Essays on the Rank-Wealth Hypothesis

by

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Abstract

This thesis comprises two manuscripts which sequentially develop and test the Rank-Wealth Model (RWM).

The first manuscript constructs the RWM from basic economic principles by assuming consumer good indivisibility. If consumer goods are indivisible, and one also assumes finite supply and homogeneous preferences, the resultant derived utility function of each individual will become their rank in society. This is an important result for it can explain 'Keeping up with the Joneses' motives as well as generate a value function that in the aggregate closely resembles Kahneman and Tversky's Prospect Theory (1979). The RWM can therefore explain a number of financial anomalies including the endowment effect, simultaneous gambling and insuring, lottery regressivity and sub-optimal diversification.

The second manuscript tests some of the predictions that arise from RWM. Using methodology similar to Kumar (2009) the second study begins by confirming the previously documented observation that poor individuals hold more lottery-type-stocks (LTS) than the rich. Next, tests of the RWM are conducted using a proxy variable that measures individual rank as well as the Gini measure of wealth concentration. As expected, high LTS portfolios do underperform low LTS portfolios using standard risk metrics but that dominance is reversed when rank is considered. The second manuscript provides empirical support for the RWM by showing that it may be fully rational for the poorest individuals to concentrate their portfolio value into a few stocks that have higher idiosyncratic risk and skewness even if the (conventional) risk-adjusted expected return of those stocks is negative.

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Dedication

To Emmanuelle, Zoë and my parents, all of whom gave me the love and support to make this possible.

Chapter 1

Introduction

At the very heart of the discipline of Finance lie the intermingled concepts of risk and return. The relationship between risk and return, with the added element of investors who flee the former and seek the latter, defines virtually every concept of note in Finance. Whether one studies corporate policies, executive compensation, asset pricing or portfolio allocation the same trade-off between risk and return is at the core of the discussion. It is no exaggeration to say that the analysis of the risk-return relationship *is* Finance.

Given that the risk-reward relationship drives the discipline of Finance it is critically important that each of these two variables, risk and return, are each theorized and measured as accurately as possible. The notion of reward is fortunately a relatively easy concept to conceive and measure, a reward is having more than before. The theoretical concept of reward is so intuitive that the academic focus can be on the measure of reward. Risk unfortunately is not as simple a concept to theorize nor to measure. What is risk? Is it simply the probability weighted expected loss or is it a loss in a critical state? How much reluctance to investing in risky assets comes from simple risk and how much from an uncertainty of what there is to be uncertain about, such as the case with Knightian Uncertainty (Knight, 1921)? The struggle to understand the impact of risk on asset pricing is at least as old as Bernouilli's attempt to resolve it (Bernouilli, 1738).

For the most part, Finance has borrowed the conceptualization of risk from the older and more established field of economics. For the purposes of most economic models, risk can simply be measured as the variation of possible rewards. Thus the most frequent measures of risk in Finance are either the standard deviation of returns or the covariance of returns and some risk-factor(s). These measures of risk have served the discipline well allowing for the development of a multitude of asset pricing models. Unfortunately many of these models do not perform particularly well out-of-sample (Starmer, 2000) in estimating returns. Possible explanations for the poor performance of these models include either poor measurement of a risk-factor and/or neglecting consideration of a priced factor.

In this thesis I posit that a very important risk-factor, namely the rank of an individual's wealth in the cross sectional distribution of wealth, has been neglected too often in Financial models. The suggestion that the wealth rank of an individual is an important variable that affects the utility of an individual is not new. To name just a few authors, Veblen (1899), Friedman and Savage (1948), Duesenberry (1949), Frank (1985), Roussanov (2009) and Krasny (2009) have all argued that rank may be an important factor that affects the utility of an individual. What I present in this thesis that is new is a theory as to why rank *should* affect (derived) utility in a rational framework. Many papers in the tournament literature (Green and Stokey, 1983; Rosen, 1986; Ehrenberg and Bognanno, 1990; Scharfstein and Stein, 1990) and a few in finance (Brown et al., 1996; DeMarzo et al., 2008; Kempf and Ruenzi, 2008) have shown that rank may be at least a partial determinant of risky choice. The theory I propose goes further than this earlier literature by showing that for the majority of a population utility is not simply affected by rank, utility is determined almost entirely by rank.

This thesis comprises two essays on the Rank-wealth hypothesis. In the first essay I construct the theoretical Rank-wealth model. I achieve the rank relevance result through a few assumptions, the most notable of which is the assumption that consumer goods are indivisible. The result of this model is that individuals who remain risk-averse in terms of consumption can make financial decisions that would give the impression that they are in fact risk-loving. Both the tournament theory literature and a paper by Kwang (1965) demonstrate that reward indivisibility can increase the risk-tolerance of investors however this thesis extends that analysis by considering the effect for the entire population. This is an important advancement as the rank wealth model (RWM) I propose has richer implications than Kwang

(1965) and can potentially explain the empirical regularities that gave rise to Prospect theory (Kahneman and Tversky, 1979) as well as rationalizing the Endowment effect.[1] In a financial context, the theory I propose can explain cross-sectional variation in sub-optimal diversification, lottery regressivity[2] and why two-fund separation is not often observed. The theory I propose is also compelling intuitively as it rationalizes the 'Keeping-up-with-the-Joneses' motive explored in Constantinides (1990) and Abel (1990).

In the second manuscript of this thesis I empirically test the Rank-wealth hypothesis using data kindly provided by Dr. Terry Odean who is presently at UC Berkeley. To test the empirical predictions of the Rank-wealth model I borrow from the methodology first introduced in Kumar (2009) and classify some securities as lottery-type-stocks (LTS). Similar to Barberis and Huang (2008), I consider stocks as lottery instruments but differ in the explanation of why individuals elect to hold idiosyncratic risks. I show that the relative allocation to LTS in an individual portfolio is considerably conditional on an individual's rank. The most fascinating result of this manuscript is that despite portfolios with large LTS allocations not performing well relative to a diversified portfolio using classic risk-metrics, when rank is the measure of performance these same portfolios perform very well. In fact, the findings of the second manuscript offer a great deal of support to the first theoretical manuscript by clearly demonstrating that the poor actually rank-maximize by holding portfolios with relatively high LTS allocations. The empirical tests conducted in this manuscript not only validate the first manuscript's theoretical model but also provide some additional insight into the motives for gambling. Classical economics often finds it difficult to explain intuitively why an individual would voluntarily play a game of chance that both increases the variance of their net worth and simultaneously reduces the expected value of that worth. The Rank-wealth model I propose and test offers just such a rational argument as to why the poor find it most advantageous

[1]The endowment effect is the propensity for individuals to value a good more once they own it.

[2]Lottery regressivity is the tendency for poorer individuals to allocate more of their investment budget to the purchasing of lottery tickets.

to play the lottery and the rich find it the least.

The common thread that unites these two manuscripts is the Rank-wealth model. The first manuscript deals with the theoretical development of the model, provides a market friction that generates the model and explores the implications of the model. The first manuscript also foreshadows theoretical extensions that can be performed to enhance the generality of the model such as including intrinsic economic risk into state realizations. Following on the theoretical results of the first manuscript, the second manuscript tests some of the key predictions of the Rank-wealth model. For the most part, the findings of the empirical study support the theoretical derivations of the Rank-wealth model. United, these two manuscripts both postulate and test an intuitive, parsimonious and powerful model that can help to reduce or entirely eliminate a number of beguiling anomalies that exist in financial theory today. Perhaps the greatest contribution of this thesis however is that it demonstrates how modifying just a handful of classic economic assumptions, such as the divisibility of goods, can lead to markedly different results. The thesis therefore offers a potential foundation for new research into the domain of decision making under risk.

The remainder of this introduction is used to survey both recent and seminal results in the area of decision under risk and portfolio allocation. This discussion is used to provide context for the two following manuscripts as well as provide reference and comparison to competing alternative theories of decision making. The versed reader may wish to only skim the rest of the introduction or refer to it as questions arise from the rest of the thesis. [3]

[3]The reader should immediately clarify the difference between what I will call the Rank-wealth model and a much better known rank-dependent expected utility theory (RDEU) analyzed by Quiggin (1982); Yaari (1987). The RDEU model adjusts the probability of unlikely events conditional on the magnitude of the realizable outcome of that event. In so doing RDEU can address some anomalies such as the Allais paradox (Allais, 1953). While the Rank-wealth model that I propose can be generalized to allow for subjective probability adjustments conditioning on some feature, as RDEU does, this thesis does not explore that possible extension. Instead the Rank-wealth theory I propose refers not with the ranking of payoffs from a risky security (as RDEU does) but rather the ranking in the cross-sectional distribution of wealth that an individual attains. It is important to note that by making certain assumptions about the supplies of indivisible consumer goods available the Rank-wealth model I propose can give similar predictions as RDEU

1.1 Anomalies

Expected utility theory (EUT) and other theories of decision making under risk are one of the richest and most enduring fields of financial research (Starmer, 2000). Much of this research has been motivated by the desire to reduce or eliminate discrepancies between predicted and observed individual behaviour. In this section I summarize some of the anomalies that the Rank-wealth theory will resolve, argue as to why resolution is important and describe why the proposed theory is superior to some of the better known alternatives.

The anomaly that this thesis focuses most upon is the prevalence of gambling. First, I make the term *gambling* more precise. For the my purposes, gambling is defined as engaging in a project or investment that increases the riskiness of terminal wealth but has a negative expected return after all costs are considered.

After the advent of Von-Neumann and Morgenstern (1944) seminal work[4], it has almost always been assumed in economic models that investors are risk averse and that the marginal utility of consumption should decline in the level of consumption.

However, the truth of the matter is that individuals frequently gamble, taking on expected loss games of chance that increase the volatility of their terminal wealth. The difficulty in reconciling the evidence of individuals taking on gambles and conventional EUT is that gambling implies a Second-Order-Stochastic Dominance (SSD) violation. This is because SSD is assured only if individuals are globally risk-averse.

in terms of decision under risk. In particular, the Rank-wealth model may elicit an apparent overweighting of certain outcomes. RDEU conjectures that this overweighting is a result of probability adjustments whereas the Rank-wealth model conjectures that the overweighting stems from 'jumps' in the marginal utility of wealth resulting from good indivisibility. It is important to note that in no way does this thesis refute the findings of RDEU. In truth, elements of the two models could be blended into a third, and much more general, theory if one wished though this task is beyond the scope of this thesis.

[4]Von-Neumann and Morgenstern (1944) does not necessarily imply that individuals are risk averse but this framework simply allowed for the use of a cardinal system to represent ordinal preferences. This work therefore permits the framework introduced by Bernouilli (1738), which does suppose risk aversion, to be used as a representation of individual decisions under risk.

To resolve this anomaly, it seems that most economists have accepted a hypothesis known as the *gambling effect* (Diecidue et al., 2004). In essence, the gambling effect is the notion that individuals derive some form of enjoyment from the act of gambling and therefore are willing to take on risk with a negative expected return.

The major failing of the gambling effect explanation for lottery behavior is the fact that such a framework *must* systematically violate first order stochastic dominance (FSD), as shown by Diecidue et al. (2004). This means that economists must surrender the notion of FSD in order to maintain the gambling effect hypothesis. Since FSD is intuitively compelling and is critical to most mathematical models of risky decision making, I argue that it must be the gambling effect that is to be abandoned and not FSD. As I will show, the Rank-wealth model I introduce in the following section allows for a replacement of the gambling effect without sacrificing FSD.

Even if economists could formalize the gambling effect whilst maintaining FSD, this would only solve theoretical issues with the concept. Empirical evidence suggests that gambling is not strictly entertainment but is often construed as an investment vehicle. For example, an alarming report by the Consumer Federation of America in 2008 showed that 21% of the general American population, and 38% of those with incomes below $25,000, believe that winning the lottery is the most practical way to accumulate several hundred thousand dollars (Gillis and Almand, 2006). Recently, Blalock et al. (2007) also suggest gambling is not strictly for entertainment. The authors argue that if during financial hardship individuals opt for cheaper forms of entertainment, such as lotteries, then the demand for other cheap forms of entertainment, such as movies, should positively correlate with lottery sales. However lottery sales increase with increasing poverty but movie ticket sales do not, therefore rebutting the notion of the lottery as primarily an entertainment source.

Without the gambling effect though, classic economics is left with the difficulty that conventional EUT, when incorporating the assumption of risk aversion, cannot explain gambling. One solution to this problem is to assume heterogeneity of risk aversion amongst individuals, so that there are both risk-lovers and risk-avoiders. This unfortunately is a poor solution as will be shown in discussing the next anomaly.

Recognizing that individuals do in fact gamble, researchers such as Friedman and Savage (1948) explored the possibility that utility over wealth has convex regions. However, these authors realized that to have heterogenous risk aversion amongst individuals while still maintaining the monotonicity condition of utility solves the gambling anomaly but only at the cost of introducing another anomaly. Namely, the anomaly introduced is that risk-seekers should not buy insurance any more than risk-averters should avoid gambling. Thus, having two static types of individuals can explain the existence of lotteries and insurance but it cannot explain the observed behavior that the same individual will both gamble and insure simultaneously.

The Friedman and Savage (1948) solution to the gambling and insurance anomaly is to have a utility function that has both convex and concave regions. While this approach will indeed solve the anomaly of simultaneous gambling and insurance, the framework is open to intuitive criticism. It is not apparent at what levels of income/wealth the utility function should be convex and moreover it is not intuitive as to why utility over consumption should be convex at all.

The Rank-wealth theory I introduce is reminiscent of the Friedman and Savage (1948) model in that both approaches have convex and concave regions of utility over wealth. Where my approach differs from the former is that the framework I propose considers how wealth is converted into consumption and so is analyzing the *derived* utility of wealth. In contrast, Friedman and Savage (1948) simply makes the *a priori* assumption about the underlying utility over consumption function. As a result, my model can be distinguished from FS by its' theoretically derived conditional local convexities. Unlike FS, the Rank-wealth model I propose derives the convex regions of the derived utility function endogenously and thus circumvents the critique of not knowing at what levels of wealth convexity should occur at. Furthermore, the model I propose allows for an individual to remain at least weakly risk-averse with respect to consumption and is therefore much more appealing than the FS model.

Another anomaly that the Rank-wealth model addresses is that individuals, on average, are not very well diversified. Conventional EUT, with the concavity and monotonicity, assumption implies that individuals should diversify their risk. Although transaction costs, information asymmetry and market frictions may limit diversification a little, the Survey of Consumer Finances suggests that most Americans hold almost all of their wealth in just two assets: their car and their house. This kind of concentration of individual wealth into just a few assets is hard to justify by market frictions.

It is true that DeMarzo et al. (2004) has successfully theoretically modeled low levels of diversification by making diversification appear to the investor as a public good. The mechanism driving this result is that if individuals are unable to completely diversify their portfolio because of trading restrictions then the presence of a few irrational investors can create a price pressure for goods demanded by the irrational investors which prompts the rational investors to diversify less than would be implied by standard models. Though the suboptimal diversification result of DeMarzo et al. (2004) is a major contribution, it does not predict any differences in diversification behavior across individuals as a function of their wealth. It is generally known, however, that the wealthy tend to be better diversified than the poor (Polkovnichenko, 2005). The Rank-wealth model I propose can partially fill this gap by giving an explanation as to why economists observe cross-sectional variation in diversification and why the wealthy tend to be the best diversified.

In contrast to the most conventional decision models, the Rank-wealth model predicts that the convex region over derived utility will be for low levels of endowment.[5] Thus, the model presented here predicts that the poor will diversify less than the wealthy, as is consistent with observation.

Since the publication of Kahneman and Tversky (1979), it has become apparent that any model that hopes to accurately describe individual risky decision behavior requires at least two key ingredients. The first is that individuals should exhibit *loss aversion* and the second is that individuals

[5] This result also requires the assumption that the wealth distribution is unimodal and non-monotonic.

should be risk-seeking in losses.

Loss aversion is the tendency for losses to loom larger on risky decisions than potential gains. While the theory I propose does not have a reference point from which to distinguish gains from losses, the framework still generates behaviors that appear in aggregate as loss aversion. The reason for this is that if the initial wealth distribution is skewed,[6] then poorly endowed individuals' derived utility will be greatly affected by the rank of their endowment as compared to other individuals. As a result of skewness in the endowment distribution, the entire derived utility of wealth will be an 'S' shaped curve with a greater incline to the left of the mean wealth than to the right. An econometrician compiling individual selections over risky choices into an average would find that individuals are on average more sensitive to losses than to gains, as is documented by Kahneman and Tversky (1979) and defined as loss aversion.

The second characteristic required for a theory to accurately describe investor behavior is to have risk-seeking in losses. To be risk-seeking in losses means that an individual prefers to suffer one large loss rather than two smaller losses that equal the larger loss. In a rank wealth framework the fact that the derived utility over wealth is 'S' shaped ensures that the average individual is risk seeking in losses.

The RWM can also address an anomaly that is of interest to financial economists, namely the Endowment effect. This effect is the tendency for individuals to have very different WTP (willing-to-pay) and WTS (willing-to-sell) prices for the exact same good. For example, if an individual is asked the maximum price they would be willing to pay for a coffee mug they may say $3. However, if you endow that same individual with the same coffee mug and ask the minimum price which they are willing to accept to surrender the mug the response may be $5. Empirical evidence of the Endowment effect has been documented numerous times and by various researchers (Knetsch and Sinden, 1984; Knetsch, 1989; Kahneman et al., 1990) and is often portrayed as conflicting with classical economic theory.

[6]Such as a log-normal distribution which is generally used to characterize the wealth distribution

The Rank-wealth model, in a world of uncertainty, can potentially explain the endowment effect as discussed in section 2. RWM will achieve this result through the model of a Sunspot equilibrium. It will then be shown that in any particular state, consumers of a good are not indifferent to the choice of either consuming an indivisible good or instead recovering its price. In fact, the clearing price of indivisible goods in terms of divisible good will be *too low* relative to the utility generated. As will be explored later, this result can potentially explain the anomaly of the endowment effect.

The endowment effect is not the only unusual individual decision making behaviour that is consistently found in experimental studies. In the words of Barberis and Xiong (2009):

> "The disposition effect is one of the most striking features of individual investor trading. Its underlying cause, however, is still unclear."

The disposition effect is the tendency for individuals to sell stocks that have enjoyed gains and a tendency to hold stocks that have suffered losses. While explaining this unusual behavior with theory may be challenging, this nonetheless changes the fact that it has been empirically well documented Odean (1998). An often purported explanation of the disposition effect is the characteristic of loss aversion in Prospect Theory. As I will show later though, the Rank-wealth model can also produce loss aversion. Though it is beyond the scope of this work to explore in detail the disposition effect, it is noteworthy that in their rigorous analysis of the causes of the effect, Barberis and Huang (2008) find that the effect is difficult to produce with paper gains and losses; however, it is much easier to generate the effect if one considers realized gains an losses. The finding of Barberis and Xiong is pertinent to the Rank-wealth model because individuals are not assumed to derive utility through the accumulation of wealth but rather through consumption that such wealth provides. Thus, the Rank-wealth model would predict that the greatest changes in utility are from realized gains and losses, consistent with loss aversion generating a disposition effect.

1.2 Theories of rank relevance

The discussion thus far has demonstrated the validity in seeking alternative theories of decision making by citing failures of classic models and listing anomalies that are not yet fully understood. The RWM is however just one possible alternative theory and joins a reasonably large set of pre-existing models which either directly or indirectly make use of a ranking argument. To best understand where the Rank-wealth model fits relative to related literature I will coarsely cut the literature into two branches: exogenous models of rank-relevance and endogenous models. The exogenous models are those that assume *a priori* that individuals maximize their wealth-rank for psychological reasons.

The most venerable of the exogenous branch of theories is that of conspicuous consumption, proposed by Veblen (1899). In essence, the theory of conspicuous consumption is that individuals garner some utility by flaunting expensive status symbols to other people. Similar concepts, like the psychological benefit one might enjoy by being a 'big fish in a small pond' are proposed by Duesenberry (1949) and Frank (1985) to explain non-efficiencies in the labor market.[7] Other approaches are more agnostic as to the exact source of psychological pleasure for being of high wealth-rank. Literature in this vein include Abel (1990), Robson (1992), Roussanov (2009) and Krasny (2009). These exogenous rank models seek to explain either patterns in aggregate economic data or individual decision making behaviour when Rank-wealth relevance is assumed.

All of the exogenous models satisfy a common intuition that individuals care about the rank of their wealth. There is good reason for this intuition as empirical research has shown (Gali, 1989; Clark, 1995; Kumar, 2009). The difficulty with the exogenous models is that they can leave the research with an unanswered question as to *why* wealth rank should matter.

The endogenous models try to answer that question. In some models,

[7]Specifically, the models are used to explain why some individuals with high productivity are not compensated substantially more in monetary terms than their less productive coworkers. The argument is that individuals derive some utility by being the most productive worker and that this serves as a psychological compensation.

such as Cole et al. (1992) or Cole et al. (2001), it is assumed that there is another market besides the consumer market that can give individuals utility. A common example of such a market is the marriage market, where individuals' marriage prospects are a function of their relative wealth. In this way, the marriage market might serve as a tournament competition making relative standing important. Alternatively, other models continue to assume that individuals derive utility from conventional consumer markets but that market frictions result in relative ranking being important. Papers in this vein are DeMarzo et al. (2004) and DeMarzo et al. (2008). Either geographic or temporal barriers limit the trading of goods between various sub-populations and thus relative wealth in one's own sub-population becomes an important variable to terminal consumption.

In many regards, the Rank-wealth model I propose is similar to DeMarzo et al. (2004), but the friction I suggest is not geographic (or temporal) trade-limitations but good indivisibility. Thus, the proposed Rank-wealth model follows a path suggested by Rosen (1997). What distinguishes my model from either DeMarzo et al's work or Rosen's is that I note how the heterogeneity of endowments will produce a derived utility function that is uncannily similar to the empirically derived Prospect Theory (Kahneman and Tversky, 1979). An example of this can be seen in Appendix Figure A.1 where I overlay the PT value function overtop of my derived utility function.

The critical difference between the model I propose and the other endogenous rank models is that I analyze the effects of the wealth distribution on derived utility. Other endogenous models assume common endowment within particular cohorts.

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Chapter 2

Developing the Rank-wealth hypothesis[8]

2.1 Introduction

In this manuscript I show that if goods are indivisible then the rank of an individual's wealth partly determines their derived utility. Wealth rank, as I will define it, does not directly affect derived utility in classic economic models and yet it can be used to explain a variety of real-world violations of risky choice theory. In particular, I show that suboptimal diversification, lottery regressivity, the disposition effect and the endowment effect can all be explained using the 'Rank-wealth' model. I also show that the Rank-wealth model can nest some of the observed behaviors cited in Prospect Theory. This manuscript contributes to the literature of financial economics by endogenizing wealth rank concerns in such a way as to create numerous localized regions of convexity that can be used to explain observed investor choice.

To create the Rank-wealth effect I use the intuitive idea that consumer goods, such as houses, televisions and automobiles, cannot be divided and still produce consumer utility. The indivisibility constraint creates the effect that derived utility is the upper envelope of piece-wise concave functions. The location where any two concave sections intersect is determined jointly by the wealth population density function and the available supply of consumer goods. The intersection of two concave functions creates a region of local convexity in the derived utility function and it is these convex regions

[8] A version of this chapter will be submitted for publication. Newton, David I. Developing the Rank-Wealth Hypothesis

that cause conditional risk-seeking behavior. Loosely, I label the effect that intersection points cluster more densely in zones of large population mass as the 'Rank-wealth effect'. The Rank-wealth model I propose gives a new channel through which the distribution of endowment across the population will impact prices. In some cases the Rank-wealth effect will be sufficiently strong to create sizable intervals over the initial endowment where risk-seeking behavior is optimal.

To demonstrate the model's ability to nest some of the results of prospect theory, I assume a unimodal and non-monotonic distribution of wealth.[9] This means that there are more individuals with intermediate levels of wealth than individuals with either low or high levels of wealth. The larger population mass at intermediate levels of wealth generates the rank-effect, clustering intersection points, and elevates the derived incremental utility of wealth. The elevated derived incremental utility of wealth implies an increased willingness to gamble which can be used to explain the anomalies I previously listed.

As classic Expected Utility Theory (EUT) often fails to predict individual choice, a number of alternative theories of risky choice have been proposed. One of the recurring themes, seemingly driven by intuition, in these alternatives is the notion that wealth rank somehow matters to individuals. Of the theories that consider rank relevance to utility, there exist two kinds: exogenously derived models of rank relevance and endogenously derived models.

In the exogenously derived models of rank relevance, it is assumed *a priori* that individuals get utility from rank wealth. Examples of such models include Veblen (1899), Duesenberry (1949), Frank (1985), Abel (1990) and Roussanov (2009). While these models can help to resolve differences between observed individual choice and theory predicted choice, they are not fully satisfying for they offer no theoretical explanation as to why rank

[9] The assumption of a particular wealth distribution will only exacerbate/attenuate the impact of the Rank-wealth effect. Assuming unimodality is not essential for the qualitative results of the model but non-monotonicity is required to get the most interesting effect of convexification of the value function.

should matter.

In the endogenously derived models, economic mechanisms such as trading restrictions or tournament prizes can be used to create rank relevance within a conventional economic framework. Examples of these models include Cole et al. (1992, 2001); Robson (1992); DeMarzo et al. (2004, 2008). I argue that these models are much more satisfying than the exogenously derived models as they do not assume *a priori* that wealth rank affects derived utility.

Yet, even the endogenously derived models have thus far failed to capture the full breadth of their possible predictions. The reason for this is because earlier endogenous models have not made full use of the heterogeneity of wealth to create risk-taking incentives within cohorts of individuals. In contrast, the Rank-wealth model I propose makes full use of the wealth density distribution to resolve many anomalies simultaneously. Moreover, the Rank-wealth model is general, parsimonious and very intuitive; relying mainly on good indivisibility as the necessary market friction to create the rank-effect.[10]

The remainder of this manuscript is structured as follows: Section 2.2 defines the 'Rank-wealth' effect and briefly justifies the intuition of good indivisibility as this assumption is crucial to derive the main results, Section 2.4 describes the competitive market model that delivers the rank relevance result in a partial equilibrium setting. This section also discusses some of the central results to the Rank-wealth model. In Section 2.5 I pose the rational expectations version of the problem, give the solution technique, and derive and discuss the solution to a low dimension problem. Section 1.1 compares various other theories of Rank-wealth to my model and Section 2.6 then concludes.

