Path Integrals for Multiply Connected Spaces

by

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Abstract

We derive the propagator for a particle constrained to a torus and to a Klein Bottle. This is accomplished by considering relative symmetries between the desired system and a system for which the propagator is known. This result is checked against the propagator derived via the method of stationary state construction, for which the entire spectrum of the Hamiltonian is required. We also briefly consider the application of further constraints to the systems, and the implications of different symmetries on the same constraint.

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Dedication

To Oma & Opa

Chapter 1

Introduction

The propagator is a powerful tool in quantum mechanics, as knowing the propagator for a system means knowing the solution to the time dependent Schrodinger equation. If we know the state $\psi(x_0, t_0)$ at some initial point (x_0, t_0), then the state at some later point (x, t) is given by:

$$\psi(x,t) = G(x,t;x_0,t_0)\psi(x_0,t_0) \tag{1.1}$$

where $G(x,t;x_0,t_0)$ is the propagator associated with the system.

Path integrals arise in determining the form of the propagator: Schulman [6] derives the propagator for a free particle with time independent Hamiltonian (which allows us to set $t_0 = 0$) to be:

$$G(x,t;x_0,0) = {}_{N\to\infty}^{lim} (\frac{m}{2\pi i\hbar\epsilon})^{(N+1)/2} \int dx_1 \ldots dx_N exp[\frac{im}{2\hbar\epsilon}\sum_{j=0}^{N}(x_{j+1}-x_j)^2] \tag{1.2}$$

where $x_{N+1} = x$ and $\epsilon = t/(N+1)$. This can be simplified to the closed form:

$$G(x,t;x_0,0) = \sqrt{\frac{m}{2\pi i\hbar t}} exp[\frac{im}{2\hbar t}(x-x_0)^2] \tag{1.3}$$

where the phase is the classical action divided by $\hbar$.

Determining the propagator becomes more complicated on multiply connected spaces, as it is possible to generate two paths having the same endpoints which cannot be continuously deformed into each other. In this situation, it is useful to define a winding number which separates these paths into homotopy classes, within which it is possible to perform this deformation continuously. However, care must be taken in determining the propagator to account for these winding numbers. These complications mean that one must often return to the definition in order to account for them properly.

One method of determining the propagator in any space is to perform a sum over states. If $(\psi_n(x), E_n)$ is a pair of stationary states and energies satisfying the Hamiltonian for the space, then the propagator can be expressed as:

$$G(x_f, t; x_i, 0) = \sum_{n=-\infty}^{\infty} \psi_n(x_f)\psi_n^*(x_i)exp(-iE_nt) \quad (1.4)$$

However, it is possible that simplifying the propagator to a useful form from this expression can still be quite complicated, as performing the sum on multiply connected spaces is not straight-forward. One might wish to obtain a simpler procedure to construct the propagator more quickly, without having to determine explicitly the entire spectrum for the Hamiltonian.

Alternatively, one can make use of the homotopy classes as outlined in various papers [2, 4, 5]. If one can find a projection p to the desired space M from a simply connected covering space $\tilde{M}$, the problem becomes much simpler. Paths in M from m to n ($m, n \in M$) correspond to paths in $\tilde{M}$ from some fixed $\tilde{m} \in p^{-1}(m)$ to each $\tilde{n}_i \in p^{-1}(n)$, where i runs over all pre-images of n. Then we carry out the path integral in $\tilde{M}$ (using the projection p to carry over the Lagrangian from M) and sum over i, including phase factors determined by the system. This is one method we will use to determine the propagator for the circle.

If we can relate the desired space to a simpler one (not necessarily a simply connected covering space) through various symmetries that are either preserved or lost between the two spaces, perhaps it is possible to construct the propagator without solving for the entire spectrum or concerning ourselves with mapping to a covering space. In this fashion, we will construct the propagator for a particle constrained to a torus and to a Klein Bottle, beginning with the propagator for a particle constrained to a circle. As a check, we will also explicitly construct the propagator from the spectrum using the method of stationary states (1.4). We will also examine briefly the question of applying further constraints to these systems which correspond to the particle model of a gauge theory.

Chapter 2

Circle Propagator

2.1 Circle Stationary State Propagator

We begin by considering the derivation of the propagator for a particle on the circle S^1 presented by Schulman [5]. This will facilitate the symmetry construction of the torus propagator, as the torus is the tensor product of two circles. Construction of the circle is achieved by identification of $x = 0$ and $x = 2\pi$ on the interval $[0, 2\pi]$. Alternatively, the circle is constructed via the identification $x = x + 2\pi$ on the real line $\Re$, which is more convenient as it preserves the geometry of $\Re$. On the circle, the local Schrodinger equation for a free particle is (taking $\hbar$=1, M=1):

$$-\frac{1}{2}\partial_x^2\psi(x) = E\psi(x) \tag{2.1}$$

The condition that the system is on the circle is enforced by requiring the state $\psi(x)$ also obey the boundary condition:

$$S^{S^1}\psi(x) = \psi(x + 2\pi) \tag{2.2}$$

where S^{S^1} is a unitary symmetry operator which commutes with the Hamiltonian, $[S^{S^1}, H] = 0$. Hence, $\psi(x)$ can be simultaneously diagonalized with respect to both operators.

If we write $S\psi = c\psi$ (where superscripts and arguments have been suppressed), then since:

$$S^n\psi(x) = \psi(x + 2n\pi) \tag{2.3}$$

and:

$$S^n S^m = S^{n+m} \tag{2.4}$$

it follows that $c = e^{i\sigma}$ (for σ a real constant) and Bloch's theorem states that:

$$\psi(x) = exp(\frac{i\sigma x}{2\pi})u(x) \tag{2.5}$$

where $u(x) = u(x+2\pi)$ is a periodic function. The solution of Schrodinger's equation (2.1) for $\psi(x)$ of this form yields the complete set of eigenstates:

$$\psi_n^{S^1}(x) = \frac{1}{\sqrt{2\pi}} exp(inx + \frac{i\sigma x}{2\pi}) \tag{2.6}$$

$$E_n^{S^1} = \frac{1}{2}(n + \frac{\sigma}{2\pi})^2 \tag{2.7}$$

Thus, instead of a unique quantization, one obtains a set of inequivalent quantizations parameterized by σ, and perfect periodicity corresponds to the unique quantization $\sigma = 0$. As a result, stationary state construction of the propagator (1.4) takes the form:

$$G^{S^1}(x_f, t; x_i, 0) = \sum_n \frac{1}{2\pi} exp(inx + \frac{i\sigma x}{2\pi} - \frac{it}{2}(n + \frac{\sigma}{2\pi})^2) \tag{2.8}$$

where $x = x_f - x_i$, and the propagator argument shall henceforth be suppressed.

