha)^\beta y'\right)' + cr^{n-1}(a+br^\alpha)^{\beta-\frac{2}{\alpha}}y = 0 \tag{3.52}$$

has a positive solution on $(0,+\infty)$, then $c \leq b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$. If now $\alpha > 0$ and $\beta < 0$, then we have

$$\lim_{r\to\infty} cr^{2(n-1)}(a+br^\alpha)^{2\beta-\frac{2}{\alpha}}\left(\int_r^\infty \frac{1}{r^{n-1}(a+br^\alpha)^\beta}dr\right)^2 = \frac{c}{b^{\frac{2}{\alpha}}(n+\alpha\beta-2)^2}.$$

Hence, by Theorem 2.1 in [29] again, the non-oscillatory aspect of the equation holds for $c \leq \frac{b^{\frac{2}{\alpha}}(n+\alpha\beta-2)^2}{4}$ which completes the proof of the first part.
A similar argument applies in the case where $\alpha\beta > 0$ to prove that (3.51) holds for all $u \in C_0^\infty(R^n)$ and $b^{\frac{2}{\alpha}}(\frac{n-2}{2})^2 \leq C \leq b^{\frac{2}{\alpha}}(\frac{n+\alpha\beta-2}{2})^2$. The best constants are estimated by carefully studying the existence of positive solutions for the ODE (3.52).

Remark 3.2.2. Recently, Blanchet et al. in [4] studied a special case of inequality (3.50) ($a = b = 1$, and $\alpha = 2$) under the additional condition:

$$\int_{R^n}(1+|x|^2)^{\beta-1}u(x)dx = 0, \quad for \quad \beta < \frac{n-2}{2}. \tag{3.53}$$

Note that we do not assume (3.53) in Theorem 3.2.2, and that we have found the best constants for $\beta \leq 0$, a case that was left open in [4].

3.2.3 Improved Hardy and Caffarelli-Kohn-Nirenberg Inequalities

In [11] Caffarelli-Kohn-Nirenberg established a set inequalities of the following form:

$$\Big(\int_{R^n}|x|^{-bp}|u|^p dx\Big)^{\frac{2}{p}} \le C_{a,b}\int_{R^n}|x|^{-2a}|\nabla u|^2 dx \text{ for all } u\in C_0^\infty(R^n), \tag{3.54}$$

where for $n\ge 3$,

$$-\infty < a < \tfrac{n-2}{2},\ a\le b\le a+1, \text{ and } p=\tfrac{2n}{n-2+2(b-a)}. \tag{3.55}$$

For the cases $n=2$ and $n=1$ the conditions are slightly different. For $n=2$

$$-\infty < a < 0,\ a < b\le a+1, \text{ and } p=\tfrac{2}{b-a}, \tag{3.56}$$

and for $n=1$

$$-\infty < a < -\tfrac{1}{2},\ a+\tfrac{1}{2} < b\le a+1, \text{ and } p=\tfrac{2}{-1+2(b-a)}. \tag{3.57}$$

Let $D_a^{1,2}$ be the completion of $C_0^\infty(R^n)$ for the inner product

$$(u,v)=\int_{R^n}|x|^{-2a}\nabla u.\nabla v dx$$

and let

$$S(a,b)=\inf_{u\in D_a^{1,2}\setminus\{0\}}\frac{\int_{R^n}|x|^{-2a}|\nabla u|^2 dx}{\big(\int_{R^n}|x|^{-bp}|u|^p dx\big)^{2/p}} \tag{3.58}$$

denote the best embedding constant. We are concerned here with the "Hardy critical" case of the above inequalities, that is when $b=a+1$. In this direction, Catrina and Wang [14] showed that for $n\ge 3$ we have $S(a,a+1)=(\frac{n-2a-2}{2})^2$ and that $S(a,a+1)$ is not achieved while $S(a,b)$ is always achieved for $a<b<a+1$. For the case $n=2$ they also showed that $S(a,a+1)=a^2$, and that $S(a,a+1)$ is not achieved, while for $a<b<a+1$, $S(a,b)$ is again achieved. For $n=1$, $S(a,a+1)=(\frac{1+2a}{2})^2$ is also not achieved.

In this section we give a necessary and sufficient condition for improvement of (3.54) with $b=a+1$ and $n\ge 1$. Our results cover also the critical case when $a=\frac{n-2}{2}$ which is not allowed by the methods of [11].

Let W be a positive radial function on the ball B in R^n ($n\ge 1$) with radius R and centered at zero. Assume $a\le\frac{n-2}{2}$. The following two statements are then equivalent:

1. W is a Bessel potential on $(0,R)$.
2. There exists $c>0$ such that the following inequality holds for all $u\in C_0^\infty(B)$

$$(\mathrm{H}_{a,cW})\qquad \int_B|x|^{-2a}|\nabla u(x)|^2dx \ \ge\ (\frac{n-2a-2}{2})^2\int_B|x|^{-2a-2}u^2dx + c\int_B|x|^{-2a}W(x)u^2dx,$$

Moreover, $(\frac{n-2a-2}{2})^2$ is the best constant and $\beta(W;R) = \sup\{c; (H_{a,cW}) holds\}$, where $\beta(W;R)$ is the weight of W on $(0,R)$.

On the other hand, there is no strictly positive $W \in C^1(0,\infty)$, such that the following inequality holds for all $u \in C_0^\infty(R^n)$,

$$\begin{aligned} \int_{R^n} |x|^{-2a}|\nabla u(x)|^2 dx &\geq (\frac{n-2a-2}{2})^2 \int_{R^n} |x|^{-2a-2}u^2 dx \\ &+ c\int_{R^n} W(|x|)u^2 dx. \end{aligned} \quad (3.59)$$

$$(3.60)$$

Proof: It suffices to use Theorems 3.2 and 3.2.2 with $V(r) = r^{-2a}$ to get that W is a Bessel function if and only if the pair $\left(r^{-2a}, W_{a,c}(r)\right)$ is a Bessel pair on $(0,R)$ for some $c > 0$, where

$$W_{a,c}(r) = (\frac{n-2a-2}{2})^2 r^{-2-2a} + cr^{-2a}W(r).$$

For the last part, assume that (3.59) holds for some W. Then it follows from Theorem 3.2.3 that for $V = cr^{2a}W(r)$ the equation $y''(r) + \frac{1}{r}y' + v(r)y = 0$ has a positive solution on $(0,\infty)$. From Lemma 3.37 we know that y is strictly decreasing on $(0,+\infty)$. Hence, $\frac{y''(r)}{y'(r)} \geq -\frac{1}{r}$ which yields $y'(r) \leq \frac{b}{r}$, for some $b > 0$. Thus $y(r) \to -\infty$ as $r \to +\infty$. This is a contradiction and the proof is complete. □

Remark 3.2.3. Theorem 3.2.3 characterizes the best constant only when Ω is a ball, while for general domain Ω, it just gives a lower and upper bounds for the best constant corresponding to a given Bessel potential W. It is indeed clear that

$$C_{B_R}(W) \leq C_\Omega(W) \leq C_{B_\rho}(W),$$

where B_R is the smallest ball containing Ω and B_ρ is the largest ball contained in it. If now W is a Bessel potential such that $\beta(W,R)$ is independent of R, then clearly $\beta(W,R)$ is also the best constant in inequality $(H_{a,cW})$ for any smooth bounded domain. This is clearly the case for the potentials $W_{k,\rho}$ and $\tilde{W}_{k,\rho}$ where $\beta(W,R) = \frac{1}{4}$ for all R, while for $W \equiv 1$ the best constant is still not known for general domains even for the simplest case $a = 0$.

Using the integral criteria for Bessel potentials, we can also deduce immediately the following.

Let Ω be a bounded smooth domain in R^n with $n \geq 1$, and let W be a non-negative function in $C^1(0, R =: \sup_{x\in\partial\Omega}|x|]$ and $a \leq \frac{n-2}{2}$.

1. If $\liminf_{r\to 0} \ln(r)\int_0^r sW(s)ds > -\infty$, then there exists $\alpha := \alpha(\Omega) > 0$ such that an improved Hardy inequality $(\mathrm{H}_{a,W_\alpha})$ holds for the scaled potential $W_\alpha(x) := \alpha^2 W(\alpha|x|)$.
2. If $\lim_{r\to 0} \ln(r)\int_0^r sW(s)ds = -\infty$, then there are no $\alpha, c > 0$, for which $(\mathrm{H}_{a,W_{\alpha,c}})$ holds with $W_{\alpha,c} = cW(\alpha|x|)$.

By applying the above to various examples of Bessel potentials, we can now deduce several old and new inequalities. The first is an extension of a result established by Brezis and Vázquez [10] in the case where $a = 0$, and $b = 0$.

Let Ω be a bounded smooth domain in R^n with $n \geq 1$ and $a \leq \frac{n-2}{2}$. Then, for any $b < 2a+2$ there exists $c > 0$ such that for all $u \in C_0^\infty(\Omega)$

$$\int_\Omega |x|^{-2a}|\nabla u|^2 dx \geq (\tfrac{n-2a-2}{2})^2 \int_\Omega |x|^{-2a-2}u^2 dx + c\int_\Omega |x|^{-b}u^2 dx. \tag{3.61}$$

Moreover, when Ω is a ball B of radius R the best constant c for which (3.61) holds is equal to the weight $\beta(r^{2a-b}; R)$ of the Bessel potential $W(r) = r^{2a-b}$ on $(0, R]$. In particular,

$$\int_B |x|^{-2a}|\nabla u|^2 dx \geq (\tfrac{n-2a-2}{2})^2 \int_B |x|^{-2a-2}u^2 dx + \lambda_B \int_B |x|^{-2a}u^2 dx, \tag{3.62}$$

where the best constant λ_B is equal to $z_0\omega_n^{2/n}|\Omega|^{-2/n}$, where ω_n and $|\Omega|$ denote the volume of the unit ball and Ω respectively, and $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$.

Proof: It suffices to apply Theorem 3.2.3 with the function $W(r) = r^{b+2a}$ which is a Bessel potential whenever $b > -2a-2$ since then

$$\liminf_{r\to 0} \ln(r) \int_0^r s^{2a+1}W(s)ds > -\infty$$

. In the case where $b = -2a$ and therefore $W \equiv 1$, we use the fact that $\beta(1; R) = \frac{z_0^2}{R^2}$ (established in section 3.5) to deduce that the best constant is then equal to $z_0\omega_n^{2/n}|\Omega|^{-2/n}$. □

The following corollary is an extension of a recent result by Adimurthi et all [1] established in the case where $a = 0$, and of another result by Wang and Willem in [34] (Theorem 2) in the case $k = 1$. We also provide here the value of the best constant. Let B be a bounded smooth domain in R^n with $n \geq 1$ and $a \leq \frac{n-2}{2}$. Then for every integer k, and $\rho = (\sup_{x\in\Omega}|x|)(e^{e^{e^{\cdot^{\cdot^{e((k-1)-times)}}}}})$, we have for any $u \in H_0^1(\Omega)$,

$$\begin{aligned}\int_\Omega |x|^{-2a}|\nabla u|^2 dx \quad &\geq \quad (\frac{n-2a-2}{2})^2 \int_\Omega \frac{u^2}{|x|^{2a+2}}dx \\ &+ \quad \frac{1}{4}\sum_{j=1}^k \int_\Omega \frac{|u|^2}{|x|^{2a+2}}\Big(\prod_{i=1}^j log^{(i)}\frac{\rho}{|x|}\Big)^{-2}dx. \end{aligned}\tag{3.63}$$

Moreover, $\frac{1}{4}$ is the best constant which is not attained in $H_0^1(\Omega)$.

Proof: As seen in section 3.5, $W_{k,\rho}(r) = \sum_{j=1}^k \frac{1}{r^2}\big(\prod_{i=1}^j log^{(i)}\frac{\rho}{|x|}\big)^{-2}dx$ is a Bessel potential on $(0, R)$ where $R = \sup_{x\in\Omega}|x|$, and $\beta(W_{k,\rho}; R) = \frac{1}{4}$. □

The very same reasoning leads to the following extension of a result established by Filippas and Tertikas [17] in the case where $a = 0$. Let Ω be a bounded smooth domain in R^n with $n \geq 1$ and $a \leq \frac{n-2}{2}$. Then for every integer k, and any $D \geq \sup_{x\in\Omega}|x|$, we have for $u \in H_0^1(\Omega)$,

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^2}{|x|^{2a}}dx &\geq (\frac{n-2a-2}{2})^2\int_\Omega \frac{u^2}{|x|^{2a+2}}dx \\ &+ \frac{1}{4}\sum_{i=1}^{\infty}\int_\Omega \frac{1}{|x|^{2a+2}}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})...X_i^2(\frac{|x|}{D})|u|^2dx, \end{aligned} \quad (3.64)$$

and $\frac{1}{4}$ is the best constant which is not attained in $H_0^1(\Omega)$.

The classical Hardy inequality is valid for dimensions $n \geq 3$. We now present optimal Hardy type inequalities for dimension two in bounded domains, as well as the corresponding best constants. Let Ω be a smooth domain in R^2 and $0 \in \Omega$. Then we have the following inequalities.

- Let $D \geq \sup_{x\in\Omega}|x|$, then for all $u \in H_0^1(\Omega)$,

  $$\int_\Omega |\nabla u|^2dx \geq \frac{1}{4}\sum_{i=1}^{\infty}\int_\Omega \frac{1}{|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})...X_i^2(\frac{|x|}{D})|u|^2dx \quad (3.65)$$

  and $\frac{1}{4}$ is the best constant.

- Let $\rho = (\sup_{x\in\Omega}|x|)(e^{e^{e^{\cdot^{\cdot^{e((k-1)-times)}}}}})$, then for all $u \in H_0^1(\Omega)$

  $$\int_\Omega |\nabla u|^2dx \geq \frac{1}{4}\sum_{j=1}^{k}\int_\Omega \frac{|u|^2}{|x|^2}\left(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{|x|}\right)^{-2}dx, \quad (3.66)$$

  and $\frac{1}{4}$ is the best constant for all $k \geq 1$.

- If $\alpha < 2$, then there exists $c > 0$ such that for all $u \in H_0^1(\Omega)$,

  $$\int_\Omega |\nabla u|^2dx \geq c\int_\Omega \frac{u^2}{|x|^\alpha}\,dx, \quad (3.67)$$

  and the best constant is larger or equal to $\beta(r^\alpha; \sup_{x\in\Omega}|x|)$.

An immediate application of Theorem 3.2 coupled with Hölder's inequality gives the following duality statement, which should be compared to inequalities dual to those of Sobolev's, recently obtained via the theory of mass transport [3, 13].

Suppose that Ω is a smooth bounded domain containing 0 in R^n $(n \geq 1)$ with $R := \sup_{x\in\Omega}|x|$. Then, for any $a \leq \frac{n-2}{2}$ and $0 < p \leq 2$, we have the following dual inequalities:

$$\begin{aligned}\inf\left\{\int_\Omega |x|^{-2a}|\nabla u|^2dx - (\frac{n-2a-2}{2})^2\int_\Omega |x|^{-2a-2}|u|^2dx;\ u \in C_0^\infty(\Omega), ||u||_p = 1\right\} \\ \geq \sup\left\{\left(\int_\Omega (\frac{|x|^{-2a}}{W(x)})^{\frac{p}{p-2}}\,dx\right)^{\frac{2-p}{p}};\ W \in \mathcal{B}(0,R)\right\}.\end{aligned}$$

3.3 General Hardy-Rellich inequalities

Let $0 \in \Omega \subset R^n$ be a smooth domain, and denote

$$C^k_{0,r}(\Omega) = \{v \in C^k_0(\Omega) : \text{v is radial and supp v } \subset \Omega\},$$

$$H^m_{0,r}(\Omega) = \{u \in H^m_0(\Omega) : \text{u is radial}\}.$$

We start by considering a general inequality for radial functions.

Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in R^n $(n \geq 1)$ and centered at zero. Assume $\int_0^R \frac{1}{r^{n-1}V(r)}dr = \infty$ and $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < n-2$. Then the following statements are equivalent:

1. (V, W) is a Bessel pair on $(0, R)$.
2. There exists $c > 0$ such that the following inequality holds for all radial functions $u \in C^\infty_{0,r}(B)$

$$\begin{aligned}(\text{HR}_{V,cW}) \int_B V(x)|\Delta u|^2 dx \quad &\geq \quad c\int_B W(x)|\nabla u|^2 dx \\ &+ \quad (n-1)\int_B \Big(\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|}\Big)|\nabla u|^2 dx.\end{aligned}$$

Moreover, the best constant is given by

$$\beta(V, W; R) = \sup\big\{c;\ (\text{HR}_{V,cW}) \text{ holds for radial functions}\big\}. \tag{3.68}$$

Proof: Assume $u \in C^\infty_{0,r}(B)$ and observe that

$$\begin{aligned}\int_B V(x)|\Delta u|^2 dx \quad &= \quad n\omega_n\{\int_0^R V(r)u_{rr}^2 r^{n-1}dr \\ &+ \quad (n-1)^2\int_0^R V(r)\frac{u_r^2}{r^2}r^{n-1}dr + 2(n-1)\int_0^R V(r)uu_r r^{n-2}dr\}.\end{aligned}$$

Setting $\nu = u_r$, we then have

$$\int_B V(x)|\Delta u|^2 dx = \int_B V(x)|\nabla \nu|^2 dx + (n-1)\int_B \Big(\frac{V(|x|)}{|x|^2} - \frac{V_r(|x|)}{|x|}\Big)|\nu|^2 dx.$$

Thus, $(\text{HR}_{V,W})$ for radial functions is equivalent to

$$\int_B V(x)|\nabla \nu|^2 dx \geq \int_B W(x)\nu^2 dx.$$

Letting $x(r) = \nu(x)$ where $|x| = r$, we then have

$$\int_0^R V(r)(x'(r))^2 r^{n-1}dr \geq \int_0^R W(r)x^2(r)r^{n-1}dr. \tag{3.69}$$

It therefore follows from Theorem 3.2 that 1) and 2) are equivalent. □

By applying the above theorem to the Bessel pair

$$V(x) = |x|^{-2m} \quad \text{and} \quad W_m(x) = V(x)\big[(\tfrac{n-2m-2}{2})^2|x|^{-2} + W(x)\big]$$

where W is a Bessel potential, and by using Theorem 3.2.2, we get the following result in the case of radial functions.

Suppose $n \geq 1$ and $m < \frac{n-2}{2}$. Let $B_R \subset R^n$ be a ball of radius $R > 0$ and centered at zero. Let W be a Bessel potential on $(0, R)$. Then we have for all $u \in C^\infty_{0,r}(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq (\frac{n+2m}{2})^2 \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W;R) \int_{B_R} W(x) \frac{|\nabla u|^2}{|x|^{2m}} dx. \quad (3.70)$$

Moreover, $(\frac{n+2m}{2})^2$ and $\beta(W;R)$ are the best constants.

3.3.1 The non-radial case

The decomposition of a function into its spherical harmonics will be one of our tools to prove the corresponding result in the non-radial case. This idea has also been used in [31]. Any function $u \in C^\infty_0(\Omega)$ could be extended by zero outside Ω, and could therefore be considered as a function in $C^\infty_0(R^n)$. By decomposing u into spherical harmonics we get

$$u = \Sigma^\infty_{k=0} u_k \text{ where } u_k = f_k(|x|)\varphi_k(x)$$

and $(\varphi_k(x))_k$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues $c_k = k(n+k-2)$, $k \geq 0$. The functions f_k belong to $C^\infty_0(\Omega)$ and satisfy $f_k(r) = O(r^k)$ and $f'(r) = O(r^{k-1})$ as $r \to 0$. In particular,

$$\varphi_0 = 1 \text{ and } f_0 = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} u ds = \frac{1}{n\omega_n} \int_{|x|=1} u(rx) ds. \quad (3.71)$$

We also have for any $k \geq 0$, and any continuous real valued functions v and w on $(0, \infty)$,

$$\int_{R^n} V(|x|)|\Delta u_k|^2 dx = \int_{R^n} V(|x|)\big(\Delta f_k(|x|) - c_k \frac{f_k(|x|)}{|x|^2}\big)^2 dx, \quad (3.72)$$

and

$$\int_{R^n} W(|x|)|\nabla u_k|^2 dx = \int_{R^n} W(|x|)|\nabla f_k|^2 dx + c_k \int_{R^n} W(|x|)|x|^{-2} f_k^2 dx. \quad (3.73)$$

Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in R^n $(n \geq 1)$ and centered at zero. Assume $\int_0^R \frac{1}{r^{n-1}V(r)} dr = \infty$ and $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < (n-2)$. If

$$W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0 \quad for \quad 0 \leq r \leq R, \quad (3.74)$$

then the following statements are equivalent.

1. (V,W) is a Bessel pair with $\beta(V,W;R)\geq 1$.
2. The following inequality holds for all $u\in C_0^\infty(B)$,

$$
\begin{aligned}
(\mathrm{HR}_{V,W}) \qquad \int_B V(x)|\Delta u|^2dx &\geq \int_B W(x)|\nabla u|^2dx \\
&+ (n-1)\int_B \Big(\frac{V(x)}{|x|^2}-\frac{V_r(|x|)}{|x|}\Big)|\nabla u|^2dx.
\end{aligned}
$$

Moreover, if $\beta(V,W;R)\geq 1$, then the best constant is given by

$$\beta(V,W;R)=\sup\big\{c;\ (\mathrm{HR}_{V,cW}) \text{ holds}\big\}. \tag{3.75}$$

Proof: That 2) implies 1) follows from Theorem 3.3 and does not require condition (3.74). To prove that 1) implies 2) assume that the equation $(B_{V,W})$ has a positive solution on $(0,R]$. We prove that the inequality $(HR_{V,W})$ holds for all $u\in C_0^\infty(B)$ by frequently using that

$$\textstyle\int_0^R V(r)|x'(r)|^2r^{n-1}dr\geq\int_0^R W(r)x^2(r)r^{n-1}dr \text{ for all } x\in C^1(0,R]. \tag{3.76}$$

Indeed, for all $n\geq 1$ and $k\geq 0$ we have

$$
\begin{aligned}
&\frac{1}{nw_n}\int_{R^n}V(x)|\Delta u_k|^2dx=\frac{1}{nw_n}\int_{R^n}V(x)\big(\Delta f_k(|x|)-c_k\frac{f_k(|x|)}{|x|^2}\big)^2dx\\
= &\int_0^R V(r)\big(f_k''(r)+\frac{n-1}{r}f_k'(r)-c_k\frac{f_k(r)}{r^2}\big)^2r^{n-1}dr\\
= &\int_0^R V(r)(f_k''(r))^2r^{n-1}dr+(n-1)^2\int_0^R V(r)(f_k'(r))^2r^{n-3}dr\\
+ &\ c_k^2\int_0^R V(r)f_k^2(r)r^{n-5}+2(n-1)\int_0^R V(r)f_k''(r)f_k'(r)r^{n-2}\\
- &\ 2c_k\int_0^R V(r)f_k''(r)f_k(r)r^{n-3}dr-2c_k(n-1)\int_0^R V(r)f_k'(r)f_k(r)r^{n-4}dr.
\end{aligned}
$$

Integrate by parts and use (3.71) for $k=0$ to get

$$
\begin{aligned}
&\frac{1}{n\omega_n}\int_{R^n}V(x)|\Delta u_k|^2dx \qquad (3.77)\\
= &\int_0^R V(r)(f_k''(r))^2r^{n-1}dr+(n-1+2c_k)\int_0^R V(r)(f_k'(r))^2r^{n-3}dr\\
+ &\ (2c_k(n-4)+c_k^2)\int_0^R V(r)r^{n-5}f_k^2(r)dr-(n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr\\
- &\ c_k(n-5)\int_0^R V_r(r)f_k^2(r)r^{n-4}dr-c_k\int_0^R V_{rr}(r)f_k^2(r)r^{n-3}dr.
\end{aligned}
$$

Now define $g_k(r) = \frac{f_k(r)}{r}$ and note that $g_k(r) = O(r^{k-1})$ for all $k \geq 1$. We have

$$\begin{aligned}
&\int_0^R V(r)(f_k'(r))^2 r^{n-3} \\
= &\int_0^R V(r)(g_k'(r))^2 r^{n-1} dr + \int_0^R 2V(r) g_k(r) g_k'(r) r^{n-2} dr + \int_0^R V(r) g_k^2(r) r^{n-3} dr \\
= &\int_0^R V(r)(g_k'(r))^2 r^{n-1} dr - (n-3)\int_0^R V(r) g_k^2(r) r^{n-3} dr - \int_0^R V_r(r) g_k^2(r) r^{n-2} dr
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^R V(r)(f_k'(r))^2 r^{n-3} &\geq \int_0^R W(r) f_k^2(r) r^{n-3} dr \\
- \quad (n-3)\int_0^R V(r) f_k^2(r) r^{n-5} dr &- \int_0^R V_r(r) f_k^2(r) r^{n-4} dr. \qquad (3.78)
\end{aligned}$$

Substituting $2c_k \int_0^R V(r)(f_k'(r))^2 r^{n-3}$ in (3.78) by its lower estimate in the last inequality (3.78), we get

$$\begin{aligned}
&\frac{1}{n\omega_n}\int_{R^n} V(x)|\Delta u_k|^2 dx \\
\geq \quad &\int_0^R W(r)(f_k'(r))^2 r^{n-1} dr + \int_0^R W(r)(f_k(r))^2 r^{n-3} dr \\
+ \quad &(n-1)\int_0^R V(r)(f_k'(r))^2 r^{n-3} dr + c_k(n-1)\int_0^R V(r)(f_k(r))^2 r^{n-5} dr \\
- \quad &(n-1)\int_0^R V_r(r) r^{n-2} (f_k')^2(r) dr - c_k(n-1)\int_0^R V_r(r) r^{n-4} (f_k)^2(r) dr \\
+ \quad &c_k(c_k-(n-1))\int_0^R V(r) r^{n-5} f_k^2(r) dr \\
+ \quad &c_k \int_0^R (W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r)) f_k^2(r) r^{n-3} dr.
\end{aligned}$$

The proof is now complete since the last term is non-negative by condition (3.74). Note also that because of this condition, the formula for the best constant requires that $\beta(V, W; R) \geq 1$, since if W satisfies (3.74) then cW satisfies it for any $c \geq 1$. □

Remark 3.3.1. In order to apply the above theorem to the Bessel pair

$$V(x) = |x|^{-2m} \quad \text{and} \quad W_{m,c}(x) = V(x)\big[(\tfrac{n-2m-2}{2})^2|x|^{-2} + cW(x)\big]$$

where W is a Bessel potential, we see that even in the simplest case $V \equiv 1$ and $W_{m,c}(x) = (\frac{n-2}{2})^2|x|^{-2} + W(x)$, condition (3.74) reduces to $(\frac{n-2}{2})^2|x|^{-2} + W(x) \geq 2|x|^{-2}$, which is then guaranteed only if $n \geq 5$.

More generally, if $V(x) = |x|^{-2m}$, then in order to satisfy (3.74) we need to have

$$\frac{-(n+4)-2\sqrt{n^2-n+1}}{6} \le m \le \frac{-(n+4)+2\sqrt{n^2-n+1}}{6}, \tag{3.79}$$

and in this case, we have for $m < \frac{n-2}{2}$ and any Bessel potential W on B_R, that for all $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \ge (\frac{n+2m}{2})^2 \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W;R) \int_{B_R} W(x) \frac{|\nabla u|^2}{|x|^{2m}} dx. \tag{3.80}$$

Moreover, $(\frac{n+2m}{2})^2$ and $\beta(W;R)$ are the best constant.

Therefore, inequality (3.80) in the case where $m = 0$ and $n \ge 5$, already includes Theorem 1.5 in [31] as a special case. It also extends Theorem 1.8 in [31] where it is established under the condition

$$0 \le m \le \frac{-(n+4)+2\sqrt{n^2-n+1}}{6} \tag{3.81}$$

which is more restrictive than (3.79). We shall see however that this inequality remains true without condition (3.79), but with a constant that is sometimes different from $(\frac{n+2m}{2})^2$ in the cases where (3.79) is not valid. For example, if $m = 0$, then the best constant is 3 in dimension 4 and $\frac{25}{36}$ in dimension 3.

We shall now give a few immediate applications of the above in the case where $m = 0$ and $n \ge 5$. Actually the results are true in lower dimensions, and will be stated as such, but the proofs for $n < 5$ will require additional work and will be postponed to the next section.

Assume W is a Bessel potential on $B_R \subset R^n$ with $n \ge 3$, then for all $u \in C_0^\infty(B_R)$ we have

$$\int_{B_R} |\Delta u|^2 dx \ge C(n) \int_{B_R} \frac{|\nabla u|^2}{|x|^2} dx + \beta(W;R) \int_{B_R} W(x)|\nabla u|^2 dx, \tag{3.82}$$

where $C(3) = \frac{25}{36}$, $C(4) = 3$ and $C(n) = \frac{n^2}{4}$ for all $n \ge 5$. Moreover, $C(n)$ and $\beta(W;R)$ are the best constants.

In particular, the following holds for any smooth bounded domain Ω in R^n with $R = \sup_{x\in\Omega} |x|$, and any $u \in H_0^2(\Omega)$.

- For any $\alpha < 2$,

$$\int_\Omega |\Delta u|^2 dx \ge C(n) \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx + \beta(|x|^\alpha; R) \int_\Omega \frac{|\nabla u|^2}{|x|^\alpha} dx, \tag{3.83}$$

  and for $\alpha = 0$,

$$\int_\Omega |\Delta u|^2 dx \ge C(n) \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx + \frac{z_0^2}{R^2} \int_\Omega |\nabla u|^2 dx, \tag{3.84}$$

  the constants being optimal when Ω is a ball.

- For any $k \geq 1$, and $\rho = R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$, we have

$$\int_\Omega |\Delta u(x)|^2 dx \geq C(n)\int_\Omega \frac{|\nabla u|^2}{|x|^2}dx + \frac{1}{4}\sum_{j=1}^{k}\int_\Omega \frac{|\nabla u|^2}{|x|^2}(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{|x|})^{-2}dx, \quad (3.85)$$

- For $D \geq R$, and X_i is defined as (3.113) we have

$$\int_\Omega |\Delta u(x)|^2 dx \geq C(n)\int_\Omega \frac{|\nabla u|^2}{|x|^2}dx + \frac{1}{4}\sum_{i=1}^{\infty}\int_\Omega \frac{|\nabla u|}{|x|^2}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})...X_i^2(\frac{|x|}{D})dx, \quad (3.86)$$

Moreover, all constants appearing in the above two inequality are optimal.

Let $W(x) = W(|x|)$ be radial Bessel potential on a ball B of radius R in R^n with $n \geq 4$, and such that $\frac{W_r(r)}{W(r)} = \frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. If $\lambda < n-2$, then the following Hardy-Rellich inequality holds:

$$\int_B |\Delta u|^2 dx \geq \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx + (\frac{n^2}{4} + \frac{(n-\lambda-2)^2}{4})\beta(W;R)\int_B \frac{W(x)}{|x|^2}u^2 dx, \quad (3.87)$$

Proof: Use first Theorem 3.3.1 with the Bessel potential W, then Theorem 3.2.3 with the Bessel pair $(|x|^{-2}, |x|^{-2}(\frac{(n-4)^2}{4}|x|^{-2} + W)$, then Theorem 3.2.2 with the Bessel pair $(W, \frac{(n-\lambda-2)^2}{4})|x|^{-2}W)$ to obtain

$$\begin{aligned}
\int_B |\Delta u|^2 dx &\geq C(n)\int_B \frac{|\nabla u|^2}{|x|^2}dx + \beta(W,R)\int_B W(x)|\nabla u|^2 dx \\
&\geq C(n)\frac{(n-4)^2}{4}\int_B \frac{u^2}{|x|^4}dx \\
&+ C(n)\beta(W,R)\int_B \frac{W(x)}{|x|^2}u^2 + \beta(W,R)\int W(x)|\nabla u|^2 dx \\
&\geq C(n)\frac{(n-4)^2}{4}\int_B \frac{u^2}{|x|^4}dx + (C(n) \\
&+ \frac{(n-\lambda-2)^2}{4})\beta(W,R)\int_B \frac{W(x)}{|x|^2}u^2 dx.
\end{aligned}$$

Recall that $C(n) = \frac{n^2}{4}$ for $n \geq 5$, giving the claimed result in these dimensions. This is however not the case when $n = 4$, and therefore another proof will be given in the next section to cover these cases.

The following is immediate from Theorem 3.3.1 and from the fact that $\lambda = 2$ for the Bessel potential under consideration.

Remark 3.3.2. *After we wrote this paper, we learnt that Beckner [5] has also computed the values of the constants $C(3)$, $C(4)$, and $C(n)$.*

Let Ω be a smooth bounded domain in R^n, $n \geq 4$ and $R = \sup_{x\in\Omega} |x|$. Then the following holds for all $u \in H_0^2(\Omega)$

1. If $\rho = R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$ and $log^{(i)}(.)$ is defined as (3.112), then

$$\int_\Omega |\Delta u(x)|^2 dx \geq \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}dx$$
$$+ \quad (1+\frac{n(n-4)}{8})\sum_{j=1}^{k}\int_\Omega \frac{u^2}{|x|^4}(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{|x|})^{-2}dx. \tag{3.88}$$

2. If $D \geq R$ and X_i is defined as (3.113), then

$$\int_\Omega |\Delta u(x)|^2 dx \geq \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}dx$$
$$+ \quad (1+\frac{n(n-4)}{8})\sum_{i=1}^{\infty}\int_\Omega \frac{u^2}{|x|^4}X_1^2(\frac{|x|}{D})X_2^2(\frac{|x|}{D})...X_i^2(\frac{|x|}{D})dx. \tag{3.89}$$

Let $W_1(x)$ and $W_2(x)$ be two radial Bessel potentials on a ball B of radius R in R^n with $n \geq 4$. If $a < 1$, then there exists $c(a,R) > 0$ such that for all $u \in H_0^2(B)$

$$\int_B |\Delta u|^2 dx \quad \geq \quad \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx + \frac{n^2}{4}\beta(W_1;R)\int_B W_1(x)\frac{u^2}{|x|^2}dx$$
$$+c(\frac{n-2a-2}{2})^2\int_B \frac{u^2}{|x|^{2a+2}}dx + c\beta(W_2;R)\int_B W_2(x)\frac{u^2}{|x|^{2a}}dx,$$

Proof: Here again we shall give a proof when $n \geq 5$. The case $n = 4$ will be handled in the next section. We again first use Theorem 3.3.1 (for $n \geq 5$) with the Bessel potential $|x|^{-2a}$ where $a < 1$, then Theorem 3.2.3 with the Bessel pair $(|x|^{-2}, |x|^{-2}(\frac{(n-4)^2}{4}|x|^{-2}+W))$, then again Theorem 3.2.3 with the Bessel pair

$(|x|^{-2a}, |x|^{-2a}((\frac{n-2a-2}{2})^2|x|^{-2} + W)$ to obtain

$$\begin{aligned}
\int_B |\Delta u|^2 dx &\geq \frac{n^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2}dx + \beta(|x|^{-2a};R)\int_B \frac{|\nabla u|^2}{|x|^{-2a}}dx \\
&\geq \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx \\
&+ \frac{n^2}{4}\beta(W_1;R)\int_B W_1(x)\frac{u^2}{|x|^2}dx + \beta(|x|^{-2a};R)\int_B \frac{|\nabla u|^2}{|x|^{-2a}}dx \\
&\geq \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx + \frac{n^2}{4}\beta(W_1;R)\int_B W_1(x)\frac{u^2}{|x|^2}dx \\
&+ \beta(|x|^{-2a};R)(\frac{n-2a-2}{2})^2\int_B \frac{u^2}{|x|^{2a+2}}dx \\
&+ \beta(|x|^{-2a};R)\beta(W_2;R)\int_B W_2(x)\frac{u^2}{|x|^{2a}}dx.
\end{aligned}$$

The following theorem will be established in full generality (i.e with $V(r) = r^{-m}$) in the next section.

Let $W(x) = W(|x|)$ be a radial Bessel potential on a smooth bounded domain Ω in R^n, $n \geq 4$. Then,

$$\int_\Omega |\Delta u(x)|^2 dx - \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}dx - \frac{n^2}{4}\int_\Omega W(x)u^2 dx \geq \frac{z_0^2}{R^2}||u||^2_{W_0^{1,2}(\Omega)},$$

u $\in H_0^2(\Omega)$.

3.3.2 The case of power potentials $|x|^m$

The general Theorem 3.3.1 allowed us to deduce inequality (3.90) below for a restricted interval of powers m. We shall now prove that the same holds for all $m < \frac{n-2}{2}$. The following theorem improves considerably Theorem 1.7, Theorem 1.8, and Theorem 6.4 in [31]. Suppose $n \geq 1$ and $m < \frac{n-2}{2}$, and let W be a Bessel potential on a ball $B_R \subset R^n$ of radius R. Then for all $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m}\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}}dx + \beta(W;R)\int_{B_R} W(x)\frac{|\nabla u|^2}{|x|^{2m}}dx, \tag{3.90}$$

where

$$a_{n,m} = \inf\left\{\frac{\int_{B_R}\frac{|\Delta u|^2}{|x|^{2m}}dx}{\int_{B_R}\frac{|\nabla u|^2}{|x|^{2m+2}}dx};\ u \in C_0^\infty(B_R)\setminus\{0\}\right\}.$$

Moreover, $\beta(W;R)$ and $a_{m,n}$ are the best constants to be computed in section 3.6.

Proof: Assuming the inequality

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m}\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}}dx,$$

holds for all $u \in C_0^\infty(B_R)$, we shall prove that it can be improved by any Bessel potential W. We will use the following inequality frequently in the proof which follows directly from Theorem 3.2.3 with n=1.

$$\begin{aligned}\int_0^R r^\alpha (f'(r))^2 dr &\geq (\frac{\alpha-1}{2})^2 \int_0^R r^{\alpha-2} f^2(r) dr \\ &+ \beta(W;R) \int_0^R r^\alpha W(r) f^2(r) dr, \quad \alpha \geq 1, \end{aligned} \tag{3.91}$$

for all $f \in C^\infty(0,R)$, where both $(\frac{\alpha-1}{2})^2$ and $\beta(W;R)$ are best constants.

Decompose $u \in C_0^\infty(B_R)$ into its spherical harmonics $\Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$. We evaluate $I_k = \frac{1}{nw_n}\int_{R^n} \frac{|\Delta u_k|^2}{|x|^{2m}} dx$ in the following way

$$\begin{aligned} I_k &= \int_0^R r^{n-2m-1}(f_k''(r))^2 dr + [(n-1)(2m+1)+2c_k]\int_0^R r^{n-2m-3}(f_k')^2 dr \\ &\quad + c_k[c_k+(n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\ &\geq \beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2 dr + [(\frac{n+2m}{2})^2+2c_k]\int_0^R r^{n-2m-3}(f_k')^2 dr \\ &\quad + c_k[c_k+(n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\ &\geq \beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2 dr + a_{n,m}\int_0^R r^{n-2m-3}(f_k')^2 dr \\ &\quad + \beta(W)[(\frac{n+2m}{2})^2+2c_k-a_{n,m}]\int_0^R r^{n-2m-3}W(x)(f_k)^2 dr \\ &\quad + ((\frac{n-2m-4}{2})^2[(\frac{n+2m}{2})^2+2c_k-a_{n,m}] \\ &\quad + c_k[c_k+(n-2m-4)(2m+2)])\int_0^R r^{n-2m-5}(f_k(r))^2 dr. \end{aligned}$$

Now by (3.122) we have

$$((\frac{n-2m-4}{2})^2[(\frac{n+2m}{2})^2+2c_k-a_{n,m}]+c_k[c_k+(n-2m-4)(2m+2)] \geq c_k a_{n,m},$$

for all $k \geq 0$. Hence, we have

$$\begin{aligned}
I_k &\geq a_{n,m}\int_0^R r^{n-2m-3}(f_k')^2dr + a_{n,m}c_k\int_0^R r^{n-2m-5}(f_k(r))^2dr \\
&+ \beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2dr \\
&+ \beta(W)[(\frac{n+2m}{2})^2 + 2c_k - a_{n,m}]\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
&\geq a_{n,m}\int_0^R r^{n-2m-3}(f_k')^2dr + a_{n,m}c_k\int_0^R r^{n-2m-5}(f_k(r))^2dr \\
&\quad +\beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2dr + \beta(W)c_k\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
&= a_{n,m}\int_{B_R}\frac{|\nabla u|^2}{|x|^{2m+2}}dx + \beta(W)\int_{B_R}W(x)\frac{|\nabla u|^2}{|x|^{2m}}dx.
\end{aligned}$$

Moreover, it is easy to see from Theorem 3.2 and the above calculation that $\beta(W;R)$ is the best constant.

