EQUIVARIANT CHOW GROUPS AND MULTIPLICITIES

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Abstract

We propose a definition of equivariant Chow groups for schemes with a torus action and develop the intersection theory related to it. The equivariant intersection theories that have been considered in the past have been the Chow groups and the K-theory of the quotient scheme, as well as the equivariant K-groups of the original scheme. The equivariant Chow groups are related to all of these. At first glance, we would expect a strong relationship with the equivariant K-groups. As it turns out, the equivariant Chow groups are more closely related to the Chow groups of the quotient scheme.

We chose to restrict to tori since for them the equivariant cycles are of a particularly nice form. For general groups the equivariant cycles are harder to describe, and so the intersection theory is far messier, if it even exists. By restricting to tori, we are able to define an equivariant multiplicity that behaves similarly to the degree in the projective case. In particular, we are able to show that for certain schemes, the equivariant multiplicity of an equivariant cycle in the equivariant Chow group is defined and is an invariant of that cycle.

While much of this work involves generalizing the work of others, in particular the work of Fulton, Rossmann and Borho, Brylinski and Macpherson, our approach is new. The equivariant Chow groups have not been considered in the past and relating the equivariant multiplicities to the equivariant Chow groups is new as well.

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Chapter 1

Introduction

In this work, we propose a definition of equivariant Chow groups and work out the intersection theory associated to it. For varieties with a group action, the equivariant intersection theories considered in the past have involved the Chow and K-groups of the quotient varieties as well as equivariant K-theory on the original variety. The Chow group we propose is related to all of these. With respect to $K_T(X)$, the equivariant Chow groups provide a geometric interpretation similar to that existing between the usual Chow and K-groups. Unfortunately, the analogue is not quite as strong as we would like. As it turns out, the equivariant Chow groups are more closely related to the Chow groups of the quotient variety. For these, the main advantage is that the equivariant Chow groups are defined on the original variety rather than the quotient, and so are easier to calculate. In defining the equivariant Chow groups, we have followed Fulton [6] very closely. By and large, the only difference between his work and ours is that we need to check for equivariance.

The groups we consider almost exclusively are tori. The reason for this is that cycles stable under a torus action are of a particularly nice form, and the intersection theory is fairly clean. In the last chapter we do consider briefly the case of reductive groups. The problem with a group G with maximal torus T is that the G stable are not particularly nice, so that any type of rational equivalence that respects the group action is quite messy. We also find that these groups are subgroups of the equivariant Chow groups with respect to the torus.

By restricting to tori, we are also able to avail ourselves of an equivariant multiplicity similar to the degree in the projective case. In particular, for varieties of a certain form we are able to show that the equivariant multiplicity of a cycle at a fixed point is an invariant of that cycle.

So, along with the Chow groups, we also develop equivariant multiplicities. Given a projective variety X stable under a torus action and an equivariant vector bundle E defined on X, we are able to obtain in a purely combinatorial manner a characteristic number formula relating the weights of E over the fixed points to the geometric multiplicity.

Since we are proposing a new definition, very little has been done on equivariant Chow groups. The individual pieces have been covered extensively, though. Quotient varieties have been considered in the past by many authors. In particular, for intersection theory, we have Kraft [11] and Knopf who have considered vector bundles on the quotient varieties, Danilov [4] and Ellingsrud and Stromme [5] who have found the Chow groups of various quotient varieties. Equivariant multiplicities have been considered by Rossmann [17] who has calculated them on subvarieties of smooth varieties with torus action, and by Joseph and Borho, Brylinski and Macpherson [1] who consider the quotient variety $X = G/B$, where G is a Lie group. In fact, Borho, Brylinski and Macpherson relate the equivariant multiplicity to $K_T(X)$. All these references work on the tangent space of the variety though. We, however, define the equivariant multiplicity on the variety itself. Finally, intersection theory as developed by Fulton [6] plays an important role in our work. In fact, except for the equivariant multiplicity results, we have essentially translated his results from the non-equivariant setting to the equivariant one. With respect to $K_T(X)$, we have Nielsen [16] and Iversen and Nielsen [9]. The former has shown the localization theorem for non-singular projective varieties, and the latter has arrived at a formula valid for 1-dimensional tori involving the Chern classes of a vector bundle and the equivariant multiplicity of a zero cycle. A similar result has been shown by Brion [3].

In the second chapter, we introduce the notation, definitions, results, and constructions we shall be using throughout. These are the same as those of usual intersection theory, but they are done in the equivariant setting.

We define the notion of equivariant Chow group in the third chapter. We show that the properties Fulton considers in [6] Chapter 1 hold in the equivariant case. We also consider how the equivariant Chow groups change as we change tori. This allows us to relate the equivariant

Chow groups for different tori.

In the fourth chapter, we consider formal characters and equivariant multiplicities of modules. Since we need equivariant multiplicity in fairly great generality, we consider a definition of the formal characters that is valid only under certain conditions, but which is valid independent of the sign (or zero-ness) of the weights. This allows us to define the equivariant multiplicity under certain conditions independent of what the weights may be.

In the fifth chapter, we consider equivariant multiplicities on varieties. We show that for projective varieties this is defined for T-stable cycles and is an invariant of the cycle in the equivariant Chow groups. We also see how multiplicities behave with respect to various maps, and we derive a few equations concerning multiplicities on vector bundles. For non-projective varieties of a particular form, this allows us to show that the equivariant multiplicity is also an invariant of the cycle in the equivariant Chow group.

We define equivariant intersection with an equivariant line bundle in the sixth chapter. In the third chapter, the generalization of Fulton [6] was fairly straightforward. In this chapter, the differences between equivariant and usual intersection theory begin to show. We define the intersection as in Fulton [6]: to a line bundle we associate a section, a Cartier divisor, and a Weil divisor. We have several possibilities of definitions of equivariant intersection, depending on what conditions we place on the section. We chose to demand that it be equivariant. The advantage of this choice is that equivariant intersections then have equivariant properties. In particular, we are able to find the equivariant multiplicity of the intersection in terms of the weights of the line bundle and the equivariant multiplicity of the cycle. The problem we have is that an equivariant section need not exist. This means that the properties that Fulton shows in [6] need not hold in all generality. We show they do hold, subject to the existence of sections on the appropriate varieties.

In the seventh chapter, we consider intersections with equivariant vector bundles. As in the previous chapter, we have problems with the existence of sections. This forces us to restrict the vector bundles we can intersect with. Because of the conditions we must place on these bundles, it does not seem worthwhile to define an equivariant intersection of cycles on a variety with T-action. We also consider equivariant multiplicities with respect to these intersections. We derive some characteristic number formulae for these.

In the eighth chapter, we consider previous work that relates to our work. We show that if the action is free, the Chow group of the quotient is the equivariant Chow group, after tensoring wih $\mathbf{Q}$.We also consider how the equivariant Chow group is related to the equivariant K-groups. This shows that equivariant multiplicity is also an invariant of the quotient variety.

Chapter 2

Basic Constructions

We collect the basic definitions, results and constructions we shall be using throughout this work. These are the usual ones of algebraic geometry, but we do them in the equivariant setting. The constructions are cones, blowing-up and vector bundles. These have all been done in Fulton [6] appendices A and B.

2.1 Notation

Notation:

k is an algebraically closed field

T is an r-dimensional torus with Lie algebra τ

$\chi : T \to G_m$ is a character of T

$\mathbf{X}(T)$ is the set of characters

$R(T) = \mathbf{Z}[\mathbf{X}(T)]$ is the representation ring

$\lambda = d\chi : \tau \to k$ is the differential of χ, called the weight of χ

If T$= \oplus_{i=1}^{r} G_m$, we will occasionally write λ as $(\lambda_1, \lambda_2, \ldots, \lambda_r)$.

$\Lambda(T)$ is the lattice of weights

In general, if f is a weight vector of a module M with T-action, we will write its weight as λ_f, and character as χ_f.

The main schemes we will be considering are subschemes of $\mathbf{P}^n$ and $\mathbf{A}^n$.

Let $\mathbf{P}^n$ be the projectivization of $\mathbf{A}^{n+1} = Spec(k[x_0, \ldots, x_n])$, where x_i is a weight vector of weight $-\lambda_i$. Let $P_i = (0 : \ldots : 0 : 1 : 0 \ldots : 0)$ be the point in $\mathbf{P}^n$ with a 1 in the i-th position. Let $U_i = U_{x_i}$. We also assume that $x \in (\mathbf{P}^n)^T$ is P_0.

If $\mathbf{P}^1$ has the T-action defined by the weight λ, we will write it as $\mathbf{P}^1_\lambda$. Unless otherwise stated, we will be assuming that $\mathbf{P}^1_\lambda = Proj(k[x_0, x_1])$, where x_0 has weight 0 and x_1 has weight λ. Unless otherwise stated, $\mathbf{P}^1$ will be $\mathbf{P}^1_0$.

If $\mathbf{A}^n$ has weights $\lambda_i = (\lambda_{1i}, \ldots, \lambda_{ni})$, we will occasionally write these as

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_{r1} & \lambda_{r2} & \ldots & \lambda_{rn}. \end{pmatrix} \tag{2.1}$$

While the form of the matrix may seem odd, it turns out that the kernel of the linear transformation defined by this matrix is the cycles T-rationally equivalent to 0 in $\mathbf{A}^n$.

2.2 Elementary Definitions and Results

Definition: X is a T-scheme if X has a T-action defined on it. We write the T-action as

$$\begin{aligned} \sigma : T \times X &\to X \\ (t, x) &\mapsto t \cdot x. \end{aligned} \tag{2.2}$$

In general, we assume that all varieties are defined over the field k and are irreducible and reduced. We will also be assuming that there is a cover of X by open affine T-subsets U such that U is a T-subscheme of $\mathbf{A}^m$ for some m. We make this assumption so that $\mathcal{O}_U$ has a T-grading. Note that if X is normal, then Sumihiro's Theorem shows that X is of this form.

Remark: The T-action on $\mathbf{P}^1$ is defined by a character. So, the set of T-actions on $\mathbf{P}^1$ is in 1-1 correspondence with $\mathbf{X}(T)$.

Definition: $f : X \to Y$ is a T-morphism if $f(t \cdot x) = t \cdot f(x)$ in Y. We occasionally call such morphisms equivariant morphisms.

Remark: Note that the set of weight vectors $f \in R(X)^*$ of weight λ is in 1-1 correspondence with the maps $f : X \to \mathbf{P}^1_\lambda$. In fact, if $f^* : \mathcal{O}_{\mathbf{P}^1_\lambda} \to \mathcal{O}_X$ is the morphism on structure sheaves, then $f \in R(X)^*$ is defined by $f^*(x_1/x_0)$.

Proposition 2.2.1 *If X is a T-scheme, then its normalization is a T-scheme.*

Proof: This is a local assertion. Suppose that X is an affine T-scheme and $A = \mathcal{O}_X$. Let K be the quotient field of A and let A' be the integral closure of A in K. Since A has a T-action, K does as well. We want to show that A' is invariant under the T-action. Suppose that $r \in A'$ solves the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0. \tag{2.3}$$

Since

$$(t \cdot r)^n + t \cdot a_{n-1}(t \cdot r)^{n-1} + \cdots + t \cdot a_0 = t \cdot (r^n + a_{n-1}r^{n-1} + \cdots + a_0) = 0, \tag{2.4}$$

$t \cdot r \in K$ solves the equation

$$x^n + t \cdot a_{n-1}x^{n-1} + \cdots + t \cdot a_0 = 0. \tag{2.5}$$

So, $t \cdot r \in A'$, and A' is T invariant. □

Lemma 2.2.2 *Let $f \in R(X)^*$ be a weight vector of T. If U is an open affine T-subset of X, there are weight vectors $a, b \in \mathcal{O}_X(U)$ such that $f = a/b$.*

Proof: We proceed by induction. Note that for the trivial torus, the result is obvious. Suppose the result is true for $\dim T' = n - 1$, and $T = T' \times G_m$. Then, locally $f = a/b$ where a, $b \in \mathcal{O}_X(U)$ are weight vectors of T'. G_m induces a grading on $\mathcal{O}_X(U)$, so $a = \sum_{i \in \mathbf{Z}} a_i$, $b = \sum_{i \in \mathbf{Z}} b_i$, and all but a finite number of the a_i and b_i are zero. Let i_0 and j_0 be the smallest integers such that $a_{i_0} \neq 0$ and $b_{j_0} \neq 0$. Then,

$$\frac{a}{b} = \frac{a_{i_0} + \sum_{i > i_0} a_i}{b_{j_0} + \sum_{j > j_0} b_j}. \tag{2.6}$$

So, if f has weight m with respect to G_m,

$$t \cdot \frac{a}{b} = t^m \frac{a}{b} = \frac{t^{i_0} a_{i_0} + \sum t^i a_i}{t^{j_0} b_{j_0} + \sum t^j b_j} = t^{i_0 - j_0} \frac{a_{i_0} + \sum t^{i-i_0} a_i}{b_{j_0} + \sum t^{j-j_0} b_j}. \tag{2.7}$$

So,

$$t^m (a b_{j_0} + a \sum t^{j-j_0} b_j) = t^{i_0 - j_0} (b a_{i_0} + b \sum t^{i-i_0} a_i). \tag{2.8}$$

Since $\mathcal{O}_X(U)[t]$ has the usual $\mathbf{Z}$-grading, we have $t^m = t^{i_0 - j_0}$ and $ab_{j_0} = ba_{i_0}$, or $a/b = a_{i_0}/b_{j_0}$. □

The reason we demand that X be locally a subscheme of affine space is so that this lemma holds. If we could ensure that f was always locally a ratio of weight vectors of $\mathcal{O}_U$, we would not need the assumption.