In order to facilitate comparison with the propagator as derived from symmetry considerations, it is useful to introduce the Jacobi theta function defined by:

$$\Theta_3(Z, T) = \sum_{n=-\infty}^{\infty} exp(i\pi n^2 T + 2inZ) \tag{2.9}$$

As a trivial result of the definition, $\Theta_3(-Z, T) = \Theta_3(Z, T)$. Furthermore, two useful identities of the theta function are:

$$\Theta_3(Z + a\pi + b\pi T, T) = exp(-i\pi b^2 T - 2ibZ)\Theta_3(Z, T) \tag{2.10}$$

$$\Theta_3(Z, T) = (-iT)^{-1/2} exp(Z^2/i\pi T)\Theta_3(Z/T, -1/T) \tag{2.11}$$

for $(a, b) \in \mathbb{Z}$, the proof of which can be found in Appendix A.

Substituting the theta function (2.9) yields:

$$G^{S^1} = \frac{1}{2\pi} exp(i(\frac{\sigma x}{2\pi} - \frac{\sigma^2 t}{8\pi^2}))\Theta_3(\frac{x}{2} - \frac{\sigma t}{4\pi}, -\frac{t}{2\pi}) \tag{2.12}$$

which, if we apply the Jacobi identity (2.11) becomes:

$$\begin{aligned}
G^{S^1} =&\frac{1}{2\pi}(i\frac{t}{2\pi})^{-1/2}exp(i(\frac{\sigma x}{2\pi}-\frac{\sigma^2 t}{8\pi^2}))* \\
&exp((\frac{x}{2}-\frac{\sigma t}{4\pi})^2/i\pi(-\frac{t}{2\pi}))\Theta_3((\frac{x}{2}-\frac{\sigma t}{4\pi})/(-\frac{t}{2\pi}),\frac{2\pi}{t}) \\
=&(\frac{1}{2\pi it})^{1/2}exp(\frac{ix^2}{2t})\Theta_3(\frac{\sigma}{2}-\frac{\pi x}{t},\frac{2\pi}{t}) \qquad (2.13)
\end{aligned}$$

2.2 Circle Symmetry Propagator

As a comparison with the stationary state construction, we can also use symmetry considerations to construct the propagator. Since two paths having the same winding number and fixed endpoints can be continuously deformed into each other, we write the full propagator as a sum over the winding number n:

$$G^{S^1}(x_f,t;x_i,0)=\sum_{n=-\infty}^{\infty}A_n G_n^{S^1}(x_f,t;x_i,0) \qquad (2.14)$$

for some non-trivial set of A_n's. By varying one endpoint (say x_f) through a full circle, we find that the full propagator is unchanged up to a phase factor, while we have translated $G_n^{S^1} \to G_{n-1}^{S^1}$. This, combined with (2.14), tells us that:

$$A_{n+1}=e^{i\sigma}A_n \qquad (2.15)$$

Furthermore, we know that $\|A_0\|=1$, and so fixing the arbitrary phase of A_0 to 0 yields:

$$G^{S^1}=\sum_n e^{in\sigma}G_n^{S^1} \qquad (2.16)$$

where the argument of the propagator shall again be suppressed.

Since one can (locally) define a smooth mapping $f:\Re\to S^1$ and inverse between the real line and circle for fixed endpoints, for each value of the winding number the propagator is that of a free particle (1.3). Substituting for $G_n^{S^1}$ into (2.16) yields:

$$G^{S^1}=\sum_n(\frac{1}{2\pi it})^{1/2}exp(in\sigma+\frac{i(x-2\pi n)^2}{2t}) \qquad (2.17)$$

Substituting the theta function (2.9) into our expression for the propagator on the circle (2.17) yields:

$$G^{S^1} = (\frac{1}{2\pi it})^{1/2} exp(\frac{ix^2}{2t})\Theta_3(\frac{\sigma}{2} - \frac{\pi x}{t}, \frac{2\pi}{t}) \tag{2.18}$$

in agreement with the stationary state propagator (2.13).

Chapter 3

Torus and Klein Bottle Foundations

3.1 Torus Foundations

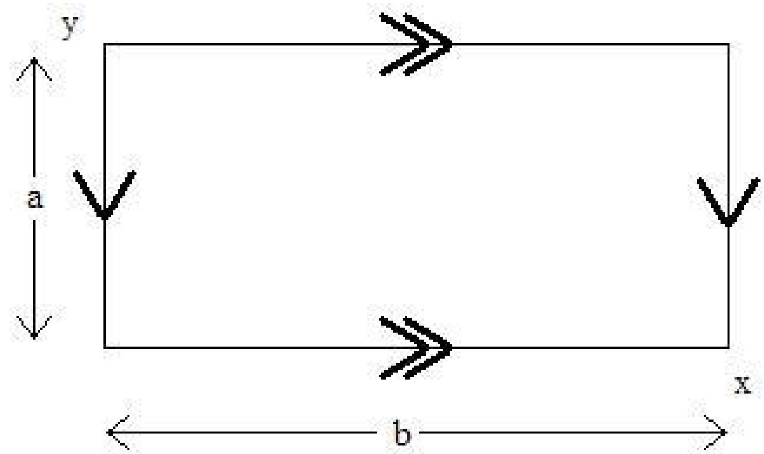

Figure 3.1: Torus Construction

We begin by generating the torus T^2 from a rectangle of side lengths a and b, seen in Fig. 3.1. Opposite sides (of equal length) are identified and connected, with the x-direction denoting motion parallel to side length b and y-direction parallel to side length a. In this way we construct a torus with boundary conditions:

$$
\begin{aligned}
(x, y) &= (x, y + a) \\
(x, y) &= (x + b, y)
\end{aligned}
\tag{3.1}
$$

Locally, Schrodinger's equation for a free particle is just:

$$-\frac{1}{2}(\partial_x^2 + \partial_y^2)\psi = E\psi \tag{3.2}$$

Eigenstates (non-normalized) and corresponding eigenvalues obeying the boundary conditions (3.1) are:

$$\psi_{rs}^{T^2} = exp(2\pi i(rx/b + sy/a)) \tag{3.3}$$

$$E_{rs}^{T^2} = \frac{1}{2}((\frac{2\pi r}{b})^2 + (\frac{2\pi s}{a})^2) \tag{3.4}$$

for $(r, s) \in \mathbb{Z}$, where $\psi_{rs}^{T^2}(x, y) = \psi_{rs}^{T^2}(x + nb, y + ma)$ for $(n, m) \in \mathbb{Z}$.