Let Ω be a smooth domain in R^n with $n \geq 1$ and let $V \in C^2(0, R =: \sup_{x\in\Omega}|x|)$ be a non-negative function that satisfies the following conditions:

$$V_r(r) \leq 0 \quad \text{and} \quad \int_0^R \tfrac{1}{r^{n-3}V(r)}dr = -\int_0^R \tfrac{1}{r^{n-4}V_r(r)}dr = +\infty. \tag{3.92}$$

There exists $\lambda_1, \lambda_2 \in R$ such that

$$\tfrac{rV_r(r)}{V(r)} + \lambda_1 \geq 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0}\tfrac{rV_r(r)}{V(r)} + \lambda_1 = 0, \tag{3.93}$$

$$\tfrac{rV_{rr}(r)}{V_r(r)} + \lambda_2 \geq 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0}\tfrac{rV_{rr}(r)}{V_r(r)} + \lambda_2 = 0, \tag{3.94}$$

and

$$\left(\tfrac{1}{2}(n-\lambda_1-2)^2 + 3(n-3)\right)V(r) - (n-5)rV_r(r) - r^2V_{rr}(r) \geq 0, \tag{3.95}$$

for all $r \in (0,R)$. Then the following inequality holds:

$$\begin{aligned}
\int_\Omega V(|x|)|\Delta u|^2dx &\geq (\frac{(n-\lambda_1-2)^2}{4} + (n-1))\frac{(n-\lambda_1-4)^2}{4}\int_\Omega\frac{V(|x|)}{|x|^4}u^2dx \\
&\quad -\frac{(n-1)(n-\lambda_2-2)^2}{4}\int_\Omega\frac{V_r(|x|)}{|x|^3}u^2dx.
\end{aligned} \tag{3.96}$$

Proof: We have by Theorem 3.2.2 and condition (3.95),

$$
\begin{aligned}
&\frac{1}{n\omega_n}\int_{R^n} V(x)|\Delta u_k|^2 dx = \int_0^R V(r)(f_k''(r))^2 r^{n-1} dr \\
+ \quad & (n-1+2c_k)\int_0^R V(r)(f_k'(r))^2 r^{n-3} dr \\
+ \quad & (2c_k(n-4)+c_k^2)\int_0^R V(r) r^{n-5} f_k^2(r) dr - (n-1)\int_0^R V_r(r) r^{n-2}(f_k')^2(r) dr \\
- \quad & c_k(n-5)\int_0^R V_r(r) f_k^2(r) r^{n-4} dr - c_k\int_0^R V_{rr}(r) f_k^2(r) r^{n-3} dr \\
\geq \quad & \int_0^R V(r)(f_k''(r))^2 r^{n-1} dr + (n-1)\int_0^R V(r)(f_k'(r))^2 r^{n-3} dr \\
- \quad & (n-1)\int_0^R V_r(r) r^{n-2}(f_k')^2(r) dr \\
+ \quad & c_k\int_0^R \left(\left(\frac{1}{2}(n-\lambda_1-2)^2 + 3(n-3)\right)V(r) - (n-5)rV_r(r) - r^2 V_{rr}(r)\right) f_k^2(r) r^{n-5}
\end{aligned}
$$

The rest of the proof follows from the above inequality combined with Theorem 3.2.2. □

Remark 3.3.3. Let $V(r) = r^{-2m}$ with $m \leq \frac{n-4}{2}$. Then in order to satisfy condition (3.95) we must have $-1 - \frac{\sqrt{1+(n-1)^2}}{2} \leq m \leq \frac{n-4}{2}$. Under this assumption the inequality (3.96) gives the following weighted second order Rellich inequality:

$$\int_B \frac{|\Delta u|^2}{|x|^{2m}} dx \geq \left(\frac{(n+2m)(n-4-2m)}{4}\right)^2 \int_B \frac{u^2}{|x|^{2m+4}} dx.$$

In the following theorem we will show that the constant appearing in the above inequality is optimal. Moreover, we will see that if $m < -1 - \frac{\sqrt{1+(n-1)^2}}{2}$, then the best constant is strictly less than $\left(\frac{(n+2m)(n-4-2m)}{4}\right)^2$. This shows that inequality (3.96) is actually sharp.

Let $m \leq \frac{n-4}{2}$ and define

$$\beta_{n,m} = \inf_{u\in C_0^\infty(B)\setminus\{0\}} \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}} dx}{\int_B \frac{u^2}{|x|^{2m+4}} dx}. \tag{3.97}$$

Then

$$
\begin{aligned}
\beta_{n,m} \quad = \quad & \left(\frac{(n+2m)(n-4-2m)}{4}\right)^2 \\
+ \quad & \min_{k=0,1,2,\ldots} \{k(n+k-2)[k(n+k-2) + \frac{(n+2m)(n-2m-4)}{2}]\}.
\end{aligned}
$$

Consequently the values of $\beta_{n,m}$ are as follows.

1. If $-1-\frac{\sqrt{1+(n-1)^2}}{2} \le m \le \frac{n-4}{2}$, then
$$\beta_{n,m} = (\frac{(n+2m)(n-4-2m)}{4})^2.$$

2. If $\frac{n}{2}-3 \le m \le -1-\frac{\sqrt{1+(n-1)^2}}{2}$, then
$$\beta_{n,m} = (\frac{(n+2m)(n-4-2m)}{4})^2 + (n-1)[(n-1)+\frac{(n+2m)(n-2m-4)}{2}].$$

3. If $k := \frac{n-2m-4}{2} \in N$, then
$$\beta_{n,m} = (\frac{(n+2m)(n-4-2m)}{4})^2 + k(n+k-2)[k(n+k-2)+\frac{(n+2m)(n-2m-4)}{2}].$$

4. If $k < \frac{n-2m-4}{2} < k+1$ for some $k \in N$, then
$$\beta_{n,m} = \frac{(n+2m)^2(n-2m-4)^2}{16} + a(m,n,k)$$

where
$$\begin{aligned} a(m,n,k) &= \min\{k(n+k-2)[k(n+k-2)+\frac{(n+2m)(n-2m-4)}{2}], \\ &\quad (k+1)(n+k-1)[(k+1)(n+k-1)+\frac{(n+2m)(n-2m-4)}{2}]\}. \end{aligned}$$

Proof: Decompose $u \in C_0^\infty(B_R)$ into spherical harmonics $\Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$. we have
$$\begin{aligned} \frac{1}{n\omega_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx &= \int_0^R r^{n-2m-1}(f_k''(r))^2 dr \\ &+ [(n-1)(2m+1)+2c_k]\int_0^R r^{n-2m-3}(f_k')^2 dr \\ &+ c_k[c_k+(n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\ &\ge ((\frac{(n+2m)(n-4-2m)}{4})^2 \\ &+ c_k[c_k+\frac{(n+2m)(n-2m-4)}{2}])\int_0^R r^{n-2m-5}(f_k(r))^2 dr, \end{aligned}$$

by Hardy inequality. Hence,
$$\begin{aligned} \beta_{n,m} \ge B(n,m,k) &:= (\frac{(n+2m)(n-4-2m)}{4})^2 + \min_{k=0,1,2,\ldots}\{k(n+k-2)[k(n+k-2) \\ &+ \frac{(n+2m)(n-2m-4)}{2}]\}. \end{aligned}$$

To prove that $\beta_{n,m}$ is the best constant, let k be such that

$$\begin{aligned} \beta_{n,m} &= \frac{(n+2m)(n-4-2m)}{4})^2 \\ &+ k(n+k-2)[k(n+k-2)+\frac{(n+2m)(n-2m-4)}{2}]. \end{aligned} \tag{3.98}$$

Set

$$u = |x|^{-\frac{n-4}{2}+m+\epsilon}\varphi_k(x)\varphi(|x|),$$

where $\varphi_k(x)$ is an eigenfunction corresponding to the eigenvalue c_k and $\varphi(r)$ is a smooth cutoff function, such that $0 \le \varphi \le 1$, with $\varphi \equiv 1$ in $[0, \frac{1}{2}]$. We have

$$\frac{\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx}{\int_{B_R} \frac{u^2}{|x|^{2m+4}} dx} = (-\frac{(n+2m)(n-4-2m)}{4} - c_k + \epsilon(2+2m+\epsilon))^2 + O(1).$$

Let now $\epsilon \to 0$ to obtain the result. Thus the inequality

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \ge \beta_{n,m} \int_{B_R} \frac{u^2}{|x|^{2m+4}} dx,$$

holds for all $u \in C_0^\infty(B_R)$.

To calculate explicit values of $\beta_{n,m}$ we need to find the minimum point of the function

$$f(x) = x(x + \frac{(n+2m)(n-2m-4)}{2}), \quad x \ge 0.$$

Observe that

$$f'(-\frac{(n+2m)(n-2m-4)}{4}) = 0.$$

To find minimizer $k \in N$ we should solve the equation

$$k^2 + (n-2)k + \frac{(n+2m)(n-2m-4)}{4} = 0.$$

The roots of the above equation are $x_1 = \frac{n+2m}{2}$ and $x_2 = \frac{n-2m-4}{2}$. 1) follows from Theorem 3.3.2. It is easy to see that if $m \le -1 - \frac{\sqrt{1+(n-1)^2}}{2}$, then $x_1 < 0$. Hence, for $m \le -1 - \frac{\sqrt{1+(n-1)^2}}{2}$ the minimum of the function f is attained in x_2. Note that if $m \le -1 - \frac{\sqrt{1+(n-1)^2}}{2}$, then $B(n, m1) \le B(n, m, 0)$. Therefore claims 2), 3), and 4) follow. □

The following theorem extends Theorem 1.6 of [31] in many ways. First, we do not assume that $n \ge 5$ or $m \ge 0$, as was assumed there. Moreover, inequality (3.99) below includes inequalities (1.17) and (1.22) of [31] as special cases.

Let $m \leq \frac{n-4}{2}$ and let $W(x)$ be a Bessel potential on a ball B of radius R in R^n with radius R. Assume $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. Then the following inequality holds for all $u \in C_0^\infty(B)$

$$\begin{aligned}\int_B \frac{|\Delta u|^2}{|x|^{2m}}dx \quad &\geq \quad \beta_{n,m}\int_B \frac{u^2}{|x|^{2m+4}}dx \\ &+ \quad \beta(W;R)(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4})\int_B \frac{W(x)}{|x|^{2m+2}}u^2 dx.\end{aligned} \tag{3.99}$$

Proof: Again we will frequently use inequality (3.91) in the proof. Decomposing $u \in C_0^\infty(B_R)$ into spherical harmonics $\Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$, we can write

$$\begin{aligned}&\frac{1}{n\omega_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx \\ = \quad &\int_0^R r^{n-2m-1}(f_k''(r))^2 dr + [(n-1)(2m+1) \\ &+2c_k]\int_0^R r^{n-2m-3}(f_k')^2 dr \\ &+c_k[c_k + (n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\ \geq \quad &(\frac{n+2m}{2})^2\int_0^R r^{n-2m-3}(f_k')^2 dr + \beta(W;R)\int_0^R r^{n-2m-1}W(x)(f_k')^2 dr \\ &+c_k[c_k + 2(\frac{n-\lambda-4}{2})^2 + (n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr,\end{aligned}$$

where we have used the fact that $c_k \geq 0$ to get the above inequality. We have

$$\begin{aligned}
& \frac{1}{n\omega_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx \\
\geq\ & \beta_{n,m}\int_0^R r^{n-2m-5}(f_k)^2dr \\
& +\beta(W;R)\frac{(n+2m)^2}{4}\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
& +\beta(W;R)\int_0^R r^{n-2m-1}W(x)(f_k')^2dr \\
\geq\ & \beta_{n,m}\int_0^R r^{n-2m-5}(f_k)^2dr \\
& +\beta(W;R)(\frac{(n+2m)^2}{4}+\frac{(n-2m-\lambda-2)^2}{4})\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
\geq\ & \frac{\beta_{n,m}}{n\omega_n}\int_B\frac{u_k^2}{|x|^{2m+4}}dx \\
& +\frac{\beta(W;R)}{n\omega_n}(\frac{(n+2m)^2}{4}+\frac{(n-2m-\lambda-2)^2}{4})\int_B\frac{W(x)}{|x|^{2m+2}}u_k^2dx,
\end{aligned}$$

by Theorem 3.2.2. Hence, (3.99) holds and the proof is complete. □

Assume $-1 < m \leq \frac{n-4}{2}$ and let $W(x)$ be a Bessel potential on a ball B of radius R and centered at zero in R^n $(n \geq 1)$. Then the following holds for all $u \in C_0^\infty(B)$:

$$\begin{aligned}
\int_B\frac{|\Delta u|^2}{|x|^{2m}}dx \quad \geq \quad & \frac{(n+2m)^2(n-2m-4)^2}{16}\int_B\frac{u^2}{|x|^{2m+4}}dx \qquad (3.100)\\
+ \quad & \beta(W;R)\frac{(n+2m)^2}{4}\int_B\frac{W(x)}{|x|^{2m+2}}u^2dx+\beta(|x|^{2m};R)||u||_{H_0^1}.
\end{aligned}$$

Proof: Decomposing again $u \in C_0^\infty(B_R)$ into its spherical harmonics $\Sigma_{k=0}^\infty u_k$

where $u_k = f_k(|x|)\varphi_k(x)$, we calculate

$$
\begin{aligned}
& \frac{1}{n\omega_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx \\
= \quad & \int_0^R r^{n-2m-1}(f_k''(r))^2dr + [(n-1)(2m+1)+2c_k]\int_0^R r^{n-2m-3}(f_k')^2dr \\
+ \quad & c_k[c_k+(n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2dr \\
\geq \quad & (\frac{n+2m}{2})^2\int_0^R r^{n-2m-3}(f_k')^2dr + \beta(|x|^{2m};R)\int_0^R r^{n-1}(f_k')^2dr \\
+ \quad & c_k\int_0^R r^{n-2m-3}(f_k')^2dr \\
\geq \quad & \frac{(n+2m)^2(n-2m-4)^2}{16}\int_0^R r^{n-2m-5}(f_k)^2dr \\
& +\beta(W;R)\frac{(n+2m)^2}{4}\int_0^R W(r)r^{n-2m-3}(f_k)^2dr \\
+ \quad & \beta(|x|^{2m};R)\int_0^R r^{n-1}(f_k')^2dr + c_k\beta(|x|^{2m};R)\int_0^R r^{n-3}(f_k)^2dr \\
= \quad & \frac{(n+2m)^2(n-2m-4)^2}{16n\omega_n}\int_{R^n}\frac{u_k^2}{|x|^{2m+4}}dx \\
+ \quad & \frac{\beta(W;R)}{n\omega_n}(\frac{(n+2m)^2}{4})\int_{R^n}\frac{W(x)}{|x|^{2m+2}}u_k^2dx + \beta(|x|^{2m};R)||u_k||_{W_0^{1,2}}.
\end{aligned}
$$

Hence (3.100) holds. □

We note that even for $m=0$ and $n\geq 4$, Theorem 3.3.2 improves considerably Theorem A.2. in [2].

3.4 Higher order Rellich inequalities

In this section we will repeat the results obtained in the previous section to derive higher order Rellich inequalities with corresponding improvements. Let W be a Bessel potential, $\beta_{n,m}$ be defined as in Theorem 3.3.2 and

$$\sigma_{n,m} = \beta(W;R)(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4}).$$

For the sake of convenience we make the following convention: $\prod_{i=1}^{0} a_i = 1$. Let B_R be a ball of radius R and W be a Bessel potential on B_R such that $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r)\geq 0$ and $\lim_{r\to 0} rf(r) = 0$. Assume $m\in N$, $1\leq l\leq m$, and $2k+4m\leq n$.

Then the following inequality holds for all $u \in C_0^\infty(B_R)$

$$\begin{aligned}\int_{B_R} \frac{|\Delta^m u|^2}{|x|^{2k}} dx &\geq \prod_{i=0}^{l-1} \beta_{n,k+2i} \int_{B_R} \frac{|\Delta^{m-l} u|^2}{|x|^{2k+4l}} dx \\ &+ \sum_{i=0}^{l-1} \sigma_{n,k+2i} \prod_{j=1}^{l-1} \beta_{n,k+2j-2} \int_{B_R} \frac{W(x)|\Delta^{m-i-1} u|^2}{|x|^{2k+4i+2}} dx \end{aligned} \tag{3.101}$$

Proof: Follows directly from theorem 3.3.2. □

Let B_R be a ball of radius R and W be a Bessel potential on B_R such that $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. Assume $m \in N$, $1 \leq l \leq m$, and $2k + 4m + 2 \leq n$. Then the following inequality holds for all $u \in C_0^\infty(B_R)$

$$\begin{aligned}&\int_{B_R} \frac{|\nabla\Delta^m u|^2}{|x|^{2k}} dx \geq (\frac{n-2k-2}{2})^2 \prod_{i=0}^{l-1} \beta_{n,k+2i+1} \int_{B_R} \frac{|\Delta^{m-l} u|^2}{|x|^{2k+4l+2}} dx \\ &+ (\frac{n-2k-2}{2})^2 \sum_{i=0}^{l-1} \sigma_{n,k+2i+1} \prod_{j=1}^{l-1} \beta_{n,k+2j-1} \int_{B_R} \frac{W(x)|\Delta^{m-i-1} u|^2}{|x|^{2k+4i+4}} dx \\ &+ \beta(W;R) \int_{B_R} W(x) \frac{|\Delta^m u|^2}{|x|^{2k}} dx \end{aligned} \tag{3.102}$$

Proof: Follows directly from Theorem 3.2.3 and the previous theorem. □

Remark 3.4.1. *For $k = 0$ Theorems 3.4 and 3.4 include Theorem 1.9 in [31] as a special case.*

Let B_R be a ball of radius R and W be a Bessel potential on B_R such that $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. Assume $m \in N$, $1 \leq l \leq m-1$, and $2k + 4m \leq n$. Then the following inequality holds for all $u \in C_0^\infty(B_R)$

$$\begin{aligned}&\int_{B_R} \frac{|\Delta^m u|^2}{|x|^{2k}} dx \geq a_{n,k}(\frac{n-2k-4}{2})^2 \prod_{i=0}^{l-1} \beta_{n,k+2i+2} \int_{B_R} \frac{|\Delta^{m-l-1} u|^2}{|x|^{2k+4l+4}} dx \\ &+ a_{n,k}(\frac{n-2k-4}{2})^2 \sum_{i=0}^{l-1} \sigma_{n,k+2i+2} \prod_{j=1}^{l-1} \beta_{n,k+2j} \int_{B_R} \frac{W(x)|\Delta^{m-i-2} u|^2}{|x|^{2k+4i+6}} dx \\ &+ \beta(W;R) a_{n,k} \int_{B_R} W(x) \frac{|\Delta^{m-1} u|^2}{|x|^{2k+2}} dx \\ &+ \beta(W;R) \int_{B_R} W(x) \frac{|\nabla\Delta^{m-1} u|^2}{|x|^{2k}} dx, \end{aligned} \tag{3.103}$$

where $a_{n,m}$ is defined in Theorem 3.3.2. **Proof:** Follows directly from Theorem 3.3.2 and the previous theorem. □

The following improves Theorem 1.10 in [31] in many ways, since it is assumed there that $l \leq \frac{-n+8+2\sqrt{n^2-n+1}}{12}$ and $4m < n$. Even for $k = 0$, Theorem 3.4 below shows that we can drop the first condition and replace the second one by $4m \leq n$.

Let B_R be a ball of radius R and W be a Bessel potential on B_R such that . Assume $m \in N$, $1 \leq l \leq m$, and $2k + 4m \leq n$. Then the following inequality holds for all $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta^m u|^2}{|x|^{2k}} dx \tag{3.104}$$

$$\geq \prod_{i=1}^{l} \frac{a_{n,k+2i-2}(n-2k-4i)^2}{4} \int_{B_R} \frac{|\Delta^{m-l} u|^2}{|x|^{2k+4l}} dx \tag{3.105}$$

$$+ \beta(W;R) \sum_{i=1}^{l} \prod_{j=1}^{l-1} \frac{a_{n,k+2j-2}(n-2k-4j)^2}{4} \int_{B_R} W(x) \frac{|\nabla \Delta^{m-i} u|^2}{|x|^{2k+4i-4}} dx$$

$$+ \beta(W;R) \sum_{i=1}^{l} a_{n,k+2i-2} \prod_{j=1}^{l-1} \frac{a_{n,k+2j-2}(n-2k-4j)^2}{4} \int_{B_R} W(x) \frac{|\Delta^{m-i} u|^2}{|x|^{2k+4i-2}} dx,$$

where $a_{n,m}$ are the best constants in inequality (3.90).

Proof: Follows directly from Theorem 3.3.2. □

3.5 The class of Bessel potentials

The Bessel equation associated to a potential W

$$(B_W) \qquad y'' + \frac{1}{r} y' + W(r) y = 0$$

is central to all results revolving around the inequalities of Hardy and Hardy-Rellich type. We summarize in this section the various properties of these equations that were used throughout this chapter. We say that a non-negative real valued C^1-function is a *Bessel potential on* $(0, R)$ if there exists $c > 0$ such that the equation (B_{cW}) has a positive solution on $(0, R)$.

The class of Bessel potentials on $(0, R)$ will be denoted by $\mathcal{B}(0, R)$. Note that the change of variable $z(s) = y(e^{-s})$ maps the equation $y'' + \frac{1}{r} y' + W(r) y = 0$ into

$$(B'_W) \qquad z'' + e^{-2s} W(e^{-s}) z(s) = 0. \tag{3.106}$$

On the other hand, the change of variables $\psi(t) = \frac{-e^{-t} y'(e^{-t})}{y(e^{-t})}$ maps it into the nonlinear equation

$$(B''_W) \qquad \psi'(t) + \psi^2(t) + e^{-2t} W(e^{-t}) = 0. \tag{3.107}$$

This will allow us to relate the existence of positive solutions of (B_W) to the non-oscillatory behaviour of equations (B'_W) and (B''_W).

The theory of sub/supersolutions –applied to (B''_W) (See Wintner [35, 36, 20])– already yields, that if (B_W) has a positive solution on an interval $(0, R)$ for some non-negative potential $W \geq 0$, then for any W such that $0 \leq V \leq W$, the equation (B_V) has also a positive solution on $(0, R)$. This leads to the definition of the *weight* of a potential $W \in \mathcal{B}(0, R)$ as:

$$\beta(W; R) = \sup\{c > 0;\ (B_{cW}) \text{ has a positive solution on } (0, R)\}. \tag{3.108}$$

The following is now straightforward.

Proposition 3.109. *1) The class $\mathcal{B}(0, R)$ is a closed convex and solid subset of $C^1(0, R)$.*

2) For every $W \in \mathcal{B}(0, R)$, the equation

$$(B_W) \qquad y'' + \tfrac{1}{r}y' + \beta(W; R)W(r)y = 0$$

has a positive solution on $(0, R)$.

The following gives an integral criteria for Bessel potentials.

Proposition 3.110. *Let W be a positive locally integrable function on .*

1. *If $\liminf\limits_{r\to 0} \ln(r) \int_0^r sW(s)ds > -\infty$, then for every $R > 0$, there exists $\alpha := \alpha(R) > 0$ such that the scaled function $W_\alpha(x) := \alpha^2 W(\alpha x)$ is a Bessel potential on $(0, R)$.*
2. *If $\lim\limits_{r\to 0} \ln(r) \int_0^r sW(s)ds = -\infty$, then there are no $\alpha, c > 0$, for which $W_{\alpha,c} = cW(\alpha|x|)$ is a Bessel potential on $(0, R)$.*

Proof: This relies on well known results concerning the existence of non-oscillatory solutions (i.e., those $z(s)$ such that $z(s) > 0$ for $s > 0$ sufficiently large) for the second order linear differential equations

$$z''(s) + a(s)z(s) = 0, \tag{3.111}$$

where a is a locally integrable function on . For these equations, the following integral criteria are available. We refer to [20, 21, 35, 36, 37]) among others for proofs and related results.

i) If $\limsup_{t\to\infty} t \int_t^\infty a(s)ds < \frac{1}{4}$, then Eq. (3.111) is non-oscillatory.

ii) If $\liminf_{t\to\infty} t \int_t^\infty a(s)ds > \frac{1}{4}$, then Eq. (3.111) is oscillatory.

It follows that if $\liminf\limits_{r\to 0} \ln(r) \int_0^r sW(s)ds > -\infty$ holds, then there exists $\delta > 0$ such that (B_W) has a positive solution on $(0, \delta)$. An easy scaling argument then shows that there exists $\alpha > 0$ such that $W_\alpha(x) := \alpha^2 W(\alpha x)$ is a Bessel potential on $(0, R)$. The rest of the proof is similar. □

We now exhibit a few explicit Bessel potentials and compute their weights. We use the following notation.

$$log^{(1)}(.) = log(.) \quad \text{and} \quad log^{(k)}(.) = log(log^{(k-1)}(.)) \quad \text{for} \quad k \geq 2. \tag{3.112}$$

and

$$X_1(t) = (1 - \log(t))^{-1}, \quad X_k(t) = X_1(X_{k-1}(t)) \quad k = 2, 3, ..., \tag{3.113}$$

Explicit Bessel potentials

1. $W \equiv 0$ is a Bessel potential on $(0, R)$ for any $R > 0$.
2. The Bessel function J_0 is a positive solution for equation (B_W) with $W \equiv 1$, on $(0, z_0)$, where $z_0 = 2.4048...$ is the first zero of J_0. Moreover, z_0 is larger than the first root of any other solution for (B_1). In other words,

   for every $R > 0$,
$$\beta(1; R) = \frac{z_0^2}{R^2}. \tag{3.114}$$
3. If $a < 2$, then there exists $R_a > 0$ such that $W(r) = r^{-a}$ is a Bessel potential on $(0, R_a)$.
4. For each $k \geq 1$ and $\rho > R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$, the equation $(B_{\frac{1}{4}W_{k,\rho}})$ corresponding to the potential
$$W_{k,\rho}(r) = \Sigma_{j=1}^k U_j \text{ where } U_j(r) = \tfrac{1}{r^2}\big(\textstyle\prod_{i=1}^j log^{(i)} \frac{\rho}{r}\big)^{-2}$$
   has a positive solution on $(0, R)$ that is explicitly given by $\varphi_{k,\rho}(r) = (\prod_{i=1}^k log^{(i)} \frac{\rho}{r})^{\frac{1}{2}}$. On the other hand, the equation $(B_{\frac{1}{4}W_{k,\rho}+\lambda U_k})$ corresponding to the potential $\frac{1}{4}W_{k,\rho} + \lambda U_k$ has no positive solution for any $\lambda > 0$. In other words, $W_{k,\rho}$ is a Bessel potential on $(0, R)$ with
$$\beta(W_{k;\rho}, R) = \tfrac{1}{4} \text{ for any } k \geq 1. \tag{3.115}$$
5. For each $k \geq 1$ and $R > 0$, the equation $(B_{\frac{1}{4}\tilde{W}_{k,R}})$ corresponding to the potential
$$\tilde{W}_{k,R}(r) = \Sigma_{j=1}^k \tilde{U}_j \text{ where } \tilde{U}_j(r) = \tfrac{1}{r^2} X_1^2(\tfrac{r}{R}) X_2^2(\tfrac{r}{R}) \ldots X_{j-1}^2(\tfrac{r}{R}) X_j^2(\tfrac{r}{R})$$
   has a positive solution on $(0, R)$ that is explicitly given by
$$\varphi_k(r) = (X_1(\frac{r}{R}) X_2(\frac{r}{R}) \ldots X_{k-1}(\frac{r}{R}) X_k(\frac{r}{R}))^{-\frac{1}{2}}.$$
   On the other hand, the equation $(B_{\frac{1}{4}\tilde{W}_{k,R}+\lambda\tilde{U}_k})$ corresponding to the potential $\frac{1}{4}\tilde{W}_{k,R} + \lambda\tilde{U}_k$ has no positive solution for any $\lambda > 0$. In other words, $\tilde{W}_{k,R}$ is a Bessel potential on $(0, R)$ with
$$\beta(\tilde{W}_{k,R}; R) = \tfrac{1}{4} \quad \text{for any } k \geq 1. \tag{3.116}$$

Proof: 1) It is clear that $\phi(r) = -log(\frac{e}{R}r)$ is a positive solution of (B_0) on $(0, R)$ for any $R > 0$.

2)The best constant for which the equation $y'' + \frac{1}{r}y' + cy = 0$ has a positive solution on $(0, R)$ is $\frac{z_0^2}{R^2}$, where $z_0 = 2.4048...$ is the first zero of Bessel function $J_0(z)$. Indeed if α is the first root of the an arbitrary solution of the Bessel equation $y''+\frac{y'}{r}+y(r) = 0$, then we have $\alpha \leq z_0$. To see this let $x(t) = aJ_0(t)+bY_0(t)$, where J_0 and Y_0 are the two standard linearly independent solutions of Bessel equation, and a and b are constants. Assume the first zero of $x(t)$ is larger than z_0. Since the first zero of Y_0 is smaller than z_0, we have $a \geq 0$. Also $b \leq 0$, because $Y_0(t) \to -\infty$ as $t \to 0$. Finally note that $Y_0(z_0) > 0$, so if $b < 0$, then $x(z_0 + \epsilon) < 0$ for ϵ sufficiently small. Therefore, $b = 0$ which is a contradiction.

3) follows directly from the integral criteria.

4) That ϕ_k is an explicit solution of the equation $(B_{\frac{1}{4}W_k})$ is straightforward. Assume now that there exists a positive function φ such that

$$-\frac{\varphi'(r) + r\varphi''(r)}{\varphi(r)} = \frac{1}{4}\sum_{j=1}^{k-1}\frac{1}{r}(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{r})^{-2} + (\frac{1}{4} + \lambda)\frac{1}{r}(\prod_{i=1}^{k} log^{(i)}\frac{\rho}{r})^{-2}.$$

Define $f(r) = \frac{\varphi(r)}{\varphi_k(r)} > 0$, and calculate,

$$\frac{\varphi'(r) + r\varphi''(r)}{\varphi(r)} = \frac{\varphi_k'(r) + r\varphi_k''(r)}{\varphi_k(r)} + \frac{f'(r) + rf''(r)}{f(r)} - \frac{f'(r)}{f(r)}\sum_{i=1}^{k}\frac{1}{\prod_{j=1}^{i}\log^j(\frac{\rho}{r})}.$$

Thus,

$$\frac{f'(r) + rf''(r)}{f(r)} - \frac{f'(r)}{f(r)}\sum_{i=1}^{k}\frac{1}{\prod_{j=1}^{i}\log^j(\frac{\rho}{r})} = -\lambda\frac{1}{r}(\prod_{i=1}^{k} log^{(i)}\frac{\rho}{r})^{-2}. \tag{3.117}$$

If now $f'(\alpha_n) = 0$ for some sequence $\{\alpha_n\}_{n=1}^{\infty}$ that converges to zero, then there exists a sequence $\{\beta_n\}_{n=1}^{\infty}$ that also converges to zero, such that $f''(\beta_n) = 0$, and $f'(\beta_n) > 0$. But this contradicts (3.117), which means that f is eventually monotone for r small enough. We consider the two cases according to whether f is increasing or decreasing:

Case I: Assume $f'(r) > 0$ for $r > 0$ sufficiently small. Then we will have

$$\frac{(rf'(r))'}{rf'(r)} \leq \sum_{i=1}^{k}\frac{1}{r\prod_{j=1}^{i}\log^j(\frac{\rho}{r})}.$$

Integrating once we get

$$f'(r) \geq \frac{c}{r\prod_{j=1}^{k}\log^j(\frac{\rho}{r})},$$

for some $c > 0$. Hence, $\lim_{r\to 0} f(r) = -\infty$ which is a contradiction.

Case II: Assume $f'(r) < 0$ for $r > 0$ sufficiently small. Then

$$\frac{(rf'(r))'}{rf'(r)} \geq \sum_{i=1}^{k} \frac{1}{r\prod_{j=1}^{i} \log^{j}(\frac{\rho}{r})}.$$

Thus,

$$f'(r) \geq -\frac{c}{r\prod_{j=1}^{k} \log^{j}(\frac{\rho}{r})}, \tag{3.118}$$

for some $c > 0$ and $r > 0$ sufficiently small. On the other hand

$$\frac{f'(r) + rf''(r)}{f(r)} \leq -\lambda \sum_{j=1}^{k} \frac{1}{r}\Big(\prod_{i=1}^{j} log^{(i)}\frac{R}{r}\Big)^{-2} \leq -\lambda\Big(\frac{1}{\prod_{j=1}^{k} \log^{j}(\frac{\rho}{r})}\Big)'.$$

Since $f'(r) < 0$, there exists l such that $f(r) > l > 0$ for $r > 0$ sufficiently small. From the above inequality we then have

$$bf'(b) - af'(a) < -\lambda l\Big(\frac{1}{\prod_{j=1}^{k} \log^{j}(\frac{\rho}{b})} - \frac{1}{\prod_{j=1}^{k} \log^{j}(\frac{\rho}{a})}\Big).$$

From (3.118) we have $\lim_{a\to 0} af'(a) = 0$. Hence,

$$bf'(b) < -\frac{\lambda l}{\prod_{j=1}^{k} \log^{j}(\frac{\rho}{b})},$$

for every $b > 0$, and

$$f'(r) < -\frac{\lambda l}{r\prod_{j=1}^{k} \log^{j}(\frac{\rho}{r})},$$

for $r > 0$ sufficiently small. Therefore,

$$\lim_{r\to 0} f(r) = +\infty,$$

and by choosing l large enouph (e.g., $l > \frac{c}{\lambda}$) we get to contradict (3.118).

The proof of 5) is similar and is left to the interested reader. □

3.6 The evaluation of $a_{n,m}$

Here we evaluate the best constants $a_{n,m}$ which appear in Theorem 3.3.2. Suppose $n \geq 1$ and $m \leq \frac{n-2}{2}$. Then for any $R > 0$, the constants

$$a_{n,m} = \inf\left\{ \frac{\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx}{\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx};\ u \in C_0^{\infty}(B_R) \setminus \{0\} \right\}$$

are given by the following expressions.

1. For $n = 1$

   - if $m \in (-\infty, -\frac{3}{2}) \cup [-\frac{7}{6}, -\frac{1}{2}]$, then

$$a_{1,m} = (\frac{1+2m}{2})^2$$

   - if $-\frac{3}{2} < m < -\frac{7}{6}$, then

$$a_{1,m} = \min\{(\frac{n+2m}{2})^2, \frac{(\frac{(n-4-2m)(n+2m)}{4} + 2)^2}{(\frac{n-4-2m}{2})^2 + 2}\}.$$

2. If $m = \frac{n-4}{2}$, then

$$a_{m,n} = \min\{(n-2)^2, n-1\}.$$

3. If $n \geq 2$ and $m \leq \frac{-(n+4)+2\sqrt{n^2-n+1}}{6}$, then $a_{n,m} = (\frac{n+2m}{2})^2$.

4. If $2 \leq n \leq 3$ and $\frac{-(n+4)+2\sqrt{n^2-n+1}}{6} < m \leq \frac{n-2}{2}$, or $n \geq 4$ and $\frac{n-4}{2} < m \leq \frac{n-2}{2}$, then

$$a_{n,m} = \frac{(\frac{(n-4-2m)(n+2m)}{4} + n - 1)^2}{(\frac{n-4-2m}{2})^2 + n - 1}.$$

5. For $n \geq 4$ and $\frac{-(n+4)+2\sqrt{n^2-n+1}}{6} < m < \frac{n-4}{2}$, define $k^* = [(\frac{\sqrt{3}}{3} - \frac{1}{2})(n-2)]$.

   - If $k^* \leq 1$, then

$$a_{n,m} = \frac{(\frac{(n-4-2m)(n+2m)}{4} + n - 1)^2}{(\frac{n-4-2m}{2})^2 + n - 1}.$$

   - For $k^* > 1$ the interval $(m_0^1 := \frac{-(n+4)+2\sqrt{n^2-n+1}}{6}, m_0^2 := \frac{n-4}{2})$ can be divided in $2k^* - 1$ subintervals. For $1 \leq k \leq k^*$ define

$$m_k^1 := \frac{2(n-5) - \sqrt{(n-2)^2 - 12k(k+n-2)}}{6},$$

$$m_k^2 := \frac{2(n-5) + \sqrt{(n-2)^2 - 12k(k+n-2)}}{6}.$$

   If $m \in (m_0^1, m_1^1] \cup [m_1^2, m_0^2)]$, then

$$a_{n,m} = \frac{(\frac{(n-4-2m)(n+2m)}{4} + n - 1)^2}{(\frac{n-4-2m}{2})^2 + n - 1}.$$

- For $k \geq 1$ and $m \in (m^1_k, m^1_{k+1}] \cup [m^2_{k+1}, m^2_k)$, then

$$\begin{aligned} a_{n,m} &= \min\{\frac{(\frac{(n-4-2m)(n+2m)}{4} + k(n+k-2))^2}{(\frac{n-4-2m}{2})^2 + k(n+k-2)}, \\ &\quad \frac{(\frac{(n-4-2m)(n+2m)}{4} + (k+1)(n+k-1))^2}{(\frac{n-4-2m}{2})^2 + (k+1)(n+k-1)}\}. \end{aligned}$$

For $m \in (m^1_{k^*}, m^2_{k^*})$, then

$$\begin{aligned} a_{n,m} &= \min\{\frac{(\frac{(n-4-2m)(n+2m)}{4} + k^*(n+k^*-2))^2}{(\frac{n-4-2m}{2})^2 + k^*(n+k^*-2)}, \\ &\quad \frac{(\frac{(n-4-2m)(n+2m)}{4} + (k^*+1)(n+k^*-1))^2}{(\frac{n-4-2m}{2})^2 + (k^*+1)(n+k^*-1)}\}. \end{aligned}$$

Proof: Letting $V(r) = r^{-2m}$ then,

$$W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) = ((\frac{n-2m-2}{2})^2 - 2 - 4m - 2m(2m+1))r^{-2m-2}.$$

In order to satisfy condition (3.74) we should have

$$\frac{-(n+4) + 2\sqrt{n^2-n+1}}{6} \leq m \leq \frac{-(n+4) + 2\sqrt{n^2-n+1}}{6}. \tag{3.119}$$

So, by Theorem 3.3.1 under the above condition we have $a_{n,m} = (\frac{n+2m}{2})^2$ as in the radial case.
For the rest of the proof we will use an argument similar to that of Theorem 6.4 in [31] who computed $a_{n,m}$ in the case where $n \geq 5$ and for certain intervals of m.
Decomposing again $u \in C^\infty_0(B_R)$ into spherical harmonics; $u = \Sigma^\infty_{k=0} u_k$, where $u_k = f_k(|x|)\varphi_k(x)$, one has

$$\begin{aligned} \int_{R^n} \frac{|\Delta u_k|^2}{|x|^{2m}} dx &= \int_{R^n} |x|^{-2m} (f''_k(|x|))^2 dx \\ &+ ((n-1)(2m+1) + 2c_k) \int_{R^n} |x|^{-2m-2} (f'_k)^2 dx \\ &+ c_k(c_k + (n-4-2m)(2m+2)) \int_{R^n} |x|^{-2m-4} (f_k)^2 dx, \end{aligned} \tag{3.120}$$

and

$$\int_{R^n} \frac{|\nabla u_k|^2}{|x|^{2m+2}} dx = \int_{R^n} |x|^{-2m-2} (f'_k)^2 dx + c_k \int_{R^n} |x|^{-2m-4} (f_k)^2 dx. \tag{3.121}$$

One can then prove as in [31] that

$$a_{n,m} = \min\{A(k,m,n);\ k \in\} \tag{3.122}$$

where

$$A(k,m,n) = \frac{(\frac{(n-4-2m)(n+2m)}{4}+c_k)^2}{(\frac{n-4-2m}{2})^2+c_k} \text{ if } m = \frac{n-4}{2} \tag{3.123}$$

and

$$A(k,m,n) := c_k \text{ if } m = \frac{n-4}{2} \text{ and } n+k > 2. \tag{3.124}$$

Note that when $m = \frac{n-4}{2}$ and $n+k > 2$, then $c_k \neq 0$. Actually, this also holds for $n+k \leq 2$, in which case one deduces that if $m = \frac{n-4}{2}$, then

$$a_{n,m} = \min\{(n-2)^2 = (\frac{n+2m}{2})^2, (n-1) = c_1\}$$

which is statement 2).

The rest of the proof consists of computing the infimum especially in the cases not considered in [31]. For that we consider the function

$$f(x) = \frac{(\frac{(n-4-2m)(n+2m)}{4}+x)^2}{(\frac{n-4-2m}{2})^2+x}.$$

It is easy to check that $f'(x) = 0$ at x_1 and x_2, where

$$x_1 = -\frac{(n-4-2m)(n+2m)}{4} \tag{3.125}$$

$$x_2 = \frac{(n-4-2m)(-n+6m+8)}{4}. \tag{3.126}$$

Observe that for for $n \geq 2$, $\frac{n-8}{6} \leq \frac{n-4}{2}$. Hence, for $m \leq \frac{n-8}{6}$ both x_1 and x_2 are negative and hence $a_{n,m} = (\frac{n+2m}{2})^2$. Also note that

$$\frac{-(n+4) - 2\sqrt{n^2-n+1}}{6} \leq \frac{n-8}{6} \quad for \ \ all \ \ n \geq 1.$$

Hence, under the condition in 3) we have $a_{n,m} = (\frac{n+2m}{2})^2$.
Also for $n = 1$ if $m \leq -\frac{3}{2}$ both critical points are negative and we have $a_{1,m} \leq (\frac{1+2m}{2})^2$. Comparing $A(0,m,n)$ and $A(1,m,n)$ we see that $A(1,m,n) \geq A(0,m,n)$ if and only if (3.119) holds.

For $n = 1$ and $-\frac{3}{2} < m < -\frac{7}{6}$ both x_1 and x_2 are positive. Consider the equations

$$x(x-1) = x_1 = \frac{(2m+3)(2m+1)}{4},$$

and

$$x(x-1) = x_2 = -\frac{(2m+3)(6m+7)}{4}.$$

By simple calculations we can see that all four solutions of the above two equations are less that two. Since, $A(1,m,1) < A(0,m,1)$ for $m < -\frac{7}{6}$, we have $a_{1,m} \leq \min\{A(1,m,1), A(2,m,1)\}$ and 1) follows.