[10] A second requirement for the rank effect to be present is one of i) satiation or ii) decreasing substitutability. One or both of these conditions will ensure that the indivisibility assumption is binding. Further discussion on this point is deferred until the model is clear to the reader.

2.2 Rank-wealth effect and indivisibility

2.2.1 Rank-wealth effect

I define the the 'Rank-wealth' effect as an effect where the derived marginal utility of wealth, at some endowment e, is increased if the mass of individuals with endowment slightly greater than e increases.

The mechanism driving this effect is as follows. Imagine a group of N individuals with heterogenous supplies of a divisible good, such as leisure-hours or gasoline, bidding on a fixed supply of N-1 identical indivisible goods, such as condominiums or automobiles. Each of the N individuals bid for the indivisible good (condominium) using their divisible good (leisure-hours) endowment. For the market to clear it must be that the N-1 individuals awarded the condominiums paid a price in divisible good (leisure-hours) that is sufficiently low such that they are at least indifferent to being awarded the condominium. For the N^{th} individual who did not have enough divisible endowment to secure an indivisible condominium their marginal utility of the divisible good is a function of the cross-sectional distribution of divisible endowment. This effect arises because the N^{th} individual has two potential uses for divisible endowment, direct consumption and indirect consumption through the acquisition of an indivisible good. If the N-1 individuals awarded the indivisible goods are substantially more endowed than the N^{th} individual then the marginal utility for that poorer individual is predominantly a function of direct divisible good consumption. However, if the N-1 individuals awarded the indivisible goods are only marginally more endowed than the N^{th} individual then that individual now has an alternative, and perhaps much more appealing, use for their next marginal unit of divisible consumption; namely they may be able to outbid one of the N-1 individuals and secure an indivisible good for themselves.

To give this mechanism a concrete example consider the situation where you are the only individual of a cohort unable to buy a condominium. If you anticipate being outbid in the condominium market by a margin of $10,000 on the least favorable condominium then the marginal increase of your wealth by $1 will only give you utility through some other consumption (perhaps buying a coffee). If however, the margin by which you anticipate being outbid shrinks, to $1000, or $100, or perhaps even $1 then your marginal dollar may have substantial marginal utility. You could still elect to buy a coffee but if the condominium is desirable enough then you will gain greater utility from your marginal dollar by using it to increase your condominium bid and become a homeowner. The margin by which you are outbid for various indivisible goods, and thus the marginal utility of your divisible good, is determined by the supply and quality of indivisible goods, the preferences of competing individuals and the cross-sectional distribution of divisible good endowment. Literally, how does your divisible good endowment rank relative to everyone else's endowment? I label the Rank-wealth effect the result that the cross-sectional distribution of another's divisible goods has an impact on the marginal utility of your own divisible good.

In the following sections I will show that the degree to which my model can increase the derived marginal utility of wealth is such that the derived utility function can become locally convex even if the essential utility function underlying the model is concave.

The model I propose allows for convexification of the derived utility function, but it also predicts that locally convex preferences will occur at endowment levels where population mass is quickly rising. With a sufficiently rich model, where there exists many consumer goods, the effect of the clustering of these locally convex areas is to produce a derived utility function that appears as concern for wealth rank. By further introducing reasonable assumptions about the shape of the endowment distribution, my model can show that a variety of financial anomalies are not anomalous at all. Thus, my theory contributes by producing convex sections of the derived utility function, which in turn can explain anomalies and does so without the *a priori* assumption that wealth rank matters.

2.2.2 Intuition for indivisibility

The force that produces the rank-effect is the assumption that some goods are indivisible and scarce, at least in the short run. In this section, I argue that good indivisibility is both intuitive and commonplace. On this matter, it is easiest to make the case by asking rhetorical questions. Such questions include: How useful is half of an automobile, a third of a computer, a fourth of a microwave oven, a fifth of an I-pod or a sixth of house? How easily can a consumer split a pair of jeans in half and receive half the utility of a full pair.[11] The evidence is all around us that goods are fundamentally indivisible, very often physically and almost always practically. It is the rare exception where goods are easily divisible, as may be argued for gasoline or electricity. The truth is that most goods are indivisible in nature.

Good indivisibility is only part of the conditions for deriving Rank-wealth effects. The second component is the short-run scarcity of goods. For goods such as seaside property, where construction is limited, it is apparent that goods are effectively in fixed supply at any horizon. For goods such as fine wines it may be that the supplies are elastic but only at longer horizons.[12]

As a brief aside, it is also interesting to note that in competitive production economies the inventories of various goods will often only be increased in response to price increases. This means that production will respond to demands rather than preempt demands and as a result it is reasonable to assume that contemporaneous supplies and demands will not match ensuring the rank effect will always be present. [13]

[11]As an aside, even for those goods where physical division does not affect function the practicality of division may be of concern. For example if an individual wanted to bake a single pancake they would need to purchase more milk, eggs, flour and sugar than they need. This is because certain items, such as eggs, can only ever come in integer units and other items, such as milk, are available only in certain sized containers. This means that the utility of one pancake cannot be had cheaply whereas the marginal utility of the second pancake is easily achieved.

[12]It should be noted that only the most desirable quality levels need be in scarce supply to generate the Rank-wealth effect. If more indivisible good is available than can be consumed then the least desirable quality levels will not be consumed.

[13]It is also interesting to note that one of the key results of Shaked and Sutton (1983) is that in a world with indivisible goods natural oligopolies will form. Those oligopoly firms will have non-zero profits in expectation, contrary to classic models of pure competition but consistent with empirical observation. I argue that this is further support for the non-divisibility assumption of the Rank-wealth model (RWM).

2.3 Definition of equilibrium and rank effect

For the purposes of this manuscript the definition of an equilibrium is a stable allocation of indivisible good of varying qualities to individuals. To be stable it must be that the allocation is feasible with respect to market clearing conditions and all budget constraints. It is also necessary that each individual's incentive compatibility and rationality constraints are met. I state the objective function of each individual and the related constraints here but will discuss the details of each element in greater detail in the following section.[14]

$$sup_{q_k} u(w - p(q_k)) + q_k \tag{2.1}$$

$$u(w - p(q_k)) + q_k > u(w) \quad \text{(IR constraint)} \tag{2.2}$$

$$u(w - p(q_k)) + q_k > u(w - p(q_{-k})) + q_{-k} \quad \text{(IC constraint)} \tag{2.3}$$

$$w \geq p(q_k) \quad \text{(Budget constraint)} \tag{2.4}$$

Where w is the divisible endowment that an individual begins the game with. The price of the indivisible good of quality k is given by the schedule $p(q_k)$. An individual optimally selects which quality level of an indivisible good they wish to consume. Where q_{-k} is $q_j \forall j \neq k$.

Within the context of this thesis I do not model the production side of the indivisible good. The game is that a competitive market has fixed supplies of varying quality levels of an indivisible good. The market suppliers have no use for the indivisible good but do have use for the divisible good and thus will trade their stock of indivisible good for the divisible endowment that individuals possess. One can liken this transaction to the trade of a merchant's wares for the cash or labor possessed by individual consumers.

In section E the incentive compatibility and individual rationality constraints of the individuals consumers are verified.

I will later formalize the exact effects of rank on portfolio allocation but I

[14] In the following specification IR is the individual rationality constraint and IC is the individual compatibility constraint

define here the rank-effect in a more general sense. Namely, *the rank-effect is the difference between the canonical investment policy as derived in a world of strictly divisible goods and the investment policy derived in a world with an indivisible good.*

For the purposes of this thesis, attention will be most focused on when the rank-effect is sufficiently strong to cause convexity in the value function and thus dramatically different results than delivered by the canonical investment policy found in the world of divisible goods. The rank-effect may however still exist even when the value function remains concave but this will not be of as much interest for the purpose of this thesis as convexity is required to help resolve documented financial anomalies.

2.4 Behaviour in a certain world and partial equilibria

To demonstrate the rank effect I first consider a world of certainty. To keep the model simple I assume homogeneous preferences amongst consumers, exogenously specified quantities and qualities of indivisible goods, and exogenously specified distribution of endowments. This economy has no uncertainty. In the appendix I will briefly discuss production in this partial equilibrium setting but will not go into great detail as it is not central to the Rank-wealth result I wish to demonstrate. In section 2.5 I will introduce and analyze extrinsic uncertainty in the form of a sunspot equilibrium. For now, however, the purpose of identifying the competitive market clearing prices in this simple economy is to demonstrate the Rank-wealth effect on an individual in a larger group of consumers when faced with good indivisibility.

To maintain parsimony, I do not model the supply side of the economy. I only assume that there is a set of indivisible goods and that each of these goods has a finite supply and a commonly known quality. Quality is expressed in terms of utiles for simplicity but one could as easily use a hedonic function of quality over some set of product characteristics (location, colour, horse-power, etc...).[15] The supply side is assumed to be endowed with the indivisible goods for which they have little value and will trade to the consumer population for the populations' endowed divisible good. The model does not specifically address the objective of the endowed owners but one could assume that the divisible good is labour or dollars and could be used to pay dividends to the endowed owners. As consumers trade their divisible good for an indivisible good of varying quality, I denote the prices of the indivisible good in terms of the divisible good which is used as a numeraire.

There is one indivisible good in the economy that has K quality levels and a single divisible good. The quantity of each of the indivisible good quality levels available for trade is exogenously specified. I assume a continuum of consumers who have identical preferences to each other but differ in the amount of endowed divisible good. Individuals do not start endowed with any indivisible good. The distribution of divisible good endowment across the economy is common knowledge to all consumers.

Individuals face the choice of either directly consuming their entire divisible endowment or using some portion of that endowment to purchase an indivisible good which can also be consumed. For the moment, it is assumed that individuals may consume only one unit of one particular quality level of the indivisible good. Later I shall address the possibility that the individuals may consume a finite number of many indivisible goods. As utility is assumed to be strictly increasing in the amount of divisible good consumed, any surplus divisible good which is not used to purchase an indivisible good is eaten. The utility function which is common for all consumers is:

$$U(w, q_k) = u(w - p_k) + q_k \tag{2.5}$$

Where w is the endowment of divisible good of the consumer in question, q_k is the quality of indivisible good that the consumer has chosen to consume

[15] While I do not specify a particular hedonic model a natural fit would be with the framework proposed by Rosen (1974). In such models the quality of a good, q, can be thought of as a function of the characteristics of that good. In the case of a car for example the quality of a car could be a function of mileage, seats, year, fuel efficiency and safety rating, or q = f(mileage, seats, year, fuel, safety).

and p_k is the price for the indivisible good of quality q_k. Again, note that the indivisible good prices are expressed in terms of divisible good. Consumers are bound by the budget constraint that the price of the indivisible good they choose to consume, p_k, is less or equal to their endowment w.

The function $u(\cdot)$ is concave and strictly increases in the argument. I have not yet imposed any restrictions on $u(\cdot)$ but later I will find it useful to assume a log utility. One may also opt to use a class of functions where $u(0) = -\infty$ to ensure an interior solution.

For an individual with endowment w to choose to consume the k^{th} good it must be that:

$$u(w - p_k) + q_k > u(w - p_j) + q_j \qquad \forall j \neq k \tag{2.6}$$

This equation simply states that, given a system of prices $(p_1, p_2, ...p_K)$, each individual will choose to consume the indivisible good that maximizes their utility subject to the prices of the qualities of the indivisible good and to their budget constraint. Individuals may always choose to consume their entire divisible endowment without purchasing any indivisible good. In this case, it is assumed that they have chosen to consume the 0^{th} quality indivisible good. This quality level of good has, without loss of generality, a utility normalized to 0 and will of course have a price of 0 as a result.

The next step to demonstrate the Rank-wealth result is to find the competitive market clearing prices of the various indivisible quality levels. In order to do this, I first introduce some additional notation. Let $f(w)$ denote the density for the wealth distribution which is assumed to be continuous on its support $[0, \bar{w}]$ where $\bar{w} > 0$. The resulting cumulative is denoted by $F(w) = \int_0^w f(x)dx$.

2.4.1 Wealth bounds

To determine the market clearing prices in order to show the rank wealth effect I create a series of 'wealth-bounds'. These are levels of endowment which mark the indifference point between consuming and not-consuming a particular indivisible good. The indifference expression with the wealth-bounds will be used to find the prices of quality levels of the indivisible good where individuals are just indifferent to ownership and will thus solve the indivisible good allocation problem. In a concrete sense a wealth bound is the amount of divisible good the poorest (marginal) owner of a particular indivisible good has. If that person had just slightly less divisible good they would opt not to consume the indivisible good they have marginally acquired because, given the fixed cost of that indivisible good, it would be utility decreasing to do so.

To generate the wealth bounds, first assume that all indivisible good qualities are ordered in their qualities so that good 1 is of inferior quality to good 2 which is of inferior quality to good 3 and so forth until good K, which is the highest quality good. This means that:

$$q_1 < q_2 < ... < q_{K-1} < q_K \tag{2.7}$$

Let w_k be the level of wealth for which an individual is just indifferent to consuming good k and less divisible good or consuming good $k-1$ and more divisible good. The indifference condition at w_k can be expressed as:

$$u(w_k - p_{k-1}) + q_{k-1} = u(w_k - p_k) + q_k \tag{2.8}$$

The difference in the amount of divisible good consumed arises from the fact that in a competitive market with scarce supplies of indivisible good, the price of the higher quality good will be greater than the price of the lower quality good.

Thus, the wealth bounds for the individuals who consume good k are $[w_k, w_{k+1})$. If k is equal to K (the highest quality good) then the bounds are $[w_K, \bar{w}]$.

2.4.2 Indivisibility

I now formalize the notion of *good indivisibility.* Individuals are restricted to consume either 0 or 1 unit of indivisible good. As individuals are on a continuum, this restriction is translated into the restriction that some mass of individuals M (M $= F(w_2) - F(w_1)$, $w_2 > w_1$), can consume at most M units of a good of quality k. As a less-general example, suppose that there exists a cumulative mass of individuals equal to 0.1 between wealth levels w_1 and w_2. Then this group of individuals with cumulative mass 0.1 may consume no more than 0.1 units of indivisible good.

The good indivisibility restriction will mean that two units of lesser quality good cannot substitute for a good of twice the quality. The restriction also means that half a unit of double quality good cannot be consumed to substitute for a full unit of good with standard quality.

2.4.3 Market clearing

Recall that the maximum quality good is q_K. I let the supply of the i^{th} good be s_i. In the case of the K^{th} good, the supply is s_K. Consumers may always opt to consume the 0^{th} good which is defined as having infinite supply, zero quality and consequently, zero price.

All the consumers of varying divisible endowment have no indivisible good to begin with. The individuals with the least endowment have the highest marginal utility of endowment because the utility derived from the divisible good is a concave function. Similarly, the individuals with the greatest initial endowment have the lowest marginal utility of endowment.

Now consider how the highest quality good of quality q_K must be allocated. The individuals with the greatest endowment have the least marginal utility over the divisible good and thus will exhibit the strongest preference for the highest quality indivisible good relative to some fixed amount of endowment. Thus, the wealthiest will be willing to pay more for the highest quality good than any poorer individual. Furthermore, the wealthiest individuals are by definition the most capable to pay for this highest quality good. As the wealthy have both the greatest capacity for payment and the greatest incentive to pay, it is certain that whatever price they pay for the highest quality good, the poorer individuals will either be unable or unwilling to match this price. It is therefore known that the highest quality good will be allocated in a competitive market to the wealthiest individuals.

Let w_K be the individual with the least endowment who purchases the highest quality good, good K. I let s_i denote the supply of the indivisible good of quality i. In a competitive market the price of good K must be such that the entire supply s_K clears the market.

$$\int_{w_K}^{\bar{w}} f(w)dw = s_K \tag{2.9}$$

Recalling the finite consumer mass assumption, the above expression becomes:

$$1 - F(w_K) = s_K \tag{2.10}$$

Rearranging and inverting yields,

$$w_K = F^{-1}(1 - s_K) \tag{2.11}$$

In a world of certainty the marginal consumer of the highest quality good K can be identified with the expression above. Once the supply of highest quality good is allocated, the allocation procedure can be repeated for the next highest quality good, namely good k-1. Thus, the process can then be repeated iteratively down the quality level of the good so that one may write the general expression:

$$w_i = F^{-1}(F(w_{i+1}) - s_i) \tag{2.12}$$

The general expression above gives the minimum endowed individual (w_i) who purchases the good of quality i.

Notice that the wealth boundaries are fully identified solely as a function of the supplies of indivisible quality levels available and the cumulative density function of population endowments. I can therefore solve this problem sequentially, first identifying the wealth boundaries and afterwards isolating the prices in equilibrium. The expression I derive above will be used in the next subsection in which I identify the prices of all indivisible quality levels.

2.4.4 Price identification

Assuming an interior optimum of consumption of the divisible good, then one can state that the individual with endowment w_K is just indifferent to purchasing the highest quality good or instead purchasing the second highest quality good and consuming the divisible price differential. The general expression of the indifference equations is:

$$u(w_i - p_i) + q_i = u(w_i - p_{i-1}) + q_{i-1} \tag{2.13}$$

In latter sections I will refer to this equation as the *within-state indifference equation*. By corollary 1, the price of a particular good can be found by rearranging the indifference equation:

Corollary 1.

$$p_i = w_i - u^{-1}\left[u(w_i - p_{i-1}) + q_{i-1} - q_i\right] \tag{2.14}$$

To solve for prices, I assume a world of *scarcity* so that the aggregate supply of all non-zero quality indivisible good cannot meet the aggregate demand of the entire population. Thus, there must be some individuals who receive no indivisible good whatsoever. Formally, the scarcity condition is:

$$1 = \int_0^{\bar{w}} f(w)dw > \sum_{i=1}^{K} s_i \tag{2.15}$$

To identify prices I begin by noting that the zero quality good must have zero price as any price higher than this is irrational for the consumer consuming the zero quality good.

Formally, in the case of the 1^{st} good with quality q_1, one can isolate the price of the good using the indifference equation and find the expression:

$$p_1 = w_1 - u^{-1}\left[u(w_1 - 0) - q_1\right] \tag{2.16}$$

Where $w_1 = F^{-1}(F(w_2) - s_1)$. Once the price of good 1 is identified, it can be used, with the market clearing conditions, to find the price of good

2. The expression for the price of good two is:

$$p_2 = w_2 - u^{-1}\left[u(w_2 - p_1) - q_2 + q_1\right] \tag{2.17}$$

This iterative process can be repeated K times until all the market clearing prices of all quality levels of the good are identified. Through the iterative process, all the unknown wealth boundaries ($\overrightarrow{w_i}$) and unknown good prices ($\overrightarrow{p_i}$) can be identified where $\overrightarrow{w_i} = (w_1, ..., w_k)$ and $\overrightarrow{p_i} = (p_1, ..., p_k$ respectively.[16] It is now possible to analyze the derived utility over wealth and, as will be shown in the following subsection, demonstrate the Rank-wealth effect.

2.4.5 Rank-wealth effect

Using the iterative procedure outlined above it is possible to find the price for each good the and endowment intervals ($[w_i, w_{i+1})$ for i = 1,...K-1) of the individuals that consume each good. The derived utility of initial endowment and the price of good as a function of quality in a competitive market can each be determined.

Assuming that an individual is not exactly at one of the wealth bounds then their derived marginal utility over a divisible good will simply be the first derivative of the utility function with respect to the divisible good at the optimally selected level of divisible good.[17] Thus, the marginal utility of wealth within the interior space of two wealth bounds is:

$$\frac{dU(w, q_k^*)}{dw} = u'(w - p_k) \tag{2.18}$$

If however the individual is at one of the wealth bounds $(w_1, w_2 ... w_{K-1}, w_K)$, then the first derivative is not defined. This is because there is a 'jump' in

[16] Alternate notation for $\overrightarrow{w_i}$ is $(w_i)_{i=1,..,k}$. Similarly, alternate notation for $\overrightarrow{p_i}$ is $(p_i)_{i=1,..,k}$

[17] This is because, between any two wealth bounds, increasing endowment only increases divisible consumption as the same indivisible good is optimal for all individuals within a wealth bound interval. When a new wealth bound is crossed however, the optimal strategy for the consumer is to adjust their divisible consumption and consume a different indivisible good.

the marginal utility as an individual transitions from consuming one kind of indivisible good to the next. The indifference condition in the previous section only precludes jumps in the derived utility and not the derived marginal utility.

Because of the jumps in marginal utility the changes in derived utility are better examined in large, measurable changes rather than infinitesimal changes. Thus, I write the expression for the rate of change in derived utility for an increase of Δ in the original endowment.[18]

$$\begin{gathered}\frac{U(w+\Delta,q_k^*)-U(w,q_j^*)}{(w+\Delta)-w}=\\ \frac{u(w+\Delta-p_{k+1})-u(w-p_k)+q_{k+1}-q_k}{\Delta}\end{gathered} \tag{2.19}$$

Assume that the change in wealth is exactly the same as the difference in price between two sequential quality levels of good. This means $\Delta = p_{k+1} - p_k$. I select this Δ for two reasons. The first is that a Δ of this value is by definition just sufficient to switch to the next quality of indivisible good. This makes the analysis easier than having to cope with many possible quality changes which could arise from a larger change in endowment. The second is that by choosing this particular value, the expressions below will simplify and allow the rank-effect to become more evident. With these assumptions, the rate of change in the utility for a change of Δ divisible endowment becomes:

$$\frac{U(w+\Delta,q_{k+1}^*)-U(w,q_k^*)}{(w+\Delta)-w}=\frac{q_{k+1}-q_k}{p_{k+1}-p_k} \tag{2.20}$$

To demonstrate the rank effect, I show how the discrete change in utility

[18] I assume that Δ is at least large enough to cause a change of indivisible consumption from good i to good i+1, for the poorest individual currently consuming good i. If Δ is not this large, then the poorest individual will not adjust their indivisible good consumption and will instead use the Δ increase in endowment strictly to consume more divisible good. Thus, the first derivative of derived utility with respect to wealth would simply the first derivative of divisible utility.

is affected by shifting mass in the wealth distribution.

To show this, let $F_1(\cdot)$ and $F_2(\cdot)$ denote, respectively, the cumulative distribution function of the wealth distribution before and after the mass transportation. Allow w_i to denote the i^{th} wealth boundary in the original equilibrium and w_i' denote the i^{th} wealth boundary under the new wealth distribution $F_2(\cdot)$.

Now consider what happens if a mass of individuals with a time zero divisible endowment between the range of w_{i+1} and w_{i+2} have their endowment reduced to an amount between w_i and w_{i+1}. Figure 2.1 depicts the mass transport.

Mathematically, the effect of the mass transport is:

$$\begin{cases} F_2(w) = F_1(w) \ \ \forall \ \ w \in [0, w_i] \cup [w_{i+2}, \bar{w}] \\ F_2(w_{i+1}) > F_1(w_{i+1}) \end{cases}$$

Equivalently, this means that $w_j = w_j'$ for all $j \neq i+1$. In other words, only the wealth boundary w_{i+1} is affected by the mass transport. In particular, $w_{i+1} > w_{i+1}'$.

As the prices of quality levels are found iteratively from the wealth boundaries, this also means that the prices for quality i and lower are unchanged by changing from distribution $F_1(\cdot)$ and $F_2(\cdot)$. Letting p_j be the price of good j in equilibrium under $F_1(\cdot)$ and p_j' be the price of good j in equilibrium under $F_2(\cdot)$, I write the expression:

$$p_j = p_j', \quad \forall j \leq i \tag{2.21}$$

As none of the supplies (s_i) are changed from this mass transportation, then $F_1(w_{i+1}) = F_2(w_{i+1}')$. Next, recall equation (2.14) which I rewrite in the form:

$$p_i = u^{-1}[u(w_i)] - u^{-1}[u(w_i - p_{i-1}) - q_{i+1} + q_i] \tag{2.22}$$

As we know that $u(u^{-1}(x)) = x$ then the derivative of price with respect to the wealth boundary can be expressed as:

$$\frac{dp_i}{dw_i} = 1 - \frac{du^{-1}[u(w_i - p_{i-1}) - q_{i+1} + q_i]}{dw_i} \tag{2.23}$$

And because u' is decreasing and $p_{i-1} < p_i$ it must be that

$$\frac{dp_i}{dw_i} > 0 \tag{2.24}$$

Therefore, as $w_{i+1} > w'_{i+1}$ then $p_{i+1} > p'_{i+1}$. As $w_i = w'_i$ then $p_i = p'_i$. These relationships combine to give the result below which is generalized from a particular i^{th} wealth bound to any boundary denoted by k:

$$\frac{q_{k+1} - q_k}{p_{k+1} - p_k} < \frac{q_{k+1} - q_k}{p'_{k+1} - p'_k} \tag{2.25}$$

Thus, if one transports population mass from one wealth bound interval to a lower wealth bound interval the price of the good will adjust in such a way as to increase the discrete change in utility for a fixed (Δ) increase in wealth. I have therefore established that the population wealth distribution is important in determining the derived utility of an individual.

The presence of a large population mass that has just slightly more wealth than an individual will enhance the incremental utility gain (of that individual) for a fixed increase in wealth. Conversely, when there is a smaller population mass that has just slightly greater endowment than said individual, the gain in utility for a given discrete change in wealth is reduced.

Notice that the individual has not had any change in wealth themselves, nor has the incremental change (Δ) been changed. The incremental utility of a Δ in endowment for an individual is partially determined by the Rank-wealth that individual has. This is what I deem the 'Rank-wealth' effect.