However, as seen on the circle, perfect periodicity corresponds to the specific quantization in which $\sigma = 0$. The more general quantization is implemented by unitary symmetry operators generated by the fundamental region:

$$\begin{aligned} S_x^{T^2} f(x, y) &= f(x + b, y) \\ S_y^{T^2} f(x, y) &= f(x, y + a) \end{aligned} \tag{3.5}$$

Simultaneous normalized eigenstates and eigenvalues of H, $S_x^{T^2}$, and $S_y^{T^2}$ are then:

$$\psi_{rs}^{T^2} = \frac{1}{\sqrt{ab}} exp(2\pi i(rx/b + sy/a) + \frac{i\sigma x}{b} + \frac{i\delta y}{a}) \tag{3.6}$$

$$E_{rs}^{T^2} = \frac{1}{2}((\frac{2\pi r + \sigma}{b})^2 + (\frac{2\pi s + \delta}{a})^2) \tag{3.7}$$

for $(r, s) \in \mathbb{Z}$, where σ and δ are the phases picked up in completing a full circuit in the x and y directions respectively (eg: x goes to x+b).

3.2 Klein Bottle Foundations

Beginning with the same identifications on the rectangle as for the torus, we join opposite ends. However, to obtain the Klein Bottle K^2 we introduce a twist before joining the ends, as shown in Fig. 3.2. Note that the construction used here is non-standard: the second twist has been included to introduce symmetry between x and y, whereas the standard (one twist) construction is asymmetrical (the untwisted direction behaves similarly to

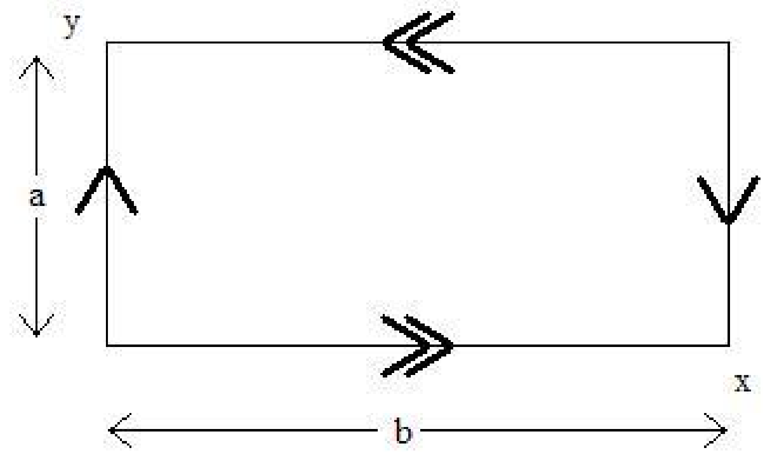

Figure 3.2: Klein Bottle Construction

the torus). For this construction, the boundary conditions are:

$$\begin{aligned}(x, y) &= (-x, y + a)\\ (x, y) &= (x + b, -y)\end{aligned} \tag{3.8}$$

Locally, Schrodinger's equation is the same for the Klein Bottle as for the torus (3.2). However, solutions are much different as a result of the boundary conditions. In terms of restricted integers (k,l) there are four groups of solutions given in Table 3.1.

$\psi_{kl}^{K^2}$	$E_{kl}^{K^2}$	k parity	l parity
$cos(\frac{\pi kx}{b})cos(\frac{\pi ly}{a})$	$\frac{\pi^2}{2}((\frac{k}{b})^2 + (\frac{l}{a})^2)$	even	even
$sin(\frac{\pi kx}{b})cos(\frac{\pi ly}{a})$	$\frac{\pi^2}{2}((\frac{k}{b})^2 + (\frac{l}{a})^2)$	even	odd
$cos(\frac{\pi kx}{b})sin(\frac{\pi ly}{a})$	$\frac{\pi^2}{2}((\frac{k}{b})^2 + (\frac{l}{a})^2)$	odd	even
$sin(\frac{\pi kx}{b})sin(\frac{\pi ly}{a})$	$\frac{\pi^2}{2}((\frac{k}{b})^2 + (\frac{l}{a})^2)$	odd	odd

Table 3.1: Klein Bottle eigenstates and eigenvalues.

Comparison of the Klein Bottle and torus states is easiest if we rewrite the torus states in terms of the restricted integers (k,l):

$$\psi_{kl}^{T^2} = exp(\pi i(kx/b + ly/a)) \tag{3.9}$$

$$E_{kl}^{T^2} = \frac{\pi^2}{2}((\frac{k}{b})^2 + (\frac{l}{a})^2) \tag{3.10}$$

where k and l are both even. Comparing this to Table 3.1, we see that the energy spectrum has the same form for both spaces with a denser spectrum for the Klein Bottle. Expanding the exponential states for the torus, we see that the four states of the Klein Bottle are all present, but with different energies. From this, we can see that the particle on the Klein Bottle is the same as on the torus, but with the degeneracy in energy lifted by the symmetric twists introduced in the construction.

Before continuing, we first simplify the Klein Bottle states into a compact form, allowing us to lift the restrictions on k and l. In order to do so, we first note that $1 + (-1)^k$ evaluates to 0 if k is odd, and 2 if k is even. Applying this, the states can be written as:

$$\psi_{kl}^{K^2} = sin(\frac{\pi kx}{b} + \frac{\pi}{4}(1 + (-1)^l))sin(\frac{\pi ly}{a} + \frac{\pi}{4}(1 + (-1)^k)) \tag{3.11}$$

while the energies remain unchanged.

As with the circle and torus before, this is the specific quantization assuming perfect periodicity. The unitary symmetry operators generated by the fundamental region are:

$$\begin{aligned} S_x^{K^2} f(x, y) &= f(x + b, -y) \\ S_y^{K^2} f(x, y) &= f(-x, y + a) \end{aligned} \tag{3.12}$$

Simultaneous normalized eigenstates and eigenvalues of H, $S_x^{K^2}$, and $S_y^{K^2}$ are then:

$$\begin{aligned} \psi_{kl}^{K^2} = \sqrt{\frac{4}{ab}} sin(\frac{\pi kx}{b} + \frac{\sigma x}{2b} + \frac{\pi}{4}(1 + (-1)^l))* \\ sin(\frac{\pi ly}{a} + \frac{\delta y}{2a} + \frac{\pi}{4}(1 + (-1)^k)) \end{aligned} \tag{3.13}$$

$$E_{kl}^{K^2} = \frac{1}{2}((\frac{\pi k}{b} + \frac{\sigma}{2b})^2 + (\frac{\pi l}{a} + \frac{\delta}{2a})^2) \tag{3.14}$$

for $(k, l) \in \mathbb{Z}$, where σ and δ are the phases picked up in completing two full circuits in the x and y directions respectively (eg: x goes to x+2b).