For $n \geq 2$ and $\frac{n-4}{2} < m < \frac{n-2}{2}$ we have $x_1 > 0$ and $x_2 < 0$. Consider the equation

$$x(x+n-2) = x_1 = -\frac{(n-4-2m)(n+2m)}{4}.$$

Then $\frac{2m+4-n}{2}$ and $-\frac{(2m+n)}{2}$ are solutions of the above equation and both are less than one. Since, for $n \geq 4$

$$\frac{n-2}{2} > \frac{-(n+4)+2\sqrt{n^2-n+1}}{6},$$

and $A(1,m,n) \leq A(0,m,n)$ for $m \geq \frac{-(n+4)+2\sqrt{n^2-n+1}}{6}$, the best constant is equal to what 4) claims.
5) follows from an argument similar to that of Theorem 6.4 in [31].

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Chapter 4

Optimal weighted Hardy-Rellich inequalities on $H^2 \cap H_0^1$ [3]

4.1 Introduction

Let Ω be a smooth bounded domain in ^{n} and $0 \in \Omega$. Let us recall that the classical Hardy-Rellich inequality assets that

$$\int_\Omega |\Delta u|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx, \quad \text{for} \quad u \in H_0^2(\Omega), \tag{4.1}$$

where the constant appearing in the above inequality is the best constant and it is never achieved in H_0^2. Recently there has been a flurry of activity about possible improvements of the following type

$$\text{If } n \geq 5 \quad \text{then} \quad \int_\Omega |\Delta u|^2 dx - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx \geq \int_\Omega W(x) u^2 dx, \tag{4.2}$$

for $u \in H_0^2(\Omega)$ as well as

$$\text{If } n \geq 3 \quad \text{then} \quad \int_\Omega |\Delta u|^2 dx - C(n) \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq \int_\Omega V(x) |\nabla u|^2 dx, \tag{4.3}$$

for all $u \in H_0^2(\Omega)$, where V, W are certain explicit radially symmetric potentials of order lower than $\frac{1}{r^2}$ (for V) and $\frac{1}{r^4}$ (for W) (see [2], [3], [8], [10], [11], [15], and [18].

The inequality (4.1) was first proved by Rellich [17] for $u \in H_0^2(\Omega)$ and then it was extended to functions in $H^2(\Omega) \cap H_0^1(\Omega)$ by Donal et al. in [11]. So far most of the results about improved Hardy-Rellich inequalities and the inequalities of the form (4.3) are proved for $u \in H_0^2(\Omega)$ (see [8], [15], and [18]). The goal of this paper is to provide a general approach to prove optimal weighted Hardy-Rellich inequalities on $H^2(\Omega) \cap H_0^1(\Omega)$ and inequalities of type (4.3) on $H^2(\Omega)$ which are important in the study of fourth order elliptic equations with Navier boudary condition and systems of second order elliptic equations (see [16]).

[3] A version of this chapter has been submitted for publication; A. Moradifam, Optimal weighted Hardy-Rellich inequalities on $H^2 \cap H_0^1$ (2009).

We start – in section 2 – by giving necessary and sufficient conditions on positive radial functions V and W on a ball B in R^n, so that the following inequality holds for some $c > 0$ and $b < 0$:

$$\int_B V(x)|\nabla u|^2 dx \geq c \int_B W(x)u^2 dx + b \int_{\partial B} u^2 \text{ for all } u \in H^1(B). \tag{4.4}$$

Assuming that the ball B has radius R and that $\int_0^R \frac{1}{r^{n-1}V(r)} dr = +\infty$, the condition is simply that the ordinary differential equation

$$(\mathrm{B}_{V,cW}) \qquad y''(r) + (\tfrac{n-1}{r} + \tfrac{V_r(r)}{V(r)})y'(r) + \tfrac{cW(r)}{V(r)} y(r) = 0$$

has a positive solution ϕ on the interval $(0, R)$ with $V(R)\frac{\phi'(R)}{\phi(R)} = b$. As in [15], we shall call such a couple (V, W) a *Bessel pair on* $(0, R)$. The *weight* of such a pair is then defined as

$$\beta(V, W; R) = \sup\left\{c;\ (\mathrm{B}_{V,cW}) \text{ has a positive solution on } (0, R)\right\}. \tag{4.5}$$

We call W a Bessel potential if $(1, W)$ is a Bessel pair. This characterization makes an important connection between Hardy-type inequalities and the oscillatory behavior of the above equations. For a detailed analysis of Bessel pairs see [15]. The above theorem in the general form of improved Hardy-type inequalities which recently has been of interest for many authors (see [1], [4], [5], [6], [7], [9], [12], [13], [19], and [20]).

Here is the main result of this paper.

Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball centered at zero with radius R in ^{n} $(n \geq 1)$ such that $\int_0^R \frac{1}{r^{n-1}V(r)} dr = +\infty$ and $\int_0^R r^{n-1}V(r)dr < +\infty$. The following statements are then equivalent:

1. (V, W) is a Bessel pair on $(0, R)$ with $\theta := V(R)\frac{\phi'(R)}{\phi(R)}$, where ϕ is the corresponding solution of $(B_{(V,W)})$.
2. $\int_B V(x)|\nabla u|^2 dx \geq \int_B W(x)u^2 dx + \theta \int_{\partial B} u^2 ds$ for all $u \in C^\infty(\bar{B})$.
3. If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < n - 2$, then the above are equivalent to

$$\begin{aligned}\int_B V(x)|\Delta u|^2 dx \quad &\geq \quad \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\frac{V(x)}{|x|^2} \\ &- \quad \frac{V_r(|x|)}{|x|})|\nabla u|^2 dx + (\theta + (n-1)V(R))\int_{\partial B} |\nabla u|^2,\end{aligned}$$

for all radial $u \in C^\infty(\bar{B})$.

4. If in addition, $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ on $(0, R)$, then the above are equivalent to

$$\begin{aligned}\int_B V(x)|\Delta u|^2 dx &\geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\frac{V(x)}{|x|^2} \\ &- \frac{V_r(|x|)}{|x|})|\nabla u|^2 dx + (\theta + (n-1)V(R))\int_{\partial B} |\nabla u|^2,\end{aligned}$$

for all $u \in C^\infty(\bar{B})$.

Appropriate combinations of 4) and 2) in the above theorem and lead to a myriad of Hardy-Rellich type inequalities on $H^2(\Omega) \cap H_0^1(\Omega)$.

Remark 4.1.1. *The condition* $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ *in the above theorem guarantees that the minimizing sequences are radial functions. We shall see in section 3 that even with out this condition our approach is applicable, although the minimizing sequences are no longer radial functions.*

Remark 4.1.2. *To see the importance and generality of the above theorem, notice that inequalities (7) and (8) in [16] which are the author's main tools to prove singularity of the extremal solutions in dimensions* $n \geq 9$ *(see [16]) are an immediate consequence of the above theorem combined with (4.4). This theorem will also allow us to extend most of the results about Hardy and Hardy-Rellich type inequalities on* $C_0^\infty(\Omega)$ *to corresponding inequalities on* $C^\infty(\bar{\Omega})$ *such as those in [15] and [18].*

We shall show that for $-\frac{n}{2} \leq m \leq \frac{n-2}{2}$

$$H_{n,m} = \inf_{u \in H^2(B)\setminus\{0\}} \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}}}{\int_B \frac{|\nabla u|^2}{|x|^{2m+2}}} = \inf_{u \in H_0^2(B)\setminus\{0\}} \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}}}{\int_B \frac{|\nabla u|^2}{|x|^{2m+2}}}, \tag{4.6}$$

and for $-\frac{n}{2} \leq m \leq \frac{n-4}{2}$

$$a_{n,m} = \inf_{u \in H^2(B)\cap H_0^1(B)\setminus\{0\}} \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}}}{\int_B \frac{u^2}{|x|^{2m+4}}} = \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}}}{\int_B \frac{u^2}{|x|^{2m+4}}}, \tag{4.7}$$

where the constants $H_{n,m}$ and $a_{n,m}$ have been computed in [18] and then more generally in [15]. For example $a_{n,0} = \frac{n^2}{4}$ for $n \geq 5$, $a_{4,0} = 3$, and $a_{3,0} = \frac{25}{36}$.

The above general theorem also allows us to obtain improved Hardy-Rellich inequalities on $H^2(B) \cap H_0^1(B)$. For instance, assume W is a Bessel potential on $(0, R)$ and ϕ is the corresponding solution of $(B_{(1,W)})$ with $R\frac{\phi'(R)}{\phi(R)} \geq -\frac{n}{2}$. If $r\frac{W_r(r)}{W(r)}$ decreases to $-\lambda$ and $\lambda \leq n-2$, then we have for all $H^2(B) \cap H_0^1(B)$

$$\int_B |\Delta u|^2 dx - \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4} dx \geq (\frac{n^2}{4} + \frac{(n-\lambda-2)^2}{4})\beta(W; R)\int_B \frac{W(x)}{|x|^2} u^2 dx. \tag{4.8}$$

By applying (4.8) to the various examples of Bessel functions, we can various improved Hardy-Rellich inequalities on $H^2(B)\cap H_0^1(B)$. Here are some basic examples of Bessel potentials, their corresponding solution ϕ of $(B_{(1,W)})$.

- $W\equiv 0$ is a Bessel potential on $(0,R)$ for any $R>0$ and $\phi=1$.

- $W\equiv 1$ is a Bessel potential on $(0,R)$ for any $R>0$, $\phi(r)=J_0(\frac{\mu r}{R})$, where J_0 is the Bessel function and $z_0=2.4048...$ is the first zero of the Bessel function J_0. Moreover $R\frac{\phi'(R)}{\phi(R)}=-\frac{n}{2}$.

- For $k\geq 1$, $R>0$, let $W_{k,\rho}(r)=\Sigma_{j=1}^k\frac{1}{r^2}\big(\prod_{i=1}^j log^{(i)}\frac{\rho}{r}\big)^{-2}$ where the functions $log^{(i)}$ are defined iteratively as follows: $log^{(1)}(.)=log(.)$ and for $k\geq 2$, $log^{(k)}(.)=log(log^{(k-1)}(.))$. $W_{k,\rho}$ is then a Bessel potential on $(0,R)$ with the corresponding solution
$$\phi_k=(\prod_{i=1}^j log^{(i)}\frac{\rho}{r})^{-\frac{1}{2}}.$$
It is easy to see that for $\rho\geq R(e^{e^{e^{\cdot^{\cdot^{e((k-1)-times)}}}}})$ large enough we have $R\frac{\phi_k'(R)}{\phi_k(R)}\geq -\frac{n}{2}$.

- For $k\geq 1$, and $R>0$, define
$$\tilde{W}_{k;\rho}(r)=\Sigma_{j=1}^k\frac{1}{r^2}X_1^2(\frac{r}{R})X_2^2(\frac{r}{R})\ldots X_{j-1}^2(\frac{r}{R})X_j^2(\frac{r}{R})$$
where the functions X_i are defined iteratively as follows: $X_1(t)=(1-\log(t))^{-1}$ and for $k\geq 2$, $X_k(t)=X_1(X_{k-1}(t))$. Then again $\tilde{W}_{k,\rho}$ is a Bessel potential on $(0,R)$ with $\phi_k=(X_1(\frac{r}{R})X_2(\frac{r}{R})\ldots X_{j-1}(\frac{r}{R})X_k(\frac{r}{R}))^{\frac{1}{2}}$. Moreover, $R\frac{\phi_k'(R)}{\phi_k(R)}=-\frac{k}{2}$.

As an example, let $k\geq 1$ and choose $\rho\geq R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$ large enough so that $R\frac{\phi'(R)}{\phi(R)}\geq -\frac{n}{2}$, where
$$\phi=(\prod_{i=1}^j log^{(i)}\frac{\rho}{|x|})^{\frac{1}{2}}. \tag{4.9}$$
Then we have
$$\begin{aligned}\int_B|\Delta u(x)|^2dx &\geq \frac{n^2(n-4)^2}{16}\int_B\frac{u^2}{|x|^4}dx \\ &+ (1+\frac{n(n-4)}{8})\sum_{j=1}^k\int_B\frac{u^2}{|x|^4}(\prod_{i=1}^j log^{(i)}\frac{\rho}{|x|})^{-2}dx,\end{aligned} \tag{4.10}$$
for all $H^2(B)\cap H_0^1(B)$ which corresponds to the result od Adimurthi et al. [2].

More generally, we show that for any $-\frac{n}{2} \le m < \frac{n-2}{2}$, and any W Bessel potential on a ball $B_R \subset R^n$ of radius R, if for the corresponding solution ϕ of $(B_{(1,W)})$ we have $R\frac{\phi'(R)}{\phi(R)} \ge -\frac{n}{2} - m$, then the following inequality holds for all $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \ge a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W;R) \int_{B_R} W(x) \frac{|\nabla u|^2}{|x|^{2m}} dx. \tag{4.11}$$

We also establish a more general version of equation (4.8). Assuming again that $\frac{rW'(r)}{W(r)}$ decreases to $-\lambda$ on $(0,R)$, and provided $m \le \frac{n-4}{2}$ and $\frac{n}{2} + m \ge \lambda \ge n - 2m - 4$, we then have for all $u \in C_0^\infty(B_R)$,

$$\begin{aligned} \int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx \ &\ge \ \frac{(n+2m)^2(n-2m-4)^2}{16} \int_{B_R} \frac{u^2}{|x|^{2m+4}} dx \\ &+ \ \beta(W;R)\Big(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4}\Big) \int_{B_R} \frac{W(x)}{|x|^{2m+2}} u^2 dx. \end{aligned} \tag{4.12}$$

4.2 General Hardy Inequalities

Here is the main result of this section. Let V and W be positive radial C^1-functions on $B_R \backslash \{0\}$, where B_R is a ball centered at zero with radius R $(0 < R \le +\infty)$ in ^{n} $(n \ge 1)$. Assume that $\int_0^a \frac{1}{r^{n-1}V(r)} dr = +\infty$ and $\int_0^a r^{n-1} V(r) dr < \infty$ for some $0 < a < R$. Then the following two statements are equivalent:

1. The ordinary differential equation

   $(\mathrm{B}_{V,W})$ $\qquad y''(r) + (\frac{n-1}{r} + \frac{V_r(r)}{V(r)}) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$

   has a positive solution on the interval $(0, R]$ with $\theta := V(R)\frac{\phi'(R)}{\phi(R)}$.

2. For all $u \in H^1(B_R)$

   $(\mathrm{H}_{V,W})$ $\qquad \int_{B_R} V(x)|\nabla u(x)|^2 dx \ge \int_{B_R} W(x) u^2 dx + \theta \int_{\partial B} u^2 \text{ ds}.$

The above theorem allows to generalize all Hardy type inequalities on $H_0^1(\Omega)$ to a corresponding inequality on $H^1(\Omega)$. For instance we can get the following general form of the Caffarelli-Kohn-Nirenberg inequalities. Assume B is the ball of radius R and and centered at zero in ^{n}. If $a \le \frac{n-2}{2}$, then

$$\begin{aligned} \int_B |x|^{-2a} |\nabla u(x)|^2 dx \ &\ge \ \Big(\frac{n-2a-2}{2}\Big)^2 \int_B |x|^{-2a-2} u^2 dx \\ &- \ \frac{(n-2a-2)R^{-2a-1}}{2} \int_{\partial B} u^2 dx, \end{aligned} \tag{4.13}$$

for all $u \in H^1(B)$.

To prove Theorem 4.2 we shall need the following lemma.

Lemma 4.14. *Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in ^{n} $(n \geq 1)$ and centered at zero. Assume*

$$\int_B \left(V(x)|\nabla u|^2 - W(x)|u|^2\right) dx - \theta \int_{\partial B} u^2 ds \geq 0 \text{ for all } u \in H^1(B),$$

for some $\theta < 0$. Then there exists a C^2-supersolution to the following linear elliptic equation

$$\begin{aligned} -\mathrm{div}(V(x)\nabla u) - W(x)u &= 0, \quad \text{in } B, && (4.15)\\ u &> 0 \quad \text{in } B \setminus \{0\}, && (4.16)\\ V\nabla u.\nu &= \theta u \quad \text{in } \partial B. && (4.17) \end{aligned}$$

Proof: Define

$$\lambda_1(V) := \inf\{\frac{\int_B V(x)|\nabla\psi|^2 - W(x)|\psi|^2 - \theta\int_{\partial B} u^2}{\int_B |\psi|^2}; \quad \psi \in C_0^\infty(B \setminus \{0\})\}.$$

By our assumption $\lambda_1(V) \geq 0$. Let (ϕ_n, λ_1^n) be the first eigenpair for the problem

$$\begin{aligned} (L - \lambda_1(V) - \lambda_1^n)\phi_n &= 0 \ \ on \ \ B \setminus B_{\frac{R}{n}}\\ \phi_n &= 0 \ \ on \ \ \partial B_{\frac{R}{n}}\\ V\nabla\phi_n.\nu &= \theta\phi_n \ \ on \ \ \partial B, \end{aligned}$$

where $Lu = -\mathrm{div}(V(x)\nabla u) - W(x)u$, and $B_{\frac{R}{n}}$ is a ball of radius $\frac{R}{n}$, $n \geq 2$. The eigenfunctions can be chosen in such a way that $\phi_n > 0$ on $B \setminus B_{\frac{R}{n}}$ and $\varphi_n(b) = 1$, for some $b \in B$ with $\frac{R}{2} < |b| < R$.

Note that $\lambda_1^n \downarrow 0$ as $n \to \infty$. Harnak's inequality yields that for any compact subset K, $\frac{\max_K \phi_n}{\min_K \phi_n} \leq C(K)$ with the later constant being independent of ϕ_n. Also standard elliptic estimates also yields that the family (ϕ_n) have also uniformly bounded derivatives on the compact sets $B - B_{\frac{R}{n}}$.
Therefore, there exists a subsequence $(\varphi_{n_{l_2}})_{l_2}$ of $(\varphi_n)_n$ such that $(\varphi_{n_{l_2}})_{l_2}$ converges to some $\varphi_2 \in C^2(B \setminus B(\frac{R}{2}))$. Now consider $(\varphi_{n_{l_2}})_{l_2}$ on $B \setminus B(\frac{R}{3})$. Again there exists a subsequence $(\varphi_{n_{l_3}})_{l_3}$ of $(\varphi_{n_{l_2}})_{l_2}$ which converges to $\varphi_3 \in C^2(B \setminus B(\frac{R}{3}))$, and $\varphi_3(x) = \varphi_2(x)$ for all $x \in B \setminus B(\frac{R}{2})$. By repeating this argument we get a supersolution $\varphi \in C^2(B \setminus \{0\})$ i.e. $L\varphi \geq 0$, such that $\varphi > 0$ on $B \setminus \{0\}$ and $V\nabla\phi.\nu = \theta\phi$ on ∂B. □

Proof of Theorem 4.2: First we prove that 1) implies 2). Let $\phi \in C^1(0, R]$ be a solution of $(B_{V,W})$ such that $\phi(x) > 0$ for all $x \in (0, R)$. Define $\frac{u(x)}{\varphi(|x|)} = \psi(x)$. Then

$$|\nabla u|^2 = (\varphi'(|x|))^2\psi^2(x) + 2\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi + \varphi^2(|x|)|\nabla\psi|^2.$$

Hence,

$$V(|x|)|\nabla u|^2 \geq V(|x|)(\varphi'(|x|))^2\psi^2(x) + 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi(x).$$

Thus, we have

$$\begin{aligned}\int_B V(|x|)|\nabla u|^2 dx &\geq \int_B V(|x|)(\varphi'(|x|))^2\psi^2(x)dx\\ &+ \int_B 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx.\end{aligned}$$

Let B_ϵ be a ball of radius ϵ centered at the origin. Integrate by parts to get

$$\begin{aligned}&\int_B V(|x|)|\nabla u|^2 dx \geq \int_B V(|x|)(\varphi'(|x|))^2\psi^2(x)dx\\ +& \int_{B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx\\ +& \int_{B\backslash B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx\\ =& \int_{B_\epsilon} V(|x|)(\varphi'(|x|))^2\psi^2(x)dx + \int_{B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx\\ -& \int_{B\backslash B_\epsilon}\left\{\big(V(|x|)\varphi''(|x|)\varphi(|x|) + (\frac{(n-1)V(|x|)}{r} + V_r(|x|))\varphi'(|x|)\varphi(|x|)\big)\psi^2(x)\right\}dx\\ +& \int_{\partial(B\backslash B_\epsilon)} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)ds\end{aligned}$$

Let $\epsilon \to 0$ and use Lemma 2.3 in [15] and the fact that ϕ is a solution of $(D_{v,w})$ to get

$$\begin{aligned}\int_B V(|x|)|\nabla u|^2 dx &\geq -\int_B [V(|x|)\varphi''(|x|) + (\frac{(n-1)V(|x|)}{r}\\ &+ V_r(|x|))\varphi'(|x|)]\frac{u^2(x)}{\varphi(|x|)}dx\\ &= \int_B W(|x|)u^2(x)dx - \theta\int_{\partial B} u^2 ds.\end{aligned}$$

To show that 2) implies 1), we assume that inequality $(\mathrm{H}_{V,W})$ holds on a ball B of radius R, and then apply Lemma 4.14 to obtain a C^2-supersolution for the equation (4.15). Now take the surface average of u, that is

$$y(r) = \frac{1}{n\omega_w r^{n-1}}\int_{\partial B_r} u(x)dS = \frac{1}{n\omega_n}\int_{|\omega|=1} u(r\omega)d\omega > 0, \tag{4.18}$$

where ω_n denotes the volume of the unit ball in R^n. We may assume that the unit ball is contained in B (otherwise we just use a smaller ball). It is easy to see that $V(R)\frac{y'(R)}{y(R)} = \theta$. We clearly have

$$y''(r) + \frac{n-1}{r}y'(r) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r} \Delta u(x)dS. \tag{4.19}$$

Since $u(x)$ is a supersolution of (4.15), we have

$$\int_{\partial B_r} div(V(|x|)\nabla u)ds - \int_{\partial B} W(|x|)udx \geq 0,$$

and therefore,

$$V(r)\int_{\partial B_r} \Delta u dS - V_r(r)\int_{\partial B_r} \nabla u.x ds - W(r)\int_{\partial B_r} u(x)ds \geq 0.$$

It follows that

$$V(r)\int_{\partial B_r} \Delta u dS - V_r(r)y'(r) - W(r)y(r) \geq 0, \tag{4.20}$$

and in view of (4.18), we see that y satisfies the inequality

$$V(r)y''(r) + (\frac{(n-1)V(r)}{r} + V_r(r))y'(r) \leq -W(r)y(r), \quad for \quad 0<r<R, \tag{4.21}$$

that is it is a positive supersolution y for $(B_{V,W})$ with $V(R)\frac{y'(R)}{y(R)} = \theta$. Standard results in ODE now allow us to conclude that $(B_{V,W})$ has actually a positive solution on $(0,R)$, and the proof of theorem 4.2 is now complete. □

An immediate application of Theorem 2.6 in [15] and Theorem 4.2 is the following very general Hardy inequality.

Let $V(x) = V(|x|)$ be a strictly positive radial function on a smooth domain Ω containing 0 such that $R = \sup_{x\in\Omega}|x|$. Assume that for some $\lambda \in$

$$\tfrac{rV_r(r)}{V(r)} + \lambda \geq 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0} \tfrac{rV_r(r)}{V(r)} + \lambda = 0. \tag{4.22}$$

If $\lambda \leq n-2$, then the following inequality holds for any Bessel potential W on $(0,R)$:

$$\begin{aligned}\int_\Omega V(x)|\nabla u|^2 dx \quad &\geq \quad (\frac{n-\lambda-2}{2})^2\int_\Omega \frac{V(x)}{|x|^2}u^2dx + \beta(W;R)\int_\Omega V(x)W(x)u^2dx \\ &+ \quad V(R)(\frac{\phi'(R)}{\phi(R)} - \frac{n-\lambda-2}{2R})\int_{\partial B} u^2 \qquad for \quad u\in H^1(\Omega),\end{aligned}$$

where ϕ is the corresponding solution of $(B_{1,W})$.

Proof: Under our assumptions, it is easy to see that $y = r^{\frac{n-\lambda-2}{2}}\phi(r)$ is a positive super-solution of $B_{(V,V(\frac{n-\lambda-2}{2})^2r^{-2}+W)}$. Now apply Theorem 2.6 in [15] and Theorem 4.2 to complete the proof. □

4.3 General Hardy-Rellich inequalities

Let $0\in\Omega\subset R^n$ be a smooth domain, and denote

$$C^k_r(\bar\Omega) = \{v \in C^k(\bar\Omega) : \text{v is radial }\}.$$

We start by considering a general inequality for radial functions.

Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in ^{n} $(n \geq 1)$ and centered at zero. Assume $\int_0^R \frac{1}{r^{n-1}V(r)}dr = \infty$ and $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < n-2$. Then the following statements are equivalent:

1. (V, W) is a Bessel pair on $(0, R)$ with $\theta := V(R)\frac{\phi'(R)}{\phi(R)}$, where ϕ is the corresponding solution of $(B_{(V,W)})$.

2. If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < n-2$, then the above are equivalent to

$$\begin{aligned}
\int_B V(x)|\Delta u|^2 dx \quad &\geq \quad \int_B W(x)|\nabla u|^2 dx \\
&+ \quad (n-1)\int_B \Big(\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|}\Big)|\nabla u|^2 dx \\
&+ \quad (\theta + (n-1)V(R))\int_{\partial B} |\nabla u|^2,
\end{aligned}$$

for all radial $u \in C^\infty(\bar{B})$.

Proof: Assume $u \in C_r^\infty(\bar{B})$ and observe that

$$\begin{aligned}
\int_B V(x)|\Delta u|^2 dx \quad &= \quad n\omega_n\Big\{\int_0^R V(r)u_{rr}^2 r^{n-1}dr + (n-1)^2\int_0^R V(r)\frac{u_r^2}{r^2}r^{n-1}dr \\
&+ \quad 2(n-1)\int_0^R V(r)uu_r r^{n-2}dr\Big\}.
\end{aligned}$$

Setting $\nu = u_r$, we then have

$$\begin{aligned}
\int_B V(x)|\Delta u|^2 dx \quad &= \quad \int_B V(x)|\nabla \nu|^2 dx \\
&+ \quad (n-1)\int_B \Big(\frac{V(|x|)}{|x|^2} \\
&- \quad \frac{V_r(|x|)}{|x|}\Big)|\nu|^2 dx + (n-1)V(R)\int_{\partial B} |\nu|^2 ds.
\end{aligned}$$

Thus, $(\mathrm{HR}_{V,W})$ for radial functions is equivalent to

$$\int_B V(x)|\nabla \nu|^2 dx \geq \int_B W(x)\nu^2 dx.$$

It therefore follows from Theorem 4.2 that 1) and 2) are equivalent. $\square$

4.3.1 The non-radial case

The decomposition of a function into its spherical harmonics will be one of our tools to prove our results. This idea has also been used in [18] and [15]. Let $u \in C^\infty(\bar{B})$. By decomposing u into spherical harmonics we get

$$u = \Sigma_{k=0}^{\infty} u_k \text{ where } u_k = f_k(|x|)\varphi_k(x)$$

and $(\varphi_k(x))_k$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues $c_k = k(N+k-2)$, $k \geq 0$. The functions f_k belong to $u \in C^\infty([0,R])$, $f_k(R) = 0$, and satisfy $f_k(r) = O(r^k)$ and $f'(r) = O(r^{k-1})$ as $r \to 0$. In particular,

$$\varphi_0 = 1 \text{ and } f_0 = \tfrac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} u ds = \tfrac{1}{n\omega_n} \int_{|x|=1} u(rx) ds. \tag{4.23}$$

We also have for any $k \geq 0$, and any continuous real valued functions v and w on $(0, \infty)$,

$$\int_{R^n} V(|x|)|\Delta u_k|^2 dx = \int_{R^n} V(|x|)\big(\Delta f_k(|x|) - c_k \frac{f_k(|x|)}{|x|^2}\big)^2 dx, \tag{4.24}$$

and

$$\int_{R^n} W(|x|)|\nabla u_k|^2 dx = \int_{R^n} W(|x|)|\nabla f_k|^2 dx + c_k \int_{R^n} W(|x|)|x|^{-2} f_k^2 dx. \tag{4.25}$$

Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in ^{n} $(n \geq 1)$ and centered at zero. Assume $\int_0^R \frac{1}{r^{n-1}V(r)} dr = \infty$ and $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < (n-2)$. If

$$W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0 \quad for \quad 0 \leq r \leq R, \tag{4.26}$$

and the ordinary differential equation $(B_{V,W})$ has a positive solution ϕ on the interval $(0, R]$ such that

$$(n - 1 + R\frac{\phi'(R)}{\phi(R)})V(R) \geq 0, \tag{4.27}$$

then the following inequality holds for all $u \in H^2(B)$.

$$(\mathrm{HR}_{V,W}) \int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B \big(\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|}\big)|\nabla u|^2.$$

Moreover, if $\beta(V, W; R) \geq 1$, then the best constant is given by

$$\beta(V, W; R) = \sup\big\{c;\ (\mathrm{HR}_{V,cW}) \text{ holds}\big\}. \tag{4.28}$$

Proof: Assume that the equation $(B_{V,W})$ has a positive solution on $(0,R]$. We prove that the inequality $(HR_{V,W})$ holds for all $u \in C_0^\infty(B)$ by frequently using that

$$\begin{aligned}\int_0^R V(r)|x'(r)|^2 r^{n-1}dr &\geq \int_0^R W(r)x^2(r)r^{n-1}dr\\ &+ V(R)\frac{\phi'(R)}{\phi(R)}R^{n-1}(x(R))^2, \end{aligned} \tag{4.29}$$

for all $x \in C^1(0,R]$. Indeed, for all $n \geq 1$ and $k \geq 0$ we have

$$\begin{aligned} &\frac{1}{nw_n}\int_{R^n} V(x)|\Delta u_k|^2dx = \frac{1}{nw_n}\int_{R^n} V(x)\big(\Delta f_k(|x|) - c_k\frac{f_k(|x|)}{|x|^2}\big)^2dx\\ = &\int_0^R V(r)\big(f_k''(r) + \frac{n-1}{r}f_k'(r) - c_k\frac{f_k(r)}{r^2}\big)^2 r^{n-1}dr\\ = &\int_0^R V(r)(f_k''(r))^2 r^{n-1}dr + (n-1)^2\int_0^R V(r)(f_k'(r))^2 r^{n-3}dr\\ &+c_k^2\int_0^R V(r)f_k^2(r)r^{n-5} + 2(n-1)\int_0^R V(r)f_k''(r)f_k'(r)r^{n-2}\\ &-2c_k\int_0^R V(r)f_k''(r)f_k(r)r^{n-3}dr - 2c_k(n-1)\int_0^R V(r)f_k'(r)f_k(r)r^{n-4}dr. \end{aligned}$$

Integrate by parts and use (4.23) for $k=0$ to get

$$\begin{aligned} &\frac{1}{n\omega_n}\int_{R^n} V(x)|\Delta u_k|^2dx = \int_0^R V(r)(f_k''(r))^2 r^{n-1}dr\\ + &(n-1+2c_k)\int_0^R V(r)(f_k'(r))^2 r^{n-3}dr\\ + &(2c_k(n-4)+c_k^2)\int_0^R V(r)r^{n-5}f_k^2(r)dr - (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr\\ - &c_k(n-5)\int_0^R V_r(r)f_k^2(r)r^{n-4}dr - c_k\int_0^R V_{rr}(r)f_k^2(r)r^{n-3}dr.\\ + &(n-1)V(R)(f_k'(R))^2R^{n-2} \end{aligned}$$

Now define $g_k(r) = \frac{f_k(r)}{r}$ and note that $g_k(r) = O(r^{k-1})$ for all $k \geq 1$. We have

$$\begin{aligned}
\int_0^R V(r)(f_k'(r))^2 r^{n-3} &= \int_0^R V(r)(g_k'(r))^2 r^{n-1} dr + \int_0^R 2V(r)g_k(r)g_k'(r)r^{n-2}dr \\
&+ \int_0^R V(r)g_k^2(r)r^{n-3}dr \\
&= \int_0^R V(r)(g_k'(r))^2 r^{n-1}dr - (n-3)\int_0^R V(r)g_k^2(r)r^{n-3}dr \\
&- \int_0^R V_r(r)g_k^2(r)r^{n-2}dr
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^R V(r)(f_k'(r))^2 r^{n-3} &\geq \int_0^R W(r)f_k^2(r)r^{n-3}dr \qquad (4.30)\\
&- (n-3)\int_0^R V(r)f_k^2(r)r^{n-5}dr - \int_0^R V_r(r)f_k^2(r)r^{n-4}dr.
\end{aligned}$$

Substituting $2c_k \int_0^R V(r)(f_k'(r))^2 r^{n-3}$ in (4.30) by its lower estimate in the last inequality (4.30), we get

$$\begin{aligned}
&\frac{1}{n\omega_n}\int_{R^n} V(x)|\Delta u_k|^2 dx \geq \int_0^R W(r)(f_k'(r))^2 r^{n-1}dr \\
&+ \int_0^R W(r)(f_k(r))^2 r^{n-3}dr \\
&+ (n-1)\int_0^R V(r)(f_k'(r))^2 r^{n-3}dr + c_k(n-1)\int_0^R V(r)(f_k(r))^2 r^{n-5}dr \\
&- (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr - c_k(n-1)\int_0^R V_r(r)r^{n-4}(f_k)^2(r)dr \\
&+ c_k(c_k-(n-1))\int_0^R V(r)r^{n-5}f_k^2(r)dr \\
&+ c_k\int_0^R (W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r))f_k^2(r)r^{n-3}dr \\
&+ (n-1)V(R)(f_k'(R))^2 R^{n-2} + V(R)\frac{\phi'(R)}{\phi(R)}R^{n-1}(f_k'(R))^2
\end{aligned}$$

The proof is now complete since the last two terms are non-negative by our assumptions. □

Remark 4.3.1. In order to apply the above theorem to

$$V(x) = |x|^{-2m}$$

we see that even in the simplest case $V \equiv 1$ condition (4.26) reduces to $(\frac{n-2}{2})^2|x|^{-2} \geq 2|x|^{-2}$, which is then guaranteed only if $n \geq 5$. More generally, if $V(x) = |x|^{-2m}$, then in order to satisfy (4.26) we need to have

$$\frac{-(n+4) - 2\sqrt{n^2-n+1}}{6} \leq m \leq \frac{-(n+4) + 2\sqrt{n^2-n+1}}{6}. \tag{4.31}$$

Also to satisfy the condition (4.27) we need to have $m > -\frac{n}{2}$. Thus for m satisfying (4.31) the inequality

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq (\frac{n+2m}{2})^2 \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx. \tag{4.32}$$

for all $u \in H^2(B_R)$. Moreover, $(\frac{n+2m}{2})^2$ is the best constant. We shall see however that this inequality remains true without condition (4.31), but with a constant that is sometimes different from $(\frac{n+2m}{2})^2$ in the cases where (4.31) is not valid. For example, if $m = 0$, then the best constant is 3 in dimension 4 and $\frac{25}{36}$ in dimension 3.

4.3.2 The case of power potentials $|x|^m$

The general Theorem 4.3.1 allowed us to deduce inequality (4.36) below for a restricted interval of powers m. We shall now prove that the same holds for all $-\frac{n}{2} \leq m < \frac{n-2}{2}$. We start with the following result.

Assume $-\frac{n}{2} \leq m < \frac{n-2}{2}$ and Ω be a smooth domain in ^{n}, $n \geq 1$. Then

$$a_{n,m} = \inf \left\{ \frac{\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx}{\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx}; \ H^2(\Omega) \setminus \{0\} \right\} =$$

$$\inf \left\{ \frac{\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx}{\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx}; \ u \in H_0^2(\Omega) \setminus \{0\} \right\}$$

Proof. Decomposing again $u \in C^\infty(\bar{B}_R)$ into spherical harmonics; $u = \Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$, one has

$$\begin{aligned}
\int_{n} \frac{|\Delta u_k|^2}{|x|^{2m}} dx &= \int_{n} |x|^{-2m} (f_k''(|x|))^2 dx \\
&+ ((n-1)(2m+1) + 2c_k) \int_{n} |x|^{-2m-2} (f_k')^2 dx \\
&+ c_k(c_k + (n-4-2m)(2m+2)) \int_{n} |x|^{-2m-4} (f_k)^2 dx \\
&+ (n-1)R^{n-2m-2} (f_k'(R))^2,
\end{aligned}$$

(4.34)

and

$$\int_{n} \frac{|\nabla u_k|^2}{|x|^{2m+2}} dx = \int_{n} |x|^{-2m-2}(f_k')^2 dx + c_k \int_{n} |x|^{-2m-4}(f_k)^2 dx. \tag{4.35}$$

The rest of the proof follows from the inequality (4.13) and an argument similar to that of Theorem 6.1 in [15]. □

Remark 4.3.2. *The constant $a_{n,m}$ has been computed explicitly in [15] (Theorem 6.1).*

Suppose $n \geq 1$ and $-\frac{n}{2} \leq m < \frac{n-2}{2}$, and W is a Bessel potential on $B_R \subset R^n$ with $n \geq 3$ and ϕ is the corresponding solution for the $(B_{1,W})$. If

$$R\frac{\phi'(R)}{\phi(R)} \geq -\frac{n}{2} - m,$$

then for all $u \in H^2(B_R)$ we have

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W;R) \int_{B_R} W(x) \frac{|\nabla u|^2}{|x|^{2m}} dx, \tag{4.36}$$

where

$$a_{n,m} = \inf \left\{ \frac{\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx}{\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx}; \, u \in H^2(B_R) \setminus \{0\} \right\}.$$

Moreover $\beta(W;R)$ and $a_{m,n}$ are the best constants. **Proof:** Assuming the inequality

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx,$$

holds for all $u \in C^\infty(\bar{B}_R)$, we shall prove that it can be improved by any Bessel potential W. We will use the following inequality in the proof which follows directly from the inequality (4.13) with n=1.

$$\begin{aligned} \int_0^R r^\alpha (f'(r))^2 dr \quad &\geq \quad (\frac{\alpha-1}{2})^2 \int_0^R r^{\alpha-2} f^2(r) dr \\ &+ \quad \beta(W;R) \int_0^R r^\alpha W(r) f^2(r) dr + (\frac{\phi'(R)}{\phi(R)} - \frac{\alpha-1}{2R}) R^\alpha, \end{aligned} \tag{4.37}$$

for $\alpha \geq 1$ and for all $f \in C^\infty(0,R]$, where both $(\frac{\alpha-1}{2})^2$ and $\beta(W;R)$ are best constants. Decompose $u \in C^\infty(\bar{B}_R)$ into its spherical harmonics $\Sigma_{k=0}^\infty u_k$, where

$u_k = f_k(|x|)\varphi_k(x)$. We evaluate $I_k = \frac{1}{nw_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx$ in the following way

$$\begin{aligned}
I_k &= \int_0^R r^{n-2m-1}(f_k''(r))^2dr + [(n-1)(2m+1)+2c_k]\int_0^R r^{n-2m-3}(f_k')^2dr \\
&\quad +c_k[c_k+(n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2dr \\
&+ (n-1)R^{n-2m-2}(f_k'(R))^2 \\
&\geq \beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2dr + [(\frac{n+2m}{2})^2+2c_k]\int_0^R r^{n-2m-3}(f_k')^2dr \\
&\quad +c_k[c_k+(n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2dr \\
&\geq \beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2dr + a_{n,m}\int_0^R r^{n-2m-3}(f_k')^2dr \\
&\quad +\beta(W)[(\frac{n+2m}{2})^2+2c_k-a_{n,m}]\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
&\quad +((\frac{n-2m-4}{2})^2[(\frac{n+2m}{2})^2+2c_k-a_{n,m}] \\
&+ c_k[c_k+(n-2m-4)(2m+2)])\int_0^R r^{n-2m-5}(f_k(r))^2dr.
\end{aligned}$$

Now by (115) in [15] we have

$$((\frac{n-2m-4}{2})^2[(\frac{n+2m}{2})^2+2c_k-a_{n,m}]+c_k[c_k+(n-2m-4)(2m+2)] \geq c_k a_{n,m},$$

for all $k \geq 0$. Hence, we have

$$\begin{aligned}
I_k &\geq a_{n,m}\int_0^R r^{n-2m-3}(f_k')^2dr + a_{n,m}c_k\int_0^R r^{n-2m-5}(f_k(r))^2dr \\
&\quad +\beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2dr \\
&\quad +\beta(W)[(\frac{n+2m}{2})^2+2c_k-a_{n,m}]\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
&\geq a_{n,m}\int_0^R r^{n-2m-3}(f_k')^2dr + a_{n,m}c_k\int_0^R r^{n-2m-5}(f_k(r))^2dr \\
&\quad +\beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2dr + \beta(W)c_k\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
&= a_{n,m}\int_{B_R}\frac{|\nabla u|^2}{|x|^{2m+2}}dx + \beta(W)\int_{B_R}W(x)\frac{|\nabla u|^2}{|x|^{2m}}dx.
\end{aligned}$$

□

In the following theorem we prove a very general class of weighted Hardy-Rellich inequalities on $H^2(\Omega)\cap H_0^1$.