We have the following basic lemma on T-schemes:

Lemma 2.2.3 *If X is a T-scheme with components X_i, then the X_i are T invariant.*

Proof: This is just the statement that the minimal associated primes of a module with T-action are T stable. For a more geometric proof, we consider the map,

$$\begin{aligned} \sigma : T \times X_i &\to X \\ (t, x) &\mapsto t \cdot x. \end{aligned} \tag{2.9}$$

Since $T \times X_i$ is a variety, $\sigma(T \times X_i)$ is as well. Since X_i is a maximal subvariety of X and $X_i \subset \sigma(T \times X_i)$, we have $\sigma(T \times X_i) = X_i$. □

2.3 Cones and Blowing-Up

The constructions of this section are found in Fulton [6] appendix B.5 and B.6. Let $S = \oplus_{i=0}^{\infty} S_i$ be a graded $\mathcal{O}_X$-module with a T-action such that

1. S_0 is T-isomorphic to $\mathcal{O}_X$,
2. S_1 is locally generated by a finite set of elements $\{f_1, \ldots, f_r\}$ such that $t \cdot f_i \in S_1$,

3. S_1 generates S as an $\mathcal{O}_X$-algebra.

We associate to S its cone and its projective cone,

$$C(S) = Spec(\oplus_{n=0}^{\infty} S_n), \tag{2.10}$$

$$P(C(S)) = Proj(\oplus_{n=0}^{\infty} S_n). \tag{2.11}$$

Since S has a T-action, these schemes are T-schemes.

The cones we are particularly interested in are those defined by T-subvarieties and blow-ups.

Let X be a T-subscheme of a T-scheme Y. $\mathcal{O}_X$ is defined by a sheaf of T invariant ideals I of $\mathcal{O}_Y$. The cone to X in Y is defined as

$$C_X(Y) = Spec(\oplus_{n=0}^{\infty} I^n/I^{n+1}). \tag{2.12}$$

$C_X(Y)$ is a T-scheme. Since the map $\mathcal{O}_X \to I^0/I$ is a T-isomorphism, the induced morphism on schemes $p : C_X(Y) \to X$ is a T-morphism.

If Y is a T-subscheme of $\mathbf{A}^n$ for some n, then the generators of I can be chosen to be weight vectors. These weight vectors determine the action on $C_X(Y)$. When Y is locally a subspace of $\mathbf{A}^n$, this construction glues together to define the cone.

Note that if Y is nonsingular, $x \in Y^T$ and $m_x = (f_1, \ldots, f_n)$ is the maximal ideal defining x where the f_i are weight vectors, then

$$C_x(Y) = T_x(Y) = Spec(Symm(m_x/m_x^2)^{\vee}) \tag{2.13}$$

and

$$\mathcal{O}_{T_x(Y)} = Symm(m_x/m_x^2) = \mathcal{O}_{N_x(Y)}. \tag{2.14}$$

With the same notation, the projective cone is

$$P(C_X(Y)) = Proj(\oplus_{n=0}^{\infty} I^n/I^{n+1}). \tag{2.15}$$

Again, this is a T-scheme and the morphism $p : P(C_X(Y)) \to X$ induced by $\mathcal{O}_X \simeq I^0/I$ is a T-morphism.

If Y is a T-subscheme of $\mathbf{A}^n$ for some $n \in \mathbf{Z}$, we can construct $P(C_X(Y))$ more explicitly. Let $I = (f_1, \ldots, f_r)$ where the f_i are weight vectors of weight $-\lambda_i$. $P(C_X(Y))$ is a T-subscheme of $X \times \mathbf{P}^{r-1}$ where $\mathbf{P}^{r-1} = Proj(k[f_1, \ldots, f_r]) = Proj(k[x_1, \ldots, x_r])$. Glueing this local construction together, we get $P(C_X(Y))$ for the non-affine case.

Note that if we set the weight of x_i to be $-\lambda_i - \lambda$ for all i and for some $\lambda \in \mathbf{X}(T)$, then the action induced on $P(C_X(Y))$ is the same as the original one.

$P(C_X(Y))$ has a canonical line bundle defined on it. Locally, we can define this as the pull back of $\mathcal{O}(1)$ from $\mathbf{P}^{r-1}$ to $P(C_X(Y))$.

We can also close a cone off in a projective space. Let $S[z]$ be the graded algebra with graded pieces

$$S[z]_n = S_n \oplus (S_{n-1} \otimes z) \oplus (S_{n-2} \otimes z^2) \oplus \ldots \oplus z^n. \tag{2.16}$$

We set

$$P(S \oplus 1) = Proj(S[z]). \tag{2.17}$$

If we set $\lambda_z = 0$, then the open set defined by inverting z is T-isomorphic to the cone $C(S)$.

If $X \hookrightarrow Y$ is a T-subscheme of Y, the blow-up of Y along X is defined by

$$Bl_X(Y) = Proj(\oplus_{n=0}^{\infty} I^n). \tag{2.18}$$

The map $\pi : Bl_X(Y) \to Y$ induced by $\mathcal{O}_Y \simeq I^0$ is a T-morphism, and it is proper, birational, and surjective.

As before, if Y is a T-subscheme of $\mathbf{A}^n$, $I = (f_1, \ldots, f_r)$ where the f_i are weight vectors of weight $-\lambda_i$. $Bl_X(Y)$ is a subspace of $Y \times \mathbf{P}^{r-1}$ where $\mathbf{P}^{r-1} = Proj(k[f_1, \ldots, f_r]) = Proj(k[x_1, \ldots, x_r])$. If we set the weight of x_i to be $-\lambda_i - \lambda$, then the action on $Bl_X(Y)$ is unchanged. As before, this glues together to give $Bl_X(Y)$ in the non affine case.

Let E be the pullback of X to $Bl_X(Y)$. This is an effective Cartier divisor and is defined locally by pairs (U_α, f_α), where $f_\alpha \in \mathcal{O}^*_{U_\alpha}$ is a weight vector. In fact, E is the Cartier divisor associated to the canonical line bundle on $Bl_X(Y)$ obtained by pulling back $\mathcal{O}(1)$ from $\mathbf{P}^{r-1}$.

2.4 Equivariant Bundles

The contents of this section are to be found mostly in Fulton [6] apppendix B 3.

A T-vector bundle $p : E \to X$ of rank e on a T-scheme X is defined by the following conditions:

1. There is a collection of open T-subsets $\{U_i\}$ of X such that $\varphi_i : E\,|_{U_i} \simeq U_i \times F_i$, φ_i is a T-morphism and F_i is a representation space for T of dimension e,
2. $$\varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \otimes F \to U_i \cap U_j \otimes F' \tag{2.19}$$

   is the identity when restricted to $\mathcal{O}_{U_i \cap U_j} \times 0$ and is linear on the F's. By this we mean that on the structure sheaves, the $\varphi_i \circ \varphi_j^{-1}$ are defined by $e \times e$ invertible matrices with coefficients in $\mathcal{O}_{U_i \cap U_j}$. We denote these maps by $g_{ij} : \mathcal{O}_{U_i \cap U_j} \times \mathcal{O}_F \to \mathcal{O}_{U_i \cap U_j} \times \mathcal{O}_{F'}$.

E can be determined by either the φ_i, or the g_{ij}.

Note that the actions on F and F' need not be the same. So, while the $\varphi_i \circ \varphi_j^{-1}$ are equivariant, the g_{ij} need not be given by T invariant matrices.

Definition: We define a weight section s of E of weight $\lambda = d\chi$ to be a morphism $s : X \to E$ such that $p \circ s = id$, and $t^{-1} \cdot s(t \cdot x) = \chi(t)s(x)$. s is a T-section if $\lambda = 0$.

A weight section is defined by a collection of weight vectors $f_i \in R(U_i)^*$ such that $f_i = g_{ij} f_j$.

Since we will be especially concerned with T-line bundles L, we consider this case explicitly.

If the F_i are spanned by the weight vectors X_i, then $F_i = Spec(k[x_i])$ are 1-dimensional representation spaces, so the action defined on the F_i is defined by a single character χ_i with associated weight λ_i. The g_{ij} are invertible functions in $\mathcal{O}_{U_i \cap U_j}$. Since the $\varphi_i \circ \varphi_j^{-1}$ are equivariant, the g_{ij} are weight vectors of weight $\lambda_j - \lambda_i$. A weight section s of L of weight λ determines weight vectors $f_i \in R(U_i)^*$ of weight $-\lambda_i - \lambda$. To see this, let $s : \mathcal{O}_L \to \mathcal{O}_X$ be the morphism on structure sheaves. Locally, we require that on the scheme, $t \cdot s(t^{-1} \cdot x_i) = \chi(t^{-1})s(x_i)$. This translates as $t \cdot \chi_i(t^{-1})s(x_i) = \chi(t^{-1})s(x_i)$ on the structure sheaves. So, $s(x_i) \in R(U_i)^*$ is a weight vector of weight $-\lambda_i - \lambda$.

Definition: We say a T-line bundle has weight λ at $x \in X^T$ if the weight of the representation space in L_x is λ.

To projectivize T-vector bundles, we consider them as equivariant $\mathcal{O}_X$-modules and take

$$\mathbf{P}(E) = Proj(SymmE^\vee). \tag{2.20}$$

$\mathbf{P}(E)$ is a T-scheme. If $E = U \times F$ locally where F has a basis of weight vectors $X_1, \ldots, X_e$, with related functions $x_1, \ldots, x_e$, then on the open subset $U \times U_k$ of $\mathbf{P}(U \times F)$ defined by inverting x_k, the x_j/x_k have weight $-\lambda_j + \lambda_k$, and the associated vectors X_j/X_k have weight $\lambda_j - \lambda_k$. If we set the weight of X_i to be $\lambda_i + \lambda$ for all i and some λ, then the action on $\mathbf{P}(E)$ is unchanged.

$\mathbf{P}(E)$ has a canonical T-line bundle $\mathcal{O}(1)$. On $U \times U_k$ it is defined by the free $\mathcal{O}_{U \times U_k}$-module generated by $1/x_k$. So, over U, $\mathcal{O}(1)$ has weight λ_k.

Chapter 3

Formal Characters and Equivariant Multiplicity

We define the formal character and equivariant multiplicity associated to a finitely generated module over a polynomial ring, both with T-action, and develop some of its properties. While most of the material in this chapter can be found elsewhere, it seems that none of it can be found in a single reference. In particular, the results concerning changes of torus seem to have only been used implicitly. In the first section, we define the formal character of a finitely generated module with T-action for non-zero weights and then extend this definition under certain circumstances to the arbitrary weight case. In the second section, we use formal characters to define the notion of equivariant multiplicity. References for the material in this chapter are Borho, Brylinski, and Macpherson [1] and Rossmann [17] . We follow mainly the presentation of Rossmann.

3.1 Characters

In the past, formal characters have been defined mainly for the "un-mixed" weight case. Since we need characters for arbitrary weights, we follow the presentation of Rossmann [17] and define them indirectly for the non-zero weight case. We then show how the formal characters behave as we change tori. Finally, we consider bi-graded modules and extend the definition of formal characters to this case. Using the change of torus property, we can then define the formal character for the "mixed" and zero weight cases.

Let $R = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring on which T acts diagonally. So, $t \cdot x_i = \chi_i(t) x_i$ for some $\chi_i \in \mathbf{X}(T)$, for all i.

Definition: M is a R, T module if M is a finitely generated R-module with T-action such that it has a finite set of generators $f_1, f_2, \ldots, f_m$ that are also weight vectors. Recall that if M has a T-action, then $t \cdot (rm) = (t \cdot r)(t \cdot m)$ for $t \in T$, $r \in R$ and $m \in M$.

If M is a R, T module, let $\chi \in \mathbf{X}(T)$ and $d\chi = \lambda$.

$$M_\chi = M_\lambda = \{f \in M : t \cdot f = \chi(t)f \text{ for all } t\}. \tag{3.1}$$

We have

$$M = \oplus_{\lambda \in \Lambda(T)} M_\lambda. \tag{3.2}$$

If all the M_λ are finite dimensional, the usual character as defined by Borho, Brylinski and Macpherson [1], say, is

$$ch_T(M) = \sum_{\lambda \in \Lambda(T)} (\dim M_\lambda) e^\lambda. \tag{3.3}$$

$ch_T(M) \in S^{-1}R(\mathbf{X}(T))$, where S is the multiplicative set generated by the $1 - e^\lambda$ for all $\lambda \in \Lambda(T)$.

We wish to consider the more general case where $\dim M_\lambda$ may be infinite. To do this, we have to define the character indirectly. Rossmann [17] has shown

Theorem 3.1.1 *If M is a finitely generated R, T module such that $\lambda_i \neq 0$ for any i, then $ch_T(M) \in S^{-1}R(\mathbf{X}(T))$ is defined uniquely by:*

1. If M is finite dimensional as a k-module, then

$$ch_T(M) = \sum_{\lambda \in \Lambda(T)} (\dim M_\lambda) e^\lambda. \tag{3.4}$$

2. If

$$0 \to M' \to M \to M'' \to 0 \tag{3.5}$$

is an exact sequence of R, T modules, then

$$ch_T(M) = ch_T(M') ch_T(M''). \tag{3.6}$$

3. If F is a finite dimensional T-module, then

$$ch_T(M \otimes F) = ch_T(M)ch_T(F). \tag{3.7}$$

Furthermore, $ch_T(M)$ is of the form,

$$ch_T(M) = \Delta_T(R)^{-1} \sum_{\lambda \in \Lambda(T)} a_\lambda e^\lambda \tag{3.8}$$

where $\Delta_T(R) = \prod_{i=1}^n (1 - e^{\lambda_i})$, $a_\lambda \in \mathbf{Z}$, all but a finite number of the a_λ are zero, and if $a_\lambda \neq 0$, then λ is a sum of the λ_i and the λ_{f_i}.