2.4.6 Continuum of good qualities

The Rank-wealth model also results in divisible-good risk-seeking behavior of an individual who is fundamentally risk-averse in terms of consumption even when there is a continuum of infinite good qualities. To show individual risk-seeking behavior it is sufficient to show that the second derivative of the value function is greater than zero Pratt (1964).[19]

To begin, I define the value-function as

$$V(w) = u[w - p(D(w))] + D(w) \tag{2.26}$$

Where w is the quantity of divisible endowment an individually initially has, D(w) is the quality of the optimal indivisible good consumed and p(D(w)) is the equilibrium price of the consumed optimal quality indivisible good.

Proposition 1. *If and only if the product of the marginal price of the optimal indivisible good consumed times the marginal quality of the optimal indivisible good consumed is greater than one will an individual will be locally risk-seeking. That is,*

$$p'(D(w))D'(w) > 1 \Rightarrow V"(w) > 0 \tag{2.27}$$

To show this, I begin by identifying the optimization problem faced by a single consumer as:

$$\max_{q} U(w, q) = u(w - p(q)) + q \tag{2.28}$$

Let $G(q)$ denote the cumulative distribution of quality and $F(w)$ denote the cumulative distribution of wealth. In equilibrium there exists a demand function, $D(w)$, that gives the optimal quality demanded by an individual with w wealth. Thus, $D(w)$ maps from $G(q)$ to $F(w)$, matching supply with demand.

$$D = G^{-1} \circ F \tag{2.29}$$

Substituting the demand function into the value function that solves

[19] An extended proof showing each algebraic step is given in the appendix C.

(2.28) gives:

$$V(w) = u[w - p(D(w))] + D(w) \tag{2.30}$$

The second derivative of the value function with respect to w is therefore:

$$V''(w) = u''[w - p(D(w))](1 - p'(D(w))D'(w)) \tag{2.31}$$

Recall that as $u(\cdot)$ is concave and strictly increasing then $u'(\cdot) > 0$ and $u''(\cdot) < 0$.

Notice that the value function may have a positive second derivative, implying convexity, if and only if

$$p'(D(w))D'(w) > 1 \quad \square \tag{2.32}$$

Thus far the optimization problem I have posed has remained completely general with the exception of $u(\cdot)$ being an increasing concave function and the utility function being separable as given. This is important as it implies that risk-seeking can potentially be elicited from a very broad class of underlying demand and price functions.

Proposition 5 (in the appendices) is sufficient to demonstrate risk-taking in the small but the effects of the distribution of wealth are not immediately apparent. In the above expression the convexity of the value function was determined by the price and demand function of the indivisible good. While these two functions are indeed jointly determined by the supply distribution of indivisible good qualities and distribution of divisible endowment (wealth) one does not immediately see the relationship between the distribution of wealth and risk-taking. To make eminently clear the Rank-wealth effect I apply the above proposition to a simplified case for which a closed form solution can be readily found in terms of the wealth distribution. For the particular application I assume that G(q), the supply distribution of indivisible good qualities, is uniformly distributed over the interval $[0, \bar{q}]$ with mass one so that $G(q) = q/\bar{q}$. For this application I make the further simplifying assumption that $u(x) = \ln(x)$.[20]

These assumptions allow the demand function to be written as:

$$D(w) = G^{-1} \circ F = \bar{q}F(w) \tag{2.33}$$

Substituting $D^{-1}(q)$ into equation (C.4) gives

$$p'(q)u'(D^{-1}(q) - p(q)) = 1 \tag{2.34}$$

The equation now becomes:

$$\frac{p'(q)}{D^{-1}(q) - p(q)} = 1 \Rightarrow p'(q) = D^{-1}(q) - p(q) \tag{2.35}$$

Which yields the differential equation:

$$p + p' = D^{-1} \tag{2.36}$$

This differential equation can be solved by the method of variation of parameters. The solution of the differential is given in Appendix D. The solution to the price function for this particular application is:

$$p(q) = \left[\int_0^q F^{-1}(x)e^x dx\right] e^{-q} \tag{2.37}$$

Next find the expression of the multiple of the marginal price of the optimal indivisible good consumed and the marginal quality of the optimal indivisible good consumed. For this application that expression is:

$$p'D' = \underbrace{\left[w - \left[\int_0^{F(w)} F^{-1}(x) \underbrace{e^{x-F(w)}}_{>0,\, \leq 1} dx\right]\right]}_{>0} F'(w) \tag{2.38}$$

As the first term on the right-hand-side is strictly positive, the entire

[20] It should be noted that these simplifying assumptions are not required for the Rank-wealth effect to be present but only facilitate a closed-form solution where the impact of the wealth distribution can be most readily observed.

expression is greater than one for sufficiently large values of $F'(w)$.[21]. That is, if the population mass is increasing quickly enough in wealth, convexity in the Value function is certain and the agent is risk-seeking in terms of divisible good (wealth). The Rank-wealth effect is now quite apparent. If a large mass of consumers have the same wealth level w, then that entire mass will be more risk-tolerant. The Rank-wealth effect can simultaneously explain risk-taking behavior in portfolio allocation, even when an individual is fundamentally risk-averse in terms of consumption, as well as a keeping up with the Joneses motive. The model can even describe why 'misery loves company'. For example, if all individuals lose a common fraction of their divisible good the allocation of indivisible good qualities in equilibrium is not perturbed. However if a single individual suffers a negative divisible good shock while others are unaffected, that individual stands to lose not only a small amount of divisible consumption but potentially a dramatic amount of indivisible good consumption conditional on their rank.

2.4.7 Partial equilibrium behaviour

In many fields of Economics, there exists a broad literature analyzing risk taking incentives when the consumption or reward set is non-convex, as is a recurring theme in the literature of mutual fund tournaments (Brown et al., 1996; Busse, 2001; Taylor, 2003; Loranth and Sciubba, 2006). More closely related to the model I propose are the works of Kwang (1965) and Rosen (1997). Each of these authors demonstrate that individual's risk-taking incentives can be increased if the consumption set is non-convex. However, because the focus and contribution of these papers is not identical to mine they do not complicate their models by introducing a many quality levels of indivisible good. For my purposes, by increasing the diversity of the consumption set the effect I choose to focus on becomes apparent and the derived utility of individuals becomes their approximate wealth rank.

[21]Recall that because $F(w)$ is a general function with non-atomic distribution, it is possible that F'(w) can have any value while F(w) retains the same value for an arbitrarily selected function. We can therefore consider the effects of altering just the immediate local mass on the value function without altering the expression $\int_0^{F(w)} F^{-1}(x)e^{x-F(w)dx}$

As an individual can always consume their endowment in the divisible good, it must be that the lower bound of the derived utility over endowment is simply the utility function over the divisible good. The presence of indivisible good qualities will potentially increase the derived utility above this lower bound. The expected result is that individuals' derived marginal utility over endowment will be increased above the lower bounds marginal utility when the population is locally dense. Over the entire endowment range, this result will mean that individuals' derived utility will be affected by how their endowment ranks against other individuals. To illustrate this point I present a very simplified specification of the model.

Imagine that there is one unit of indivisible good with quality $q_1 = 1$ utile. Further imagine that there are two individuals with identical preferences but different endowments of divisible good. Let those endowments be respectively 1 and 1.5. To simplify the example, I assume the two individuals have square root utility over the divisible good. Now consider the derived utility of each individual.

If each individual were simply to consume their divisible endowment they would respectively achieve utilities of $\sqrt{1} = 1$ and $\sqrt{1.5} = 1.23$. Note though that the wealthiest individual will purchase the indivisible good. Furthermore, she will pay 1 unit of divisible good to purchase the indivisible good. If she were to pay less, the poorer individual would be indifferent to parting with their entire endowment so as to have the indivisible good. Because of the competitive clearing mechanism, the richer individual will not be willing to pay more than one unit of divisible good for the indivisible good as there are no other consumers that are willing and able to pay higher than one. Thus, if the richer individual paid a price higher than one, she would only reduce her derived utility. Therefore the derived utilities of each individual is not 1 and 1.23 but rather 1 and 1.71.

As I have assumed that all individuals have the same utility function, I can analyze the derived utility of one individual by examining the schedule of derived utilities for all individuals. Notice that an endowment of zero will produce zero derived utility. From the perspective of an outside observer, the derived utility over the endowment is convex despite the function over the divisible good being concave. Table 2.1 illustrates this point.

The derived utility over the endowment has a slope increasing in endowment, implying local convexity. However, the degree to this convexity is a function of how densely packed individuals are with respect to their endowment. If the wealthier individual had an endowment of 3 rather than of 1.5 then the derived utility of wealth shifts to the situation depicted in Table 2.2.

In the case presented in Table 2.2 the slope of derived utility decreases with respect to endowment, implying concavity. If individuals have closer endowments the convexity is increased, if the reverse is true then the rank effect will be present but will not overcome the concavity of the utility function over the divisible good. Therefore, it is important to note that the Rank-wealth effect can create convexity in the derived utility function but that it does not necessarily guarantee convexity.

The previous example is crude but it does help to illustrate the intuition behind the rank effect. Consider now the implication in a broader population where there are not two individuals but many individuals. Where many individuals have common endowments the derived marginal utility of the endowment will be elevated. At every point over the endowment distribution, individuals' derived marginal utility is in part affected by the local density of individuals. Cumulatively, this is the same as saying that the derived utility (no longer marginal) is a function of the cumulative mass of individuals with lesser endowments. In other words, the rank of an individuals endowment relative to others is important in determining their derived utility in a competitive market.

2.4.8 Comparative statics

There are a number of factors which can affect the behavior of the derived utility over wealth in the rank model I propose. In the discrete case, these factors are the number of good quality levels (K), the quality interval between the various good ($\Delta_q = q_{i+1} - q_i$), and the distribution of wealth (F(w)) and the supply of good. In the continuous case the distributions of good supplies and wealth remain relevant but the number of good quality levels, K, becomes infinite.

I begin by illustrating graphically the effect of varying these parameters on the discrete model. For the base case, I assume a normal distribution of endowment with $\mu = 50$ and $\sigma = 15$. In addition, log utility over the divisible good is assumed and it is assumed that goods are available in equal supply and that the total supply of indivisible goods can satisfy total demand.

By introducing more goods (increasing K) the derived utility over the endowment becomes smoother, with smaller jumps in the marginal utility of endowment, as observable in Figure 2.2.[22] In the limit, the differential jumps in quality becomes infinitesimal and the derived utility converges to the continuous expression of section 2.4.6.

Increasing the endowment of an individual when they are between any two optimal cutoffs for consuming indivisible goods, only increases their consumption of the divisible good. Thus the derived utility is a piece-wise composition of the log function. The location of where the pieces 'attach' is determined by the local population density. If there are more individuals clustered at one endowment level then more of the indivisible good's total stock will be priced to clear in this market. This has the effect of bunching the log functions and, in turn, elevating the derived marginal utility over wealth where there are more individuals. By having more goods, the rank effect remains but the derived utility function smoothes out, appearing more and more like a combination of the cumulative distribution of endowments and the log function.

In the situation where the wealth distribution of the population is distributed as the distribution of good qualities, so that $F(w) = G(D(w))$ then the rank-effect is to linearize the utility function as can be seen in Figure 2.3.

In Figure 2.4 one can observe the prices of the various goods as a function

[22] The total change in lowest to highest quality, $q_K - q_0$, is equal to 14.1 for all choices of K. As K increases though, the number of increments changes decreasing the magnitude of Δ_q. For example if K=1, then there are two intervals of consumption including the 0^{th} good and Δ_q = 14.1/(K+1) = 14.1/2 = 7.05. If K = 20, then Δ_q = 14.1/(20+1) = 0.67.

of their qualities. Initially, a large price increase is required to achieve even a modest increase in quality (high slope) but as the quality increases the slope declines. Thus, there are many goods of varied quality but with differing prices available in the 'mid price' range. As the quality further increases though, the slope once more increases and the cost of even greater quality becomes prohibitive. This pattern arises from the unimodal and non-monotonic pattern of initial endowment. In a sense, there is intense price competition where the price is within reach of a large mass of individuals. This pattern of price and quality is arguably true in the real-world. For the argument, consider the dispersion of quality-levels of homes that are available in twenty to thirty thousand dollar range. Next, consider the dispersion of qualities for homes in the five hundred to five hundred and ten thousand dollar range. Finally, consider the dispersion of quality in homes from one million to one million and ten thousand dollars. I argue that in the first and the third categories the number of various qualities available is small. In the first case, there is unlikely to be any quality available at all. In the third case, the increasing price is unlikely to increase the quality of home substantially. This is because there is less price competition at this range of incomes and thus the consumer surplus in this region is reduced.

Also notice that as the number of qualities of goods increases towards the continuous case the price to quality function becomes smoother and approaches a shape that is the inverse of the cumulative endowment distribution.

2.4.9 Uncertainty with idiosyncratic investment

In this subsection I explore the implications of the Rank-wealth hypothesis on individual behaviour when uncertainty is introduced, an individual believes herself to have a unique investment opportunity (idiosyncratic), and the individual is atomistic with respect to prices. As the individual is atomistic and is assumed to have a unique opportunity to invest, there are no strategic effects[23] of that individual's investment that cause prices to change.

[23] In Section 2.5 I explore the optimal actions of individuals when investment is not idiosyncratic and strategic effects do affect prices in a Sunspot equilibrium. The Sunspot

In this case, the individual's objective function is simply to maximize expected terminal derived utility.

To illustrate the difference between the optimal policies of conventional utility and Rank-wealth utility, I make use of two kinds of idiosyncratic investment opportunities. The first kind of investment opportunity I use is where the individual optimally invests α proportion of their time zero endowment into a security that pays R with h probability and 0 otherwise. [24] I vary the gross return, R, of the asset and the probability of their payoff to contrast the optimal policy against other conventional models.

The second kind of investment opportunity I use is that of a portfolio optimization problem where an individual invests in two normally distributed assets and one risk free asset. I use this example to demonstrate failure of the two-fund separation result.

In the first analysis the expected wealth of the individual who begins with w_0 endowment will take on the value of $w_{(h)igh} = w_0((1-\alpha)+\alpha R)$ with h probability and $w_{(l)ow} = w_0(1-\alpha)$ with $1-h$ probability.

The objective function of the individual is:

$$\begin{aligned} \max_{\alpha} h[u(w_h - p(D(w_h))) + D(w_h)]... \\ +(1-h)[u(w_l - p(D(w_l)) + D(w_l)] \\ s.t. \quad 0 \leq \alpha \leq 1 \end{aligned} \tag{2.39}$$

To simplify the analysis, I use the continuum of goods case where $u(\cdot)$ is log allowing for an analytical expression of both $p(D(\cdot))$ and $D(\cdot)$ to be used. This simplifies the task for the solver in that the derived utility of wealth in each state can be determined simply from exogenous parameters of the wealth distribution.

I assume that each individual uses her personal assessment of the distri-

version of the model is a complete general equilibrium, unlike the current consideration which is partial, but is more difficult to solve for large dimension problems.

[24]For simplicity I assume that the individual cannot consume her time zero endowment. A more general model where time zero endowment could be consumed could be as easily solved but would only complicate the discussion of the rank effect.

bution of wealth in each state i to forecast indivisible good prices. Forecasting prices, she can then make an optimal allocation of her income across each state. I also assume that the distribution of wealth is non-varying across states to better demonstrate the Rank-wealth effect in isolation of intrinsic volatility.

First, consider the case of a two-point lottery as described above. To examine the effects of payoff magnitude, I fix the probability of payoff for the simple security at 80% but vary R. I select R values of 1.25, 1.5, 1.75, 2 and 3 which produce, respectively, expected returns of 0%, 20%, 40%, 60%, 140% on the asset. The density of good qualities, $G'(q)$, is assumed to be uniform and the density of wealth, $F'(w)$, is assumed to be truncated normally distributed.[25]

Figure 2.5 plots the optimal proportion of wealth dedicated to the simple security as a function of initial endowment. The dotted line plots the optimal holdings of an individual with log utility and who has no opportunity to purchase indivisible goods. Each pair of colored lines is associated with a particular R. Notice that for the log utility, only when R becomes large enough to produce a high expected return does the log utility investor hold a non-zero proportion of the risky asset (red dotted line). Yet, in the Rank-wealth model, despite the utility of divisible consumption remaining fundamentally log, the derived utility of wealth is affected by rank in a dramatic way. Though individuals with large time zero endowments do hold a proportion of the risky asset that exceeds what log utility would predict, the poor individuals hold dramatically more than the log benchmark. Of particular interest is the solid blue line which shows the optimal holdings when the simple security has 0% expected return. A conventional risk-averse investor would avoid holding any of this asset at all as it increases variance of terminal consumption but does not increase the mean. However in the Rank-wealth model individuals in the middle of the endowment distribution

[25] A truncated normal distribution is selected as it is simple to model numerically and exhibits the desired characteristics of non-monotonicity and unimodality. It has the further advantage of being relatively symmetrical which allows for easier identification of the rank effect in visual plots.

hold dramatically more of the risky asset than would normally be predicted. This is because the region of intermediate endowments has a very high local population density and creates a substantially convex derived utility over wealth.

In Figure 2.6 the probability of payoff varies rather than the rate of return. In this way, I can explore the effects of probability on asset demand.

In this analysis I now fix the security to payoff 300% return but allow the probability of payoff, h, to vary from 20% to 60% in 10% increments. Notice that for the lowest probability of 20%, this means that the gamble is an expected loss (E(R) = 0.2(3)+0.8(-1) = -20%).

As before, the dotted line shows the optimal holdings of a purely log investor.

Notice that for the poor the variation of the probability of payoff only has a modest effect on optimal holdings. Increasing the probability of payoff does increase demand for the risky asset but not dramatically.

The Rank-wealth can thus explain some lottery behaviour and design. Large jackpots, with near zero probability, will be especially appealing to low levels of endowment and thus may increase lottery revenues when compared to small jackpots with higher probability of payoff. The model therefore predicts behaviour that is consistent with observation that the least affluent in society tend to gamble more in long-shot games (Kallick, 1979; Clotfelter, 1979; Welte et al., 2002; Jones, 2004). Consider as well that the predicted behaviour does not suggest a 'sickness' or gambling addiction prevalence in the less affluent but rather a perfectly rational attempt to rank maximize.

The second kind of investment situation I use to demonstrate the Rank-wealth implications is that of two normally distributed assets, where again the investor has a unique opportunity to invest in these assets and need not concern him or herself with the strategic effects of their optimal policy choice. In this case the assets have a mean returns of μ_A and μ_B, and respectively, variance σ_A and σ_B. The correlation between the returns of assets A and B is assumed to be 0.4115 so that there are some benefits to diversification.[26]

[26]This particular value of correlation is completely arbitrary and does not have any

In this situation, the investor has the choice of two normally distributed assets and a risk free security. The variable α continues to represents the proportion of time zero endowment put into the riskier of the two normally distributed assets. The optimization problem in this case is:

$$\begin{aligned} \max_{\alpha,\beta} & \, E\left[U(w_T, q_{k,i})\right] \\ s.t. & \quad 0 \leq \alpha \leq 1 \\ & \quad 0 \leq \beta \leq 1 \end{aligned} \tag{2.40}$$

Where w_T is the terminal wealth and is given by the expression:

$$w_T = w_0 \left(\beta + (1-\beta) \left[\alpha \widetilde{R_A} + (1-\alpha)\widetilde{R_B} \right] \right) \tag{2.41}$$

Where $R_A \sim N(\mu_A, \sigma_A)$ and $R_B \sim N(\mu_B, \sigma_B)$ are the returns to each of the respective normally distributed assets.

The result of two-fund separation, Tobin (1958), ensures that if risk tolerances are linear then individuals will agree on the optimal proportion of risky assets to constitute their optimal risky portfolio. With this result, the only decision that differs from one individual to the next is how to split income amongst the riskless asset and the optimal risky portfolio. In a Rank-wealth framework however, individuals do not agree on a single optimal risky portfolio and the two-fund separation result is lost. I show this in Proposition 3 in but will first demonstrate it in the partial equilibrium setting of this section.

Next, I consider the effects of Rank-wealth utility on the two fund separation result. The optimal mix of two risky, normally distributed assets, is given in Figure 2.7.

The setup consists of two securities, each with normally distributed returns and the two assets are set to have the following characteristics listed in table 2.6.

In a classic framework, when utility has linear risk tolerances, then two-

qualitative effect on the results.

fund separation ensures that all individuals will hold the same risky portfolio (Cass and Stiglitz, 1970). Each individual holds a combination of the risk free asset and the optimal 'market' portfolio. The tangency portfolio for this case has approximately 30% of the investment in the higher risk security (Asset B) and 70% of the investment in the lower risk security (Asset A).

In the Rank-wealth model however, the situation changes dramatically. For very low levels of endowment the high risk security is optimally 100% of the portfolio. This effect arises because those with low levels of endowment are in a very convex region of the derived utility function and thus could be seen as risk-takers. In truth, they are still risk averse in consumption terms but because they stand to gain so much rank, and thus quality of indivisible consumption, they take on more risk.

As endowment increases, the effects of rank increase are diminished and the individual is now in a region of high marginal utility of wealth for even tiny changes of income. As a result the 'lower middle-class' are predominantly concerned about a loss of rank and therefore reduce their riskier asset B holding well below the implied tangency portfolio holdings (the dotted green line).

Further increasing endowment reduces the derived marginal utility of wealth for the investor and thus she can tolerate greater variation in her income once again. The effect of this is that the proportion of the riskier asset B steadily increases, eventually surpassing the tangency portfolio proportion. For extremely high levels of endowment, the superior return characteristics of asset B result in total allocation to this asset. This is because for such high levels of endowment, modest fluctuations in income have very little effect in either divisible or indivisible marginal utility.

It is apparent that the Rank-wealth model can readily describe why individuals who can diversify may choose not to. The Rank-wealth model also explains far greater variation in individual portfolios than a model with two-fund separation.

2.5 Sunspot equilibrium

The previous section explored both a world of certainty and a world where the investment set was either idiosyncratic or the individual generated their expectations of terminal wealth distributions without considering strategic interactions amongst consumers investment choices. I restricted the analysis to illustrate the Rank-wealth effect in the simplest way. I now consider a world of uncertainty in the context of a Sunspot equilibrium where individuals rationally predict the effects of strategic interactions in their investment decisions.

For the following analysis I assume no *intrinsic* volatility.[27] This means that preferences, initial endowments, total indivisible goods and good qualities are all invariant to the realization of a particular Sunspot state. The only source of volatility in this model is *extrinsic*. An example of this volatility might be Sunspot or random dice rolls that have no influence on the real economy. Individuals can however contract on outcomes of the extrinsic random variables. It is therefore assumed that the contracts are enforceable, satisfy the rationality constraints of all contracting parties ex ante and that the outcome of the extrinsic variable is common knowledge.

2.5.1 Role of extrinsic volatility in imperfect markets

Normally, if all individuals are risk-averse and markets are perfect, then extrinsic random variables will not be used in contracts between risk-averse individuals. The rationale for this is that the extrinsic variable can only serve to increase the volatility of the consumption of one or both individuals that use it to contract upon. Recall that the extrinsic variable has no information content and is not correlated with any intrinsic randomness that is present in real economic factors. Thus, in a common equilibrium model, extrinsic random variables will be treated as white noise and will be ignored.

[27]The model I propose and its solution technique will work equally well if intrinsic volatility is present. I only model extrinsic (sunspot) risk here as to better illustrate the results of the Rank-wealth effect in isolation but it would be a trivial matter to extend the model above to allow for intrinsic risk that impacted the actual supply of consumer goods in particular states.

However, in the presence of market inefficiencies extrinsic random variables can serve a useful purpose in contract writing between risk-averse individuals. The use of extrinsic volatility in such cases is shown in Cass and Shell (1983) and Shell and Wright (1993). When individuals are able to contract on extrinsic volatility, in the presence of certain market frictions, they can be made better off than would be achieved in a Walrasian equilibrium.

The source of market friction I assume is the indivisibility of some goods available in the consumption set. The ability to contract on extrinsic volatility is useful to investors because it can (partially) convert an indivisible good into a divisible one, allowing for a Pareto optimal allocation of resources that is superior to the Walrasian market allocation. For example, because of indivisibility individuals may not be able to trade 30% of a house but they could trade a claim that transfers home ownership with a 30% chance. A simple example illustrates:

Consider a single period trade economy with two agents who have identical preferences and differing endowments. As per most economic models, assume that the marginal utility of corn is decreasing in the amount consumed. Suppose that one individual has 20 tonnes of corn, and the other individual has an indivisible house and 1 tonne of corn. Consistent with classical models, let the marginal utility of corn be declining in the quantity available for consumption. Suppose further that the utility derived from the house is equivalent to 25 tonnes of corn. The current allocation is Pareto optimal. The only plausible trade (which would not be corn for corn) would be if individual one traded some of his corn for individual 2's house. This outcome obviously will not occur though as individual two values the house to the equivalent of 25 tonnes of corn, a price individual one cannot pay because of budget constraints. Despite this, assuming concave utility over corn consumption, the marginal utility of corn for individual two is far higher than the marginal utility for individual one. In a sense, if the house could be made divisible then there would be gains from trade and a superior Pareto optimum could be achieved.

Now suppose the two individuals could trade on an extrinsic random variable, such as Sunspot patterns or on a game of chance unrelated to the real economy. Individual two could contract to surrender the house if some random event occurred, such as a sunspot, and not if no event occurred. Individual one could promise to pay some corn unconditional on the event and would receive the house in compensation only if the agreed upon random event occurred. Both individuals are made better off by this trade. Individual one has a low marginal utility of corn so surrendering some is not particularly painful and the possible reward of a house may justify the trade. Similarly, person two would not like to lose the house but the probability of such an event could be quite low and they could be compensated by receiving corn unconditionally, for which they have a high marginal utility. In this way an extrinsic variable has partially circumvented the restriction of the house indivisibility to improve the expected welfare of all individuals.

