On The Existence of Jet Schemes Logarithmic Along Families of Divisors

by

Andrew Philippe Staal

B.Sc., University of Ottawa, 2006

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

The Faculty of Graduate Studies

(Mathematics)

The University of British Columbia
(Vancouver)

December 2008

Abstract

A section of the total tangent space of a scheme X of finite type over a field k, *i.e.* a vector field on X, corresponds to an X-valued 1-jet on X. In the language of jets the notion of a vector field becomes functorial, and the total tangent space constitutes one of an infinite family of jet schemes $J_m(X)$ for $m \geq 0$. We prove that there exist families of “logarithmic” jet schemes $J_m^{\mathcal{D}}(X)$ for $m \geq 0$, in the category of k-schemes of finite type, associated to any given X and its family of divisors $\mathcal{D} = (D_1, \ldots, D_r)$. The sections of $J_1^{\mathcal{D}}(X)$ correspond to so-called vector fields on X with logarithmic poles along the family of divisors $\mathcal{D} = (D_1, \ldots, D_r)$. To prove this, we first introduce the categories of pairs $(X, \mathcal{D})$ where $\mathcal{D}$ is as mentioned, an r-tuple of (effective Cartier) divisors on the scheme X. The categories of pairs provide a convenient framework for working with only those jets that pull back families of divisors.

Contents

Acknowledgments

I would like to thank all those people whose support, encouragement, and assistance has lead and inspired me through the successful completion of this thesis. First, I wish to thank the various friends and colleagues of mine who have enriched my personal pursuit of mathematical knowledge. Especially among these, I thank Jesse Collingwood, Chris Dionne, Samantha Marion, Matthew Mazowita, Slava Pestov, and Eric Serré at the University of Ottawa; Alex Ustian during our visit at the Independent University of Moscow; and Adam Clay, Alexander Duncan, Benjamin Purcell, Simon Rose and the rest of the motivated group of algebra and algebraic geometry graduate students at the University of British Columbia.

Further I would like to express sincere gratitude to the professional mathematicians who have nurtured my interests, especially Professors Thierry Giordano, Barry Jessup, Monica Nevins, and Vladimir Pestov at the University of Ottawa, and Professor Julia Gordon at the University of British Columbia. I would like to extend special thanks to Professor Kai Behrend at the University of British Columbia for taking the time to read and evaluate my thesis. Finally, I wish to express my deepest appreciation to Professor Kalle Karu, my research supervisor at the University of British Columbia; without the constant guidance, instruction, ideas, suggestions, and encouragement of Professor Karu this work simply could not have been possible.

On a personal note, I would like to thank my family; it is impossible to sum up all the reasons for which they deserve my utmost gratitude with regards to this thesis, so I simply thank Mom, Dad, and Nico with all my heart. Thanks also to Gladys for her company and encouragement (and food!) after my move to Vancouver.

To Ralph and Isabel,

en honor de Salomón y AuraMaría

1 Introduction

Let X and Z denote schemes of finite type over $\mathbb{C}$. A *Z-valued m-jet* in X is a morphism $\gamma : Z \times \operatorname{Spec} \mathbb{C}[t]/(t^{m+1}) \to X$. Let L_m^X denote the contravariant functor $\operatorname{Hom}(- \times \operatorname{Spec}\mathbb{C}[t]/(t^{m+1}), X)$. The functor L_m^X is *representable*; that is, the functor $- \times \operatorname{Spec}\mathbb{C}[t]/(t^{m+1})$ has a right adjoint, which we denote $J_m(-)$. Thus, any Z-valued m-jet γ in X corresponds uniquely, and functorially, to a *Z-valued point* $\widetilde{\gamma} : Z \to J_m(X)$ of the scheme $J_m(X)$. The scheme $J_m(X)$ is referred to as the m^{th} *jet scheme* of X, and is of finite type over $\mathbb{C}$. Let us sketch a constructive proof that $J_m(X)$ exists for such a scheme X; for more thorough treatments on jet schemes see for example the articles [Mus01], [EM08], [Ish07].

First, we may assume that Z and X are affine; in the first case this follows after refining Yoneda's lemma applied to the category $\mathbb{C}$-**Schemes** of schemes of finite type over $\mathbb{C}$, and in the second case from the gluing construction on schemes. Thus, let $Z = \operatorname{Spec} A$, where A is a finitely generated $\mathbb{C}$-algebra, and let $X = \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_s)$ be a closed immersion of X into complex affine n-space. Then a Z-valued m-jet γ is determined by a homomorphism $\gamma^* : \mathbb{C}[X_1, \ldots, X_n] \to A[t]/(t^{m+1})$ such that $\gamma^*(f_j) = 0$ for each $1 \le j \le s$. Let $\gamma^*(f_j) = f_{j0} + f_{j1}t + \cdots + f_{jm}t^m$; the condition $\gamma^*(f_j) = 0$ translates to $f_{jl'} = 0$ for every $0 \le l' \le m$. Note that each $f_{jl'}$ is a polynomial in the coefficients $(a_{il})_{i,l}$ of the elements $\gamma^*(X_i) = a_{i0} + a_{i1}t + \cdots + a_{im}t^m$. Thus, consider the homomorphism

$$\mathbb{C}[X_1^{(0)}, \ldots, X_n^{(0)}, X_1^{(1)}, \ldots, X_n^{(1)}, \ldots, X_1^{(m)}, \ldots, X_n^{(m)}] \to A$$

mapping $X_i^{(l)} \mapsto a_{il}$; this Z-valued point of affine $n(m+1)$-space determines and is determined by γ^* if and only if the condition $f_{jl'} \mapsto 0$ holds for every $1 \le j \le s$, $0 \le l' \le m$ when we consider $f_{jl'}$ as a polynomial in the variables $X_i^{(l)}$. Moreover, this correspondence is functorial; hence $J_m(X)$ is the closed immersion

$$J_m(X) = \operatorname{Spec} \mathbb{C}[X_1^{(0)}, \ldots, X_n^{(0)}, X_1^{(1)}, \ldots, X_n^{(1)}, \ldots, X_1^{(m)}, \ldots, X_n^{(m)}]/(f_{jl'})_{j,l'}$$

in complex affine $n(m+1)$-space.

We can describe the equations $f_{jl'}$ explicitly. First, let m' and m be integers, $m' > m$. The *projection morphism* $\pi_{m',m} : J_{m'}(X) \to J_m(X)$ is the morphism of schemes induced by the truncation homomorphism $\mathbb{C}[t]/(t^{m'+1}) \to \mathbb{C}[t]/(t^{m+1})$. The jet schemes of X with their projection morphisms form a projective system

$$\cdots \to J_m(X) \xrightarrow{\pi_{m,m-1}} J_{m-1}(X) \to \cdots \to X,$$

whose projective limit $J_\infty(X)$ is called the *arc space of* X. Further, there are *projection morphisms* $\rho_m : J_\infty(X) \to J_m(X)$ (it will always be clear to which "projection" we refer). Similarly to the jet schemes, over $\mathbb{C}$ the arc space of an affine scheme $X = \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_s)$ immerses into

$$J_\infty(\mathbb{A}^n_{\mathbb{C}}) = \operatorname{Spec} \mathbb{C}[X_1^{(0)}, \ldots, X_n^{(0)}, \ldots, X_n^{(l)}, X_1^{(l+1)} \ldots].$$

Notice that $J_\infty(X)$ is not generally a scheme of finite type over $\mathbb{C}$. One obtains explicit equations for $J_\infty(X)$ as follows (borrowing notation from [EM08]): let

$$S_\infty = \mathbb{C}[X_1^{(0)}, \ldots, X_n^{(0)}, \ldots, X_n^{(l)}, X_1^{(l+1)}, \ldots]$$

denote the polynomial ring in infinitely and denumerably many variables. There is a derivation d on S_∞, mapping $X_i^{(l)} \mapsto X_i^{(l+1)}$. For any $f \in \mathbb{C}[X_1, \ldots, X_n]$, consider f as an element of S_∞ by substituting $X_i^{(0)}$ for X_i. Denote $f^{(0)} \stackrel{\text{def}}{=} f$ and let $f^{(l+1)} \stackrel{\text{def}}{=} df^{(l)}$ recursively. Writing I_∞ for the ideal $(f_j^{(l)} | 1 \leq j \leq s, l \in \mathbb{N} \cup \{0\})$, it is straight-forward to prove that $\operatorname{Spec} S_\infty / I_\infty = J_\infty(X)$; that is, the equations $f_j, df_j, d^2 f_j, \ldots$ over all $1 \leq j \leq s$ cut out the scheme $J_\infty(X)$ from (infinite-dimensional) affine space. Under the truncation homomorphism $S_\infty \to S_m = \mathbb{C}[X_1^{(0)}, \ldots, X_n^{(m)}]$ we obtain the equations for $J_m(X)$. Namely, $J_m(X) = \operatorname{Spec} S_m / I_m$, where $I_m = (f_j, df_j, \ldots, d^m f_j : 1 \leq j \leq s)$.

Let us shift focus momentarily to the complex-analytic setting. Let X now be a smooth complex-analytic variety. Recall that a *divisor* Y in X is a formal, locally finite linear combination $Y = \sum a_i Y_i$ of irreducible analytic hypersurfaces Y_i in X. So, each Y_i is locally the zero-locus of a single irreducible holomorphic function f_i on X such that for any open $U \subset X$, $Y_i \cap U \neq \emptyset$ for only finitely many i. Such a divisor is said to have *normal crossings* if $Y = \sum Y_i$ and the components Y_i meet transversally; that is, when k of the components, say $Y_{i_1}, \ldots, Y_{i_k}$, pass through $x \in X$, one can always choose local coordinates $x_1, \ldots, x_n$ on some U containing x such that $x = (0, \ldots, 0)$ and $f_{i_j} = x_j$ for $1 \leq j \leq k$. In particular $Y \cap U$ is geometrically the zero-locus of $x_1 \cdots x_k$.

Let Y be a divisor with normal crossings on X, let $X^* = X - Y$, and let $\vartheta : X^* \to X$ be the inclusion morphism. Recall that $\Omega^1_X(\log Y)$ denotes the locally free sub-$\mathcal{O}_X$-module of the direct image sheaf $\vartheta_* \Omega^1_{X^*}$, locally generated by elements $\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}, dx_{r+1}, \ldots, dx_n$ in a neighbourhood U as above. This sheaf is called the *sheaf of differential* 1*-forms on* X *with logarithmic poles along* Y. (See [GH94] and especially the papers of Deligne such as [Del71] for more on this structure.)

Now, let X denote the algebraic variety $\mathbb{A}^n_{\mathbb{C}}$, and let $\Omega_{X/\mathbb{C}}$ be its sheaf of Kähler differentials over $\mathbb{C}$. $\Omega_{X/\mathbb{C}}$ is locally free; the (geometric) vector bundle associated to $\Omega_{X/\mathbb{C}}$, $T_X := \mathit{Spec}\,(\mathit{Sym}\,\Omega_{X/\mathbb{C}})$, is called the *total tangent space of* X. It is straightforward to show that $J_1(X) \cong T_X$. We refer to a section v of the structure morphism $T_X \to X$ as a *vector field* on X; note that such a v is an X-valued 1-jet in X.

Let D be the (effective) Cartier divisor on X defined by the global section $X_1 \cdots X_r$. There is a sheaf $\Omega_{X/\mathbb{C}}(\log D)$ on X associated to D analogous to the sheaf $\Omega^1_X(\log Y)$ described above in the analytic context. This is the sheaf $\widetilde{B}$ of $\mathcal{O}_X$-modules associated to the free $\mathbb{C}[X] = \mathbb{C}[X_1, \ldots, X_n]$-module

$$B = \mathbb{C}[X] \cdot \frac{dX_1}{X_1} \oplus \ldots \oplus \mathbb{C}[X] \cdot \frac{dX_r}{X_r} \oplus \mathbb{C}[X] \cdot dX_{r+1} \oplus \ldots \oplus \mathbb{C}[X] \cdot dX_n;$$

that is, $\Omega_{X/k}(\log D) = \widetilde{B}$. Further, there is a vector bundle

$$T_X(\log D) := \mathit{Spec}\,(\mathit{Sym}\,\Omega_{X/\mathbb{C}}(\log D))$$

over X called the *logarithmic total tangent space of X with respect to the divisor D*. There is a canonical injection $\Omega_{X/\mathbb{C}} \to \Omega_{X/\mathbb{C}}(\log D)$ from which we obtain a morphism $T_X(\log D) \to T_X$ factoring the structure morphism $T_X(\log D) \to X$. One refers to a section of $T_X(\log D) \to X$ as a *vector field on X with logarithmic poles along D* (though we may simply use *vector field logarithmic along D*, or *logarithmic vector field*).

A vector field v on X is written $v = f_1 \cdot \frac{\partial}{\partial X_1} + \cdots + f_n \cdot \frac{\partial}{\partial X_n}$, with $f_i \in \mathbb{C}[X]$ for each i. Here $\frac{\partial}{\partial X_i}$ is the homomorphism $\Omega_{X/\mathbb{C}} \to \mathbb{C}[X]$ determined by the Kronecker delta: $\frac{\partial}{\partial X_i}(dX_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. Supposing that D is a divisor with normal crossings as above, a vector field logarithmic along D has the form

$$v = g_1 \cdot X_1 \cdot \frac{\partial}{\partial X_1} + \cdots + g_r \cdot X_r \cdot \frac{\partial}{\partial X_r} + f_{r+1} \cdot \frac{\partial}{\partial X_{r+1}} + \cdots + f_n \cdot \frac{\partial}{\partial X_n},$$

with $g_i \in \mathbb{C}[X]$; that is for such a vector field $X_i | f_i$ for every $1 \leq i \leq r$. The isomorphism between $J_1(X)$ and T_X lets us reformulate this in terms of jets. Namely, the vector field v is an X-valued point of $J_1(X)$ corresponding to a homomorphism $\mathbb{C}[X] \to \mathbb{C}[X][t]/(t^2)$. Since v is a section of the projection π_1, this homomorphism is the one determined by $X_i \mapsto X_i + f_i t$. When v is logarithmic along D this becomes $X_i \mapsto X_i + (g_i \cdot X_i)t = X_i(1 + g_i t)$ for $1 \leq i \leq r$; that is, X_i maps to X_i times a unit of the ring $\mathbb{C}[X][t]/(t^2)$. Equivalently, the equation $X_1 \cdots X_r$ defining D maps to $X_1 \cdots X_r$ times a unit.