Chapter 4

Torus Propagator

4.1 Torus Symmetry Propagator

Consider a particle on the torus, which is subject to no further constraints. We wish to find the propagator describing that particle's motion: as with the circle it is possible to construct this propagator by a purely symmetry-based argument. Namely, consider the particle to be further constrained to travel in only the x-direction. The propagator, under this further constraint, should be identical to that of the particle on a circle, for this new constraint physically replicates exactly that. Likewise, constraining the particle to travel in the y-direction should also yield the propagator for a circle. This is reflected in the decomposition of the torus space, which can be expressed as $T^2 = S^1 \times S^1$.

Generalizing the derivation of the circle propagator with this in mind, we can express the torus propagator as a sum over winding numbers in each direction:

$$G^{T^2} = \sum_{r,s} A_r B_s G_{rs}^{T^2} \tag{4.1}$$

where the coefficients, as for the circle, become phase factors and the individual propagators those of a free particle. Since we are now dealing with the general period b, we must rescale the coordinate by $\frac{2\pi}{b}$ in the free particle state, carrying it through to the final form of the propagator. The modified states (2.6), energies (2.7), and finally propagator (2.18) become:

$$\psi_n^{S^1} = \frac{1}{\sqrt{b}} exp(i\frac{2\pi}{b}nx + \frac{i\sigma x}{b}) \tag{4.2}$$

$$E_n^{S^1} = \frac{1}{2}(\frac{2\pi n}{b} + \frac{\sigma}{b})^2 \tag{4.3}$$

$$G^{S^1} = (\frac{1}{2\pi it})^{1/2} exp(\frac{ix^2}{2t})\Theta_3(\frac{\sigma}{2} - \frac{xb}{2t}, \frac{b^2}{2\pi t}) \tag{4.4}$$

Making these substitutions into the torus propagator (4.1):

$$G^{T^2} = \sum_{r,s} \frac{1}{2\pi it} exp(\frac{i}{2t}(x^2+y^2) + ir(\sigma - \frac{xb}{t}) + is(\delta - \frac{ya}{t}) + \frac{ib^2r^2}{2t} + \frac{ia^2s^2}{2t}) \tag{4.5}$$

which, when we substitute the theta function (2.9), becomes:

$$G^{T^2} = \frac{1}{2\pi it} exp(\frac{i}{2t}(x^2+y^2))\Theta_3(\frac{\sigma}{2} - \frac{xb}{2t}, \frac{b^2}{2\pi t})\Theta_3(\frac{\delta}{2} - \frac{ya}{2t}, \frac{a^2}{2\pi t}) \tag{4.6}$$

We recognize this as simply being the product of two propagators for the circle, as we expect given the relationship between the two spaces.

4.2 Torus Stationary State Propagator

Now, as with the circle, we can check our result for the symmetry propagator (4.6) against the stationary state form. Beginning from the definition for the propagator as a sum over stationary states:

$$G(x_f, y_f, t; x_i, y_i, 0) = \sum_{r,s=-\infty}^{\infty} \psi_{rs}(x_f, y_f)\psi^*_{rs}(x_i, y_i) exp(-iE_{rs}t) \tag{4.7}$$

and substituting the normalized states (3.6) and energies (3.7), we find that the propagator for the torus is:

$$\begin{aligned} G^{T^2} = \sum_{r,s} \frac{1}{ab} exp(2\pi i(rx/b + sy/a) + \frac{i\sigma x}{b} + \frac{i\delta y}{a} \\ - \frac{it}{2}((\frac{2\pi r + \sigma}{b})^2 + (\frac{2\pi s + \delta}{a})^2)) \end{aligned} \tag{4.8}$$

where $x = x_f - x_i$, $y = y_f - y_i$, and the argument of the propagator shall once more be suppressed. Substituting the theta function (2.9) into the above propagator (4.8) yields:

$$G^{T^2} = \frac{1}{ab} exp(\frac{i\sigma x}{b} + \frac{i\delta y}{a} - \frac{it\sigma^2}{2b^2} - \frac{it\delta^2}{2a^2}) *$$
$$\Theta_3(\frac{\pi x}{b} - \frac{\pi\sigma t}{b^2}, -\frac{2\pi t}{b^2})\Theta_3(\frac{\pi y}{a} - \frac{\pi\delta t}{a^2}, -\frac{2\pi t}{a^2}) \quad (4.9)$$

Applying the Jacobi identity (2.11), we find:

$$G^{T^2} = \frac{1}{ab}(\frac{2\pi it}{b^2})^{-1/2}(\frac{2\pi it}{a^2})^{-1/2} exp(\frac{i\sigma x}{b} + \frac{i\delta y}{a} - \frac{it\sigma^2}{2b^2} - \frac{it\delta^2}{2a^2}) *$$
$$exp((\frac{\pi x}{b} - \frac{\pi\sigma t}{b^2})^2 / - \frac{2\pi^2 it}{b^2}) exp((\frac{\pi y}{a} - \frac{\pi\delta t}{a^2})^2 / - \frac{2\pi^2 it}{a^2}) *$$
$$\Theta_3((\frac{\pi x}{b} - \frac{\pi\sigma t}{b^2}) / - \frac{2\pi t}{b^2}, \frac{b^2}{2\pi t})\Theta_3((\frac{\pi y}{a} - \frac{\pi\delta t}{a^2}) / - \frac{2\pi t}{a^2}, \frac{a^2}{2\pi t}) \quad (4.10)$$

which simplifies to:

$$G^{T^2} = \frac{1}{2\pi it} exp(\frac{i}{2t}(x^2 + y^2))\Theta_3(\frac{\sigma}{2} - \frac{xb}{2t}, \frac{b^2}{2\pi t})\Theta_3(\frac{\delta}{2} - \frac{ya}{2t}, \frac{a^2}{2\pi t}) \quad (4.11)$$

in agreement with the symmetry propagator (4.6).