Note that if individuals are free to contract on white noise variables, and the number of those independent variables is infinite, then good indivisibility will cease to be a market friction and the Rank-wealth effect would vanish. All locally convex areas of derived utility would become concave once more for all individuals. I argue however that the conditions for this to occur are not realistic. Even if there are an infinite number of white noise variables, they must be variables which individuals can legally contract upon and that courts would enforce the agreement. Given the strict restrictions in most industrialized nations on gambling then individuals will be forced into legally agreed upon means of randomizing their wealth. These methods include casinos, lotteries, security markets and the like. To the degree that an individual is willing to break the law, criminal gambling or even criminal activity may be another method to circumvent this market friction but it is implausible that most individuals would pursue this course and thus the rank effect will remain.

2.5.2 Sunspot equilibrium

Having established why extrinsic volatility will be useful in an uncertain Rank-wealth world, I next discuss the solution technique for this enriched model. To solve the sunspot equilibrium, a similar approach is used as in the partial equilibrium. The major difference between the partial and sunspot equilibria is that in the latter there is now an element of chance. Individuals must optimally allocate their time zero divisible endowment amongst a set of Arrow-Debreu (AD) securities which will payoff a divisible consumption commodity which can be used to purchase indivisible goods. To proceed I introduce additional required notation.

Let ζ be a countable, finite set of sunspot states with non-zero support. Further, let h_ς be the probability of sunspot state ς, where $\varsigma \in \zeta$. Let π_ς be the time zero price of an Arrow-Debreu security that pays 1 unit of divisible consumption, at time 1, if sunspot state ς occurs and 0 if any other state occurs.

As the only source of volatility in this model is extrinsic, then the supplies of each indivisible good across states is invariant. Similarly the quality of the goods are invariant but the price of each indivisible good may differ from one state to the next. I use the notion of an Arrow-Debreu (AD) commodity for each good so that the same indivisible good in each state is considered as if it were a different good. This assumption only helps clarify the expressions that will follow and has no impact on the results of the Sunspot equilibrium.

Thus, as before there are K quality types of goods, but now ζ states for each quality level. Let $q_{k,\varsigma}$ be the quality of the k^{th} good in sunspot state ς. Let $p_{k,\varsigma}$ be the price of the k^{th} commodity in sunspot state ς. There are, in total, $(\zeta \times K)$ AD commodities.

The final concept I introduce before demonstrating the solution technique is what I label as a *consumption strategy*. First, I rank order in increasing value the product of the probability of a state and the quality of each indivisible good; or in other words rank-order the utility of each AD commodity. So for example, if there was a single indivisible good and two Sunspot states that might be realized there would be two values of probability by indivisible good utility. A concrete example might be that there is an indivisible Corvette automobile and two states: sunny-days and rainy-days. In this example there would be two AD commodities, sunny-day Corvettes and rainy-day Corvettes, each with their own associated utility. To achieve rank-ordering I suppose that one of the two events is more likely. For example, suppose that the probability of a rainy-day occurrence is 1/3 while the probability of a sunny-day is 2/3.[28]

Next, the concept of a consumption strategy is which combination of AD commodities will be consumed. The logically exhaustive list in for the former example is given in the table below:

Strategy	(Indivisible goods) Consumption Plan
1	Consume neither rainy-day nor sunny-day Corvettes
2	Consume rainy-day Corvette (inferior utility to sunny-day)
3	Consume sunny-day Corvette
4	Consume both rainy-day and sunny-day Corvettes

Similar to the logic in the partial equilibrium setting, it is known that the poorest individuals will follow strategy 1, the slightly wealthier will follow strategy 2, the near-wealthy will follow strategy 3 and the very wealthy will follow strategy 4.

Unlike the partial equilibrium setting, individuals at each of the wealth bounds are not indifferent to the consumption of a good in certainty but are rather ex ante indifferent between two consumption strategies in an uncertain world. The number of wealth bounds is no longer K-1 but rather $(\zeta \times K) - 1$. Retaining i as the index for the wealth bounds, the expected utility is given by the expression:

$$EU = \sum_{\varsigma \in \zeta} h_{\varsigma} \left[u \left(\frac{r_{\varsigma}}{\pi_{\varsigma}} \left(w_i - \sum_{\varsigma \in \zeta} \pi_{\varsigma} p_{\varsigma,k} \right) \right) + q_{\varsigma,k} \right] \tag{2.42}$$

[28]While I do not explore changes in the utility derived from an indivisible good conditional on the sunspot state (as this would be a form of intrinsic volatility), the same system of equations described here would yield the solution to this more general problem. Furthermore, this specification is pretty intuitive, recalling the example of the sunny and rainy-day Corvettes, it may be that a Corvette sports car gives more utility on sunny-days (when the soft-top can be lowered) than on rainy-days.

The variable r_ς is the proportion of time zero endowment not dedicated to individual good consumption that is consumed in Sunspot state ς. As everyone has homogenous preferences and because the utility function is assumed to be linearly separable (between divisible and indivisible utility) then each individual will allocate the same amount of non-dedicated[29] endowment in the same proportion amongst states as all other individuals. Furthermore, the optimal allocation is easily found by optimizing the expected utility of any individual with respect to choices of $\overrightarrow{r_\varsigma}$ subject to the constraint $\sum_\zeta r_\varsigma = 1$. I also prohibit the short sales of divisible good in each sunspot state as they are non-credible because there is no means by which an individual can produce divisible good to meet short-sale obligations. This restriction mathematically is $r_\varsigma \geq 0 \forall \varsigma \in \zeta$.

Solving the Lagrangian of the problem above will produce the conventional expression:

$$\frac{h_\upsilon \pi_\eta}{h_\eta \pi_\upsilon} = \frac{u'\left(\frac{r_\eta}{\pi_\eta}\left(w_i - \sum_\zeta \pi_\varsigma p_{\varsigma,k}\right)\right)}{u'\left(\frac{r_\upsilon}{\pi_\upsilon}\left(w_i - \sum_\zeta \pi_\varsigma p_{\varsigma,k}\right)\right)} \tag{2.43}$$

Where $\eta, \upsilon \in \zeta$ and are particular states. While the expression above may not be analytically solvable for all utility functions, it will be trivial to find numerical solutions when $u(\cdot)$ is monotone, as is assumed throughout this thesis.

Also note that r_ς is a function of $\overrightarrow{h_\varsigma}, \overrightarrow{p_{k,\varsigma}}$ and $\overrightarrow{\pi_\varsigma}$. Thus, once those vectors are known $\overrightarrow{r_\varsigma}$ is also known. This is noteworthy when considering the identification of the system of equations that follows as the proportion of non-dedicated endowments for divisible good consumption do not contribute to the unknowns that must be found.

Next, I let $q'_{\varsigma,k}$ denote the next sequential consumption strategy after

[29] I use the term *dedicated divisible endowment* to be the amount of divisible endowment which is intended to be used for the purchase of indivisible goods. The *non-dedicated divisible endowment* is that which is intended for direct consumption (through the $u(\cdot)$ function). As a concrete example an individual might buy sufficient AD securities to achieve a wealth level of 5 in state 1. If state 1 is realized then that individual might consume 2 units of their wealth directly and use the remaining 3 units of wealth to purchase an indivisible good.

strategy $q_{\varsigma,k}$. The original indifference equations are therefore replaced by the expression:

$$\sum_{\varsigma\in\zeta} h_\varsigma \left[u\left(\frac{r_\varsigma}{\pi_\varsigma} \left(w_i - \sum_{\varsigma\in\zeta} \pi_\varsigma p_{\varsigma,k} \right) \right) + q_{\varsigma,k} \right] = \sum_{\varsigma\in\zeta} h_\varsigma \left[u\left(\frac{r_\varsigma}{\pi_\varsigma} \left(w_i - \sum_{\varsigma\in\zeta} \pi_\varsigma p'_{\varsigma,k} \right) \right) + q'_{\varsigma,k} \right] \tag{2.44}$$

Later, I will refer to the equation above as the *cross-state indifference equation.* In addition to the $\zeta \times K - 1$ cross-state indifference equations there are also ζ AD price equations. The price of each AD security is simply the aggregate amount of time zero endowment spent on said security. The aggregate spending includes the allocation for both divisible and indivisible consumption in the respective Sunspot state, by all individuals. The AD security price conditions are:

$$\pi_\eta = \int_0^{\bar{w}} f(w) \left[r^*_\varsigma \left(w - \sum_{\varsigma\in\zeta} \pi_\varsigma p_\varsigma(q_{k,\varsigma}(w)^*) \right) + \frac{p_\eta(q_{k,\eta}(w)^*)}{\pi_\eta} \right] dw \tag{2.45}$$

Where $q_{k,\varsigma}(w)^*$ is the optimally selected quality of indivisible good to be consumed by individual with wealth w in sunspot state ς. Recall from the partial equilibrium discussion that f(w) is simply the partial density function of the cumulative wealth mass function, F(w).

Finally there are $K \times \zeta$ market clearing conditions. Expressing the market clearing condition equations is facilitated by the use of matrix and vector notation. I first introduce two vectors, the first is the cumulative density of the population with wealth less than w_i and the second vector of the supplies of the K indivisible goods. I define a vector of masses, $\overrightarrow{M(w)}$, to be the mass of population between two adjacent wealth boundaries.

$$\text{Let } \overrightarrow{F(w)} = \begin{bmatrix} F(w_1) \\ F(w_2) \\ .. \\ .. \\ F(w_{\zeta \times K-1}) \\ 1 \end{bmatrix}, \vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ .. \\ .. \\ s_K \end{bmatrix},$$

$$\overrightarrow{M(w)} = \begin{bmatrix} F(w_2) - F(w_1) \\ F(w_3) - F(w_2) \\ .. \\ .. \\ F(w_{\zeta \times K-2}) - F(w_{\zeta \times K-1}) \\ 1 - F(w_{\zeta \times K-2}) \end{bmatrix}$$

Lastly, I introduce a matrix that expressed the various consumption strategies. Each row represents a consumption strategy. Each column represents an AD commodity, with the first column representing the highest rank-ordered AD commodity and the last column the lowest ranked commodity. A zero indicates that the associated AD commodity is not consumed while an entry of 1 indicates that the AD commodity is consumed. Thus, **C** will be of the following form:

$$\mathbf{C} = \begin{bmatrix} 0 & ... & 0 & 0 \\ 0 & ... & 0 & 1 \\ 0 & ... & 1 & 0 \\ 0 & ... & 1 & 1 \\ ... & & & \\ 1 & .. & 1 & 0 \\ 1 & ... & 1 & 1 \end{bmatrix}$$

The market clearing conditions can now be expressed as:

$$\mathbf{C^T}\overrightarrow{M(w)} = \vec{s_k} \tag{2.46}$$

In the unknowns, there are $\zeta \times K - 1$ wealth boundaries, ζ AD prices, and $K \times \zeta$ good prices to be identified. Thus, the system is exactly identified regardless of the selection of ζ or K.

Solving this system of equations analytically will not be tractable for most interesting specifications of $F(\cdot)$ and $u(\cdot)$ but solving this same system numerically is often feasible.

To solve the system of equations, an optimizer selects a vector of AD security prices, AD commodity prices and wealth boundaries. The squared difference between the left-hand side and right-hand side of each of the equations described above is then found. The optimizer objective function is to minimize the sum of square differences of the conditions listed.

As the sunspot equilibrium is of such a higher dimension than the partial or idiosyncratic equilibrium given in section 2.4.9, it is not as easy to solve. To limit the dimensionality of the problem I arbitrarily restrict the problem to two states and one indivisible good. This is the simplest possible sunspot problem which can be solved.

Examining Figure 2.8 one can see the same essential characteristics as were identified in the partial equilibrium setting. Namely, the very poor do not invest a large amount of their time zero endowment in the less probable state for they would not be able to purchase the indivisible good even if they did. As endowment hits the first wealth boundary, w_1, these individuals shift a large proportion of their time zero endowment into the lower probability state because they can purchase the indivisible good in that state. For individuals with endowment in the bounds of w_2 and w_3, they can purchase the indivisible good in the higher probability state and dramatically decrease their consumption in the low probability state for this purpose. Beyond w_3, individuals can afford both indivisible goods and allocate their endowment to each state in a fashion that converges to the same value as if no indivisible goods were present.

2.5.3 Different indivisible goods and anomalies

The results of the analysis will assume that individuals can only consume one type of indivisible good per Sunspot state. For example, an individual might consume a large house if their lottery ticket wins and a small house if it doesn't but they will not consume both a small and large house in the same state.

The restriction of an individual consuming only a single indivisible good per state is not essential to achieving the rank-result, nor is it necessary for the solution technique I described in section 2.5.2 but it will simplify the discussion.

In the event that one wished to model simultaneous consumption of various indivisible goods the consumption matrix and supply conditions could easily be modified to represent this design. An example of such a situation would be if there were two qualities of cars, two quality of houses and two states (sunny/rainy) but each individual could only have at most one house and one car in each state.

Having numerous goods will reduce the rank of the consumption matrix as every possible combination of 1 and zero will no longer be permissible. The system of equations will remain fully identified however because as each consumption strategy is removed, reducing **C** by one rank, there will be exactly one new consumption constraint condition to exactly offset the reduction in rank.

Though the complete Sunspot equilibrium is harder to solve than the partial equilibrium it has a number of advantages over the latter model. As listed previously the Sunspot model produces general equilibrium prices for the Arrow-Debreu securities and can model intrinsic uncertainty. Two more advantages of the Sunspot model are that it can potentially explain the Endowment effect and why individuals do not hold the same risky portfolio. The interesting part of disagreement upon which risky-portfolio investors choose to hold is that it does not require information asymmetry to achieve this result. As the rank-model predicts failure of two-fund portfolio separation then the theory may also offer a partial theoretical explanation as to why the CAPM is not well-supported empirically.

I proceed by showing the endowment effect and the two-fund separation failure as propositions. For the first proposition, I will prove by contradiction but before continuing I first clarify what a Sunspot state is and define discount for this context. A Sunspot state is a particular realization of the extrinsic random variable that governs which AD securities will payoff. For example, a coin can land heads or tails. The realization of 'heads' is one Sunspot state while the realization of 'tails' is the alternative Sunspot realization. This is titled a Sunspot realization because the coin toss does not have any impact on actual economic production and thus does not affect the supply of indivisible goods in each realizable state of the world.

I define discount to mean that an indivisible good trades at a price in divisible good such that if that amount of divisible good were consumed it would produce less utility than consuming the indivisible good. For example, if an indivisible good produces one utile and if consumers all have one unit of divisible corn which produces square root utility then the *non-discounted* price of the indivisible good is one unit of corn. A discounted price, in this case, is any amount less than one unit of divisible corn.

Another way of thinking about the discount is to instead say that there are 'jumps' in the derived utility function within each Sunspot state with respect to increasing wealth. In other words, once a state is realized no individual exists who is just indifferent to consuming the indivisible good or consuming that good's price in divisible consumption instead.

Proposition 2. *In the Sunspot specification of the Rank-wealth model, indivisible goods in each realized sunspot state are priced at a discount.*

Proof. I prove the proposition by contradiction. First, recall that an individual at a wealth bound that transitions from one consumption strategy to the next should be just indifferent to the two strategies, let w_s denote this wealth bound. Suppose that the quality of good in each state q_k is optimally chosen given the price of said good in state i being $p_i(q_k)$. The *cross-state* indifference condition between the two strategies is:

$$\sum_i h_i \left[u\left(\left(\frac{r_i}{\pi_i}\right)(w_s - \sum_i \pi_i p_i(q_{k,i})) \right) + q_{k,i} \right] = \sum_i h_i \left[u\left(\left(\frac{r_i}{\pi_i}\right)(w_s - \sum_i \pi_i p'_i(q'_{k,i})) \right) + q'_{k,i} \right] \tag{2.47}$$

However, if there is no discount the *within-state* indifference also holds; that is:

$$u\left(\left(\frac{r_\varsigma}{\pi_\varsigma}\right)(w_s - \sum_{i\neq\varsigma}\pi_i p_i(q_{k,i})) - \pi_\varsigma p_{\varsigma,k}\right) + q_\varsigma = u\left(\left(\frac{r_\varsigma}{\pi_\varsigma}\right)(w_s - \sum_{i\neq\varsigma}\pi_i p_i(q'_{k,i})) - \pi_\varsigma p'_{\varsigma,k}\right) + q'_\varsigma \tag{2.48}$$

Substituting the latter equation into the right hand side of the former equation yields the expression:

$$\sum_{i\neq\varsigma} h_i\left[u\left(\left(\frac{r_i}{\pi_i}\right)(w_s - \sum_i \pi_i p_i(q_k))\right) + q_i\right] = \sum_{i\neq\varsigma} h_i\left[u\left(\left(\frac{r_i}{\pi_i}\right)(w_s - \sum_i \pi_i p_i(q'_k))\right) + q'_i\right] \tag{2.49}$$

However, because the only change in indivisible good was for state ς, then $q_i = q'_i \ \forall \ i \neq \varsigma$ which implies:

$$\sum_{i\neq\varsigma} h_i\left[u\left(\left(\frac{r_i}{\pi_i}\right)(w_s - \sum_i \pi_i p_i(q_k))\right)\right] = \sum_{i\neq\varsigma} h_i\left[u\left(\left(\frac{r_i}{\pi_i}\right)(w_s - \sum_i \pi_i p_i(q'_k))\right)\right] \tag{2.50}$$

However, because the individual bought more ς AD securities to purchase the improved quality of indivisible good in that state they would have had reduced budget for divisible consumption in all other states, or equivalently:

$$\sum_i \pi_i p_i(q_k)) < \sum_i \pi_i p_i(q'_k)) \tag{2.51}$$

Yet, as $u(\cdot)$ is monotonically increasing in the interior argument, this implies that:

$$\sum_{i \neq \varsigma} h_i \left[u\left(\left(\frac{r_i}{\pi_i} \right) (w_s - \sum_i \pi_i p_i(q_k)) \right) \right] >$$
$$\sum_{i \neq \varsigma} h_i \left[u\left(\left(\frac{r_i}{\pi_i} \right) (w_s - \sum_i \pi_i p_i(q'_k)) \right) \right] \tag{2.52}$$

Which contradicts the finding from the earlier substitution. Thus it must be that the *within state indifference equation* for any arbitrary Sunspot state does not hold. In fact, the only way by which the *across state indifference equations* could hold is if:

$$u\left(\left(\frac{r_\varsigma}{\pi_\varsigma} \right) (w_s - \sum_{i \neq \varsigma} \pi_i p_i(q_{k,i})) - \pi_\varsigma p_{\varsigma,k} \right) + q_\varsigma <$$
$$u\left(\left(\frac{r_\varsigma}{\pi_\varsigma} \right) (w_s - \sum_{i \neq \varsigma} \pi_i p_i(q'_{k,i})) - \pi_\varsigma p'_{\varsigma,k} \right) + q'_\varsigma \tag{2.53}$$

□

The corollary of the existence of a quality premium is the existence of an *Endowment effect*. As has just been demonstrated, the Rank-wealth model implies that in any particular realized state their must be 'jumps' in the derived utility of endowment. Moreover, in each Sunspot state there are certain AD payoffs which have zero population support; that is nobody has that level of AD securities. Both of these factors contribute to the explanation of the Endowment effect.

If there are jumps in the derived utility of consuming an indivisible good then, it is evident that an individual will demand a premium of divisible good to surrender that good. In contrast, those individuals who have less divisible good and are not endowed with the indivisible good have such a high marginal utility for the divisible good they will only offer a small sum to acquire the indivisible good.

Figure 2.9 demonstrates the endowment effect by illustrating the derived utility over AD payoffs received. Notice that the support for certain AD security payoff levels is zero. Also illustrated on the graph is the WTS and WTP of the marginal individuals which have or do not have, respectively, the indivisible good.

The gaps of zero-level support mean that there are no individuals who are just indifferent to consuming the indivisible good or instead receiving the market price of that good in divisible consumption. The individuals who would be so indifferent do not exist (zero-support) and thus a researcher would rarely, if ever observe WTS=WTP.

It is interesting to note that the Rank-wealth model explains the Endowment effect only by good fortune. The model was derived with the purpose of explaining lottery regressivity and suboptimal diversification, the fact that it also explains another behavioural anomaly is surprising and may hint at the efficacy of the Rank-wealth in describing real individual behaviour.

The Rank-wealth model also yields an interesting hypothesis that individuals who are endowed with a good they do not presently own, but value, will have a higher WTS than those who already have that same good. This seems a rather simple conjecture to test and would be markedly different than what classic economics would conjecture. Namely, in classic economics, those individuals who have the means to acquire an object and have not yet done so would be assumed to have low marginal use for that good. In the Rank-wealth model, those individuals with the means but who have not yet acquired the object of discussion could garner more utility from that good than those who already own it.

The other listed advantage of the Sunspot equilibrium is that it can partially explain the failed empirical support for the CAPM. The partial explanation is achieved because a critical assumption required for the CAPM derivation, namely two-fund separation, fails in the presence of the Rank-wealth model.

Proposition 3. *The Rank-wealth framework rejects portfolio separation demonstrated in Tobin (1958).*

Proof. I prove that the Rank-wealth framework rejects the portfolio separation result by contradiction. Cass and Stiglitz (1970) show that for portfolio separation to hold a necessary condition is that:

$$u'(w) = (a + bw)^c \tag{2.54}$$

Where a,b and c are constants. In the case of the Rank-wealth model, the analog of this necessary condition is:

$$u'(w - p_k) = (a + bw)^c \tag{2.55}$$

This condition must also hold for a larger wealth, w^*.

$$u'(w^* - p_{k+1}) = (a + bw^*)^c \tag{2.56}$$

However, letting $w^* = w + p_{k+1} - p_k$ gives the expression:

$$u'(w - p_k) = u'(w - p_k) \Rightarrow (a + bw^*) = (a + bw) \tag{2.57}$$

Which reduces to:

$$bw^* = bw \Rightarrow 0 = b(w^* - w) \tag{2.58}$$

Recall $w^* - w = w + p_{k+1} - p_k - w = p_{k+1} - p_k$ and $p_{k+1} - p_k \neq 0 \Rightarrow b = 0$
However, if b=0 then:

$$\frac{d(a + bw)^c}{dw} = 0 \Rightarrow \frac{du'(w - p_k)}{dw} = 0 \tag{2.59}$$

However, this is not true by the assumption that $u(\cdot)$ is non-linear. Thus, it must be that Rank-wealth utility does not exhibit linear risk tolerance. By the result in Cass and Stiglitz (1970) then portfolio separation does not hold.

□

The Sunspot model is compelling because it allows the nesting of the Rank-wealth mechanism presented in the partial equilibrium model as well as explaining two other 'anomalies' that the partial equilibrium cannot. Those two anomalies are the endowment effect and the failure of the CAPM to find empirical support. The former anomaly is resolved by demonstrating that in any realized state all indivisible goods must be trading at a discount implying that there will be a difference between a willingness to sell and a willingness to buy. This bid-ask gap can explain why experimental studies that assign an indivisible good to an individual affects the perceived value of that good. With respect to the CAPM, by implying failure of portfolio separation the Sunspot model ensures that the CAPM in its canonical specification may not be correct. Further research into a modified form of the CAPM which accounts for the Rank-wealth effect is warranted but is beyond the scope of this thesis.

2.6 Conclusion

The fact that most classic models of investor choice under risk fail to predict observed individual behavior is both a challenge and an opportunity. The challenges exists as while economists cannot predict individual choices, it remains questionable if macroeconomic theories can be correct. The challenge of explaining individual behavior is enhanced by the desire to find a parsimonious, and rational explanation for the patterns in observed individual investment decisions. Yet, there is also a remarkable opportunity made available to researchers by the present lack of a complete theory of choice. The opportunity is that if a superior model of risky decision making can be found, entirely new testable implications will become evident.

While it is true that Prospect Theory can explain a great deal of individual choice already, it is unclear as to why this behavior is rational and it is often not apparent how to formalize the model to derive new testable implications. In contrast, the Rank-wealth model I proposed in this manuscript nests much of the explanatory power of Prospect Theory but does so by starting with a rational framework that allows for further implication derivation. The Rank-wealth model also gives rise to a theoretical justification for a "keeping up with the joneses" type of preference without having to assume it *a priori*.

Moreover, the model I propose is gratifying in that it does not rely on a direct assumption of rank relevance in the utility function to explain individual, as would be the case of Veblen's or Frank's theories of conspicuous consumption. Instead, using the intuitive assumption that goods are indivisible and available only in finite supply, the Rank-wealth model can resolve many financial anomalies simultaneously. With the Rank-wealth model, theorists can believe that marginal utility is strictly decreasing in consumption and yet still be able to explain financial anomalies by noting that derived marginal utility in wealth need not be strictly decreasing. The Rank-wealth model may be a particularly useful way to think about the nature of the consumption set as many 'anomalies' may not be so anomalous when a dash of realism is adopted.

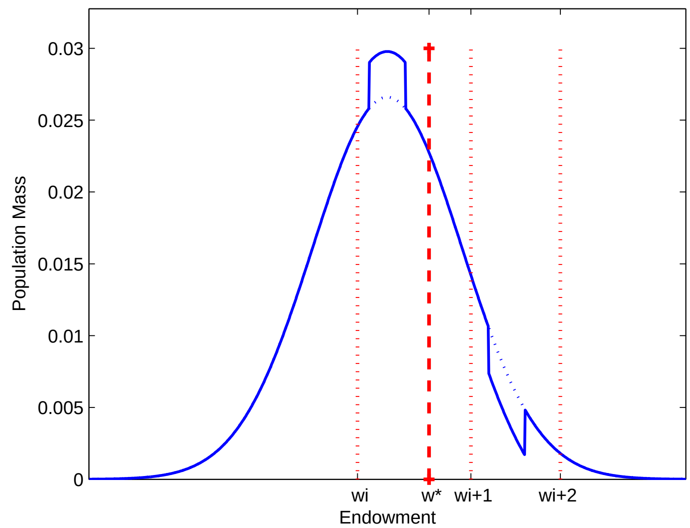
Table 2.1: Derived utility by endowment: similar endowments.

Endowment	Derived utility -Indivisible good	Derived utility -Divisible good	Derived utility -Total	$\frac{\Delta \text{Derived utility}}{\Delta \text{Endowment}}$
0	0	$\sqrt{0}$	0+0=0	N.A.
1	0	$\sqrt{1}$	0+1=1	$\frac{1-0}{1-0} = 1$
1.5	1	$\sqrt{1.5-1}$	1+0.71=1.71	$\frac{1.71-1}{1.5-1} = 1.42$

Table 2.2: Derived utility by endowment: dissimilar endowments.