We can restate this last condition as follows: an equation of the (normally crossing, effective Cartier) divisor D pulls back to an equation for the divisor

$$D^1 := \{(X \times \mathrm{Spec}\, \mathbb{C}[t]/(t^2), X_1 \cdots X_r)\}.$$

That is, if $v_\gamma : X \times \mathrm{Spec}\, \mathbb{C}[t]/(t^2) \to X$ is the X-valued jet corresponding to v, then the morphism of sheaves $(v_\gamma)^* : \mathcal{O}_X \to (v_\gamma)_* \mathcal{O}_{X \times \mathrm{Spec}\, \mathbb{C}[t]/(t^2)}$ takes a local equation for D to a local equation for D^1. We observe here that the condition "v is a logarthmic vector field along D" can be stated functorially. That is, given any schemes X and Z of finite type over any (algebraically closed) field, and any $m \geq 0$ we may ask which Z-valued m-jets on X pull back some fixed divisor D on X. These m-jets will comprise the Z-valued points of a scheme that we call the m^{th} *logarithmic jet scheme of X with respect to the divisor D*. In fact, we can work in more generality, parametrizing the m-jets in X that pull back a fixed family of effective Cartier divisors $(D_1, \ldots, D_r)$. In the case of a normally crossing divisor on affine space X, it will then follow that a section of the logarithmic jet scheme of X with respect to D corresponds exactly to a vector field on X logarithmic along the divisor.

In the proceeding section we will recall some notions that are basic in algebraic geometry, but essential to our study. Namely we will recall the functor of points of a scheme, and Cartier divisors on schemes. Following this we will fix a framework, the categories of pairs, for working with divisors on schemes and morphisms pulling their local equations back to the domain. Following these two sections we will move on to use this framework to prove the existence of logarithmic jet schemes associated to any pair $(X, (D_1, \ldots, D_r))$ as above. Here, results about the jet schemes generalize to such pairs, allowing a constructive proof; we obtain equations for the logarithmic jet schemes similarly to the method above in the case of ordinary jet schemes. Finally, we will conclude with a short summary of our work, and then make some closing remarks on likely improvements to the choice of categories in which we can prove our results.

2 Preliminaries

In this section we collect some definitions and results that we will use in the following sections. We will begin by recalling the definition of the functor of points of an object in a general category, and stating Yoneda's Lemma. This notion is basic, but fundamental; our main result uses Yoneda's lemma to prove that a certain functor is representable. We also include the definitions of Cartier and effective Cartier divisors on schemes, as we will use these throughout.

2.1 Functors of Points and Yoneda's Lemma

From the formulation of scheme-theoretic algebraic geometry in terms of prime ideal spectra and their Zariski topologies on arbitrary commutative rings, situations arise in which not all the information encapsulated by a scheme is captured by the underlying point-sets. As a simple example, the points of a (fibered) product scheme are not necessarily in direct correspondence with the points of the Cartesian product of the underlying sets. Further, the Zariski-topology on a scheme is defined in such a way that generic points of a scheme do not relate exactly the geometry of what one usually considers to be a point.

Thanks to Grothendieck, the present language of algebraic geometry includes an alterate notion of points on a scheme. Namely to any scheme we have an associated "functor of points". Though seemily set at a high level of abstraction (defined as a *process* rather than as an *object*), this functorial notion of points retains, and effectively describes in terms of sets, (universal) properties we expect in geometry. With regards to the example above, the functor of points of a (fibered) product of schemes *is* canonically isomorphic to the fibered product of the functors of points of the factors, which is in essence a Cartesian product of sets.

Let us recall the definition of functors of points. It is possible, and it will be more efficient for us, to define functors of points of objects in a general category. In particular, we will talk of the functor of points of a "pair" in the next chapter.

Let X and Y denote objects in a category $\mathbf{\Gamma}$. Recall that the *functor of points of X*, denoted h_X, is the (contravariant) functor defined as follows: let $h_X : (\mathbf{\Gamma})^\circ \to \mathbf{Sets}$ take any Y to the set $h_X(Y) = \mathrm{Hom}_{\mathbf{\Gamma}}(Y, X)$, where $(\mathbf{\Gamma})^\circ$ denotes the opposite category to $\mathbf{\Gamma}$. In this context, a morphism $\phi : Y \to X$ is referred to as a *Y-valued point of X*. Further, recall that the mapping $h : \mathbf{\Gamma} \to \mathrm{Fun}((\mathbf{\Gamma})^\circ, \mathbf{Sets})$ taking X to h_X is a (covariant) functor. We include the following fundamental fact from category theory:

Lemma (Yoneda) 2.1. *Let X and Y be objects in a category $\mathbf{\Gamma}$ as above. Then,*

(i) if $\mathsf{F} : (\mathbf{\Gamma})^\circ \to \mathbf{Sets}$ is a functor, the natural transformations from $h_X = \mathrm{Hom}_{\mathbf{\Gamma}}(-, X)$ to F are in natural correspondence with the elements of $\mathsf{F}(X)$.

(ii) if $h_X = \mathrm{Hom}_{\mathbf{\Gamma}}(-, X) \cong \mathrm{Hom}_{\mathbf{\Gamma}}(-, Y) = h_Y$, $X \cong Y$. That is, $h : X \to h_X$ is fully faithful.

When $\mathbf{\Gamma}$ is the category of k-schemes, this result can be refined to the following:

Proposition 2.2. *The functor*

$$h : k - \mathbf{Schemes} \to \mathrm{Fun}(k - \mathbf{Algebras}, \mathbf{Sets})$$

*is a fully faithful functor from the category of schemes over k to the category of functors from k-***Algebras** *to* **Sets** *. That is, a scheme over k is determined by the restriction of its functor of points to the category of affine schemes over k.*

One may actually replace k with any commutative ring R here; however, we do not need this generality. We shall not provide proofs here as they are easily found elsewhere (see *e.g.* [EH00] or [FmI+05]).

These results are crucial in obtaining a suitable parameter space via a concise functorial definition (*e.g.* the jet scheme $J_m(X)$ parametrizing jets on a scheme X). The first part of Yoneda's lemma tells us in particular that natural transformations from h_X to h_Y naturally correspond to morphisms from X to Y. The second part tells us that X is uniquely determined by h_X. Hence, rather than simply studying the objects X and Y and the morphisms between them in $\mathbf{\Gamma}$, we may alternatively study their functors of points, transferring our inquiry into the broader context of natural transformations between functors.

Finally, the notion of a representable functor will be important for us. Recall that a functor $\mathsf{F} : (\mathbf{\Gamma})^\circ \to \mathbf{Sets}$ is a *representable functor* if there is some object X in $\mathbf{\Gamma}$ such that $h_X \cong \mathsf{F}$. Such an object X is unique by the second part of Yoneda's lemma. In this case we also say that X *represents the functor* F.

We shall return to functors of points later on; in particular, proposition 2.2 has an analogue in the categories of pairs to be defined.

2.2 Divisors

In the first part of 2.2 we will collect some necessary definitions, particularly those of Cartier divisors and predivisors on a scheme. The second part of this section will link these two notions, and state a useful result relating divisors and predivisors on X to those on a scheme Y, given a morphism $Y \to X$.

2.2.1 Definitions of Divisors

We recall here the notion of a Cartier divisor. We will almost exclusively work with effective Cartier divisors in the following sections, and we define these as well. For a thorough introduction to divisors, see [Har06]. Let us first agree on the notation $a \in \mathrm{nzd}(R)$ for "a is a non-zerodivisor in the ring R". Given a scheme X and an open neighbourhood $U \subseteq X$, let $S_X(U) \subseteq \Gamma(U, \mathcal{O}_X)$ denote the set of sections over U that are non-zerodivisors in every local ring $\mathcal{O}_{X,p}$ with $p \in U$. The mapping $U \mapsto S_X(U)^{-1}\Gamma(U, \mathcal{O}_X)$ is a pre-sheaf on X; its associated sheaf is named the *sheaf of total quotient rings of* X, denoted by $\mathcal{M}_X$. Further, let the sheaf of multiplicative groups of invertible elements of a sheaf of rings $\mathcal{G}$ be denoted by $\mathcal{G}^*$. Cartier divisors are defined as follows:

Definition 2.2.1. A *Cartier divisor D on X* is a global section of the sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. Thus, we may specify a Cartier divisor D on X with an open covering $\{U_\alpha : \alpha \in A\}$ of X and an element $f_\alpha \in \Gamma(U_\alpha, \mathcal{M}_X^*)$ for each α, such that for every $\alpha, \beta \in A$ the quotient $f_\alpha/f_\beta \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*)$. We write $D = \{(U_\alpha, f_\alpha) : \alpha \in A\}$ for such an object.

Remark 2.1. Let us unravel this definition for the case X is of finite type, and in particular is *locally Noetherian*. Remember that if R is a Noetherian ring, then $r \in \operatorname{nzd}(R)$ if and only if $r/1 \in \operatorname{nzd}(R_\mathfrak{p})$ for all prime ideals $\mathfrak{p} \leq R$. When specifying a Cartier divisor $D = \{(U_\alpha, f_\alpha) : \alpha \in A\}$ on a locally Noetherian X, we may assume that every $U_\alpha = \operatorname{Spec} R_\alpha$ is an affine open subscheme such that R_α is a Noetherian ring. Then, $f_\alpha \in \Gamma(U_\alpha, \mathcal{M}_X^*)$ is equivalent to $f_\alpha = g_\alpha/h_\alpha$ for $g_\alpha, h_\alpha \in \operatorname{nzd}(R_\alpha)$.

From the definition, we see that $D = \{(U_\alpha, f_\alpha) : \alpha \in A\}$ and $D' = \{(W_\gamma, h_\gamma) : \gamma \in G\}$ determine the same Cartier divisor if for any α and γ such that $U_\alpha \cap W_\gamma \neq \emptyset$ the restrictions of f_α and h_γ differ by a unit in the ring $\Gamma(U_\alpha \cap W_\gamma, \mathcal{O}_X)$; that is, $f_\alpha|_{U_\alpha \cap W_\gamma} = u_{\alpha,\gamma} \cdot h_\gamma|_{U_\alpha \cap W_\gamma}$ for some $u_{\alpha,\gamma} \in \Gamma(U_\alpha \cap W_\gamma, \mathcal{O}_X^*)$.

We may restrict a Cartier divisor D to any open subscheme U of X. The *restriction of D to U* is the Cartier divisor $\{(U_\alpha \cap U, f_\alpha|_{U_\alpha \cap U}) : \alpha \in A\}$ on U, denoted $D|_U$.

Effective Cartier divisors, defined as follows, are those Cartier divisors that correspond to closed subschemes whose sheaf of ideals can locally be generated by a single section that is a non-zerodivisor (*i.e. locally principal* proper closed subschemes):

Definition 2.2.2. A Cartier divisor $D = \{(U_\alpha, f_\alpha) : \alpha \in A\}$ on X is called *effective* if $f_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X)$ for every $\alpha \in A$.

In other words, D is an effective Cartier divisor on X if and only if D defines a closed subscheme $X_D \to X$ whose sheaf of ideals $\mathcal{J}_D$ is *invertible* (*i.e.* locally isomorphic to $\mathcal{O}_X$). The set of Cartier divisors on a scheme X forms a group $\operatorname{Div}(X)$, and in "nice" situations, such as for subschemes of projective space over a field, $\operatorname{Div}(X)$ is generated by the effective Cartier divisors. We will not dwell on properties of divisors here; for a detailed treatment see [EH00], [Har06], [Gro67] or any of the litany of references that exists touching on the subject.

Though we will work mainly with effective Cartier divisors in the following sections, the condition that $f_{\alpha,p} \in \operatorname{nzd}(\mathcal{O}_{X,p})$ for every $p \in U_\alpha$, on an effective Cartier divisor D as above, is quite restrictive. Since it may be desirable from a geometric perspective to loosen this condition, we include the following definition:

Definition 2.2.3. A *predivisor D' on X* is a collection $\{(U_\alpha, f_\alpha) : \alpha \in A\}$ satisfying the following:

- $\{U_\alpha : \alpha \in A\}$ is an open cover of X,
- $f_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X)$ for all $\alpha \in A$, and
- for every $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, there exists some $u_{\alpha,\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*)$ such that $f_\alpha|_{U_\alpha \cap U_\beta} = u_{\alpha,\beta} \cdot f_\beta|_{U_\alpha \cap U_\beta}$.

Remark 2.2. It is important to note that every effective Cartier divisor D on X defines many predivisors on X; to every equivalent presentation of D, there is an associated predivisor. Conversely, given a predivisor $D' = \{(U_\alpha, f_\alpha) : \alpha \in A\}$ such that $f_{\alpha,p} \in \mathrm{nzd}(\mathcal{O}_{X,p})$ for all $p \in U_\alpha$ and all $\alpha \in A$, there is an effective Cartier divisor associated to D', and this Cartier divisor is associated to any predivisor whose presentation satisfies the natural equivalence condition with D'. In the following subsection, we will elaborate on the link between predivisors on X and effective Cartier divisors on X.