Proof: The proof is contained in Rossmann [17]. The method of proof is to resolve M into free R, T modules and then take the characters of these. $\Delta_T(R)$ comes from setting $R_i = k[x_i, \ldots, x_n]$ and considering the exact sequence

$$0 \to x_i R_i \to R_i \to R_{i+1} \to 0. \tag{3.9}$$

We have $ch_T(R_{i+1}) = (1 - e^{\lambda_i})ch_T(R_i)$, and using induction then yields

$$ch_T(R) = \prod_{i=1}^n (1 - e^{\lambda_i})^{-1}. \tag{3.10}$$

□

Remark: If $\dim M_\lambda < \infty$, for all λ, Borho, Brylinski and Macpherson show that the usual character $\sum_{\lambda \in \Lambda(T)} (dim M_\lambda) e^\lambda$ satisfies all three properties. So $ch_T(M)$ is really an extension of the usual character.

Note that if $T = G_m$ and all x_i have weight 1, then we get the usual Poincaré series $\sum_{n=0}^{\infty} \dim M_n t^n$ for M.

The theorem has the following corollary:

Corollary 3.1.2 *If $R' = k[y_1, \ldots, y_k]$ with the y_i as weight vectors, then $M \otimes R'$ is a $R \otimes R'$, T-module, and*

$$ch_T(M \otimes R') = ch_T(M)ch_T(R'). \tag{3.11}$$

Proof: If M has resolution

$$0 \to M_N \to \cdots \to M_0 \to M \to 0, \tag{3.12}$$

then $M \otimes R'$ has resolution

$$0 \to M_N \otimes R' \to \cdots \to M_0 \otimes R' \to M \otimes R' \to 0. \tag{3.13}$$

The result then comes from $ch_T(R \otimes R') = \Delta_T(R)^{-1}\Delta_T(R')^{-1}$. □

As with R-modules, R, T modules have composition series of R, T submodules. To show this, we start with the following lemma:

Lemma 3.1.3 *If v is a weight vector of M, then its annihilator* $\mathrm{Ann}(v)$ *is a T invariant ideal.*

Proof: Note that

$$av = 0 \Leftrightarrow \chi_v(t)(av) = 0 \Leftrightarrow a(t \cdot v) = 0. \tag{3.14}$$

So, $\mathrm{Ann}(v) = \mathrm{Ann}(t \cdot v)$. Also,

$$av = 0 \Leftrightarrow (t \cdot a)(t \cdot v) = t \cdot (av) = 0. \tag{3.15}$$

So, $\mathrm{Ann}(t \cdot v) = t \cdot \mathrm{Ann}(v)$. □

Theorem 3.1.4 *1. If M is a R, T module, then M has a composition series of R, T modules:*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M \tag{3.16}$$

where M_i/M_{i-1} is T-isomorphic to the 1-dimensional R, T module $R/P_i \cdot v_i$, where P_i is a T invariant prime ideal of R and $v_i \in M$ is a weight vector of weight μ_i .

2.

$$ch_T(M) = \sum_i e^{\mu_i} ch_T(R/P_i). \tag{3.17}$$

Proof: For 1, the proof is exactly as in Lang [12]. All that needs to be remarked is that in choosing a vector in M, we can choose it to be a weight vector, so that its annihilator is a T invariant ideal. Note that if all the weights are of one sign, this is just be the decomposition into highest weight modules.

2. This follows from the exactness property of $ch_T(-)$. □

We will also need to know how $ch_T(M)$ behaves as T changes. As far as we know, these results have only been used implicitly, but have never been proved explicitly.

Suppose $\varphi : T' \to T$ is a (homo)-morphism of tori. Let $t' \in T'$. If $T = \otimes_{i=1}^{r} G_m$, we can decompose φ into the characters $\varphi_i : T' \to G_m$. So, $\varphi = (\varphi_1, \ldots, \varphi_r)$. We set

$$d\varphi = (d\varphi_1 dt', \ldots d\varphi_r dt'). \tag{3.18}$$

Suppose $\chi : T \to G_m$ is a character. Considering $\chi \circ \varphi : T' \to G_m$ as a function, we see that

$$d(\chi \circ \varphi) = (d\chi) \circ (d\varphi). \tag{3.19}$$

Theorem 3.1.5 *Let $\varphi : T' \to T$ be a morphism of tori, and define the T'-action on R by $t' \cdot x = \varphi(t') \cdot x$ for all $x \in R$. Let M be a R, T and a R, T' module such that*

$$\begin{tikzcd} T' \otimes M \arrow[r, "\varphi\otimes 1"] \arrow[dr] & T \otimes M \arrow[d] \\ & M \end{tikzcd} \tag{3.20}$$

commutes. Then $ch_{T'}(M) = ch_T(M) \circ d\varphi$.

Proof: We prove the results by considering the properties of $ch_T(M) \circ d\varphi$. Suppose M is finite dimensional as a k-module. If λ' is a weight of T', then $M_{\lambda'}$ is generated by T' weight vectors $v_1, \ldots, v_l$. Let $d\chi' = \lambda'$. Since each v_i has a decomposition into T weight vectors, we get,

$$M_{\lambda'} \subset \oplus_{j=1}^{k} M_{\lambda_k} \tag{3.21}$$

for some set of weights $\lambda_k \in \Lambda(T)$. We assume this set is minimal. Now, if

$$v_i = w_1 + \cdots + w_k, \tag{3.22}$$

$$t' \cdot v_i = \chi'(t')v_i = \chi_1 \circ \varphi(t')w_1 + \cdots + \chi_k \circ \varphi(t')w_k, \tag{3.23}$$

and $\chi_j \circ \varphi = \chi'$ for all j. So, $M_{\lambda_j} \subset M_{\lambda'}$ for all j, and

$$\oplus_{j=1}^{k} M_{\lambda_j} = M_{\lambda'}. \tag{3.24}$$

So, if M is finite dimensional as a k-module,

$$ch_{T'}(M) = \sum_{\lambda' \in \Lambda(T')} (\sum_{j=1}^{k} \dim M_{\lambda_j})e^{\lambda'} = \sum_{\lambda \in \Lambda(T)} (\dim M_\lambda)e^{\lambda \circ d\varphi} = ch_T(M) \circ d\varphi. \tag{3.25}$$

The other two properties, exactness and multiplicativity, follow from the exactness and multiplicativity properties of $ch_T(M)$. □

This restriction property allow us to define $ch_T(M)$ under certain conditions when some of the λ_i are zero. Suppose R has its usual grading by degree and M is a graded R,T module. We will call such modules bi-graded R,T modules. The usual grading determines a G_m-action on R and on M, which we label by T_1. So, if M is bi-graded, then it is a R, $T \times T_1$ module. If $\lambda_i \neq 0$ for any i, let

$$\begin{aligned} \varphi : T &\to T \times T_1 \\ t &\mapsto (t, 1). \end{aligned} \tag{3.26}$$

Then,

$$ch_T(M) = ch_{T\times T_1}(M) \circ d\varphi \tag{3.27}$$

and since $\Delta_{T\times T_1}(R) \circ d\varphi = \Delta_T(R)$,

$$\Delta_T(R)ch_T(M) = (\Delta_{T\times T_1}(R)ch_{T\times T_1}(M)) \circ d\varphi. \tag{3.28}$$

Since $d\varphi = 1 \oplus 0$, we write this as

$$\Delta_T(R)ch_T(M) = (\Delta_{T\times T_1}(R)ch_{T\times T_1}(M)) \mid_T \tag{3.29}$$

Definition: Suppose M is a bi-graded R, T module. Then

$$\Delta_T(R)ch_T(M) = \Delta_{T\times T_1}(R)ch_{T\times T_1}(M)\mid_T . \tag{3.30}$$

Remark: We could have extended the formal character in several ways: for the multiplicity, we really only need the numerator of the formal character, so we could have ignored the denominator even when some of the λ_i were 0. Alternatively, we could have taken a residue. The former has the problem that its formal properties are not easy to prove, while the latter has the problem that the restriction property of tori becomes hard to state. Since all modules we consider are bi-graded R, T modules, we decided to use the restriction property even though greater generality could have been obtained using residues.

In future, if M is a bi-graded R, T module, and we do not specify $\lambda_i \neq 0$ for all i, we shall be using this definition. For completeness, we list its properties.

Proposition 3.1.6 *Suppose that M is a bi-graded R,T module.*

1. *If $\lambda_i \neq 0$ for all i then the two definitions of $\Delta_T(R)ch_T(M)$ agree.*
2. *If*

$$0 \to M' \to M \to M'' \to 0 \tag{3.31}$$

is an exact sequence of bi-graded R, T modules, then

$$\Delta_T(R)ch_T(M) = \Delta_T(R)ch_T(M') + \Delta_T(R)ch_T(M''). \tag{3.32}$$

3. *If F is finite dimensional as a k-module, and is a bi-graded R, T module, then*

$$\Delta_T(R)ch_T(M \otimes F) = \Delta_T(R)ch_T(M)ch_T(F). \tag{3.33}$$

where $ch_T(F)$ is the usual character of F.

4.

$$\Delta_T(R)ch_T(M) = \sum_{\lambda\in\Lambda(T)} a_\lambda e^\lambda, \tag{3.34}$$

where $a_\lambda \in \mathbf{Z}$, and all but a finite number of them are zero. Also, $a_\lambda \neq 0$ implies λ is a sum of the λ_i and the λ_{f_i}.

5. If M is a bi-graded R, T module, $R' = k[y_1, \ldots, y_m]$ has a T-action, then

$$\Delta_T(R\otimes R')ch_T(M\otimes R') = \Delta_T(R)ch_T(M). \tag{3.35}$$

6. Let $\varphi : T' \to T$ be a morphism of tori, and define the action of T' on R by $t'\cdot x = \varphi(t')\cdot x$. If M is a bi-graded R, T module, and

$$\begin{array}{ccc} T'\times M & \xrightarrow{\varphi\times 1} & T\times M \\ & \searrow & \downarrow \\ & & M \end{array} \tag{3.36}$$

commutes, then

$$(\Delta_T(R)\circ d\varphi)\Delta_{T'}(R)ch_{T'}(M) = (\Delta_{T'}(R))(\Delta_T(R)ch_T(M))\circ d\varphi. \tag{3.37}$$

7. If $f \in R$ is a weight vector and is homogeneous in the usual sense and f acts as a non-zero divisor on M, then

$$\Delta_T(R)ch_T(M/fM) = (1 - e^{\lambda_f})\Delta_T(R)ch_T(M). \tag{3.38}$$

8. If M is a bi-graded R, T module, then M has a composition series

$$0 = M_0 \to M_1 \to \cdots \to M_k = M \tag{3.39}$$

of bi-graded R, T modules such that

$$M_i/M_{i-1} \simeq R/P_i \cdot v_i \tag{3.40}$$

for P_i a $T \times T_1$ invariant prime ideal of R and v_i a $T \times T_1$ weight vector of M. Under these conditions,

$$\Delta_T(R)ch_T(M) = \sum_{i=1}^{k} e^{\lambda_{v_i}} \Delta_T(R)ch_T(R/P_i). \tag{3.41}$$

Proof: Only 6 is not a direct consequence of previous results. For 5, recall that

$$ch_{T\times T_1}(M \otimes R') = ch_{T\times T_1}(M)\Delta_{T\times T_1}(R')^{-1}. \tag{3.42}$$

and $\Delta_{T\times T_1}(R \otimes R') = \Delta_{T\times T_1}(R)\Delta_{T\times T_1}(R')$. So,

$$\Delta_{T\times T_1}(R \otimes R')ch_{T\times T_1}(M \otimes R') = \Delta_{T\times T_1}(R)ch_{T\times T_1}(M) \tag{3.43}$$

Restriction to T now yields the result.

For 6,

$$(\Delta_{T\times T_1}(R) \circ (d\varphi \times id))\Delta_{T'\times T_1}(R)ch_{T'\times T_1}(M) = \Delta_{T'\times T_1}(R)(\Delta_{T\times T_1}(R)ch_{T\times T_1}(M) \circ (d\varphi \times id)) \tag{3.44}$$

and on restricting,

$$\Delta_T(R) \circ d(\varphi)(\Delta_{T'}(R)ch_{T'}(M)) = \Delta_{T'}(R)(\Delta_T(R)ch_T(M) \circ d\varphi). \tag{3.45}$$

For 7, while we have not stated it explicitly for non bi-graded modules before, we have used it in Theoren 3.1.1 to show $ch_T(R) = \Delta_T(R)^{-1}$. □

Remark: Note that 6 does not yield much if $\lambda_i = 0$ for some i. In this case, all it yields is $0 = 0$. Also note that if M and R have trivial T-action, but are $\mathbf{Z}$ graded, then $ch_{T\times T_1}(M)$ is the usual Poincaré series, $\sum(\dim M_n)e^n$.

Definition: If M is a bi-graded R,T module, or R has no $\lambda_i = 0$ and M is a R,T module, we will say that M is quasi bi-graded.

3.2 Equivariant Multiplicity

Using the formal character defined for quasi bi-graded modules, we define the equivariant multiplicity of a quasi bi-graded module. While equivariant multiplicities have been considered for the non-zero weight case, the zero weight case has not to have been treated in the past.

Definition: Suppose M is quasi bi-graded. If we consider $\Delta_T(R)ch_T(M)$ as a function in $\Lambda(T)$, where $e^\lambda = 1 + \lambda + \frac{\lambda^2}{2} + \cdots$, then we define the equivariant multiplicity $mult_T(M, R)$ of M as the first non-identically zero term in $\Delta_T(R)ch_T(M)$.