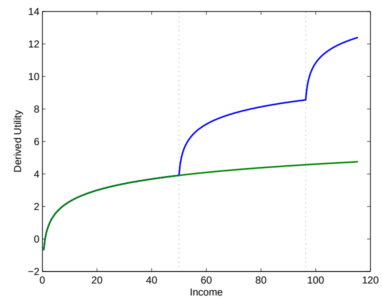
Endowment	Derived utility -Indivisible good	Derived utility -Divisible good	Derived utility -Total	$\frac{\Delta \text{Derived utility}}{\Delta \text{Endowment}}$
0	0	$\sqrt{0}$	0+0=0	N.A.
1	0	$\sqrt{1}$	0+1=1	$\frac{1-0}{1-0} = 1$
3	1	$\sqrt{3-1}$	1+1.41=2.41	$\frac{2.41-1}{3-1} = 0.71$

Figure 2.1: Depiction of mass transportation

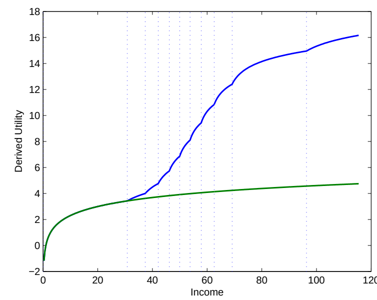


Characteristic	Asset A	Asset B
Expected Return	+5%	+10%
Standard Deviation	3.93%	8.86%
Correlation(A,B)	-0.4115	

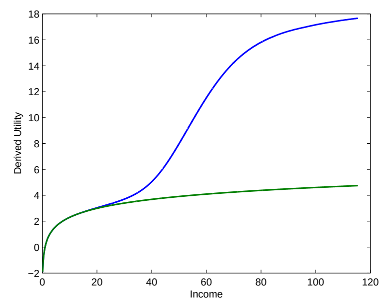
Figure 2.2: Derived utility as a function of endowment, varying number of different qualities of indivisible good



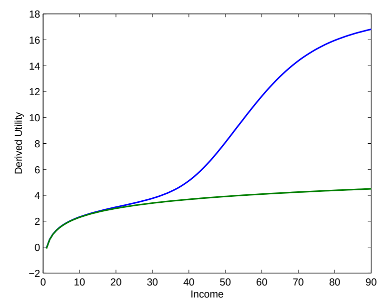
K $= 2$, $q_{i+1} - q_i = 4.71$



K $= 10$, $q_{i+1} - q_i = 1.28$

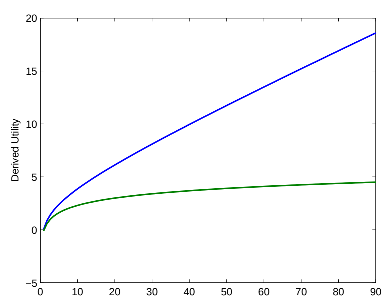


K $= 100$, $q_{i+1} - q_i = 0.14$

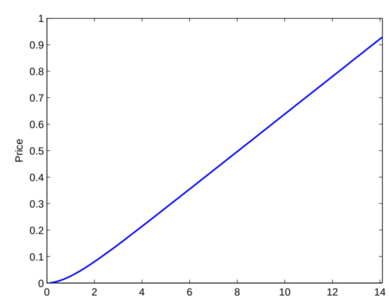


Continuous

Figure 2.3: Derived utility and price of quality when wealth and quality distributions match

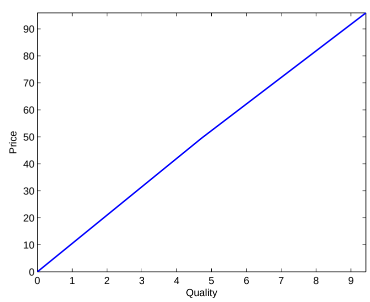


Derived utility over endowment

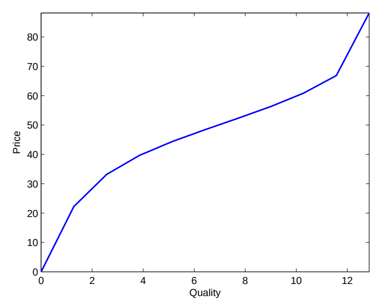


Price as a function of quality

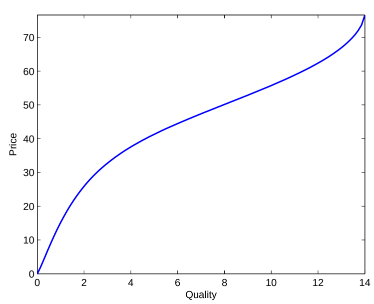
Figure 2.4: Price as a function of quality, varying number of indivisible goods



K = 2



K = 10



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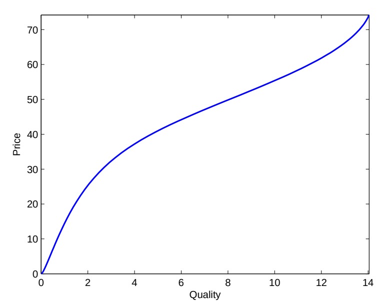


Continuous

Figure 2.5: Payoff effect on gambling tendency

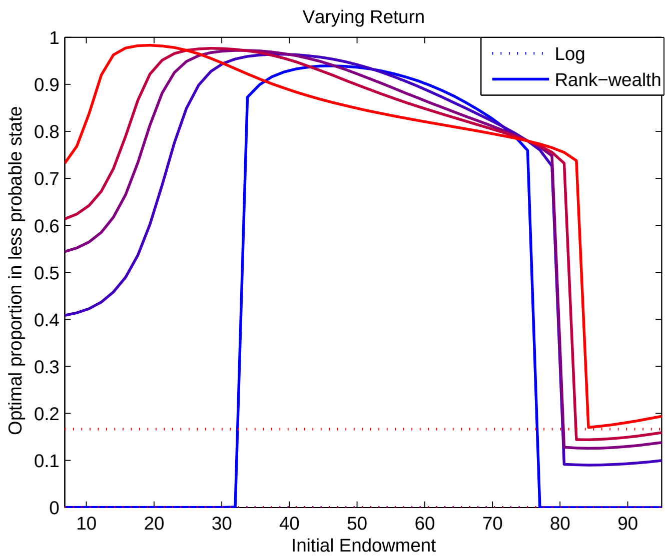


Figure 2.6: Probability effect on gambling tendency

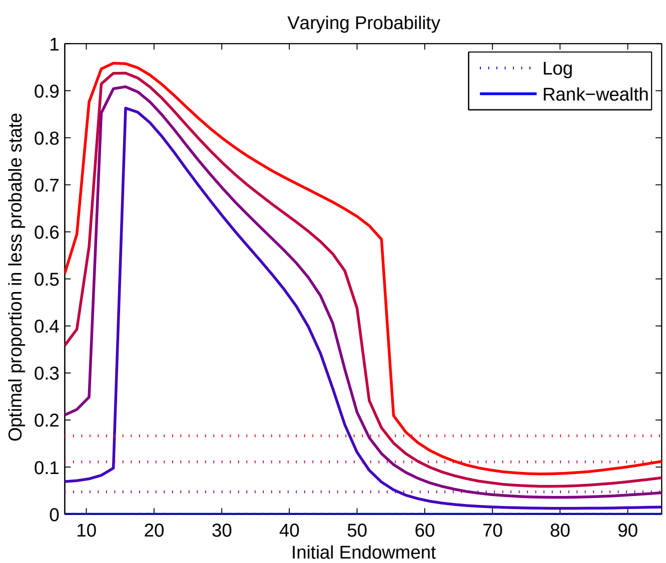


Figure 2.7: Optimal portfolio allocations for Tobin-separation and Rank-wealth

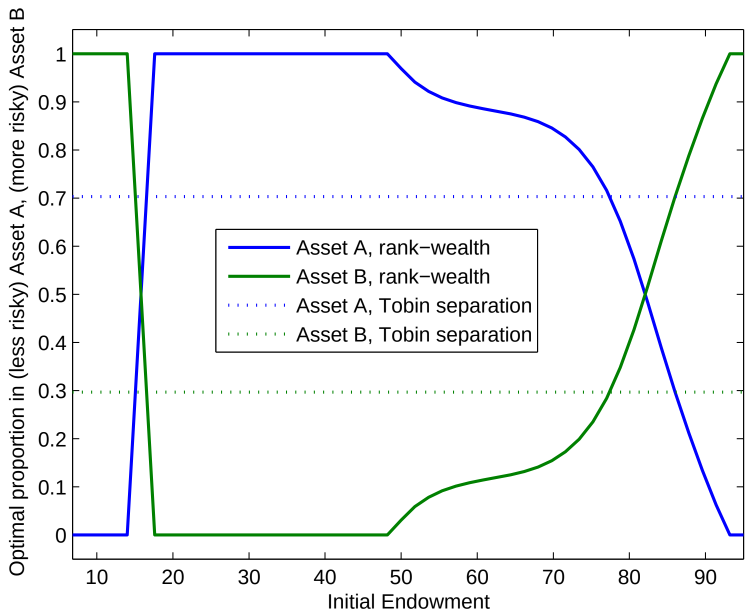


Figure 2.8: Optimal risky allocation by endowment: Sunspot-equilibrium

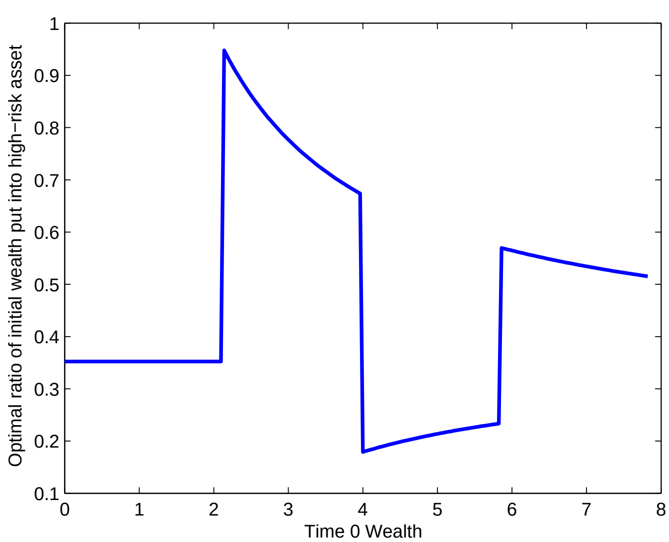
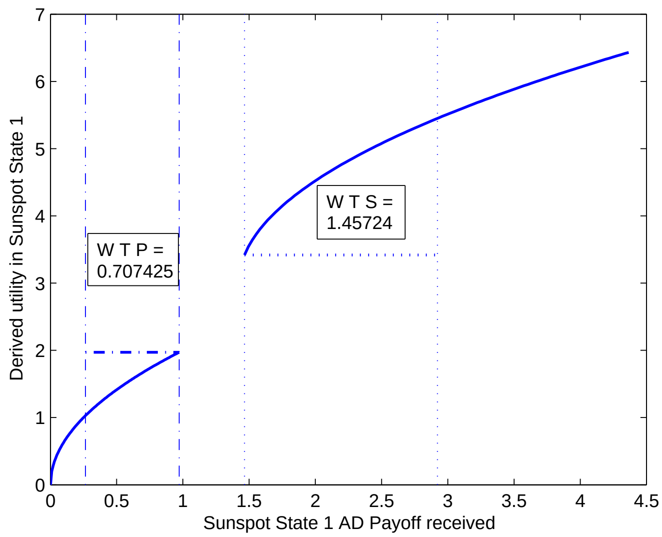


Figure 2.9: Realized Sunspot-state derived utility and Endowment effect



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Chapter 3

Testing the Rank-wealth hypothesis[30]

3.1 Introduction

There is mounting evidence that individuals' risk-aversion, absolute at least and relative possibly, is positively correlated with wealth.[31] Whether we consider lottery regressivity, suboptimal diversification or preference for 'lottery-type stocks', empirical observations are quickly undermining faith in classic economic models in which investors have non-increasing risk aversion. One possible explanation for increasing risk aversion in wealth is that individuals vary in type and the risk-averse individual accumulates more wealth than the risk-seeking individual. Another possibility is that the poor rationally decide to take on more risk than the rich for reasons not explicit in classic economic models. In this manuscript, I consider the second of these two possible explanations for increasing risk aversion and demonstrate support for what I term the Rank-wealth model.

Classic economic theories of individual behavior often suggest that the marginal utility of wealth declines in the absolute level of wealth. This result arises from two underlying and joint assumptions that are often made together and less frequently made explicit. The first of these joint assumptions is that the marginal utility of consumption declines in the level of consumption. The second assumption, and the one which this manuscript

[30] A version of this chapter will be submitted for publication. Newton, David I. Testing the Rank-Wealth Hypothesis

[31] Goetzmann and Kumar (2008); Polkovnichenko (2005); Kelly (1995); Blume and Friend (1975)

will challenge, is that consumption and wealth are linearly related. For if wealth and consumption are not linearly related then the situation can arise where the marginal utility of consumption declines in the level of consumption but the marginal utility of wealth could increase in the level of wealth.

In the model that I propose the quality of goods consumed, and thus utility derived from those goods, will be a nonlinear function of the wealth expended on consumer goods. In particular, the prices of various goods in the economy will be endogenously determined in such a way that it is the *rank* of an individual's wealth which determines their bundle of consumer goods. As a result of this nonlinear-mapping between wealth and consumption individuals in my model will have the incentive to take on financial wealth risk and will potentially even be risk-seeking in wealth. Yet, these same individuals will remain fundamentally risk-averse in the context of consumption.

To demonstrate how this model works a short example of a tournament or auction can be used to easily convey the concept of a non-linear relation between consumption and wealth. Imagine, for example, an auction where two individual bidders have a common and additive utility function given by the expression:

$$U(H, w_i) = u(w_i - P_i(H)) + H \tag{3.1}$$

Where w is the wealth of the individual, $u(\cdot)$ is a concave downwards function and H is the utility gained by owning a home of quality H. Suppose also that the two individuals start with different amounts of wealth, $w_A < w_B$, and may use their wealth in an auction to bid on a single available home of quality H where each pay their bid $P_i(H)$.

It is apparent that the clearing price, p, of the house will be given by the expression:

$$p = w_A - u^{-1}\left[u(w_A) - H\right] \tag{3.2}$$

This expression above is the price of the house because at this level the less endowed individual is just indifferent to the actions of purchasing the house and not purchasing it. If the house traded at any price lower than this level the less-endowed individual would have the incentive to bid up the price of the house. At any price higher than this level the well-endowed individual would be effectively overpaying for the house as there is no competition for the asset. At any price between the clearing price p and w_A the poorer individual would be able to bid more on the house but would find it an irrational act that reduces utility. As both individuals desire the house and because the wealthier individual has a lower marginal utility of wealth (w), then the clearing price will be just sufficient so that the wealthy individual acquires the house in equilibrium.

Retaining this equilibrium, now consider the actions of the poorer individual if he or she were offered a unique risky bet, unavailable or unknown to the wealthier individual. To make this example concrete imagine the poorer individual was given the bet $[0.5, 0.5 : +\epsilon, -2\epsilon]$, and that $w_A = 1$, $H = w_B = 1.1^2$ and $u(x) = \sqrt{x}$.

If the poorer individual does not bet then he or she would receive $u(w_A) = \sqrt{1} = 1$ with certainty. Second order stochastic dominance (SSD) then ensures that the individual require a consumption risk-premium to take on any bet that increases the variance of the consumption payoff (Hanoch and Levy, 1969). However, by introducing an indivisible good the derived utility function over wealth can become non-concave and nullify the implication of SSD. To demonstrate this effect, consider the expected utility if the poorer individual did take the bet. One of two possible situations could arise from the bet being taken. The two possibilities are:

$$E[u(\cdot)|w_A + \epsilon > w_B] = 0.5(u(w_A + \epsilon - p_{new}) + H) + 0.5(u(w_A - 2\epsilon)) \quad (3.3)$$

where $p_{new} = u(w_B) - u^{-1}\left[u(w_B) - H\right]$ or

$$E[u(\cdot)|w_A + \epsilon < w_B] = 0.5(u(w_A + \epsilon)) + 0.5(u(w_A - 2\epsilon)) \quad (3.4)$$

If the second case arose, then because $u(x) = \sqrt{x}$ is concave down and because the bet increases risk and has a negative expected payoff $(0.5(+\epsilon)+0.5(-2\epsilon)=-0.5\epsilon)$ it is clear that according to classic expected utility theory (EUT) the poorer individual should not take the bet. However if the first case arose then:

$$E[u(\cdot)|w_A+\epsilon>w_B]=0.5\left[\sqrt{\epsilon-0.1}+1\right]+0.5\left[\sqrt{1-2\epsilon}\right] \tag{3.5}$$

Which is greater than the certain utility of 1 if $\epsilon\in[0.11505,0.3961]$. Thus, despite that the decision to avoid gambling dominating the choice to gamble when there exists no auction, in the presence of an auction the choice to gamble can stochastically dominate the safe choice. The poorer individual may be prompted to gamble if they can gain in *rank* against other consumers competing for the same goods.

This manuscript explores the effect that rank in a wealth distribution has on an investors' propensity to take on risk. The above simple model demonstrates that in the presence of indivisible consumer goods, individual wealth-rank is an important determinant of utility. I term this enriched expected utility model, which includes indivisible goods, the wealth-rank model.

3.1.1 Circumstance or type

Numerous studies (Goetzmann and Kumar, 2008; Polkovnichenko, 2005; Kelly, 1995; Blume and Friend, 1975) have documented that poor individuals tend to diversify less and gamble more than the rich. Yet, a natural question that follows is which causes which? Is it that people who like to gamble and diversify less become poor, or is it that people who are poor choose optimally to gamble more and diversify less?

Answering this question is important in Finance as it sheds light on both risk-taking behavior and on portfolio decisions. Fama and French (1996b) demonstrates that asset pricing anomalies, which often preoccupies academic attention, may be resolved by simply finding the 'correct' pricing factors. A long-line of literature, including Fama and French (1996a), has shown that the classic CAPM (Sharpe, 1964; Litner, 1965) is often inadequate in describing asset returns. Various alternative asset pricing models have been proposed, most notably Fama and French (1992), which have identified numerous factors that can better describe cross-sectional returns than the CAPM. Though good theoretical progress has also been made to explain why these factors might be priced (Carlson et al., 2004), this manuscript takes a different approach. I will challenge the fundamental assumptions about derived individual preferences and then show that the implications of my new assumptions can partially explain individual portfolio allocation. This manuscript contributes by testing the founding assumptions of what I term the Rank-wealth model and thereby builds a case for further research into this explanation of pricing anomalies.

Determining why individuals gamble and how they make portfolio allocation decisions is also a very important topic from a public policy perspective. Many states derive notable portions of their budget from lottery sales but because the poor tend to spend a greater proportion of their incomes on lotteries than the rich, state lotteries are often criticized as being regressive (Combs et al., 2008; Beckert and Lutter, 2008; Mobilia, 1992). If buying lottery tickets can be shown to be a rational action rather than a symptom of limited education or poor cognitive abilities then states may elect to regulate lottery playing in much the same fashion as they regulate alcohol consumption; allowing for some indulgence but not so much as to be destructive.

In addition, this question is also salient to sociology, criminology and psychology as answering it may offer insights into risk seeking behavior. In particular, if individuals gamble not because of their intrinsic type but because of their external circumstance then the implied methods to treat gambling addictions may be dramatically different than those currently employed. The results of this analysis also bear on social policy as they suggest that criminal elements are caused by social structure and not by a persistent occurrence of criminal types being born into society. The theory I test may help explain why some socially progressive countries often have low incarceration rates and per capita violent crimes (Soares, 2004; Fajnzylber et al., 2002b,a; Chiu and Madden, 1998; Freeman, 1996).

Within the context of portfolio optimization, the theory I test may explain why gambling is a popular pastime despite clearly having a negative expected rate of return and increasing the variance of individual wealth. Though some economists may consider that gambling is a form of leisure, the empirical evidence does not support this view. Blalock et al. (2007) demonstrates that when economic hardship, as measured by unemployment rates, increases within states in the USA lottery ticket sales increase but movie-ticket sales decline. Given that the two activities, gambling and movie-going, have comparably low cost participation fees it is not clear why individuals would strongly favor one leisure activity over another during economic recessions. However, if we allow for gambling to serve not only a purely leisure role, but also serve as a form of 'investment', it may be fully rational for individuals to gamble more during an economic downturn.

Extending from the theory that I introduce in the previous manuscript, I empirically test five core hypotheses. The first two hypotheses I test relate to individual risk-taking and I will refer to them as gambling hypotheses. The remaining hypotheses will be referred to as portfolio hypotheses and they explore the relative performance of individuals' portfolios.

Of the gambling hypotheses, the first predicts that individuals residing in regions with higher degrees of wealth concentration should have greater demand for Lottery-type-stocks (LTS). The classification of stocks into lottery-type and non lottery-type follows the methodology of Kumar (2009) and is discussed in the following section. The second gambling hypothesis I test is that proclivity to lottery-type-stock holdings[32] is a function of an individual's rank in the wealth distribution; those of lowest rank should have the greatest demand for LTS. This second hypothesis differs markedly from the common economic perspective that individuals are of specific risk-types and either enjoy or dislike gambling because of their nature.

In addition to the gambling hypotheses, the Rank-wealth theory also predicts portfolio characteristics of various individuals. The first portfolio hypothesis I test is that individuals of lower rank should tend to hold

[32] Considered a measure of risk-taking

portfolios with greater risk-exposure, identified either by CAPM Beta or Fama-French 3 factor loadings. The rank model I conjecture predicts this relationship because very low ranked individuals should have rapidly increasing marginal utility in the level of wealth. This will occur in a society where wealth is approximately unimodally and non-monotonically distributed and thus low ranked individuals will be relatively insensitive to dollar losses but highly motivated by potential dollar gains. The second portfolio hypothesis predicted by my model is that the portfolios of high rank individuals should highly correlate with one another and that the portfolios of low rank individuals should not correlate with one another. This hypothesis stems from the fact that highly ranked individuals will attempt to maintain their rank and will have little to gain by small increases in their net worth. As such, those that are highly ranked should all hold portfolios which do not frequently suffer large losses and should correlate with each-other. In contrast, low ranked individuals are hypothesized to construct portfolios that generate large idiosyncratic shocks to portfolio value that would allow for maximal rank gain. The presence of large idiosyncratic shocks in the portfolios of low ranked individuals would imply that they have a low degree of correlation to the portfolio returns of higher ranked individuals.

To summarize the key hypotheses I test are:

Hypothesis	Type	Conjecture
1	Gambling	Individuals living in regions with high degrees of wealth concentration should have greater proportional LTS holdings
2	Gambling	Individual LTS demand should be highest for low-ranked individuals in a society with unimodally distributed wealth
3	Portfolio	Low ranked individuals should have high risk portfolios as measured by CAPM Beta or Fama-French loadings
4	Portfolio	Portfolios with higher than median LTS proportion should have relatively poor raw returns but provide rank gain
5	Portfolio	Portfolios returns of high ranked individuals should be positively correlated with eachother. Portfolio returns of low rank individuals should not positively correlate with other portfolios

3.2 Data and methodology

In order to identify the risk taking of individuals I use the same data and approximately the same filters as Kumar (2009). Namely, I analyze the holdings of over 80,000 individual trading accounts at a large U.S. retail brokerage from the years 1991-1996. The average account dollar value is $55,240, with $41,036 in common equity and has on average 5.49 individual securities. For a sub-sample of the data there is additional demographic data which was collected in 1996 by Infobase. The additional demographic data includes information on home-ownership, investor gender, investor age, career type. This additional information spans 16,255 accounts. A complete description of the data is available in Odean (1998). I use the proportion of funds allocated to lottery-type stocks (LTS) as a measure of risk taking. I compare individual LTS holdings to relative to various benchmarks and compute the LTS proportion at each month's end. The classification of a security into lottery-type and non lottery-type follows the methodology introduced by Kumar (2009) and is described in the following subsection.

The central hypotheses tested in this manuscript require measures of an individual's wealth and of their rank in the distribution of wealth in the population which they share consumer markets. Unfortunately, even the data provided by Odean (1998) data is not detailed enough to include this information.[33] Recall that the brokerage data only captures a part of an individual's total wealth, namely the value of all accounts at one brokerage. I am therefore forced to use the data available to construct proxies of the desired measures. To proxy for wealth I use the aggregate market value of all securities in an individual's portfolio at the end of month. Though it is almost certain that individuals have large components of their personal wealth that are not described in the data set it is reasonable that over a large data set, such as the one analyzed, there would exist at least a positive correlation between total net wealth and trading account value. This means that the analysis I conduct should identify underlying relationships between LTS holdings and approximate wealth which is sufficient to make a case for the Rank-wealth model. As individual total net worth and trading account value at one brokerage are probably not perfectly correlated, it is reasonable to expect a great deal of unexplained variation in the regression models that follow and thus the expectation is that the models testing the relationship between LTS holdings and wealth will have a relatively weak fit as measured by the adjusted R^2. My analysis will therefore concentrate mostly on the significance and sign of various correlations rather than the overall fit of the model.