2.2.2 Pullbacks of Divisors

In this subsection we will describe the effects of morphisms on predivisors and Cartier divisors (in the case of Cartier divisors, we refer the reader to [Gro67] for further details). Let us agree that, when unmodified, *divisor* shall mean effective Cartier divisor in all that follows. First, let $\phi : Y \to X$ be a k-morphism between k-schemes Y and X; by definition this is a continuous map $\phi : Y \to X$ of topological spaces, coupled with a morphism $\phi^* : \mathcal{O}_X \to \phi_*\mathcal{O}_Y$ of sheaves on X (which behaves "nicely" with regard to localisation). Further, let $D' = \{(U_\alpha, f_\alpha) : \alpha \in A\}$ be a predivisor on X. Let

$$\phi_\alpha^* \stackrel{\text{def}}{=} \phi_{U_\alpha}^* : \Gamma(U_\alpha, \mathcal{O}_X) \to \Gamma(\phi^{-1}(U_\alpha), \mathcal{O}_Y),$$

so that $\phi_\alpha^*(f_\alpha) \in \Gamma(\phi^{-1}(U_\alpha), \mathcal{O}_Y)$. We find that the collection $\{(\phi^{-1}(U_\alpha), \phi_\alpha^*(f_\alpha)) : \alpha \in A\}$ is a predivisor on Y. In particular note that the third condition for predivisors holds, as

$$\phi_\alpha^*(f_\alpha)|_{\phi^{-1}(U_\alpha)\cap\phi^{-1}(U_\beta)} = \phi_{U_\alpha\cap U_\beta}^*(u_{\alpha,\beta}) \cdot \phi_\beta^*(f_\beta)|_{\phi^{-1}(U_\alpha)\cap\phi^{-1}(U_\beta)},$$

for every $\alpha, \beta \in A$ such that $\phi^{-1}(U_\alpha) \cap \phi^{-1}(U_\beta) \neq \emptyset$. Note that $\phi_{U_\alpha\cap U_\beta}^*(u_{\alpha,\beta}) \in \Gamma(\phi^{-1}(U_\alpha) \cap \phi^{-1}(U_\beta), \mathcal{O}_Y^*)$ is invertible, as $u_{\alpha,\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*)$ is. We make the following definition:

Definition 2.2.1. Given a predivisor D' on X, and a morphism $Y \to X$ as above, the predivisor $\{(\phi^{-1}(U_\alpha), \phi_\alpha^*(f_\alpha)) : \alpha \in A\}$ on Y is called the *pullback of D' by ϕ*, denoted $\phi^*(D')$. We say that ϕ *pulls back D' to $\phi^*(D')$*.

Recall that for every $p \in X$ and any $q \in \phi^{-1}(p)$ there is an induced morphism $\phi_q^* : \mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$ of local rings. Suppose that $p \in U_\alpha \subseteq X$; this morphism takes $f_{\alpha,p}$ to the element $\phi_q^*(f_{\alpha,p}) = (\phi_\alpha^*(f_\alpha))_q$. Hence, if $\phi_q^*(f_{\alpha,p}) \in \mathrm{nzd}(\mathcal{O}_{Y,q})$ for every such p, q, and α, then the pullback $\phi^*(D') = \{(\phi^{-1}(U_\alpha), \phi_\alpha^*(f_\alpha)) : \alpha \in A\}$ will define an effective Cartier divisor on Y, by remark 2.2. We obtain the following lemma:

Lemma 2.1. *Suppose that $D' = \{(U_\alpha, f_\alpha) : \alpha \in A\}$ and $D'' = \{(W_\gamma, h_\gamma) : \gamma \in G\}$ are predivisors obtained from equivalent presentations of the effective Cartier divisor D. Moreover, suppose that D' satisfies the condition above, so that $\phi^*(D')$ yields an effective Cartier divisor on Y. Then, the pullback $\phi^*(D'')$ also yields an effective Cartier divisor on Y, equal to the one obtained from $\phi^*(D')$.*

Proof. To verify this, we must show that the pullback $\phi^*(D'')$ is locally defined by sections that differ by a unit from those defining $\phi^*(D')$ on the intersections of their respective domains. Now, as $\phi^*_\alpha(f_\alpha)$ is locally a non-zerodivisor, so is $\phi^*_{\alpha,\gamma}(f_\alpha|_{U_\alpha\cap W_\gamma})$ for all α and γ such that $U_\alpha \cap W_\gamma \neq \emptyset$, where $\phi^*_{\alpha,\gamma} = \phi^*_{U_\alpha\cap W_\gamma}$. Then,

$$\phi^*_{\alpha,\gamma}(f_\alpha|_{U_\alpha\cap W_\gamma}) = \phi^*_{\alpha,\gamma}(u_{\alpha,\gamma}\cdot h_\gamma|_{U_\alpha\cap W_\gamma}) = \phi^*_{\alpha,\gamma}(u_{\alpha,\gamma})\cdot\phi^*_{\alpha,\gamma}(h_\gamma|_{U_\alpha\cap W_\gamma}),$$

for some $u_{\alpha,\gamma} \in \Gamma(U_\alpha \cap W_\gamma, \mathcal{O}^*_X)$. Since $u_{\alpha,\gamma} \in \Gamma(U_\alpha \cap W_\gamma, \mathcal{O}^*_X)$ is invertible, we have $\phi^*_{\alpha,\gamma}(u_{\alpha,\gamma}) \in \Gamma(\phi^{-1}(U_\alpha \cap W_\gamma), \mathcal{O}^*_Y) = \Gamma(\phi^{-1}(U_\alpha) \cap \phi^{-1}(W_\gamma), \mathcal{O}^*_Y)$; that is, $\phi^*_{\alpha,\gamma}(u_{\alpha,\gamma})$ is invertible. Hence $\phi^*_{\alpha,\gamma}(h_\gamma|_{U_\alpha\cap W_\gamma})$ is locally a non-zerodivisor, showing that $\phi^*_\gamma(h_\gamma)$ is itself locally a non-zerodivisor. Also, we see that on $\phi^{-1}(U_\alpha \cap W_\gamma) = \phi^{-1}(U_\alpha) \cap \phi^{-1}(W_\gamma)$ the local equations $\phi^*_\alpha(f_\alpha)$ and $\phi^*_\gamma(h_\gamma)$ differ by a unit. Hence, the predivisors $\phi^*(D')$ and $\phi^*(D'')$ yield equivalent effective Cartier divisors. □

We formalize this case with the following definition:

Definition 2.2.2. Let D be an effective Cartier divisor on the scheme X and let $\phi : Y \to X$ be a morphism, as in lemma 2.1. We define the *pullback of D by ϕ* to be the effective Cartier divisor on Y obtained from the pullback $\phi^*(D')$ of the predivisor $D' = \{(U_\alpha, f_\alpha) : \alpha \in A\}$, where $\{(U_\alpha, f_\alpha) : \alpha \in A\}$ is any presentation of D. We denote the pullback of D as $\phi^*(D)$, and say in this case that *ϕ pulls back the effective Cartier divisor D to $\phi^*(D)$.*

The following definition is convenient in the context of the categories $\mathbf{Pairs}_r$, described in the following section:

Definition 2.2.3. Let $r \geq 1$ be an integer, and let $\mathcal{D}' = (D'_1, D'_2, \ldots, D'_r)$ be an r-tuple of predivisors on the scheme X. Given a morphism $\phi : Y \to X$ as above, we define the *pullback of $\mathcal{D}'$ by ϕ* to be the r-tuple $(\phi^*(D'_1), \phi^*(D'_2), \ldots, \phi^*(D'_r))$, denoted $\phi^*(\mathcal{D}')$. In the case that each D'_i is a presentation of a divisor D_i, then as above each pullback $\phi^*(D'_i)$ is a presentation of a divisor determined by D_i. Letting $\mathcal{D}$ denote the r-tuple of divisors $(D_1, D_2, \ldots, D_r)$, we define the *pullback of $\mathcal{D}$ by ϕ* to be the r-tuple $(\phi^*(D_1), \phi^*(D_2), \ldots, \phi^*(D_r))$ of divisors on Y. In this case, we say that *ϕ pulls back the family $\mathcal{D}$ of divisors on X to the family $\phi^*(\mathcal{D})$ on Y.*

We will end the section with a description of how, by "removing components" from a scheme X of finite type over k, one can force a predivisor D' on X to describe an effective Cartier divisor D on some maximal closed immersion $X' \to X$. Thus, any morphism $Y \to X$ of schemes of finite type such that $\phi^*(D')$ defines an effective Cartier divisor on Y will factor through X'.

Lemma 2.2. *Given a predivisor $D' = \{(U_\alpha, f_\alpha) : \alpha \in A\}$ on a scheme X of finite type over k, there exists a unique closed immersion $i : X' \to X$ such that*

(i) *$i^*(D')$ yields an effective Cartier divisor on X', and* *(ii) for any k-morphism $\phi : Y \to X$ such that $\phi^*(D')$ yields an effective Cartier divisor on Y, there exists a morphism $\phi' : Y \to X'$ such that $\phi = i \circ \phi'$. Consequently, the pullback by ϕ' of the effective Cartier divisor defined by $i^*(D')$ is the effective Cartier divisor defined by $\phi^*(D')$.*

Proof. Let $g_{i_\alpha} = f_\alpha|_{V_{i_\alpha}} \in R_{i_\alpha}$, where $\{V_{i_\alpha} = \operatorname{Spec} R_{i_\alpha} : i_\alpha \in I_\alpha\}$ is an open cover of U_α by affines for each $\alpha \in A$. Suppose that g_{i_α} is a zerodivisor in R_{i_α}. We begin by letting $S_{i_\alpha} = \{g_{i_\alpha}^\gamma : \gamma \geq 0\}$ be the multiplicative subset in R_{i_α} generated by g_{i_α}, and we consider the canonical ring homomorphism $h_{i_\alpha} : R_{i_\alpha} \to R_{i_\alpha}[S_{i_\alpha}^{-1}]$. Let $K_{i_\alpha} = \ker(h_{i_\alpha})$. Notice that $a \in K_{i_\alpha}$ if and only if $\exists\, \gamma$ such that $g_{i_\alpha}^\gamma \cdot a = 0$ in R_{i_α}. Thus, the ring $R_{i_\alpha}/K_{i_\alpha}$ is the largest quotient ring of R_{i_α} in which every zerodivisor of g_{i_α} equals zero; that is, K_{i_α} is the minimal ideal containing all such elements.

Now let ϕ be a morphism as in the statement of the lemma. The homomorphism $(\phi|_{\phi^{-1}(V_{i_\alpha})})^* : R_{i_\alpha} \to B_{i_\alpha} = \Gamma(\phi^{-1}(V_{i_\alpha}), \mathcal{O}_Y)$ must factor through $R_{i_\alpha}/K_{i_\alpha}$; *i.e.* the morphism $\phi|_{\phi^{-1}(V_{i_\alpha})} : \phi^{-1}(V_{i_\alpha}) \to V_{i_\alpha} = \operatorname{Spec} R_{i_\alpha}$ factors through $\operatorname{Spec} R_{i_\alpha}/K_{i_\alpha}$. Thus the ideals K_{i_α} indexed over all $\alpha \in A$ and $i_\alpha \in I_\alpha$ define a closed immersion $i : X' \to X$ through which $\phi : Y \to X$ must factor. By construction, we see that the pullback $i^*(D)$ yields an effective Cartier divisor on X'. Moreover, it is immediate that the pullback of this effective Cartier divisor is defined by $\phi^*(D)$. □

3 Categories of Pairs

In the first part of this section, we define the categories of pairs. This allows us to use some categorical arguments in studying scheme morphisms that pull back divisors. Following this, we collect some definitions and examples in the categories of pairs that will be used in the next section.

3.1 Defining the Categories of Pairs

The *categories of pairs over k* provide us primarily with a convenient framework for studying jets on a scheme that pull back divisors. In any category of pairs we will define the "jet pairs" analogously to jet schemes, in terms of the representability of a certain functor. For any $r \geq 0$ there is a category of pairs k-$\mathbf{Pairs}_r$, though we will usually work in a general category **Pairs**, specifying r and k only as needed.

An object in **Pairs** consists of a scheme X of finite type over k coupled with an ordered r-tuple $\mathcal{D} = (D_1, D_2, \ldots, D_r)$ of its effective Cartier divisors D_i, $1 \leq i \leq r$. This forms the *pair* $\mathbf{X} = (X, \mathcal{D})$ (again, one may wish specify *r-pair*, *pair over k*, *r-pair over k*, etc.).

We must describe the morphisms in **Pairs**. Given two pairs $\mathbf{X} = (X, \mathcal{D})$ and $\mathbf{Y} = (Y, \mathcal{E}) \in$ **Pairs**, we define

$$\mathrm{Hom}_{\mathbf{Pairs}}(\mathbf{Y}, \mathbf{X}) = \{\phi \in \mathrm{Hom}_{k-\mathbf{Schemes}}(Y, X) : \phi^*(\mathcal{D}) = \mathcal{E}\}.$$

Thus, for $\phi : Y \to X$ to be considered as a morphism of pairs, the pullback of $\mathcal{D}$ by ϕ must exist and equal $\mathcal{E}$. It is clear from the definitions that the identity $\mathrm{id}_X : (X, \mathcal{D}) \to (X, \mathcal{D})$ is a morphism of pairs; simply note that $\mathrm{id}_X^* = \mathrm{id}_{\mathcal{O}_X}$. Further, pullbacks behave well with regards to composition; that is, given two morphisms of pairs $\psi : (Z, \mathcal{F}) \to (Y, \mathcal{E})$ and $\phi : (Y, \mathcal{E}) \to (X, \mathcal{D})$, their composition $\phi \circ \psi$ is a morphism of pairs. To verify this, note that if $D_i = \{(U_{\alpha_i}, f_{\alpha_i}) : \alpha_i \in A_i\}$ is a presentation of D_i, then $E_i = \{(\phi^{-1}(U_{\alpha_i}), \phi^*_{\alpha_i}(f_{\alpha_i})) : \alpha_i \in A_i\}$ is a presentation of E_i, and so

$$\begin{aligned} F_i &= \{(\psi^{-1}(\phi^{-1}(U_{\alpha_i})), \psi^*_{\phi^{-1}(U_{\alpha_i})}(\phi^*_{\alpha_i}(f_{\alpha_i}))) : \alpha_i \in A_i\} \\ &= \{((\phi \circ \psi)^{-1}(U_{\alpha_i}), (\phi \circ \psi)^*_{\alpha_i}(f_{\alpha_i})) : \alpha_i \in A_i\}, \end{aligned}$$

is a presentation of F_i, proving that the pullback of D_i under $\phi \circ \psi$ exists and equals F_i. Hence k-$\mathbf{Pairs}_r$ is a category. We choose to define k-$\mathbf{Pairs}_0$ to be the category of schemes of finite type over k, k-**Schemes**.