From the properties of $\Delta_T(R)ch_T(M)$ we get:

Proposition 3.2.1 *Suppose M is a quasi bi-graded module.*

1.

$$mult_T(M, R) = \frac{1}{N!} \sum_{\lambda \in \Lambda(T)} a_\lambda \lambda^N \tag{3.46}$$

for some $N \in \mathbf{Z}$, $a_\lambda \in \mathbf{Z}$, all but a finite number of the a_λ are zero, and $a_\lambda \neq 0$ implies that λ is a sum of the λ_i and the λ_{f_i}.

2. If R' is a polynomial ring with T-action, then

$$mult_T(M \otimes R', R \otimes R') = mult_T(M, R). \tag{3.47}$$

3. Let $\varphi : T' \to T$ be a morphism of tori, and define the T'- action on R by $t' \cdot x = \varphi(t') \cdot x$ for all $x \in R$. If

$$\begin{tikzcd} T' \times M \arrow[r, "\varphi \times 1"] \arrow[dr] & T \times M \arrow[d] \\ & M \end{tikzcd} \tag{3.48}$$

commutes, then

$$\left(\prod(-\lambda_i) \circ d\varphi\right) mult_{T'}(M, R) = \prod(-\lambda_i')(mult_T(M, R) \circ d\varphi) \tag{3.49}$$

4. Suppose $f \in R$ is a weight vector which is homogeneous in the usual sense if R has some $\lambda_i = 0$. If f acts as a non-zero divisor on M, then

$$mult_T(M/fM, R) = -\lambda_f mult_T(M, R). \tag{3.50}$$

Furthermore, if $d(M)$ is the Krull dimension of M, then

$$d(M/fM) = d(M) - 1. \tag{3.51}$$

5. M has a filtration by quasi bi-graded modules with quotients isomorphic to $R/P_i \cdot v_i$ where the P_i are equivariant prime ideals of R and the v_i are weight vectors of M. With this notation, if $\mathcal{J}$ is the set of i where $\Delta_T(R)ch_T(R/P_i)$ has its leading term of minimal degree, then

$$mult_T(M, R) = \sum_{i \in \mathcal{J}} mult_T(R/P_i, R) \tag{3.52}$$

Proof: Except for 5, these are all consequences of Proposition 3.1.6. The second part of 4 is a property of Krull dimension. For 5, M has a composition series with factor modules R/P_i. So

$$\Delta_T(R)ch_T(M) = \sum_{i=1}^{k} e^{\mu_i} \Delta_T(R)ch_T(R/P_i, R) \tag{3.53}$$

for some weights μ_i. Let $\mathcal{J}$ be the set of i where the leading term of $\Delta_T(R)ch_T(R/P_i)$ is of minimal degree. Since e^{μ_i} does not affect the multiplicity, we have

$$mult_T(M, R) = \sum_{i \in \mathcal{J}} mult_T(R/P_i, R). \tag{3.54}$$

□

Note that 2 and 4 have the following corollary:

Corollary 3.2.2 *Suppose that $R' = k[y_1 \ldots y_l]$, where the y_i are weight vectors of weight μ_i and M is a quasi bi-graded module. We can consider M as a quasi bi-graded $R \otimes R'$, T module, where $(1 \otimes y)m = 0$ for all $y \in R'$ and $m \in M$. Then $M \simeq M \otimes R'/(1 \otimes y_1, \ldots, 1 \otimes y_{l-1})(M \otimes R')$, and identifying M with this module,*

$$mult_T(M, R \otimes R') = \prod_{i=1}^{l} (-\mu_i) mult_T(M, R). \tag{3.55}$$

Proof: $1 \otimes y_i$ acts as a non-zero divisor of $M \otimes R'/(1 \otimes y_1, \ldots, 1 \otimes y_{i-1})(M \otimes R')$ □

We also have

Lemma 3.2.3 *If M is a zero-dimensional quasi bi-graded R, T module, then*

$$mult_T(M, R) = \prod(-\lambda_i) \dim M. \tag{3.56}$$

Proof: M is a finite dimensional k-vector space, so,

$$ch_{T\times T_1}(M) = \sum_{(\lambda\times\lambda_1)\in\Lambda(T\times T_1))} \dim M_{(\lambda\times\lambda_1)} e^{\lambda\times\lambda_1}, \tag{3.57}$$

and

$$\Delta_{T\times T_1}(R) ch_{T\times T_1}(M) = \prod_{i=1}^{n}(1 - e^{\lambda_i\times 1}) \sum_{(\lambda\times\lambda_1)\in\Lambda(T\times T_1)} \dim M_{(\lambda\times\lambda_1)} e^{\lambda\times\lambda_1}. \tag{3.58}$$

On restricting to T, the first term of $(1 - e^{\lambda_i\times 1})\,|_T$ is $-\lambda_i$. On restricting, since the e^{λ} in the sum do not affect the first term, we get the result. □

By Proposition 3.2.1 part 1, we can consider $mult_T(M, R)$ as a polynomial in $\Lambda(T)$ of degree N. If M is a quasi bi-graded R, T factor module of R, then $mult_T(M, R)$ is a polynomial in n variables of degree N.

we can determine N precisely:

Theorem 3.2.4 *In Proposition 3.2.1 part 1, $N = n - d(M)$.*

Proof: The proof is a slight variation of that in Borho, Brylinski and Macpherson [1]. We prove this by induction on $d(M)$. Our hypothesis is that for $d(M') < d(M)$, $mult_T(M, R)$ is a polynomial of degree $N' = n - d(M')$, and $(-1)^{N'} mult_T(M', R)$ as a polynomial is positive on positive weights.

If $d(M) = 0$, then the previous lemma shows that $(-1)^N mult_T(M, R)$ is positive on positive weights. Suppose $d(M) > 0$. Let $f \in R - P_i$ be a non-constant weight vector. If $L = R/P_i$,

$$mult_T(L/fL, R) = (-\lambda_f) mult_T(L, R) \tag{3.59}$$

and

$$d(L/fL) = d(L) - 1. \tag{3.60}$$

Since $(-1)^{N+1} mult_T(L/fL, R)$ is positive on positive weights, $(-1)^N mult_T(L, R)$ is as well. Since $mult_T(M, R)$ is a sum of the $mult_T(L, R)$ with positive coefficients, $(-1)^N mult_T(M, R)$ is also positive on positive weights, and is a polynomial in $\Lambda(T)$ of degree N. □

Note that the theorem implies that while $mult_T(M, R)$ may be zero, considering it as a polynomial on $\Lambda(T)$, it is non-zero.

If $T = G_m$ and all the x_i have weight 1, then $mult_T(M, R)$ is the usual multiplicity of a module over a polynomial ring.

We also have the following Bezout theorem:

Proposition 3.2.5 *If $f \in R$ is a weight vector acting as a non-zero divisor on a quasi bi-graded module M, then M/fM has a composition series with factor modules R/P_i. Let $\mathcal{J}$ be the set of i with $d(R/P_i)$ maximal, and m_i the number of times each R/P_i occurs as a factor module. Then*

$$mult_T(M/fM, R) = (-\lambda_f) mult_T(M, R) = \sum_{i \in \mathcal{J}} m_i mult_T(R/P_i, R). \tag{3.61}$$

Proof: This is a direct consequence of Proposition 3.2.1 parts 4 and 5 and Theorem 3.2.4. □

Chapter 4

Equivariant Chow Groups

If X is defined over an algebraically closed field k, has a torus action defined on it and is locally isomorphic to a representation space, we define the equivariant Chow groups $A_k^T(X)$, and prove some of the basic results of $A_k^T(X)$. The results we are particularly interested in are those considered in Fulton [6] and the change of torus properties. Most of this chapter concerns determining the former. In doing this, we have followed Fulton very closely. In fact, the results we present are really rephrasings of those of Fulton in the equivariant setting. For the change of torus properties, we need to develop the theory of Chow schemes. We do this in section 4.7. The results of that section are mainly extensions of results of Brion [3].

Throughout the rest of this work, we will be assuming that all T-schemes X can be covered by open affine T-subsets that are T-isomorphic to T-subschemes of T-representation spaces.

4.1 Equivariant Chow Groups

We define the equivariant Chow groups. The definitions are the same as in Fulton [6], with the exception that we demand that the subvarieties be T invariant and that the functions be weight vectors of weight 0.

Definition: We denote the free group on the set of k-dimensional T-subvarieties by $Z_k^T(X)$.

Definition: Suppose X is an n-dimensional affine T-scheme and $f \in \mathcal{O}_X$ is a weight vector. We define

$$\mathrm{div} f = \sum_W \mathrm{ord}_W(f)[W], \tag{4.1}$$

where the sum is over all codimension 1 subvarieties of X and $\mathrm{ord}_W(f) = l_{\mathcal{O}_{W,X}}(\mathcal{O}_{W,X}/f\mathcal{O}_{W,X})$.

When X is not affine this local definition glues together to give a global definition of $\mathrm{div}f$.

Note that if $a \in \mathcal{O}_X(U)$ is a weight vector, the closure of the scheme defined by a has T-varieties as components. So, the set for which $\mathrm{ord}_V f \neq 0$ consists of T-subvarieties and $\mathrm{div}f \in Z^T_{n-1}(X)$.

If X is an affine T-scheme and if $f = a/b \in R(X)^*$, we set

$$\mathrm{div}f = \mathrm{div}a - \mathrm{div}b. \tag{4.2}$$

When X is not affine, this local definition glues together to give a global definition. Since there are weight vectors a' and b' in $\mathcal{O}_X$ such that $f = a'/b'$, $\mathrm{div}a'$ and $\mathrm{div}b'$ are T-cycles, $\mathrm{div}f$ is a T-cycle as well.

Definition: $\alpha \in Z^T_k(X)$ is T-rationally equivalent to 0 if there exists a collection of $k+1$-dimensional T-subvarieties $V_1, \dots, V_n$ of X and weight vectors $f_i \in R(V_i)^*$ of weight 0 such that

$$\alpha = \sum_{i=1}^{n} \mathrm{div}f_i. \tag{4.3}$$

We define $A^T_k(X)$ as $Z^T_k(X)$ modulo this relation.

4.2 Proper Pushforward

Let $p : X \to Y$ be a proper T-morphism of T-schemes. We define $p_* : Z^T_k(X) \to Z^T_k(Y)$. If V is a k-dimensional T-subvariety of X, let $d = [R(V) : R(p(V))]$. We set

$$p_*(V) = \begin{cases} 0 & \text{if } \dim p(V) < \dim V \\ d[p(V)] & \text{if } \dim p(V) = \dim V. \end{cases} \tag{4.4}$$

We extend this definition by linearity to $p_* : Z^T_k(X) \to Z^T_k(Y)$.

Definition: Let L be a finite field extension of the field K. We consider L as a finite vector space over K. For $f \in L$, we define the norm $N_K^L(f)$ of f to be the determinant of the K-linear morphism defined by multiplication by f in L.

This definition agrees with the definition of the norm as a product of conjugates, as found in, say, Lang [12]. This is shown in Godement [7] chapter 26, exercises 4 and 5.

Theorem 4.2.1 *Let $p : X \to Y$ be a proper T-morphism of T-schemes. If $\alpha \in Z_k^T(X)$ is T-rationally equivalent to 0, then $p_*(\alpha)$ is T-rationally equivalent to 0 in $Z_k^T(Y)$.*

Proof: We can restrict to the case where X is a $k+1$-dimensional T-variety, $Y = p(X)$, $f \in R(X)^*$ is a weight vector of weight 0 and $\alpha = \operatorname{div} f$. The result is then a consequence of the following proposition.

Proposition 4.2.2 *Let $p : X \to Y$ be a proper surjective T-morphism of T-varieties, and $f \in R(X)^*$ be a weight vector. Then*

$$p_*(\operatorname{div} f) = \begin{cases} 0 & \textit{if} \dim X > \dim Y \\ [\operatorname{div} N_K^L(f)] & \textit{if} \dim X = \dim Y \end{cases} \tag{4.5}$$

where $K = R(Y)$ and $L = R(X)$.

Proof: The proof is as in Fulton, Proposition 1.4. The only thing we have to check is that if X is a T-variety, then its normalization is also a T-variety. This, however, was shown in Proposition 2.2.1. □

To complete the proof of Theorem 4.2.1, note that if f is a weight vector in $R(X)^*$, then

$$N_K^L(t \cdot f) = N_K^L(\chi_f(t) f) = \chi_f(t)^d N_K^L(f), \tag{4.6}$$

where $d = [R(Y) : R(X)]$. So, if $\lambda_f = 0$, $N_K^L(f)$ is a weight vector in $R(p(V))^*$ of weight 0. □

4.3 Cycle Associated to a Scheme

If X is a T-scheme with components X_i, let $m_i = l_{\mathcal{O}_{X_i,X}}(\mathcal{O}_{X_i,X})$ and set

$$[X] = \sum m_i[X_i] \in Z_*^T(X). \tag{4.7}$$

If X is a T-subscheme of a T-scheme Y, then $[X] \in Z_*^T(X) \subset Z_*^T(Y)$, and we write $[X]$ for the asociated cycle in $Z_*^T(Y)$ as well.