Let Ω be a smooth domain in R^n with $n \geq 1$ and let $V \in C^2(0, R =: \sup_{x\in\Omega}|x|)$ be a non-negative function that satisfies the following conditions:

$$V_r(r) \leq 0 \quad \text{and} \quad \int_0^R \tfrac{1}{r^{n-3}V(r)}dr = -\int_0^R \tfrac{1}{r^{n-4}V_r(r)}dr = +\infty. \tag{4.38}$$

There exists $\lambda_1, \lambda_2 \in R$ such that

$$\tfrac{rV_r(r)}{V(r)} + \lambda_1 \geq 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0} \tfrac{rV_r(r)}{V(r)} + \lambda_1 = 0, \tag{4.39}$$

$$\tfrac{rV_{rr}(r)}{V_r(r)} + \lambda_2 \geq 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0} \tfrac{rV_{rr}(r)}{V_r(r)} + \lambda_2 = 0, \tag{4.40}$$

and

$$\left(\tfrac{1}{2}(n-\lambda_1-2)^2 + 3(n-3)\right)V(r) - (n-5)rV_r(r) - r^2V_{rr}(r) \geq 0 \text{ for all } r \in (0,R). \tag{4.41}$$

If $\lambda_1 \leq n$, then the following inequality holds:

$$\begin{aligned}\int_\Omega V(|x|)|\Delta u|^2dx \quad &\geq \quad \Big(\frac{(n-\lambda_1-2)^2}{4} + (n-1)\Big)\frac{(n-\lambda_1-4)^2}{4}\int_\Omega \frac{V(|x|)}{|x|^4}u^2dx \\ &\quad - \frac{(n-1)(n-\lambda_2-2)^2}{4}\int_\Omega \frac{V_r(|x|)}{|x|^3}u^2dx. \end{aligned}\tag{4.42}$$

Proof: We have by Theorem 4.2 and condition (4.41),

$$\begin{aligned}
&\frac{1}{n\omega_n}\int_{R^n} V(x)|\Delta u_k|^2dx = \int_0^R V(r)(f_k''(r))^2r^{n-1}dr \\
+ \quad & (n-1+2c_k)\int_0^R V(r)(f_k'(r))^2r^{n-3}dr \\
+ \quad & (2c_k(n-4)+c_k^2)\int_0^R V(r)r^{n-5}f_k^2(r)dr - (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr \\
- \quad & c_k(n-5)\int_0^R V_r(r)f_k^2(r)r^{n-4}dr - c_k\int_0^R V_{rr}(r)f_k^2(r)r^{n-3}dr \\
+ \quad & (n-1)V(R)(f_k'(R))^2R^{n-2} \\
\geq \quad & \int_0^R V(r)(f_k''(r))^2r^{n-1}dr + (n-1)\int_0^R V(r)(f_k'(r))^2r^{n-3}dr \\
- \quad & (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr \\
+ \quad & c_k\int_0^R\left(\left(\frac{1}{2}(n-\lambda_1-2)^2+3(n-3)\right)V(r)-(n-5)rV_r(r)-r^2V_{rr}(r)\right)f_k^2(r)r^{n-5}dr \\
+ \quad & (n-1)V(R)(f_k'(R))^2R^{n-2}
\end{aligned}$$

The rest of the proof follows from the above inequality combined with Theorem 4.2. □

Remark 4.3.3. Let $V(r) = r^{-2m}$ with $-\frac{n}{2} \leq m \leq \frac{n-4}{2}$. Then in order to satisfy condition (4.41) we must have $-1-\frac{\sqrt{1+(n-1)^2}}{2} \leq m \leq \frac{n-4}{2}$. Since $-1-\frac{\sqrt{1+(n-1)^2}}{2} \leq -\frac{n}{2}$, if $-\frac{n}{2} \leq m \leq \frac{n-4}{2}$ the inequality (4.42) gives the following weighted second order Rellich inequality:

$$\int_B \frac{|\Delta u|^2}{|x|^{2m}} dx \geq H_{n,m} \int_B \frac{u^2}{|x|^{2m+4}} dx \quad u \in H^2(\Omega) \cap H_0^1(\Omega),$$

where

$$H_{n,m} := (\frac{(n+2m)(n-4-2m)}{4})^2. \tag{4.43}$$

The following theorem includes a large class of improved Hardy-Rellich inequalities as special cases.

Let $-\frac{n}{2} \leq m \leq \frac{n-4}{2}$ and let $W(x)$ be a Bessel potential on a ball B of radius R in R^n with radius R. Assume $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. If $\lambda \leq \frac{n}{2} + m$, then the following inequality holds for all $u \in H^2 \cap H_0^1(B)$

$$\begin{aligned} \int_B \frac{|\Delta u|^2}{|x|^{2m}} dx \quad &\geq \quad H_{n,m} \int_B \frac{u^2}{|x|^{2m+4}} dx \\ &+ \quad \beta(W;R)(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4}) \int_B \frac{W(x)}{|x|^{2m+2}} u^2 dx. \end{aligned} \tag{4.44}$$

Moreover, both constants are the best constants.

Proof: Again we will frequently use inequality (4.37) in the proof. Decomposing $u \in C^\infty(\bar{B}_R)$ into spherical harmonics $\Sigma_{k=0}^{\infty} u_k$, where $u_k = f_k(|x|)\varphi_k(x)$, we can write

$$\begin{aligned} \frac{1}{n\omega_n} \int_{R^n} \frac{|\Delta u_k|^2}{|x|^{2m}} dx \quad &= \quad \int_0^R r^{n-2m-1} (f_k''(r))^2 dr \\ &+ \quad [(n-1)(2m+1) + 2c_k] \int_0^R r^{n-2m-3} (f_k')^2 dr \\ &+ \quad c_k[c_k + (n-2m-4)(2m+2)] \int_0^R r^{n-2m-5} (f_k(r))^2 dr \\ &+ \quad (n-1)(f_k'(R))^2 R^{n-2m-2} \\ &\geq \quad (\frac{n+2m}{2})^2 \int_0^R r^{n-2m-3} (f_k')^2 dr \\ &+ \quad \beta(W;R) \int_0^R r^{n-2m-1} W(x) (f_k')^2 dr \\ &+ \quad c_k[c_k + 2(\frac{n-\lambda-4}{2})^2 \\ &+ \quad (n-2m-4)(2m+2)] \int_0^R r^{n-2m-5} (f_k(r))^2 dr \\ &+ \quad (n-1)(f_k'(R))^2 R^{n-2m-2}, \end{aligned}$$

where we have used the fact that $c_k \geq 0$ to get the above inequality. We have

$$\begin{aligned}
&\frac{1}{n\omega_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx \geq \beta_{n,m}\int_0^R r^{n-2m-5}(f_k)^2dr \\
&+\beta(W;R)\frac{(n+2m)^2}{4}\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
&+\beta(W;R)\int_0^R r^{n-2m-1}W(x)(f_k')^2dr \\
\geq\ & \beta_{n,m}\int_0^R r^{n-2m-5}(f_k)^2dr \\
&+\beta(W;R)(\frac{(n+2m)^2}{4}+\frac{(n-2m-\lambda-2)^2}{4})\int_0^R r^{n-2m-3}W(x)(f_k)^2dr \\
\geq\ & \frac{\beta_{n,m}}{n\omega_n}\int_B\frac{u_k^2}{|x|^{2m+4}}dx \\
&+\frac{\beta(W;R)}{n\omega_n}(\frac{(n+2m)^2}{4}+\frac{(n-2m-\lambda-2)^2}{4})\int_B\frac{W(x)}{|x|^{2m+2}}u_k^2dx,
\end{aligned}$$

by Theorem 4.2. Hence, (4.44) holds and the proof is complete. □

We shall now give a few immediate applications of the above in the case where $m = 0$ and $n \geq 3$.

Assume W is a Bessel potential on $B_R \subset R^n$ with $n \geq 3$ and ϕ is the corresponding solution for the $(B_{1,W})$. If

$$R\frac{\phi'(R)}{\phi(R)} \geq -\frac{n}{2},$$

then for all $u \in H^2(B_R)$ we have

$$\int_{B_R}|\Delta u|^2dx \geq C(n)\int_{B_R}\frac{|\nabla u|^2}{|x|^2}dx + \beta(W;R)\int_{B_R}W(x)|\nabla u|^2dx, \tag{4.45}$$

where $C(3) = \frac{25}{36}$, $C(4) = 3$ and $C(n) = \frac{n^2}{4}$ for all $n \geq 5$. Moreover, $C(n)$ and $\beta(W;R)$ are best constants.

The following holds for any smooth bounded domain Ω in R^n with $R = \sup_{x\in\Omega}|x|$, and any $u \in H^2(\Omega)$.

1. Let z_0 be the first zero of the Bessel function $J_0(z)$ and choose $0 < \mu < z_0$ so that

$$\mu\frac{J_0'(\mu)}{J_0(\mu)} = -\frac{n}{2}. \tag{4.46}$$

   Then

$$\textstyle\int_\Omega|\Delta u|^2dx \geq C(n)\int_\Omega\frac{|\nabla u|^2}{|x|^2}dx + \frac{\mu^2}{R^2}\int_\Omega|\nabla u|^2dx \tag{4.47}$$

2. For any $k \geq 1$, choose $\rho \geq R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$ large enough so that $R\frac{\phi'(R)}{\phi(R)} \geq -\frac{n}{2}$, where

$$\phi = (\prod_{i=1}^{j} log^{(i)} \frac{\rho}{|x|})^{\frac{1}{2}}. \tag{4.48}$$

Then we have

$$\begin{aligned} \int_\Omega |\Delta u(x)|^2 dx &\geq C(n) \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \qquad (4.49) \\ &+ \frac{1}{4} \sum_{j=1}^{k} \int_\Omega \frac{|\nabla u|^2}{|x|^2} (\prod_{i=1}^{j} log^{(i)} \frac{\rho}{|x|})^{-2} dx, \end{aligned}$$

3. We have

$$\begin{aligned} \int_\Omega |\Delta u(x)|^2 dx &\geq C(n) \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \qquad (4.50) \\ &+ \frac{1}{4} \sum_{i=1}^{n} \int_\Omega \frac{|\nabla u|}{|x|^2} X_1^2(\frac{|x|}{R}) X_2^2(\frac{|x|}{R}) \ldots X_i^2(\frac{|x|}{R}) dx. \end{aligned}$$

The following is immediate from Theorem 4.3.2 and from the fact that $\lambda = 2$ for the Bessel potential under consideration.

Let Ω be a smooth bounded domain in ^{n}, $n \geq 4$ and $R = \sup_{x \in \Omega} |x|$. Then the following holds for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$

1. Choose $\rho \geq R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$ so that $R\frac{\phi'(R)}{\phi(R)} \geq -\frac{n}{2}$. Then

$$\begin{aligned} \int_\Omega |\Delta u(x)|^2 dx &\geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx \qquad (4.51) \\ &+ (1 + \frac{n(n-4)}{8}) \sum_{j=1}^{k} \int_\Omega \frac{u^2}{|x|^4} (\prod_{i=1}^{j} log^{(i)} \frac{\rho}{|x|})^{-2} dx. \end{aligned}$$

2. Let X_i is defined as in the introduction, then

$$\begin{aligned} \int_\Omega |\Delta u(x)|^2 dx &\geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx \qquad (4.52) \\ &+ (1 + \frac{n(n-4)}{8}) \sum_{i=1}^{n} \int_\Omega \frac{u^2}{|x|^4} X_1^2(\frac{|x|}{R}) X_2^2(\frac{|x|}{R}) \ldots X_i^2(\frac{|x|}{R}) dx. \end{aligned}$$

Moreover, all constants in the above inequalities are best constants.

Let $W_1(x)$ and $W_2(x)$ be two radial Bessel potentials on a ball B of radius R in R^n with $n \geq 4$. Then for all $u \in H^2(B) \cap H_0^1(B)$

$$\begin{aligned}\int_B |\Delta u|^2 dx \quad &\geq \quad \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx + \frac{n^2}{4}\beta(W_1;R)\int_B W_1(x)\frac{u^2}{|x|^2}dx \\ &+\mu(\frac{n-2}{2})^2\int_B \frac{u^2}{|x|^2}dx + \mu\beta(W_2;R)\int_B W_2(x)u^2 dx,\end{aligned}$$

where μ is defined by (4.46). **Proof:** Here again we shall give a proof when $n \geq 5$. The case $n = 4$ will be handled in the next section. We again first use Theorem 4.3.2 (for $n \geq 5$), then Theorem 2.15 in [15] with the Bessel pair $(|x|^{-2}, |x|^{-2}(\frac{(n-4)^2}{4}|x|^{-2}+W))$, then again Theorem 4.2 with the Bessel pair $(1, (\frac{n-2}{2})^2|x|^{-2}+ W)$ to obtain

$$\begin{aligned}\int_B |\Delta u|^2 dx \quad &\geq \quad \frac{n^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2}dx + \mu\int_B |\nabla u|^2 dx \\ &\geq \quad \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx + \frac{n^2}{4}\beta(W_1;R)\int_B W_1(x)\frac{u^2}{|x|^2}dx \\ &+\mu\int_B |\nabla u|^2 dxdx \\ &\geq \quad \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx + \frac{n^2}{4}\beta(W_1;R)\int_B W_1(x)\frac{u^2}{|x|^2}dx \\ &+\mu(\frac{n-2}{2})^2\int_B \frac{u^2}{|x|^2}dx + \mu\beta(W_2;R)\int_B W_2(x)u^2 dx.\end{aligned}$$

Assume $n \geq 4$ and let $W(x)$ be a Bessel potential on a ball B of radius R and centered at zero in R^n. Then the following holds for all $u \in H^2(B) \cap H_0^1(B)$:

$$\int_B |\Delta u|^2 dx \quad \geq \quad \frac{n^2(n-4)^2}{16}\int_B \frac{u^2}{|x|^4}dx \tag{4.53}$$

$$+\beta(W;R)\frac{n^2}{4}\int_B \frac{W(x)}{|x|^2}u^2 dx + \frac{\mu^2}{R^2}||u||_{H_0^1}, \tag{4.54}$$

where $\frac{\mu^2}{R^2}$ is defined by (4.46). **Proof:** Decomposing again $u \in C^\infty(\bar{B}_R)$ into its spherical harmonics $\Sigma_{k=0}^{\infty} u_k$ where $u_k = f_k(|x|)\varphi_k(x)$, we calculate

$$\begin{aligned}
\frac{1}{n\omega_n}\int_{R^n}|\Delta u_k|^2 dx &= \int_0^R r^{n-1}(f_k''(r))^2 dr + [n-1+2c_k]\int_0^R r^{n-3}(f_k')^2 dr \\
&+ c_k[c_k+n-4]\int_0^R r^{n-5}(f_k(r))^2 dr \\
&+ (n-1)(f_k'(R))^2 R^{n-2m-2} \\
&\geq \frac{n^2}{4}\int_0^R r^{n-3}(f_k')^2 dr + \frac{\mu^2}{R^2}\int_0^R r^{n-1}(f_k')^2 dr \\
&+ c_k\int_0^R r^{n-3}(f_k')^2 dr \\
&\geq \frac{n^2(n-4)^2}{16}\int_0^R r^{n-5}(f_k)^2 dr \\
&\quad +\beta(W;R)\frac{n^2}{4}\int_0^R W(r)r^{n-3}(f_k)^2 dr \\
&+ \frac{\mu^2}{R^2}\int_0^R r^{n-1}(f_k')^2 dr + c_k\frac{\mu^2}{R^2}\int_0^R r^{n-3}(f_k)^2 dr \\
&= \frac{n^2(n-4)^2}{16n\omega_n}\int_{R^n}\frac{u_k^2}{|x|^{2m+4}}dx \\
&+ \frac{\beta(W;R)}{n\omega_n}(\frac{n^2}{4})\int_{R^n}\frac{W(x)}{|x|^2}u_k^2 dx + \frac{\mu^2}{n\omega_n R^2}||u_k||_{W_0^{1,2}}.
\end{aligned}$$

Hence (4.53) holds. □

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Part II

Fourth Order Nonlinear Eigenvalue Problems

Chapter 5

The critical dimension for a fourth order elliptic problem with singular nonlinearity [4]

5.1 Introduction

The following model has been proposed for the description of the steady-state of a simple Electrostatic MEMS device:

$$\begin{cases} \alpha\Delta^2 u = \left(\beta\int_\Omega |\nabla u|^2 dx + \gamma\right)\Delta u + \frac{\lambda f(x)}{(1-u)^2\left(1+\chi\int_\Omega \frac{dx}{(1-u)^2}\right)} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = \alpha\partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\alpha, \beta, \gamma, \chi \geq 0$, $f \in C(\overline{\Omega}, [0,1])$ are fixed, Ω is a bounded domain in $\mathbb{R}^N$ and $\lambda \geq 0$ is a varying parameter (see for example Bernstein and Pelesko [19]). The function $u(x)$ denotes the height above a point $x \in \Omega \subset \mathbb{R}^N$ of a dielectric membrane clamped on $\partial\Omega$, once it deflects torwards a ground plate fixed at height $z = 1$, whenever a positive voltage – proportional to λ – is applied.

In studying this problem, one typically makes various simplifying assumptions on the parameters $\alpha, \beta, \gamma, \chi$, and the first approximation of (5.1) that has been studied extensively so far is the equation

$$\begin{cases} -\Delta u = \lambda \frac{f(x)}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (S)_{\lambda,f}$$

where we have set $\alpha = \beta = \chi = 0$ and $\gamma = 1$ (see for example [6], [8], [9], and the monograph [7]) . This simple model, which lends itself to the vast literature on second order semilinear eigenvalue problems, is already a rich source of interesting mathematical problems. The case when the "permittivity profile" f is constant ($f = 1$) on a general domain was studied in [16], following the pioneering work of

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Joseph and Lundgren [13] who had considered the radially symmetric case. The case for a non constant permittivity profile f was advocated by Pelesko [18], taken up by [11], and studied in depth in [6],[8], [9]. The starting point of the analysis is the existence of a pull-in voltage $\lambda^*(\Omega, f)$, defined as

$$\lambda^*(\Omega, f) := \sup\Big\{\lambda > 0 : \text{there exists a classical solution of } (S)_{\lambda,f}\Big\}.$$

It is then shown that for every $0 < \lambda < \lambda^*$, there exists a smooth minimal (smallest) solution of $(S)_{\lambda,f}$, while for $\lambda > \lambda^*$ there is no solution even in a weak sense. Moreover, the branch $\lambda \mapsto u_\lambda(x)$ is increasing for each $x \in \Omega$, and therefore the function $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ can be considered as a generalized solution that corresponds to the pull-in voltage λ^*. Now the issue of the regularity of this extremal solution – which, by elliptic regularity theory, is equivalent to whether $\sup_\Omega u^* < 1$ – is an important question for many reasons, not the least of which being the fact that it decides whether the set of solutions stops there, or whether a new branch of solutions emanates from a bifurcation state (u^*, λ^*). This issue turned out to depend closely on the dimension and on the permittivity profile f. Indeed, it was shown in [9] that u^* is regular in dimensions $1 \le N \le 7$, while it is not necessarily the case for $N \ge 8$. In other words, the dimension $N = 7$ is critical for equation $(S)_\lambda$ (when $f = 1$, we simplify the notation $(S)_{\lambda,1}$ into $(S)_\lambda$). On the other hand, it is shown in [8] that the regularity of u^* can be restored in any dimension, provided we allow for a power law profile $|x|^\eta$ with η large enough.

The case where $\beta = \gamma = \chi = 0$ (and $\alpha = 1$) in the above model, that is when we are dealing with the following fourth order analog of $(S)_\lambda$

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)_\lambda$$

was also considered by [4] and [14] but with limited success. One of the reasons is the lack of a "maximum principle" which plays such a crucial role in developing the theory for the Laplacian. Indeed, it is a well known fact that such a principle does not normally hold for general domains Ω (at least for the clamped boundary conditions $u = \partial_\nu u = 0$ on $\partial\Omega$) unless one restricts attention to the unit ball $\Omega = B$ in $\mathbb{R}^N$, where one can exploit a positivity preserving property of Δ^2 due to T. Boggio [3]. This is precisely what was done in the references mentioned above, where a theory of the minimal branch associated with $(P)_\lambda$ is developed along the same lines as for $(S)_\lambda$. The second obstacle is the well-known difficulty of extracting energy estimates for solutions of fourth order problems from their stability properties. This means that the methods used to analyze the regularity of the extremal solution for $(S)_\lambda$ could not carry to the corresponding problem for $(P)_\lambda$.

This is the question we address in this paper as we eventually show the following result.

Theorem 5.2. *The unique extremal solution u^* for $(P)_{\lambda^*}$ in B is regular in dimension $1 \le N \le 8$, while it is singular (i.e, $\sup_B u^* = 1$) for $N \ge 9$.*

In other words, the critical dimension for $(P)_\lambda$ in B is $N = 8$, as opposed to being equal to 7 in $(S)_\lambda$. We add that our methods are heavily inspired by the recent paper of Davila et al. [5] where it is shown that $N = 12$ is the critical dimension for the fourth order nonlinear eigenvalue problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B \\ u = \partial_\nu u = 0 & \text{on } \partial B, \end{cases}$$

while the critical dimension for its second order counterpart (i.e., the Gelfand problem) is $N = 9$. There is, however, one major difference between our approach and the one used by Davila et al. [5]. It is related to the most delicate dimensions – just above the critical one – where they use a computer assisted proof to establish the singularity of the extremal solution, while our method is more analytical and relies on improved and non standard Hardy-Rellich inequalities recently established by Ghoussoub-Moradifam [10] (See the lase section).

Throughout this paper, we will always consider problem $(P)_\lambda$ on the unit ball B. We start by recalling some of the results from [4] concerning $(P)_\lambda$, that will be needed in the sequel. We define

$$\lambda^* := \sup\Big\{\lambda > 0 : \text{ there exists a classical solution of } (P)_\lambda\Big\},$$

and note that we are not restricting our attention to radial solutions. We will deal also with weak solutions:

Definition 5.3. *We say that u is a weak solution of $(P)_\lambda$ if $0 \le u \le 1$ a.e. in B, $\frac{1}{(1-u)^2} \in L^1(B)$ and*

$$\int_B u\Delta^2\phi = \lambda \int_B \frac{\phi}{(1-u)^2}, \qquad \forall \phi \in C^4(\bar{B}) \cap H_0^2(B).$$

We say that u is a weak super-solution (resp. weak sub-solution) of $(P)_\lambda$ if the equality is replaced with the inequality $\ge$ (resp. $\le$) for all $\phi \in C^4(\bar{B}) \cap H_0^2(B)$ with $\phi \ge 0$.

We also introduce notions of regularity and stability.

Definition 5.4. *Say that a weak solution u of $(P)_\lambda$ is regular (resp. singular) if $\|u\|_\infty < 1$ (resp. $=$) and stable (resp. semi-stable) if*

$$\mu_1(u) = \inf\left\{\int_B (\Delta\phi)^2 - 2\lambda \int_B \frac{\phi^2}{(1-u)^3} : \phi \in H_0^2(B), \|\phi\|_{L^2} = 1\right\}$$

is positive (resp. non-negative).

The following extension of Boggio's principle will be frequently used in the sequel (see [2, Lemma 16] and [5, Lemma 2.4]):

Lemma 5.5 (Boggio's Principle)**.** *Let $u \in L^1(B)$. Then $u \geq 0$ a.e. in B, provided one of the following conditions hold:*

1. $u \in C^4(\overline{B})$, $\Delta^2 u \geq 0$ *on* B, *and* $u = \frac{\partial u}{\partial n} = 0$ *on* ∂B.
2. $\int_B u \Delta^2 \phi \, dx \geq 0$ *for all* $0 \leq \phi \in C^4(\overline{B}) \cap H_0^2(B)$.
3. $u \in H^2(B)$, $u = 0$ *and* $\frac{\partial u}{\partial n} \leq 0$ *on* ∂B, *and* $\int_B \Delta u \Delta \phi \geq 0$ *for all* $0 \leq \phi \in H_0^2(B)$.

Moreover, either $u \equiv 0$ or $u > 0$ a.e. in B.

The following theorem summarizes the main results in [4] that will be needed in the sequel:

Theorem 5.6. *The following assertions hold:*

1. *For each $0 < \lambda < \lambda^*$ there exists a classical minimal solution u_λ of $(P)_\lambda$. Moreover u_λ is radial and radially decreasing.*
2. *For $\lambda > \lambda^*$, there are no weak solutions of $(P)_\lambda$.*
3. *For each $x \in B$ the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$.*
4. *The pull-in voltage λ^* satisfies the following bounds:*

$$\max\left\{\frac{32(10N - N^2 - 12)}{27}, \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3})\right\} \leq \lambda^* \leq \frac{4\nu_1}{27}$$

where ν_1 denotes the first eigenvalue of Δ^2 in $H_0^2(B)$.

5. *For each $0 < \lambda < \lambda^*$, u_λ is a stable solution (i.e., $\mu_1(u_\lambda) > 0$).*

Using the stability of u_λ, it can be shown that u_λ is uniformly bounded in $H_0^2(B)$ and that $\frac{1}{1-u_\lambda}$ is uniformly bounded in $L^3(B)$. Since now $\lambda \mapsto u_\lambda(x)$ is increasing, the function $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ is well defined (in the pointwise sense), $u^* \in H_0^2(B)$, $\frac{1}{1-u^*} \in L^3(B)$ and u^* is a weak solution of $(P)_{\lambda^*}$. Moreover u^* is the unique weak solution of $(P)_{\lambda^*}$.

The second result we list from [4] is critical in identifying the extremal solution.

Theorem 5.7. *If $u \in H_0^2(B)$ is a singular weak solution of $(P)_\lambda$, then u is semi-stable if and only if $(u, \lambda) = (u^*, \lambda^*)$.*

5.2 The effect of boundary conditions on the pull-in voltage

As in [5], we are led to examine problem $(P)_\lambda$ with non-homogeneous boundary conditions such as

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } B \\ \alpha < u < 1 & \text{in } B \\ u = \alpha\,,\ \partial_\nu u = \beta & \text{on } \partial B, \end{cases} \qquad (P)_{\lambda,\alpha,\beta}$$

where α, β are given.

Notice first that some restrictions on α and β are necessary. Indeed, letting $\Phi(x) := (\alpha - \frac{\beta}{2}) + \frac{\beta}{2}|x|^2$ denote the unique solution of

$$\begin{cases} \Delta^2 \Phi = 0 & \text{in } B \\ \Phi = \alpha\,,\ \partial_\nu \Phi = \beta & \text{on } \partial B, \end{cases} \tag{5.8}$$

we infer immediately from Lemma 5.5 that the function $u - \Phi$ is positive in B, which yields to

$$\sup_B \Phi < \sup_B u \leq 1.$$

To insure that Φ is a classical sub-solution of $(P)_{\lambda,\alpha,\beta}$, we impose $\alpha \neq 1$ and $\beta \leq 0$, and condition $\sup_B \Phi < 1$ rewrites as $\alpha - \frac{\beta}{2} < 1$. We will then say that the pair (α, β) is *admissible* if $\beta \leq 0$, and $\alpha - \frac{\beta}{2} < 1$.

This section will be devoted to obtaining results for $(P)_{\lambda,\alpha,\beta}$ when (α, β) is an admissible pair, which are analogous to those for $(P)_\lambda$. To cut down on notation, we shall sometimes drop α and β from our expressions whenever such an emphasis is not needed. For example in this section u_λ and u^* will denote the minimal and extremal solution of $(P)_{\lambda,\alpha,\beta}$.

We now introduce a notion of weak solution for $(P)_{\lambda,\alpha,\beta}$.

Definition 5.9. *We say that u is a weak solution of $(P)_{\lambda,\alpha,\beta}$ if $\alpha \leq u \leq 1$ a.e. in B, $\frac{1}{(1-u)^2} \in L^1(B)$ and if*

$$\int_B (u - \Phi)\Delta^2 \phi = \lambda \int_B \frac{\phi}{(1-u)^2}, \qquad \forall \phi \in C^4(\bar{B}) \cap H_0^2(B),$$

where Φ is given in (5.8). We say that u is a weak super-solution (resp. weak sub-solution) of $(P)_{\lambda,\alpha,\beta}$ if the equality is replaced with the inequality $\geq$ (resp. $\leq$) for $\phi \geq 0$.

We now define as before

$$\lambda^* := \sup\{\lambda > 0 : (P)_{\lambda,\alpha,\beta} \text{ has a classical solution}\}$$

and

$$\lambda_* := \sup\{\lambda > 0 : (P)_{\lambda,\alpha,\beta} \text{ has a weak solution}\}.$$

Observe that by the Implicit Function Theorem, one can always solve $(P)_{\lambda,\alpha,\beta}$ for small λ's. Therefore, λ^* (and also λ_*) is well defined.

Let now U be a weak super-solution of $(P)_{\lambda,\alpha,\beta}$. Recall the following standard existence result.

Theorem 5.10 ([2]). *For every $0 \le f \in L^1(B)$, there exists a unique $0 \le u \in L^1(B)$ which satisfies*

$$\int_B u\Delta^2\phi = \int_B f\phi$$

for all $\phi \in C^4(\bar{B}) \cap H_0^2(B)$.

We can now introduce the following "weak iterative scheme": Start with $u_0 = U$ and (inductively) let u_n, $n \ge 1$, be the solution of

$$\int_B (u_n - \Phi)\Delta^2\phi = \lambda \int_B \frac{\phi}{(1-u_{n-1})^2} \qquad \forall\, \phi \in C^4(\bar{B}) \cap H_0^2(B)$$

given by Theorem 5.10. Since 0 is a sub-solution of $(P)_{\lambda,\alpha,\beta}$, one can easily show inductively by using Lemma 5.5 that $\alpha \le u_{n+1} \le u_n \le U$ for every $n \ge 0$. Since

$$(1-u_n)^{-2} \le (1-U)^{-2} \in L^1(B),$$

we get by Lebesgue Theorem, that the function $u = \lim_{n\to+\infty} u_n$ is a weak solution of $(P)_{\lambda,\alpha,\beta}$ such that $\alpha \le u \le U$. In other words, the following result holds.

Proposition 5.2.1. *Assume the existence of a weak super-solution U of $(P)_{\lambda,\alpha,\beta}$. Then there exists a weak solution u of $(P)_{\lambda,\alpha,\beta}$ so that $\alpha \le u \le U$ a.e. in B.*

In particular, we can find a weak solution of $(P)_{\lambda,\alpha,\beta}$ for every $\lambda \in (0, \lambda_*)$. Now we show that this is still true for regular weak solutions.

Proposition 5.2.2. *Let (α,β) be an admissible pair and let u be a weak solution of $(P)_{\lambda,\alpha,\beta}$. Then for every $0 < \mu < \lambda$, there is a regular solution for $(P)_{\mu,\alpha,\beta}$.*

Proof. Let $\varepsilon \in (0,1)$ be given and let $\bar{u} = (1-\varepsilon)u + \varepsilon\Phi$, where Φ is given in (5.8). We have that

$$\sup_B \bar{u} \le (1-\varepsilon) + \varepsilon \sup_B \Phi < 1\,, \quad \inf_B \bar{u} \ge (1-\varepsilon)\alpha + \varepsilon \inf_B \Phi = \alpha,$$

and for every $0 \le \phi \in C^4(\bar{B}) \cap H_0^2(B)$ there holds:

$$\begin{aligned} \int_B (\bar{u}-\Phi)\Delta^2\phi &= (1-\varepsilon)\int_B (u-\Phi)\Delta^2\phi = (1-\varepsilon)\lambda \int_B \frac{\phi}{(1-u)^2} \\ &= (1-\varepsilon)^3\lambda \int_B \frac{\phi}{(1-\bar{u}+\varepsilon(\Phi-1))^2} \ge (1-\varepsilon)^3\lambda \int_B \frac{\phi}{(1-\bar{u})^2}. \end{aligned}$$

Note that $0 \le (1-\varepsilon)(1-u) = 1-\bar{u}+\varepsilon(\Phi-1) < 1-\bar{u}$. So $\bar{u}$ is a weak super-solution of $(P)_{(1-\varepsilon)^3\lambda,\alpha,\beta}$ satisfying $\sup_B \bar{u} < 1$. From Proposition 5.2.1 we get the existence of a weak solution w of $(P)_{(1-\varepsilon)^3\lambda,\alpha,\beta}$ so that $\alpha \le w \le \bar{u}$. In particular, $\sup_B w < 1$ and w is a regular weak solution. Since $\varepsilon \in (0,1)$ is arbitrarily chosen, the proof is complete. $\square$

Proposition 5.2.2 implies in particular the existence of a regular weak solution U_λ for every $\lambda \in (0, \lambda_*)$. Introduce now a "classical" iterative scheme: $u_0 = 0$ and (inductively) $u_n = v_n + \Phi$, $n \ge 1$, where $v_n \in H_0^2(B)$ is the (radial) solution of

$$\Delta^2 v_n = \Delta^2(u_n - \Phi) = \frac{\lambda}{(1-u_{n-1})^2} \qquad \text{in } B. \tag{5.11}$$

Since $v_n \in H_0^2(B)$, u_n is also a weak solution of (5.11), and by Lemma 5.5 we know that $\alpha \le u_n \le u_{n+1} \le U_\lambda$ for every $n \ge 0$. Since $\sup_B u_n \le \sup_B U_\lambda < 1$ for $n \ge 0$, we get that $(1-u_{n-1})^{-2} \in L^2(B)$ and the existence of v_n is guaranteed. Since v_n is easily seen to be uniformly bounded in $H_0^2(B)$, we have that $u_\lambda := \lim_{n\to+\infty} u_n$ does hold pointwise and weakly in $H^2(B)$. By Lebesgue Theorem, we have that u_λ is a radial weak solution of $(P)_{\lambda,\alpha,\beta}$ so that $\sup_B u_\lambda \le \sup_B U_\lambda < 1$. By elliptic regularity theory [1] $u_\lambda \in C^\infty(\bar{B})$ and $u_\lambda - \Phi = \partial_\nu(u_\lambda - \Phi) = 0$ on ∂B. So we can integrate by parts to get

$$\int_B \Delta^2 u_\lambda \phi = \int_B \Delta^2(u_\lambda - \Phi)\phi = \int_B (u_\lambda - \Phi)\Delta^2\phi = \lambda \int_B \frac{\phi}{(1-u_\lambda)^2}$$

for every $\phi \in C^4(\bar{B}) \cap H_0^2(B)$. Hence, u_λ is a radial classical solution of $(P)_{\lambda,\alpha,\beta}$ showing that $\lambda^* = \lambda_*$. Moreover, since Φ and $v_\lambda := u_\lambda - \Phi$ are radially decreasing in view of [20], we get that u_λ is radially decreasing too. Since the argument above shows that $u_\lambda < U$ for any other classical solution U of $(P)_{\mu,\alpha,\beta}$ with $\mu \ge \lambda$, we have that u_λ is exactly the minimal solution and u_λ is strictly increasing as $\lambda \uparrow \lambda^*$. In particular, we can define u^* in the usual way: $u^*(x) = \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$.

Finally, we show the finiteness of the pull-in voltage.

Lemma 5.12. *If (α, β) is an admissible pair, then $\lambda^*(\alpha,\beta) < +\infty$.*

Proof. Let u be a classical solution of $(P)_{\lambda,\alpha,\beta}$ and let (ψ, ν_1) denote the first eigenpair of Δ^2 in $H_0^2(B)$ with $\psi > 0$. Now, let C be such that

$$\int_{\partial B} (\beta\Delta\psi - \alpha\partial_\nu\Delta\psi) = C \int_B \psi.$$

Multiplying $(P)_{\lambda,\alpha,\beta}$ by ψ and then integrating by parts one arrives at

$$\int_B \left(\frac{\lambda}{(1-u)^2} - \nu_1 u - C \right) \psi = 0.$$

Since $\psi > 0$ there must exist a point $\bar{x} \in B$ where $\frac{\lambda}{(1-u(\bar{x}))^2} - \nu_1 u(\bar{x}) - C \leq 0$. Since $\alpha < u(\bar{x}) < 1$, one can conclude that $\lambda \leq \sup_{\alpha<u<1}(\nu_1 u + C)(1-u)^2$, which shows that $\lambda^* < +\infty$. □

The following summarizes what we have shown so far.

Theorem 5.13. *If (α, β) is an admissible pair, then $\lambda^* := \lambda^*(\alpha, \beta) \in (0, +\infty)$ and the following hold:*

1. *For each $0 < \lambda < \lambda^*$ there exists a classical, minimal solution u_λ of $(P)_{\lambda,\alpha,\beta}$. Moreover u_λ is radial and radially decreasing.*
2. *For each $x \in B$ the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$.*
3. *For $\lambda > \lambda^*$ there are no weak solutions of $(P)_{\lambda,\alpha,\beta}$.*

5.2.1 Stability of the minimal branch of solutions

This section is devoted to the proof of the following stability result for minimal solutions. We shall need yet another notion of $H^2(B)-$weak solutions, which is an intermediate class between classical and weak solutions.

Definition 5.14. *We say that u is a $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$ if $u - \Phi \in H_0^2(B)$, $\alpha \leq u \leq 1$ a.e. in B, $\frac{1}{(1-u)^2} \in L^1(B)$ and if*

$$\int_B \Delta u \Delta \phi = \lambda \int_B \frac{\phi}{(1-u)^2}, \qquad \forall \phi \in C^4(\bar{B}) \cap H_0^2(B),$$

where Φ is given in (5.8). We say that u is a $H^2(B)-$weak super-solution (resp. $H^2(B)-$weak sub-solution) of $(P)_{\lambda,\alpha,\beta}$ if for $\phi \geq 0$ the equality is replaced with $\geq$ (resp. $\leq$) and $u \geq \alpha$ (resp. $\leq$), $\partial_\nu u \leq \beta$ (resp. $\geq$) on ∂B.

Theorem 5.15. *Suppose (α, β) is an admissible pair.*

1. *The minimal solution u_λ is then stable and is the unique semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$.*
2. *The function $u^* := \lim_{\lambda \nearrow \lambda^*} u_\lambda$ is a well-defined semi-stable $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$.*
3. *When u^* is classical solution, then $\mu_1(u^*) = 0$ and u^* is the unique $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$.*
4. *If v is a singular, semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$, then $v = u^*$ and $\lambda = \lambda^*$*

The crucial tool is a comparison result which is valid exactly in this class of solutions.

Lemma 5.16. *Let (α, β) be an admissible pair and u be a semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$. Assume U is a $H^2(B)-$weak super-solution of $(P)_{\lambda,\alpha,\beta}$ so that $U - \Phi \in H_0^2(B)$. Then*

1. *$u \le U$ a.e. in B;*
2. *If u is a classical solution and $\mu_1(u) = 0$ then $U = u$.*

Proof. (i) Define $w := u - U$. Then by the Moreau decomposition [17] for the biharmonic operator, there exist $w_1, w_2 \in H_0^2(B)$, with $w = w_1 + w_2$, $w_1 \ge 0$ a.e., $\Delta^2 w_2 \le 0$ in the $H^2(B)-$weak sense and $\int_B \Delta w_1 \Delta w_2 = 0$. By Lemma 5.5, we have that $w_2 \le 0$ a.e. in B.
Given now $0 \le \phi \in C_c^\infty(B)$, we have that

$$\int_B \Delta w \Delta \phi \le \lambda \int_B (f(u) - f(U))\phi,$$

where $f(u) := (1-u)^{-2}$. Since u is semi-stable, one has

$$\lambda \int_B f'(u) w_1^2 \le \int_B (\Delta w_1)^2 = \int_B \Delta w \Delta w_1 \le \lambda \int_B (f(u) - f(U)) w_1.$$

Since $w_1 \ge w$ one also has

$$\int_B f'(u) w w_1 \le \int_B (f(u) - f(U)) w_1,$$

which once re-arranged gives

$$\int_B \tilde{f} w_1 \ge 0,$$

where $\tilde{f}(u) = f(u) - f(U) - f'(u)(u - U)$. The strict convexity of f gives $\tilde{f} \le 0$ and $\tilde{f} < 0$ whenever $u \ne U$. Since $w_1 \ge 0$ a.e. in B one sees that $w \le 0$ a.e. in B. The inequality $u \le U$ a.e. in B is then established.

(ii) Since u is a classical solution, it is easy to see that the infimum in $\mu_1(u)$ is attained at some ϕ. The function ϕ is then the first eigenfunction of $\Delta^2 - \frac{2\lambda}{(1-u)^3}$ in $H_0^2(B)$. Now we show that ϕ is of fixed sign. Using the above decomposition, one has $\phi = \phi_1 + \phi_2$ where $\phi_i \in H_0^2(B)$ for $i = 1, 2$, $\phi_1 \ge 0$, $\int_B \Delta\phi_1 \Delta\phi_2 = 0$ and $\Delta^2 \phi_2 \le 0$ in the $H_0^2(B)-$weak sense. If ϕ changes sign, then $\phi_1 \not\equiv 0$ and $\phi_2 < 0$ in B (recall that either $\phi_2 < 0$ or $\phi_2 = 0$ a.e. in B). We can write now:

$$\begin{aligned} 0 = \mu_1(u) &\le \frac{\int_B (\Delta(\phi_1 - \phi_2))^2 - \lambda f'(u)(\phi_1 - \phi_2)^2}{\int_B (\phi_1 - \phi_2)^2} \\ &< \frac{\int_B (\Delta\phi)^2 - \lambda f'(u)\phi^2}{\int_B \phi^2} = \mu_1(u) \end{aligned}$$

in view of $\phi_1\phi_2 < -\phi_1\phi_2$ in a set of positive measure, leading to a contradiction. So we can assume $\phi \geq 0$, and by the Boggi's principle we have $\phi > 0$ in B. For $0 \leq t \leq 1$ define

$$g(t) = \int_B \Delta\left[tU + (1-t)u\right]\Delta\phi - \lambda \int_B f(tU + (1-t)u)\phi,$$

where ϕ is the above first eigenfunction. Since f is convex one sees that

$$g(t) \geq \lambda \int_B \left[tf(U) + (1-t)f(u) - f(tU + (1-t)u)\right]\phi \geq 0$$

for every $t \geq 0$. Since $g(0) = 0$ and

$$g'(0) = \int_B \Delta(U-u)\Delta\phi - \lambda f'(u)(U-u)\phi = 0,$$

we get that

$$g''(0) = -\lambda \int_B f''(u)(U-u)^2\phi \geq 0.$$

Since $f''(u)\phi > 0$ in B, we finally get that $U = u$ a.e. in B. □

Based again on Lemma 5.5(3), we can show a more general version of the above Lemma 5.16.