At times we will want to focus our attention exclusively on pairs with affine underlying schemes, and (families of) divisors defined by global sections. Just as the affine schemes form a category, we let k-**Aff** $\mathbf{Pairs}_r$ denote the category whose objects are pairs $\mathbf{X} = (X, \mathcal{D})$ in k-$\mathbf{Pairs}_r$ such that X is an affine scheme, and $\mathcal{D} = (\{(X, f_1)\}, \ldots, \{(X, f_r)\})$ for some choice of presentations $D_i = \{(X, f_i)\}$. The morphisms between two objects are all those between the objects in **Pairs**. Hence, k-**Aff** $\mathbf{Pairs}_r$ forms a full subcategory of k-$\mathbf{Pairs}_r$; we call this the *category of affine pairs.* Again, we will almost exclusively write **Aff Pairs** and work in the general category.

3.2 Definitions and Examples in the Categories of Pairs

Here we collect some examples of morphisms and constructions in the categories of pairs that will be useful to us later on.

3.2.1 Open Subpairs and a Gluing Construction for Pairs

The simplest examples of morphisms of pairs come from open subschemes. Let $\mathbf{X} = (X, \mathcal{D})$ be a pair, and $U \subseteq X$ an open subscheme. The divisors D_i for $1 \leq i \leq r$ naturally restrict to U (as mentioned in section 2.2) as the local equations for D_i are non-zerodivisors in the stalks $\mathcal{O}_{U,p} = \mathcal{O}_{X,p}$ for all $p \in U$. So the open immersion $U \to X$ naturally defines an *open immersion of pairs* $\mathbf{U} \to \mathbf{X}$, where $\mathbf{U} = (U, \mathcal{D}|_U)$ and $\mathcal{D}|_U = (D_1|_U, \ldots, D_r|_U)$. We will refer to $\mathbf{U}$ as an *open subpair* of the pair $\mathbf{X}$.

We will define the *intersection of two open subpairs* $\mathbf{U}$ *and* $\mathbf{V}$ of $\mathbf{X}$, whose underlying schemes $U, V \subseteq X$ have non-empty intersection $U \cap V \neq \emptyset$, to be the open subpair $\mathbf{U} \cap \mathbf{V} = (U \cap V, \mathcal{D}|_{U \cap V})$ of $\mathbf{X}$.

Suppose that $\{(X_\alpha, \mathcal{D}_\alpha) : \alpha \in A\}$ is a collection of open subpairs of $(X, \mathcal{D})$ indexed by the set A. We will call $\{(X_\alpha, \mathcal{D}_\alpha) : \alpha \in A\}$ an *open cover of* $(X, \mathcal{D})$ if $\{X_\alpha : \alpha \in A\}$ is an open cover of X. Note that by assumption the families of divisors $\mathcal{D}_\alpha|_{X_\alpha \cap X_\beta}$ and $\mathcal{D}_\beta|_{X_\alpha \cap X_\beta}$ are equivalent on the open subscheme $X_\alpha \cap X_\beta$ for all $\alpha, \beta \in A$, as they are both equivalent to $\mathcal{D}|_{X_\alpha \cap X_\beta}$.

Given another pair $\mathbf{Y} = (Y, \mathcal{E})$, and a morphism $\phi : \mathbf{Y} \to \mathbf{X}$, we would also like to speak of the preimage of an open subpair $\mathbf{U} \subseteq \mathbf{X}$. We let the *preimage of* $\mathbf{U}$ *under* ϕ refer to the open subpair $\phi^{-1}(\mathbf{U}) = (\phi^{-1}(U), \mathcal{E}|_{\phi^{-1}(U)})$ of $\mathbf{Y}$.

Notice that we have restricted our attention especially to open subschemes here. Suppose that $Y \to X$ is a closed immersion, and that $(X, \mathcal{D})$ is a pair. In general it is certainly not true that the local sections defining a Cartier divisor on X will pull back to non-zerodivisors in the stalks of the structure sheaf of Y. Hence working with closed immersions is considerably more subtle; in general we must "remove components" in order to ensure we are always working in the category **Pairs** (see lemma 2.2).

Example (Gluing Construction) 3.1. We now remind the reader of the gluing construction for schemes, giving the analogous construction in **Pairs**. The reason that gluing works in **Pairs** is simple; gluing together a scheme from a collection of schemes with divisors, whose local equations coincide (up to multiplication by an invertible section) via the local isomorphisms defining the gluing, yields divisors on the new scheme that are locally defined by the original equations. That is, suppose first that we are given a collection of pairs $\{\mathbf{X}_\alpha = (X_\alpha, \mathcal{D}_\alpha) : \alpha \in A\}$ indexed by a set A and for every $\alpha, \beta \in A$ such that $\beta \neq \alpha$ an open subpair $\mathbf{U}_{\alpha\beta}$ of $\mathbf{X}_\alpha$. Suppose further that we have isomorphisms $\psi_{\alpha\beta} : \mathbf{U}_{\alpha\beta} \to \mathbf{U}_{\beta\alpha}$ for all such α, β, that satisfy $\psi_{\beta\alpha} = \psi_{\alpha\beta}^{-1}$ for all α, β; that

$$\psi_{\alpha\beta}(\mathbf{U}_{\alpha\beta} \cap \mathbf{U}_{\alpha\gamma}) = \mathbf{U}_{\beta\alpha} \cap \mathbf{U}_{\beta\gamma} \quad \forall \alpha, \beta, \gamma;$$

and that

$$\psi_{\alpha\beta} \circ \psi_{\beta\gamma}|_{\mathbf{U}_{\alpha\beta} \cap \mathbf{U}_{\alpha\gamma}} = \psi_{\alpha\gamma}|_{\mathbf{U}_{\alpha\beta} \cap \mathbf{U}_{\alpha\gamma}} \quad \forall \alpha, \beta, \gamma.$$

Then we may glue together a pair along the isomorphisms $\psi_{\alpha\beta}$ analogously to the gluing of schemes. In fact, the underlying scheme is obtained by gluing along the morphisms $\psi_{\alpha\beta}$ considered as morphisms of schemes, while the r-tuple of divisors on the glued scheme will have local equations exactly those on any $\mathbf{X}_\alpha$.

The following example shows that flat morphisms of schemes are morphisms of pairs:

Example 3.2. Let $\mathbf{X} = (X, \mathcal{D})$ be a pair, and let $\phi : Y \to X$ be a morphism of schemes, such that Y is flat over X. This means that for every $y \in Y$ and $x = \phi(y)$ the morphism $\phi_y^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ makes $\mathcal{O}_{Y,y}$ into a flat $\mathcal{O}_{X,x}$-module.

Let $U \subseteq X$ be an open subscheme, and suppose that for each $1 \leq i \leq r$ the local equation of D_i on U is f_i. We pull the local equations for $\mathcal{D}$ back to $\phi^*(f_i) \in \phi^{-1}(U) \subseteq Y$. Since ϕ makes Y flat over X, for every point $y \in \phi^{-1}(U)$ we deduce that $(\phi^*(f_i))_y = \phi_y^*(f_{i,x})$ is a non-zerodivisor in $\mathcal{O}_{Y,y}$. (This follows from a basic property of flat modules; namely, since $f_{i,x} \in \mathrm{nzd}(\mathcal{O}_{X,x})$ and the morphism ϕ_y^* makes $\mathcal{O}_{Y,y}$ into a flat $\mathcal{O}_{X,x}$-module, $f_{i,x}$ remains a non-zerodivisor on $\mathcal{O}_{Y,y}$ (see *e.g.* [Eis04]).)

Hence the pullback $\phi^*(\mathcal{D})$ exists on Y, and so any flat morphism from a scheme to a scheme with attached family of effective Cartier divisors automatically pulls back the family, yielding a morphism of pairs. Of course, this implies that smooth and étale morphisms also always pull back families of effective Cartier divisors. We shall refer to a morphism $\mathbf{Y} \to \mathbf{X}$ of pairs as *flat* (resp. *étale*, *smooth*) if the underlying scheme-morphism is flat (resp. étale, smooth).

3.2.2 m-jets in Pairs

The reason for formulating **Pairs** as we have done stems from the following example. Let Y and X be schemes of finite type over k. Morphisms from the fibred product $Y \times \operatorname{Spec} k[t]/(t^{m+1})$ to X are referred to as *Y-valued m-jets*, and are thought of as order m germs of arcs on X. We would like to study m-jets that pull back divisors on X. That is, given r-tuples $\mathcal{D}$ on X and $\mathcal{E}'$ on $Y \times \operatorname{Spec} k[t]/(t^{m+1})$ we will study the morphisms of pairs $(Y \times \operatorname{Spec} k[t]/(t^{m+1}), \mathcal{E}') \to (X, \mathcal{D})$.

As a particular example, let us consider the projection $Y \times \operatorname{Spec} k[t]/(t^{m+1}) \to Y$. Let $\mathcal{E} = (E_1, \ldots, E_r)$ be an r-tuple of effective Cartier divisors on Y. For any $1 \leq i \leq r$ suppose that $V_i = \operatorname{Spec} B_i \subseteq Y$ is an open affine subscheme on which E_i is defined by $g_i \in B_i$. The preimage of V_i under the projection is isomorphic to $V_i \times \operatorname{Spec} k[t]/(t^{m+1}) = \operatorname{Spec} B_i[t]/(t^{m+1})$, and the morphism of structure sheaves takes the section g_i on V_i to g_i as an element in the ring $B_i[t]/(t^{m+1})$. Since g_i is a non-zerodivisor in B_i, it is a non-zerodivisor in $B_i[t]/(t^{m+1})$. Further, as the rings B_i and $B_i[t]/(t^{m+1})$ are finitely generated k-algebras, and hence Noetherian, this implies that $g_i/1$ is an element in $\mathcal{M}^*_{Y \times \operatorname{Spec} k[t]/(t^{m+1})}(V_i \times \operatorname{Spec} k[t]/(t^{m+1}))$. Thus g_i is locally the equation of an effective Cartier divisor on $V_i \times \operatorname{Spec} k[t]/(t^{m+1}) = \operatorname{Spec} B_i[t]/(t^{m+1}) \subseteq Y \times \operatorname{Spec} k[t]/(t^{m+1})$. We denote this divisor E_i^m, and let $\mathcal{E}^m \overset{\text{def}}{=} (E_1^m, E_2^m, \ldots, E_r^m)$.

Examining further, we find that $g_i \in \mathrm{nzd}(B_i)$ if and only if $g_i + g_{i1}t + \cdots + g_{im}t^m \in \mathrm{nzd}(B_i[t]/(t^{m+1}))$ for any $g_{i1}, \ldots, g_{im} \in B_i$. Therefore a divisor E' on $V_i \times \operatorname{Spec} k[t]/(t^{m+1})$

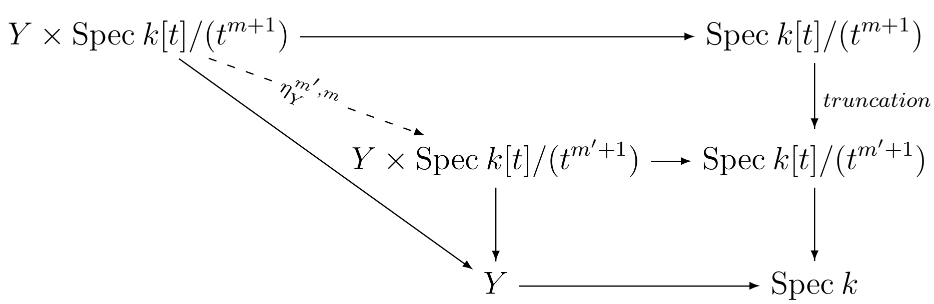
defined by $g_i + g_{i1}t + \cdots + g_{im}t^m$ is of the form above if and only if $g_i | g_{il}$ for all $1 \leq l \leq m$. Referring back to the introduction, this condition matches exactly the one we derived for 1-jets that guarantees they are logarithmic along a divisor with normal crossings. Thus, we make the following definition:

Definition 3.2.1. Given pairs $(Y, \mathcal{E})$ and $(X, \mathcal{D})$ as above, a morphism of pairs from $(Y \times \operatorname{Spec} k[t]/(t^{m+1}), \mathcal{E}^m)$ to $(X, \mathcal{D})$ is called a *$(Y, \mathcal{E})$-valued m-jet in $(X, \mathcal{D})$*.

Alternatively, we may use the terms *m-jets logarithmic along $\mathcal{D}$*, or simply *m-jets* when the context is clear. Obviously, in $\mathbf{Pairs}_0 = k$-**Schemes**, a logarithmic m-jet is the same thing as a usual m-jet.