Note that the subscheme defined by a single function $f \in \mathcal{O}_Y$ has $[\mathrm{div} f]$ as its associated cycle. This is a local result. If Y is an affine T-scheme, let $A = \mathcal{O}_Y$ and let the prime ideal defining X_i be P. If we identify the image of f in A_P with f and if we identify the image of P in A/fA with P, then $\mathcal{O}_{X_i,Y} = A_P$, $\mathcal{O}_{X_i,X} = (A/fA)_P$ and $A_P/fA_P \simeq (A/fA)_P$. This yields,

$$\mathrm{ord}_{X_i} f = l_{A_P}(A_P/fA_P) = l_{(A/fA)_P}((A/fA)_P) = m_i. \tag{4.8}$$

Example: This is Example 1.5.1 of Fulton [6]. Suppose $f : X \to \mathbf{P}^1_\lambda$ is a dominant T-morphism where X is a $k+1$-dimensional T-variety. f defines a weight vector in $R(X)^*$ of weight λ. We denote this weight vector by f as well. We have,

$$[f^{-1}(0)] - [f^{-1}(\infty)] = [\mathrm{div} f]. \tag{4.9}$$

If $\mathbf{P}^1$ has the trivial action, then the weight vector f has weight 0.

4.4 Alternate Definition of T-Rational Equivalence

Suppose X is a T-scheme. Let V be a T-subvariety of $X \times \mathbf{P}^1_\lambda$ such that the T-morphism induced by the projection onto $\mathbf{P}^1_\lambda$ is dominant. Label this map $f : V \to \mathbf{P}^1_\lambda$. f defines a weight vector $f \in R(V)^*$ of weight λ. Let $p : X \times \mathbf{P}^1_\lambda \to X$ be the projection onto X. If $P \in {\mathbf{P}^1_\lambda}^T$, then $f^{-1}(P)$ is mapped isomorphically by p onto a T-subscheme of X which we call $V(P)$. So, $[V(P)] = p_*[f^{-1}(P)]$,

$$[f^{-1}(0)] - [f^{-1}(\infty)] = [\mathrm{div} f] \tag{4.10}$$

in $Z_k^T(X)$ and,

$$[V(0)] - [V(\infty)] = p_*[\mathrm{div} f] \tag{4.11}$$

in $Z_k^T(X)$.

If $\mathbf{P}^1$ has trivial T-action, then $f \in R(V)^*$ has weight 0. So, $[V(0)] - [V(\infty)]$ is T-rationally equivalent to 0.

We claim that all cycles T-rationally equivalent to 0 arise in this way.

Proposition 4.4.1 *Let $\alpha \in Z_k^T(X)$. α is T-rationally equivalent to 0 if and only if there exists a collection of $k+1$-dimensional T-subvarieties, $V_1, \ldots, V_n$ of $X \times \mathbf{P}^1$ such that the T-morphisms $f_i : V_i \to \mathbf{P}^1$ induced by the projection onto $\mathbf{P}^1$ are dominant and*

$$\sum_{i=1}^{n} [V_i(0)] - [V_i(\infty)] = \alpha. \tag{4.12}$$

Proof: The proof is as in Fulton. We have shown the backward implication. For the other direction, we need only consider one T-subvariety W of X and a single weight zero vector $g \in R(W)^*$ with $\alpha = \mathrm{div} g$. g defines a T-morphism which we label $g : W \to \mathbf{P}^1$. Let V be the closure of the image of the graph morphism,

$$\begin{aligned} gr : W &\to X \times \mathbf{P}^1 \\ w &\mapsto (w, g(w)). \end{aligned} \tag{4.13}$$

Let $f : V \to \mathbf{P}^1$ be the dominant T-morphism induced by the projection $X \times \mathbf{P}^1 \to \mathbf{P}^1$. f defines a weight vector $f \in R(V)^*$ of weight 0. From the example of the previous section, we have

$$[f^{-1}(0)] - [f^{-1}(\infty)] = [\mathrm{div} f]. \tag{4.14}$$

The T-morphism $p : V \to W$ induced by the projection $X \times \mathbf{P}^1 \to X$ is birational, so,

$$[V(0)] - [V(\infty)] = p_*[\mathrm{div} f] = [\mathrm{div} N_{R(W)}^{R(V)}(f)] = [\mathrm{div} g]. \tag{4.15}$$

□

This proposition allows us to move a subvariety with respect to a different torus action. The following has been proved in the case of a connected solvable group and $T' = 1$ in a different manner by Brion [3] in part 1 of the Theorem of section 1.3.

Proposition 4.4.2 *Suppose that $\varphi : T' \to T$ is a morphism of tori. Let X be a T and a T'-scheme such that*

$$\begin{tikzcd} T' \times X \arrow[r, "\varphi\times 1"] \arrow[dr] & T \times X \arrow[d] \\ & X \end{tikzcd} \tag{4.16}$$

commutes. If $\alpha \in Z_k^{T'}(X)$, then α is T'-rationally equivalent to a cycle $\beta \in Z_k^T(X)$. In particular, $\alpha \in Z_k(X)$ is rationally equivalent to a cycle $\beta \in Z_k^T(X)$.

Proof: We prove this by induction. First, we can restrict to the case where $\alpha = [V] \in Z_k^{T'}(X)$. If $\varphi(T') = T$, there is nothing to prove. Suppose that $T = \varphi(T') \times G_m$. We move V with respect to the G_m-action. Consider the graph morphism,

$$\begin{aligned} \sigma : G_m \times V &\to (G_m \times V) \times X \\ (t, v) &\mapsto (t, v, t \cdot v). \end{aligned} \tag{4.17}$$

Let T' act on $G_m \times V$ by $t' \cdot (t, v) = (t, t' \cdot v)$, and on $G_m \times V \times X$ by $t' \cdot (t, v, x) = (t, t' \cdot v, t' \cdot x)$. If G_m acts on $G_m \times V$ by $t' \cdot (t, v) = (t't, v)$, and on $G_m \times V \times X$ by $t' \cdot (t, v, x) = (t't, v, (t't) \cdot x)$, then σ is a T-morphism. Consider the projection p_{13} onto the first and third factors. Let $\lambda = 1$ and let T' act trivially on $\mathbf{P}^1_\lambda$. We inject $p_{13}(\sigma(G_m \times V))$ into $\mathbf{P}^1_\lambda \times X$ and close it to get a T-subvariety W of $\mathbf{P}^1_\lambda \times X$. The projection $f : W \to \mathbf{P}^1_\lambda$ is a dominant T-morphism. Now, the cycle $[W(\infty)]$ is T'-rationally equivalent to $[V]$ and is G_m invariant. Since $T = T' \times G_m$, $[W(\infty)]$ is T invariant as well. □

Note that the method of this proposition moves an effective cycle in $Z_k^{T'}(X)$ to an effective cycle in $Z_k^T(X)$.

4.5 Flat Pullback

Suppose that $f : X \to Y$ is a flat T-morphism of relative dimension n. For V a k-dimensional T-subvariety of Y, let

$$f^*[V] = [f^{-1}(V)] \tag{4.18}$$

in $Z_{k+n}^T(X)$.

This extends by linearity to give a morphism,

$$f^* : Z_k^T(Y) \to Z_{k+n}^T(X). \tag{4.19}$$

The proofs of the next three results are contained in Fulton [6].

Lemma 4.5.1 *If $f : X \to Y$ is a flat T-morphism, then for any T-subscheme Z of Y,*

$$f^*[Z] = [f^{-1}(Z)]. \tag{4.20}$$

Proof: The proof is exactly as in Fulton [6] Lemma 1.7.1. □

Lemma 4.5.2 *If*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array} \tag{4.21}$$

is a fibre square with f a proper T-morphism and g a flat T-morphism, then f' is a proper T-morphism, and g' is a flat T-morphism. If $\alpha \in Z_k^T(X)$, then $g^* f_*(\alpha) = f'_* g'^*(\alpha)$ in $Z_{k+n}^T(Y')$.

Proof: Again, the lemma is as in Fulton [6], Proposition 1.7. □

Theorem 4.5.3 *If $f : X \to Y$ is a flat T-morphism of relative dimension n and $\alpha \in Z_k^T(Y)$ is T-rationally equivalent to 0, then $f^*(\alpha)$ is T-rationally equivalent to 0 in $Z_{k+n}^T(X)$.*

Proof: The proof is in Fulton [6], Theorem 1.7. The proof depends on the alternate definition of T-rational equivalence and uses the two lemmas above, as well as the results of section 4.3.□

4.6 An Exact Sequence

Theorem 4.6.1 *Let X be a closed T-subscheme of a T-scheme Y. Let $U = Y - X$. Then*

$$A_k^T(X) \xrightarrow{i_*} A_k^T(Y) \xrightarrow{j^*} A_k^T(U) \to 0 \tag{4.22}$$

is an exact sequence, where i and j are the inclusion morphisms.

Proof: The proof is exactly as in Fulton [6]. First, note that

$$Z_k^T(X) \xrightarrow{i_*} Z_k^T(Y) \xrightarrow{j^*} Z_k^T(U) \to 0 \tag{4.23}$$

is exact. Let $\alpha \in Z_k^T(Y)$. Suppose $j^*\alpha$ is T-rationally equivalent to 0 in $Z_k^T(U)$. Then, there in a set of $k+1$-dimensional T-subvarieties $V_1, \ldots, V_n$ of U and weight vectors $f_i \in R(V_i)^*$ of weight 0 such that $\alpha = \sum_{i=1}^n [\mathrm{div}(f_i)]$. Let $\bar{V}_i$ be the closure of $j(V)$ in Y. j induces an isomorphism $R(V_i) \simeq R(\bar{V}_i)$. Let $\bar{f}_i$ be the function in $R(\bar{V}_i)$ associated to f_i. Then, $\bar{f}_i$ is a weight vector of weight 0, and $j^*(\alpha - \sum[\mathrm{div}\bar{f}_i)] = 0$ in $Z_k^T(U)$. So,

$$\alpha - \sum_{i=1}^{n}[\mathrm{div}\bar{f}_i] = i_*\beta \tag{4.24}$$

for some $\beta \in Z_k^T(X)$. On passing to T-rational equivalence, we get

$$\alpha = i_*\beta \tag{4.25}$$

in $A_k^T(Y)$. □

4.7 Chow Schemes and Changing Tori

We want to know how $A_k^T(X)$ changes as T changes. For this, we need to develop the theory of Chow schemes. The theory is contained entirely in Samuel [18]. Our main result relating the Chow groups for different tori is a re-writing of a result of Brion [3]. We only give an outline of the construction of the Chow scheme. In this section, we do not assume that k is algebraically closed.

Let $V \subset \mathbf{P}^n$ be an r-dimensional subvariety (not necessarily a T-subvariety) with field of definition k. We consider the generic projections $f : \mathbf{P}^n \to \mathbf{P}^{r+1}$. These are defined by equations

$$\sum_{j=0}^{n} c_{ij} X_j = Y_i \tag{4.26}$$

where $(X_0, \ldots, X_n)$ is a point in $\mathbf{P}^n$ and $(Y_0, \ldots, Y_{r+1})$ is a point in $\mathbf{P}^{r+1}$, and the c_{ij} are algebraically independent over k. Ker$f$ is then an $(n-r-2)$-dimensional linear subspace of $\mathbf{P}^n$. Since the c_{ij} are algebraically independent over k, $\ker f$ and V do not intersect. So, $f(V)$ is an r-dimensional subvariety of $\mathbf{P}^{r+1}$. $f(V)$ is defined by a single equation $G_V(Y_i, c_{ij}) \in k[Y_0, \ldots, Y_{r+1}]$ depending only on the c_{ij}. We call this the Chow form of V. The coefficients of $G_V(Y_i, c_{ij})$ form the Chow coordinate.

If $\alpha = \sum n_V[V] \in Z_k(X)$ is effective, we define its Chow form to be

$$G_\alpha(Y_i, c_{ij}) = \prod G_V(Y_i, c_{ij})^{n_V}. \tag{4.27}$$

The coefficients of the Chow form of α form the Chow coordinate of α.

The set of all Chow coordinates of effective cycles in $Z_k(X)$ form the Chow scheme $\mathrm{Chow}_k(X)$ of X. If $\alpha \in Z_k(X)$, we will label its Chow coordinate by $\alpha \in \mathrm{Chow}_k(X)$.

The Chow scheme has an addition defined on it. If $\alpha,\ \beta \in Z_k(X)$, $\alpha + \beta \in \mathrm{Chow}_k(X)$ is the Chow coordinate of $\alpha + \beta \in Z_k(X)$.

The T-action on $\mathbf{P}^n$ induces a T-action on the Chow scheme. Suppose T acts on $\mathbf{P}^n$ by $t \cdot X_j = \chi_j(t)X_j$. We define $(t \cdot f)$ by

$$\sum \chi_j(t)c_{ij}X_j = Y_i. \tag{4.28}$$

Since $t \cdot - : V \to t \cdot V$ is an isomorphism, the field of definition of $t \cdot V$ is k. So, the $\chi_j(t)c_{ij}$ are algebraically independent over k, $G_{t \cdot V}(Y_i, \chi_j(t)c_{ij}) = G_V(Y_i, c_{ij})$ and the Chow form of $t \cdot V$ is $G_{t \cdot V}(Y_i, c_{ij}) = G_V(Y_i, \chi_j(t^{-1})c_{ij})$. This induces the T-action on $\text{Chow}_k(X)$.

Rational equivalence of cycles can also be defined in the Chow scheme. The statement of the following is in Fulton [6], Example 1.6.3.

Theorem 4.7.1 *Let $X \in \mathbf{P}^n$. If α, $\alpha' \in Z_k(X)$ are two rationally equivalent effective cycles in X, then there exists an effective cycle $\beta \in Z_k(X)$ and a map $g : \mathbf{P}^1 \to \text{Chow}_k(X)$ such that $g(0) = \alpha + \beta$ and $g(\infty) = \alpha' + \beta$.*

Proof: As usual, we need only consider the case of a single $k+1$-dimensional variety V and a single function $f \in R(V)^*$ such that $\text{div} f = \alpha - \alpha'$. If the zero cycle of $\text{div} f$ is γ, let $\beta = \gamma - \alpha$. The pole cycle of $\text{div} f$ is then $\alpha' + \beta$. f defines a function $f : V \to \mathbf{P}^1$, with $\alpha - \alpha' = [\text{div} f] = [f^{-1}(0)] - [f^{-1}(\infty)]$. We consider $f^{-1}(P)$ for $P \in \mathbf{P}^1$. We associate the Chow point of $[f^{-1}(P)]$ to P. This defines the morphism $g : \mathbf{P}^1 \to \text{Chow}_k(X)$ with $g(0) = \alpha + \beta$ and $g(\infty) = \alpha' + \beta$.