Lemma 5.17. *Let (α, β) be an admissible pair and $\beta' \leq 0$. Let u be a semi-stable $H^2(B)-$weak sub-solution of $(P)_{\lambda,\alpha,\beta}$ with $u = \alpha$, $\partial_\nu u = \beta' \geq \beta$ on ∂B. Assume that U is a $H^2(B)-$weak super-solution of $(P)_{\lambda,\alpha,\beta}$ with $U = \alpha$, $\partial_\nu U = \beta$ on ∂B. Then $U \geq u$ a.e. in B.*

Proof. Let $\tilde{u} \in H_0^2(B)$ denote a weak solution to $\Delta^2\tilde{u} = \Delta^2(u-U)$ in B. Since $\tilde{u} - u + U = 0$ and $\partial_\nu(\tilde{u} - u + U) \leq 0$ on ∂B, by Lemma 5.5 one has that $\tilde{u} \geq u - U$ a.e. in B. Again by the Moreau decomposition [17], we may write $\tilde{u}$ as $\tilde{u} = w + v$, where $w, v \in H_0^2(B)$, $w \geq 0$ a.e. in B, $\Delta^2 v \leq 0$ in a $H^2(B)-$weak sense and $\int_B \Delta w \Delta v = 0$. Then for $0 \leq \phi \in C^4(\bar{B}) \cap H_0^2(B)$ one has

$$\int_B \Delta\tilde{u}\Delta\phi = \int_B \Delta(u-U)\Delta\phi \leq \lambda \int_B (f(u) - f(U))\phi.$$

In particular, we have that

$$\int_B \Delta\tilde{u}\Delta w \leq \lambda \int_B (f(u) - f(U))w.$$

Since by semi-stability of u

$$\lambda \int_B f'(u)w^2 \leq \int_B (\Delta w)^2 = \int_B \Delta\tilde{u}\Delta w,$$

we get that

$$\int_B f'(u)w^2 \le \int_B (f(u) - f(U))w.$$

By Lemma 5.5 we have $v \le 0$ and then $w \ge \tilde{u} \ge u - U$ a.e. in B. So we see that

$$0 \le \int_B (f(u) - f(U) - f'(u)(u - U))\, w.$$

The strict convexity of f implies as in Lemma 5.16 that $U \ge u$ a.e. in B. □

We shall need the following a-priori estimates along the minimal branch u_λ.

Lemma 5.18. *Let (α, β) be an admissible pair. Then one has*

$$2\int_B \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3} \le \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^2},$$

where Φ is given in (5.8). In particular, there is a constant $C > 0$ so that for every $\lambda \in (0, \lambda^*)$, we have

$$\int_B (\Delta u_\lambda)^2 + \int_B \frac{1}{(1-u_\lambda)^3} \le C. \tag{5.19}$$

Proof. Testing $(P)_{\lambda,\alpha,\beta}$ on $u_\lambda - \Phi \in C^4(\bar{B}) \cap H_0^2(B)$, we see that

$$\lambda \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^2} = \int_B \Delta u_\lambda \Delta(u_\lambda - \Phi) = \int_B (\Delta(u_\lambda - \Phi))^2 \ge 2\lambda \int_B \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3}$$

in view of $\Delta^2 \Phi = 0$. In particular, for $\delta > 0$ small we have that

$$\begin{aligned}
\int_{\{|u_\lambda - \Phi| \ge \delta\}} \frac{1}{(1-u_\lambda)^3} &\le \frac{1}{\delta^2} \int_{\{|u_\lambda - \Phi| \ge \delta\}} \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3} \le \frac{1}{\delta^2} \int_B \frac{1}{(1-u_\lambda)^2} \\
&\le \delta \int_{\{|u_\lambda - \Phi| \ge \delta\}} \frac{1}{(1-u_\lambda)^3} + C_\delta
\end{aligned}$$

by means of Young's inequality. Since for δ small,

$$\int_{\{|u_\lambda - \Phi| \le \delta\}} \frac{1}{(1-u_\lambda)^3} \le C'$$

for some $C' > 0$, we can deduce that for every $\lambda \in (0, \lambda^*)$,

$$\int_B \frac{1}{(1-u_\lambda)^3} \le C$$

for some $C > 0$. By Young's and Hölder's inequalities, we now have

$$\begin{aligned}
\int_B (\Delta u_\lambda)^2 &= \int_B \Delta u_\lambda \Delta \Phi \\
&+ \lambda \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^2} \le \delta \int_B (\Delta u_\lambda)^2 + C_\delta + C\left(\int_B \frac{1}{(1-u_\lambda)^3}\right)^{\frac{2}{3}}
\end{aligned}$$

and estimate (5.19) is therefore established. □

We are now ready to establish Theorem 5.15.

Proof (of Theorem 5.15): (1) Since $\|u_\lambda\|_\infty < 1$, the infimum defining $\mu_1(u_\lambda)$ is achieved at a first eigenfunction for every $\lambda \in (0, \lambda^*)$. Since $\lambda \mapsto u_\lambda(x)$ is increasing for every $x \in B$, it is easily seen that $\lambda \mapsto \mu_1(u_\lambda)$ is an increasing, continuous function on $(0, \lambda^*)$. Define

$$\lambda_{**} := \sup\{0 < \lambda < \lambda^* : \mu_1(u_\lambda) > 0\}.$$

We have that $\lambda_{**} = \lambda^*$. Indeed, otherwise we would have that $\mu_1(u_{\lambda_{**}}) = 0$, and for every $\mu \in (\lambda_{**}, \lambda^*)$ u_μ would be a classical super-solution of $(P)_{\lambda_{**},\alpha,\beta}$. A contradiction arises since Lemma 5.16 implies $u_\mu = u_{\lambda_{**}}$.
Finally, Lemma 5.16 guarantees uniqueness in the class of semi-stable $H^2(B)-$weak solutions.
(2) By estimate (5.19) it follows that $u_\lambda \to u^*$ in a pointwise sense and weakly in $H^2(B)$, and $\frac{1}{1-u^*} \in L^3(B)$. In particular, u^* is a $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$ which is also semi-stable as limiting function of the semi-stable solutions $\{u_\lambda\}$.
(3) Whenever $\|u^*\|_\infty < 1$, the function u^* is a classical solution, and by the Implicit Function Theorem we have that $\mu_1(u^*) = 0$ to prevent the continuation of the minimal branch beyond λ^*. By Lemma 5.16 u^* is then the unique $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$. An alternative approach –which we do not pursue here– based on the very definition of the extremal solution u^* is available in [4] when $\alpha = \beta = 0$ (see also [15]) to show that u^* is the unique weak solution of $(P)_{\lambda^*}$, regardless of whether u^* is regular or not.
(4) If $\lambda < \lambda^*$, by uniqueness $v = u_\lambda$. So v is not singular and a contradiction arises.

By Theorem 5.13(3) we have that $\lambda = \lambda^*$. Since v is a semi-stable $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$ and u^* is a $H^2(B)-$weak super-solution of $(P)_{\lambda^*,\alpha,\beta}$, we can apply Lemma 5.16 to get $v \le u^*$ a.e. in B. Since u^* is a semi-stable solution too, we can reverse the roles of v and u^* in Lemma 5.16 to see that $v \ge u^*$ a.e. in B. So equality $v = u^*$ holds and the proof is done.

5.3 Regularity of the extremal solution for $1 \le N \le 8$

We now return to the issue of the regularity of the extremal solution in problem $(P)_\lambda$. Unless stated otherwise, u_λ and u^* refer to the minimal and extremal solutions of $(P)_\lambda$. We shall show that the extremal solution u^* is regular provided $1 \le N \le 8$. We first begin by showing that it is indeed the case in small dimensions:

Theorem 5.20. *u^* is regular in dimensions $1 \le N \le 4$.*

Proof. As already observed, estimate (5.19) implies that $f(u^*) = (1 - u^*)^{-2} \in L^{\frac{3}{2}}(B)$. Since u^* is radial and radially decreasing, we need to show that $u^*(0) < 1$

to get the regularity of u^*. The integrability of $f(u^*)$ along with elliptic regularity theory shows that $u^* \in W^{4,\frac{3}{2}}(B)$. By the Sobolev imbedding Theorem we get that u^* is a Lipschitz function in B.
Now suppose $u^*(0) = 1$ and $1 \le N \le 3$. Since

$$\frac{1}{1-u} \ge \frac{C}{|x|} \qquad \text{in } B$$

for some $C > 0$, one sees that

$$+\infty = C^3 \int_B \frac{1}{|x|^3} \le \int_B \frac{1}{(1-u^*)^3} < +\infty.$$

A contradiction arises and hence u^* is regular for $1 \le N \le 3$.
For $N = 4$ we need to be more careful and observe that $u^* \in C^{1,\frac{1}{3}}(\bar{B})$ by the Sobolev Imbedding Theorem. If $u^*(0) = 1$, then $\nabla u^*(0) = 0$ and

$$\frac{1}{1-u^*} \ge \frac{C}{|x|^{\frac{4}{3}}} \qquad \text{in } B$$

for some $C > 0$. We now obtain a contradiction exactly as above. □

We now tackle the regularity of u^* for $5 \le N \le 8$. We start with the following crucial result:

Theorem 5.21. *Let $N \ge 5$ and (u^*, λ^*) be the extremal pair of $(P)_\lambda$. When u^* is singular, then*

$$1 - u^*(x) \le C_0 |x|^{\frac{4}{3}} \qquad \text{in } B,$$

where $C_0 := \left(\frac{\lambda^*}{\bar{\lambda}}\right)^{\frac{1}{3}}$ and $\bar{\lambda} = \bar{\lambda}_N := \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3})$.

Proof. First note that Theorem 5.6(4) gives the lower bound:

$$\lambda^* \ge \bar{\lambda} = \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3}). \tag{5.22}$$

For $\delta > 0$, we define $u_\delta(x) := 1 - C_\delta |x|^{\frac{4}{3}}$ with $C_\delta := \left(\frac{\lambda^*}{\bar{\lambda}} + \delta\right)^{\frac{1}{3}} > 1$. Since $N \ge 5$, we have that $u_\delta \in H^2_{loc}(\mathbb{R}^N)$, $\frac{1}{1-u_\delta} \in L^3_{loc}(\mathbb{R}^N)$ and u_δ is a H^2-weak solution of

$$\Delta^2 u_\delta = \frac{\lambda^* + \delta \bar{\lambda}}{(1-u_\delta)^2} \qquad \text{in } \mathbb{R}^N.$$

We claim that $u_\delta \le u^*$ in B, which will finish the proof by just letting $\delta \to 0$.

Assume by contradiction that the set $\Gamma := \{r \in (0,1) : u_\delta(r) > u^*(r)\}$ is non-empty, and let $r_1 = \sup \Gamma$. Since

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

we have that $0 < r_1 < 1$ and one infers that

$$\alpha := u^*(r_1) = u_\delta(r_1)\,, \quad \beta := (u^*)'(r_1) \geq u'_\delta(r_1).$$

Setting $u_{\delta,r_1}(r) = r_1^{-\frac{4}{3}}(u_\delta(r_1 r) - 1) + 1$, we easily see that u_{δ,r_1} is a $H^2(B)-$weak super-solution of $(P)_{\lambda^*+\delta\bar{\lambda}_N,\alpha',\beta'}$, where

$$\alpha' := r_1^{-\frac{4}{3}}(\alpha - 1) + 1\,, \quad \beta' := r_1^{-\frac{1}{3}}\beta.$$

Similarly, let us define $u^*_{r_1}(r) = r_1^{-\frac{4}{3}}(u^*(r_1 r) - 1) + 1$. The dilation map

$$w \to w_{r_1}(r) = r_1^{-\frac{4}{3}}(w(r_1 r) - 1) + 1 \tag{5.23}$$

is a correspondence between solutions of $(P)_\lambda$ on B and of $(P)_{\lambda,1-r_1^{-\frac{4}{3}},0}$ on $B_{r_1^{-1}}$ which preserves the H^2-integrability. In particular, $(u^*_{r_1}, \lambda^*)$ is the extremal pair of $(P)_{\lambda,1-r_1^{-\frac{4}{3}},0}$ on $B_{r_1^{-1}}$ (defined in the obvious way). Moreover, $u^*_{r_1}$ is a singular semi-stable $H^2(B)-$ weak solution of $(P)_{\lambda^*,\alpha',\beta'}$.

Since u^* is radially decreasing, we have that $\beta' \leq 0$. Define the function w as $w(x) := (\alpha' - \frac{\beta'}{2}) + \frac{\beta'}{2}|x|^2 + \gamma(x)$, where γ is a solution of $\Delta^2\gamma = \lambda^*$ in B with $\gamma = \partial_\nu\gamma = 0$ on ∂B. Then w is a classical solution of

$$\begin{cases} \Delta^2 w = \lambda^* & \text{in } B \\ w = \alpha'\,, \quad \partial_\nu w = \beta' & \text{on } \partial B. \end{cases}$$

Since $\frac{\lambda^*}{(1-u^*_{r_1})^2} \geq \lambda^*$, by Lemma 5.5 we have $u^*_{r_1} \geq w$ a.e. in B. Since $w(0) = \alpha' - \frac{\beta'}{2} + \gamma(0)$ and $\gamma(0) > 0$, the bound $u^*_{r_1} \leq 1$ a.e. in B yields to $\alpha' - \frac{\beta'}{2} < 1$. Namely, (α', β') is an admissible pair and by Theorem 5.15(4) we get that $(u^*_{r_1}, \lambda^*)$ coincides with the extremal pair of $(P)_{\lambda,\alpha',\beta'}$ in B.

Since (α', β') is an admissible pair and u_{δ,r_1} is a $H^2(B)-$weak super-solution of $(P)_{\lambda^*+\delta\bar{\lambda}_N,\alpha',\beta'}$, by Proposition 5.2.1 we get the existence of a weak solution of $(P)_{\lambda^*+\delta\bar{\lambda}_N,\alpha',\beta'}$. Since $\lambda^* + \delta\bar{\lambda}_N > \lambda^*$, we contradict the fact that λ^* is the extremal parameter of $(P)_{\lambda,\alpha',\beta'}$. □

Thanks to this lower estimate on u^*, we get the following result.

Theorem 5.24. *If $5 \leq N \leq 8$, then the extremal solution u^* of $(P)_\lambda$ is regular.*

Proof. Assume that u^* is singular. For $\varepsilon > 0$ set $\psi(x) := |x|^{\frac{4-N}{2}+\varepsilon}$ and note that

$$(\Delta\psi)^2 = (H_N + O(\varepsilon))|x|^{-N+2\varepsilon}$$

where

$$H_N := \frac{N^2(N-4)^2}{16}.$$

Given $\eta \in C_0^\infty(B)$, and since $N \geq 5$, we can use the test function $\eta\psi \in H_0^2(B)$ into the stability inequality to obtain

$$2\lambda \int_B \frac{\psi^2}{(1-u^*)^3} \leq \int_B (\Delta\psi)^2 + O(1),$$

where $O(1)$ is a bounded function as $\varepsilon \searrow 0$. By Theorem 5.21 we find that

$$2\bar{\lambda}_N \int_B \frac{\psi^2}{|x|^4} \leq \int_B (\Delta\psi)^2 + O(1),$$

and then

$$2\bar{\lambda}_N \int_B |x|^{-N+2\varepsilon} \leq (H_N + O(\varepsilon)) \int_B |x|^{-N+2\varepsilon} + O(1).$$

Computing the integrals one arrives at

$$2\bar{\lambda}_N \leq H_N + O(\varepsilon).$$

As $\varepsilon \to 0$ finally we obtain $2\bar{\lambda}_N \leq H_N$. Graphing this relation one sees that $N \geq 9$. □

We can now slightly improve the lower bound (5.22).

Corollary 5.25. *In any dimension $N \geq 1$, we have*

$$\lambda^* > \bar{\lambda}_N = \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3}). \tag{5.26}$$

Proof. The function $\bar{u} := 1 - |x|^{\frac{4}{3}}$ is a $H^2(B)-$ weak solution of $(P)_{\bar{\lambda}_N, 0, -\frac{4}{3}}$. If by contradiction $\lambda^* = \bar{\lambda}_N$, then $\bar{u}$ is a $H^2(B)-$weak super-solution of $(P)_\lambda$ for every $\lambda \in (0, \lambda^*)$. By Lemma 5.16 we get that $u_\lambda \leq \bar{u}$ for all $\lambda < \lambda^*$, and then $u^* \leq \bar{u}$ a.e. in B.

If $1 \leq N \leq 8$, u^* is then regular by Theorems 5.20 and 5.24. By Theorem 5.15(3) there holds $\mu_1(u^*) = 0$. Lemma 5.16 then yields that $u^* = \bar{u}$, which is a contradiction since then u^* will not satisfy the boundary conditions.

If now $N \geq 9$ and $\bar{\lambda} = \lambda^*$, then $C_0 = 1$ in Theorem 5.21, and we then have $u^* \geq \bar{u}$. It means again that $u^* = \bar{u}$, a contradiction that completes the proof. □

5.4 The extremal solution is singular for $N \geq 9$

We prove in this section that the extremal solution is singular for $N \geq 9$. For that we have to distinguish between three different ranges for the dimension. For each range, we will need a suitable Hardy-Rellich type inequality that will be established in the last section, by using the recent results of Ghoussoub-Moradifam [10]. As in the previous section (u^*, λ^*) denotes the extremal pair of $(P)_\lambda$.

• **Case** $N \geq 17$**:** To establish the singularity of u^* for these dimensions we shall need the following well known improved Hardy-Rellich inequality, which is valid for $N \geq 5$. There exists $C > 0$, such that for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2\,dx \geq \frac{N^2(N-4)^2}{16}\int_B \frac{\phi^2}{|x|^4}\,dx + C\int_B \phi^2\,dx. \tag{5.27}$$

• **Case** $10 \leq N \leq 16$**:** For this case, we shall need the following inequality valid for all $\phi \in H_0^2(B)$

$$\begin{aligned}\int_B (\Delta\phi)^2 \quad &\geq \quad \frac{(N-2)^2(N-4)^2}{16}\int_B \frac{\phi^2}{(|x|^2-|x|^{\frac{N}{2}+1})(|x|^2-|x|^{\frac{N}{2}})} \\ &+\frac{(N-1)(N-4)^2}{4}\int_B \frac{\phi^2}{|x|^2(|x|^2-|x|^{\frac{N}{2}})}.\end{aligned} \tag{5.28}$$

• **Case** $N = 9$**:** This case is the trickiest and will require the following inequality for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2 \geq \int_B Q(|x|)\left(P(|x|) + \frac{N-1}{|x|^2}\right)\phi^2, \tag{5.29}$$

where

$$P(r) = \tfrac{\Delta_N \varphi}{\varphi} \quad \text{ and } \quad Q(r) = \tfrac{\Delta_{N-2}\psi}{\psi},$$

with φ and ψ being two appropriately chosen polynomials, namely

$$\varphi(r) := r^{-\frac{N}{2}+1} + r - 1.9$$

and

$$\psi(r) := r^{-\frac{N}{2}+2} + 20r^{-1.69} + 10r^{-1} + 10r + 7r^2 - 48.$$

Recall that for a radial function φ, we set $\Delta_N\varphi(r) = \varphi''(r) + \frac{(N-1)}{r}\varphi'(r)$.

We shall first show the following upper bound on u^*.

Lemma 5.29. *If $N \geq 9$, then $u^* \leq 1 - |x|^{\frac{4}{3}}$ in B.*

Proof. Recall from Corollary 5.25 that $\bar{\lambda} := \frac{8}{9}(N-\frac{2}{3})(N-\frac{8}{3}) < \lambda^*$. We now claim that $u_\lambda \leq \bar{u}$ for all $\lambda \in (\bar{\lambda}, \lambda^*)$. Indeed, fix such a λ and assume by contradiction that

$$R_1 := \inf\{0 \leq R \leq 1 : u_\lambda < \bar{u} \text{ in } (R,1)\} > 0.$$

From the boundary conditions, one has that $u_\lambda(r) < \bar{u}(r)$ as $r \to 1^-$. Hence, $0 < R_1 < 1$, $\alpha := u_\lambda(R_1) = \bar{u}(R_1)$ and $\beta := u'_\lambda(R_1) \leq \bar{u}'(R_1)$. Introduce, as in the proof of Theorem 5.21, the functions u_{λ,R_1} and $\bar{u}_{R_1}$. We have that u_{λ,R_1} is a classical super-solution of $(P)_{\bar{\lambda}_N,\alpha',\beta'}$, where

$$\alpha' := R_1^{-\frac{4}{3}}(\alpha - 1) + 1\,, \quad \beta' := R_1^{-\frac{1}{3}}\beta.$$

Note that $\bar{u}_{R_1}$ is a $H^2(B)-$weak sub-solution of $(P)_{\bar{\lambda}_N,\alpha',\beta'}$ which is also semi-stable in view of the Hardy-Rellich inequality (5.27) and the fact that

$$2\bar{\lambda}_N \leq H_N := \frac{N^2(N-4)^2}{16}.$$

By Lemma 5.17, we deduce that $u_{\lambda,R_1} \geq \bar{u}_{R_1}$ in B. Note that, arguing as in the proof of Theorem 5.21, (α',β') is an admissible pair. We have therefore shown that $u_\lambda \geq \bar{u}$ in B_{R_1} and a contradiction arises in view of the fact that $\lim\limits_{x\to 0} \bar{u}(x) = 1$ and $\|u_\lambda\|_\infty < 1$. It follows that $u_\lambda \leq \bar{u}$ in B for every $\lambda \in (\bar{\lambda}_N, \lambda^*)$, and in particular $u^* \leq \bar{u}$ in B. □

The following lemma is the key for the proof of the of u^* in higher dimensions.

Lemma 5.30. *Let $N \geq 9$. Suppose there exist $\lambda' > 0$, $\beta > 0$ and a singular radial function $w \in H^2(B)$with $\frac{1}{1-w} \in L^\infty_{loc}(\bar{B} \setminus \{0\})$ such that*

$$\begin{cases} \Delta^2 w \leq \frac{\lambda'}{(1-w)^2} & \text{for } 0 < r < 1, \\ w(1) = 0, \ w'(1) = 0, \end{cases} \tag{5.31}$$

and

$$2\beta \int_B \frac{\phi^2}{(1-w)^3} \leq \int_B (\Delta\phi)^2 \ \text{ for all } \ \phi \in H^2_0(B), \tag{5.32}$$

1. *If $\beta \geq \lambda'$, then $\lambda^* \leq \lambda'$.*
2. *If either $\beta > \lambda'$ or if $\beta = \lambda' = \frac{H_N}{2}$, then the extremal solution u^* is necessarily singular.*

Proof: 1) First, note that (5.32) and $\frac{1}{1-w} \in L^\infty_{loc}(\bar{B}\setminus\{0\})$ yield to $\frac{1}{(1-w)^2} \in L^1(B)$. By a density argument, (5.31) implies now that w is a $H^2(B)-$weak sub-solution of $(P)_{\lambda'}$ whenever $N \geq 4$. If now $\lambda' < \lambda^*$, then by Lemma 5.17 w would necessarily be below the minimal solution $u_{\lambda'}$, which is a contradiction since w is singular while $u_{\lambda'}$ is regular.
2) Suppose first that $\beta = \lambda' = \frac{H_N}{2}$ and that $N \geq 9$. Since by part 1) we have $\lambda^* \leq \frac{H_N}{2}$, we get from Lemma 5.29 and the improved Hardy-Rellich inequality (5.27) that there exists $C > 0$ so that for all $\phi \in H^2_0(B)$

$$\int_B (\Delta\phi)^2 - 2\lambda^* \int_B \frac{\phi^2}{(1-u^*)^3} \geq \int_B (\Delta\phi)^2 - H_N \int_B \frac{\phi^2}{|x|^4} \geq C \int_B \phi^2.$$

It follows that $\mu_1(u^*) > 0$ and u^* must therefore be singular since otherwise, one could use the Implicit Function Theorem to continue the minimal branch beyond λ^*.
Suppose now that $\beta > \lambda'$, and let $\frac{\lambda'}{\beta} < \gamma < 1$ in such a way that

$$\alpha := (\frac{\gamma\lambda^*}{\lambda'})^{1/3} < 1. \tag{5.33}$$

Setting $\bar{w} := 1 - \alpha(1-w)$, we claim that

$$u^* \leq \bar{w} \quad \text{in} \quad B. \tag{5.34}$$

Note that by the choice of α we have $\alpha^3\lambda' < \lambda^*$, and therefore to prove (5.34) it suffices to show that for $\alpha^3\lambda' \leq \lambda < \lambda^*$, we have $u_\lambda \leq \bar{w}$ in B. Indeed, fix such λ and note that

$$\Delta^2 \bar{w} = \alpha \Delta^2 w \leq \frac{\alpha\lambda'}{(1-w)^2} = \frac{\alpha^3\lambda'}{(1-\bar{w})^2} \leq \frac{\lambda}{(1-\bar{w})^2}.$$

Assume that $u_\lambda \leq \bar{w}$ does not hold in B, and consider

$$R_1 := \sup\{0 \leq R \leq 1 \mid u_\lambda(R) > \bar{w}(R)\} > 0.$$

Since $\bar{w}(1) = 1 - \alpha > 0 = u_\lambda(1)$, we then have $R_1 < 1$, $u_\lambda(R_1) = \bar{w}(R_1)$ and $(u_\lambda)'(R_1) \leq (\bar{w})'(R_1)$. Introduce, as in the proof of Theorem 5.21, the functions u_{λ,R_1} and $\bar{w}_{R_1}$. We have that u_{λ,R_1} is a classical solution of $(P)_{\lambda,\alpha',\beta'}$, where

$$\alpha' := R_1^{-\frac{4}{3}}(u_\lambda(R_1) - 1) + 1\,, \quad \beta' := R_1^{-\frac{1}{3}}(u_\lambda)'(R_1).$$

Since $\lambda < \lambda^*$ and then

$$\frac{2\lambda}{(1-\bar{w})^3} \leq \frac{2\lambda^*}{\alpha^3(1-w)^3} = \frac{2\lambda'}{\gamma(1-w)^3} < \frac{2\beta}{(1-w)^3},$$

by (5.32) $\bar{w}_{R_1}$ is a stable $H^2(B)-$weak sub-solution of $(P)_{\lambda,\alpha',\beta'}$. By Lemma 5.17, we deduce that $u_\lambda \geq \bar{w}$ in B_{R_1} which is impossible, since $\bar{w}$ is singular while u_λ is regular. Note that, arguing as in the proof of Theorem 5.21, (α', β') is an admissible pair. This establishes claim (5.34) which, combined with the above inequality, yields

$$\frac{2\lambda^*}{(1-u^*)^3} \leq \frac{2\lambda^*}{\alpha^3(1-w)^3} < \frac{2\beta}{(1-w)^3},$$

and therefore

$$\inf_{\phi \in H_0^2(B)} \frac{\int_B (\Delta\phi)^2 - \frac{2\lambda^*\phi^2}{(1-u^*)^3}}{\int_B \phi^2} > 0.$$

It follows that again $\mu_1(u^*) > 0$ and u^* must be singular, since otherwise, one could use the Implicit Function Theorem to continue the minimal branch beyond λ^*.

Consider for any $m > 0$ the following function:

$$w_m := 1 - \frac{3m}{3m-4} r^{4/3} + \frac{4}{3m-4} r^m, \tag{5.35}$$

which satisfies the right boundary conditions: $w_m(1) = w_m'(1) = 0$. We can now prove that the extremal solution is singular for $N \geq 9$.

Theorem 5.36. *Let $N \geq 9$. The following upper bounds on λ^* hold:*

1. *If $N \geq 31$, then Lemma 5.30 holds with $w := w_2$, $\lambda' = 27\bar{\lambda}_N$ and $\beta = \frac{H_N}{2}$, and therefore $\lambda^*(N) \leq 27\bar{\lambda}_N$.*
2. *If $17 \leq N \leq 30$, then Lemma 5.30 holds with $w := w_3$, $\lambda' = \beta = \frac{H_N}{2}$, and therefore $\lambda^*(N) \leq \frac{H_N}{2}$.*
3. *If $10 \leq N \leq 16$, then Lemma 5.30 holds with $w := w_3$, $\lambda'_N < \beta_N$ given in Table 5.1, and therefore $\lambda^*(N) \leq \lambda'_N$.*
4. *If $N = 9$, then Lemma 5.30 holds with $w := w_{2.8}$, $\lambda'_9 := 366 < \beta_9 := 368.5$, and therefore $\lambda^*(9) \leq 366$.*

The extremal solution is therefore singular for dimension $N \geq 9$.

Table 5.1: Summary 2

N	w	λ'_N	β_N
9	$w_{2.8}$	366	366.5
10	w_3	450	487
11	w_3	560	739
12	w_3	680	1071
13	w_3	802	1495
14	w_3	940	2026
15	w_3	1100	2678
16	w_3	1260	3469
$17 \leq N \leq 30$	w_3	$H_N/2$	$H_N/2$
$N \geq 31$	w_2	$27\bar{\lambda}_N$	$H_N/2$

Proof. 1) Assume first that $N \geq 31$, then $27\bar{\lambda} \leq \frac{H_N}{2}$. We shall show that w_2 is a singular $H^2(B)-$weak sub-solution of $(P)_{27\bar{\lambda}}$ so that (5.32) holds with $\beta = \frac{H_N}{2}$. Indeed, write

$$w_2 := 1 - |x|^{\frac{4}{3}} - 2(|x|^{\frac{4}{3}} - |x|^2) = \bar{u} - \phi_0,$$

where $\phi_0 := 2(|x|^{\frac{4}{3}} - |x|^2)$, and note that $w_2 \in H_0^2(B)$, $\frac{1}{1-w_2} \in L^3(B)$, $0 \leq w_2 \leq 1$ in B, and

$$\Delta^2 w_2 = \frac{3\bar{\lambda}}{r^{\frac{8}{3}}} \leq \frac{27\bar{\lambda}}{(1-w_2)^2} \qquad \text{in } B \setminus \{0\}.$$

So w_2 is $H^2(B)-$weak sub-solution of $(P)_{27\bar{\lambda}}$. Moreover, by $\phi_0 \geq 0$ and (5.27) we get that

$$H_N \int_B \frac{\phi^2}{(1-w_2)^3} = H_N \int_B \frac{\phi^2}{(|x|^{\frac{4}{3}} + \phi_0)^3} \leq H_N \int_B \frac{\phi^2}{|x|^4} \leq \int_B (\Delta\phi)^2$$

for all $\phi \in H_0^2(B)$. It follows from Lemma 5.30 that u^* is singular and that $\lambda^* \leq 27\bar{\lambda} \leq \frac{H_N}{2}$.
2) Assume $17 \leq N \leq 30$ and consider the function

$$w_3 := 1 - \frac{9}{5}r^{\frac{4}{3}} + \frac{4}{5}r^3.$$

We show that w_3 is a semi-stable singular $H^2(B)-$weak sub-solution of $(P)_{\frac{H_N}{2}}$. Indeed, we clearly have that $0 \leq w_3 \leq 1$ in B, $w_3 \in H_0^2(B)$ and $\frac{1}{1-w_3} \in L^3(B)$. To show the stability condition, we consider $\phi \in H_0^2(B)$ and write

$$\begin{aligned} H_N \int_B \frac{\phi^2}{(1-w_3)^3} &= 125 H_N \int_B \frac{\phi^2}{(9r^{\frac{4}{3}} - 4r^3)^3} \leq 125 H_N \sup_{0<r<1} \frac{1}{(9-4r^{\frac{5}{3}})^3} \int_B \frac{\phi^2}{r^4} \\ &= H_N \int_B \frac{\phi^2}{r^4} \leq \int_B (\Delta\phi)^2 \end{aligned}$$

by virtue of (5.27). An easy computation shows that

$$\begin{aligned} \frac{H_N}{2(1-w_3)^2} - \Delta^2 w_3 &= \frac{25 H_N}{2(9r^{\frac{4}{3}} - 4r^3)^2} - \frac{9\bar{\lambda}}{5r^{\frac{8}{3}}} - \frac{12}{5}\frac{N^2-1}{r} \\ &= \frac{25N^2(N-4)^2}{32(9r^{\frac{4}{3}} - 4r^3)^2} - \frac{8(N-\frac{2}{3})(N-\frac{8}{3})}{5r^{\frac{8}{3}}} - \frac{12}{5}\frac{N^2-1}{r}. \end{aligned}$$

By using Maple one can verify that this final quantity is nonnegative on $(0,1)$ whenever $17 \leq N \leq 30$, and hence w_3 is a $H^2(B)-$weak sub-solution of $(P)_{\frac{H_N}{2}}$. It follows from Lemma 5.30 that u^* is singular and that $\lambda^* \leq \frac{H_N}{2}$.
3) Assume $10 \leq N \leq 16$. We shall prove that again $w := w_3$ satisfies the assumptions of Lemma 5.30. Indeed, using Maple, we show that for each dimension $10 \leq N \leq 16$, inequality (5.31) holds with λ'_N given by Table 5.1. Then, by using Maple again, we show that for each dimension $10 \leq N \leq 16$, the following inequality holds

$$\begin{aligned} &\frac{(N-2)^2(N-4)^2}{16} \frac{1}{(|x|^2 - |x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} \\ + \quad &\frac{(N-1)(N-4)^2}{4} \frac{1}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})} \geq \frac{2\beta_N}{(1-w_3)^3}. \end{aligned}$$

where β_N is again given by Table 5.1. The above inequality and the Hardy-Rellich inequality (5.28) guarantee that the stability condition (5.32) holds with $\beta := \beta_N$. Since $\beta_N > \lambda'_N$, we deduce from Lemma 5.30 that the extremal solution is singular for $10 \leq N \leq 16$.
4) Suppose now $N = 9$ and consider $w := w_{2.8}$. Using Maple on can see that

$$\Delta^2 w \leq \frac{366}{(1-w)^2} \quad \text{in} \quad B$$

and

$$\frac{723}{(1-w)^3} \leq Q(r)\left(P(r) + \frac{N-1}{r^2}\right) \quad \text{for all} \quad r \in (0,1),$$

where P and Q are given in (5.29). Since $723 > 2 \times 366$, by Lemma 5.30 the extremal solution u^* is singular in dimension $N = 9$. □

5.5 Improved Hardy-Rellich Inequalities

We now prove the improved Hardy-Rellich inequalities used in section 4. They rely on the results of Ghoussoub-Moradifam in [10] which provide necessary and sufficient conditions for such inequalities to hold. At the heart of this characterization is the following notion of a Bessel pair of functions.

Definition 5.37. *Assume that B is a ball of radius R in $\mathbb{R}^N$, $V, W \in C^1(0,1)$, and $\int_0^R \frac{1}{r^{N-1}V(r)}dr = +\infty$. Say that the couple (V, W) is a Bessel pair on $(0, R)$ if the ordinary differential equation*

$$(\mathrm{B}_{V,W}) \qquad y''(r) + \left(\tfrac{N-1}{r} + \tfrac{V_r(r)}{V(r)}\right)y'(r) + \tfrac{W(r)}{V(r)}y(r) = 0$$

has a positive solution on the interval $(0, R)$.

The space of radial functions in $C_0^\infty(B)$ will be denoted by $C_{0,r}^\infty(B)$. The needed inequalities will follow from the following result.

Theorem 5.38. (Ghoussoub-Moradifam [10]) *Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball centered at zero with radius R in $\mathbb{R}^N$ ($N \geq 1$) such that $\int_0^R \frac{1}{r^{N-1}V(r)}dr = +\infty$ and $\int_0^R r^{N-1}V(r)dr < +\infty$. The following statements are then equivalent:*

1. *(V, W) is a Bessel pair on $(0, R)$.*
2. *$\int_B V(|x|)|\nabla\phi|^2 dx \geq \int_B W(|x|)\phi^2 dx$ for all $\phi \in C_0^\infty(B)$.*
3. *If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < N-2$, then the above are equivalent to*

$$\int_B V(|x|)(\Delta\phi)^2 dx \geq \int_B W(|x|)|\nabla\phi|^2 dx + (N-1)\int_B \left(\frac{V(|x|)}{|x|^2} - \frac{V_r(|x|)}{|x|}\right)|\nabla\phi|^2 dx$$

for all $\phi \in C_{0,r}^\infty(B)$.

4. *If in addition, $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ on $(0, R)$, then the above are equivalent to*

$$\int_B V(|x|)(\Delta\phi)^2 dx \geq \int_B W(|x|)|\nabla\phi|^2 dx + (N-1)\int_B \left(\frac{V(|x|)}{|x|^2} - \frac{V_r(|x|)}{|x|}\right)|\nabla\phi|^2 dx$$

for all $\phi \in C_0^\infty(B)$.

We shall now deduce the following corollary.

Corollary 5.39. *Let $N \geq 5$ and B be the unit ball in $\mathbb{R}^N$. Then the following improved Hardy-Rellich inequality holds for all $\phi \in C_0^\infty(B)$:*

$$\begin{aligned} \int_B (\Delta\phi)^2 \quad &\geq \quad \frac{(N-2)^2(N-4)^2}{16} \int_B \frac{\phi^2}{(|x|^2 - |x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} \\ &\quad + \frac{(N-1)(N-4)^2}{4} \int_B \frac{\phi^2}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})} \end{aligned} \tag{5.40}$$

Proof. Let $0 < \alpha < 1$ and define $y(r) := r^{-\frac{N}{2}+1} - \alpha$. Since

$$-\frac{y'' + \frac{(N-1)}{r} y'}{y} = \frac{(N-2)^2}{4} \frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}},$$

the couple $\left(1, \frac{(N-2)^2}{4} \frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}}\right)$ is a Bessel pair on $(0,1)$. By Theorem 5.38(4) the following inequality then holds:

$$\int_B (\Delta\phi)^2 dx \geq \frac{(N-2)^2}{4} \int_B \frac{|\nabla\phi|^2}{|x|^2 - \alpha|x|^{\frac{N}{2}+1}} + (N-1) \int_B \frac{|\nabla\phi|^2}{|x|^2} \tag{5.41}$$

for all $\phi \in C_0^\infty(B)$. Set $V(r) := \frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}}$ and note that

$$\frac{V_r}{V} = -\frac{2}{r} + \frac{\alpha(N-2)}{2} \frac{r^{\frac{N}{2}-2}}{1 - \alpha r^{\frac{N}{2}-1}} \geq -\frac{2}{r}.$$

The function $y(r) = r^{-\frac{N}{2}+2} - 1$ is decreasing and is then a positive super-solution on $(0,1)$ for the ODE

$$y'' + (\frac{N-1}{r} + \frac{V_r}{V}) y'(r) + \frac{W_1(r)}{V(r)} y = 0,$$

where

$$W_1(r) = \frac{(N-4)^2}{4(r^2 - r^{\frac{N}{2}})(r^2 - \alpha r^{\frac{N}{2}+1})}.$$

Hence, by Theorem 5.38(2) we deduce

$$\int_B \frac{|\nabla\phi|^2}{|x|^2 - \alpha|x|^{\frac{N}{2}+1}} \geq (\frac{N-4}{2})^2 \int_B \frac{\phi^2}{(|x|^2 - \alpha|x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})}$$

for all $\phi \in C_0^\infty(B)$. Similarly, for $V(r) = \frac{1}{r^2}$ we have that

$$\int_B \frac{|\nabla\phi|^2}{|x|^2} \geq (\frac{N-4}{2})^2 \int_B \frac{\phi^2}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}$$

for all $\phi \in C_0^\infty(B)$. Combining the above two inequalities with (5.41) and letting $\alpha \to 1$ we get inequality (5.40). □

Corollary 5.42. *Let $N = 9$ and B be the unit ball in $\mathbb{R}^N$. Define $\varphi(r) := r^{-\frac{N}{2}+1} + r - 1.9$ and $\psi(r) := r^{-\frac{N}{2}+2} + 20r^{-1.69} + 10r^{-1} + 10r + 7r^2 - 48$. Then the following improved Hardy-Rellich inequality holds for all $\phi \in C_0^\infty(B)$:*

$$\int_B (\Delta\phi)^2 \geq \int_B Q(|x|) \left(P(|x|) + \frac{N-1}{|x|^2} \right) \phi^2, \tag{5.43}$$

where

$$P(r) := -\frac{\varphi''(r) + \frac{N-1}{r}\varphi'(r)}{\varphi(r)} \quad \text{and} \quad Q(r) := -\frac{\psi''(r) + \frac{N-3}{r}\psi'(r)}{\psi(r)}.$$

Proof. By definition $(1, P(r))$ is a Bessel pair on $(0,1)$. One can easily see that $P(r) \geq \frac{2}{r^2}$. Hence, by Theorem 5.38(4) the following inequality holds:

$$\int_B (\Delta\phi)^2 dx \geq \int_B P(|x|)|\nabla\phi|^2 + (N-1)\int_B \frac{|\nabla\phi|^2}{|x|^2} \tag{5.44}$$

for all $\phi \in C_0^\infty(B)$. Using Maple it is easy to see that

$$\frac{P_r}{P} \geq -\frac{2}{r} \quad \text{in} \quad (0,1),$$

and therefore $\psi(r)$ is a positive super-solution for the ODE

$$y'' + (\frac{N-1}{r} + \frac{P_r(r)}{P(r)})y'(r) + \frac{P(r)Q(r)}{P(r)}y = 0,$$

on $(0,1)$. Hence, by Theorem 5.38(2) we have for all $\phi \in C_0^\infty(B)$

$$\int_B P(|x|)|\nabla\phi|^2 \geq \int_B P(|x|)Q(|x|)\phi^2,$$

and similarly

$$\int_B \frac{|\nabla\phi|^2}{|x|^2} \geq \int_B \frac{Q(|x|)}{|x|^2}\phi^2,$$

since $\psi(r)$ is a positive solution for the ODE

$$y'' + \frac{N-3}{r}y'(r) + Q(r)y = 0.$$

Combining the above two inequalities with (5.44) we get (5.43). □

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Chapter 6

On the critical dimension of a fourth order elliptic problem with negative exponent [5]

6.1 Introduction

Consider the fourth order elliptic problem

$$\begin{cases} \beta\Delta^2 u - \tau\Delta u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ 0 < u \le 1 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (G_\lambda)$$

where $\lambda > 0$ is a parameter, $\tau > 0$, $\beta > 0$ are fixed constants, and $\Omega \subset^N$ $(N \geq 2)$ is a bounded smooth domain. This problem with $\beta = 0$ models a simple electrostatic Micro-Electromechanical Systems (MEMS) device which has been recently studied by many authors. For instance, see [3], [5], [7], [8], [9], [10], [11], [14], [15], [16], and the references cited therein.