A relevant critique to the use of a single brokerage's data as a proxy for total wealth is that the holdings at a single brokerage may not be re-

[33]Numerous large financial institutions in Canada were contacted requesting and an enriched data set that combined banking, brokerage and mortgage information. This data set would have, at least partially, addressed the concern that a fraction of total individual wealth is unobservable. Although the Canadian digital privacy law allows the anonymized version of this information to be shared for research purposes, none of the institutions approached elected to follow through on the request.

flective of the investor's overall attitude towards risk. For example, a very wealthy individual might use a particular brokerage account as a sort of 'play-money' account and elect to take on a good deal of risk in this tiny part of their overall portfolio. In contrast a poorer individual may have just one account and thus they may elect to have less risk in that brokerage account as it represents a substantial portion of their net worth. Therefore the tests I conduct which find a negative relation between total LTS holdings and account value may result from an unobserved component of wealthy individual's portfolio which is of low risk. Given the data limitations there is no way to control for this unobserved effect and thus no way to fully refute the alternative that the unobserved effect is driving the LTS levels relationship. However, to the degree one believes that wealthier individuals tend to have a higher level of home-ownership than poor individuals, this supposed unobserved effect would imply that home-ownership should predict higher LTS allocation as home-ownership would predict the occurrence of 'play-money' accounts. As I demonstrate in table 3.3 the home-ownership variable is significantly inversely related with LTS holdings, consistent with the RWM and inconsistent with the story of an unobserved effect driving this relation.

To proxy for rank in the distribution I parse investors into their U.S. state of residence and compute the ranking of their portfolio wealth against all other investors in the same state for each month end. It must be acknowledged that by using state level clustering I am forced to make the unpalatable assumption that individuals compete on a state level for consumer goods. Unfortunately data limitations on variables important to the hypothesis tests, such as the GINI index, preclude assigning individuals to particular metropolitan areas. Another data limitation arises because the Infobase data is a single-period cross section. This limitation allows for the possibility that individuals move across state lines during the window of analysis. However, as there is no theoretical reason to suspect that the proxies used would be biased in favor of supporting the Rank-wealth model such limitations should only reduce the significance of the model. The imprecise measure of the underlying variables of interest may therefore reduce the support for the Rank-wealth model but will not likely cause type I error.

In addition to the brokerage data I also use the monthly stock file from CRSP to identify LTS. Additional data also includes the Fama and French factors, used to determine individual portfolio factor loadings, available from WRDS. As a measure of wealth inequality I make use of the GINI index given on the U.S. Census Bureau's S4 table[34]. In order to approximately match the results of Kumar to allow for comparison of results I also employ the Association of Religion Data Archives (ARDA) which gives information of religious denominations within state. US census data on race, education and foreign-born proportions by state are also used as controls for the testing regressions.

3.2.1 Defining lottery-type stocks

Lottery-type stocks (LTS) are those that exhibit, in relative terms at least, properties that are comparable to lottery ticket returns. Namely, if a stock has high idiosyncratic variance, positive idiosyncratic skewness and a low stock price it is deemed to be a lottery-type stock. The sorting process I use to identify LTS is identical to that of Kumar (2009) so as to ensure comparability of results. For completeness I reiterate the sorting procedure below:

To compute idiosyncratic variance I run the in-sample regression of excess stock returns on the excess market return, the Fama and French factors (HML and SMB) and the yearly momentum factor. The estimated regression is:

$$R_{i,t} - R_{f,t} = \alpha_0 + \beta_1 (R_{M,t} - R_{f,t}) + \beta_2 HML_t + \beta_3 SMB_t + \beta_4 UMD_t + \epsilon_{i,t} \quad (3.6)$$

To compute the idiosyncratic variance I then take the variance of the residuals, $\epsilon_{i,t}$. If a stock is in the highest 50^{th} percentile of idiosyncratic variance at each month end I assign a value of unity to a high volatility dummy.

[34]http://www.census.gov/hhes/www/income/histinc/state/state4.html

To compute the idiosyncratic skewness of each stock I use the method proposed in Harvey and Siddique (2000) where excess stock returns are regressed on the two factor model of excess stock returns and the square of excess stock returns. The regression is:

$$R_{i,t} - R_{f,t} = \alpha_0 + \beta_1(R_{M,t} - R_{f,t}) + \beta_2(R_{M,t} - R_{f,t})^2 + \upsilon_{i,t} \tag{3.7}$$

The idiosyncratic skewness of the stock is the scaled skewness of the residuals of the above regression, $\upsilon_{i,t}$. As with idiosyncratic variance, if a stock is in the highest 50^{th} percentile of idiosyncratic skewness at month's end it is given a skewness dummy variable value of 1.

Finally, if a stock is in the lowest 50^{th} percentile of stock price in the previous month then a security is given a low stock price dummy value of 1 in the given month.

If a stock has a value of one for all of the high variance dummy, high skewness dummy and low price dummy then the stock is given an LTS dummy value of one for the month in question.

3.2.2 LTS benchmarks

A key variable of interest in testing the Rank-wealth model is the proportion of individual portfolio value put into lottery-type-stocks (LTS). Kumar (2009) has demonstrated that LTS underperform non-LTS by at least 4%/year even when accounting for a variety of risk-adjustments. The Rank-wealth model offers a potential explanation for individuals choosing to hold LTS in absence of hedging motives.

I employ two distinct measures of LTS holdings, both of which replicate measures employed by Kumar. The first measure is a simple computation of dollar proportion in LTS in an individual's portfolio at a specific time. The second measure considers the fluctuation of LTS proportion in the broad market and thus finds the proportion allocated to LTS in an individual portfolio in excess of the broad market LTS allocation. Although Kumar identifies three other measures of LTS holdings, all deliver comparable results in his analysis and I therefore restrict the analysis to two of the simpler measures. The measures specifically are:

$$\text{LTS1}_{i,t} = \frac{\sum_{j=1}^{N} I_j q_{i,j,t} P_{j,t}}{\sum_{j=1}^{N} q_{i,j,t} P_{j,t}} \tag{3.8}$$

$$\text{LTS3}_{i,t} = \frac{LTS_{i,t} - LP_{\text{mkt},t}}{LP_{\text{mkt},t}}(100) \tag{3.9}$$

Where j is the index of all securities in the CRSP universe at time t, $q_{i,j,t}$ is the number of shares held by investor i, at time t in security j and $P_{j,t}$ is the price of security j at time t. The indicator variable, I_j, takes on a value of one if security j is identified as being a LTS as described in the previous subsection. The indicator variable takes a value of zero in all other cases. The variable $LP_{mkt,t}$ is the proportion of the broad market portfolio that is allocated into LTS at time t.

3.3 Results

3.3.1 Effects of wealth concentration

If the Rank-wealth model I propose is a valid representation of real-world decision making processes then a critical determinant of investor risk preferences will be the distribution of wealth in society. If wealth is highly concentrated in society then for the median individual to reach the upper echelons of rank would require a dramatic increase in their financial wealth. Thus, if individuals are attempting to rank-maximize it would be expected that as wealth becomes more concentrated the relatively poorer individuals would seek securities with greater and greater windfall payoffs, even at the expense of mean return. To test this hypothesis I estimate the parameters of a regression where the dependent variable is the proportion of lottery-type-stock holdings an individual has and include GINI[35] as an independent

[35] GINI is a measure of wealth concentration which varies from 0, when wealth is perfectly equally distributed, and 1, when wealth is concentrated entirely into a single individual's hands. The GINI measure is available only at multi-year intervals and thus is a fixed value for the regressions throughout the sample but does vary cross-sectionally by

variable. The independent variables include obvious explanatory factors for risk-taking such as age, home-ownership and wealth. In addition, I include the controls proposed by Kumar (2009) which include, but are not limited to, proportion of population foreign-born or college educated in the same geographic region, gender, predominant religion in the same geographic region and a marriage dummy.

For the purpose of my analysis I introduce not only the GINI measure of wealth concentration but two other new variables as well: Rank and Immobility. The rank measure is computed by finding the rank of an individual's financial wealth relative to all other individuals in the sample for the same month and for the same state. It is expected that as an individual's rank increases they will be more concerned with preserving their position in society rather than increasing wealth further. As such, those individuals of high-rank will seek securities that give positive rates of return in many states of the world. The high-ranked individuals will also have little to gain by dramatic increases in their wealth and thus will favor non-LTS over LTS. I therefore predict that the rank variable have a negative parameter estimate in the specified regression.

The last variable I introduce is Immobility. This measure computes the ratio of dollar gain necessary to increase one rank to the dollar loss necessary to lose one rank for each individual in each state and each month. Using the subscript i to denote a particular individual and i-1 to denote the individual of one lower rank then:

$$\text{Immobility}_{i,t} = \frac{\text{Value portfolio}_{i+1,t} - \text{Value of portfolio}_{i,t}}{\text{Value portfolio}_{i,t} - \text{Value of portfolio}_{i-1,t}} \tag{3.10}$$

If the immobility measure is high then a fixed dollar increase in wealth will translate into a smaller increase in rank than the same fixed dollar decrease in wealth would translate to a loss of rank. The measure is computed only 'in the small' which considers single rank adjustments. Computing immobility for any larger increment than one rank adjustment would require

U.S. state.

the imposition of an arbitrarily selected weighting. The rank-model would predict that low immobility should decrease the demand for LTS. If the dollar gain necessary to gain a rank is small relative to the dollar loss that would result in a loss of rank then an individual can earn a modest rate of return, without taking risk of rank loss, and still gain rank. If however, immobility is high and a large dollar gain is necessary to gain rank then modest rates of return are far less likely to result in rank increases. Thus the sign of this parameter is expected to be positive.

The full regression to test the signs of the parameter estimates of GINI, Rank and Immobility is given by:

$$\begin{aligned} \text{LTS}_{i,t} = \beta_0 + \beta_1 \text{Wealth}_{i,t} + \beta_2 \text{Age}_{i,t} + \beta_3 \text{Rank}_{i,t} + ... \\ \beta_4 \text{GINI}_i + \beta_5 \text{Immobility}_{i,t} + \beta_j \text{Controls} \end{aligned} \tag{3.11}$$

It is likely that individuals have unobserved characteristics which determine LTS holdings and thus I test the hypotheses using cluster regression analysis and cluster on each account. While there may be some serial correlation across time as well, I refrain from clustering on time as I have so few years of observations. Clustering across a dimension with too few observations can lead to bias in the standard error estimates as demonstrated in Petersen (2008). In the analysis conducted on LTS changes, first-differencing will partially account for serial correlation.

Table 3.3 is split into four specifications for two measures of LTS. The two LTS measures are consistent with the first and third definitions provided in Kumar (2009). However, unlike Kumar, I do not normalize or shift the independent variables. While this will mean that some parameter estimates will be in power notation because of scale, it also preserves direct interpretation of the estimated effect of each independent variable.

Considering that Kumar's various definitions of LTS holding generated similar results and given that the analysis of LTS1 and LTS3 demonstrates similar parameter estimates, further tables will only report the results of LTS1 for the sake of brevity. Analysis of LTS3 generates similar qualitative results to LTS1, both preserving sign and significance of various estimates.

The first specification I test considers only classically proposed determinants of risk-taking. Consistent with findings in Kumar (2009), the results indicate that increasing wealth reduces the proportion of LTS in an individual's portfolio. However, the parameter estimate of the wealth effect on LTS proportion, though highly statistically significant, indicates that for every one million dollars worth of wealth increase LTS proportion decreases by only 2.46E-5%, an economically trivial sum. Age has a more profound effect, where LTS proportion declines by approximately 0.219% per year. The home-owner dummy is also significant and estimates that home owners have 1.615% less of their portfolio in LTS than individuals who do not own their own home. Though the economic insignificance of wealth is a little surprising, thus far the tests suggest a story of increasing risk-aversion both with age and wealth. It should be noted however that this particular specification can only account for 1.58% of the variation in LTS1 holdings across all observations.

In contrast, the simplest test of the rank model which has just the GINI and rank variables considered can explain 3.48% of the variation of LTS1 holdings across observations. The wealth concentration measure (GINI) is highly significant and has an estimate of 1.5593. At this estimate level, if the GINI increases by 0.082, the difference between wealth concentration in the Canada and the U.S., the model predicts an average increase in LTS proportion of 12.77%, or of approximately eight times the magnitude effect that home-ownership had in the previous specification.

Of particular interest is the highly significant estimate of the rank measure. The sign is negative, as predicted by the model, and has a magnitude of 0.00016. Thus by increasing rank by a single step, the empirical model predicts a reduction of LTS proportion by 0.016%. To interpret this result consider table 3.2 which shows that 90% of all rank gaps in sample are smaller than $187.61. This implies that for 90% of the sample, conditional on a rank-adjustment having occurred, the effect of a $1 gain in wealth is to reduce LTS proportion by 8.53E-5 or nearly 3.5 times the effect of a wealth increase without conditioning on rank adjustment.

The third specification of LTS1 is a similar regression as conducted by Kumar, with some variations noted previously in normalization procedure and sample filtering. Similar to Kumar's findings, this specification has a respectable adjusted R^2 of 7.92% and largely corroborates with those earlier findings on signs for most variables. Two notable differences of the signs in this study and Kumar's is that this analysis suggests that being married increases LTS proportion and that Catholic states discourage LTS holdings. Kumar finds the opposite sign for these variables though his marriage-dummy estimates are not significant for all of his specifications.

The fourth specification of my analysis includes all variables. Note that the adjusted R^2 is 11.73%, quite a bit higher than the model Kumar proposes. Also of interest is that in the full model the wealth variable is insignificant, with a 0.98 t-stat. The rank variable remains highly significant in the model and its estimate is actually increased in this specification by 18.75%. Rank immobility is also significant and the sign is as expected from the Rank-wealth model. The wealth concentration measure surprisingly flips in sign but its magnitude is reduced dramatically as well as its significance, though it remains significant.

The estimates from the LTS3 measure are very similar, which is not particularly surprising given the similarity of the two measures. Noteworthy is the fact that adjusted R^2's, F-test values, estimate signs and magnitude[36] of the LTS3 specifications remain comparable to the LTS1 specifications.

Differential concentration effects

Although I propose that the effect of wealth concentration should be to increase the overall LTS allocation the truth of the matter is that it should have a differential effect conditional on the rank of the individual in question. Those that are lowly ranked will certainly have an increasing demand for LTS as wealth concentrates as they have little chance of gaining rank by modest portfolio gains. In contrast, those with high rank will become increasingly secure in their rank as wealth is concentrated, as the gaps between them and

[36] After accounting for the fact that LTS1 and LTS3 scales differ by a factor of 100.

lower ranked individuals grow. The impetus for the high ranked individuals then will be to reduce LTS holdings as they have been shown to reduce portfolio performance and instead follow a strategy of 'sure and steady' gains, steadily increasing the gap between them and the poor.

The GINI coefficient was significant in all specifications of the LTS level regressions but it unexpectedly flipped sign between the pared down model and the full model with all controls. I therefore determine in the model with all controls if individuals of differing rank have differing responses to wealth concentration. To check this I first run the complete regression on LTS1 using all variables except for the GINI. I then compute the residuals between the observed and the estimated LTS for each individual. Finally, I separate all accounts into rank deciles by month and state. I then run the following regression of the estimated LTS residuals on the GINI coefficient which was not included in the first regression. The exact test is:

$$\mathrm{LTS}_{i,t} - \widehat{\mathrm{LTS}_{i,t}} = \beta_0 + \beta_1 \mathrm{GINI}_i \tag{3.12}$$

If the parameter estimate on the GINI coefficient is positive for a given rank decile then the implication is that the residual LTS is positively correlated with GINI. That is, increasing GINI increases the LTS that would be estimated in a full model. Conversely, a negative alpha in the above regression implies that increasing GINI will decrease the proportion of the portfolio allocated to LTS for a given rank decile.

Table 3.4 reports the results of the regression for each decile. As predicted, for higher rank individuals wealth concentration decreases the allocation to LTS whereas for low rank individuals the effect is reverse, increasing the allocation to LTS.

3.3.2 Effects of rank shocks

The second gambling hypothesis predicted by the Rank-wealth model is that as an individual enjoys/suffers gains/loses in rank they will have decreased/increased proportion of their portfolio in LTS. In the case of low-rank individuals, increasing the probability of dramatic gains is the only way to gain substantial rank. In the case of high-ranked individuals, taking on risk of losses, which would reduce rank, cannot be justified by the small rank gains associated with increased mean returns provided by taking on risk.

To test the adjustment in LTS hypothesis, I first compute the month-by-month change in LTS for each investor which is triggered by a trade and not by a gradual shift in the portfolio components value. I use the regression specification estimates from the previous section to first deduct changes in LTS which are attributable to monthly increases in the investor's age. I also address the gradual shift in LTS that can be attributed to fluctuations in value of portfolio components. For example, it is conceivable that a LTS has a particularly high return in one period which would increase its relative weight in an individual's portfolio. However, the resultant increase in the LTS measure would not be through any direct action of the investor. I therefore only include observations where at least one security's number of shares in an investor's portfolio increased or decreased from one month to the next so as to include situations where rebalancing is actively increasing or decreasing LTS proportion. While I do not specifically control for share splits these should be a small part of the sample and will only reduce the magnitude and significance of the independent parameter estimates as they would be flagged a trade but when in fact LTS proportion nor wealth-rank should be affected.

Not all investors will re-balance their portfolio in response to a small shock in rank or other independent variables. I therefore compute the wealth and rank shocks as the difference between the current month wealth/rank and the average of wealth/rank over the previous one month, one quarter, half-year and full-year horizons. The longer time horizon measures will pick-up shocks that result in wealth/rank levels very different than the investor is accustomed to.

I also control for the fact that all investors may exhibit some time trend in LTS by no actions of their own. As Kumar's definition of LTS is the joint set of high volatility, low price and high skewness securities it is possible that the market amount of LTS classified stocks is time variant. To control for this I identify all LTS stocks in the CRSP dataset and then compute the market value of both LTS and non-LTS securities for each month. With this information I then determine the Market LP drift, or the percentage increase in LTS in the broad market month by month. A group of passive buy-and-hold investors who have diversified portfolios should exhibit a drift equivalent to the Market LP drift if they are not actively seeking or avoiding LTS. The regression I run to test the second gambling hypothesis is:

$$\begin{aligned} \text{LTS Adjustment}_{i,t} &= \beta_0 + \beta_1 \text{Wealth Shock}_{i,t} + ... \\ &\beta_2 \text{Market drift}_{i,t} + \beta_3 \text{Rank Shock}_{i,t} \end{aligned} \tag{3.13}$$

or, equivalently, the full expression of the regression is:

$$\begin{aligned} LTS_{i,t} - LTS_{i,t-1} + \frac{\text{Annual Age Adjustment*}(j-1)}{12} &= \beta_0 + ... \\ \beta_1 \left(\text{Wealth}_{i,t} - \frac{\sum_{s=t-1}^{t-j} \text{Wealth}_{i,s}}{j-1} \right) + \beta_2 \left(\frac{\text{Proportion CRSP LTS}_t}{\text{Proportion CRSP LTS}_{t-1}} - 1 \right) &+ ... \\ \beta_3 \left(\text{Rank}_{i,t} - \frac{\sum_{s=t-1}^{t-j} \text{Rank}_{i,s}}{j-1} \right) & \end{aligned} \tag{3.14}$$

Table 3.5 Panel A reports the findings of the cluster regression analysis of LTS adjustments. In Panel A I perform the basic regression for each of the four time intervals (month, quarter, half-year, year). The resultant adjusted R^2 from the regression find that the models which do not include rank-shocks have extremely low explanatory power, varying from 0.29% of LTS adjustment variation to 0.42%. In contrast, the models that include the rank shock have a much greater capacity to explain inter-temporal variation of LTS holdings. The adjusted R^2 of these specifications vary from a low of 2.83% to 3.12%. Furthermore, the wealth shock is statistically insignificant for the one month horizon and remains economically insignificant for longer periods. The rank shock in contrast has the expected negative sign (an increase in rank will result in a decrease in LTS holdings) is significant for all time horizons and is economically about three times as significant per dollar than the wealth shock. It should be noted that in all time horizons the intercept is positive and significant for all model specifications.

Theoretically, once passive market drift and time variant factors, such as age[37], are accounted for the intercept term should be statistically insignificant from zero. Thus, Panel B reports the same respective regressions for each time period as panel A but restricts the intercept term to be zero. Possible explanations for the significant and positive intercept include the possibility that the wealth proxy is inappropriate or that a higher-order relationship exists which is not modeled and could erroneously identify the intercept as being non-zero. Another possible explanation for the intercept estimate in Panel A is the lack of temporal variation in the GINI coefficient, which over time has tended to increase. Considering the positive sign on the GINI for some specifications of the LTS levels analysis, it may be that increasing wealth concentration could result in a non-zero intercept for the changes regression as well. I therefore conduct an analysis in Panel B where the intercept is arbitrarily restricted to be zero as would be theoretically predicted if there were no missing variables. The findings of Panel B generally confirm the variable sign and significance as well as overall model fit reported in Panel A.

In Panel C, I compute the variation of the estimates at each time horizon between panels A and B. For short horizon (one month, one quarter) the difference in estimates for wealth shock are modestly affected by restricting the intercept to zero. Thus, for short horizons at least, even if the model in A is neglecting some explanatory variable which would eliminate the intercept term it is unlikely to dramatically adjust the estimate of wealth shock influence. The rank shock estimates vary even less and even at a full year horizon, the variation between panel A and panel B estimates is only 13.51%. It is therefore reasonable to conclude that including variables to explain the unusual intercept in panel A will be unlikely to have any

[37]It is assumed that investors' other personal characteristics are time invariant. These characteristics include religion, state of residence and marital status. Although some of these variables may in fact change over time the limitations of the data, in that it has only a single period assessment of such variables, precludes testing the change in LTS portfolio proportion attributable in investor characteristic changes.

bearing on the estimates of the rank-shock. The exception to this analysis is the Market LP drift, which even for short horizons can vary by more than 30% between panel A and B. It is therefore quite likely that including additional explanatory variables in the model which eliminate the intercept would have dramatic effect on the estimate and possibly the significance of the drift factor.

In summary, the analysis of the LTS adjustments suggest that rank shocks are both economically and statistically significant, both at long and short horizons, and would likely remain so if additional explanatory variables that eliminated the intercept weighting were introduced. This is an important distinction between the classic-economic model and the Rank-wealth model as the former is far less able to explain temporal variation in individual LTS holdings than the latter. If wealth really was the determining factor of LTS preference then it should be a more important factor in these regressions.

3.3.3 Factor loadings

Although a structural relationship between conventional risk factors and rank is not fully developed in this thesis, it remains intuitive that those of lower rank would take on greater risk in their portfolio in hopes of gaining rank as demonstrated in Bothner et al. (2007). To test this hypothesis I compute the monthly return for all investors' portfolios and then determine the 12-month factor loading of those portfolio returns on both the CAPM Beta and the Fama and French 3 factor model. Once the factor loadings are determined I then run a simple OLS regression on the loadings using either lag rank or lagged mean rank over a window and the dependent variable. The regression for the CAPM Beta for example is:

$$\beta_{i,t} = \alpha_0 + \alpha_1 \text{Rank}_{i,t} \quad (3.15)$$

Where i is the index of the individual investor and t is the month time index.

The results of the various regressions are reported in table 3.7. I report only the results for the one-month lag rank dependent variable though the tests using mean-rank over longer periods up to and including a year generate qualitatively similar results. As can be quickly surmised, none of the regressions have any notable degree of explanatory power regardless of the factor loading selection. In none of the cases analyzed is lag rank a significant variable and in no case can zero be reliably rejected as the parameter estimate value. Figure 3.1 in the appendices illustrates the factor loadings on portfolios by lagged rank. The first subfigure displays the CAPM Beta loadings for the CAPM model alone. The figure of the excess market portfolio loadings, a component of the Fama and French 3 factor model, is not displayed but is visually comparable to the CAPM Beta presented.

It is apparent that rank is not a significant determinant in portfolio factor loadings, contrary to my original expectations. A possible explanation for this is that the risks associated with LTS are by design orthogonal to other risk-factors. A complete general equilibrium of the Rank-wealth model is not yet developed but future theoretical research in this direction may shed light on why conventional risk factor loadings do not correlate with rank. I offer as an early conjecture that it may be that individuals, when attempting to gain rank, are intentionally seeking idiosyncratic payoffs. For example, if the low ranked individuals wished to gain rank by increasing CAPM Beta it may be that in general equilibrium the high ranked individual would be able to anticipate this action and increase their own portfolio Beta. As my model proposes that rank gain is the objective of the lowly ranked, a market that either 'lifts' or 'sinks all boats' does not benefit low ranked individuals as their rank would remain unchanged. Thus these investors may have to allocate portfolio funds into securities that do not load on well-known risk factors in order to gain rank.

3.3.4 LTS effect on rank-gain

The Rank-wealth model I test predicts that individuals should rationally hold LTS in their portfolios with the purposes of gaining rank. To test this hypothesis I compare the performance of individuals' portfolios in two dimensions. First I compute the one-month lagged percentile rank within the appropriate US state for all accounts and then divide the sample into quartiles of low to high past ranking. I then divide each quartile sub-sample into two groups. The first group are those individuals, within each quartile, who have the median or lower amount of LTS proportion in their portfolio. The second group conversely has higher LTS proportion than the median. For each sub-sample of LTS proportion and past quartile ranking I then have a distribution of portfolio returns and rank returns.

Next, I test to determine if for each quartile ranking the individuals with relatively high LTS proportion have a different rate of either rank or portfolio returns. As I do not know the fundamental distribution underlying either portfolio or rank returns and because I do not want to arbitrarily impose structure I use the non-parametric Wilcoxon-Whitney-Mann (WWM) test to determine if each low LTS proportion distribution is sampled from a different population distribution than its respective high LTS proportion distribution. The WWM test utilizes the U statistic and is an analog to the independent sample t-test but does not impose structure on the underlying distributions. A complete discussion of the WWM test procedure is available in Hollander and Wolfe (1999).

For all past rank quartiles, except the third, I reject, at better than 0.1% confidence, the null hypothesis that high LTS portfolio and rank returns are drawn from the same sample as low LTS portfolio and rank returns. For the third quartile the null can only be rejected with 41.99% confidence for portfolio returns and 35.81% for rank returns. Confident that three of the four quartiles have different population distributions for low and high LTS proportionate holdings, I next compute the first four moments and the median return for each LTS category and quartile.