For the opposite direction, let $g : \mathbf{P}^1 \to \text{Chow}_k(X)$ with $g(0) = \alpha + \beta$, and $g(\infty) = \alpha' + \beta$. If $P \in \mathbf{P}^1$, we associate the scheme in $\mathbf{P}^n$ given by the cycle associated to the Chow point $f(P)$. We take the diagonal morphism $1 \times g : \mathbf{P}^1 \to \mathbf{P}^1 \times X$. Let $W = (1 \times g)(\mathbf{P}^1)$. The projection $\mathbf{P}^1 \times X \to \mathbf{P}^1$ induces a morphism $f : W \to \mathbf{P}^1$ which is a dominant, and $f^{-1}(0) = g(0) = \alpha + \beta$, $f^{-1}(\infty) = g(\infty) = \alpha' + \beta$. □

We now consider what happens if we demand that the cycles be T invariant. If V is T invariant, then its Chow point is as well. In the above, if the $k+1$-dimensional subvariety V is T invariant and $f \in R(V)^*$ is a weight vector of weight 0, then the morphism $f : V \to \mathbf{P}^1$

defined by f is equivariant where $\mathbf{P}^1$ has the trivial action. So, each $f^{-1}(P)$ is T invariant and the map $g : \mathbf{P}^1 \to \mathrm{Chow}_k(X)$ is a T-map.

Note that the curve $g(\mathbf{P}^1)$ is pointwise T invariant. So, $g(\mathbf{P}^1) \subset \mathrm{Chow}_k(X)^T$.

Proposition 4.7.2 *Two effective cycles α, $\alpha' \in Z_k^T(X)$ are T-rationally equivalent if there is a T-map $g : \mathbf{P}^1 \to \mathrm{Chow}_k(X)$, $g(0) = \alpha + \beta$ and $g(\infty) = \alpha' + \beta$.*

Proof: This is a result of the Theorem and discussion above. □

The following has been proved by Brion [3] in part 2 of the Theorem of section 1.3 in the case of solvable connected groups, where $T' = 1$. The proof we use is a re-writing of his proof.

Theorem 4.7.3 *Suppose $\varphi : T' \to T$ is a morphism of tori, $\mathbf{P}^n$ has a T and a T'-action defined on it and*

$$\begin{tikzcd} T' \times \mathbf{P}^n \arrow[r, "\varphi\times 1"] \arrow[dr] & T \times \mathbf{P}^n \arrow[d] \\ & \mathbf{P}^n \end{tikzcd} \tag{4.29}$$

commutes. Suppose that $X \subset \mathbf{P}^n$ is a normal T-scheme and hence a normal T'-scheme. Then the set of effective cycles in $Z_k^{T'}(X)$ that are T'-rationally equivalent to 0 is generated by the cycles $[\mathrm{div}(f)]$ where V is a T invariant $k+1$-dimensional subvariety of X and $f \in R(V)^*$ is a T weight vector of weight 0 with respect to T'.

Proof: By Proposition 4.4.2, we can assume that α, $\alpha' \in Z_k^T(X)$ and are T' rationally equivalent. So, there is a $\beta' \in Z_k^{T'}(X)$ such that in $\mathrm{Chow}_k(X)$ there is a T'-map $g' : \mathbf{P}^1 \to \mathrm{Chow}_k(X)$ with $g'(0) = \alpha + \beta'$ and $g'(\infty) = \alpha' + \beta'$. Using Proposition 4.4.2, we can connect β' to a T invariant point β by a T' invariant curve. So, there is a $\beta \in Z_k^T(X)$ and a T-map $g : \mathbf{P}^1 \to \mathrm{Chow}_k(X)$ such that $g(0) = \alpha + \beta$ and $g(\infty) = \alpha' + \beta$. We can move this with respect to T to a T invariant curve. Since $g(\mathbf{P}^1) \in \mathrm{Chow}_k(X)^{T'}$ and since the actions of T and T' commute, we move the curve within $\mathrm{Chow}_k(X)^{T'}$ and the new curve is also pointwise T' invariant. Each component of the curve has a T-action, and has at least two fixed points. We normalize each component to get a collection of curves isomorphic to $\mathbf{P}^1$. The action on each component extends to one on its normalization, and the action on $\mathbf{P}^1$ is by a character $\chi(t)$ with $\chi \circ \varphi(t') = 1$. So from the proposition above, the two cycles are T'-rationally equivalent and the T'-rational functions involved are weight vectors of T of weight zero with respect to T'. □

Corollary 4.7.4 *If X is a normal quasi-projective variety and $\bar{X} \subset \mathbf{P}^n$, with notation as above, the obvious map $A_k^T(X) \to A_k^{T'}(X)$ is surjective with kernel generated by the cycles described above. In particular, if $T' = 1$, $A_k(X)$ is generated by T invariant cycles, and those rationally equivalent to 0 can be obtained as described above.*

Proof: If X is quasi-projective, we can close it in $\mathbf{P}^n$ to get $\bar{X}$. The result is true for effective cycles on $\bar{X}$, and from the exact sequence of section 4.6, the result is true for effective cycles on X. Since the effective cycles generate $A_k^T(X)$, the result is true in general. □

Example: We find $A_k^T(F)$ where F is an n-dimensional representation space for T.

We assume the T-action is diagonal. Let $F = Spec(k[x_1, \ldots, x_n])$, and let the vector in F related to x_i be X_i. Let $F_{(i_1,\ldots,i_k)}$ be the subspace of F spanned by $X_{i_1}, \ldots, X_{i_k}$. Let $T' = \oplus_{i=1}^n G_m$, and let $(t_1, \ldots, t_n) \cdot X_i = t_i X_i$ for all i. The T'-subvarieties of F are the $F_{(i_1,\ldots,i_k)}$ and the weight vectors of $R(F)^*$ are of the form $\prod x_i^{n_i}$ for $n_i \in \mathbf{Z}$. The weight zero vectors are just the constants. So,

$$A_k^{T'}(F) \simeq \oplus_{i=1}^{\binom{n}{k}} \mathbf{Z} \tag{4.30}$$

is the free group generated by the $F_{(i_1,\ldots,i_k)}$.

Suppose $T \neq T'$. Then $T \subset T'$. Consider the cycles generated by restricting the $f = \prod x_i^{n_i}$ that are weight vectors of weight 0 with respect to T to the $F_{(i_1,\ldots,i_{k+1})}$. Let B be the submodule of $A_k^T(F)$ generated by all these $\mathrm{div} f$. Then,

$$A_k^T(F) = A_k^{T'}(F)/B. \tag{4.31}$$

We can also describe this using linear equations. Recall that if $\dim T = r$ and the T-action on F was defined by weights $\lambda_1, \dots, \lambda_n$ where $\lambda_i = (\lambda_{1i}, \dots, \lambda_{ri})$ we can represent the action by

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ \vdots & \vdots & & \vdots \\ \lambda_{r1} & \lambda_{r2} & \dots & \lambda_{rn}. \end{pmatrix} \tag{4.32}$$

To get a T' weight vector of $R(F_{(i_1,\dots,i_{k+1})})^*$ that is of weight 0 with respect to T, we replace all but the $i_1, \dots, i_{k+1}$ columns by columns of 0's to get a matrix M. Let Y be a solution of $MY = 0$ where $Y \in \mathbf{Z}^n$ and $Y_{i_j} = 0$ for $1 \leq j \leq k$. Since $MY = 0$ if and only if $\prod x_i^{Y_i}$ is a weight zero vector with respect to T these Y correspond to the T' weight vectors in $R(F_{(i_1,\dots,i_k)})^*$ that are of weight zero with respect to T. Let B be the subgroup of $\oplus_{i=1}^{\binom{n}{k}} \mathbf{Z}$ generated by the $\ker M$ for all choices of $i_1, \dots, i_{k+1}$ as above. Then,

$$A_k^T(F) = \oplus_{i=1}^{\binom{n}{k}} \mathbf{Z}/B. \tag{4.33}$$

Remark: If $X = \mathbf{P}^n$ has a T-action, the same method of this example shows that

$$A_k^T(\mathbf{P}^n) = \oplus_{i=1}^{\binom{n}{k}} \mathbf{Z}/B, \tag{4.34}$$

where we define B as above and we also require that Y be such that $\sum Y_i = 0$.

Example: Let F^{ss} be the semi-simple points of F. Then F^{ss} is F less a few linear subspaces each of which is spanned by a subset of the X_i. Let B' be the B of the previous example union the free group on the set of k-dimensional T'-subspaces contained in the deleted linear subspaces. Then,

$$A_k^T(X) = \mathbf{Z}^{\binom{n}{k}}/B'. \tag{4.35}$$

Note that this is the group Ellingsrud and Stromme [5] have calculated for $A_{k-r}(F^{ss}//T)$ when this is non-singular.

Example: We work out a concrete example of the above. This example also shows that the equivariant Chow group and the usual Chow group are not necessarily the same.

Let $F = \mathbf{A}^3$ have the $T = G_m \otimes G_m$-action defined by

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 3 \end{pmatrix}. \tag{4.36}$$

The T-vectors of weight zero correspond to the solutions of

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -3 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = MY = 0. \tag{4.37}$$

$\ker M = (1,3,2)\mathbf{Z}$, and

$$\begin{aligned} A_2^T(F) &= \mathbf{Z}^3/(1,3,2)\mathbf{Z} \\ A_2^T(F^{ss}) &= \mathbf{Z}^3/((1,3,2),(0,1,0)) = \mathbf{Z} \oplus \mathbf{Z}. \end{aligned} \tag{4.38}$$

4.8 Affine T-Bundles

Theorem 4.8.1 *Suppose X is a normal quasi-projective scheme. If $E \simeq X \times F$ where F is a T-representation space, then there is a surjective map*

$$\oplus_{j=0}^{e} (A_{k-j}^T(X) \oplus A_j^T(F)) \to A_k^T(E). \tag{4.39}$$

In particular, if X has the trivial T-action, then the map is an isomorphism.

Proof: We assume that T acts diagonally on $F = Spec(k[x_1, \ldots, x_e])$. Let $T' = \oplus_{i=1}^{e} G_m$. Let T' act trivially on X, and let $(t_1, \ldots, t_e) \cdot x_i = t_i^{-1} x_i$ for all i. We consider E with $T \times T'$ action where T acts trivially on F. The k-dimensional $T \times T'$ invariant subvarieties of E are of the form $W = V \times F_{(i_1,\ldots,i_j)}$, where V is a $k-j$-dimensional T-subvariety of X. So,

$$Z_{k-j}^T(X) \oplus Z_j^{T'}(F) = Z_{k-j}^{T \times T'}(X) \oplus Z_j^{T \times T'}(F) = Z_k^{T \times T'}(E). \tag{4.40}$$

We need to show that the map is defined with respect to T-rational equivalence and is surjective. Since any T-cycle in $Z_k^T(F)$ and $Z_k^T(E)$ is T-rationally equivalent to a T'-cycle, we need only consider T'-cycles. If $W = V \times F_{(i_1,\dots,i_j)}$, the $T \times T'$ weight vectors of $R(W)^*$ are of the form $f \prod_{i=1}^{e} x_i^{n_i}$, where $f \in R(V)^*$ and $n_i \in \mathbf{Z}$. If the weight of x_i with respect to T is $-\lambda_i$, we need $\lambda_f = \sum n_i \lambda_i$. Since the weight vectors with $\lambda_f = 0$ and $\sum n_i \lambda_i = 0$ are in this set, the map passes to T-rational equivalence and it is a surjection.

To get the isomorphism in the case of a trivial T-action on X, let $W = V \times F_{(i_1,\dots,i_j)}$ and let $f \prod_{i=1}^{e} x_i^{n_i} \in R(W)^*$ be a weight zero vector. Since $\lambda_f = 0$ for all $f \in R(V)^*$, we have $\sum n_i \lambda_i = 0$, and the cycle $\mathrm{div}(\prod_{i=1}^{e} x_i^{n_i})$ in $Z_j^T(F)$ must have been T-rationally equivalent to zero. So, we have an injection and hence an isomorphism. □

Remark: In the theorem, if T is the 1-dimensional torus, X has trivial T-action and the weights of F are all 1, then we get

$$\oplus_{j=0}^{e} A_{k-j}^T(X) \simeq A_k^T(E). \tag{4.41}$$

So, in this case, $A_k^T(E)$ is the same as $A_k(\mathbf{P}(E))$.

Proposition 4.8.2 *Supppose $p : X \to Y$ is a T-morphism of T-schemes, Y has trivial T-action, and there is a covering of Y by open sets U such that $X = U \times T$. If $\dim T = r$, then*

$$p^* : A_k(Y) \to A_{k+r}^T(X) \tag{4.42}$$

exists, and is an isomorphism.

Proof: Since p is a flat T-morphism, we have existence.

The T-subvarieties of X are of the form $W = p^{-1}(V)$ where V is a subvariety of X. To see this, note that the result is certainly true locally, and glueing together gives the global result. This shows that $p^* : Z_k(Y) \to Z_{k+r}^T(X)$ is a surjection. So, on passing to T-rational equivalence, $p^* : A_k(Y) \to A_{k+r}^T(X)$ is a surjection as well.