Recently, Lin and Yang [18] derived the equation (G_λ) in the study of the charged plates in electrostatic actuators. They showed that there exists $0 < \lambda^* < \infty$ such that for $\lambda \in (0, \lambda^*)$ (G_λ) has a minimal regular solutions u_λ $(\sup_B u_\lambda < 1)$ while for $\lambda > \lambda^*$, (G_λ) does not have any regular solution. Moreover, the branch $\lambda \to u_\lambda(x)$ is increasing for each $x \in B$, and therefore the function $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ can be considered as a generalized solution that corresponds to the pull-in voltage λ^*. Now the important question is whether the extremal solution u^* is regular or not. In a recent paper Guo and Wei [17] proved that the extremal solution u^* is regular for dimensions $N \leq 4$. In this paper we consider the problem (G_λ) on the unit ball in ^{N}:

$$\begin{cases} \beta\Delta^2 u - \tau\Delta u = \frac{\lambda}{(1-u)^2} & \text{in } B, \\ 0 < u \le 1 & \text{in } B, \\ u = \Delta u = 0 & \text{on } \partial B, \end{cases} \qquad (P_\lambda)$$

[5] A version of this chapter has been accepted for publication. A. Moradifam, The critical dimension for a fourth order elliptic problem with singular nonlinearity, Journal of Differential Equations, 248 (2010), 594-616.

and show that the critical dimension for (P_λ) is $N = 9$. Indeed we prove that the extremal solution of (P_λ) is regular ($\sup_B u^* < 1$) for $N \leq 8$ and $\beta, \tau > 0$ and it is singular ($\sup_B u^* = 1$) for $N \geq 9$, $\beta > 0$, and $\tau > 0$ with $\frac{\tau}{\beta}$ small. Our proof of regularity of the extremal solution in dimensions $5 \leq N \leq 8$ is heavily inspired by [4] and [6]. On the other hand we shall use certain improved Hardy-Rellich inequalities to prove that the extremal solution is singular in dimensions $N \geq 9$. Our improve Hardy-Rellich inequalities follow from the recent result of Ghoussoub-Moradifam [12] about Hardy and Hardy-Rellich inequalities.

We now start by recalling some of the results from [17] concerning (P_λ) that will be needed in the sequel. Define

$$\lambda^*(B) := \sup\{\lambda > 0 : (P_\lambda) \text{ has a classical solution}\}.$$

We now introduce the following notion of solution.

We say that u is a *weak solution* of (G_λ), if $0 \leq u \leq 1$ a.e. in Ω, $\frac{1}{(1-u)^2} \in L^1(\Omega)$ and if

$$\int_\Omega u(\beta\Delta^2\phi - \tau\Delta\phi)\, dx = \lambda \int_\Omega \frac{\phi}{(1-u)^2} dx, \qquad \forall \phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega),$$

Say that u is a *weak super-solution* (resp. *weak sub-solution*) of (G_λ), if the equality is replaced with $\geq$ (resp. $\leq$) for $\phi \geq 0$.

We now introduce the notion of stability. First, we equip the function space $\mathcal{H} := H^2(\Omega) \cap H_0^1(\Omega) = W^{2,2}(\Omega) \cap H_0^1(\Omega)$ with the norm

$$\|\psi\| = \Big(\int_\Omega [\tau|\nabla\psi|^2 + \beta|\Delta\psi|^2] dx\Big)^{1/2}.$$

We say that a weak solution u_λ of (G_λ) is stable (respectively semi-stable) if the first eigenvalue $\mu_{1,\lambda}(u_\lambda)$ of the problem

$$-\tau\Delta h + \beta\Delta^2 h - \frac{2\lambda}{(1-u_\lambda)^3} h = \mu h \text{ in } \Omega, \quad h = \Delta h = 0 \text{ on } \partial\Omega \tag{6.1}$$

is positive (resp., nonnegative).

The operator $\beta\Delta^2 u - \tau\Delta u$ satisfies the following maximum principle which will be frequently used in the sequel.

Lemma 6.2. *([17]) Let $u \in L^1(\Omega)$. Then $u \geq 0$ a.e. in Ω, provided one of the following conditions hold:*

1. *$u \in C^4(\overline{\Omega})$, $\beta\Delta^2 u - \tau\Delta u \geq 0$ on Ω, and $u = \Delta u = 0$ on $\partial\Omega$.*
2. *$\int_\Omega u(\beta\Delta^2\phi - \tau\Delta\phi)\, dx \geq 0$ for all $0 \leq \phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$.*
3. *$u \in W^{2,2}(\Omega)$, $u = 0$, $\Delta u \leq 0$ on ∂B, and $\int_\Omega \big[\beta\Delta u\Delta\phi + \tau\nabla u\nabla\phi\big]dx \geq 0$ for all $0 \leq \phi \in W^{2,2}(\Omega) \cap H_0^1(\Omega)$.*

Moreover, either $u \equiv 0$ or $u > 0$ a.e. in Ω.

6.2 The pull-in voltage

As in [6] and [4], we are led here to examine problem (P_λ) with non-homogeneous boundary conditions such as

$$\begin{cases} \beta\Delta^2 u - \tau\Delta u = \frac{\lambda}{(1-u)^2} & \text{in } B, \\ \alpha < u \le 1 & \text{in } B, \\ u = \alpha,\ \Delta u = \gamma & \text{on } \partial B, \end{cases} \qquad (P_\lambda, \alpha, \gamma)$$

where α, γ are given. Whenever we need to emphasis the parameters β and τ we will refer to problem $(P_{\lambda,\alpha,\gamma})$ as $(P_{\lambda,\beta,\tau,\alpha,\gamma})$. In this section and Section 3 we will obtain several results for the following general form of $(P_\lambda, \alpha, \gamma)$

$$\begin{cases} \beta\Delta^2 u - \tau\Delta u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ \alpha < u \le 1 & \text{in } \Omega, \\ u = \alpha,\ \Delta u = \gamma & \text{on } \partial\Omega, \end{cases} \qquad (G_\lambda, \alpha, \gamma)$$

which are analogous to the results obtained by Gui and Wei for (G_λ) in [17].

Let Φ denote the unique solution of

$$\begin{cases} \beta\Delta^2\Phi - \tau\Delta\Phi = 0 & \text{in } \Omega, \\ \Phi = \alpha\,,\ \ \Delta\Phi = \gamma & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

We will say that the pair (α, γ) is admissible if $\gamma \le 0$, $\alpha < 1$, and $\sup_\Omega \Phi < 1$. We now introduce a notion of weak solution. We say that u is a *weak solution* of $(P_{\lambda,\alpha,\gamma})$, if $\alpha \le u \le 1$ a.e. in Ω, $\frac{1}{(1-u)^2} \in L^1(\Omega)$ and if

$$\int_\Omega (u - \Phi)(\beta\Delta^2\phi - \tau\Delta\phi) = \lambda \int_\Omega \frac{\phi}{(1-u)^2} \qquad \forall \phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega),$$

where Φ is given in (6.3). We say u is a *weak super-solution* (resp. *weak sub-solution*) of $(P_{\lambda,\alpha,\gamma})$, if the equality is replaced with $\ge$ (resp. $\le$) for $\phi \ge 0$. We say a weak solution u of $(P_{\lambda,\alpha,\gamma})$ is regular (resp. singular) if $\|u\|_\infty < 1$ (resp. $\|u\|_\infty = 1$). We now define

$$\lambda^*(\alpha, \gamma) := \sup\Big\{\lambda > 0 : (P_{\lambda,\alpha,\gamma}) \text{ has a classical solution}\Big\}$$

and

$$\lambda_*(\alpha, \gamma) := \sup\Big\{\lambda > 0 : (P_{\lambda,\alpha,\gamma}) \text{ has a weak solution}\Big\}.$$

Observe that by the Implicit Function Theorem, we can classically solve $(P_{\lambda,\alpha,\gamma})$ for small λ's. Therefore, $\lambda^*(\alpha,\gamma)$ and $\lambda_*(\alpha,\gamma)$ are well defined for any admissible pair (α,γ). To cut down on notations we won't always indicate α and γ. For example, λ_* and λ^* will denote the "weak and strong critical voltages" of $(P_{\lambda,\alpha,\gamma})$.

Now let U be a weak super-solution of $(P_{\lambda,\alpha,\gamma})$ and recall the following existence result. ([17]) For every $0 \le f \in L^1(\Omega)$ there exists a unique $0 \le u \in L^1(\Omega)$ which satisfies

$$\int_\Omega u(\beta\Delta^2\phi - \tau\Delta\phi)\,dx = \int_\Omega f\phi\,dx,$$

for all $\phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$.

We can introduce the following "weak" iterative scheme: $u_0 = U$ and (inductively) let u_n, $n \geq 1$, be the solution of

$$\int_\Omega (u_n - \Phi)(\beta\Delta^2\phi - \tau\Delta\phi) = \lambda \int_\Omega \frac{\phi}{(1-u_{n-1})^2} \qquad \forall\, \phi \in W^{4,2}(\bar{\Omega}) \cap H_0^1(\Omega)$$

given by Theorem 6.2. Since 0 is a sub-solution of $(P_{\lambda,\alpha,\gamma})$, inductively it is easily shown by Lemma 6.2 that $\alpha \leq u_{n+1} \leq u_n \leq U$ for every $n \geq 0$. Since

$$(1-u_n)^{-2} \leq (1-U)^{-2} \in L^1(\Omega),$$

by Lebesgue Theorem the function $u = \lim\limits_{n\to+\infty} u_n$ is a weak solution of $(P_{\lambda,\alpha,\gamma})$ so that $\alpha \leq u \leq U$. We therefore have the following result.

Lemma 6.4. *Assume the existence of a weak super-solution U of $(P_{\lambda,\alpha,\gamma})$. Then there exists a weak solution u of $(P_{\lambda,\alpha,\gamma})$ so that $\alpha \leq u \leq U$ a.e. in Ω.*

In particular, for every $\lambda \in (0, \lambda_*)$, we can find a weak solution of $(P_{\lambda,\alpha,\gamma})$. In the same range of λ's, this is still true for regular weak solutions as shown in the following lemma.

Lemma 6.5. *Let (α, γ) be an admissible pair and u be a weak solution of $(P_{\lambda,\alpha,\gamma})$. Then, there exists a regular solution for every $0 < \mu < \lambda$.*

Proof: Let $\epsilon \in (0,1)$ be given and let $\bar{u} = (1-\epsilon)u + \epsilon\Phi$, where Φ is given in (6.3). By Lemma 6.2 $\sup_\Omega \Phi < \sup_\Omega u \leq 1$. Hence

$$\sup_\Omega \bar{u} \leq (1-\epsilon) + \epsilon \sup_\Omega \Phi < 1\,, \quad \inf_\Omega \bar{u} \geq (1-\epsilon)\alpha + \epsilon \inf_\Omega \Phi = \alpha,$$

and for every $0 \leq \phi \in W^{4,2}(\bar{\Omega}) \cap H_0^1(\Omega)$ there holds:

$$\begin{aligned} \int_\Omega (\bar{u} - \Phi)(\beta\Delta^2\phi - \tau\Delta\phi) &= (1-\epsilon)\int_\Omega (u-\Phi)(\beta\Delta^2\phi - \tau\Delta\phi) \\ &= (1-\epsilon)\lambda \int_\Omega \frac{\phi}{(1-u)^2} \\ &= (1-\epsilon)^3\lambda \int_\Omega \frac{\phi}{(1-\bar{u}+\epsilon(\Phi-1))^2} \\ &\geq (1-\epsilon)^3\lambda \int_\Omega \frac{\phi}{(1-\bar{u})^2}. \end{aligned}$$

Note that $0 \leq (1-\epsilon)(1-u) = 1-\bar{u}+\epsilon(\Phi-1) < 1-\bar{u}$. So $\bar{u}$ is a weak super-solution of $(P_{(1-\epsilon)^3\lambda,\alpha,\gamma})$ so that $\sup\limits_\Omega \bar{u} < 1$. By Lemma 6.4 we get the existence of a weak solution w of $(P_{(1-\epsilon)^3\lambda,\alpha,\gamma})$ so that $\alpha \leq w \leq \bar{u}$. In particular, $\sup\limits_\Omega w < 1$ and w is a regular weak solution. Since $\epsilon \in (0,1)$ is arbitrarily chosen, the proof is done. □

Lemma 6.5 implies the existence of a regular weak solution U_λ for every $\lambda \in (0, \lambda_*)$. Introduce now a "classical" iterative scheme: $u_0 = 0$ and (inductively) $u_n = v_n + \Phi$, $n \geq 1$, where $v_n \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$ is the solution of

$$\beta\Delta^2 v_n - \tau\Delta v_n = \beta\Delta^2 u_n - \tau\Delta u_n = \frac{\lambda}{(1-u_{n-1})^2} \quad \text{in } \Omega \quad \text{and} \quad \Delta v_n = 0 \quad \text{on } \partial\Omega. \tag{6.6}$$

Since $v_n \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$, u_n is also a weak solution of (6.6), and by Lemma 6.2 we know that $\alpha \leq u_n \leq u_{n+1} \leq U_\lambda$ for every $n \geq 0$. Since $\sup\limits_\Omega u_n \leq \sup\limits_\Omega U_\lambda < 1$ for $n \geq 0$, we get that $(1-u_{n-1})^{-2} \in L^2(\Omega)$ and the existence of v_n is guaranteed. Since v_n is easily seen to be uniformly bounded in $H^2(\Omega)$, we have that $u_\lambda := \lim\limits_{n\to+\infty} u_n$ does hold pointwise and weakly in $H^2(\Omega)$. By Lebesgue theorem, we have that u_λ is a radial weak solution of (P_λ) so that $\sup\limits_\Omega u_\lambda \leq \sup\limits_\Omega U_\lambda < 1$. By elliptic regularity theory [1], $u_\lambda \in C^\infty(\bar\Omega)$ and $u_\lambda = \Delta u_\lambda = 0$ on $\partial\Omega$. So we can integrate by parts to get

$$\int_\Omega \beta(\Delta^2 u_\lambda - \tau\Delta u_\lambda)\phi\, dx = \int_\Omega u_\lambda(\beta\Delta^2\phi - \tau\Delta\phi)\, dx = \lambda\int_\Omega \frac{\phi}{(1-u_\lambda)^2}$$

for every $\phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$. Hence, u_λ is a classical solution of (P_λ) showing that $\lambda^* = \lambda_*$.

Since the argument above shows that $u_\lambda < U$ for any other classical solution U of (P_μ, α, γ) with $\mu \geq \lambda$, we have that u_λ is exactly the minimal solution and u_λ is strictly increasing as $\lambda \uparrow \lambda^*$. In particular, we can define u^* in the usual way: $u^*(x) = \lim\limits_{\lambda \nearrow \lambda^*} u_\lambda(x)$.

Lemma 6.7. $\lambda^*(\Omega) < +\infty$.

Proof: Let u be a classical solution of $(P_{\lambda,\alpha,\gamma})$ and let (ψ, μ_1) with $\Delta\psi = 0$ on $\partial\Omega$ denote the first eigenpair of $\beta\Delta^2 - \tau\Delta$ in $H^2(\Omega) \cap H_0^1(\Omega)$ with $\psi > 0$. Now let C be such that

$$\int_{\partial\Omega} ((\tau\alpha - \beta\gamma)\partial_\nu\psi - \beta\alpha\partial_\nu(\Delta\psi)) = C\int_\Omega \psi.$$

Multiplying $(P_{\lambda,\alpha,\gamma})$ by ψ and then integrating by parts one arrives at

$$\int_\Omega \left(\frac{\lambda}{(1-u)^2} - \mu_1 u - C\right)\psi = 0.$$

Since $\psi > 0$ there must exist a point $\bar x \in \Omega$ where $\frac{\lambda}{(1-u(\bar x))^2} - \mu_1 u(\bar x) - C \leq 0$. Since $\alpha < u(\bar x) < 1$, hence one can conclude that $\lambda \leq \sup_{0<u<1}(\mu_1 u + C)(1-u)^2$, which shows that $\lambda^* < +\infty$. □

In conclusion, we have shown the following description of the minimal branch. $\lambda^* \in (0, +\infty)$ and the following holds:

1. For each $0 < \lambda < \lambda^*$ there exists a regular and minimal solution u_λ of $(P_{\lambda,\alpha,\gamma})$.

2. For each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$.

3. For $\lambda > \lambda^*$ there are no weak solutions of $(P_{\lambda,\alpha,\gamma})$.

6.3 Stability of the minimal solutions

This section is devoted to the proof of the following stability result for minimal solutions. We shall need the following notion of $\mathcal{H}-$weak solutions, which is an intermediate class between classical and weak solutions.

We say that u is an $\mathcal{H}-$weak solution of $(P_{\lambda,\alpha,\gamma})$ if $u - \Phi \in H^2(\Omega) \cap H_0^1(\Omega)$, $0 \le u \le 1$ a.e. in Ω, $\frac{1}{(1-u)^2} \in L^1(\Omega)$ and

$$\int_\Omega \left[\beta \Delta u \Delta \phi + \tau \nabla u \nabla \phi\right] dx = \lambda \int_\Omega \frac{\phi}{(1-u)^2}, \qquad \forall \phi \in W^{2,2}(\Omega) \cap H_0^1(\Omega).$$

where Φ is given by (6.3). We say that u is an $\mathcal{H}-$weak super-solution (resp. an $\mathcal{H}-$weak sub-solution) of $(P_{\lambda,\alpha,\gamma})$ if for $\phi \ge 0$ the equality is replaced with $\ge$ (resp. $\le$) and $u \ge 0$ (resp. $\le$), $\Delta u \le 0$ (resp. $\ge$) on $\partial\Omega$.

Suppose that (α, γ) is an admissible pair.

1. The minimal solution u_λ is stable, and is the unique semi-stable $\mathcal{H}-$weak solution of $(P_{\lambda,\alpha,\gamma})$.

2. The function $u^* := \lim_{\lambda \nearrow \lambda^*} u_\lambda$ is a well-defined semi-stable $\mathcal{H}-$weak solution of $(P_{\lambda^*,\alpha,\gamma})$.

3. u^* is the unique $\mathcal{H}-$weak solution of $(P_{\lambda^*,\alpha,\gamma})$, and when u^* is classical solution, then $\mu_1(u^*) = 0$.

4. If v is a singular, semi-stable $\mathcal{H}-$weak solution of $(P_{\lambda,\alpha,\gamma})$, then $v = u^*$ and $\lambda = \lambda^*$

The main tool is the following comparison lemma which is valid exactly in the class $\mathcal{H}$.

Lemma 6.8. *Let (α, γ) be an admissible pair and u be a semi-stable $\mathcal{H}-$weak solution of $(P_{\lambda,\alpha,\gamma})$. Assume U is a $\mathcal{H}-$weak super-solution of $(P_{\lambda,\alpha,\gamma})$. Then*

1. *$u \le U$ a.e. in Ω;*
2. *If u is a classical solution and $\mu_1(u) = 0$ then $U = u$.*

Proof: (i) Define $w := u - U$. Then by means of the Moreau decomposition for the biharmonic operator (see [19] and [2]), there exist w_1 and $w_2 \in H^2(\Omega) \cap H_0^1(\Omega)$, with $w = w_1 + w_2$, $w_1 \geq 0$ a.e., $\beta\Delta^2 w_2 - \tau\Delta w_2 \leq 0$ in the $\mathcal{H}-$weak sense and $\int_\Omega \beta\Delta w_1 \Delta w_2 + \tau\nabla w_1.\nabla w_2 = 0$. Lemma 6.2 gives that $w_2 \leq 0$ a.e. in Ω.
Given $0 \leq \phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \beta\Delta w\Delta\phi + \tau\nabla w.\nabla\phi \leq \lambda\int_\Omega (f(u) - f(U))\phi,$$

where $f(u) := (1-u)^{-2}$. Since u is semi-stable, one has

$$\begin{aligned}\lambda\int_\Omega f'(u)w_1^2 \leq \int_\Omega \beta(\Delta w_1)^2 + \tau|\nabla w_1|^2 &= \int_\Omega \beta\Delta w\Delta w_1 + \tau\nabla w.\nabla w_1 \\ &\leq \lambda\int_\Omega (f(u) - f(U))w_1.\end{aligned}$$

Since $w_1 \geq w$ one has

$$\int_\Omega f'(u)ww_1 \leq \int_\Omega (f(u) - f(U))w_1,$$

which re-arranged gives

$$\int_\Omega \tilde{f}w_1 \geq 0,$$

where $\tilde{f}(u) = f(u) - f(U) - f'(u)(u - U)$. The strict convexity of f gives $\tilde{f} \leq 0$ and $\tilde{f} < 0$ whenever $u \neq U$. Since $w_1 \geq 0$ a.e. in Ω, one sees that $w \leq 0$ a.e. in Ω. The inequality $u \leq U$ a.e. in Ω is then established.

(ii) Since u is a classical solution, it is easy to see that the infimum of $\mu_1(u)$ is attained at some ϕ. The function ϕ is then the first eigenfunction of $\beta\Delta^2 - \tau\Delta - \frac{2\lambda}{(1-u)^3}$ in $H^2(\Omega) \cap H_0^1(\Omega)$. Now we show that ϕ is of fixed sign. Using the above decomposition, one has $\phi = \phi_1 + \phi_2$ where $\phi_i \in H^2(\Omega) \cap H_0^1(\Omega)$ for $i = 1, 2$, $\phi_1 \geq 0$, $\int_\Omega \beta\Delta\phi_1\Delta\phi_2 + \tau\nabla\phi_1.\nabla\phi_2 = 0$ and $\beta\Delta^2\phi_2 - \tau\Delta\phi_2 \leq 0$ in the $\mathcal{H}-$weak sense. If ϕ changes sign, then $\phi_1 \not\equiv 0$ and $\phi_2 < 0$ in Ω (recall that either $\phi_2 < 0$ or $\phi_2 = 0$ a.e. in Ω). We can write now

$$\begin{aligned}0 = \mu_1(u) &\leq \frac{\int_\Omega \beta(\Delta(\phi_1 - \phi_2))^2 + \tau|\nabla(\phi_1 - \phi_2)|^2 - \lambda f'(u)(\phi_1 - \phi_2)^2}{\int_\Omega (\phi_1 - \phi_2)^2} \\ &< \frac{\int_\Omega \beta(\Delta\phi)^2 + \tau|\nabla\phi|^2 - \lambda f'(u)\phi^2}{\int_\Omega \phi^2} = \mu_1(u),\end{aligned}$$

in view of $\phi_1\phi_2 < -\phi_1\phi_2$ in a set of positive measure, leading to a contradiction.

So we can assume $\phi \geq 0$, and by Lemma 6.2 we have $\phi > 0$ in Ω. For $0 \leq t \leq 1$, define

$$g(t) = \int_\Omega \beta\Delta\left[tU + (1-t)u\right]\Delta\phi + \tau\nabla\left[tU + (1-t)u\right].\nabla\phi - \lambda\int_\Omega f(tU + (1-t)u)\phi,$$

where ϕ is the above first eigenfunction. Since f is convex one sees that

$$g(t) \geq \lambda \int_\Omega [tf(U) + (1-t)f(u) - f(tU + (1-t)u)]\,\phi \geq 0$$

for every $t \geq 0$. Since $g(0) = 0$ and

$$g'(0) = \int_\Omega \beta\Delta(U-u)\Delta\phi + \tau\nabla(U-u).\nabla\phi - \lambda f'(u)(U-u)\phi = 0,$$

we get that

$$g''(0) = -\lambda \int_\Omega f''(u)(U-u)^2\phi \geq 0.$$

Since $f''(u)\phi > 0$ in Ω, we finally get that $U = u$ a.e. in Ω. □

A more general version of Lemma 6.8 is available in the following.

Lemma 6.9. *Let (α, γ) be an admissible pair and $\gamma' \leq 0$. Let u be a semi-stable $\mathcal{H}-$weak sub-solution of $(P_{\lambda,\alpha,\gamma})$ with $u = \alpha' \leq \alpha$, $\Delta u = \beta' \geq \beta$ on $\partial\Omega$. Assume that U is a $\mathcal{H}-$weak super-solution of $(P_{\lambda,\alpha,\gamma})$ with $U = \alpha$, $\Delta U = \beta$ on $\partial\Omega$. Then $U \geq u$ a.e. in Ω.*

Proof: Let $\tilde{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ denote a weak solution of $\beta\Delta^2\tilde{u} - \tau\Delta\tilde{u} = \beta\Delta^2(u-U) - \tau\Delta(u-U)$ in Ω and $\tilde{u} = \Delta\tilde{u} = 0$ on $\partial\Omega$. Since $\tilde{u}-u+U \geq 0$ and $\Delta(\tilde{u}-u+U) \leq 0$ on $\partial\Omega$, by Lemma 6.2 one has that $\tilde{u} \geq u - U$ a.e. in Ω. By means of the Moreau decomposition (see [19] and [2]) we write $\tilde{u}$ as $\tilde{u} = w+v$, where $w, v \in H_0^2(\Omega)$, $w \geq 0$ a.e. in Ω, $\beta\Delta^2 v - \tau\Delta v \leq 0$ in a $\mathcal{H}-$weak sense and $\int_\Omega \beta\Delta w\Delta v + \tau\nabla w.\nabla v = 0$. Then for $0 \leq \phi \in W^{4,2}(\bar{\Omega}) \cap H_0^1(\Omega)$, one has

$$\int_\Omega \beta\Delta\tilde{u}\Delta\phi + \tau\nabla\tilde{u}.\nabla\phi \leq \lambda \int_\Omega (f(u) - f(U))\phi.$$

In particular, we have

$$\int_\Omega \beta\Delta\tilde{u}\Delta w + \tau\nabla\tilde{u}.\nabla w \leq \lambda \int_\Omega (f(u) - f(U))w.$$

Since the semi-stability of u gives that

$$\lambda \int_\Omega f'(u)w^2 \leq \int_\Omega \beta(\Delta w)^2 + \tau|\nabla w|^2 = \int_\Omega \beta\Delta\tilde{u}\Delta w + \tau\nabla\tilde{u}.\nabla w,$$

we get that

$$\int_\Omega f'(u)w^2 \leq \int_\Omega (f(u) - f(U))w.$$

By Lemma 6.2 we have $v \leq 0$ and then $w \geq \tilde{u} \geq u - U$ a.e. in Ω. So we obtain that

$$0 \leq \int_\Omega (f(u) - f(U) - f'(u)(u-U))\,w.$$

The strict convexity of f implies that $U \geq u$ a.e. in Ω. □

We need also some a-priori estimates along the minimal branch u_λ.

Lemma 6.10. *Let (α, γ) be an admissible pair. Then for every $\lambda \in (0, \lambda^*)$, we have*

$$2\int_\Omega \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3} \leq \int_\Omega \frac{u_\lambda - \Phi}{(1-u_\lambda)^2},$$

where Φ is given by (6.3). In particular, there is a constant $C > 0$ independent of λ so that

$$\int_\Omega (\tau|\nabla u_\lambda|^2 + \beta|\Delta u_\lambda|^2)dx + \int_\Omega \frac{1}{(1-u_\lambda)^3} \leq C, \tag{6.11}$$

for every $\lambda \in (0, \lambda^*)$.

Proof: Testing $(P_{\lambda,\alpha,\gamma})$ on $u_\lambda - \Phi \in W^{4,2}(\Omega) \cap H_0^1(\Omega)$, we see that

$$\lambda \int_\Omega \frac{u_\lambda - \Phi}{(1-u_\lambda)^2} = \int_\Omega (\tau|\nabla(u_\lambda - \Phi)|^2 + \beta(\Delta(u_\lambda - \Phi))^2)dx \geq 2\lambda \int_\Omega \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3}.$$

In the view of $\beta\Delta^2\Phi - \tau\Delta\Phi = 0$. In particular, for $\delta > 0$ small we have that

$$\begin{aligned}\int_{\{|u_\lambda| \geq \delta\}} \frac{1}{(1-u_\lambda)^3} &\leq \frac{1}{\delta^2}\int_{\{|u_\lambda - \Phi| \geq \delta\}} \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3} \leq \frac{1}{\delta^2}\int_\Omega \frac{1}{(1-u_\lambda)^2} \\ &\leq \delta \int_{\{|u_\lambda - \Phi| \geq \delta\}} \frac{1}{(1-u_\lambda)^3} + C_\delta\end{aligned}$$

by means of Young's inequality. Since for δ small

$$\int_{\{|u_\lambda - \Phi| \leq \delta\}} \frac{1}{(1-u_\lambda)^3} \leq C,$$

for some $C > 0$, we get that

$$\int_\Omega \frac{1}{(1-u_\lambda)^3} \leq C,$$

for some $C > 0$ and for every $\lambda \in (0, \lambda^*)$. Since

$$\begin{aligned}\int_\Omega (\tau|\nabla u_\lambda|^2 + \beta|\Delta u_\lambda|^2)dx &= \int_\Omega (\beta\Delta u_\lambda \Delta\Phi + \tau\nabla u_\lambda.\nabla\Phi) + \lambda\int_\Omega \frac{u_\lambda - \Phi}{(1-u_\lambda)^2} \\ &\leq \delta \int_\Omega (\tau|\nabla u_\lambda|^2 + \beta|\Delta u_\lambda|^2)dx \\ &+ C_\delta + C\left(\int_\Omega \frac{1}{(1-u_\lambda)^3}\right)^{\frac{2}{3}}\end{aligned}$$

in view of Young's and Hölder's inequalities, estimate (6.11) is finally established. □

Proof of Theorem 6.3: (1) Since $\|u_\lambda\|_\infty < 1$, the infimum defining $\mu_1(u_\lambda)$ is achieved at a first eigenfunction for every $\lambda \in (0, \lambda^*)$. Since $\lambda \mapsto u_\lambda(x)$ is increasing for every $x \in \Omega$, it is easily seen that $\lambda \mapsto \mu_1(u_\lambda)$ is a decreasing and continuous function on $(0, \lambda^*)$. Define

$$\lambda_{**} := \sup\{0 < \lambda < \lambda^* : \mu_1(u_\lambda) > 0\}.$$

We have that $\lambda_{**} = \lambda^*$. Indeed, otherwise we would have $\mu_1(u_{\lambda_{**}}) = 0$, and for every $\mu \in (\lambda_{**}, \lambda^*)$, u_μ would be a classical super-solution of $(P_{\lambda_{**},\alpha,\gamma})$. A contradiction arises since Lemma 6.8 implies $u_\mu = u_{\lambda_{**}}$. Finally, Lemma 6.8 guarantees the uniqueness in the class of semi-stable $\mathcal{H}-$weak solutions.

(2) It follows from (6.11) that $u_\lambda \to u^*$ in a pointwise sense and weakly in $H^2(\Omega)$, and $\frac{1}{1-u^*} \in L^3(\Omega)$. In particular, u^* is a H^2-weak solution of $(P_{\lambda^*,\alpha,\gamma})$ which is also semi-stable as the limiting function of the semi-stable solutions $\{u_\lambda\}$.

(3) Whenever $\|u^*\|_\infty < 1$, the function u^* is a classical solution, and by the Implicit Function Theorem we have that $\mu_1(u^*) = 0$ to prevent the continuation of the minimal branch beyond λ^*. By Lemma 6.8, u^* is then the unique $\mathcal{H}-$weak solution of $(P_{\lambda^*,\alpha,\gamma})$.

(4) If $\lambda < \lambda^*$, we get by uniqueness that $v = u_\lambda$. So v is not singular and a contradiction arises. Now, by Theorem 6.2(3) we have that $\lambda = \lambda^*$. Since v is a semi-stable $\mathcal{H}-$ weak solution of $(P_{\lambda^*,\alpha,\gamma})$ and u^* is a $\mathcal{H}-$ weak super-solution of $(P_{\lambda^*,\alpha,\gamma})$, we can apply Lemma 6.8 to get $v \leq u^*$ a.e. in Ω. Since u^* is also a semi-stable solution, we can reverse the roles of v and u^* in Lemma 6.8 to see that $v \geq u^*$ a.e. in Ω. So equality $v = u^*$ holds and the proof is done. □

6.4 Regularity of the extremal solutions in dimensions $N \leq 8$

In this section we shall show that the extremal solution is regular in small dimensions. Let us begin with the following lemma.

Lemma 6.12. *Let $N \geq 5$ and (u^*, λ^*) be the extremal pair of (P_λ). If u^* is singular, and he set*

$$\Gamma := \{r \in (0,1) : u_\delta(r) > u^*(r)\} \tag{6.13}$$

is non-empty, where $u_\delta(x) := 1 - C_\delta |x|^{\frac{4}{3}}$ and $C_\delta > 1$ is a constant. Then there exists $r_1 \in (0,1)$ such that $u_\delta(r_1) \geq u^*(r_1)$ and $\Delta u_\delta(r_1) \leq \Delta u^*(r_1)$.

Proof. Assume by contradiction that for every r with $u_\delta(r_1) \geq u^*(r_1)$ one has $\Delta u_\delta(r_1) > \Delta u^*(r_1)$. Since Γ is non-empty and

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

there exists $s_1 \in (0,1)$ such that $u_\delta(s_1) = u^*(s_1)$. We claim that

$$u_\delta(s) > u^*(s),$$

for $0 < s < s_1$. Assume that there exist $s_3 < s_2 \leq s_1$ such that $u^*(s_2) = u_\delta(s_2)$, $u^*(s_3) = u_\delta(s_3)$ and $u_\delta(s) \geq u^*(s)$ for $s \in (s_3, s_2)$. By our assumption $\Delta u_s > \Delta u^*(s)$ for $s \in (s_3, s_2)$ which contradicts the maximum principle and justifies the claim. Therefore $u_\delta(s) > u^*(s)$ for $0 < s < s_1$. Now set $w := u_\delta - u^*$. Then $w \geq 0$ on B_{s_1} and $\Delta w \leq 0$ in B_{s_1}. Since $w(0) = 0$, by strong maximum principle we get $w \equiv 0$ on B_{s_1}. This is a contradiction and completes the proof. □

Let $N \geq 5$ and (u^*, λ^*) be the extremal pair of (P_λ). When u^* is singular, then

$$1 - u^* \leq C|x|^{\frac{4}{3}} \quad in \quad B,$$

where $C := (\frac{\lambda^*}{\beta\bar{\lambda}})^{\frac{1}{3}}$ and $\bar{\lambda} := \frac{8(N-\frac{2}{3})(N-\frac{8}{3})}{9}$. **Proof.** For $\delta > 0$, define $u_\delta(x) := 1 - C_\delta|x|^{\frac{4}{3}}$ with $C_\delta := (\frac{\lambda^*}{\beta\bar{\lambda}} + \delta)^{\frac{1}{3}} > 1$. Since $N \geq 5$, we have that $u_\delta \in H^2_{loc}(^N)$ and u_δ is a $\mathcal{H}-$weak solution of

$$\beta\Delta^2 u_\delta - \tau\Delta u_\delta = \frac{\lambda^* + \beta\delta\bar{\lambda}}{(1-u_\delta)^2} + \frac{4}{3}\tau C_\delta(N - \frac{2}{3})|x|^{-\frac{2}{3}} \quad in \quad ^N.$$

We claim that $u_\delta \leq u^*$ in B, which will finish the proof by just letting $\delta \to 0$.

Assume by contradiction that the set $\Gamma := \{r \in (0,1) : u_\delta(r) > u^*(r)\}$ is non-empty. By Lemma 6.12 the set

$$\Lambda := \{r \in (0,1) : u_\delta(r) \geq u^*(r) \ \text{ and } \ \Delta u_\delta(r) \leq \Delta u^*(r)\}$$

is non-empty. Let $r_1 \in \Lambda$. Since

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

we have that $0 < r_1 < 1$. Define

$$\alpha := u_*(r_1) \leq u_\delta(r_1), \quad \gamma := \Delta u^*(r_1) \geq \Delta u_\delta(r_1).$$

Setting $u_{\delta,r_1} = r_1^{-\frac{4}{3}}(u_\delta(r_1 r) - 1) + 1$, we see that u_{δ,r_1} is a $\mathcal{H}-$weak super-solution of $(P_{\lambda^*+\delta\lambda,\beta,r_1^{-2}\tau,\alpha',\gamma'})$, where

$$\alpha' := r_1^{-\frac{4}{3}}(\alpha - 1) + 1, \quad \gamma' = r_1^{\frac{2}{3}}\gamma.$$

Similarly, define $u^*_{r_1}(r) = r_1^{-\frac{4}{3}}(u^*(r_1 r) - 1) + 1$. Note that $\Delta^2 u^* - \alpha\Delta u^* \geq 0$ in B and $\Delta u^* = 0$ on ∂B. Hence by maximum principle we have $\Delta u^* \leq 0$ in B and therefore $\gamma' \leq 0$. Also obviously $\alpha' < 1$. So, (α', γ') is an admissible pair and by Theorem 6.3(4) we get that $(u^*_{r_1}, \lambda^*)$ coincides with the extremal pair of $(P_{\lambda,\beta,r_1^{-2}\tau,\alpha',\gamma'})$ in B. Also by Lemma 6.4 we get the existence of a week solution of $(P_{\lambda^*+\delta\lambda,\beta,r_1^{-2}\tau,\alpha',\gamma'})$. Since $\lambda^* + \delta\lambda > \lambda^*$, we contradict the fact that λ^* is the extremal parameter of $(P_{\lambda,\beta,r_1^{-2}\tau,\alpha',\gamma'})$. □

Now we are ready to prove the following result.

If $5 \leq N \leq 8$, then the extremal solution u^* of $(P)_\lambda$ is regular.

Proof. Assume that u^* is singular. For $\epsilon > 0$ define $\varphi(x) := |x|^{\frac{4-N}{2}+\epsilon}$ and note that

$$(\Delta\varphi)^2 = (H_N + O(\epsilon))|x|^{-N+2\epsilon}, \quad \text{where} \quad H_N := \frac{N^2(N-4)^2}{16}.$$

Given $\eta \in C_0^\infty(B)$, and since $N \geq 5$, we can use the test function $\eta\varphi \in H_0^2(B)$ into the stability inequality to obtain

$$2\lambda^* \int_B \frac{\varphi^2}{(1-u^*)^3} \leq \beta \int_B (\Delta\varphi)^2 + \tau \int_B |\nabla\varphi|^2 + O(1),$$

where $O(1)$ is a bounded function as $\epsilon \to 0$. By Theorem 6.4 we find

$$2\bar{\lambda} \int_B \frac{\varphi^2}{|x|^4} \leq \int_B (\Delta\varphi)^2 + O(1),$$

and then

$$2\bar{\lambda} \int_B |x|^{-N+2\epsilon} \leq (H_N + O(\epsilon)) \int_B |x|^{-N+2\epsilon} + O(1).$$

Computing the integrals on obtains

$$2\bar{\lambda} \leq H_N + O(\epsilon).$$

Letting $\epsilon \to 0$ we get $2\bar{\lambda} \leq H_N$. Graphing this relation we see that $N \geq 9$. □

6.5 The extremal solution is singular in dimensions $N \geq 9$

In this section we will show that the extremal solution u^* of $(P_{\lambda,\beta,\tau,0,0})$ in dimensions $N \geq 9$ is singular for $\tau > 0$ sufficiently small. To do this, first we shall show that the extremal solution of $(P_{\lambda,1,0,0,0})$ is singular in dimensions $N \geq 9$. Again to cut down the notation we won't always indicate that $\beta = 1$ and $\tau = 0$.

We have to distinguish between three different ranges for the dimension. For each range, we will need a suitable Hardy-Rellich type inequality that will be established in the last section, by using the recent results of Ghoussoub-Moradifam [12].