Often, the way we work in the category **Pairs** is as follows: we begin with some pair $(X, \mathcal{D})$ and a scheme-morphism $Y \to X$, and then examine whether the pullback of $\mathcal{D}$ exists on Y. The way we defined logarithmic m-jets provides an example of this. This process seems to be in contrast to the way the category is defined; namely, the morphisms of pairs are those that *a priori* have well-defined pullbacks that coincide with an r-tuple of divisors on the domain. This is not a problem; the real advantage of the categorical formalism lies in the arguments and results it enables us to use, as in the next section.

Now, consider the truncation homomorphism $k[t]/(t^{m'+1}) \to k[t]/(t^{m+1})$, where m' and m are integers such that $m' \geq m \geq 0$. This morphism induces a morphism of affine schemes $\operatorname{Spec} k[t]/(t^{m+1}) \to \operatorname{Spec} k[t]/(t^{m'+1})$. Given a scheme Y consider the following diagram, which commutes:



So $\eta_Y^{m',m}$ is the unique morphism of schemes guaranteed by the universal mapping property of the fibred product $Y \times \operatorname{Spec} k[t]/(t^{m'+1})$ making the diagram commute. As in the previous example, suppose that $(Y, \mathcal{E})$ is a pair. Then we may pull $\mathcal{E}$ back to both fibred products $Y \times \operatorname{Spec} k[t]/(t^{m+1})$ and $Y \times \operatorname{Spec} k[t]/(t^{m'+1})$, making $\eta_Y^{m',m}$ into a morphism of pairs. (Note that the local equations of the r-tuples remain essentially unchanged; only the ring they live in changes.) Thus, denoting $(Y, \mathcal{E})$ by $\mathbf{Y}$, let us write $\eta_{\mathbf{Y}}^{m',m}$ for the morphism of pairs $(Y \times \operatorname{Spec} k[t]/(t^{m+1}), \mathcal{E}^m) \to (Y \times \operatorname{Spec} k[t]/(t^{m'+1}), \mathcal{E}^{m'})$. We refer to such a morphism as a *truncation morphism*; note that we can pull back $(Y, \mathcal{E})$-valued m'-jets to $(Y, \mathcal{E})$-valued m-jets via $\eta_{\mathbf{Y}}^{m',m}$. We will write $\eta_{\mathbf{Y}}^{m'}$ rather than $\eta_{\mathbf{Y}}^{m',0}$ when $m = 0$. Later on, we will use these morphisms to define "projection morphisms" between jet pairs.

Moving on, suppose that $\phi : \mathbf{Z} \to \mathbf{Y}$ is a morphism, where $\mathbf{Z} = (Z, \mathcal{F})$ and $\mathbf{Y} = (Y, \mathcal{E})$. Then there is a morphism $\phi^m : Z \times \operatorname{Spec} k[t]/(t^{m+1}) \to Y \times \operatorname{Spec} k[t]/(t^{m+1})$ of schemes guaranteed by the universal mapping property for the fibred product $Y \times \operatorname{Spec} k[t]/(t^{m+1})$. ϕ^m is in fact a morphism of pairs $\phi^m : (Z \times \operatorname{Spec} k[t]/(t^{m+1}), \mathcal{F}^m) \to (Y \times \operatorname{Spec} k[t]/(t^{m+1}), \mathcal{E}^m)$. This follows since pulling back the local equations of $\mathcal{E}^m$ by ϕ^m is equivalent to pulling back the local equations of $\mathcal{E}$ by the composition $Z \times \operatorname{Spec} k[t]/(t^{m+1}) \to Z \xrightarrow{\phi} Y$. Thus for every ϕ as above we obtain an induced morphism ϕ^m such that the following commutes:

$$\begin{array}{ccc}
(Z \times \operatorname{Spec} k[t]/(t^{m+1}), \mathcal{F}^m) & \xrightarrow{\phi^m} & (Y \times \operatorname{Spec} k[t]/(t^{m+1}), \mathcal{E}^m) \\
\downarrow{\scriptstyle projection} & & \downarrow{\scriptstyle projection} \\
(Z, \mathcal{F}) & \xrightarrow{\phi} & (Y, \mathcal{E})
\end{array}$$

We will use morphisms of this form in the results in the following section. To finish this section, let us explicitly give the definition of the functor of points of a pair, as we will work with these immediately in what follows.

Example 3.3. Recall that in section 2.1 we defined the functor of points of an object in a general category. Let $\mathbf{X} \in \mathbf{Pairs}$; the *functor of points of the pair* $\mathbf{X}$, denoted $h_{\mathbf{X}}$, is defined to be the functor

$$h_{\mathbf{X}} : (\mathbf{Pairs})^{\circ} \to \mathbf{Sets}$$

such that

$$h_{\mathbf{X}}(\mathbf{Y}) = \operatorname{Hom}_{\mathbf{Pairs}}(\mathbf{Y}, \mathbf{X})$$

for $\mathbf{Y} \in \mathbf{Pairs}$.

4 Jet Pairs and Logarithmic Jet Schemes

We prove some results about pairs that enable us to demonstrate the existence of a parameter space for certain $(Y, \mathcal{E})$-valued m-jets on a pair $(X, \mathcal{D})$. We call this parameter space, which lives in a category of pairs, the "jet pair" associated to the pair $(X, \mathcal{D})$. The underlying scheme of the jet pair is referred to as the "logarithmic jet scheme of X with respect to $\mathcal{D}$". The preliminary results generalize analogous results in the category of schemes, which are employed to give a constructive proof of the existence of the jet schemes associated to a chosen scheme. Indeed, the scheme case is subsumed within ours by the case $\mathbf{Pairs}_0$.

4.1 The Main Construction

Let $\mathbf{X} = (X, \mathcal{D}) \in \mathbf{Pairs}$, and let j_m denote $\operatorname{Spec} k[t]/(t^{m+1})$. The mapping

$$\mathsf{L}_m^{\mathbf{X}} : (Y, \mathcal{E}) \mapsto \operatorname{Hom}_{\mathbf{Pairs}}((Y \times j_m, \mathcal{E}^m), (X, \mathcal{D}))$$

defines a covariant functor $(\mathbf{Pairs})^\circ \to \mathbf{Sets}$ (in other words a contravariant functor from **Pairs** to **Sets**). We will prove that this functor is representable for all $m > 0$, *i.e.* that there exists a pair $J_m(\mathbf{X}) = (J_m^{\mathcal{D}}(X), J_m(\mathcal{D}))$ such that $h_{J_m(\mathbf{X})} \cong \mathsf{L}_m^{\mathbf{X}}$. (Note in addition that $\mathsf{L}_0^{\mathbf{X}}$ is always representable by $\mathbf{X}$). To do this we will use two helpful facts; first, that one can determine whether such a functor is representable from its action on affine pairs, and second, that the functors $h_{J_m(\mathbf{X})}$ can be obtained by "gluing together affine pieces" in the sense of the gluing construction of example 3.1. Once these facts are established we will only need to work in the category of affine pairs to prove representability; it will then be true for all pairs.

We begin with the first claim, noting that this is only an adjustment of the analogous fact, lemma 2.2 on page 6, in which the roles of the categories of pairs and affine pairs (with arrows reversed) are taken by the categories of k-schemes and k-algebras respectively.

Proposition 4.1. *Let $\mathbf{X} = (X, \mathcal{D})$ be a pair over k. The restriction of the functor of points $h_{\mathbf{X}}$ of X to the category of affine pairs over k determines $\mathbf{X}$. That is, the functor*

$$h : \mathbf{Pairs} \longrightarrow \operatorname{Func}((\mathbf{Aff\,Pairs})^\circ, (\mathbf{Sets}))$$

$$\mathbf{X} \longmapsto h_{\mathbf{X}}|_{(\mathbf{Aff\,Pairs})^\circ}$$

is fully faithful.

Proof. Let $\mathbf{Y} = (Y, \mathcal{E})$ also be an element of **Pairs**, and let $h_{\mathbf{X}}$ and $h_{\mathbf{Y}}$ denote the restrictions $h_{\mathbf{X}}|_{(\mathbf{Aff\,Pairs})^\circ}$ and $h_{\mathbf{Y}}|_{(\mathbf{Aff\,Pairs})^\circ}$ respectively for the remainder of the proof. Any morphism $\mathrm{t} : \mathbf{Y} \to \mathbf{X}$ defines a natural transformation $h_{\mathrm{t}} : h_{\mathbf{Y}} \to h_{\mathbf{X}}$ by composition of t with morphisms $\phi : \mathbf{Z} \to \mathbf{Y}$ for any $\mathbf{Z} = (Z, \mathcal{F}) \in \mathbf{Aff\,Pairs}$. That is,

$$h_{\mathrm{t}}(\mathbf{Z}) : \quad h_{\mathbf{Y}}(\mathbf{Z}) \longrightarrow h_{\mathbf{X}}(\mathbf{Z})$$

$$\operatorname{Hom}_{\mathbf{Pairs}}(\mathbf{Z}, \mathbf{Y}) \longrightarrow \operatorname{Hom}_{\mathbf{Pairs}}(\mathbf{Z}, \mathbf{X})$$

$$\phi \longmapsto t \circ \phi$$

Hence, it is sufficient to prove that any natural transformation $\tau : h_{\mathbf{Y}} \to h_{\mathbf{X}}$ is equal to h_{t} for some unique morphism $\mathrm{t} : \mathbf{Y} \to \mathbf{X}$.

Let τ be such a natural transformation. We shall obtain the desired morphism t from τ. First, let $\{\mathbf{Y}_\alpha = (Y_\alpha, \mathcal{E}_\alpha) : \alpha \in A\}$ be an open cover of $\mathbf{Y}$ by affines. Let $\imath_\alpha : \mathbf{Y}_\alpha \to \mathbf{Y}$ denote the inclusion for each α. Then, there is a unique morphism $t_\alpha \stackrel{\text{def}}{=} \tau_{\mathbf{Y}_\alpha}(\imath_\alpha) : \mathbf{Y}_\alpha \to \mathbf{X}$ corresponding to each inclusion. We claim that the t_α's glue together to define the desired morphism $t : \mathbf{Y} \to \mathbf{X}$. To show this, first let $\mathbf{Y}_{\alpha\beta} = \mathbf{Y}_\alpha \cap \mathbf{Y}_\beta$ for every $\alpha, \beta \in A$. Further, let $\imath_{\alpha\beta} : \mathbf{Y}_{\alpha\beta} \to \mathbf{Y}_\alpha$ denote the inclusion of the intersection into $\mathbf{Y}_\alpha$. Then, by naturality of τ we see that $\tau_{\mathbf{Y}_{\alpha\beta}}(\imath_\alpha \circ \imath_{\alpha\beta}) = t_\alpha \circ \imath_{\alpha\beta}$, and that $\tau_{\mathbf{Y}_{\beta\alpha}}(\imath_\alpha \circ \imath_{\alpha\beta}) = t_\beta \circ \imath_{\beta\alpha}$, using the fact that $\imath_\beta \circ \imath_{\beta\alpha} = \imath_\alpha \circ \imath_{\alpha\beta}$. But $\mathbf{Y}_{\alpha\beta} = \mathbf{Y}_{\beta\alpha}$, therefore $t_\alpha \circ \imath_{\alpha\beta} = t_\beta \circ \imath_{\beta\alpha}$; that is, the restrictions of the t_α's to the intersections $\mathbf{Y}_\alpha \cap \mathbf{Y}_\beta$ are equal, and we may glue the morphisms to define $t : \mathbf{Y} \to \mathbf{X}$.

Now we will prove $h_{\mathrm{t}} = \tau$ by showing that $h_{\mathrm{t}}(\mathbf{Z})(\phi) = \tau_{\mathbf{Z}}(\phi)$ for any $\mathbf{Z}$ and any element $\phi : \mathbf{Z} \to \mathbf{Y}$ of $h_{\mathbf{Y}}(\mathbf{Z})$. Letting $\mathbf{Z}_\alpha = \phi^{-1}(\mathbf{Y}_\alpha)$, and remebering that $h_{\mathrm{t}}(\mathbf{Z})(\phi) = t \circ \phi$, we see that it suffices to prove $(t \circ \phi)|_{\mathbf{Z}_\alpha} = \tau_{\mathbf{Z}}(\phi)|_{\mathbf{Z}_\alpha}$ for all $\alpha \in A$. Let $\phi_\alpha = \phi|_{\mathbf{Z}_\alpha} : \mathbf{Z}_\alpha \to \mathbf{Y}_\alpha$ and let $\jmath_\alpha : \mathbf{Z}_\alpha \to \mathbf{Z}$ be the inclusion for each $\alpha \in A$. Then by naturality of τ we find that $\tau_{\mathbf{Z}_\alpha}(\imath_\alpha \circ \phi_\alpha) = t_\alpha \circ \phi_\alpha$, and using $\imath_\alpha \circ \phi_\alpha = \phi \circ \jmath_\alpha$ that $\tau_{\mathbf{Z}_\alpha}(\imath_\alpha \circ \phi_\alpha) = \tau_{\mathbf{Z}}(\phi) \circ \jmath_\alpha$. Thus, $(t \circ \phi)|_{\mathbf{Z}_\alpha} = \phi_\alpha \circ t_\alpha = \tau_{\mathbf{Z}}(\phi) \circ \jmath_\alpha = \tau_{\mathbf{Z}}(\phi)|_{\mathbf{Z}_\alpha}$ for any α. Hence indeed $t \circ \phi = \tau_{\mathbf{Z}}(\phi)$, and we are done.