For injectivity, T weight vectors $g \in R(W)^*$ of weight 0 are locally of the form $f \otimes 1 \in R(U \times T)^*$ for $f \in R(U)^*$. However, $[\mathrm{div}(f \otimes 1)] = p^*[\mathrm{div} f]$. So, $\ker p^* = 0$ in $A_k(X)$ and p^* is injective. □

4.9 Notes

We could consider different types of equivariant Chow groups. For example, we could demand that the f_i be weight vectors, but not necessarily of weight 0. For this form of equivariant Chow group, all the properties we have considered in this section hold. In particular, proper pushforward, flat pullback, the alternate definition of rational equivalence where $\mathbf{P}^1$ is replaced by $\mathbf{P}^1_\lambda$ for some unspecified λ and the exact sequence result hold. However, as we have seen in the change of torus section, for normal projective varieties, this form of equivariant Chow group is just the usual Chow group. We could also define $A_k^T(X)$ as $Z_k^T(X)$ modulo rational equivalence. The problem with this group is that its properties are too hard to determine. In particular, pushforward and pullback become hard to show. Also, for normal projective varieties this is only the usual Chow group.

Chapter 5

Multiplicities on Varieties

We consider the results of the last two chapters applied to varieties. We start by considering projective varieties. On a projective variety X, it is possible to define equivariant degrees and multiplicities that are invariants of $A_k^T(X)$. As such, the equivariant degree is related to the usual degree. The relationship between the equivariant multiplicity and the usual multiplicity is not quite as strong. We consider these two equivariant objects in the first section. In the second section, we consider the usual equivariant multiplicity as defined by Rossmann [17], or Borho, Brylinski and Macpherson [1] in a particular case.

Since there seems to be some confusion in the literature concerning signs of characters, we state the following convention explicitly:

Convention:

> If $\mathbf{A}^n$ is the vector space with basis $X_1, \ldots, X_n$ then its structure sheaf is $\mathcal{O}_{\mathbf{A}^k} = Spec(k[x_1, \ldots, x_n])$. We assume the X_i are weight vectors of weight λ_i. This means that the related functions x_i have weight $-\lambda_i$. The net effect of this is that the equivariant multiplicity is no longer alternating in sign.

5.1 Multiplicity on Projective Varieties

In this section we develop the basic properties of equivariant multiplicities of projective T-varieties. We define an equivariant degree as well as an equivariant multiplicity. Unfortunately, this multiplicity is not defined in the generality we would like. A lot of this section consists of showing that we can extend the equivariant multiplicity to all cases. We accomplish this by using the equivariant version of a result of Mumford. We also show that these are invariants of $A_k^T(X)$. We end this section with a useful result involving the equivariant multiplicities of all the fixed points of $\mathbf{P}^n$. The definitions and results of this section are new.

Unless otherwise stated we are assuming that $x = P_0$.

Definition: If X is a T-scheme in $\mathbf{P}^n$, it is defined by a homogeneous T invariant ideal $I \subset k[x_0, \ldots, x_n]$. Let $deg_T(X, \mathbf{P}^n)$ be the equivariant multiplicity of the origin in the affine cone defined by I in $\mathbf{A}^{n+1}$.

Proposition 5.1.1 *1.* *$deg_T(X, \mathbf{P}^n)$ is a polynomial over $\mathbf{Q}$ in $\lambda_0, \ldots, \lambda_n$ of degree* $\operatorname{codim}(X, \mathbf{P}^n)$.

Proof: this results from Proposition 3.2.1 and Theorem 3.2.4. □

Definition: Let x be a fixed point of $\mathbf{P}^n$, U_0 be the open affine T-subset of $\mathbf{P}^n$ defined by inverting x_0 and let X be a T-subscheme of $\mathbf{P}^n$. $X\,|_{U_0}$ is defined by a T invariant ideal $J \subset k[x_1/x_0, \ldots x_n/x_0]$. If x is an isolated fixed point, let

$$mult_{T,\mathbf{P}^n}(x, X) = mult_T(k[x_1/x_0, \ldots, x_n/x_0]/J, k[x_1/x_0, \ldots, x_n/x_0]). \tag{5.1}$$

If x is not isolated, we consider $deg_T(X, \mathbf{P}^n)$ as a polynomial and define

$$mult_{T,\mathbf{P}^n}(x, X) = deg_T(0, \lambda_1 - \lambda_0, \ldots, \lambda_n - \lambda_0). \tag{5.2}$$

Remark: Note that J need not be homogeneous in the usual sense. This is the reason we have a different definition for non-isolated fixed points. We show later that the definition is consistent.

Also, note that the definition makes sense even if $x \notin X$. We show later that in this case the equivariant multiplicity is 0.

As for the equivariant degree,

Corollary 5.1.2 *$mult_{T,\mathbf{P}^n}(x,X)$ is a polynomial over $\mathbf{Q}$ in $\lambda_1-\lambda_0,\ldots,\lambda_n-\lambda_0$ of degree* $\operatorname{codim}(X,\mathbf{P}_n)$.

Remark: If $T=G_m$ and we consider the G_m-action on $\mathbf{A}^{n+1}$ where x_i has weight 1 for all i, then $deg_T(X,\mathbf{P}^n)$ is just the usual degree of a projective scheme. With this action, the weight of the x_i/x_0 in U_0 are all 0. So, $mult_{T,\mathbf{P}^n}(x,X)=0$. Needless to say, this differs from the usual multiplicity for a point in a projective scheme. We can obtain the usual one by using the G_m-action where x_0 has weight 0 and x_i has weight 1 if $i\neq 0$. If the tangent cone $C_x(X)$ is considered as a subscheme of $\mathbf{P}^n$, the usual multiplicity of x in X is then $mult_{T,\mathbf{P}^n}(x,C_x(X))$.

From our results on equivariant multiplicities of modules, we easily get:

Proposition 5.1.3 *1. For $0\leq m\leq n$, let $\mathbf{P}^m=Proj(k[x_0,\ldots,x_m])$, and $\mathbf{A}^{n-m}=Spec(k[x_{m+1},\ldots,x_n])$, let $X\subset\mathbf{P}^m$ and let $x\in(\mathbf{P}^m)^T$. Let $f:\mathbf{P}^m\times\mathbf{A}^{n-m}\to\mathbf{P}^m$ be the projection, and let i be the inclusion $\mathbf{P}^m\hookrightarrow\mathbf{P}^n$. We have the diagram:*

$$\begin{array}{ccccc} f^{-1}(X) & \longrightarrow & \mathbf{P}^m\times\mathbf{A}^{n-m} & \subset & \mathbf{P}^n \\ \Big\downarrow & & \Big\downarrow{\scriptstyle f} & & \\ X & \longrightarrow & \mathbf{P}^m. & & \end{array} \tag{5.3}$$

Let $Y=\overline{f^{-1}(X)}\subset\mathbf{P}^n$. Then

$$deg_T(Y,\mathbf{P}^n)=deg_T(X,\mathbf{P}^m). \tag{5.4}$$

If $i(x)$ is an isolated fixed point of $\mathbf{P}^n$,

$$mult_{T,\mathbf{P}^n}(i(x),Y)=mult_{T,\mathbf{P}^m}(x,X). \tag{5.5}$$

2. With notation as in 1,

$$deg_T(i(X),\mathbf{P}^n)=\Big(\prod_{i=m+1}^{n}\lambda_i\Big)deg_T(X,\mathbf{P}^m) \tag{5.6}$$

and

$$mult_{T,\mathbf{P}^n}(i(x), i(X)) = \prod_{i=m+1}^{n} (\lambda_i - \lambda_0) mult_{T,\mathbf{P}^m}(x, X). \tag{5.7}$$

3. Let $\varphi : T' \to T$ be a morphism of tori, $\mathbf{P}^n$ be T and T' invariant, and let the action of T' on $\mathbf{P}^n$ be defined by the weights $\lambda'_0, \ldots, \lambda'_n$. If X is a T and a T'-subvariety of $\mathbf{P}^n$ and

$$\begin{tikzcd} T' \times \mathbf{P}^n \arrow[r, "\varphi \times 1"] \arrow[dr] & T \times \mathbf{P}^n \arrow[d] \\ & \mathbf{P}^n \end{tikzcd} \tag{5.8}$$

commutes, then

$$\prod (\lambda_i \circ d\varphi) deg_{T'}(X, \mathbf{P}^n) = \prod \lambda'_i (deg_T(X, \mathbf{P}^n) \circ d\varphi). \tag{5.9}$$

If x is an isolated fixed point of $\mathbf{P}^n$, then,

$$\prod (\lambda_i - \lambda_0) \circ d\varphi mult_{T,\mathbf{P}^n}(x, X) = \prod (\lambda'_i - \lambda'_0 (mult_{T,\mathbf{P}^n}(x, X) \circ d\varphi). \tag{5.10}$$

4. Let H be the T hypersurface of $\mathbf{P}^n$ defined by the function f of weight $-\lambda_H$. If H intersects X in codimension one, then

$$deg_T(X \cap H, \mathbf{P}^n) = \lambda_H deg_T(X, \mathbf{P}^n). \tag{5.11}$$

Suppose x is an isolated fixed point of $\mathbf{P}^n$. If $H|_{U_0}$ is defined by a function of weight $-\lambda_{H|_{U_0}}$ and H intersects X in codimension one, then

$$mult_{T,\mathbf{P}^n}(x, X \cap H) = \lambda_{H|_{U_0}} mult_{T,\mathbf{P}^n}(x, X). \tag{5.12}$$

5. Let X be a T-subscheme of $\mathbf{P}^n$ with components $V_1, \ldots, V_m$. If the geometric multiplicity of V_i in X is m_i, then

$$deg_T(X, \mathbf{P}^n) = \sum_{i=1}^{m} m_i deg_T(V_i, \mathbf{P}^n)). \tag{5.13}$$

6. With notation as in 4 and 5,

$$\lambda_H deg_T(X \cap H, \mathbf{P}^n) = \sum_{i=1}^{m} m_i deg_T(X, \mathbf{P}^n). \tag{5.14}$$

Proof: These are all direct consequences of the results in section 3.2. □

Note that we have not stated 6 and 7 for $mult_{T,\mathbf{P}^n}(x, X)$. These are true, but we need to know that the definition of $mult_{T,\mathbf{P}^n}(x, X)$ is consistent.

We can extend $deg_T(x, \mathbf{P}^n)$ linearly to cycles $\alpha \in \mathbf{Z}_k^T(\mathbf{P}^n)$.

Corollary 5.1.4 *1. If X is a subscheme of $\mathbf{P}^n$, then*

$$deg_T(X, \mathbf{P}^n) = deg_T([X], \mathbf{P}^n). \tag{5.15}$$

2. If $\alpha \in Z_k^T(\mathbf{P}^n)$ is T-rationally equivalent to 0, then

$$deg_T(\alpha, \mathbf{P}^n) = 0. \tag{5.16}$$

In particular, $deg_T(\alpha, \mathbf{P}^n)$ is an invariant of the cycle $\alpha \in A_k^T(X)$.

Proof: The first statement is a rephrasing of 6 above. For the second, α is T-rationally equivalent to 0 if there is a collection of $k+1$-dimensional T-subvarieties V_i and weight vectors $f_i \in R(V_i)^*$ of weight 0 such that $\alpha = \sum \mathrm{div} f_i$. Since the f_i are quotients of weight vectors of $\mathcal{O}_X$, 4 yields the result. □

We now relate $deg_T(X, \mathbf{P}^n)$ to $mult_{T,\mathbf{P}^n}(x, X)$.

Proposition 5.1.5 *If $x \in (\mathbf{P}^n)^T$ is isolated, then*

$$deg_T(X, \mathbf{P}^n)(0, \lambda_1 - \lambda_0, \ldots, \lambda_n - \lambda_0) = mult_{T,\mathbf{P}^n}(x, X). \tag{5.17}$$

Proof: Suppose $X|_{U_0}$ is defined by the ideal J. We can homogenize the generators of J to get an ideal $J' \subset k[x_0, \ldots, x_n]$. J' defines a scheme $X' \in \mathbf{P}^n$, each component of X extends to one of X' with the same geometric multiplicity, and all other components of X' are contained in $\mathbf{P}^n - U_0$. Resolving $k[x_0 \ldots, x_n]/J'$ and $k[x_1/x_0, \ldots, x_n/x_0]/J$ yield the same sequences, but with weights $\lambda_0, \ldots \lambda_n$ and $\lambda_1 - \lambda_0, \ldots, \lambda_n - \lambda_0$ respectively. So viewing deg_T as a polynomial in $\Lambda(T)$, we get,

$$deg_T(X', \mathbf{P}^n)(0, \lambda_1 - \lambda_0, \ldots, \lambda_n - \lambda_0) = mult_{T,\mathbf{P}^n}(x, X). \tag{5.18}$$

Now, if $V \subset \mathbf{P}^n - U_0 = \mathbf{P}^{n-1}$ we inject V into $\mathbf{P}^n$, and Proposition 5.1.3 part 2 gives us,

$$deg_T(V, \mathbf{P}^n) = \lambda_0 deg_T(V, \mathbf{P}^{n-1}). \tag{5.19}$$

So,

$$deg_T(V, \mathbf{P}^n)(0, \lambda_1 - \lambda_0, \ldots \lambda_n - \lambda_0) = 0, \tag{5.20}$$

and

$$deg_T(X, \mathbf{P}^n)(0, \lambda_1 - \lambda_0, \ldots, \lambda_n - \lambda_0) = mult_{T,\mathbf{P}^n}(x, X). \tag{5.21}$$

□

This show that defining $mult_{T,\mathbf{P}^n}(x, X) = deg_T(X, \mathbf{P}^n)(0, \lambda_1 - \lambda_0, \ldots, \lambda_n - \lambda_0)$ for non-isolated fixed points is consistent.