Chapter 5

Klein Bottle Propagator

5.1 Klein Bottle Symmetry Propagator

Now that we have a procedure for deriving the propagator through symmetry considerations, let us apply it to the Klein Bottle and compare with the stationary state result. We begin by recalling the symmetry for the Klein Bottle (3.12), and notice that two successive applications of the same symmetry operator looks like a single application of the torus symmetry (3.5) with twice the period:

$$\begin{aligned}(S_x^{K^2})^2 f(x,y) &= S_x^{K^2} f(x+b,-y) \\ &= f(x+2b,y)\end{aligned} \tag{5.1}$$

Since the Klein Bottle chosen is symmetric in x and y, it suffices to discuss only one direction: the other follows immediately. This suggests that the Klein Bottle propagator should contain a similar structure as the torus propagator with twice the period, which we will call $\acute{G^{T^2}}$. If we take as an initial guess some combination of $\acute{G^{T^2}}$ and S^{K^2}, we can check certain properties. First, the propagator should only pick up a phase when the symmetry operator is applied (namely $\sigma/2$, since that would produce the required phase for two applications, ie: x goes to x+2b):

$$S_{x_f}^{K^2} G^{K^2}(x_f, y_f, t; x_i, y_i, 0) = exp(i\sigma/2) G^{K^2}(x_f, y_f, t; x_i, y_i, 0) \tag{5.2}$$

and second, the propagator is unitary (ie: the propagator should preserve the norm of the states), since we can write:

$$\psi_{kl}^{K^2}(x_f, y_f, t) = G^{K^2}(x_f, y_f, t; x_i, y_i, 0)\psi_{kl}^{K^2}(x_i, y_i, 0) \tag{5.3}$$

In satisfying the first property (5.2), let us successively apply S^{K^2} to $\acute{G^{T^2}}$ until we are left with a phase factor:

$$
\begin{aligned}
S_{x_f}^{K^2} G^{\acute{T}^2} =& S_{x_f}^{K^2} \sum_{r,s} \frac{1}{2\pi i t} exp(\frac{i}{2t}((x-2rb)^2 + (y-2sa)^2) + ir\sigma + is\delta) \\
=& \sum_{r,s} \frac{1}{2\pi i t} exp(\frac{i}{2t}((x+b-2rb)^2 + (-\bar{y}-2sa)^2) + ir\sigma + is\delta) \\
=& \sum_{r,s} \frac{1}{2\pi i t} exp(\frac{i}{2t}((x-b-2(r-1)b)^2 + (-\bar{y}-2sa)^2) + \\
& i\sigma + i(r-1)\sigma + is\delta) \\
=& exp(i\sigma) \sum_{r,s} \frac{1}{2\pi i t} exp(\frac{i}{2t}((x-b-2rb)^2 + (-\bar{y}-2sa)^2) + \\
& ir\sigma + is\delta)
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
(S_{x_f}^{K^2})^2 G^{\acute{T}^2} =& (S_{x_f}^{K^2})^2 \sum_{r,s} \frac{1}{2\pi i t} exp(\frac{i}{2t}((x-2rb)^2 + (y-2sa)^2) + ir\sigma + is\delta) \\
=& \sum_{r,s} \frac{1}{2\pi i t} exp(\frac{i}{2t}((x+2b-2rb)^2 + (y-2sa)^2) + ir\sigma + is\delta) \\
=& \sum_{r,s} \frac{1}{2\pi i t} exp(\frac{i}{2t}((x-2(r-1)b)^2 + (y-2sa)^2) + \\
& i\sigma + i(r-1)\sigma + is\delta) \\
=& exp(i\sigma) \sum_{r,s} \frac{1}{2\pi i t} exp(\frac{i}{2t}((x-2rb)^2 + (y-2sa)^2) + ir\sigma + is\delta) \\
=& exp(i\sigma) G^{\acute{T}^2}
\end{aligned} \tag{5.5}
$$

where $\bar{y} = y_f + y_i$. Therefore, as an initial propagator, take:

$$
\begin{aligned}
G^{K^2} =& 4(G^{\acute{T}^2} + exp(-i\sigma/2) S_{x_f}^{K^2} G^{\acute{T}^2} + exp(-i\delta/2) S_{y_f}^{K^2} G^{\acute{T}^2} + \\
& exp(-i\sigma/2) exp(-i\delta/2) S_{x_f}^{K^2} S_{y_f}^{K^2} G^{\acute{T}^2})
\end{aligned} \tag{5.6}
$$

where the norm has been chosen to satisfy the unitarity property (5.3), and the symmetry property (5.2) is obeyed as a result of (5.5):

$$\begin{aligned} S_{x_f}^{K^2} G^{K^2} =& 4(S_{x_f}^{K^2} G\acute{^{T^2}} + exp(-i\sigma/2)(S_{x_f}^{K^2})^2 G\acute{^{T^2}} + exp(-i\delta/2) S_{x_f}^{K^2} S_{y_f}^{K^2} G\acute{^{T^2}} + \\ & exp(-i\sigma/2) exp(-i\delta/2)(S_{x_f}^{K^2})^2 S_{y_f}^{K^2} G\acute{^{T^2}}) \\ =& 4(S_{x_f}^{K^2} G\acute{^{T^2}} + exp(i\sigma/2) G\acute{^{T^2}} + exp(-i\delta/2) S_{x_f}^{K^2} S_{y_f}^{K^2} G\acute{^{T^2}} + \\ & exp(i\sigma/2) exp(-i\delta/2) S_{y_f}^{K^2} G\acute{^{T^2}}) \\ =& exp(i\sigma/2) G^{K^2} \end{aligned} \tag{5.7}$$

Substituting for the torus propagator (4.6) and applying the symmetry operators, the Klein Bottle propagator becomes:

$$\begin{aligned} G^{K^2} = \frac{2}{\pi i t}(& exp(\frac{i}{2t}(x^2+y^2))\Theta_3(\frac{\sigma}{2} - \frac{xb}{2t}, \frac{b^2}{2\pi t})\Theta_3(\frac{\delta}{2} - \frac{ya}{2t}, \frac{a^2}{2\pi t}) + \\ & exp(\frac{i}{2t}((x+b)^2 + (-\bar{y})^2) - \frac{i\sigma}{2}) * \\ & \Theta_3(\frac{\sigma}{2} - \frac{(x+b)b}{2t}, \frac{b^2}{2\pi t})\Theta_3(\frac{\delta}{2} + \frac{\bar{y}a}{2t}, \frac{a^2}{2\pi t}) + \\ & exp(\frac{i}{2t}((-\bar{x})^2 + (y+a)^2) - \frac{i\delta}{2}) * \\ & \Theta_3(\frac{\sigma}{2} + \frac{\bar{x}b}{2t}, \frac{b^2}{2\pi t})\Theta_3(\frac{\delta}{2} - \frac{(y+a)a}{2t}, \frac{a^2}{2\pi t}) + \\ & exp(\frac{i}{2t}((-\bar{x}+b)^2 + (-\bar{y}+a)^2) - \frac{i\sigma}{2} - \frac{i\delta}{2}) * \\ & \Theta_3(\frac{\sigma}{2} + \frac{(\bar{x}-b)b}{2t}, \frac{b^2}{2\pi t})\Theta_3(\frac{\delta}{2} + \frac{(\bar{y}-a)a}{2t}, \frac{a^2}{2\pi t})) \end{aligned} \tag{5.8}$$