Table 3.6 demonstrates, without exception or distinction by quartile, that the mean rate of portfolio return for high LTS portfolios exceed that of the mean rate of return for low LTS portfolios. The mean rate of return is highest for the high LTS portfolio in the second quartile. Because LTS holdings negatively correlate with wealth then the proportion of LTS in the lower quartile strategies is higher than the respective holdings of the third or fourth quartile strategies. The average return for the high LTS first quartile sample is 2.742%/month or an astonishing 38.349%/year. The highest median annual portfolio return is a more modest figure of 16.213%. However, considering that the average market (S&P500) return for the years 1991-1996 was 17.63% percent per year, the median return is not particularly surprising. Interestingly the median portfolio return of high LTS portfolios is lower than the respective low LTS portfolio median return for all quartiles but the fourth.

Predictably the high LTS strategy has higher variance than the low LTS strategy in both portfolio and rank returns. For all quartiles, the high LTS strategy has a less positively skewed portfolio return. This is true as well for the first two quartiles of rank returns but is not so for the third and fourth quartile of rank returns where a high LTS strategy actually increases the skew of returns. Without exception, the high LTS strategy had lower kurtosis for both rank and portfolio returns than the low LTS strategy regardless of quartile.

Not surprisingly, as the model is based on relative rankings computed in-sample, the mean rank gain for both strategies across all quartiles was 0%. However, as is predicted by the model, the high LTS strategy for the lower two quartiles generates a substantially higher mean rank gain than the low LTS strategy. This implies that of those individuals who assign more portfolio weight to LTS in the first two quartiles tend to suffer either small losses in rank or dramatic increases. This is also reported in the positive skewness of rank returns for the high LTS strategy.

In contrast, the high LTS strategy for the top two quartiles generates a lower mean rank than the low LTS strategy. Also notable is that the mean rank loss for the highest quartile is predictably largest as the individuals in this quartile have nowhere to go but down. The fact that the mean rank gain is greatest for the high LTS strategy in the second quartile is of particular interest. Note that for each percent mean portfolio return in the second quartile there is a mean rank gain of 8.42%. In contrast, the ratio of mean rank gain to mean portfolio gain for the high LTS strategy in the lowest quartile is only 7.63%. This is suggestive of a much greater density of wealth in the second quartile than the first. For the same 1% gain of portfolio value a person in the first quartile will not gain as much rank as a person in the second quartile. This pattern is similar between quartiles three and four where the ratios are, respectively, a more dramatic -4.55% and -8.33%. In the two upper quartiles rank is on average lost as there is little room to move up in the distribution and yet the density of population appears to still be higher near the median investor. This observation lends support to the model's underlying assumption that the wealth distribution is approximately unimodal and non-monotonic in shape.

Comparing the benefits of the high or low LTS strategies by each quartile, we observe that all quartiles gain a relatively modest 35 to 47 basis points (b.p.) in portfolio returns each month by following a high LTS strategy. Given that the period of time studied was notable for strong positive returns the additional return of holding much more LTS in the portfolio seems rather small. In contrast however the mean rank gain for holding high LTS in the first two quartiles is an impressive 1.87%/month for quartile one and a dramatic 4.954%/month for quartile two. This is in sharp contrast to quartiles 3 and 4 where following the high LTS holdings strategy cost 44 b.p./month or 11 b.p./month of mean rank return as compared to the low LTS strategy. In terms of portfolio returns, the high LTS strategy does not yield substantial mean portfolio returns, increases variance of the portfolio and reduces desirable skewness by as much as 32%. In terms of rank returns however, for the first two quartiles, the high LTS strategy delivers a dramatic increase in the mean rank return, more modestly increases relative variance and sacrifices only 16% of the skewness attained by the low LTS strategy. If rank gain is desirable then the high LTS strategy is perfectly reasonable for the lower two quartiles. For the individuals placing above the median in terms of rank, the high LTS strategy under-performs the rank gain of the low LTS strategy.

3.3.5 Correlation analysis

The last of the hypotheses I put forward proposes that individuals of low rank should not have portfolio returns which very positively correlate with other portfolio returns, and especially not with high ranked individual's portfolios. I would also expect that highly ranked individuals should have portfolio returns that positively correlate with other highly ranked portfolio returns. The intuition for this is that once an individual has attained high rank their primary investment motivation will be to secure that high rank. The marginal utility of wealth at a high rank is low but a large discrete drop in wealth could cause a tremendous loss in utility. To avoid a large drop in utility the high ranked individuals should invest in securities which often enjoy small positive returns and very rarely suffer large losses. In contrast, those of low rank must invest in securities which others do not hold in large proportion in order to gain rank. A wealthy individual and high ranked individual will by definition have more resources to put into securities than a low ranked individual. Thus the low ranked individual must seek out securities that few wish to invest in so that if that security did happen to gain a large amount they alone would dramatically increase their wealth and allow for an increase in rank.

To test the correlation hypothesis I break all investor portfolios into quintile rank groups. I then compute the correlation of portfolio returns for each pair of rank quintiles.[38] Table 3.8 reports the mean correlation between each rank quintile pair as well as the upper and lower comparison bounds computed by the Gabriel method.

Although the correlation boundaries do overlap for many combinations of pairs, it still remains apparent that the lowest rank quintile has a low mean correlation with other rank quintiles. The 1-1 pair for example has a mean correlation of just 0.0386 and the 5-1 pair has a mean correlation of 0.0655. By comparison the 5-5 pair has a mean correlation of 0.1204. While

[38] I compute the correlations only for a sub-sample to shorten computation time. For those wishing to replicate the analysis I use the Surveyselect procedure in SAS to draw a sub-sample of all accounts in the data and use the seed 29660. Testing of other seeds confirms that the results are not qualitatively affected by selection.

for the middle quintiles it is difficult to discern a pattern is is clear that the lowest correlation of 1-1, which has an upper boundary of 0.0523 is far short of the 5-5 lower boundary of 0.1068. This additional evidence lends further support to the rank-model conjecture.

3.4 Conclusion

This manuscript provides some support for the Rank-wealth hypothesis of investor preferences. In agreement with that theory, individuals of lower wealth rank in their state wealth distribution do appear to take on greater risks by holding larger proportions of lottery-type-stocks (LTS). Moreover, shocks to rank have a demonstrably greater effect on lottery-type holdings than shocks to wealth. Though wealth can be used to explain some of the cross-sectional LTS preference its effect is economically insignificant when compared to the effect of rank.

The Rank-wealth model also predicts that increasing wealth concentration should increase the demand for LTS as the 'slow-and-steady' form of investment yields less rank gain in a highly wealth-concentrated society. Again, the empirical findings of this manuscript demonstrate that as wealth concentration increases, measured through the state level GINI index, individuals on average do tend to hold more LTS. Even more notable is that conditioning on rank, those of lower rank tend to hold more LTS while those of high rank hold less LTS as concentration increases, exactly as predicted by the theory.

While there is no evidence that lowly ranked individuals have portfolios with great risk exposure as measured by the CAPM or Fama-French 3 factors, there is evidence that they opt to hold portfolios that have payoffs which are less correlated to other individual portfolios. Conversely, those of the highest ranking tend to hold portfolios which do have high correlations as they wish to 'lock-in' their rank.

Most importantly, the Rank-wealth model is supported by showing that the effects of holding large proportions of LTS in an individual portfolio will have comparable effects to rate of dollar return across rank deciles but will have dramatically different effects on rank gain. This is a critical observation as it offers forth an explanation as to why lotteries are regressive and why the poor tend to diversify the least though classic economic models would predict otherwise. While Kumar (2009) clearly shows that holding large amounts of LTS inside a portfolio would generally be a drag on performance[39], there is good evidence that such a strategy can help lowly-ranked individuals ‘get ahead’.

The contribution of this manuscript is important for it proposes an intuitive but novel explanation as to why poor individuals buy lottery tickets and why they do not diversify their portfolios. Until now, the only explanations for such behavior rested on assumptions of either low education, mental incapacity or other socio-demographic factors such as gender or race. The Rank-wealth model provides a wholly more satisfying explanation that allows for observed individual behavior to be fully rational and only requires the sensible assumption that consumer goods are fundamentally indivisible.

[39] After risk-adjustment and considering a different period of history which did not experience the remarkable returns of 1991-1996.

Table 3.1: Summary of sample data

This table reports the basic characteristics of the sample of brokerage data set used in this paper. The first column lists the average of monthly observations of common equity, portfolio size, turnover and proportion in lottery-type stocks. The second column reports the same information but also details the proportion of the sample that is male, employed in a professional occupation, married and is a homeowner. Note that the Infobase supplemental data is available only for a subset of the entire sample. The account information between the Infobase sub-sample and the entire sample is very comparable.

	Entire sample	Infobase subset
Number of accounts	88,907	16,255
Average common equity	$41,036	$37,844
Average portfolio value	$55,240	$48,982
Average portfolio size	5.49	5.52
Average monthly turnover	2.71	2.70
Average proportion in LTS	4.98%	4.65%
Proportion:		
-Male	No data	90.79%
-Professional	No data	60.47%
-Married	No data	72.87%
-Homeowners	No data	95.81%

Table 3.2: Summary characteristics of wealth-gaps in sample

This table reports on the distribution of wealth gap between ranks. For each month and each state individuals are assigned a descending rank of account value. Assuming account value positively correlates with total wealth then the rank variable formed in this fashion will be a noisy proxy of overall wealth-rank, a key variable for the analysis of this paper. Also of interest will be the gap, or distance, between each rank in each state and month. By computing the dollar difference between each rank a distribution of wealth gaps is identified. This table shows the cumulative proportion of the population that has a dollar gap between their rank and the next higher rank that is less than or equal to the given dollar value. Thus, the median gap, for which 50% of the population has over the entire sample history is just $1.85.

Cumulative percentage of sample investors with:	Dollar gap between ranks less than or equal to:
10%	$0.00
25%	$0.00
50% (Median)	$1.85
75%	$16.75
90%	$187.61
100%	$23,491,875.00
Mean gap	$1,748.24
Mode gap	$0.00

Table 3.3: Determinants of LTS (lottery-type-stock) allocation

This table reports the model estimates for the proportion of lottery-type-stocks (LTS) held by individuals at each month end in the tested sample. Four specifications of the model are tested and two measures of the LTS are used as the dependent variable. The model tests are conducted using cluster-regression, clustering on account. T-statistics of each parameter estimate, for each model, are given below in smaller font and bracketed. Unlike in Kumar (2009), the variables in this regression are not standardized to mean zero and standard deviation of one as this transformation makes interpretation of coefficients difficult. As a result of the lack of a transform in these regressions, one cannot directly compare the parameter estimates found by Kumar to these estimates. Measures of model fit including the adjusted R^2 and the F-test value are comparable across studies.

	LTS1	LTS1	LTS1	LTS1	LTS3	LTS3	LTS3	LTS3
Intercept	0.39873	-0.35009	0.43681	0.54273	40.17024	-33.8753	44.83231	58.08537
(t-stat)	(47.89)	(-15.07)		(11.54)	(46.43)	(-14.01)	(27.55)	(11.91)
Rank		-0.00016		-0.00019		-0.01558		-0.01829
		(-62.32)		(-60.16)		(-59.66)		(-56.22)
Wealth Concentration		1.5593		-0.61554		153.7358		-67.3723
		(28.64)		(-5.8)		(27.13)		-6.13
Rank immobility				0.000146				0.01871
				(2.93)				(3.63)
Wealth (dollars)	-2.46E-07		1.77E-08	-1.71E-08	-2.6E-05		1.62E-06	-4.2E-06
	(-31.4)		(2.16)	(-0.98)	(-32.34)		(1.9)	(-2.33)
Age (years)	-0.00219		-0.00113	-0.00079	-0.22393		-0.11056	-0.07602
	(-21.31)		(-11.12)	(-7.95)	(-20.95)		(-10.53)	(-7.35)
Owner dummy	-0.01615		-0.02087	-0.01034	-1.50195		-1.9528	-0.98586
	(-2.65)		(-3.49)	(-1.77)	(-2.37)		(-3.14)	(-1.62)
Professional Dummy			0.00601	0.00145	1.36025		0.77371	0.2919
			(2.23)	(0.55)	(4.75)		(2.77)	(1.07)
Male Dummy			0.02318	0.02669			2.2618	2.62148
			(4.82)	(5.66)			(4.53)	(5.36)
Married Dummy			0.0016	-0.00308			0.15588	-0.37817
			(0.56)	(-1.1)			(0.53)	(-1.3)
Catholic Dummy			-0.01939	-0.02255			-1.84594	-1.92619
			(-4.5)	(-5.24)			(-4.13)	(-4.32)
Protestant Dummy			-0.01476	0.04293			-1.88491	3.60776
			(-3.38)	(9.69)			(-4.16)	(7.86)
Afr-Amer:White ratio			-0.06078	-0.11072			-5.89669	-9.28506
			(-3.69)	(-5.49)			(-3.45)	(-4.44)
Hisp:White ratio			-0.01127	0.15509			-1.97786	14.67777
			(-0.66)	(8.8)			(-1.12)	(8.03)
Education ratio			0.03991	0.66531			-2.90714	55.90484
			(0.27)	(4.39)			(-0.19)	(3.56)
Proportion foreign born			0.16626	0.90556			19.78712	92.04642
			(4.53)	(20.1)			(5.2)	(19.7)
Portfolio Size			-0.01251	-0.01181			-1.32503	-1.25698
			(-84.75)	(-70.26)			(-86.47)	(-72.14)
Cluster	7434	7457	7434	7359	7434	7457	7434	7359
Adj. R^2	1.58%	3.48%	7.92%	11.73%	1.63%	3.20%	8.19%	11.68%
F-test	433.49	1942.39	710.17	864.91	447.37	1779.97	738	860.17

Table 3.4: Wealth concentration effect on LTS level conditioning on wealth decile

This table reports a secondary analysis conducted on the LTS-levels. First, a modified regression of the fourth specification given in table 3.3 is run. The difference between this regression and the fourth specification is the removal of the GINI, or wealth-concentration measure. The residuals of this regression are kept and then the sample is broken into wealth-rank deciles. The residuals are then regressed against the GINI measure for each decile and the results are reported here. This methodology allows for testing of a non-linear relationship between LTS and GINI conditionalized on wealth-decile.

Decile	Estimate	t-stat	F-test	R^2
1	1.5815	17.23	297.04	28.7440%
2	1.2341	15.31	234.27	2.1431%
3	1.0072	12.36	152.86	1.4107%
4	0.5358	6.61	43.70	0.4083%
5	-0.2573	-2.75	7.57	0.0715%
6	-0.3956	-3.88	15.05	0.1438%
7	-1.1582	-10.14	102.87	0.9793%
8	-1.7203	-13.65	186.28	1.7752%
9	-1.2955	-8.71	75.87	0.7302%
10	-1.7678	-4.96	24.60	0.2494%

Table 3.5: LTS adjustments in response to wealth and rank shocks

This table reports the estimated changes in proportional lottery-type-stock (LTS) holdings as a result of changes in wealth and rank after controlling for changes in LTS in the market portfolio. As individuals may not adjust their portfolio in response to either a small wealth or rank shock, four time horizons are used to estimate shocks from one month, one quarter, two quarter and one year horizons. Shocks are computed by taking the difference between the month t realization of either wealth or rank and deducting the rolling average of the appropriate measure for each of the time horizons. Panel A reports the results for the cluster regression clustering on account. Panel B reports the same results but restricts the regression to have a zero intercept as would be theoretically expected if the model was complete and the estimates free from bias. Panel C reports the percentage change in the parameter estimates between panels A and B and demonstrates that for short horizons the impact of including or excluding an intercept has little effect on the estimate of either wealth or rank shocks.

Panel A: Regressions with Intercept

Parameter	One Month		One Quarter		Two Quarters		One Year	
Intercept	0.012647	0.009328	0.033485	0.02909	0.067804	0.061882	0.135155	0.126173
(t-value)	(5.7563)	(2.5301)	(14.264)	(6.1006)	(22.0786)	(10.9686)	(35.5523)	(17.5643)
Wealth shock	-9.96E-07	-6.4E-07	-7.07E-07	-4.3E-07	-6.74E-07	-4.1E-07	-5.54E-07	-3.5E-07
	(-1.6627)	(-1.5107)	(-2.1429)	(-2.0013)	(-2.5426)	(-2.3242)	(-3.1355)	(-2.8338)
Market LP drift	0.055862	0.045319	0.065342	0.054641	0.059538	0.057176	0.056151	0.06227
	(3.4815)	(2.5767)	(4.5200)	(3.8829)	(3.7380)	(3.536357)	(3.539365)	(3.53004)
Rank shock		-0.0007		-0.00061		-0.00055		-0.00048
		(-2.3835)		(-2.9640)		(-3.2916)		(-3.4537)
Adj. R^2	0.42%	3.12%	0.30%	2.87%	0.32%	2.83%	0.29%	2.90%

Panel B: No Intercept Regressions

Wealth shock	-9.80E-07	-6.27E-07	-6.53E-07	-3.76E-07	-5.31E-07	-2.71E-07	-2.21E-07	-1.32E-08
(t-value)	(-1.65122)	(-1.4915)	(-2.06499)	(-1.85652)	(-2.22716)	(-1.73964)	(-1.62304)	(-0.13451)
Market LP drift	0.075212	0.059546	0.116327	0.098698	0.161995	0.150201	0.256631	0.248455
	(4.75744)	(3.146268)	(6.66265)	(6.477621)	(6.96202)	(8.625512)	(7.30972)	(9.8701)
Rank shock		-0.0007		-0.00062		-0.00057		-0.00055
		(-2.38441)		(-2.9494)		(-3.21963)		(-3.2448)
Adj. R^2	0.42%	3.13%	0.29%	2.92%	0.27%	2.98%	0.27%	3.47%

Panel C: Variation of Parameters

Wealth shock	-1.59%	-1.98%	-7.56%	-11.77%	-21.22%	-34.28%	-60.06%	-96.20%
Market LP drift	34.64%	31.39%	78.03%	80.63%	172.09%	162.70%	357.04%	298.99%
Rank shock		0.29%		1.36%		4.27%		13.51%

Table 3.6: Dollar and rank performance of high LTS portfolios

This table reports the performance of both dollar computed rate of return and rank return for each of two strategies by quartile of rank. The two strategies considered are high and low LTS where a portfolio is deemed to have high LTS if the proportion of stock in LTS for said portfolio is above the median LTS holdings of portfolios in the same quartile ranking. Quartile rankings are determined by separating the data sample into four groups for each month and each state. Panel A provides the mean, median and higher moment estimates of the distribution of portfolio return for each pairing of quartile and strategy type. Panel B similarly reports the mean, median and higher moment estimates of the distribution of rank adjustments for each pairing of quartile and strategy type. Panel C displays the difference in mean return and in mean rank adjustment between the high and low LTS strategies for each quartile. A non-parametric Wilcoxon-Whitney-Mann test is conducted on the distributions of both rank adjustment and portfolio returns contrasting the distributions of low and high LTS strategies within each quartile. For all quartiles but the third, the high-LTS distributions of both returns and rank adjustment are significantly different at the 5% confidence from the comparable distributions for low-LTS.

Panel A: Portfolio Returns

Past Rank Quartile	LTS holdings	Return Mean	Median	Std.Dev	Skewness	Kurtosis
1	Low	2.147%	0.976%	0.140	1.987	10.579
	High	2.565%	0.000%	0.197	1.336	3.956
2	Low	2.267%	1.126%	0.135	2.093	11.932
	High	2.742%	0.956%	0.166	1.628	6.759
3*	Low	1.870%	1.077%	0.131	1.725	11.347
	High	2.280%	1.043%	0.154	1.480	7.383
4	Low	1.479%	1.030%	0.125	1.343	10.624
	High	1.830%	1.103%	0.143	1.246	7.576

Panel B: Rank Adjustments

Past Rank Quartile	LTS holdings	Mean	Median	Std.Dev	Skewness	Kurtosis
			Rank			
1	Low	17.703%	0%	1.976	2.629	21.003
	High	19.580%	0%	2.305	2.230	14.757
2	Low	18.160%	0%	3.670	0.777	8.910
	High	23.113%	0%	4.560	0.642	4.778
3*	Low	-9.933%	0%	3.614	-0.729	10.247
	High	-10.374%	0%	4.243	-0.557	6.818
4	Low	-15.130%	0%	1.941	-3.498	81.497
	High	-15.243%	0%	2.122	-1.920	16.055

Panel C: Comparative Performance

Past Rank Quartile	Return benefit LTS for high LTS	Rank benefit for high LTS
1	0.418%	1.877%
2	0.475%	4.954%
3*	0.410%	-0.440%
4	0.350%	-0.114%

Table 3.7: Portfolio risk factor loadings by wealth-rank

This table reports the estimated change for each of three common risk-factor loadings by rank. Factor loadings are first determined by running a regression on individual portfolio returns using either the CAPM or the Fama-French 3 factor models. Each investor portfolio is then assigned a rank for each month and state. The estimated loadings are then regressed against the ranking. Contrary to expectation, there is no evidence at any appreciable level of significance that lower ranked individuals load more heavily on commonly cited risk-factors. Figure 3.1 depicts graphically the relationship of factor loadings by rank.

Factor	Estimate	t-stat	Adjusted R^2
CAPM Beta	0.00016610	0.02	0.0000
SMB	0.01777	0.94	0.0000
HML	0.00743	0.47	0.0001

Figure 3.1: Risk-factor loadings by lagged-rank

This figure shows the risk-factor loadings by lagged-rank. Lagged-rank is computed as the wealth rank a portfolio attained in the previous month's wealth distribution for a particular state. Factor loadings are determined by running monthly portfolio returns against either the CAPM or Fama-French 3 factor models. The rank-model proposed would suggest that poorly ranked portfolios should have higher risk-factor loadings. The plot does not suggest that this intuition is correct but it may be possible that loadings on an unaccounted set of risk-factors which are orthogonal to these do vary by rank. For display purposes the y-axis is truncated at -2 and 2 respectively.

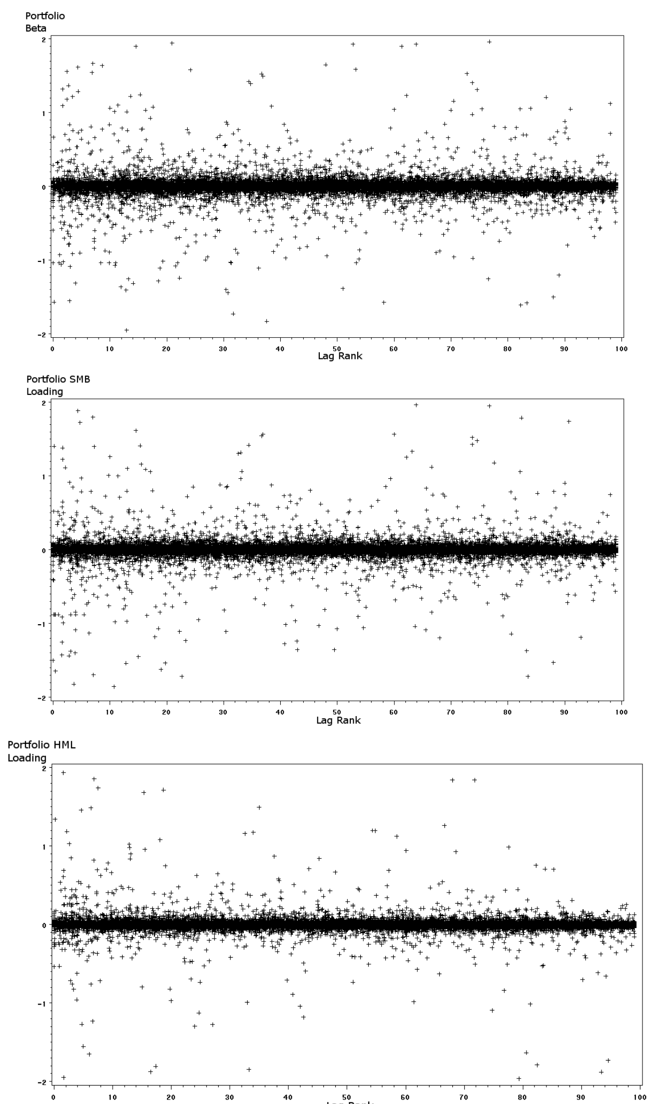

Table 3.8: Quintile wealth-rank portfolio correlations

This table reports the mean correlation between portfolio returns for pairs of quintiles of portfolio wealth rankings. The table also reports the Gabriel comparison interval for the correlations. Portfolio returns as a monthly buy-hold return and exclude effects due to deposits and withdrawals. The pattern of correlations between mid-ranked portfolios is not monotonic but the theoretical expectation that highly ranked portfolios (5-5) should have higher mean positive correlations than low ranked portfolios (1-1) is suggested in the data.

Quintile Pair	Lower comparison interval	Mean correlation	Upper comparison interval
5-5	0.1068	0.1204	0.1340
5-2	0.0998	0.1131	0.1264
2-2	0.0898	0.1034	0.1170
5-3	0.0766	0.0899	0.1032
3-2	0.0759	0.0892	0.1025
4-2	0.0729	0.0862	0.0995
5-4	0.0676	0.0809	0.0942
2-1	0.0623	0.0756	0.0889
4-3	0.0612	0.0745	0.0878
3-3	0.0581	0.0717	0.0854
5-1	0.0522	0.0655	0.0788
4-4	0.0477	0.0614	0.0750
3-1	0.0451	0.0584	0.0717
4-1	0.0448	0.0581	0.0714
1-1	0.0250	0.0386	0.0523

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Chapter 4

Conclusion

This thesis has posited a new model of decision making under risk which I have titled the Rank-wealth model. At its heart, the Rank-wealth model is an economically rational explanation for why the rank of an individual's wealth/endowment in the cross sectional distribution of wealth/endowment is relevant in decision making under risk. This relevance of decision making in the discipline of Finance cannot be overstated. Asset prices and corporate actions, two key areas of Finance, are both governed by the underlying assumptions of how individuals and institutions make financial decisions. Fully understanding how individuals come to make the decisions they do is in some ways the Holy Grail of Finance. This thesis is one of many research projects that attempt to seek out that ever elusive prize of a complete theory of financial decision making. While it is clear that the theory I propose is not a complete explanation of individual action it does contribute to the discipline of Finance by opening a rich and new area of research and it also immediately explains some empirical observations of individual decision making that other canonical models cannot.