□

So supposing that there is a pair $J_m(\mathbf{X})$ and an isomorphism of functors

$$h_{J_m(\mathbf{X})}|_{(\textbf{Aff Pairs})^\circ} \cong \mathsf{L}_m^{\mathbf{X}}|_{(\textbf{Aff Pairs})^\circ},$$

by this proposition we may conclude that $h_{J_m(\mathbf{X})} \cong \mathsf{L}_m^{\mathbf{X}}$ as functors on $(\textbf{Pairs})^\circ$. Before we move on to the second claim, let us suppose that given any pair $\mathbf{X}$ and any $m > 0$, there is a pair $J_m(\mathbf{X}) = (J_m^{\mathcal{D}}(X), J_m(\mathcal{D}))$ that represents $\mathsf{L}_m^{\mathbf{X}}$. Recall from an example in 3.2.2 that for a pair $\mathbf{Z} = (Z, \mathcal{F})$ and any $m' > m$, the truncation homomorphism $k[t]/(t^{m'}) \to k[t]/(t^m)$ induces a morphism $\eta_{\mathbf{Z}}^{m',m} : (Z \times j_m, \mathcal{F}^m) \to (Z \times j_{m'}, \mathcal{F}^{m'})$ of pairs. Using this, we can define a mapping

$$\pi_{\mathbf{Z}}^{m',m} : h_{J_{m'}(\mathbf{X})}(\mathbf{Z}) \to h_{J_m(\mathbf{X})}(\mathbf{Z})$$

by pulling back $\mathbf{Z}$-valued points of $J_{m'}(\mathbf{X})$ to $\mathbf{Z}$-valued points of $J_m(\mathbf{X})$ via $\eta_{\mathbf{Z}}^{m',m}$. That is, the $\mathbf{Z}$-valued point $\widetilde{\gamma}$ of $J_{m'}(\mathbf{X})$ corresponds to a unique m'-jet γ that we pull back with $\eta_{\mathbf{Z}}^{m',m}$ to an m-jet. This m-jet corresponds uniquely to a $\mathbf{Z}$-valued point of $J_m(\mathbf{X})$ that will be the image $\pi_{\mathbf{Z}}^{m',m}(\widetilde{\gamma})$ of $\widetilde{\gamma}$. We would like to show that these mappings on $\mathbf{Z}$-valued points define a morphism from $J_{m'}(\mathbf{X}) \to J_m(\mathbf{X})$ in the category **Pairs**. By Yoneda's lemma 2.1 this is equivalent to the following fact, which we prove:

Lemma 4.2. *The mappings $\pi_{\mathbf{Z}}^{m',m} : h_{J_{m'}(\mathbf{X})}(\mathbf{Z}) \to h_{J_m(\mathbf{X})}(\mathbf{Z})$ over all $\mathbf{Z}$ define a natural transformation $\pi^{m',m}$ between the functors $h_{J_{m'}(\mathbf{X})}$ and $h_{J_m(\mathbf{X})}$.*

Proof. Suppose that $\phi : \mathbf{Z} \to \mathbf{Y}$ is a morphism of pairs. We must show that the following diagram commutes:

$$\begin{array}{ccc} h_{J_{m'}(\mathbf{X})}(\mathbf{Z}) & \xleftarrow{h_{J_{m'}(\mathbf{X})}(\phi)} & h_{J_{m'}(\mathbf{X})}(\mathbf{Y}) \\ \downarrow{\scriptstyle \pi^{m',m}_{\mathbf{Z}}} & & \downarrow{\scriptstyle \pi^{m',m}_{\mathbf{Y}}} \\ h_{J_m(\mathbf{X})}(\mathbf{Z}) & \xleftarrow{h_{J_m(\mathbf{X})}(\phi)} & h_{J_m(\mathbf{X})}(\mathbf{Y}) \end{array}$$

This diagram will commute if and only if the next diagram commutes, since we have supposed that the pairs $J_m(\mathbf{X})$ and $J_{m'}(\mathbf{X})$ represent $\mathsf{L}^{\mathbf{X}}_m$ and $\mathsf{L}^{\mathbf{X}}_{m'}$ respectively:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Pairs}}((Z \times j_{m'}, \mathcal{F}^{m'}), \mathbf{X}) & \xleftarrow{(-)\circ\phi^{m'}} & \mathrm{Hom}_{\mathbf{Pairs}}((Y \times j_{m'}, \mathcal{E}^{m'}), \mathbf{X}) \\ \downarrow{\scriptstyle (-)\circ\eta^{m',m}_{\mathbf{Z}}} & & \downarrow{\scriptstyle (-)\circ\eta^{m',m}_{\mathbf{Y}}} \\ \mathrm{Hom}_{\mathbf{Pairs}}((Z \times j_m, \mathcal{F}^m), \mathbf{X}) & \xleftarrow{(-)\circ\phi^m} & \mathrm{Hom}_{\mathbf{Pairs}}((Y \times j_m, \mathcal{E}^m), \mathbf{X}) \end{array}$$

Recall that $\phi^m : (Z \times j_m, \mathcal{F}^m) \to (Y \times j_m, \mathcal{E}^m)$ is the morphism induced by ϕ, as we defined in section 3.2.2. Hence for any $\gamma : (Y \times j_{m'}, \mathcal{E}^{m'}) \to \mathbf{X}$, we must show that

$$\gamma \circ \phi^{m'} \circ \eta^{m',m}_{\mathbf{Z}} = \gamma \circ \eta^{m',m}_{\mathbf{Y}} \circ \phi^m,$$

which holds if $\phi^{m'} \circ \eta^{m',m}_{\mathbf{Z}} = \eta^{m',m}_{\mathbf{Y}} \circ \phi^m$. Now, both $\phi^{m'} \circ \eta^{m',m}_{\mathbf{Z}}$ and $\eta^{m',m}_{\mathbf{Y}} \circ \phi^m$ are morphisms from $Z \times j_m$ to $Y \times j_{m'}$, which are compatible with the projections $Y \times j_{m'} \to Y$ and $Y \times j_{m'} \to j_{m'}$. Hence by the universal mapping property of the fibred product $Y \times j_{m'}$, $\phi^{m'} \circ \eta^{m',m}_{\mathbf{Z}} = \eta^{m',m}_{\mathbf{Y}} \circ \phi^m$. □

So $\pi^{m',m}$ uniquely determines a morphism, which we call the *projection morphism from* $J_{m'}(\mathbf{X})$ *to* $J_m(\mathbf{X})$, and denote $\pi_{m',m}$. We may denote $\pi^{m',m}$ by $\pi^{\mathbf{X},m',m}$, and $\pi_{m',m}$ by $\pi^{\mathbf{X}}_{m',m}$ to avoid confusion when working with more than one projection. We will also denote $\pi^{m',0}$ by $\pi^{m'}$, and $\pi_{m',0}$ by $\pi_{m'}$. Note, of course, that this implies that we may alternatively express the pair $(J^{\mathcal{D}}_m(X), J_m(\mathcal{D}))$ as $(J^{\mathcal{D}}_m(X), \pi^*_m(\mathcal{D}))$. Our next result will help to prove the second claim made at the beginning of this section.

Proposition 4.3. *Let* $\mathbf{V} = (V, \mathcal{D}_V)$ *be an open subpair of the pair* $\mathbf{U} = (U, \mathcal{D}_U)$*. If there exists a pair* $J_m(\mathbf{U})$ *representing* $\mathsf{L}^{\mathbf{U}}_m$*, then there exists a pair* $J_m(\mathbf{V})$ *representing* $\mathsf{L}^{\mathbf{V}}_m$ *and* $(\pi^{\mathbf{U}}_m)^{-1}(\mathbf{V}) = J_m(\mathbf{V})$*.*

Proof. We will show that for any $\mathbf{Z} = (Z, \mathcal{F})$-valued m-jet $\gamma : (Z \times j_m, \mathcal{F}^m) \to \mathbf{U}$, γ factors through $\mathbf{V}$ if and only if the morphism $\widetilde{\gamma} : \mathbf{Z} \to J_m(\mathbf{U})$ corresponding to γ under the representation of $\mathsf{L}^{\mathbf{U}}_m$ factors through $(\pi^{\mathbf{U}}_m)^{-1}(\mathbf{V})$. First of all, we may suppose that $\mathbf{Z} = \mathrm{Spec}\, A$ is affine, given our proposition 4.1. Assume that γ factors through $\mathbf{V}$. The truncation homomorphism $A[t]/(t^{m+1}) \to A$ induces the morphism of pairs $\eta^m_{\mathbf{Z}} : \mathbf{Z} \to (\mathrm{Spec}\, A[t]/(t^{m+1}), \mathcal{F}^m)$ with which we pull back γ to $\pi^m(\gamma)$. Note that, by definition, pulling back γ to $\pi^m(\gamma)$ yields the same result as composing $\widetilde{\gamma}$ with π_m. Hence, the composition $\pi_m \circ \widetilde{\gamma}$ factors through V. From this we see that $\widetilde{\gamma}$ must factor through $(\pi_m^{\mathbf{U}})^{-1}(\mathbf{V})$.

Conversely, let $\widetilde{\gamma}$ be a $\mathbf{Z}$-valued point of $J_m(\mathbf{U})$ that factors through $(\pi_m^{\mathbf{U}})^{-1}(\mathbf{V})$. Noting that this implies that $\pi_m \circ \widetilde{\gamma}$ factors through $\mathbf{V}$, we obtain the following commutative square:

$$\begin{array}{ccc} \operatorname{Spec} A[t]/t^{m+1} & \xrightarrow{\gamma} & U \\ \uparrow{\scriptstyle \eta_{\mathbf{Z}}^m} & \searrow & \uparrow{\scriptstyle immersion} \\ \operatorname{Spec} A & \xrightarrow{\pi^m(\gamma)} & V \end{array}$$

where γ is the m-jet corresponding to $\widetilde{\gamma}$. We wish to show that γ factors through $\mathbf{V}$. But this is true because the open immersion $V \to U$ is formally étale, this property ensuring us a scheme morphism $\operatorname{Spec} A[t]/t^{m+1} \to V$ commuting with the square. Further, the pullback of $\mathcal{D}_V$ by this morphism exists and equals $\mathcal{F}^m$ (we are just pulling back $\mathcal{D}_U$ restricted to V). $\square$

This result yields the local isomorphisms needed to glue together the jet pair $J_m(\mathbf{X})$ of an arbitrary $\mathbf{X}$ from the jet pairs $J_m(\mathbf{X}_\alpha)$, where $\{\mathbf{X}_\alpha : \alpha \in A\}$ is an open cover of $\mathbf{X}$. Before we apply this result we will prove two more results. The second of these generalizes the statement of proposition 4.3 to the case of (formally) étale morphisms, hence provides another proof of proposition 4.3.

Proposition 4.4. *Let $\phi : \mathbf{Y} \to \mathbf{X}$ be a morphism of pairs, and suppose that $\mathsf{L}_m^{\mathbf{Y}}$ and $\mathsf{L}_m^{\mathbf{X}}$ are represented by $J_m(\mathbf{Y})$ and $J_m(\mathbf{X})$ respectively. Then ϕ induces a morphism $\phi_m : J_m(\mathbf{Y}) \to J_m(\mathbf{X})$ that commutes with the projections as in the following diagram:*

$$\begin{array}{ccc} J_m(\mathbf{Y}) & \xrightarrow{\phi_m} & J_m(\mathbf{X}) \\ \downarrow{\scriptstyle \pi_m^{\mathbf{Y}}} & & \downarrow{\scriptstyle \pi_m^{\mathbf{X}}} \\ \mathbf{Y} & \xrightarrow{\phi} & \mathbf{X} \end{array}$$

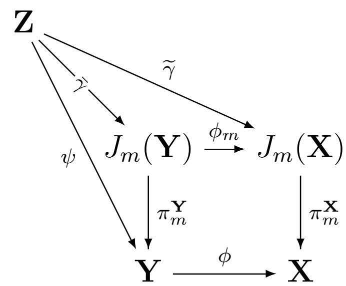
Proof. Similarly to the jet scheme case, we begin by choosing the $J_m(\mathbf{Y})$-valued point of $J_m(\mathbf{Y})$ given by $\mathrm{id}_{J_m(\mathbf{Y})}$, which corresponds to $\iota_{\mathbf{Y}} : (J_m^{\mathcal{E}}(Y) \times j_m, J_m(\mathcal{E})^m) \to \mathbf{Y}$. Composing $\iota_{\mathbf{Y}}$ with ϕ corresponds to the $J_m(\mathbf{Y})$-valued point of $J_m(\mathbf{X})$ that we denote ϕ_m. Then, because $\phi \circ \iota_{\mathbf{Y}}$ and ϕ_m correspond to each other under the representation of $\mathsf{L}_m^{\mathbf{X}}$, we know that $\pi_m^{\mathbf{X}} \circ \phi_m = \phi \circ \iota_{\mathbf{Y}} \circ \eta^m_{J_m(\mathbf{Y})}$ (the correspondence is trivial when $m = 0$). By the same reasoning $\pi_m^{\mathbf{Y}} \circ \mathrm{id}_{J_m(\mathbf{Y})} = \iota_{\mathbf{Y}} \circ \eta^m_{J_m(\mathbf{Y})}$, as $\iota_{\mathbf{Y}}$ corresponds to $\mathrm{id}_{J_m(\mathbf{Y})}$. Thus, $\pi_m^{\mathbf{X}} \circ \phi_m = \phi \circ \pi_m^{\mathbf{Y}} \circ \mathrm{id}_{J_m(\mathbf{Y})} = \phi \circ \pi_m^{\mathbf{Y}}$. So the diagram above commutes. $\square$

Proposition 4.5. *Let $\phi : \mathbf{Y} \to \mathbf{X}$ be an étale morphism of pairs, and suppose that $\mathsf{L}_m^{\mathbf{Y}}$ and $\mathsf{L}_m^{\mathbf{X}}$ are represented by $J_m(\mathbf{Y})$ and $J_m(\mathbf{X})$ respectively. Then, $J_m(\mathbf{Y}) \cong J_m(\mathbf{X}) \times_{\mathbf{X}} \mathbf{Y}$.*

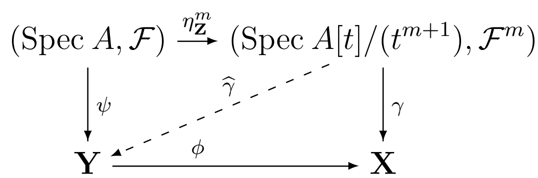
Proof. We will show that for any pair $\mathbf{Z}$, and every commutative square

$$\begin{array}{ccc} \mathbf{Z} & \xrightarrow{\widetilde{\gamma}} & J_m(\mathbf{X}) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \pi_m^{\mathbf{X}}} \\ \mathbf{Y} & \xrightarrow{\phi} & \mathbf{X} \end{array}$$

there exists a unique morphism $\bar{\gamma} : \mathbf{Z} \to J_m(\mathbf{Y})$ making the following diagram commutative:



Now, $\pi_m^{\mathbf{X}} \circ \widetilde{\gamma} = \gamma \circ \eta_{\mathbf{Z}}^m$, where γ is the jet corresponding to the point $\widetilde{\gamma}$. Without loss of generality, assume that $\mathbf{Z} = (\mathrm{Spec}\, A, \mathcal{F})$ is affine; the following diagram commutes:



Since $\phi : Y \to X$ is an étale morphism of schemes, γ factors through Y; *i.e.* there is a unique "scheme-jet" $\widehat{\gamma} : \mathrm{Spec}\, A[t]/(t^{m+1}) \to Y$ commuting with the square. This jet does indeed define a "pair-jet", since the local equations of the pullback commute around the bottom triangle in the opposite direction, and since ϕ and γ pull back the divisors. This jet $\widehat{\gamma}$ corresponds to the $\mathbf{Z}$-valued point we desire, $\bar{\gamma}$.