• **Case** $N \geq 16$**:** To establish the singularity of u^* for these dimensions we shall need the classical Hardy-Rellich inequality, which is valid for all $\phi \in H^2(B) \cap H_0^1(B)$:

$$\int_B (\Delta\phi)^2 \, dx \geq \frac{N^2(N-4)^2}{16} \int_B \frac{\phi^2}{|x|^4} \, dx. \tag{6.14}$$

• **Case** $10 \leq N \leq 16$**:** For this case, we shall need the following inequality valid for all $\phi \in H^2(B) \cap H_0^1(B)$

$$\int_B (\Delta\phi)^2 \geq \frac{(N-2)^2(N-4)^2}{16} \int_B \frac{\phi^2}{(|x|^2 - \frac{N}{2(N-1)}|x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} + \frac{(N-1)(N-4)^2}{4} \int_B \frac{\phi^2}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}. \tag{6.15}$$

• **Case** $N = 9$**:** This case is the trickiest and will require the following inequality for all $\phi \in H^2(B) \cap H_0^1(B)$, which is valid for $N \geq 7$

$$\int_B |\Delta u|^2 \geq \int_B W(|x|)u^2. \tag{6.16}$$

where where

$$W(r) = K(r)\Big(\frac{(N-2)^2}{4(r^2 - \frac{N}{2(N-1)}r^{\frac{N}{2}+1})} + \frac{(N-1)}{r^2}\Big),$$

$$K(r) = -\frac{\varphi''(r) + \frac{(n-3)}{r}\varphi'(r)}{\varphi(r)},$$

and

$$\varphi(r) = r^{-\frac{N}{2}+2} + 9r^{-2} + 10r - 20.$$

The next lemma will be our main tool to guarantee that u^* is singular for $N \geq 9$. The proof is based on an upper estimate by a singular stable sub-solution.

Lemma 6.17. *Suppose there exist* $\lambda' > 0$ *and a radial function* $u \in H^2(B) \cap W^{4,\infty}_{loc}(B \setminus \{0\})$ *such that*

$$\Delta^2 u \leq \frac{\lambda'}{(1-u)^2} \quad \textit{for} \quad 0 < r < 1, \tag{6.18}$$

$$u(1) = 0, \qquad \Delta u|_{r=1} = 0, \tag{6.19}$$

$$u \textit{ is singular}, \tag{6.20}$$

and

$$2\beta \int_B \frac{\varphi^2}{(1-u)^3} \leq \int_B (\Delta\varphi)^2 \quad \textit{for all} \quad \varphi \in H^2(B) \cap H_0^1(B), \tag{6.21}$$

for some $\beta > \lambda'$*. Then* u^* *is singular and*

$$\lambda^* \leq \lambda' \tag{6.22}$$

Proof. By Lemma 6.9 we have (6.22). Let $\frac{\lambda'}{\beta} < \gamma < 1$ and

$$\alpha := \Big(\frac{\gamma\lambda^*}{\lambda'}\Big)^{1/3}, \tag{6.23}$$

and define $\bar{u} := 1 - \alpha(1-u)$. We claim that

$$u^* \leq \bar{u} \quad \text{in} \quad B. \tag{6.24}$$

To prove this, we shall show that for $\lambda < \lambda^*$

$$u_\lambda \leq \bar{u} \quad \text{in} \quad B. \tag{6.25}$$

Indeed, we have

$$\Delta^2(\bar{u}) = \alpha\Delta^2(\bar{u}) \leq \frac{\alpha\lambda'}{(1-u)^2} = \frac{\alpha^3\lambda'}{(1-\bar{u})^2}.$$

By (6.22) and the choice of α

$$\alpha^3\lambda' < \lambda^*.$$

To prove (6.24) it suffices to prove it for $\alpha^3\lambda' < \lambda < \lambda^*$. Fix such λ and assume that (6.24) is not true. Then

$$\Lambda = \{0 \leq R \leq 1 \mid u_\lambda(R) > \bar{u}(R)\},$$

in non-empty. There exists $0 < R_1 < 1$, such that $u_\lambda(R_1) \geq u^*(R_1)$ and $\Delta u_\lambda(R_1) \leq \Delta u^*(R_1)$, since otherwise we can find $0 < s_1 < s_2 < 1$ so that $u_\lambda(s_1) = \bar{u}(s_1)$, $u_\lambda(s_2) = \bar{u}(s_2)$, $u_\lambda(R) > \bar{u}(R)$, and $\Delta u_\lambda(R_1) > \Delta u^*(R_1)$ which contradict the maximum principle. Now consider the following problem

$$\begin{aligned} \Delta^2 u &= \frac{\lambda}{(1-u)^2} \quad \text{in} \quad B \\ u &= u_\lambda(R_1) \quad \text{on} \quad \partial B \\ \Delta u &= \Delta u_\lambda \quad \text{on} \quad \partial B. \end{aligned}$$

Then u_λ is a solution to the above problem while $\bar{u}$ is a sub-solution to the same problem. Moreover $\bar{u}$ is stable since,

$$\lambda < \lambda^*$$

and hence

$$\frac{2\lambda}{(1-\bar{u})^3} \leq \frac{2\lambda^*}{\alpha^3(1-u)^3} = \frac{2\lambda'}{\gamma(1-u)^3} < \frac{2\beta}{(1-u)^3}.$$

We deduce $\bar{u} \leq u_\lambda$ in B_{R_1} which is impossible, since $\bar{u}$ is singular while u_λ is smooth. This establishes (6.24). From (6.24) and the above two inequalities we have

$$\frac{2\lambda^*}{(1-u^*)^3} \leq \frac{2\lambda'}{\gamma(1-u)^3} < \frac{\beta}{(1-u)^3}.$$

Thus

$$\inf_{\varphi \in C_0^\infty}(B)\frac{\int_B (\Delta\varphi)^2 - \frac{2\lambda^*\varphi^2}{(1-u^*)^3}}{\int_B \varphi^2} > 0.$$

This is not possible if u^* is a smooth solution. □

For any $m > \frac{4}{3}$ define

$$w_m := 1 - a_{N,m} r^{\frac{4}{3}} + b_{N,m} r^m,$$

where

$$a_{N,m} := \frac{m(N+m-2)}{m(N+m-2) - \frac{4}{3}(N-2/3)},$$

and

$$b_{N,m} := \frac{\frac{4}{3}(N-2/3)}{m(N+m-2) - \frac{4}{3}(N-2/3)}.$$

Now we are ready to prove the main result of this section.

The following upper bounds on λ^* hold in large dimensions.

1. If $N \geq 31$, then Lemma 6.17 holds with $u := w_2$, $\lambda'_N = 27\bar{\lambda}$ and $\beta = \frac{H_N}{2} > 27\bar{\lambda}$.

2. If $16 \leq N \leq 30$, then Lemma 6.17 holds with $u := w_3$, $\lambda'_N = \frac{H_N}{2} - 1, \beta_N = \frac{H_N}{2}$.

3. If $10 \leq N \leq 15$, then Lemma 6.17 holds with $u := w_3$, $\lambda'_N < \beta_N$ given in Table 6.1.

4. If $N = 9$, then Lemma 6.17 holds with $u := w_{2.8}$, $\lambda'_9 := 249 < \beta_9 := 251$.

The extremal solution is therefore singular for dimensions $N \geq 9$.

Proof. 1) Assume first that $N \geq 31$, then it is easy to see that $a_{N,2} < 3$ and $a_{N,2}^3 \bar{\lambda} \leq 27\bar{\lambda} < \frac{H_N}{2}$. We shall show that w_2 is a singular $\mathcal{H}-$weak sub-solution of $(P)_{a_{N,2}^3 \bar{\lambda}}$ which is stable. Note that $w_2 \in H^2(B)$, $\frac{1}{1-w_2} \in L^3(B)$, $0 \leq w_2 \leq 1$ in B, and

$$\Delta^2 w_2 \leq \frac{a_{N,2}^3 \bar{\lambda}}{(1-w_2)^2} \text{ in } B \setminus \{0\}.$$

So w_2 is a $\mathcal{H}-$weak sub-solution of $(P)_{27\bar{\lambda}}$. Moreover,

$$w_2 = 1 - |x|^{\frac{4}{3}} + (a_{N,2} - 1)(|x|^{\frac{4}{3}} - |x|^2) \leq 1 - |x|^{\frac{4}{3}}.$$

Since $27\bar{\lambda} \leq \frac{H_N}{2}$, we get that

$$54\bar{\lambda} \int_B \frac{\varphi^2}{(1-w_2)^3} \leq H_N \int_B \frac{\varphi^2}{(1-w_2)^3} \leq H_N \int_B \frac{\varphi^2}{|x|^4} \leq \int_B (\Delta\varphi)^2$$

for all $\varphi \in C_0^\infty(B)$. Hence, w_2 is stable. Thus it follows from Lemma 6.17 that u^* is singular and $\lambda^* \leq 27\bar{\lambda}$.

2) Assume $16 \leq N \leq 30$ and consider

$$w_3 := 1 - a_{N,3} r^{\frac{4}{3}} + b_{N,3} r^3.$$

We show that it is a singular $\mathcal{H}-$weak sub-solution of $(P_{\frac{H_N}{2}-1})$ which is stable. Indeed, we clearly have $0 \leq w_3 \leq 1$ a.e. in B, $w_3 \in H^2(B)$ and $\frac{1}{1-w_3} \in L^3(B)$. Note that

$$\begin{aligned} H_N \int_B \frac{\varphi^2}{(1-w_3)^3} &= H_N \int_B \frac{\varphi^2}{(a_{N,m} r^{\frac{4}{3}} - b_{N,m} r^m)^3} \\ &\leq \sup_{0<r<1} \frac{H_N}{(a_{N,m} - b_{N,m} r^{m-\frac{4}{3}})^3} \int_B \frac{\varphi^2}{r^4} \\ &= H_N \int_B \frac{\varphi^2}{r^4} \leq \int_B (\Delta\varphi)^2. \end{aligned}$$

Using maple one can verify that for $16 \leq N \leq 31$

$$\Delta^2 w_3 \leq \frac{\frac{H_N}{2} - 1}{(1-w_3)^2} \quad \text{on} \quad (0,1).$$

Hence w_3 is a sub-solution of $(P_{\frac{H_N}{2}-1})$. By Lemma 6.17 u^* is singular and $\lambda^* \leq \frac{H_N}{2} - 1$.

3) Assume $10 \leq N \leq 15$. We shall show that w_3 satisfies the assumptions of Lemma 6.17 for each dimension $10 \leq N \leq 15$. Using maple, for each dimension $10 \leq N \leq 15$, one can verify that inequality (6.26) holds for λ'_N given by Table 6.1. Then, by using maple again, we show that there exists $\beta_N > \lambda'_N$ such that

$$\begin{aligned} &\frac{(N-2)^2(N-4)^2}{16} \frac{1}{(|x|^2 - \frac{N}{2(N-1)}|x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} \\ &+ \frac{(N-1)(N-4)^2}{4} \frac{1}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})} \geq \frac{2\beta_N}{(1-w_3)^3}. \end{aligned}$$

The above inequality and improved Hardy-Rellich inequality (6.40) guarantee that the stability condition (6.29) holds for $\beta_N > \lambda'$. Hence by Lemma 6.17 the extremal solution is singular for $10 \leq N \leq 15$. The values of λ_N and β_N are shown in Table 6.1.

4) Let u:=$w_{2.8}$. Using Maple on can see that

$$\Delta^2 u \leq \frac{249}{(1-u)^2} \quad \text{in} \quad B$$

and

$$\frac{502}{(1-u(r))^3} \leq W(r) \quad \text{for all} \quad r \in (0,1),$$

where W is given by (6.42). Since, $502 > 2 \times 249$, by Lemma 6.17 the extremal solution u^* is singular in dimension $N = 9$. □

Remark 6.5.1. *It follows from the proof of Theorem 6.5 that for $N \geq 9$ and $\frac{\tau}{\beta}$ sufficiently small, there exists $u \in H^2(B) \cap W^{4,\infty}_{loc}(B \setminus \{0\})$ such that*

$$\Delta^2 u - \frac{\tau}{\beta}\Delta u \leq \frac{\lambda''_N}{(1-u)^2} \quad \textit{for} \quad 0 < r < 1, \tag{6.26}$$

Table 6.1: Summary 1

N	λ'_N	β_N
9	249	251
10	320	367
11	405	574
12	502	851
13	610	1211
14	730	1668
15	860	2235
$16 \leq N \leq 30$	$\frac{H_N}{2} - 1$	$\frac{H_N}{2}$
$N \geq 31$	$27\bar{\lambda}$	$\frac{H_N}{2}$

$$u(1) = 0, \qquad \Delta u|_{r=1} = 0, \tag{6.27}$$

$$u \textit{ is singular}, \tag{6.28}$$

and

$$2\beta'_N \int_B \frac{\varphi^2}{(1-u)^3} \leq \int_B (\Delta\varphi)^2 + \frac{\tau}{\beta}|\nabla\varphi|^2 \quad \textit{for all} \quad \varphi \in H^2(B) \cap H^1_0(B), \tag{6.29}$$

where $\beta'_N > \lambda''_N > 0$ are constants. Indeed, for each dimension $N \geq 9$, it is enough to take u to be the sub-solution we constructed in the proof of Theorem 6.5, $\beta'_N := \beta_N$, $\lambda' < \lambda'' < \beta$. If $\frac{\tau}{\beta}$ is sufficiently small so that $-\frac{\tau}{\beta}\Delta u < \frac{\lambda''-\lambda'}{(1-u)^2}$ on $(0,1)$, then with an argument similar to that of Lemma 6.17 we deduce that the extremal solution u^* of $(P_{\lambda,\beta,\tau,0,0})$ is singular. We believe that the extremal solution of $(P_{\lambda,\beta,\tau,0,0})$ is singular for all $\beta, \tau > 0$ in dimensions $N \geq 9$.

6.6 Improved Hardy-Rellich Inequalities

We now prove the improved Hardy-Rellich inequalities used in section 4. They rely on the results of Ghoussoub-Moradifam in [12] which provide necessary and sufficient conditions for such inequalities to hold. At the heart of this characterization is the following notion of a Bessel pair of functions. Assume that B is a ball of radius R in ^{N}, $V, W \in C^1(0,1)$, and $\int_0^R \frac{1}{r^{N-1}V(r)} dr = +\infty$. Say that the couple (V, W) is a *Bessel pair on* $(0, R)$ if the ordinary differential equation

$$(\mathrm{B}_{V,W}) \qquad y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)}\right)y'(r) + \frac{W(r)}{V(r)}y(r) = 0$$

has a positive solution on the interval $(0, R)$. The needed inequalities will follow from the following two results. **(Ghoussoub-Moradifam [12])** Let V and W

be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball centered at zero with radius R in ^{N} $(N \geq 1)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)}dr = +\infty$ and $\int_0^R r^{N-1}V(r)dr < +\infty$. The following statements are then equivalent:

1. (V, W) is a Bessel pair on $(0, R)$.
2. $\int_B V(|x|)|\nabla\phi|^2 dx \geq \int_B W(|x|)\phi^2 dx$ for all $\phi \in C_0^\infty(B)$.

Let B be the unit ball in ^{N} $(N \geq 5)$. Then the inequality

$$\int_B |\Delta u|^2 dx \geq \int_B \frac{|\nabla u|^2}{|x|^2 - \frac{N}{2(N-1)}|x|^{\frac{N}{2}+1}}dx + (N-1)\int_B \frac{|\nabla u|^2}{|x|^2}dx, \tag{6.30}$$

holds for all $u \in C_0^\infty(\bar{B})$.

We shall need the following result to prove (6.30).

Lemma 6.31. *For every $u \in C^1([0,1])$ the following inequality holds*

$$\int_0^1 |u'(r)|^2 r^{N-1} dr \geq \int_0^1 \frac{u^2}{r^2 - \frac{N}{2(N-1)}r^{\frac{N}{2}+1}} r^{N-1} dr - (N-1)(u(1))^2. \tag{6.32}$$

Proof. Let $\varphi := r^{-\frac{N}{2}+1} - \frac{N}{2(N-1)}$ and $k(r) := r^{N-1}$. Define $\psi(r) = u(r)/\varphi(r)$, $r \in [0,1]$. Then

$$\begin{aligned}
\int_0^1 |u'(r)|^2 k(r) dr &= \int_0^1 |\psi(r)|^2 |\varphi'(r)|^2 k(r) dr \\
&+ \int_0^1 2\varphi(r)\varphi'(r)\psi(r)\psi'(r)k(r)dr + \int_0^1 |\varphi(r)|^2|\psi'(r)|^2 k(r)dr \\
&= \int_0^1 |\psi(r)|^2(|\varphi'(r)|^2 k(r) - (k\varphi\varphi')'(r))dr \\
&+ \int_0^1 |\varphi(r)|^2|\psi'(r)|^2 k(r)dr + \psi^2(1)\varphi'(1)\varphi(1) \\
&\geq \int_0^1 |\psi(r)|^2(|\varphi'(r)|^2 k(r) - (k\varphi\varphi')'(r))dr + \psi^2(1)\varphi'(1)\varphi(1)
\end{aligned}$$

Note that $\psi^2(1)\varphi'(1)\varphi(1) = u^2(1)\frac{\varphi'(1)}{\varphi(1)} = -(N-1)u^2(1)$. Hence, we have

$$\int_0^1 |u'(r)|^2 k(r) dr \geq \int_0^1 -u^2(r)\frac{k'(r)\varphi'(r) + k(r)\varphi''(r)}{\varphi})dr - (N-1)u^2(1)$$

Simplifying the above inequality we get (6.32). □

The decomposition of a function into its spherical harmonics will be one of our tools to prove Theorem 6.6. Let $u \in C_0^\infty(\bar{B})$. By decomposing u into spherical harmonics we get

$$u = \Sigma_{k=0}^{\infty} u_k \text{ where } u_k = f_k(|x|)\varphi_k(x)$$

and $(\varphi_k(x))_k$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues $c_k = k(N+k-2)$, $k \geq 0$. The functions f_k belong to $u \in C^\infty([0,1])$, $f_k(1) = 0$, and satisfy $f_k(r) = O(r^k)$ and $f'(r) = O(r^{k-1})$ as $r \to 0$. In particular,

$$\varphi_0 = 1 \text{ and } f_0 = \tfrac{1}{N\omega_N r^{N-1}} \int_{\partial B_r} u ds = \tfrac{1}{N\omega_N} \int_{|x|=1} u(rx) ds. \tag{6.33}$$

We also have for any $k \geq 0$, and any continuous real valued W on $(0,1)$,

$$\int_B |\Delta u_k|^2 dx = \int_B \big(\Delta f_k(|x|) - c_k \frac{f_k(|x|)}{|x|^2}\big)^2 dx, \tag{6.34}$$

and

$$\int_B W(|x|)|\nabla u_k|^2 dx = \int_B W(|x|)|\nabla f_k|^2 dx + c_k \int_B W(|x|)|x|^{-2} f_k^2 dx. \tag{6.35}$$

Now we are ready to prove Theorem 6.6. We shall use the inequality

$$\int_0^1 |x'(r)|^2 r^{N-1} dr \geq \tfrac{(N-2)^2}{4} \int_0^1 \tfrac{x^2(r)}{r^2 - \frac{N}{2(N-1)} r^{\frac{N}{2}+1}} r^{N-1} dr \text{ for all } x \in C^1([0,1]),$$

with $x(1) = 0$.

Proof of Theorem 6.6: For all $N \geq 5$ and $k \geq 0$ we have

$$\begin{aligned}
\frac{1}{Nw_N} \int_B |\Delta u_k|^2 dx &= \frac{1}{Nw_N} \int_B \big(\Delta f_k(|x|) - c_k \frac{f_k(|x|)}{|x|^2}\big)^2 dx \\
&= \int_0^1 \big(f_k''(r) + \frac{N-1}{r} f_k'(r) - c_k \frac{f_k(r)}{r^2}\big)^2 r^{N-1} dr \\
&= \int_0^1 (f_k''(r))^2 r^{N-1} dr + (N-1)^2 \int_0^1 (f_k'(r))^2 r^{N-3} dr \\
&\quad + c_k^2 \int_0^1 f_k^2(r) r^{N-5} + 2(N-1) \int_0^1 f_k''(r) f_k'(r) r^{N-2} \\
&\quad - 2c_k \int_0^1 f_k''(r) f_k(r) r^{N-3} dr \\
&\quad - 2c_k(N-1) \int_0^1 f_k'(r) f_k(r) r^{N-4} dr.
\end{aligned}$$

Integrate by parts and use (6.33) for $k=0$ to get

$$\begin{aligned}\frac{1}{N\omega_N}\int_B |\Delta u_k|^2 dx &\geq \int_0^1 (f_k''(r))^2 r^{N-1} dr + (N-1+2c_k)\int_0^1 (f_k'(r))^2 r^{N-3} dr \\ &+ (2c_k(n-4)+c_k^2)\int_0^1 r^{n-5} f_k^2(r) dr \\ &+ (N-1)(f_k^{'}(1))^2 \end{aligned} \tag{6.36}$$

Now define $g_k(r) = \frac{f_k(r)}{r}$ and note that $g_k(r) = O(r^{k-1})$ for all $k \geq 1$. We have

$$\begin{aligned}\int_0^1 (f_k'(r))^2 r^{N-3} &= \int_0^1 (g_k'(r))^2 r^{N-1} dr \\ &+ \int_0^1 2g_k(r)g_k'(r) r^{N-2} dr + \int_0^1 g_k^2(r) r^{N-3} dr \\ &= \int_0^1 (g_k'(r))^2 r^{N-1} dr - (N-3)\int_0^1 g_k^2(r) r^{N-3} dr\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^1 (f_k'(r))^2 r^{N-3} &\geq \frac{(N-2)^2}{4}\int_0^1 \frac{f_k^2(r)}{r^2} \\ &- \frac{N}{2(N-1)} r^{\frac{N}{2}+1} r^{N-3} dr - (N-3)\int_0^1 f_k^2(r) r^{N-5} dr\end{aligned} \tag{6.37}$$

(6.39)

Substituting $2c_k\int_0^1(f_k'(r))^2r^{N-3}$ in (6.36) by its lower estimate in the last inequality (6.37), and using Lemma 6.31 we get

$$\begin{aligned}
\frac{1}{N\omega_N}\int_B|\Delta u_k|^2dx \quad &\geq \quad \frac{(N-2)^2}{4}\int_0^1\frac{(f_k'(r))^2}{r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1}}r^{N-1}dr \\
&+ \quad 2c_k\frac{(N-2)^2}{4}\int_0^1\frac{f_k^2(r)}{r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1}}r^{n-3}dr \\
&+ \quad (N-1)\int_0^1(f_k'(r))^2r^{N-3}dr \\
&+ \quad c_k(N-1)\int_0^1(f_k(r))^2r^{N-5}dr \\
&+ \quad c_k(c_k-(N-1))\int_0^1 r^{N-5}f_k^2(r)dr \\
&+ \quad c_k\int_0^1\frac{(N-2)^2}{4(r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1})}-\frac{2}{r^2})dr. \\
&\geq \quad \frac{(N-2)^2}{4}\int_0^1\frac{(f_k'(r))^2}{r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1}}r^{N-1}dr \\
&+ \quad c_k\frac{(N-2)^2}{4}\int_0^1\frac{f_k^2(r)}{r^2-\frac{N}{2(N-1)}r^{\frac{N}{2}+1}}r^{n-3}dr \\
&+ \quad (N-1)\int_0^1(f_k'(r))^2r^{N-3}dr \\
&+ \quad c_k(N-1)\int_0^1(f_k(r))^2r^{N-5}dr
\end{aligned}$$

The proof is complete in the view of (6.35). □

We shall now deduce the following corollary. Let $N\geq 5$ and B be the unit ball in ^{N}. Then the following improved Hardy-Rellich inequality holds for all $\phi\in H^2(B)\cap H_0^1(B)$:

$$\begin{aligned}
\int_B(\Delta\phi)^2 \quad &\geq \quad \frac{(N-2)^2(N-4)^2}{16}\int_B\frac{\phi^2}{(|x|^2-\frac{N}{2(N-1)}|x|^{\frac{N}{2}+1})(|x|^2-|x|^{\frac{N}{2}})} \\
&+\frac{(N-1)(N-4)^2}{4}\int_B\frac{\phi^2}{|x|^2(|x|^2-|x|^{\frac{N}{2}})}
\end{aligned} \tag{6.40}$$

Proof. Let $\alpha:=\frac{N}{2(N-1)}$ and $V(r):=\frac{1}{r^2-\alpha r^{\frac{N}{2}+1}}$ and note that

$$\frac{V_r}{V}=-\frac{2}{r}+\frac{\alpha(N-2)}{2}\frac{r^{\frac{N}{2}-2}}{1-\alpha r^{\frac{N}{2}-1}}\geq-\frac{2}{r}.$$

The function $y(r) = r^{-\frac{N}{2}+2} - 1$ is decreasing and is then a positive super-solution on $(0,1)$ for the ODE

$$y'' + (\frac{N-1}{r} + \frac{V_r}{V})y'(r) + \frac{W_1(r)}{V(r)}y = 0,$$

where

$$W_1(r) = \frac{(N-4)^2}{4(r^2 - r^{\frac{N}{2}})(r^2 - \alpha r^{\frac{N}{2}+1})}.$$

Hence, by Theorem 6.6 we deduce

$$\int_B \frac{|\nabla\phi|^2}{|x|^2 - \alpha|x|^{\frac{N}{2}+1}} \geq (\frac{N-4}{2})^2 \int_B \frac{\phi^2}{(|x|^2 - \alpha|x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})}$$

for all $\phi \in H^2(B) \cap H_0^1(B)$. Similarly, for $V(r) = \frac{1}{r^2}$ we have that

$$\int_B \frac{|\nabla\phi|^2}{|x|^2} \geq (\frac{N-4}{2})^2 \int_B \frac{\phi^2}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}$$

for all $\phi \in H^2(B) \cap H_0^1(B)$. Combining the above two inequalities with (6.32) we get (6.40). □

Let $N \geq 7$ and B be the unit ball in ^{N}. Then the following improved Hardy-Rellich inequality holds for all $\phi \in H^2(B) \cap H_0^1(B)$:

$$\int_B |\Delta u|^2 \geq \int_B W(|x|)u^2. \tag{6.41}$$

where

$$W(r) = K(r)(\frac{(N-2)^2}{4(r^2 - \frac{N}{2(N-1)}r^{\frac{N}{2}+1})} + \frac{(N-1)}{r^2}), \tag{6.42}$$

$$K(r) = -\frac{\varphi''(r) + \frac{(n-3)}{r}\varphi'(r)}{\varphi(r)},$$

and

$$\varphi(r) = r^{-\frac{N}{2}+2} + 9r^{-2} + 10r - 20.$$

Proof. Let $\alpha := \frac{N}{2(N-1)}$ and $V(r) := \frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}}$. Then φ is a sub-solution for the ODE

$$y'' + (\frac{N-1}{r} + \frac{V_r}{V})y'(r) + \frac{W_2(r)}{V(r)}y = 0,$$

where

$$W_2(r) = \frac{K(r)}{r^2 - \alpha r^{\frac{N}{2}+1}},$$

Hence by Theorem 6.6 we have

$$\int_B \frac{|\nabla u|^2}{|x|^2 - \alpha|x|^{\frac{N}{2}+1}} \geq \int_B W_2(|x|)u^2. \tag{6.43}$$

Similarly

$$\int_B \frac{|\nabla u|^2}{|x|^2} \geq \int_B W_3(|x|)u^2. \tag{6.44}$$

where

$$W_3(r) = \frac{K(r)}{r^2}.$$

Combining the above two inequalities with (6.32) we get improved Hardy-Rellich inequality (6.41). □

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Chapter 7

The singular extremal solutions of the bilaplacian with exponential nonlinearity
6

7.1 Introduction

Consider the fourth order elliptic problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases} \tag{7.1}$$

where B is the unit ball in ^{N}, $N \geq 1$, n is the exterior unit normal vector and $\lambda \geq 0$ is a parameter. This problem is the fourth order analogue of the classical Gelfand problem (see [2], [4], and [9]). Recently, many authors are intrested in fourth order equations and interesting results can be found in [1], [2], [3], [5], [8], [10], [11] and the references cited therein. In [1], Arioli et al. studied the problem (7.1) and showed that for each dimension $N \geq 1$ there exists a $\lambda^* > 0$ such that for every $0 < \lambda < \lambda^*$, there exists a smooth minimal (smallest) solution of (7.1), while for $\lambda > \lambda^*$ there is no solution even in a weak sense. Moreover, the branch $\lambda \mapsto u_\lambda(x)$ is increasing for each $x \in B$, and therefore the function $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ can be considered as a generalized solution that corresponds to λ^*. Now the important question is whether u^* is regular ($u^* \in L^\infty(B)$) or singular ($u^* \notin L^\infty(B)$). Even though there are similarities between (7.1) and the Gelfand problem, several tools which have been developed for the Gelfand problem, are no longer available for (7.1). In [5] the authors developed a new method to prove the regularity of the extremal solutions in low dimensions and showed that for $N \leq 12$, u^* is regular. But unlike the Gelfand problem the natural candidate $u = -4\ln(|x|)$, for the extremal solution, does not satisfy the boundary conditions and hence showing the singular nature of the extremal solution in large dimensions close

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to the critical dimension is challenging. Dávila et al. [5] used a computer assisted proof to show that the extremal solution is singular in dimensions $13 \leq N \leq 31$ while they gave a mathematical proof in dimensions $N \geq 32$. In this paper we introduce a unified mathematical approach to deal with this problem and show that for $N \geq 13$, the extremal solution is singular. One of our main tools is an improved Hardy-Rellich inequality that follows from the recent result of Ghoussoub-Moradifam about improved Hardy and Hardy-Rellich inequalities developed in [7] and [6].

7.2 An improved Hardy-Rellich inequality

In this section we shall prove an improvement of classical Hardy-Rellich inequality which will be used to prove our main result in Section 3. It relies on the results of Ghoussoub-Moradifam in [6] which provide necessary and sufficient conditions for such inequalities to hold. At the heart of this characterization is the following notion of a Bessel pair of functions. Assume that B is a ball of radius R in ^{N}, $V, W \in C^1(0,1)$, and $\int_0^R \frac{1}{r^{N-1}V(r)}dr = +\infty$. Say that the couple (V, W) is a *Bessel pair on* $(0, R)$ if the ordinary differential equation

$$(\mathrm{B}_{V,W}) \qquad y''(r) + (\tfrac{N-1}{r} + \tfrac{V_r(r)}{V(r)})y'(r) + \tfrac{W(r)}{V(r)}y(r) = 0$$

has a positive solution on the interval $(0, R)$.

(**Ghoussoub-Moradifam** [6]) Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball centered at zero with radius R in ^{N} $(N \geq 1)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)}dr = +\infty$ and $\int_0^R r^{N-1}V(r)dr < +\infty$. The following statements are then equivalent:

1. (V, W) is a Bessel pair on $(0, R)$ and $\beta(V, W; R) \geq 1$.
2. $\int_B V(x)|\nabla u|^2 dx \geq \int_B W(x)u^2 dx$ for all $u \in C_0^\infty(B)$.
3. If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < N-2$ and $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ on $(0, R)$, then the above are equivalent to

$$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (N-1)\int_B (\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|})|\nabla u|^2 dx,$$

for all $u \in C_0^\infty(B)$.

As an application we have the following improvement of the classical Hardy-Rellich inequality.

Let $N \geq 5$ and B be the unit ball in ^{N}. Then the following improved Hardy-Rellich inequality holds for all $u \in C_0^\infty(B)$.

$$\begin{aligned}\int_B |\Delta u|^2 \quad &\geq \quad \frac{(N-2)^2(N-4)^2}{16}\int_B \frac{u^2}{(|x|^2 - \frac{9}{10}|x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} \\ &+ \quad \frac{(N-1)(N-4)^2}{4}\int_B \frac{u^2}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}. \end{aligned} \tag{7.2}$$

As a consequence the following improvement of classical Hardy-Rellich inequality holds:

$$\int_B |\Delta u|^2 \geq \frac{N^2(N-4)^2}{16}\int_B \frac{u^2}{|x|^2(|x|^2-|x|^{\frac{N}{2}})}. \tag{7.3}$$

Proof. Let $\varphi := r^{-\frac{N}{2}+1} - \frac{9}{10}$. Since

$$-\frac{\varphi'' + \frac{(N-1)}{r}\varphi'}{\varphi} = \frac{(N-2)^2}{4}.\frac{1}{r^2 - \frac{9}{10}r^{\frac{N}{2}+1}},$$

$(1, \frac{(N-2)^2}{4}\frac{1}{r^2-\frac{9}{10}r^{\frac{N}{2}+1}})$ is a bessel pair on $(0,1)$. By Theorem 7.2 the following inequality holds for all $u \in C_0^\infty(B)$.

$$\int_B |\Delta u|^2 dx \geq \frac{(N-2)^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2 - \frac{9}{10}|x|^{\frac{N}{2}+1}} + (N-1)\int_B \frac{|\nabla u|^2}{|x|^2}. \tag{7.4}$$

Let $V(r) := \frac{1}{r^2-\frac{9}{10}r^{\frac{N}{2}+1}}$. Then

$$\frac{V_r}{V} = -\frac{2}{r} + \frac{9}{10}(\frac{N-2}{2})\frac{r^{\frac{N}{2}-2}}{1-\frac{9}{10}r^{\frac{N}{2}-1}} \geq -\frac{2}{r}, \tag{7.5}$$

and $\psi(r) = r^{-\frac{N}{2}+2} - 1$ is a positive super-solution for the ODE

$$y'' + (\frac{N-1}{r} + \frac{V_r}{V})y'(r) + \frac{W_1(r)}{V(r)}y = 0, \tag{7.6}$$

where

$$W_1(r) = \frac{(N-4)^2}{4(r^2 - r^{\frac{N}{2}})(r^2 - \frac{9}{10}r^{\frac{N}{2}+1})}.$$

Hence the ODE (7.6) has actually a positive solution and by Theorem 7.2 we have

$$\int_B \frac{|\nabla u|^2}{|x|^2 - \frac{9}{10}|x|^{\frac{N}{2}+1}} \geq (\frac{N-4}{2})^2\int_B \frac{u^2}{(|x|^2-\frac{9}{10}|x|^{\frac{N}{2}+1})(|x|^2-|x|^{\frac{N}{2}})}. \tag{7.7}$$

Similarly

$$\int_B \frac{|\nabla u|^2}{|x|^2} \geq (\frac{N-4}{2})^2\int_B \frac{u^2}{|x|^2(|x|^2-|x|^{\frac{N}{2}})}. \tag{7.8}$$

Combining the above two inequalities with (7.4) we get (7.2). □

7.3 Main results

In this section we shall prove that the extremal solution u^* of the problem (7.1) is singular in dimensions $N \geq 13$. The next lemma will be our main tool to guarantee that u^* is singular for $N \geq 13$. The proof is based on an upper estimate by a singular stable sub-solution.

Lemma 7.9. *Suppose there exist $\lambda' > 0$ and a radial function $u \in H^2(B) \cap W^{4,\infty}_{loc}(B \setminus \{0\})$ such that*

$$\Delta^2 u \leq \lambda' e^u \quad \textit{for all} \quad 0 < r < 1, \tag{7.10}$$

$$u(1) = 0, \qquad \frac{\partial u}{\partial n}(1) = 0, \tag{7.11}$$

$$u \notin L^\infty(B), \tag{7.12}$$

and

$$\beta \int_B e^u \varphi^2 \leq \int_B (\Delta\varphi)^2 \quad \textit{for all} \quad \varphi \in C_0^\infty(B), \tag{7.13}$$

for some $\beta > \lambda'$. Then u^* is singular and

$$\lambda^* \leq \lambda' \tag{7.14}$$

Proof. By Lemma 2.6 in [5] we have (7.14). Define

$$\alpha := \ln(\frac{\lambda'}{\gamma\lambda^*}), \tag{7.15}$$

where $\frac{\lambda'}{\beta} < \gamma < 1$ and let $\bar{u} := u + \alpha$. We claim that

$$u^* \leq \bar{u} \quad \text{in} \quad B. \tag{7.16}$$

To prove this, we shall show that for $\lambda < \lambda^*$

$$u_\lambda \leq \bar{u} \quad \text{in} \quad B. \tag{7.17}$$

Indeed, we have

$$\Delta^2(\bar{u}) = \Delta^2(u) \leq \lambda' e^u = \lambda' e^{-\alpha} e^{\bar{u}} = \gamma\lambda^* e^{\bar{u}}.$$

To prove (7.16) it suffices to prove it for $\gamma\lambda^* < \lambda < \lambda^*$. Fix such λ and assume that (7.16) is not true. Let

$$R_1 := \sup\{0 \leq R \leq 1 \mid u_\lambda(R) = \bar{u}(R)\}.$$

Since $\bar{u}(1) = \alpha > 0 = u_\lambda(1)$, we have $0 < R_1 < 1$, $u_\lambda(R_1) = \bar{u}(R_1)$, and $u'_\lambda(R_1) \leq \bar{u}'(R_1)$. Now consider the following problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } \Omega \\ u = u_\lambda(R_1) & \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} = u'_\lambda(R_1) & \text{on } \partial\Omega. \end{cases}$$

Obviously u_λ is a solution to the above problem while $\bar{u}$ is a sub-solution to the same problem. Moreover $\bar{u}$ is stable since,

$$\lambda < \lambda^*$$

and hence

$$\lambda e^{\bar{u}} \leq \lambda^* e^{\alpha} e^{u} = \frac{\lambda'}{\gamma} e^{u} < \beta e^{u}.$$

We deduce $\bar{u} \leq u_\lambda$ in B_{R_1} which is impossible, since $\bar{u}$ is singular while u_λ is smooth. This establishes (7.16). From (7.16) and the above two inequalities we have

$$\lambda^* e^{u^*} \leq \lambda^* e^{a} e^{u} = \frac{\lambda'}{\gamma} e^{u}.$$

Since $\frac{\lambda'}{\gamma} < \beta$,

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \lambda^* e^{u^*}}{\int_B \varphi^2} > 0.$$

This is not possible if u^* is a smooth solution. □

In the following, for each dimension $N \geq 13$, we shall construct u satisfying all the assumptions of Lemma 7.9. Define

$$w_m := -4\ln(r) - \frac{4}{m} + \frac{4}{m} r^m, \quad m > 0,$$

and let $H_N := \frac{N^2(N-4)^2}{16}$. Now we are ready to prove our main result.

The following upper bounds on λ^* hold in large dimensions.

1. If $N \geq 32$, then Lemma 7.9 holds with $u := w_2$, $\lambda'_N = 8(N-2)(N-4)e^2$ and $\beta = H_N > \lambda'_N$.
2. If $13 \leq N \leq 31$, then Lemma 7.9 holds with $u := w_{3.5}$ and $\lambda'_N < \beta_N$ given in Table 1.

The extremal solution is therefore singular for dimensions $N \geq 13$.

Proof. 1) Assume first that $N \geq 32$, then

$$8(N-2)(N-4)e^2 < \frac{N^2(N-4)^2}{16},$$

and

$$\Delta^2 w_2 = 8(N-2)(N-4)\frac{1}{r^4} \leq 8(N-2)(N-4)e^2 e^{w_2}.$$

Moreover,

$$8(N-2)(N-4)e^2 \int_B e^{w_2}\varphi^2 \leq H_n \int_B e^{-4\ln(|x|)}\varphi^2 = H_n \int_B \frac{\varphi^2}{|x|^2} \leq \int_B |\Delta\varphi|^2.$$

Thus it follows from Lemma 7.9 that u^* is singular and $\lambda^* \leq 8(N-2)(N-4)e^2$.

2) Assume $13 \leq N \leq 31$. We shall show that $u = w_{3.5}$ satisfies the assumptions of Lemma 7.9 for each dimension $13 \leq N \leq 31$. Using Maple, for each dimension $13 \leq N \leq 31$, one can verify that inequality (7.10) holds for λ'_N given by Table 7.1. Then, by using Maple again, we show that there exists $\beta_N > \lambda'_N$ such that

$$\begin{aligned}
&\frac{(N-2)^2(N-4)^2}{16}\frac{1}{(|x|^2-0.9|x|^{\frac{N}{2}+1})(|x|^2-|x|^{\frac{N}{2}})}\\
&+\frac{(N-1)(N-4)^2}{4}\frac{1}{|x|^2(|x|^2-|x|^{\frac{N}{2}})}\\
&\geq \beta_N e^{w_{3.5}}.
\end{aligned}$$

The above inequality and improved Hardy-Rellich inequality (7.2) guarantee that the stability condition (7.13) holds for $\beta_N > \lambda'$. Hence by Lemma 7.9 the extremal solution is singular for $13 \leq N \leq 31$. The values of λ_N and β_N are shown in Table 7.1.

Table 7.1: Summary 3

N	λ'_N	β_N
$N \geq 32$	$8(N-2)(N-4)e^2$	H_n
31	20000	86900
30	18500	76500
29	17000	67100
28	16000	58500
27	14500	50800
26	13500	43870
25	12200	37630
24	11100	32050
23	10100	27100
22	9050	22730
21	8150	18890
20	7250	15540
19	6400	12645
18	5650	10155
17	4900	8035
16	4230	6250
15	3610	4765
14	3050	3545
13	2525	2560

Remark 7.3.1. *The values of λ'_N and β_N in Table 7.1 are not optimal.*

Remark 7.3.2. *The improved Hardy-Rellich inequality (7.2) is crucial to prove that u^* is singular in dimensions $N \geq 13$. Indeed by the classical Hardy-Rellich inequality and $u := w_{3.5}$, Lemma 7.9 only implies that u^* is singular in dimensions $N \geq 22$.*

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Part III

Preconditioning of Nonsymmetric Linear Systems

Chapter 8

Simultaneous preconditioning and symmetrization of non-symmetric linear systems [7]

8.1 Introduction and main results

Many problems in scientific computing lead to systems of linear equations of the form,

$$Ax = b, \tag{8.1}$$

where $A \in R^{n\times n}$ is a nonsingular but sparse matrix, and b is a given vector in R^n and various iterative methods have been developed for a fast and efficient resolution of such systems. The Conjugate Gradient Method (CG) which is the oldest and best known of the nonstationary iterative methods, is highly effective in solving symmetric positive definite systems. For indefinite matrices, the minimization feature of CG is no longer an option, but the Minimum Residual (MINRES) and the Symmetric LQ (SYMMLQ) methods are often computational alternatives for CG, since they are applicable to systems whose coefficient matrices are symmetric but possibly indefinite.