In the first essay I create the Rank-wealth model from first economic principles. To generate the unique predictions of my model I introduce the assumption of consumer good indivisibility. To summarize, this assumption simply states that one cannot consume a fraction of many types of consumer goods and receive a fraction of the utility that consuming the entire good would provide. An example of this is the indivisibility of a house, car or television where a fraction of each of these units will not deliver a fraction of their utility. Arising from the indivisibility assumption is a rank-ordering of consumer goods by the population and a sequential pricing of those goods based on the marginal consumer's indifference function. Under this model, the distribution of wealth (or any divisible endowment) will have an impact on the prices of indivisible goods and thus consumers derived utility function over endowment. I demonstrate that for a non-monotonic and unimodal distribution, like those in most first world nations, where the 'middle-class' largely outnumber either the very-poor or the very-rich that the derived utility function will take on a shape that is uncannily similar to that documented by Kahneman and Tversky (1979). Given the prominence of Prospect Theory and its relative success in applications that sought to explain financial anomalies (Barberis and Huang, 2008), the rank-model contributes to the literature simply by offering a theoretical mechanism for how such preferences can arise. Yet the model I propose promises much more than that. As indirectly documented by Krasny (2009), the Rank-wealth model also contributes by eliminating two fund separation (Tobin, 1958) and thus is a theoretical explanation as to why the CAPM (Sharpe, 1964) is misspecified and can partially explain the failure of that model.

In the second essay of this thesis I test the rank-model I proposed in the first manuscript. Using the same data set as Odean (1998) and employing a methodology very similar to Kumar (2009) I show that poorly endowed individuals do indeed hold more lottery-type-stocks (LTS) in their portfolios. I also show that the proportion of LTS in a portfolio is more responsive to a dollar shock conditional on rank adjustment than on a dollar shock without a rank adjustment by an order of magnitude. This is suggestive that the rank variable is indeed an important one in determining individual risk-taking propensity. Contrary to my expectations, I do not find evidence that portfolios held by the poorer individuals load more heavily onto well recognized risk factors such as the market premium, HML or SMB (Fama and French, 1992). A possible explanation for this, in accordance with the rank-model I propose, is that the risk that poor individuals elect to hold is idiosyncratic in nature and does not project well onto canonical model's vector spaces. Consistent with Kumar (2009), I find that portfolios which have high levels of LTS underperform in a classic risk-return metric as compared to low LTS portfolios. However, my theory suggests an additional test to determine how portfolio categories perform relative to each other on a rank metric. On this basis, high LTS portfolios actually perform reasonably well and this again offers support to the proposed theory that individuals are most concerned about maximizing their rank and not their dollar-return.

The Rank-wealth model (RWM) presented in this thesis has a number of advantages in comparison to competing theories of decision making. Though the model is effectively the same as conventional expected utility theory (EUT) in terms of the axioms of decision making, the good indivisibility assumption of my model is both intuitive and delivers markedly different predictions. RWM lacks the degrees of freedom of the RDEU models of Quiggin (1982); Yaari (1987) but this is a double-edged sword for it also means that the model I propose has many more testing opportunities that can potentially invalidate the hypothesis. RMW is relatively easily supported by failing to find contrary evidence whereas such tests are not as readily identified for RDEU. The model I propose also delivers rank concerns, as fits with many economists' intuition, but does so without making an *a priori* assumption of rank relevance as some other models do (Abel, 1990; Constantinides, 1990; Robson, 1992; Roussanov, 2009; Krasny, 2009). Other models that derive rank concerns from first principles (Cole et al., 1992, 2001; DeMarzo et al., 2004, 2008) have contributed greatly to the discussion of decision making. RWM extends these models a little further by positing implications for individuals conditional on their cross-sectional wealth/endowment. The rank-model is also able to explain the endowment effect and intriguingly gives rise to Sunspot equilibria. The former ability is important because the endowment effect is well documented (Knetsch and Sinden, 1984; Knetsch, 1989; Kahneman et al., 1990) and is often not easily explained by conventional decision making theories. The delivery of Sunspot equilibria is particularly interesting as it can explain why idiosyncratic risk might be priced.

The RWM is not yet a complete theory. There are still a number of valid objections to the model as well as a number of implications which need to be explored further. Objections to the model include an assumption of homogeneous preferences amongst consumers. Though this assumption greatly simplified the analysis it is only reasonable to recognize heterogeneity in consumer tastes which would undoubtedly affect the model, especially in the case of numerous goods. One possible extension to address this concern is to create a hedonic model similar to Rosen (1974) and price good characteristics rather than indivisible goods. While the Sunspot version of the model can nest a hedonic model problem of this nature another criticism arises when one attempts to do so. Namely, the algorithm for solving the linear problem of the Sunspot equilibrium is slow for even modest sized state spaces. Obvious work is needed on the algorithm but hope exists that such efforts will be quite fruitful as a long line of extant literature in computational programming addresses concerns just such as these. Finally, though RWM can explain why the CAPM may be unsuccessful as an asset pricing model there is, as of yet, no substitute for the CAPM derived from the RWM. This is not wholly satisfying as dismantling an opposing theory, such as the CAPM, is invariably easier than providing a legitimate substitute. RWM can currently price Sunspot securities which could in fact replace the CAPM but, again because of the algorithm issues discussed, the computational requirements are prohibitive at this time. Perhaps with further theoretical development and a few more plausible assumptions a more tractable asset-pricing framework will arise from RWM and offer an alternative to the CAPM. Unfortunately this work is beyond the scope of this thesis and may require some time to fully develop.

Despite the RWM shortcomings the model is already useable to test a variety of financial economic hypotheses. Predictions about the behaviour of the endowment effect can be determined even from the relatively small state space Sunspot problem, which is solveable. Moreover, the implications of the general rank effects can be tested empirically, as in the second essay, and experimental studies can also be conducted. To that end an on-going research project has already been commenced at the John Molson School of Business where two hundred students have participated in a decision survey which can test more of the predictions of the RWM. Though the statistical analysis of this new research is not yet complete, at first glance there once again appears to be empirical support for the model I propose in the first essay. In closing, the RWM can help explain a number of anomalies that have bedeviled financial models and may even, one day, provide a new route to asset pricing. Further research is certainly required to verify the rank wealth hypothesis and to address limitations in the current version of the theory but the promise of an intuitive and improved model of decision making under risk warrants just such an exploration.

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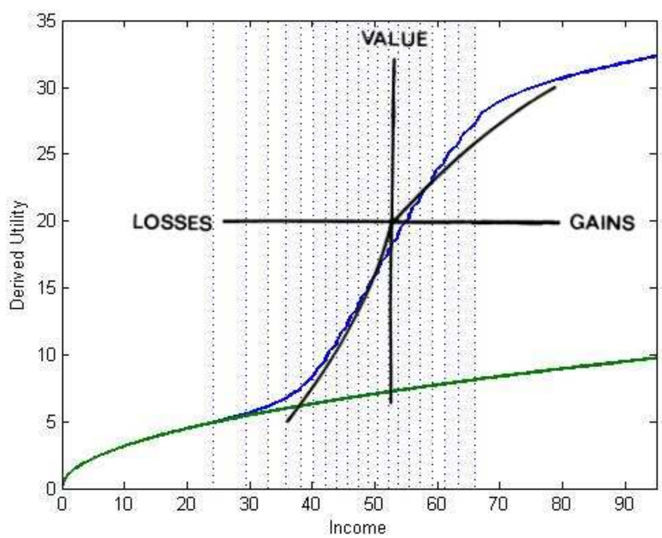
Appendix A

Prospect-theory value function and RWM value function

Figure A.1: Overlay of Prospect-theory value function on the Rank-wealth model



Appendix B

Existence proposition

Proposition 4. *The Rank-wealth model has a unique equilibrium.*

Proof. Let $\overrightarrow{c}$ be the vector of indicator variables taking values of 0 and 1 that denote consuming a particular indivisible good. Let $\overrightarrow{P}$ and $\overrightarrow{q}$ be the vectors of prices and qualities of the indivisible goods respectively. Let c_i, P_i and q_i be the consumption indicator variable, price and quality of the i^{th} indivisible good. Further allow C to be the set of all possible consumption vectors, $\overrightarrow{c}$, that satisfy the constraint that the sum of their elements is 1.

The demand of an individual with wealth w for good k is:

$$D_i(w, P_i) = \arg\max_{c_i \in C} u(w - c_i P_i) + c_i q_i \tag{B.1}$$

The aggregate demand, which equals supply to meet the market clearing condition, for good i is therefore:

$$D_i^{Aggregate}(w) = \int D_i(w, P_i) \cdot f(w) dw = s_i \tag{B.2}$$

Note that if $c_j q_j < c_i q_i$ then $\forall P_j \geq P_i, P_j \in \mathbb{R}^+$ it is the case that $D_j^{Aggregate}(w, P_j) = 0$.

In words, if good i is of superior quality to good j and of equal or lesser price, then the demand of good j will be zero as all individuals would do better to purchase good i than j.

If $s_j \neq 0$ then the market clearing condition cannot be met if $P_j \geq P_i$ for $c_j q_j < c_i q_i$ which implies that $P_i > P_j$ for $s_j \neq 0$.

This holds true for all pairwise comparisons which means that good prices must be strictly increasing in quality. Let:

$$\Omega(P_i) = \{w : u(w - c_i P_i) - c_i q_i > u(w - c_j P_j) - c_j q_j \forall j \neq i\} \tag{B.3}$$

Fixing all other prices but price P_i,

$$P_i' < P_i \Rightarrow \Omega(P_i') \supset \Omega(P_i) \tag{B.4}$$

Rank ordering the goods in ascending sequence of quality so that good K is of the highest quality and good 1 is of lowest, the aggregate demand of each good is:

$$D_i^{Aggregate}(P_{i-1}) = 1 - \sum_{j=i+1}^{K} s_i \tag{B.5}$$

$$D_i^{Aggregate}(P_{i+1}) = 0 \tag{B.6}$$

In the case of the K^{th} good, $s_{K+1} = 0$ by definition and $P_{i+1} = \overline{w}$.
As $\sum_i^K s_i < F(\overline{w}) - F(\underline{w})$ then there is one unique P_i such that:

$$D_i^{Aggregate}(P_i) = s_i \tag{B.7}$$

As each P_i is unique then the vector $\overrightarrow{P}$ is also unique for a given $\overrightarrow{q}$ and distribution $F(\cdot)$. This also means that there is a unique vector of wealth boundaries, $\overrightarrow{w}$, determined by the indifference equations. Thus, the equilibrium $(\overrightarrow{P}, \overrightarrow{w})$ that solves the set of equations is unique.

□

Appendix C

Value function convexity

This proof demonstrates that the Rank-wealth effect persists even when there is a continuum of infinitely many indivisible good qualities.

Proof.

Proposition 5. *If and only if the multiple of the marginal price of the optimal indivisible good consumed and the marginal quality of the optimal indivisible good consumed is greater than one then the individual will be locally risk-seeking. That is,*

$$p'(D(w))D'(w) > 1 \Rightarrow V''(w) > 0 \tag{C.1}$$

There exists a cumulative distribution of wealth, F(w) and let G(q) denote the supply of good of quality q where $q \in [0, \bar{q}]$. Let $p(\cdot)$ be the price function so that p(q) is the price of good quality q in equilibrium.

The maximization problem faced by a single consumer is:

$$\max_q U(w, q) = u(w - p(q)) + q \tag{C.2}$$

At her personal optimum, her FOC with respect to q must, for any given function $p(q)$, be equal to zero. That is,

$$\frac{dU(w, q)}{dq} = -u'(w - p(q))p'(q) + 1 = 0 \tag{C.3}$$

Using the FOC and isolating p'(q) yields

$$p'(q) = \frac{1}{u'(w - p(q))} \tag{C.4}$$

This expression will be used later as a substitution into the value function.

In equilibrium there exists a demand function, $D(w)$, that gives the optimal quality demanded by individual with w wealth. Thus, $D(w)$ maps from $G(q)$ to $F(w)$.

$$D = G^{-1} \circ F \tag{C.5}$$

The value function can therefore be written as:

$$V(w) = u[w - p(D(w))] + D(w) \tag{C.6}$$

The first derivative of the Value function, with respect to w, is:

$$V'(w) = u'[w - p(D(w))](1 - p'(D(w))D'(w)) + D'(w) \tag{C.7}$$

Substituting from equation (C.4) into the equation above gives:

$$V'(w) = -D'(w) + u'(w) + D'(w) = u'(w) \tag{C.8}$$

The second derivative of the value function with respect to w is therefore:

$$V"(w) = u"[w - p(D(w))](1 - p'(D(w))D'(w)) \tag{C.9}$$

Recall that as $u(\cdot)$ is concave and strictly increasing then $u'(\cdot) > 0$ and $u"(\cdot) < 0$.

Notice that the value function may have a positive second derivative, implying convexity, if and only if

$$p'(D(w))D'(w) > 1 \tag{C.10}$$

Thus far the problem posed has remained completely general with the exception of $u(\cdot)$ being an increasing concave function and the utility function being separable as given.

To determine if the continuous good case can produce convexity in the derived utility it must be determined if equation (C.10) is true or false. To proceed I first assume that G(q) is uniformly distributed over the interval $[0, \bar{q}]$ with mass one so that $G(q) = 1/\bar{q}$. This means that $G^{-1}(x) = \bar{q}x$ and:

$$D(w) = G^{-1} \circ F = \bar{q}F(w) \tag{C.11}$$

Substituting $D^{-1}(q)$ into equation (C.4) gives

$$p'(q)u'(D^{-1}(q) - p(q)) = 1 \tag{C.12}$$

To actually solve this problem I will make the further assumption that $u(x) = \ln(x)$, and so, $u'(x) = 1/x$. The equation now becomes:

$$\frac{p'(q)}{D^{-1}(q) - p(q)} = 1 \Rightarrow p'(q) = D^{-1}(q) - p(q) \tag{C.13}$$

$$p + p' = D^{-1} \tag{C.14}$$

This differential equation can be solved by the method of variation of parameters. First identify the fundamental solution to the homogenous equation as $p(q) = e^{-q}$. The solution to this particular differential equation is given by $p(q) = a(q)e^{-q}$, where a(q) is a function to be identified.

The good of zero quality clearly must have a zero price which gives the initial condition of $p(0) = 0$.

Substituting for $p(q)$ into the differential equation gives:

$$a'(q)e^{-q} = D^{-1} \Rightarrow a'(q) = D^{-1}(q)e^{q} \tag{C.15}$$

Integrating a(q) and substituting F(x) for D(x):[40]

$$a(q) = \int_0^q D^{-1}(x)e^x dx = \int_0^q F^{-1}(x)e^x dx \tag{C.16}$$

[40] A direct substitution is possible because $G(\cdot)$ was assumed to be uniformly distributed over the interval [0,1].

Thus the price is given by the expression:

$$p(q) = \left[\int_0^q F^{-1}(x)e^x dx\right] e^{-q} \tag{C.17}$$

and

$$\frac{dp(q)}{dq} = p'(q) = F^{-1}(q) - \left[\int_0^q F^{-1}(x)e^x dx\right] e^{-q} \tag{C.18}$$

Substituting w for $F^{-1}(q)$ and F(w) for q,

$$p'D' = \left[w - \left[\int_0^{F(w)} F^{-1}(x)e^x dx\right] e^{-F(w)}\right] F'(w) \tag{C.19}$$

Taking $e^{-F(w)}$ inside the integral

$$p'D' = \underbrace{\left[w - \left[\int_0^{F(w)} F^{-1}(x) \underbrace{e^{x-F(w)}}_{>0,\, \leq 1} dx\right]\right]}_{>0} F'(w) \tag{C.20}$$

Over the interval of integration the expression exponential expression under the integral is between the value of 0 and 1. As the other expression under integration, $F^{-1}(x)$, is strictly less than w over the interval of integration the entire bracketed section of the equation above is positive.

This implies that $p'D'$ can be greater than one for a sufficiently large $F'(w)$. That is, if the population mass is increasing quickly enough, convexity in the Value function is possible in the example situation where $G(\cdot)$ is uniform and $u(\cdot) = ln(\cdot)$.

□

Appendix D

Solution of application

One starts with the differential equation:

$$p + p' = D^{-1} \tag{D.1}$$

First identify the fundamental solution to the homogenous equation as $p(q) = e^{-q}$. The solution to this particular differential equation is given by $p(q) = a(q)e^{-q}$, where a(q) is a function to be identified.

The good of zero quality clearly must have a zero price which gives the initial condition of $p(0) = 0$.

Substituting for $p(q)$ into the differential equation gives:

$$a'(q)e^{-q} = D^{-1} \Rightarrow a'(q) = D^{-1}(q)e^{q} \tag{D.2}$$

Integrating a(q) and substituting F(x) for D(x):[41]

$$a(q) = \int_0^q D^{-1}(x)e^{x}dx = \int_0^q F^{-1}(x)e^{x}dx \tag{D.3}$$

Thus the price is given by the expression:

$$p(q) = \left[\int_0^q F^{-1}(x)e^{x}dx\right]e^{-q} \tag{D.4}$$

and

$$\frac{dp(q)}{dq} = p'(q) = F^{-1}(q) - \left[\int_0^q F^{-1}(x)e^{x}dx\right]e^{-q} \tag{D.5}$$

[41] A direct substitution is possible because $G(\cdot)$ was assumed to be uniformly distributed over the interval [0,1].

Substituting w for $F^{-1}(q)$ and F(w) for q,

$$p'D' = \left[w - \left[\int_0^{F(w)} F^{-1}(x) e^x dx \right] e^{-F(w)} \right] F'(w) \tag{D.6}$$

Taking $e^{-F(w)}$ inside the integral

$$p'D' = \underbrace{\left[w - \left[\int_0^{F(w)} F^{-1}(x) \underbrace{e^{x-F(w)}}_{>0,\, \leq 1} dx \right] \right]}_{>0} F'(w) \tag{D.7}$$

Appendix E

Compatibility and rationality constraints

The proof below demonstrates both the individual rationality (IR) and the individual compatibility (IC) constraints for the specification where $u(\cdot) = \ln(\cdot)$.

From D it is known that the value function is given by:

$$V(w) = \ln(w - p(D(w))) + D(w) \tag{E.1}$$

Where the quality of the indivisible good is given by the expression $q = D(w)$.

I prove both the IC and IR constraints by contradiction. Suppose $D(w) = q^*$ and suppose individual with wealth w is contemplating consuming a different quality level of q' instead. For this to be rational it must be:

$$\ln[w - p(q')] + q' - \ln(w - p(q^*)] - q^* > 0 \tag{E.2}$$

Reorganizing and employing the property of the natural logarithm $u(a) - u(b) = u(a/b)$,

$$\ln\left[\frac{w - p(q')}{w - p(q^*)}\right] > q^* - q' \tag{E.3}$$

$$\frac{w - p(q')}{w - p(q^*)} > e^{q^* - q'} \tag{E.4}$$

Isolating w yields,

$$w > \frac{p(q') - p(q^*)e^{-q'+q^*}}{1 - e^{-q'+q^*}} \tag{E.5}$$

Substituting $F^{-1}(q)$ for w and $p(q) = \left[\int_0^q F^{-1}(x)e^x dx\right] e^{-q}$ for the prices gives and canceling common coefficients gives:

$$F^{-1}(q^*) > \frac{\left[\int_0^{q'} F^{-1}(x)e^x dx\right] e^{-q'} - \left[\int_0^{q^*} F^{-1}(x)e^x dx\right] e^{-q'}}{1 - e^{-q'+q^*}} \tag{E.6}$$

Combining integral terms and taking the constant left-hand term inside the integral creates the expression:

$$0 > \frac{\left[\int_{q^*}^{q'} \left(F^{-1}(x) - F^{-1}(q^*)(1 - e^{-q'+q^*})\right) e^x dx\right] e^{-q'}}{1 - e^{-q'+q^*}} \tag{E.7}$$

For deviation to any other q' from q^* to be optimal the above expression must be true.

Suppose $q' > q^*$ then the term $\int_{q^*}^{q'} \left(F^{-1}(x) - F^{-1}(q^*)(1 - e^{-q'+q^*})\right) e^x dx$ is positive, the term $e^{-q'}$ is always greater than zero and the denominator is also positive. Which means than the entire right hand side is greater than zero, which violates the condition. It is not optimal to select $q' > q^*$.

Suppose instead $q' < q^*$ then the term $\int_{q'}^{q^*} \left(F^{-1}(x) - F^{-1}(q^*)(1 - e^{-q'+q^*})\right) e^x dx$ is negative, the term $e^{-q'}$ is always greater than zero and the denominator is now also negative. As both denominator and numerator are less than zero then again the entire right hand side is greater than zero, which violates the condition. It is not optimal to select $q' < q^*$.

This proves the incentive compatibility of the policy proposed. Substituting a particular value of q' with $q' = 0$ likewise proves the individual rationality constraint.

□

Appendix F

Producer problem

The emphasis of this paper is not on the incentives of producers but a valid concern regarding the results of this paper is whether or not indivisible goods are in finite supply. I demonstrate for a simple case that a producing firm will optimally produce too little to meet total demand in order to maximize profits.[42] What is novel in the following analysis is that firms, in a rank-world, will shift production resources to manufacture smaller amounts of luxury goods in less egalitarian societies. In the long-run this would imply that wealth accumulation will reduce the social well-being of most individuals even though the instantaneous resource allocation may be Pareto optimal.

I illustrate this effect through a simple example in partial equilibria. To simplify the world I restrict the analysis to a single good, k, with quality q_k. The good is produced by a single firm.

Suppose the firm has a maximum quality budget from which it can produce goods; label this quality budget Q_b. If the firm chooses to manufacture fewer goods it can increase the quality of the goods it produces. The firm attempts to maximize the divisible good collected for producing and selling the indivisible good. The objective function of the firm is therefore:

$$\max_{q_k, s_k} \int_{w_k}^{\bar{w}} p_k f(x) dx \tag{F.1}$$

Subject to the constraint $Q_b \geq s_k q_k$.[43] I rewrite this problem by inte-

[42]This result is simply the Cournot monopolist outcome.

[43]By specifying this constraint I am assuming a certain elasticity of substitution between quality and quantity produced. One could assume different elasticities to reflect various production cost specifications but such an analysis would only mute or accentuate the results I describe unless the elasticity was highly non-monotone and related to the wealth

grating. The problem becomes:

$$\max_{s_k} p_k \left(1 - F(w_k)\right) \tag{F.2}$$

I substitute in the expressions for the price of the good and the wealth boundary and also use the relation $q_k = Q_b/s_k$. It is known that the firm will use the entire quality budget as it is costless to do so and can increase either the quality (and hence price) or quantity of good k produced (and hence sales volume):

$$\max_{s_k} \left[F^{-1}(1 - s_k) - u^{-1}\left(u(F^{-1}(1 - s_k)) - \frac{Q_b}{s_k}\right)\right] s_k \tag{F.3}$$

The FOC with respect to s_k is the expression:

$$\frac{d\Pi}{ds_k} = \frac{\left[F^{-1}(1 - s_k) - u^{-1}\left(u(F^{-1}(1 - s_k)) - \frac{Q_b}{s_k}\right)\right] s_k}{ds_k} = 0 \tag{F.4}$$

Analytically isolating an optimal s_k will be impossible for many specifications of u(·) and F(·) but will often be very easily solved numerically. The table below gives the optimal s_k for a producer which has a quality budget of 0.9 and faces consumers with log utility of divisible good and a truncated normal distribution of endowments with mean value of 2 and a given standard deviation.

Table F.1: Monopolist profit and optimal policy by consumer wealth dispersion.

Standard Deviation	Optimal s_k^*	Optimal q_k^*	Profit (Π)
0.25	0.8232	1.0932	0.9677
0.5	0.6725	1.3383	0.8814
1	0.5059	1.7792	0.8348
2	0.3777	2.3829	0.8993
3	0.3269	2.7533	1.0239

distribution.

Examining table F.1 one can see that as wealth is concentrated (by increasing the standard deviation of the wealth distribution) the optimal quality of good increases and the optimal quantity provided declines. Furthermore, notice that as wealth is concentrated the profits of the monopolist first decline and then increase.

The reason for this pattern is because at very low wealth dispersion the monopolist can discriminate quite effectively against all consumers with a single price and quality because all consumers have nearly identical wealth and homogenous preferences.

As the wealth dispersion increases though, the monopolist finds it increasingly difficult to discriminate against all consumers and the consumers win proportionately more of the gains of trade. Eventually however, the wealth dispersion becomes so large, or equivalently wealth is so concentrated into few hands, that the monopolist chooses not to serve a large section of the population. In so doing, the monopolist can once again more effectively price discriminate against a rich subsection of the consumers and therefore his profits increase.

Imagining that the firm is a foreign producer whose profits are not returned to shareholders, the effects of social welfare are apparent. As the quality budget, or equivalently the total number of utiles the firms indivisible good can deliver, is fixed then any increase in profits is a strict cost to society. A perfectly egalitarian society, where all individuals have identical wealth and preferences would allow the firm to perfectly price discriminate such that all individuals are just indifferent to consuming the indivisible good. In this case, the firm would gain all the benefits of trade. Similarly, in a perfectly non-egalitarian society where one individual has all divisible good, the monopolist can perfectly discriminate against that one individual and once more all the gains of trade go to the firm.

I do not model a system where the profits of the firm are returned to the consuming population but if one were to do so I conjecture that the wealth accumulation of the rich would cause a temporal trend to increasing quality and decreasing quantity. Furthermore, as this trend progressed, the profits of the firm would rise accelerating the wealth accumulation and the flight to quality. This feedback loop would continue and further accelerate the process. In a practical setting this process might explain patterns of macroeconomic development where at first an extant middle-class has high comparative living standards which erode as the rich become increasingly rich. The erosion of the middle-class continues until this group all but ceases to exist. The standard of living of the rich would increase as wealth is concentrated, but at a slowing rate because the supplying firms absorb an increasing proportion of the gains of trade. In the limit, I conjecture that this system would likely cause a social revolution when the majority of the population falls below some threshold living standard but this conjecture goes beyond the scope of the analysis in this thesis.