To verify that $\phi_m \circ \bar{\gamma} = \widetilde{\gamma}$, note that the jet corresponding to $\phi_m \circ \bar{\gamma}$ is $\phi \circ \widehat{\gamma}$. But this is exactly γ, hence $\widetilde{\gamma} = \phi_m \circ \bar{\gamma}$. Similarly, $\pi_m^{\mathbf{Y}} \circ \bar{\gamma} = \widehat{\gamma} \circ \eta_{\mathbf{Z}}^m$ because $\widehat{\gamma}$ is the jet corresponding to $\bar{\gamma}$. The latter equals ψ, so $\pi_m^{\mathbf{Y}} \circ \bar{\gamma} = \psi$. Thus, the second diagram is commutative; we conclude that $J_m(\mathbf{Y}) \cong J_m(\mathbf{X}) \times_{\mathbf{X}} \mathbf{Y}$. □

Now, let us suppose momentarily that given any affine pair $\mathbf{X}_\alpha = (X_\alpha, \mathcal{D}_\alpha)$ and any $m > 0$ the functor $\mathsf{L}_m^{\mathbf{X}_\alpha}$ is represented by $J_m(\mathbf{X}_\alpha)$. For $\mathbf{X} = (X, \mathcal{D}) \in$ **Pairs** let $\{\mathbf{X}_\alpha : \alpha \in A\}$ be an open cover by affine pairs. Then, according to proposition 4.3, for every α and β such that $X_\alpha \cap X_\beta \neq \emptyset$, both $(\pi_m^{\mathbf{X}_\alpha})^{-1}(\mathbf{X}_\alpha \cap \mathbf{X}_\beta)$ and $(\pi_m^{\mathbf{X}_\beta})^{-1}(\mathbf{X}_\alpha \cap \mathbf{X}_\beta)$ yield the pair $J_m(\mathbf{X}_\alpha \cap \mathbf{X}_\beta)$ representing $\mathsf{L}_m^{\mathbf{X}_\alpha \cap \mathbf{X}_\beta}$; that is, the preimages are canonically isomorphic to each other. These isomorphisms satisfy the conditions necessary to glue together a pair $J_m(\mathbf{X})$ from the various $J_m(\mathbf{X}_\alpha)$.

We claim that the pair we obtain this way represents $\mathsf{L}_m^{\mathbf{X}}$. Indeed, letting $\mathbf{Z} = (Z, \mathcal{F})$, given any m-jet $\gamma : (Z \times j_m, \mathcal{F}^m) \to (X, \mathcal{D})$, we can break up the jet into its restrictions $\gamma^{-1}(\mathbf{X}_\alpha) \to \mathbf{X}_\alpha$. Then, we can break up the 0^{th} truncation of γ into morphisms $(\eta_{\mathbf{Z}}^m)^{-1}(\gamma^{-1}(\mathbf{X}_\alpha)) = (\gamma \circ \eta_{\mathbf{Z}}^m)^{-1}(\mathbf{X}_\alpha) \to \mathbf{X}_\alpha$. Since $(\gamma \circ \eta_{\mathbf{Z}}^m)^{-1}(\mathbf{X}_\alpha)$ is an open subpair of $\mathbf{Z}$, we know that its preimage $p_{\mathbf{Z}}^{-1}((\gamma \circ \eta_{\mathbf{Z}}^m)^{-1}(\mathbf{X}_\alpha))$ under the projection $p_{\mathbf{Z}} : (Z \times j_m, \mathcal{F}^m) \to \mathbf{Z}$ is isomorphic to $((\gamma \circ \eta_Z^m)^{-1}(X_\alpha) \times j_m, \mathcal{F}^m|_{(\gamma \circ \eta_Z^m)^{-1}(X_\alpha) \times j_m})$. Thus we get a corresponding $(\gamma \circ \eta_{\mathbf{Z}}^m)^{-1}(\mathbf{X}_\alpha)$-valued point of $J_m(\mathbf{X}_\alpha)$. Since these points must agree on overlaps $X_\alpha \cap X_\beta \neq \emptyset$ and the preimages $(\gamma \circ \eta_{\mathbf{Z}}^m)^{-1}(\mathbf{X}_\alpha)$ cover $\mathbf{Z}$, we obtain a unique $\mathbf{Z}$-valued point of $J_m(\mathbf{X})$ corresponding to γ by gluing the domain of these morphisms. Note that all the divisors pulled back locally throughout, and that funtoriality follows from the fact that these correspondences are functorial locally on the pairs.

We can now prove that $\mathsf{L}_m^{\mathbf{X}}$ is a representable functor for every $m > 0$ when $\mathbf{X}$ is any pair.

Theorem 4.6. *Let* $\mathbf{X} = (X, \mathcal{D}) \in \mathbf{Pairs}$. *For every* $m \geq 0$ *the contravariant functor* $\mathsf{L}_m^{\mathbf{X}} : (Y, \mathcal{E}) \mapsto \mathrm{Hom}_{\mathbf{Pairs}}((Y \times j_m, \mathcal{E}^m), (X, \mathcal{D}))$ *from* **Pairs** *to* **Sets** *is representable, represented by a pair* $J_m(\mathbf{X}) = (J_m^{\mathcal{D}}(X), J_m(\mathcal{D}))$.

Proof. The case $m = 0$ is trivial, so let $m > 0$. By our previous results we may restrict to the case $\mathbf{X} = (\mathrm{Spec}\, A, \mathcal{D})$ is affine and the domain of $\mathsf{L}_m^{\mathbf{X}}$ is **Aff Pairs**. So let $(Y, \mathcal{E}) = (\mathrm{Spec}\, B, \mathcal{E})$ and let $\gamma : (Y \times j_m, \mathcal{E}^m) \to (X, \mathcal{D})$ be an m-jet in $\mathbf{X}$. Thus $\gamma : \mathrm{Spec}\, B[t]/(t^{m+1}) \to \mathrm{Spec}\, A$ has corresponding homomorphism $\gamma^* : A \to B[t]/(t^{m+1})$. We wish to describe a scheme with a B-valued point corresponding uniquely to γ. We will break up the remainder of the proof into two cases.

Case 1. Assume that $A = k[X_1, \ldots, X_n]$ is affine n-space for some $n > 0$, that $r \leq n$, and that $D_i = \{(\mathrm{Spec}\, A, X_i)\}$ for $1 \leq i \leq r$. Then the homomorphism γ^* is determined exactly by the values $\gamma^*(X_1), \ldots, \gamma^*(X_n)$. Let $\gamma^*(X_i) = b_{i0} + b_{i1}t + \cdots + b_{im}t^m$. Since X_i is the local equation for D_i when $1 \leq i \leq r$, $\gamma^*(X_i)$ is the local equation for E_i^m. But this local equation is b_{i0}, hence $b_{i0} \cdot u_i = b_{i0} + b_{i1}t + \cdots + b_{im}t^m$ for some invertible regular section u_i. Writing $u_i = u_{i0} + u_{i1}t + \cdots + u_{im}t^m$ we see that $u_{i0} = 1$ and $b_{i0} \cdot u_{il} = b_{il}$ for each $1 \leq l \leq m$. Hence, the value $\gamma^*(X_i)$ is determined by the values $b_{i0}, u_{i1}, u_{i2}, \ldots, u_{im}$. For $r + 1 \leq i \leq n$, the value of $\gamma^*(X_i)$ is simply determined by $b_{i0}, b_{i1}, b_{i2}, \ldots, b_{im}$.

Now let us write the affine coordinate ring in $n(m + 1)$ variables as

$$C = k[X_1^{(0)}, \ldots, X_n^{(0)}, \frac{X_1^{(1)}}{X_1}, \ldots, \frac{X_r^{(1)}}{X_r}, X_{r+1}^{(1)}, \ldots, X_n^{(1)}, \ldots, \frac{X_1^{(m)}}{X_1}, \ldots, \frac{X_r^{(m)}}{X_r}, X_{r+1}^{(m)}, \ldots, X_n^{(m)}].$$

Then γ^* determines a unique homomorphism $\widetilde{\gamma}^* : C \to B$ sending

$$X_i^{(0)} \mapsto b_{i0}, \quad \forall 1 \leq i \leq n,$$

$$\frac{X_i^{(l)}}{X_i} \mapsto u_{il}, \quad \forall 1 \leq i \leq r, 1 \leq l \leq m,$$

and

$$X_i^{(l)} \mapsto b_{il}, \quad \forall r + 1 \leq i \leq n, 1 \leq l \leq m.$$

Thus we let $J_m^{\mathcal{D}}(X) = \mathrm{Spec}\, C$, and we let $J_m(\mathcal{D}) = (\{(\mathrm{Spec}\, C, X_1)\}, \ldots, \{(\mathrm{Spec}\, C, X_r)\})$. It is immediate that $\widetilde{\gamma} : (\mathrm{Spec}\, B, \mathcal{E}) \to (J_m^{\mathcal{D}}(X), J_m(\mathcal{D}))$ is indeed a morphism of pairs and that this correspondence is functorial; note that for any morphism $\phi : (\mathrm{Spec}\, S, \mathcal{F}) \to (\mathrm{Spec}\, B, \mathcal{E})$ the homomorpism $(\phi^m)^*$ maps $B[t]/(t^{m+1}) \to S[t]/(t^{m+1})$ such that $b_0 + b_1t + \cdots + b_mt^m \mapsto \phi^*(b_0) + \phi^*(b_1)t + \cdots + \phi^*(b_m)t^m$. This guarantees functoriality, and so we see that $\mathsf{L}_m^{\mathbf{X}} \cong h_{(J_m^{\mathcal{D}}(X), J_m(\mathcal{D}))}$ when $\mathbf{X} = (\mathbb{A}_k^n, (\{(\mathbb{A}_k^n, X_1)\}, \ldots, \{(\mathbb{A}_k^n, X_r)\}))$.

Case 2. Let $A = k[X_1, \ldots, X_n]/(f_1, \ldots, f_s)$ and suppose that for every i, $1 \leq i \leq r$, D_i is defined by $\overline{g_i}$ on X, where $g_i \in k[X_1, \ldots, X_n]$. A homomorphism from the polynomial ring in $n+r$ variables $R = k[X_1, \ldots, X_n, W_1, \ldots, W_r]$ to A sending each X_i to $\overline{X_i}$ is onto. Consider the homomorphism

$$k[X_1, \ldots, X_n, W_1, \ldots, W_r] \to k[X_1, \ldots, X_n]/(f_1, \ldots, f_s)$$

such that

$$X_1 \mapsto \overline{X_1}, \ldots, X_n \mapsto \overline{X_n}$$

and

$$W_1 \mapsto \overline{g_1}, \ldots, W_r \mapsto \overline{g_r}.$$

This makes $X = \operatorname{Spec} A$ into a closed immersion in $\operatorname{Spec} R = \mathbb{A}^{n+r}$ cut out by the ideal

$$I = (f_1, \ldots, f_s, W_1 - g_1, \ldots, W_r - g_r)$$

(here is it important to notice that $R/(f_1, \ldots, f_s) \cong A[W_1, \ldots, W_r]$). Under this k-algebra isomorphism $R/I \leftrightarrow A$, the local equations $\overline{g_1}, \ldots, \overline{g_r}$ map to $\overline{W_1}, \ldots, \overline{W_r}$ respectively, hence this isomorphism of schemes defines an isomorphism of pairs

$$(\operatorname{Spec} R/I, (\{(\operatorname{Spec} R/I, \overline{W_1})\}, \ldots, \{(\operatorname{Spec} R/I, \overline{W_r})\})) \to (X, \mathcal{D}).$$

Thus, in this case we will define the desired parameter space for X as a closed immersion in the parameter space for $(\mathbb{A}^{n+r}, (\{(\mathbb{A}^{n+r}, W_1)\}, \ldots, \{(\mathbb{A}^{n+r}, W_r)\}))$.