Applying the boundary conditions (3.8) associated with the symmetry operators to the final endpoints, we can transform

$$\left.\begin{matrix} x+b = (x_f+b) - x_i \\ -\bar{y} = -y_f - y_i \end{matrix}\right\} \rightarrow \left\{\begin{matrix} x_f - x_i = x \\ y_f - y_i = y \end{matrix}\right. \tag{5.9}$$

Applying this transformation to individual terms in the propagator (5.8) and factoring yields:

$$G^{K^2} = \frac{2}{\pi i t}(1 + exp(\frac{-i\sigma}{2}) + exp(\frac{-i\delta}{2}) + exp(-\frac{i\sigma}{2} - \frac{i\delta}{2}))*$$
$$exp(\frac{i}{2t}(x^2 + y^2))\Theta_3(\frac{\sigma}{2} - \frac{xb}{2t}, \frac{b^2}{2\pi t})\Theta_3(\frac{\delta}{2} - \frac{ya}{2t}, \frac{a^2}{2\pi t}) \tag{5.10}$$

5.2 Klein Bottle Stationary State Propagator

Now, as with the torus, we substitute the normalized states (3.13) and energies (3.14) into the stationary state expression for the propagator (4.7) which yields:

$$G^{K^2} = \sum_{k,l} \frac{4}{ab} sin(\frac{\pi k x_f}{b} + \frac{\sigma x_f}{2b} + \frac{\pi}{4}(1 + (-1)^l))*$$
$$sin(\frac{\pi k x_i}{b} + \frac{\sigma x_i}{2b} + \frac{\pi}{4}(1 + (-1)^l)) * ((Y)) * exp(-iE_{kl}^{K^2} t) \tag{5.11}$$

where ((Y)) denotes the equivalent function of x for y (y replaces x, l replaces k, δ replaces σ, etc.). Applying the product formula for sine, we have:

$$G^{K^2} = \sum_{k,l} \frac{1}{ab}(cos(\frac{\pi k \bar{x}}{b} + \frac{\sigma \bar{x}}{2b} + \frac{\pi}{2}(1 + (-1)^l))-$$
$$cos(\frac{\pi k x}{b} + \frac{\sigma x}{2b})) * ((Y)) * exp(-iE_{kl}^{K^2} t) \tag{5.12}$$

Rewriting as exponentials via Euler's formula, and using:

$$exp(i\frac{\pi}{2}(1 + (-1)^l)) = exp(i\pi(l + 1)) \tag{5.13}$$

the propagator becomes:

$$
\begin{aligned}
G^{K^2} = \sum_{k,l} \frac{1}{4ab}((&exp(i(\pi k + \frac{\sigma}{2})\frac{\bar{x}}{b} + i\pi(l+1))+ \\
&exp(-i(\pi k + \frac{\sigma}{2})\frac{\bar{x}}{b} + i\pi(l+1)))- \\
&(exp(i(\pi k + \frac{\sigma}{2})\frac{x}{b}) + exp(-i(\pi k + \frac{\sigma}{2})\frac{x}{b})))* \\
&((Y)) * exp(-iE_{kl}^{K^2} t)
\end{aligned} \tag{5.14}
$$

Applying $exp(i\pi) = -1$ and cancelling the resulting overall -1 factor on both ((X)) and ((Y)), with (3.14) substituted the propagator becomes:

$$
\begin{aligned}
G^{K^2} = \sum_{k,l} \frac{1}{4ab}(&exp(i(\pi k + \frac{\sigma}{2})\frac{\bar{x}}{b} + i\pi l) + exp(-i(\pi k + \frac{\sigma}{2})\frac{\bar{x}}{b} + i\pi l)+ \\
&exp(i(\pi k + \frac{\sigma}{2})\frac{x}{b}) + exp(-i(\pi k + \frac{\sigma}{2})\frac{x}{b}))* \\
&exp(-it(\frac{\sigma^2}{8b^2} + \frac{\sigma\pi k}{2b^2} + \frac{\pi^2 k^2}{2b^2})) * ((Y))
\end{aligned} \tag{5.15}
$$

Carrying out the cross-multiplication and substituting the theta function (2.9) yields 16 terms of similar structure, so it is useful to define the following function:

$$
\begin{aligned}
F(u,v,w,z) =& exp(i(\frac{\sigma u}{2b} - \frac{\sigma^2 t}{8b^2} + \frac{\delta v}{2a} - \frac{\delta^2 t}{8a^2}))* \\
&\Theta_3(\frac{-\sigma\pi t}{4b^2} + \frac{\pi w}{2b}, -\frac{\pi t}{2b^2}) * \Theta_3(\frac{-\delta\pi t}{4a^2} + \frac{\pi z}{2a}, -\frac{\pi t}{2a^2})
\end{aligned} \tag{5.16}
$$

Substituting this into the propagator yields:

$$
\begin{aligned}
G^{K^2} = \frac{1}{4ab}(&F(\bar{x},\bar{y},\bar{x}+b,\bar{y}+a)+F(\bar{x},-\bar{y},\bar{x}+b,-\bar{y}+a)+\\
&F(-\bar{x},\bar{y},-\bar{x}+b,\bar{y}+a)+F(-\bar{x},-\bar{y},-\bar{x}+b,-\bar{y}+a)+\\
&F(\bar{x},y,\bar{x},y+a)+F(\bar{x},-y,\bar{x},-y+a)+\\
&F(-\bar{x},y,-\bar{x},y+a)+F(-\bar{x},-y,-\bar{x},-y+a)+\\
&F(x,\bar{y},x+b,\bar{y})+F(x,-\bar{y},x+b,-\bar{y})+\\
&F(-x,\bar{y},-x+b,\bar{y})+F(-x,-\bar{y},-x+b,-\bar{y})+\\
&F(x,y,x,y)+F(x,-y,x,-y)+\\
&F(-x,y,-x,y)+F(-x,-y,-x,-y))
\end{aligned} \tag{5.17}
$$

Now, as with the symmetry propagator, we can apply the boundary conditions to transform individual terms. Recalling (5.9):

$$
\left.\begin{array}{c} x+b=(x_f+b)-x_i \\ -\bar{y}=-y_f-y_i \end{array}\right\} \rightarrow \left\{\begin{array}{c} x_f-x_i=x \\ y_f-y_i=y \end{array}\right.
$$

We can also write:

$$
\left.\begin{array}{c} x+b=x_f-(x_i-b) \\ \bar{y}=y_f+y_i \end{array}\right\} \rightarrow \left\{\begin{array}{c} x_f-x_i=x \\ y_f-y_i=y \end{array}\right. \tag{5.18}
$$