The case of non-symmetric linear systems is more challenging, and again methods such as CGNE, CGNR, GMRES, BiCG, QMR, CGS, and Bi-CGSTAB have been developed to deal with these situations (see the survey books [9] and [11]). One approach to deal with the non-symmetric case, consists of reducing the problem to a symmetric one to which one can apply the above mentioned schemes. The one that is normally used consists of simply applying CG to the normal equations

$$A^TAx = A^Tb \quad \text{or} \quad AA^Ty = b, \quad x = A^Ty. \tag{8.2}$$

[7] A version of this chapter has been accepted for publication. N. Ghoussoub and A. Moradifam, Simultaneous preconditioning and symmetrization of non-symmetric linear systems, Numer. Linear Algebra Appl., to appear (2010).

It is easy to understand and code this approach, and the CGNE and CGNR methods are based on this idea. However, the convergence analysis of these methods depends closely on the *condition number* of the matrix under study. For a general matrix A, the condition number is defined as

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|, \tag{8.3}$$

and in the case where A is positive definite and symmetric, the condition number is then equal to

$$\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}, \tag{8.4}$$

where $\lambda_{\min}(A)$ (resp., $\lambda_{\max}(A)$) is the smallest (resp., largest) eigenvalue of A). The two expressions can be very different for non-symmetric matrices, and these are precisely the systems that seem to be the most pathological from numerical point of view. Going back to the crudely symmetrized system (8.2), we echo Greenbaum's statement [9] that numerical analysts *cringe* at the thought of solving these normal equations because the *condition number* (see below) of the new matrix $A^T A$ is the square of the condition number of the original matrix A.

In this paper, we shall follow a similar approach that consists of symmetrizing the problem so as to be able to apply CG, MINRES, or SYMMLQ. However, we argue that for a large class of non-symmetric, ill-conditionned matrices, it is sometimes beneficial to replace problem (8.1) by one of the form

$$A^T M A x = A^T M b, \tag{8.5}$$

where M is a symmetric and positive definite matrix that can be chosen properly so as to obtain good convergence behavior for CG when it is applied to the resulting symmetric $A^T M A$. This reformulation should not only be seen as a symmetrization, but also as preconditioning procedure. While it is difficult to obtain general conditions on M that ensure higher efficiency by minimizing the condition number $\kappa(A^T M A)$, we shall show theoretically and numerically that by choosing M to be either the inverse of the symmetric part of A, or its resolvent, one can get surprisingly good numerical schemes to solve (8.1).

The basis of our approach originates from the selfdual variational principle developed in [6, 7] to provide a variational formulation and resolution for non self-adjoint partial differential equations that do not normally fit in the standard Euler-Lagrangian theory. Applied to the linear system (8.1), the new principle yields the following procedure. Split the matrix A into its symmetric A_a (resp., anti-symmetric part A_a)

$$A = A_s + A_a, \tag{8.6}$$

where

$$A_s := \frac{1}{2}(A + A^T) \text{ and } A_a := \frac{1}{2}(A - A^T). \tag{8.7}$$

Proposition 8.8. (Selfdual symmetrization) *Assume the matrix A is positive definite, i.e., for some $\delta > 0$,*

$$\langle Ax, x\rangle \geq \delta |x|^2 \textit{ for all } x \in R^n. \tag{8.9}$$

The convex continuous functional

$$I(x) = \frac{1}{2}\langle Ax, x\rangle + \frac{1}{2}\langle A_s^{-1}(b - A_a x), b - A_a x\rangle - \langle b, x\rangle \tag{8.10}$$

then attains its minimum at some $\bar{x}$ in R^n, in such a way that

$$I(\bar{x}) = \inf_{x\in R^n} I(x) = 0 \tag{8.11}$$

$$A\bar{x} = b. \tag{8.12}$$

Here $\langle x, y\rangle = x^T y$ and $|x|^2 = \langle x, x\rangle$.

Symmetrization and preconditioning via selfduality: Note that the functional I can be written as

$$I(x) = \frac{1}{2}\langle \tilde{A}x, x\rangle + \langle A_a A_s^{-1}b - b, x\rangle + \frac{1}{2}\langle A_s^{-1}b, b\rangle, \tag{8.13}$$

where

$$\tilde{A} := A_s - A_a A_s^{-1} A_a = A^T A_s^{-1} A. \tag{8.14}$$

By writing that $DI(\bar{x}) = 0$ (DI is the subdifferential of the functional I), one gets the following equivalent way of solving (8.1).

If both $A \in R^{n\times n}$ and its symmetric part A_s are nonsingular, then x is a solution of the equation (8.1) if and only if it is a solution of the linear symmetric equation

$$A^T A_s^{-1} A x = (A_s - A_a A_s^{-1} A_a)x = b - A_a A_s^{-1} b = A^T A_s^{-1} b. \tag{8.15}$$

One can therefore apply to (8.15) all known iterative methods for symmetric systems to solve the non-symmetric linear system (8.1). As mentioned before, the new equation (8.15) can be seen as a new symmetrization of problem (8.1) which also preserves positivity, i.e., $A^T A_s^{-1} A$ is positive definite if A is. This will then allow for the use of the Conjugate Gradient Method (CG) for the functional I. More important and less obvious than the symmetrization effect of $\tilde{A}$, is our observation that for a large class of matrices, the convergence behavior of the system (8.15) is more favorable than the original one. The Conjugate Gradient method –which can now be applied to the symmetrized matrix $\tilde{A}$– has the potential of providing an efficient algorithm for resolving non-symmetric linear systems. We shall call this scheme the *Self-Dual Conjugate Gradient for Non-symmetric matrices* and we will refer to it as SD-CGN.

As mentioned above, the convergence analysis of this method depends closely on the condition number $\kappa(\tilde{A})$ of $\tilde{A} = A^T A_s^{-1} A$. We observe in section 2.3 that for a large class of ill-conditioned matrices, $\kappa(\tilde{A})$ may be very small and hence SD-CGN can be very efficient. In other words, the inverse C of $A^T A_s^{-1}$ can be an efficient preconditioning matrix, in spite of the additional cost involved in finding the inverse of A_s. Moreover, the efficiency of C seems to surprisingly improve in many cases as the norm of the anti-symmetric part gets larger (Proposition 2.2). A typical example is when the anti-symmetric matrix A_a is a multiple of the symplectic matrix J (i.e. $JJ^* = -J^2 = I$). Consider then a matrix $A_\epsilon = A_s + \frac{1}{\epsilon}J$ which has an arbitrarily large anti-symmetric part. One can show that

$$\kappa(\tilde{A}_\epsilon) \leq \kappa(A_s) + \epsilon^2 \lambda_{\max}(A_s)^2, \tag{8.16}$$

which means that the larger the anti-symmetric part, the smaller our upper bound for $\kappa(\tilde{A}_\epsilon)$ and consequently the more efficient is our proposed selfdual preconditioning. Needless to say that this method is of practical interest only when the equation $A_s x = d$ can be solved with less computational effort than the original system, which is not always the case.

Now the relevance of this approach stems from the fact that non-symmetric Krylov subspace solvers are costly since they require the storage of previously calculated vectors. It is however worth noting that Concus and Golub [3] and Widlund [15] have also proposed another way to combine CG with a preconditioning using the symmetric part A_s, which does not need this extended storage. Their method has essentially the same cost per iteration as the preconditioning with the inverse of $A^T A_s^{-1}$ that we propose for SD-CGN and both schemes converge to the solution in at most N iterations.

Iterated preconditioning: Another way to see the relevance of A_s as a preconditioner, is by noting that the convergence of "simple iteration"

$$A_s x_k = -A_a x_{k-1} + b \tag{8.17}$$

applied to the decomposition of A into its symmetric and anti-symmetric parts, requires that the spectral radius $\rho(I - A_s^{-1}A) = \rho(A_s^{-1}A_a) < 1$. By multiplying (8.17) by A_s^{-1}, we see that this is equivalent to the process of applying simple iteration to the original system (8.1) conditioned by A_s^{-1}, i.e., to the system

$$A_s^{-1}Ax = A_s^{-1}b. \tag{8.18}$$

On the other hand, "simple iteration" applied to the decomposition of $\tilde{A}$ into A_s and $A_a A_s^{-1} A_a$ is given by

$$A_s x_k = A_a A_s^{-1} A_a x_{k-1} + b - A_a A_s^{-1} b. \tag{8.19}$$

Its convergence is controlled by $\rho(I - A_s^{-1}\tilde{A}) = \rho((A_s^{-1}A_a)^2) = \rho(A_s^{-1}A_a)^2$ which is strictly less than $\rho(A_s^{-1}A_a)$, i.e., an improvement when the latter is strictly less than one, which the mode in which we have convergence. In other words, the linear system (8.15) can still be preconditioned one more time as follows:

If both $A \in R^{n\times n}$ and its symmetric part A_s are nonsingular, then x is a solution of the equation (8.1) if and only if it is a solution of the linear symmetric equation

$$\begin{aligned} \bar{A}x := A_s^{-1}A^T A_s^{-1}Ax &= [I - (A_s^{-1}A_a)^2]x \\ &= (I - A_s^{-1}A_a)A_s^{-1}b = A_s^{-1}A^T A_s^{-1}b. \end{aligned} \tag{8.20}$$

Note however that with this last formulation, one has to deal with the potential loss of positivity for the matrix $\tilde{A}$.

Anti-symmetry in transport problems: Numerical experiments on standard linear ODEs (Example 3.1) and PDEs (Example 3.2), show the efficiency of SD-CGN for non-selfadjoint equations. Roughly speeking, discretization of differential equations normally leads to a symmetric component coming from the Laplace operator, while the discretization of the non-self-adjoint part leads to the anti-symmetric part of the coefficient matrix. As such, the symmetric part of the matrix is of order $O(\frac{1}{h^2})$, while the anti-symmetric part is of order $O(\frac{1}{h})$, where h is the step size. The coefficient matrix A in the original system (8.1) is therefore an $O(h)$ perturbation of its symmetric part. However, for the new system (8.15) we have roughly

$$\tilde{A} = A_s - A_a A_s^{-1} A_a = O(\frac{1}{h^2}) - O(\frac{1}{h})O(h^2)O(\frac{1}{h}) = O(\frac{1}{h^2}) - O(1), \qquad (8.21)$$

making the matrix $\tilde{A}$ an $O(1)$ perturbation of A_s, and therefore a matrix of the form $A_s + \alpha I$ becomes a natural candidate to precondition the new system (8.15).

Resolvents of A_s as preconditioners: One may therefore consider preconditioned equations of the form $A^T M A x = A^T M b$, where M is of the form

$$M_\alpha = \big(\alpha A_s + (1-\alpha)I\big)^{-1} \quad \text{or} \quad N_\beta = \beta A_s^{-1} + (1-\beta)I, \qquad (8.22)$$

for some $0 \leq \alpha, \beta \in R$, and where I is the unit matrix.

Note that we obviously recover (8.2) when $\alpha = 0$, and (8.15) when $\alpha = 1$. As $\alpha \to 0$ the matrix $\alpha A_s + (1-\alpha)I$ becomes easier to invert, but the matrix

$$A_{1,\alpha} = A^T(\alpha A_s + (1-\alpha)I)^{-1} A \qquad (8.23)$$

may become more ill conditioned, eventually leading (for $\alpha = 0$) to $A^T A x = A^T b$. There is therefore a trade-off between the efficiency of CG for the system (8.5) and the condition number of the inner matrix $\alpha A_s + (1-\alpha)I$, and so by an appropriate choice of the parameter α we may minimize the cost of finding a solution for the system (8.1). In the case where A_s is positive definite, one can choose –and it is sometimes preferable as shown in example (3.4)– $\alpha > 1$, as long as $\alpha < \frac{1}{1-\lambda^s_{\min}}$, where $\lambda^s_{\min}$ is the smallest eigenvalue of A_s. Moreover, in the case where the matrix A is not positive definite or if its symmetric part is not invertible, one may take α small enough, so that the matrix M_α (and hence $A_{1,\alpha}$) becomes positive definite, and therefore making CG applicable (See example 3.4). Similarly, the matrix $N_\beta = \beta A_s^{-1} + (1-\beta)I$ provides another choice for the matrix M in (8.5), for $\beta < \frac{\lambda^s_{\max}}{\lambda^s_{\max}-1}$ where $\lambda^s_{\max}$ is the largest eigenvalue of A_s. Again we may choose α close to zero to make the matrix N_β positive definite. As we will see in the last section, appropriate choices of β, can lead to better convergence of CG for equation (8.5).

One can also combine both effects by considering matrices of the form

$$L_{\alpha,\beta} = \big(\alpha A_s + (1-\alpha)I\big)^{-1} + \beta I, \qquad (8.24)$$

as is done in example (3.4).

We also note that the matrices $M'_\alpha := (\alpha A'_s + (1-\alpha)I)^{-1}$ and $N'_\beta := \beta(A'_s)^{-1} + (1-\beta)I$ can be other options for the matrix M, where A'_s is a suitable approximation of A_s, chosen is such a way that $M'_\alpha q$ and $N'_\beta q$ can be relatively easier to compute for any given vector q.

Finally, we observe that the above reasoning applies to any decomposition $A = B + C$ of the non-singular matrix $A \in R^{n\times n}$, where B and $(B - C)$ are both invertible. In this case, $B(B - C)^{-1}$ can be a preconditioner for the equation (8.1). Indeed, since $B - CB^{-1}C = (B - C)B^{-1}A$, x is a solution of (8.1) if and only of it is a solution of the system

$$(B - C)B^{-1}Ax = (B - CB^{-1}C)x = b - CB^{-1}b. \tag{8.25}$$

In the next section, we shall describe a general framework based on the ideas explained above for the use of iterative methods for solving non-symmetric linear systems. In section 3 we present various numerical experiments to test the effectiveness of the proposed methods.

8.2 Selfdual methods for non-symmetric systems

By *selfdual methods* we mean the ones that consist of first associating to problem (8.1) the equivalent system (8.5) with appropriate choices of M, then exploiting the symmetry of the new system by using the various existing iterative methods for symmetric systems such as CG, MINRES, and SYMMLQ, leading eventually to the solution of the original problem (8.1). In the case where the matrix M is positive definite and symmetric, one can then use CG on the equivalent system (8.5). This scheme (SD-CGN) is illustrated in Table (1) below, in the case where the matrix M is chosen to be the inverse of the symmetric part of A. If M is not positive definite, then one can use MINRES (or SYMMLQ) to solve the system (8.15). We will then refer to them as SD-MINRESN (i.e., Self-Dual MINRES for Nonsymmetric linear equations).

8.2.1 Exact methods

In each iteration of CG, MINRES, or SYMMLQ, one needs to compute Mq for certain vectors q. Since selfdual methods call for a preconditioner matrix M that involves inverting another one, the computation of Mq can therefore be costly, and therefore not necessarily efficient for all linear equations. But as we will see in section 3, M can sometimes be chosen so that computing Mq is much easier than solving the original equation itself. This is the case for example when the symmetric part is either diagonal or tri-diagonal, or when we are dealing with several linear systems all having the same symmetric part, but with different anti-symmetric components. Moreover, one need not find the whole matrix M, in order to compute Mq. The following scheme illustrates the exact SD-CGN method applied in the case

Given an initial guess x_0, Solve $A_s y = b$ Compute $\bar{b} = b - A_a y$. Solve $A_s y_0 = A_a x_0$ Compute $r_0 = \bar{b} - A_s x_0 + A_a y_0$ and set $p_0 = r_0$. For k=1,2, . . . , Solve $A_s z = A_a p_{k-1}$ Compute $w = A_s p_{k-1} - A_a z$. Set $x_k = x_{k-1} + \alpha_{k-1} p_{k-1}$, where $\alpha_{k-1} = \frac{<r_{k-1}, r_{k-1}>}{<p_{k-1}, w>}$. Cpmpute $r_k = r_{k-1} - \alpha_{k-1} w$. Set $p_k = r_k + b_{k-1} p_{k-1}$, where $b_{k-1} = \frac{<r_k, r_k>}{<r_{k-1}, r_{k-1}>}$. Check convergence; continue if necessary.

Table 8.1: GCGN

where the coefficient matrix A in (8.1) is positive definite, and when $A^T A_s^{-1} Aq$ can be computed exactly for any given vector q.

In the case where A is not positive definite, or when it is preferable to choose a non-positive definite conditioning matrix M, then one can apply MINRES or SYMMLQ to the equivalent system (8.5). These schemes will be then called SD-MINRESN and SD-SYMMLQN respectively.

8.2.2 Inexact methods

The SD-CGN, SD-MINRESN and SD-SYMMLQN are of practical interest when for example, the equation

$$A_s x = q \tag{8.26}$$

can be solved with less computational effort than the original equation (8.1). Actually, one can use CG, MINRES, or SYMMLQ to solve (8.26) in every iteration of SD-CGN, SD-MINRESN, or SD-SYMMLQN. But since each sub-iteration may lead to an error in the computation of (8.26), one needs to control such errors, in order for the method to lead to a solution of the system (8.1) with the desired tolerance. This leads to the Inexact SD-CGN, SD-MINRESN and SD-SYMMLQN methods (denoted below by ISD-CGN, ISD-MINRESN and ISD-SYMMLQN respectively).

The following proposition –which is a direct consequence of Theorem 4.4.3 in [9]– shows that if we solve the inner equations (8.26) "accurately enough" then ISD-CGN and ISD-MINRESN can be used to solve (8.1) with a pre-determined accuracy. Indeed, given $\epsilon > 0$, we assume that in each iteration of ISD-CGN or ISD-MINRESN, we can solve the inner equation –corresponding to A_s– accurately enough in such a way that

$$\|(A_s - A_a A_s^{-1} A_a)p - (A_s p - A_a y)\| = \|A_a A_s^{-1} A_a p - A_a y\| < \epsilon, \tag{8.27}$$

where y is the (inexact) solution of the equation

$$A_s y = A_a p. \tag{8.28}$$

In other words, we assume CG and MINRES are implemented on (8.28) in a finite precision arithmetic with machine precision ϵ. Set

$$\epsilon_0 := 2(n+4)\epsilon, \quad \epsilon_1 := 2(7 + n\frac{\| |A_s - A_a A_s^{-1} A_a| \|}{\|A_s - A_a A_s^{-1} A_a\|})\epsilon, \tag{8.29}$$

where $|D|$ denotes the matrix whose terms are the absolute values of the corresponding terms in the matrix D. Let $\lambda_1 \leq ... \leq \lambda_n$ be the eigenvalues of $(A_s - A_a A_s^{-1} A_a)$ and let $T_{k+1,k}$ be the $(k+1) \times k$ tridiagonal matrix generated by a finite precision Lanczos computation. Suppose that there exists a symmetric tridiagonal matrix T, with $T_{k+1,k}$ as its upper left $(k+1) \times k$ block, whose eigenvalues all lie in the intervals

$$S = \cup_{i=1}^{k} [\lambda_i - \delta, \lambda_i + \delta], \tag{8.30}$$

where none of the intervals contain the origin. Let d denote the distance from the origin to the set S, and let p_k denote a polynomial of degree k.

Proposition 8.31. *The ISD-MINRESN residual r_k^{IM} then satisfies*

$$\frac{||r_k^{IM}||}{||r_0||} \leq \sqrt{(1+2\epsilon_0)(k+1)} \min_{p_k} \max_{z=S} |p_k(z)| + 2\sqrt{k}(\frac{\lambda_n}{d})\epsilon_1. \tag{8.32}$$

If A is positive definite, then the ISD-CGN residual r^{IC} satisfies

$$\frac{||r_k^{IC}||}{||r_0||} \leq \sqrt{(1+2\epsilon_0)(\lambda_n + \delta)/d} \min_{p_k} \max_{z=S} |p_k(z)| + \sqrt{k}(\frac{\lambda_n}{d})\epsilon_1. \tag{8.33}$$

It is shown by Greenbaum [6] that $T_{k+1,k}$ can be extended to a larger symmetric tridiagonal matrix T whose eigenvalues all lie in tiny intervals about the eigenvalues of $(A_s - A_a A_s^{-1} A_a)$. Hence the above proposition guarantees that if we solve the inner equations accurate enough, then ISD-CGN and ISD-MINRESN converges to the solution of the system (8.1) with the desired relative residual (see the last section for numerical experiments).

8.2.3 Preconditioning

As mentioned in the introduction, the convergence of iterative methods depends heavily on the spectral properties of the coefficient matrix. Preconditioning techniques attempt to transform the linear system (8.1) into an equivalent one of the form $C^{-1}Ax = C^{-1}b$, in such a way that it has the same solution, but hopefully with more favorable spectral properties. As such the reformulation of (1) as

$$A^T A_s^{-1} Ax = A^T A_s^{-1} b, \tag{8.34}$$

can be seen as a preconditioning procedure with C being the inverse of $A^T A_s^{-1}$. The spectral radius, and more importantly the condition number of the coefficient matrix in linear systems, are crucial parameters for the convergence of iterative methods. The following simple proposition gives upper bounds on the condition number of $\tilde{A} = A^T A_s^{-1} A$.

Proposition 8.35. *Assume A is an invertible positive definite matrix, then*

$$\kappa(\tilde{A}) \leq \min\{\kappa_1, \kappa_2\}, \tag{8.36}$$

where

$$\kappa_1 := \kappa(A_s) + \tfrac{\|A_a\|^2}{\lambda_{\min}(A_s)^2} \quad \textit{and} \quad \kappa_2 := \kappa(A_s)\kappa(-A_a^2) + \tfrac{\lambda_{\max}(A_s)^2}{\lambda_{\min}(-A_a^2)}. \tag{8.37}$$

Proof: We have

$$\lambda_{min}(\tilde{A}) = \lambda_{min}(A_s - A_a A_s^{-1} A_a) \geq \lambda_{min}(A_s).$$

We also have

$$\begin{aligned} \lambda_{max}(\tilde{A}) &= \sup_{x \neq 0} \frac{x^T \tilde{A} x}{x^T x} = \sup_{x \neq 0} \frac{x^t (A_s - A_a A_s^{-1} A_a) x}{x^T x} \\ &\leq \lambda_{max}(A_s) + \frac{||A_a||^2}{\lambda_{min}(A_s)}. \end{aligned}$$

Since $\kappa(\tilde{A}) = \frac{\lambda_{max}(\tilde{A})}{\lambda_{min}(\tilde{A})}$, it follows that $\kappa(\tilde{A}) \leq \kappa_1$.

To obtain the second estimate, observe that

$$\begin{aligned} \lambda_{min}(\tilde{A}) &= \lambda_{min}(A_s - A_a A_s^{-1} A_a) > \lambda_{min}(-A_a A_s^{-1} A_a) \\ &= \inf_{x \neq 0} \frac{-x^T A_a A_s^{-1} A_a x}{x^T x} \\ &= \inf_{x \neq 0} \{ \frac{(A_a x)^T A_s^{-1} (A_a x)}{(A_a x)^T (A_a x)} \times \frac{(A_a x)^T (A_a x)}{x^T x} \} \\ &\geq \inf_{x \neq 0} \frac{(A_a x)^T A_s^{-1} (A_a x)}{(A_a x)^T (A_a x)} \times \inf_{x \neq 0} \frac{x^T (A_a)^T (A_a) x}{x^T x} \\ &= \frac{1}{\lambda_{max}(A_s)} \times \lambda_{min}((A_a)^T A_a) \\ &= \frac{1}{\lambda_{max}(A_s)} \times \lambda_{min}(-A_a^2) \end{aligned}$$

With the same estimate for $\lambda_{max}(\tilde{A})$ we get $\kappa(\tilde{A}) \leq \kappa_2$.

Remark 8.2.1. Inequality (8.36) shows that SD-CGN and SD-MINRES can be very efficient schemes for a large class of ill conditioned non-symmetric matrices, even those that are almost singular and with arbitrary large condition numbers. It suffices that either κ_1 or κ_2 be small. Indeed,

- The inequality $\kappa(\tilde{A}) \leq \kappa_1$ shows that the condition number $\kappa(\tilde{A})$ is reasonable as long as the anti-symmetric part A_a is not too large. On the other hand, even if $\|A_a\|$ is of the order of $\lambda_{\max}(A_s)$, and $\kappa(\tilde{A})$ is then as large as $\kappa(A_s)^2$, it may still be an improved situation, since this can happen for cases when $\kappa(A)$ is exceedingly large. This can be seen in example 2.2 below.
- The inequality $\kappa(\tilde{A}) \leq \kappa_2$ is even more interesting especially in situations when $\lambda_{\min}(-A_a^2)$ is arbitrarily large while remaining of the same order as $||A_a||^2$. This means that $\kappa(\tilde{A})$ can remain of the same order as $\kappa(A_s)$ regardless how large is A_a. A typical example is when the anti-symmetric matrix A_a is a multiple of the symplectic matrix J (i.e. $JJ^* = -J^2 = I$). Consider then a matrix $A_\epsilon = A_s + \frac{1}{\epsilon}J$ which has an arbitrarily large anti-symmetric part. By using that $\kappa(\tilde{A}) \leq \kappa_2$, one gets
$$\kappa(\tilde{A}_\epsilon) \leq \kappa(A_s) + \epsilon^2 \lambda_{\max}(A_s)^2. \tag{8.38}$$

Here are other examples where the larger the condition number of A is, the more efficient is the proposed selfdual preconditioning. Consider the matrix

$$A_\epsilon = \begin{bmatrix} 1 & -1 \\ 1 & -1+\epsilon \end{bmatrix} \tag{8.39}$$

which is a typical example of an ill-conditioned non-symmetric matrix. One can actually show that $\kappa(A_\epsilon) = O(\frac{1}{\epsilon}) \to \infty$ as $\epsilon \to 0$ with respect to any norm. However, the condition number of the associated selfdual coefficient matrix

$$\tilde{A}_\epsilon = A_s - A_a(A_s)^{-1}A_a = \begin{bmatrix} \frac{\epsilon}{\epsilon-1} & 0 \\ 0 & \epsilon \end{bmatrix}$$

is $\kappa(\tilde{A}_\epsilon) = \frac{1}{1-\varepsilon}$, and therefore goes to 1 as $\varepsilon \to 0$. Note also that the condition number of the symmetric part of A_ϵ goes to one as $\epsilon \to 0$. In other words, the more ill-conditioned problem (8.1) is, the more efficient the selfdual conditioned system (8.15) is.

We also observe that $\kappa(A_s^{-1}A)$ goes to ∞ as ϵ goes to zero, which means that besides making the problem symmetric, our proposed conditioned matrix $A^T A_s^{-1} A$ has a much smaller condition number than the matrix $A_s^{-1}A$, which uses A_s as a preconditioner.

Similarly, consider the non-symmetric linear system with coefficient matrix

$$A_\epsilon = \begin{bmatrix} 1 & -1+\epsilon \\ 1 & -1 \end{bmatrix}. \tag{8.40}$$

As $\epsilon \to 0$, the matrix becomes again more and more ill-conditioned, while the condition number of its symmetric part converges to one. Observe now that the condition number of $\tilde{A}_\epsilon$ also converges to 1 as ϵ goes to zero. This example shows that self-dual preconditioning can also be very efficient for non-positive definite problems.

8.3 Numerical experiments

In this section we present some numerical examples to illustrate the proposed schemes and to compare them to other known iterative methods for non-symmetric linear systems. Our experiments have been carried out on Matlab (7.0.1.24704 (R14) Service Pack 1). In all cases the iteration was started with $x_0 = 0$.

Consider the ordinary differential equation

$$-\epsilon y'' + y' = f(x), \quad \text{on} \quad [0,1], \quad y(0) = y(1) = 0. \tag{8.41}$$

By discretizing this equation with stepsize 1/65 and by using backward difference for the first order term, one obtains a nonsymmetric system of linear equations with 64 unknowns. We present in Table 2 below, the number of iterations needed for various decreasing values of the residual ϵ. We use ESD-CGN and ISD-CGN (with relative residual 10^{-7} for the solutions of the inner equations). We then compare them to the known methods CGNE, BiCG, QMR, CGS, and BiCGSTAB for solving non-symmetric linear systems. We also test preconditioned version of these methods by using the symmetric part of the corresponding matrix as a preconditioner.

Table 8.2: Number of iterations for (8.41) with the solution $y = x\sin(\pi x)$.

N=64	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-10}$	$\epsilon = 10^{-16}$
ESD-CGN	22	8	5	4	3	2
ISD-CGN	24	9	6	4	3	2
GCNE	88	64	64	64	64	64
QMR	114	> 1000	> 1000	> 1000	> 1000	> 1000
PQMR	34	51	50	52	52	52
BiCGSTAB	63.5	78.5	92.5	98.5	100.5	103.5
PBiCGSTAB	26.5	46.5	50.5	50	51.5	51.5
BiCG	125	> 1000	> 1000	> 1000	> 1000	> 1000
PBiCG	31	44	50	50	52	52
CGS	> 1000	> 1000	> 1000	> 1000	> 1000	> 1000
PCGS	27	51	46	46	46	48

As we see in Tables 2 and and 3, a phenomenon similar to Example 8.2.3 is occuring. As the problem gets harder (ϵ smaller), SD-CGN becomes more efficient. These results can be compared with the number of iterations that the HSS iteration method needs to solve equation (8.41) (Tables 3,4, and 5 in [2]).

Consider the partial differential equation

$$-\Delta u + a\frac{\partial u}{\partial x} = f(x,y), \quad 0 \le x \le 1, \quad 0 \le y \le 1, \tag{8.42}$$

with Dirichlet boundary condition. The number of iterations that ESD-CGN and ISD-CGN needed to find a solution with relative residual 10^{-6}, are presented in Table 4 below for different coefficients a.

Table 8.3: Number of iterations for equation (8.41) with the solution $y = \frac{x(1-x)}{\cos(x)}$.

N=128	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-10}$	$\epsilon = 10^{-16}$
ESD-CGN	37	11	6	4	3	2
ISD-CGN(10^{-7})	38	12	7	4	3	2
GCNE	266	140	128	128	128	128
QMR	> 1000	> 1000	> 1000	> 1000	> 1000	> 1000
PQMR	40	77	87	92	90	85
BiCGSTAB	136.5	167.5	241	226.5	233.5	237.5
PBiCGSTAB	35.5	87.5	106.5	109	110.5	110.5
BiCG	> 1000	> 1000	> 1000	> 1000	> 1000	> 1000
PBiCG	37	76	84	89	85	91
CGS	> 1000	> 1000	> 1000	> 1000	> 1000	> 1000
PCGS	34	80	96	91	94	90

Table 4 and 5 can be compared with Table 1 in [15], where Widlund had tested his Lanczos method for non-symmetric linear systems. Comparing Table 5 with Table 1 in [15] we see that for small a (1 and 10) Widlund's method is more efficient than SD-CGN, but for large values of a, SD-CGN turns out to be more efficient than Widlund's Lanczos method.

Remark 8.3.1. *As we see in Tables 2,3, and 4, the number of iterations for ESD-CGN and ISD-CGN (with relative residual* 10^{-7} *for the solutions of the inner equations) are almost the same One might choose dynamic relative residuals for the solutions of inner equations to decrease the average cost per iterations of ISD-CGN. It is interesting to figure out whether there is a procedure to determine the accuracy of solutions for the inner equations to minimize the total cost of finding a solution.*

Consider the partial differential equation

$$-\Delta u + 10\frac{\partial(\exp(3.5(x^2+y^2)u)}{\partial x} + 10\exp(3.5(x^2+y^2))\frac{\partial u}{\partial x} = f(x), \qquad (8.43)$$

on $[0,1]\times[0,1]$ with Dirichlet boundary condition, and choose f so that

$$\sin(\pi x)\sin(\pi y)\exp((x/2+y)^3)$$

is the solution of the equation. We take the stepsize $h = 1/31$ which leads to a linear system $Ax = b$ with 900 unknowns. Table 5 includes the number of iterations

Table 8.4: Number of iterations for the backward scheme method (Example 3.2)

a	N	I (ESD-CGN)	I (ISD-CGN)	Solution
100	49	18	18	random
100	225	40	37	random
100	961	44	46	random
100	961	52	51	$\sin \pi x \sin \pi y. \exp((x/2+y)^3)$
1000	49	10	10	random
1000	225	31	31	random
1000	961	36	37	random
1000	961	31	39	$\sin \pi x \sin \pi y. \exp((x/2+y)^3)$
10^6	49	4	4	random
10^6	225	6	6	random
10^6	961	6	6	random
10^6	961	6	6	$\sin \pi x \sin \pi y. \exp((x/2+y)^3)$
10^{16}	961	2	2	$\sin \pi x \sin \pi y. \exp((x/2+y)^3)$

which CG needs to converge to a solution with relative residual 10^{-6} when applied to the preconditioned matrix

$$A^T(\alpha A_s^{-1} + (1-\alpha)I)A. \tag{8.44}$$

Table 5 can be compared with Table 1 in [15], where Widlund has presented the number of iterations needed to solve equation (8.43).

Remark 8.3.2. *As we see in Table 5, for $\lambda^s_{max}(\frac{1-\alpha}{\alpha}) = -.99$ we have the minimum number of iterations. Actually, this is the case in some other experiments, but for many other system the minimum number of iterations accrues for some other α with $-1 < \lambda^s_{max}(\frac{1-\alpha}{\alpha}) \leq 0$. Our experiments show that for a well chosen $\alpha > 1$, one may considerably decrease the number of iterations. Obtaining theoretical results on how to choose parameter α in 8.44 seems to be an interesting problem.*

Note that the coefficient matrix of the linear system corresponding to (8.43) is positive definite. Hence we may also apply CG with the preconditioned symmetric system of equations

$$A^T(A_s - \alpha\lambda^s_{\min}I)^{-1}A = A^T(A_s - \alpha\lambda^s_{min}I)^{-1}b, \tag{8.45}$$

where $\lambda^s_{\min}$ is the smallest eigenvalue of A_s and $\alpha < 1$. The number of iterations function of α, that CG needs to converges to a solution with relative residual 10^{-6} are presented in Table 7.

Table 8.5: Number of iterations for the centered difference scheme method (Example 3.2)

a	N	I (ESD-CGN)	Solution	Relative Residoual
1	49	21	random	6.71×10^{-6}
1	225	73	random	9.95×10^{-6}
1	961	91	random	8.09×10^{-6}
1	961	72	$\sin \pi x \sin \pi y. \exp((x/2 + y)^3)$	9.70×10^{-6}
10	49	18	random	9.97×10^{-6}
10	225	65	random	5.90×10^{-6}
10	961	78	random	8.95×10^{-6}
10	961	65	$\sin \pi x \sin \pi y. \exp((x/2 + y)^3)$	7.78×10^{-6}
100	49	31	random	6.07×10^{-6}
100	225	42	random	5.20×10^{-6}
100	961	43	random	5.03×10^{-6}
100	961	38	$\sin \pi x \sin \pi y. \exp((x/2 + y)^3)$	4.69×10^{-6}
1000	49	65	random	4.54×10^{-6}
1000	225	130	random	8.66×10^{-6}
1000	961	140	random	2.12×10^{-6}
100	961	150	$\sin \pi x \sin \pi y. \exp((x/2 + y)^3)$	5.98×10^{-6}

Remark 8.3.3. *As we see in the above table, for $\alpha = 0.99$ in (8.45) we have the minimum number of iterations. Obtaining theoretical results on how to choose the parameter α seems to be an interesting problem to study.*

We also repeat the experiment by applying CG to the system of equations

$$A^T \left(A_s - 0.99\lambda^s_{\min} I)^{-1} - \frac{0.99}{\lambda^s_{\max}} I \right) A = A^T \left((A_s - o.99\lambda^s_{\min} I)^{-1} - \frac{0.99}{\lambda^s_{\max}} I \right) b. \tag{8.46}$$

Then CG needs 131 iterations to converge to a solution with relative residual 10^{-6}.

As another experiment we apply CG to the preconditioned linear system

$$A_s^{-1} A^T A_s^{-1} A x = A_s^{-1} A^T A_s^{-1} b,$$

to solve the non-symmetric linear system obtained from discritization of the Equation (8.43). The CG converges in 31 iterations to a solution with relative residual less than 10^{-6}. Since, we need to solve two equations with the coefficient matrix As, the cost of each iteration in this case is twice as much as SD-CGN. So, by the above preconditioning we decrease cost of finding a solution to less that 62/131 of that of SD-CGN (System (8.46)).

Consider now the following equation

$$-\Delta u+10\frac{\partial(\exp(3.5(x^2+y^2)u)}{\partial x}+10\exp(3.5(x^2+y^2))\frac{\partial u}{\partial x}-200u=f(x),\quad \text{on } [0,1]\times[0,1], \tag{8.47}$$

If we discretize this equation with stepsize 1/31 and use backward differences for the first order term, we get a linear system of equations $Ax=b$ with A being a non-symmetric and non-positive definite coefficient matrix. We then apply CG to the following preconditioned, symmetrized and positive definite matrix

$$A^T((A_s-\alpha\lambda^s_{\min}I)^{-1}+\beta I)A=A^T((A_s-\alpha\lambda^s_{\min}I)^{-1}+\beta I)b, \tag{8.48}$$

with $\alpha<1$. For different values of α the number of iterations which CG needs to converge to a solution with the relative residual 10^{-6} are presented in Table 8. We

Table 8.6: Number of iterations for SD-CGN with different values of α.

$\lambda^s_{max}(\frac{1-\alpha}{\alpha})$	I		$\lambda^s_{max}(\frac{1-\alpha}{\alpha})$	I
$\infty(\alpha=0)$	>5000		0.1	232
$0(\alpha=1)$	229		0.2	237
-0.1	221		0.4	249
-0.25	216		0.8	263
-0.5	201		1	272
-0.7	191		5	384
-0.8	186		10	474
-0.9	180		20	642
-0.95	179		50	890
-0.99	177		100	1170
-0.999	180		1000	2790
-0.9999	234		10000	4807

repeat our experiment with stepsize 1/61 and get a system with 3600 unknowns. With $\alpha=-1.00000001$ and $\beta=0$, CG converges in one single iteration to a solution with relative residual less than 10^{-6}. We also apply QMR, BiCGSTAB, BiCG, and CGS (also preconditioned with the symmetric part as well) to solve the corresponding system of linear equations with stepsize 1/31. The number of iterations needed to converge to a solution with relative residual 10^{-6} are presented in Table 9.

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Table 8.7: Number of iterations for (8.43) with different values of α.

α	I
0	229
0.5	204
0.9	177
0.99	166
0.999	168
0.9999	181
0.99999	194
0.999999	222
0.9999999	248
0.99999999	257

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Chapter 9

Conclusion

In chapters 2, 3, and 4 we presented necessary and sufficient conditions under which one can improve Hardy and Hardy-Rellich type inequalities. Indeed we made a very useful connection between Hardy-type inequalities and the oscillatory behavior of certain ordinary differential equations that allowed us to improve, extend, and unify many results about Hardy and Hardy-Rellich type inequalities such as those in [1], [3], [4], [7], [11], and [15]. In chapter 4, we developed an approach to prove various classes of optimal weighted Hardy-Rellich inequalities on $H^2 \cap H_0^1$ which are crucial in the study of fourth order nonlinear elliptic equations and systems of elliptic partial differential equations. The approach developed in [8], [9], and[14] basically finishes the problem of improving Hardy and Hardy-Rellich inequalities in R^n.

In chapters 5 and 6, we studied the critical dimension of the fourth order elliptic equation with negative exponent under Drichlet and Navier boundary conditions. In [5], and [8] we showed that under both boundary conditions the critical dimension is $N = 9$. For a general domain our problem suffers from the lack of energy estimates. Also the blow-up analysis that we use to prove the regularity of the extremal solution in dimensions $5 \leq N \leq 8$ does not work. So determining the critical dimension on general domains remains a very interesting and important open problem which probably needs new ideas and techniques. However we conjecture that the critical dimension is $N = 9$.

Improved Hardy-Rellich inequalities obtain in chapters 3 and 4 play an important role to prove the singular nature of the extremal solutions in large dimensions close to the critical dimension in chapter 5, 6, and 7. In chapter 7 these inequalities allow us to provide a unified mathematical proof for the singularity of the extremal solutions of the bi-laplacian with exponential nonlinearity in dimension $N \geq 13$. This result was first proved in [6] by a computer assisted proof. There are many open problems about the singularity of the extremal solutions of nonlinear eigenvalue problems. I believe that the above approach can be modified to prove the singularity of the extremal solutions in these problems.

In chapter 8, motivated by the theory of self-duality, we proposed new templates for solving large non-symmetric linear systems. Our approach seems to be efficient when dealing with certain ill-conditioned, and highly non-symmetric systems. Our scheme in [10] is surprisingly efficient when dealing with certain ill-conditioned systems. However, obtaining theoretical result seem to be hard. It is interesting to obtain theoretical results about the SD-CGN scheme that we developed in chapter 8.

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