By the arguments made in the first case this latter space is the pair consisting of the scheme $\mathbb{A}^{(n+r)(m+1)}$ with the r-tuple of divisors defined on $\mathbb{A}^{(n+r)(m+1)}$ by the $W_i^{(0)}$'s. To find the equations for the ideal of $J_m^{\mathcal{D}}(X)$ we must consider m-jets in $\mathbb{A}^{n+r}$ that factor through X. A $(\operatorname{Spec} B, \mathcal{E})$-valued m-jet in $(X, \mathcal{D})$ is determined by a homomorphism

$$\gamma^* : k[X_1, \ldots, X_n, W_1, \ldots, W_r] \to B[t]/(t^{m+1})$$

such that $\gamma^*(W_i)$ is the non-zerodivisor locally defining the "i^{th}" effective Cartier divisor E_i^m of $\operatorname{Spec} B[t]/(t^{m+1})$, and such that $\gamma^*(f_j) = 0$ and $\gamma^*(W_i - g_i) = 0$ for every $1 \leq j \leq s$ and $1 \leq i \leq r$. Such a homomorphism is completely determined by the coefficients of $\gamma^*(X_1), \ldots, \gamma^*(X_n)$ and $\gamma^*(W_1) \ldots, \gamma^*(W_r)$, hence γ^* defines a homomorphism as we expect from

$$k[X_1^{(0)}, \ldots, X_n^{(0)}, W_1^{(0)}, \ldots, W_r^{(0)}, X_1^{(1)}, \ldots, X_n^{(1)}, W_1^{(1)}, \ldots, W_r^{(1)}, \ldots, X_1^{(m)}, \ldots, W_r^{(m)}] \to B.$$

Given the condition on pullbacks, just as in the first case the degree 0 coefficient of $\gamma^*(W_i)$ divides the coefficients of the higher degree terms, hence the m-jet is equivalently determined by a homomorphism

$$k[X_1^{(0)}, \ldots, X_n^{(0)}, W_1^{(0)}, \ldots, W_r^{(0)}, X_1^{(1)}, \ldots, X_n^{(1)}, \frac{W_1^{(1)}}{W_1}, \ldots, \frac{W_r^{(1)}}{W_r}, \ldots, X_1^{(m)}, \ldots, \frac{W_r^{(m)}}{W_r}] \to B.$$

Let us denote the domain of this homomorphism as S. We write $\gamma^*(X_i) = b_{i0} + b_{i1}t + \cdots + b_{im}t^m$ for $1 \leq i \leq n$. Then $\gamma^*(f_j) = f_{j0} + f_{j1}t + \cdots + f_{jm}t^m$ for $1 \leq j \leq s$, where for each $0 \leq l' \leq m$ the coefficient $f_{jl'}$ is a polynomial in $(b_{il})_{1\leq i\leq n, 1\leq l\leq m}$. Thus, we consider each $f_{jl'}$ as a polynomial in $(X_i^{(l)})_{i,l}$; the condition $\gamma^*(f_j)$ translates in terms of the homomorphism from S to B into $f_{jl'} \mapsto 0$ for all $1 \leq j \leq s$ and $0 \leq l' \leq m$.

Similarly, writing $\gamma^*(W_i) = c_{i0}+c_{i0}\cdot u_{i1}t+\cdots+c_{i0}\cdot u_{im}t^m$ and $\gamma^*(g_i) = g_{i0}+g_{i1}t+\cdots+g_{im}t^m$ for $1 \leq i \leq r$, the condition on γ^* indicates that $c_{i0} = g_{i0}$ and $c_{i0} \cdot u_{il'} = g_{il'}$ for every $1 \leq l' \leq m$. This time considering $g_{il'}$ as a polynomial in $(X_i^{(l)})_{1\leq i\leq n, 1\leq l\leq m}$, we must have $g_{il'} - W_i^{(0)} \cdot \frac{W_i^{(l')}}{W_i^{(0)}} \mapsto 0$ for all $1 \leq i \leq r$ and $0 \leq l' \leq m$.

Hence, the $(\operatorname{Spec} B, \mathcal{E})$-valued m-jets on $(X, \mathcal{D})$ are parametrized by points in the closed immersion of schemes

$$\operatorname{Spec} S/(f_{jl}, g_{i0} - W_i^{(0)}, g_{il'} - W_i^{(0)} \cdot \frac{W_i^{(l')}}{W_i^{(0)}} : 1 \leq j \leq s, 0 \leq l \leq m, 1 \leq i \leq r, 1 \leq l' \leq m)$$

in $\mathbb{A}^{(n+r)(m+1)}$. Thus, for every $\mathbf{Y}$-valued point of $J_m(\mathbb{A}^{n+r})$ corresponding to an m-jet that factors through $\mathbf{X}$, its underlying scheme morphism factors through the closed immersion we have just described. However, the equations W_i on $\mathbb{A}^{(n+r)(m+1)}$ may not pull back to non-zerodivisors in this closed immersion. In order to obtain the pair $(J_m^{\mathcal{D}}(X), J_m(\mathcal{D}))$ representing $\mathsf{L}_m^{\mathbf{X}}$ we may need to remove components, as in lemma 2.2; it is immediate that doing this yields a pair with the appropriate functor of points.

□

We make the following definition:

Definition 4.1. We call the pair $J_m(\mathbf{X})$ associated to $\mathbf{X}$ from theorem 4.6 the *jet pair associated to the pair* $\mathbf{X}$. We refer to the scheme $J_m^{\mathcal{D}}(X)$ underlying $J_m(\mathbf{X})$ as the *logarithmic jet scheme of X with respect to $\mathcal{D}$.*

Remark 4.1. Let $(X, \mathcal{D})$ be a pair over the field k, and suppose that $\operatorname{char}(k) = 0$. We can describe the equations of the ideal of $J_m^{\mathcal{D}}(X)$ more explicitly. First, let S be the ring

$$S = k[X_1^{(0)}, \ldots, W_r^{(0)}, X_1^{(1)}, \ldots, \frac{W_r^{(1)}}{W_r}, \ldots, X_1^{(m)}, \ldots, \frac{W_r^{(m)}}{W_r}].$$

As in the jet scheme case outlined in the introduction, there is a k-derivation d on S determined as follows: $dX_i^{(l)} = X_i^{(l+1)}$ where $X_i^{(l)} = 0$ for all $l > m$, and $dW_i^{(l)} = W_i^{(l+1)}$ where $W_i^{(l)} \stackrel{\text{def}}{=} W_i^{(0)} \cdot \frac{W_i^{(l)}}{W_i}$ for all $1 \leq l \leq m$ and $W_i^{(l)} = 0$ for all $l > m$. By the same arguments as in the jet scheme case we find that the equations f_jl' all map to 0 if and only if the equations $d^{l'}f_j$ all map to 0, where we consider f_j as a polynomial in $X_1^{(0)}, \ldots, X_n^{(0)}$. Similarly, since $d^{l'}(g_i - W_i^{(0)}) = d^{l'}g_i - d^{l'}W_i^{(0)}$ we find that $g_{i0} - W_i^{(0)}$ and $g_{il'} - W_i^{(0)}$ all map to 0 if and only if the equations $d^{l'}(g_i - W_i^{(0)})$ all map to 0. Thus we have an equality of ideals

$$(f_{jl'}, g_{i0}-W_i^{(0)}, g_{il'}-W_i^{(0)}\cdot\frac{W_i^{(l')}}{W_i^{(0)}}) = (f_j, df_j, \ldots, d^m f_j, g_i-W_i^{(0)}, d(g_i-W_i^{(0)}), \ldots, d^m(g_i-W_i^{(0)})).$$

5 Conclusion

5.1 Summary

Following the constructive method of providing a proof for the existence of jet schemes $J_m(X)$ associated to a scheme X of finite type over an algebraically closed field k as in [EM08], [Mus01], [Ish07], we have provided a constructive proof of the existence of logarithmic jet schemes $J_m^{\mathcal{D}}(X)$ associated to X and its family of effective Cartier divisors $\mathcal{D} = (D_1, \ldots, D_r)$.

This was carried out in four major steps as follows: after providing basic definitions for the objects we would work with throughout our paper, we first formulated the categories of pairs $\mathbf{Pairs}_r$, whose objects $(X, \mathcal{D})$ consist of a scheme X of finite type over a fixed algebraically closed field k and its r-tuple of effective Cartier divisors, and whose morphisms $(X, \mathcal{D}) \to (Y, \mathcal{E})$ are those scheme morphisms $\phi : Y \to X$ that "pull back" $\mathcal{D}$ to $\mathcal{E}$; second, we defined the functors $\mathsf{L}_m^{(X,\mathcal{D})}$ taking a "pair" $(Y, \mathcal{E})$ to the set of "$(Y, \mathcal{E})$-valued m-jets in $(X, \mathcal{D})$"; third, we proved that the representability of such functors can be determined by the case of affine $(X, \mathcal{D})$ and $(Y, \mathcal{E})$; finally, we gave explicit equations for a pair representing the functor $\mathsf{L}_m^{(X,\mathcal{D})}$.

The question of representability of such functors, or equivalently of parametrizability of such families of m-jets, was motivated by the construction of the sheaf of differential 1-forms with logarithmic poles along a normally crossing divisor on a complex-analytic variety, and the possibility of framing such a construction functorially, as the sheaf of differetial 1-forms finds expression in jet schemes.

5.2 Discussion and Further Research

Referring back to the definition 2.2.3, notice that a predivisor on the scheme X is, by definition, a presentation of a global section of the quotient sheaf of commutative monoids $\mathcal{O}_X/\mathcal{O}_X^*$. It is straightforward to define the pullback $\phi^*(D)$ of a global section D of this sheaf by a morphism $Y \to X$ of schemes (of finite type over k) as we have done for effective Cartier divisors. Let us refer to such an object D temporarily as an *effective divisor*. With this notion at hand, we might choose to work in a category whose objects are pairs $(X, \mathcal{D})$, where now $\mathcal{D} = (D_1, \ldots, D_r)$ is an r-tuple of effective divisors on X, and whose morphisms $(Y, \mathcal{E}) \xrightarrow{\phi} (X, \mathcal{D})$ pull back $\mathcal{D}$ to $\phi^*(\mathcal{D}) \stackrel{\text{def}}{=} (\phi^*(D_1), \ldots, \phi^*(D_r)) = \mathcal{E}$. One may verify that the proofs supplied in section 4 carry over to this category word-for-word, with the exception of omitting some justifications that certain pullbacks of sections do not locally divide zero.

The geometric significance in this choice of a category lies in that rather than only parametrizing jets that "avoid" the family $\mathcal{D}$ of effective Cartier divisors, we parametrize also the jets that are "tangential along" the family $\mathcal{D}$ of effective divisors. For example, letting $\mathcal{D}$ consist of the single effective Cartier divisor defined on $X = \mathbb{A}^2$ globally by $X_1 \cdot X_2$, one shows in the first case that the fibre of the projection $\pi_1 : J_1^{\mathcal{D}}(\mathbb{A}^2) \to \mathbb{A}^2$ above the origin $(0, 0)$ (or above any point on the X_1 or X_2 axis) is empty, whereas in the second case we have $\pi_1^{-1}((0, 0)) \cong \mathbb{A}^2$ (while $\pi_1^{-1}((a, b)) \cong \mathbb{A}^1$ for any (a, b) with $a = 0, b \neq 0$ or $a \neq 0, b = 0$).

Though this adjustment to the categories $\mathbf{Pairs}_r$ immediately yields some interesting geometric objects, it is likely that there is an even better category in which to formulate our results and construct such objects. Namely, we expect that the natural context for the logarithmic jet schemes lies in the category of "schemes with logarithmic structures", on which foundational material was developed by Fontaine-Illusie and Kato (see for example [Kat94] and especially [Kat89]). In particular it seems that this formalism supplies a language for working with (sheaves of) monoids attached to schemes, and will hopefully carry over the idea we have just mentioned. Once the transfer to this language is complete, we hope in particular to apply the geometry of the logarithmic jet schemes to the study of singularities; we allude in particular to such results as contained in the work of Mustaţă in [Mus01]. Of course, the jet schemes are fundamental to the theory of motivic integration, and we also hope to study the logarithmic jets in this context.

References

[Del71] Pierre Deligne, *Théorie de Hodge : II*, Publications mathématiques de l'I.H.É.S. **40** (1971), 5 – 57.

[EH00] David Eisenbud and Joe Harris, *The Geometry of Schemes*, GTM, vol. 197, Springer-Verlag New York, Inc., 2000.

[Eis04] David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, GTM, vol. 150, Springer Science+Business Media, Inc., 2004.

[EM08] Lawrence Ein and Mircea Mustaţă, *Jet Schemes and Singularities*, arXiv:math/0612862v2 [math.AG], 2008.

[FmI+05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*, Mathematical Surveys and Monographs, vol. 123, The American Mathematical Society, 2005.

[GH94] Philip Griffiths and Joe Harris, *Principles of Algebraic Geometry*, Wiley Classics Library, John Wiley & Sons, Inc., 1994.

[Gro67] Alexander Grothendieck, *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : IV. Étude locale des schémas et des morphismes de schémas, Quatriéme partie*, Publications mathématiques de l'I.H.É.S. **32** (1967), 5 – 361.

[Har06] Robin Hartshorne, *Algebraic Geometry*, GTM, vol. 52, Springer Science+Business Media, LLC, 2006.

[Ish07] Shihoko Ishii, *Jet Schemes, Arc Spaces and the Nash Problem*, arXiv:0704.3327v1 [math.AG], 2007.

[Kat89] Kazuya Kato, *Logarithmic Structures of Fontaine-Illusie*, Algebraic Analysis, Geometry, and Number Theory (J. Igusa, ed.), Johns Hopkins University Press, 1989, pp. 191 – 224.

[Kat94] ______, *Toric singularities*, American Journal of Mathematics **116** (1994), no. 5, 1073 – 1099.

[Mus01] Mircea Mustaţă, *Jet Schemes of Locally Complete Intersection Canonical Singularities*, Inventiones Mathematicae **145** (2001), no. 3, 397 – 424.