Applying these transformations to the propagator (5.17) yields:

$$
\begin{aligned}
G^{K^2} = \frac{1}{4ab}(&4F(x-b,y-a,x,y)+2F(x,y-a,x,y)+\\
&2F(x,-y-a,x,-y)+2F(x-b,y,x,y)+\\
&2F(-x-b,y,-x,y)+F(x,y,x,y)+\\
&F(x,-y,x,-y)+F(-x,y,-x,y)+F(-x,-y,-x,-y))
\end{aligned} \tag{5.19}
$$

There is one last type of transformation we can generate from applying the boundary condition twice, once on each endpoint:

$$
\left.\begin{array}{c} x=(x_f+b)-(x_i+b) \\ -y=-y_f+y_i \end{array}\right\} \rightarrow \left\{\begin{array}{c} x_f-x_i=x \\ y_f-y_i=y \end{array}\right. \tag{5.20}
$$

which allows us to further collapse the propagator into:

$$G^{K^2} = \frac{1}{ab}(F(x-b, y-a, x, y) + F(x, y-a, x, y) + \\ F(x-b, y, x, y) + F(x, y, x, y)) \quad (5.21)$$

Applying the Jacobi identity (2.11) to our function F, we can rewrite it as:

$$F(u,v,w,z) =(\frac{2ab}{i\pi t})exp(i(\frac{\sigma u}{2b} - \frac{\sigma^2 t}{8b^2} + \frac{\delta v}{2a} - \frac{\delta^2 t}{8a^2}))* \\ exp(i(\frac{-\sigma\pi t}{4b^2} + \frac{\pi w}{2b})^2/\frac{\pi^2 t}{2b^2})exp(i(\frac{-\delta\pi t}{4a^2} + \frac{\pi z}{2a})^2/\frac{\pi^2 t}{2a^2})* \\ \Theta_3((\frac{-\sigma\pi t}{4b^2} + \frac{\pi w}{2b})/-\frac{\pi t}{2b^2}, \frac{2b^2}{\pi t})* \\ \Theta_3((\frac{-\delta\pi t}{4a^2} + \frac{\pi z}{2a})/-\frac{\pi t}{2a^2}, \frac{2a^2}{\pi t}) \quad (5.22)$$

which simplifies to:

$$F(u,v,w,z) =(\frac{2ab}{i\pi t})exp(i(\frac{\sigma(u-w)}{2b} + \frac{w^2}{2t} + \frac{\delta(v-z)}{2a} + \frac{z^2}{2t}))* \\ \Theta_3((\frac{\sigma}{2} - \frac{wb}{t}), \frac{2b^2}{\pi t})\Theta_3((\frac{\delta}{2} + \frac{za}{t}), \frac{2a^2}{\pi t}) \quad (5.23)$$

Substituting this into our expression for the propagator (5.21) yields:

$$G^{K^2} = \frac{2}{i\pi t}(exp(i(\frac{-\sigma}{2} + \frac{x^2}{2t} + \frac{-\delta}{2} + \frac{y^2}{2t}))+ \\ exp(i(\frac{x^2}{2t} + \frac{-\delta}{2} + \frac{y^2}{2t})) + exp(i(\frac{-\sigma}{2} + \frac{x^2}{2t} + \frac{y^2}{2t}))+ \\ exp(i(\frac{x^2}{2t} + \frac{y^2}{2t})))\Theta_3((\frac{\sigma}{2} - \frac{xb}{t}), \frac{2b^2}{\pi t})\Theta_3((\frac{\delta}{2} + \frac{ya}{t}), \frac{2a^2}{\pi t}) \quad (5.24)$$

which can be rewritten as:

$$
\begin{aligned}
G^{K^2} =& \frac{2}{\pi i t}(1 + exp(\frac{-i\sigma}{2}) + exp(\frac{-i\delta}{2}) + exp(-\frac{i\sigma}{2} - \frac{i\delta}{2}))* \\
& exp(\frac{i}{2t}(x^2 + y^2))\Theta_3(\frac{\sigma}{2} - \frac{xb}{2t}, \frac{b^2}{2\pi t})\Theta_3(\frac{\delta}{2} - \frac{ya}{2t}, \frac{a^2}{2\pi t})
\end{aligned} \tag{5.25}
$$

in agreement with the symmetry propagator (5.10).

Chapter 6

Additional Constraints

6.1 Constraint Applied to Torus

Suppose we wish to apply additional constraints to the systems, such as the constraint:

$$p_y \psi = 0 \tag{6.1}$$

Imposition of such a constraint corresponds to the particle model of a gauge theory that is a simple analog of the one in [3]. Because of the phase that appears in the states we cannot simply use the old momentum operator $p_y = -i\partial_y$, but must instead determine a new momentum operator which includes the phase. As a first step in determining the new form, consider the effect of our old momentum operator on the state:

$$p_y \psi_{rs}^{T^2} = (\frac{2\pi s}{a} + \frac{\delta}{a})\psi_{rs}^{T^2} \tag{6.2}$$

Therefore a momentum operator of the form:

$$p_y^{T^2C} = \frac{1}{i}\partial_y - \frac{\delta}{a} \tag{6.3}$$

will eliminate the constant factor and allow us to satisfy the constraint (6.1) which becomes:

$$p_y^{T^2C} \psi_{rs}^{T^2} = \frac{2\pi s}{a} \psi_{rs}^{T^2} = 0 \tag{6.4}$$

with the condition s=0. With the new momentum operator, the energy spectrum becomes:

$$E_{rs}^{T^2C} = \frac{1}{2}((\frac{2\pi r}{b})^2 + (\frac{2\pi s}{a})^2) \tag{6.5}$$

and the constrained states and energies become:

$$\psi_{r0}^{T^2C} = \frac{1}{\sqrt{ab}} exp(\frac{2\pi irx}{b} + \frac{i\sigma x}{b} + \frac{i\delta y}{a}) \tag{6.6}$$

$$E_{r0}^{T^2C} = \frac{1}{2}(\frac{2\pi r}{b})^2 \tag{6.7}$$

Note that the constrained system matches that of the circle with periodicity b, as is reasonable to expect given the nature of the constraint we've imposed. While the y coordinate still appears in the constrained state, the constraint implies that this term merely contributes an overall identical phase factor to the individual states. By redefining our coordinate system to shift the x-axis to position y, this phase factor can be eliminated.

6.2 Constraint Applied to Klein Bottle

Now consider applying the same constraint (6.1) to the Klein Bottle. The old momentum operator $p_y = -i\partial_y$ acting on our eigenstates (3.13) yields:

$$\begin{aligned} p_y\psi_{kl}^{K^2} =& -i\sqrt{\frac{4}{ab}}(\frac{\pi l}{a} + \frac{\delta}{2a}) sin(\frac{\pi kx}{b} + \frac{\sigma x}{2b} + \frac{\pi}{4}(1 + (-1)^l)) * \\ & cos(\frac{\pi ly}{a} + \frac{\delta y}{2a} + \frac{\pi}{4}(1 + (-1)^k)) \\ =& -i(\frac{\pi l}{a} + \frac{\delta}{2a})\tilde{\psi}_{kl}^{K^2} \end{aligned} \tag{6.8}$$

so we see that our eigenstates of the Hamiltonian are not eigenstates of the old momentum operator.

To see why that is, consider the commutator of p_y and $S_x^{K^2}$:

$$\begin{aligned} [p_y, S_x^{K^2}]f(x,y) =& (-i\partial_y S_x^{K^2} + S_x^{K^2} i\partial_y)f(x,y) \\ =& -i\partial_y f(x+b,-y) + iS_x^{K^2}\frac{\partial f(x,y)}{\partial y} \\ =& -i\frac{\partial f(x+b,-y)}{\partial y} - i\frac{\partial f(x+b,-y)}{\partial y} \neq 0 \end{aligned} \tag{6.9}$$

Since the operators do not commute, unlike on the torus, it is not possible to simultaneously diagonalize the states. Eigenstates of the Hamiltonian must also be eigenstates of the symmetry operator, so it is not possible to form eigenstates of the momentum operator. As such, we cannot impose the constraint (6.1) on the Klein Bottle, since any candidate for a new momentum operator would also not commute with the symmetry operator.

Chapter 7

Conclusion

By considering relative symmetries between two systems, one for which the propagator is known and one for which we wish to find the propagator, we have derived the desired propagator without applying any knowledge of the states of the target system. For the torus, we began with the circle and the knowledge that in both the x and y directions, motion around the torus replicates motion around the circle. For the non-standard (double-twist) Klein Bottle, we began with the torus and the knowledge that motion twice around the Klein Bottle in either direction replicates motion once around the torus. Armed with these relationships and the symmetries obeyed on each space, we have found the propagator for the torus and Klein Bottle. In each case, we have checked our result against the stationary state construction and found the propagators to agree.

In this way, it should be possible to construct the propagator for more complex systems, provided one can relate the desired system to a simpler one for which the propagator is known. For example, the standard (single-twist) Klein Bottle is a mix between the two systems solved here, and as such its propagator would be a blend between the torus propagator in one dimension and the non-standard Klein Bottle in the other. One can treat composite systems in much the same way, eg: the torus is composed of two circles, one in each dimension x and y.

Furthermore, we have also treated briefly the topic of applying further constraints to these systems. As a result, we have shown that not all constraints are viable candidates to be applied, they must commute with the symmetry operators of the system. On the torus, the constraint $p_y\psi = 0$ does commute with the symmetries, and the resulting constrained states resemble those of the circle as expected. On the Klein Bottle, the momentum operator does not commute with one of the symmetries (namely, $S_x^{K^2}$), and so the constraint cannot be applied. Attempting to do so exposed the fact that the eigenstates of the Hamiltonian are not eigenstates of the momentum operator.

Bibliography

[1] T. M. Apostol. *Mathematical Analysis*. Addison-Wesley, 2nd edition, 1974.

[2] J. S. Dowker. Quantum mechanics and field theory on multiply connected and on homogeneous spaces. *J. Phys. A: Gen. Phys.*, 5(936), 1972.

[3] R. Friedberg et al. A soluble gauge model with gribov-type copies. *Annals Phys.*, 246(381), 1996.

[4] M. G. G. Laidlaw and C. M. DeWitt. Feynman functional integrals for systems of indistinguishable particles. *Phys. Rev. D*, 3(1375), 1971.

[5] L. S. Schulman. A path integral for spin. *Phys. Rev.*, 176(1558), 1968.

[6] L. S. Schulman. *Techniques and Applications of Path Integration*. Wiley, 1981.

Appendix A

Jacobi Theta Function Identities

Beginning with the definition of the Jacobi theta function (2.9):

$$\Theta_3(Z,T) = \sum_{n=-\infty}^{\infty} exp(i\pi n^2 T + 2inZ) \tag{A.1}$$

we first prove the periodicity identity (2.10), where for $(a,b) \in \mathbb{Z}$:

$$\begin{aligned} \Theta_3(Z + a\pi + b\pi T, T) &= \sum_n exp(i\pi n^2 T + 2in(Z + a\pi + b\pi T)) \\ &= \sum_n exp(i\pi n^2 T + 2inZ + 2inb\pi T) exp(2\pi ina) \end{aligned} \tag{A.2}$$

The second factor in the sum is unity for all values of n and a, and completing the square on b yields:

$$\begin{aligned} \Theta_3(Z + a\pi + b\pi T, T) &= \sum_n exp(i\pi n^2 T + 2inZ + 2inb\pi T + i\pi b^2 T - i\pi b^2 T) \\ &= \sum_n exp(i\pi (n+b)^2 T + 2inZ - i\pi b^2 T) \end{aligned} \tag{A.3}$$

Finally, we re-index the sum from n to n+b, continuing to write it as being over n (since it goes from -∞ to ∞):

$$\begin{aligned}\Theta_3(Z + a\pi + b\pi T, T) &= \sum_n exp(i\pi n^2 T + 2i(n-b)Z - i\pi b^2 T) \\ &= \sum_n exp(i\pi n^2 T + 2inZ)exp(-2ibZ - i\pi b^2 T) \\ &= \Theta_3(Z, T)exp(-2ibZ - i\pi b^2 T) \qquad \text{(A.4)}\end{aligned}$$

which gives us the identity (2.10).

Proving the second required identity for the Jacobi theta function (2.11), we begin by applying the Poisson summation formula to the theta function in a more general form than presented by Apostol [1], for which only the Z=0 case is proven:

$$\Theta_3(Z, T) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 T + 2inZ} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\pi x^2 T + 2ixZ} e^{2\pi ixn} dx \qquad \text{(A.5)}$$

Recognizing the integral on the right as the Fourier transform of the theta function, we find:

$$\begin{aligned}\Theta_3(Z, T) &= \sum_{n=-\infty}^{\infty} (-iT)^{-1/2} exp(-i\frac{(Z + \pi n)^2}{\pi T}) \\ &= (-iT)^{-1/2} e^{-iZ^2/\pi T} \sum_{n=-\infty}^{\infty} exp(-\frac{2iZn}{T} - \frac{i\pi n^2}{T}) \qquad \text{(A.6)}\end{aligned}$$

which, using $\Theta_3(-Z, T) = \Theta_3(Z, T)$, gives us the second identity (2.11):

$$\Theta_3(Z, T) = (-iT)^{-1/2} exp(Z^2/i\pi T)\Theta_3(Z/T, -1/T) \qquad \text{(A.7)}$$