We can extend the definition of $mult_{T,\mathbf{P}^n}(x, X)$ linearly to cycles $\alpha \in Z_k^T(\mathbf{P}^n)$.

Corollary 5.1.6 *All the statements of Proposition 5.1.3 hold for $mult_{T,\mathbf{P}^n}(x, X)$ even when x is not an isolated fixed point of $\mathbf{P}^n$. In particular, if X is a T-subscheme of $\mathbf{P}^n$, then*

$$mult_{T,\mathbf{P}^n}(x, X) = mult_{T,\mathbf{P}^n}(x, [X]). \tag{5.22}$$

If $\alpha \in Z_k^T(\mathbf{P}^n)$ is T-rationally equivalent to 0, then

$$mult_{T,\mathbf{P}^n}(x, X) = 0. \tag{5.23}$$

Note that this implies that unlike the usual multiplicity, $mult_{T,\mathbf{P}^n}(\alpha, X)$ is an invariant of $\alpha \in A_k^T(X)$.

Consider the G_m-action on $\mathbf{P}^n$ where in $\mathbf{A}^{n+1}$, x_0 has weight 0 and x_i has weight 1 for $i \neq 0$. If X is a G_m-scheme, *i.e.* $X = C_x(X)$, then

$$deg_{G_m}(X, \mathbf{P}^n)(0, 1, \ldots, 1) = mult_{T,\mathbf{P}^n}(x, C_x(X)). \tag{5.24}$$

If T acts on $\mathbf{P}^n$ where x_i has weight 1 for all i, then since the ideal defining $C_x(X)$ in $k[x_0, \ldots, x_n]$ has its generators in $k[x_1, \ldots, x_n]$, $deg_T(C_x(X), \mathbf{P}^n) = deg_{G_m}(C_x(X), \mathbf{P}^n)$. So, the usual degree of $C_x(X)$ in $\mathbf{P}^n$ and the usual multiplicity of x in X in $\mathbf{P}^n$ agree. We have a similar result for the equivariant degree and multiplicity. This is the equivariant version of a result of Mumford.

Let X be a k-dimensional subscheme of $\mathbf{P}^n$. We consider moving X around with respect to a 1-dimensional torus acting on $\mathbf{P}^n$. Let the graph morphism be

$$\begin{aligned} \sigma : X \times G_m &\hookrightarrow \mathbf{P}^n \times \mathbf{P}^1 \\ (x, t) &\mapsto (t \cdot x, t) \end{aligned} \tag{5.25}$$

and let $p_2 : \mathbf{P}^n \times \mathbf{P}^1 \to \mathbf{P}^1$ be the projection. If $Y = \overline{\sigma(X \times G_m)}$ in $\mathbf{P}^n \times \mathbf{P}^1$, let $f : Y \to \mathbf{P}^1$ be the map induced by p_2. With notation as in section 4.4, we define $\lim_{t\to\infty} t \cdot X$ as:

$$X' = \lim_{t\to\infty} t \cdot X = Y(\infty). \tag{5.26}$$

Since $[X'] = [X] \in A_k^T(\mathbf{P}^n)$, $deg_T(X, \mathbf{P}^n) = deg_T(X', \mathbf{P}^n)$ and $mult_{T,\mathbf{P}^n}(x, X) = mult_{T,\mathbf{P}^n}(x, X')$.

Suppose we moved X with respect to the torus defined by the weights $\{0, 1, \ldots, 1\}$ in $\mathbf{A}^{n+1}$.

Theorem 5.1.7 *Let* $mult'_{T,\mathbf{P}^n}(x, X) = deg_T(C_x(X), \mathbf{P}^n)$ *and let*

$$p_x : X - x \subset Bl_x(X) \subset Bl_x(\mathbf{P}^n) \subset \mathbf{P}^n \times \mathbf{P}^{n-1} \xrightarrow{p_2} \mathbf{P}^{n-1} \tag{5.27}$$

where $\mathbf{P}^{n-1} = Proj(k[x_1, \ldots, x_n])$. *Then, if* $k = \mathbf{C}$,

$$deg_T(X, \mathbf{P}^n) - mult'_{T,\mathbf{P}^n}(x, X) = \begin{cases} \lambda_0 \deg p_x deg_T(p_x(X - \{x\}), \mathbf{P}^n) & \text{if } X \neq C_x(X) \\ 0 & \text{if } X = C_x(X)\ . \end{cases} \tag{5.28}$$

If $k \neq \mathbf{C}$,

$$deg_T(X, \mathbf{P}^n) - mult'_{T,\mathbf{P}^n}(x, X) = \begin{cases} \lambda_0 deg_T(p_{2*}[\lim_{t\to\infty} t \cdot (X - \{x\})], \mathbf{P}^{n-1}) & \text{if } X \neq C_x(X) \\ 0 & \text{if } X = C_x(X). \end{cases} \tag{5.29}$$

Proof: We can describe the map p_x using the action above. We end up moving X to the cycle with components $C_x(X)$ and a subvariety of $\mathbf{P}^n$ that is T-isomorphic to $p_x(X - \{x\})$. Consider $X - \{x\} \subset \mathbf{P}^n - \{x\}$. $p_x(X - \{x\})$ is the variety defined by $\lim_{t\to\infty} t \cdot (X - \{x\})$. We can see this by noting that $\mathbf{P}^n - \{x\} = \mathbf{A} \times \mathbf{P}^{n-1}$. The weights of the action are $\{0, 1, \ldots, 1\}$, so that on points $\lim_{t\to\infty} t \cdot (v, y) = p_2(v, y) = y$. Note that restricting to U_0 gives $\lim_{t\to\infty} t \cdot (X\,|_{U_0}) = C_x(X)|_{U_0}$. So,

$$[X] = [\lim_{t\to\infty} t \cdot X] = [C_x(X)] + [X'], \tag{5.30}$$

where $X' = \lim_{t\to\infty} t \cdot (X - \{x\})$ in $\mathbf{P}^n$. If $k \neq \mathbf{C}$, $deg_T([X'], \mathbf{P}^n) = \lambda_0 deg_T(p_{2*}([X'], \mathbf{P}^{n-1}))$ and we have the result. Since p_x is proper, if $k = \mathbf{C}$,

$$[X'] = \deg p_x [p_x(X - \{x\})] \tag{5.31}$$

and taking equivariant degrees gives

$$deg_T(X, \mathbf{P}^n) - mult'_{T,\mathbf{P}^n}(x, X) = \begin{cases} \lambda_0 \deg p_x deg_T(p_x(X - \{x\}), \mathbf{P}^{n-1}) & \text{if } X \neq C_x(X) \\ 0 & \text{if } X = C_x(X). \end{cases} \tag{5.32}$$

□

Corollary 5.1.8 *1.*

$$\begin{aligned} deg_T(X, \mathbf{P}^n)(0, \lambda_1 - \lambda_0, \ldots, \lambda_n - \lambda_0) &= mult'_{T,\mathbf{P}^n}(x, X)(0, \lambda_1 - \lambda_0, \ldots, \lambda_n - \lambda_0) \\ &= mult_{T,\mathbf{P}^n}(x, X). \end{aligned} \tag{5.33}$$

2. If $x \notin X$, then $mult_{T,\mathbf{P}^n}(x, X) = 0$.

3. If the fixed point component of X containing x does not contain the fixed point component of $\mathbf{P}^n$ containing x, then $mult_{T,\mathbf{P}^n}(x, X) = 0$.

Proof: For 1, the first equality results from the theorem, the second has already been shown.

2 results from the theorem. If $x \notin X$, $C_x(X)$ does not exist and moving X as in the theorem, we get $[X] = [X'] = [\lim_{t\to\infty} t \cdot X]$. $deg_T(X', \mathbf{P}^n)(0, \lambda_1 - \lambda_0, \dots, \lambda_n - \lambda_0) = 0$ then yields the result.

For 3, if X does not contain the fixed point component of $\mathbf{P}^n$ containing x, then there is a point y in the same fixed point component of $\mathbf{P}^n$ as x, but which is not in X. With no loss of generality, we can assume that x is P_1 and y is P_0. Moving X as in the theorem, we get

$$deg_T(X, \mathbf{P}^n) = \lambda_0 deg_T(p_{2*}(\lim_{t\to\infty} t \cdot X), \mathbf{P}^{n-1}). \tag{5.34}$$

Since $\lambda_1 = \lambda_0$,

$$deg_T(X, \mathbf{P}^n)(0, 0, \dots, \lambda_n - \lambda_0) = mult_{T,\mathbf{P}^n}(x, X) = 0. \tag{5.35}$$

□

Let G_m act on $\mathbf{A}^{n+1}$ with weights $(0, 1, \dots, 1)$. In this case, since the usual multiplicity is given by $mult_{G_m,\mathbf{P}^n}(x, C_x(X))$, the theorem states that if $deg(X)$ and $mult(x, X)$ are the usual degree and multiplicity then,

$$deg(X) - mult(x, X) = \begin{cases} \deg p_x deg(p_x(X - \{x\})) & \text{if } X \neq C_x(X) \\ 0 & \text{if } X = C_x(X). \end{cases} \tag{5.36}$$

This is precisely the result that Mumford [14] Theorem 5.11 gets.

By the example after Corollary 4.7.4 any T-cycle in $Z_k^T(\mathbf{P}^n)$ is T-rationally equivalent to a cycle whose components are defined by ideals of the form $(x_{i_1}, \dots x_{i_{n-k}})$. We can also show this more explicitly. In the theorem, we moved X with respect to the action defined by the weights $(0, 1, \dots, 1)$ in $\mathbf{A}^{n+1}$. We can move the resulting cycle with respect to the action defined by the weights $(1, 0, 1, \dots, 1)$ in $\mathbf{A}^{n+1}$. Repeating this for each new cycle, we get a cycle that is invariant under the maximal torus of $GL(\mathbf{A}^{n+1})$ acting on $\mathbf{P}^n$. This means that the components of the cycle are defined by ideals of the form $(x_{i_1}, \dots x_{i_{n-k}})$. This gives an easy way to find $deg_T(X, \mathbf{P}^n)$ and $mult_{T,\mathbf{P}^n}(x, X)$, and,

Proposition 5.1.9 *$deg_T(X, \mathbf{P}^n)$ is a polynomial over $\mathbf{Z}$ in the $\lambda_0, \ldots, \lambda_n$.*

$mult_{T,\mathbf{P}^n}(x, X)$ is a polynomial over $\mathbf{Z}$ in $\lambda_1 - \lambda_0, \ldots . \lambda_n - \lambda_0$.

Proof: if V is defined by the ideal $(x_{i_1}, \ldots x_{i_{n-k}})$, then $deg_T(V, \mathbf{P}^n) = \prod_{j=1}^{n-k} \lambda_{i_j}$ and $mult_{T,\mathbf{P}^n}(x, V) = \prod_{j=1}^{n-k}(\lambda_{i_j} - \lambda_0)$. Since X is T-rationally equivalent to a cycle whose components are of this form, we have the result. □

We now consider how the equivariant multiplicities at the fixed points of $\mathbf{P}^n$ are related.

Proposition 5.1.10 *Suppose $\mathbf{P}^n$ has only isolated fixed points. If* $\dim X > 0$*, then*

$$\sum_{i=0}^{n} \frac{1}{\prod_{j \neq i}(\lambda_j - \lambda_i)} mult_{T,\mathbf{P}^n}(P_i, X) = 0. \tag{5.37}$$

If $\dim X = 0$ *and X has only one component P_0, then*

$$mult_{T,\mathbf{P}^n}(P_0, X) = \prod(\lambda_j - \lambda_0)m, \tag{5.38}$$

where m is the geometric multiplicity of P_0 in X.

Proof: We first consider $\mathbf{P}^n$ itself. The equivariant multiplicity of $\mathbf{P}^n$ at each fixed point is 1, and after taking common denominators in the sum, we find that the numerator is:

$$\det \begin{pmatrix} 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^{n-1} & 1 \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} & 1 \end{pmatrix} = 0. \tag{5.39}$$

If $m < n$, we inject $\mathbf{P}^m \hookrightarrow \mathbf{P}^n$, where $\mathbf{P}^m = Proj(k[x_0, \ldots, x_m])$. Since $mult_{T,\mathbf{P}^n}(P_i, X) = 0$ if $P_i \notin \mathbf{P}^m$, by Corollary 5.1.6 and Proposition 5.1.3, we get,

$$\begin{aligned} \sum_{i=0}^{n} \frac{1}{\prod_{j \neq i}(\lambda_j - \lambda_i)} mult_{T,\mathbf{P}^n}(P_i, \mathbf{P}^m) &= \sum_{i \leq m} \frac{\prod_{j > m}(\lambda_j - \lambda_i)}{\prod_{j \neq i}(\lambda_j - \lambda_i)} mult_{T,\mathbf{P}^m}(P_i, \mathbf{P}^m) \\ &= \sum_{i=0}^{m} \frac{1}{\prod_{\substack{j \leq m \\ j \neq i}}(\lambda_j - \lambda_i)} mult_{T,\mathbf{P}^m}(P_i, \mathbf{P}^m) \qquad (5.40) \\ &= 0. \end{aligned}$$

If $\dim X = 0$, then the proposition results from Corolary 5.1.6, Proposition 5.1.3 part 6 and Lemma 3.2.3. □

5.2 Equivariant Multiplicities on Varieties

We define the equivariant multiplicity of a fixed point x of a T-subscheme X of a smooth T-variety Y. The definition we use is the the one used by Rossmann [17]. We show that in the case we are concerned with, namely where there is an affine neighbourhood in Y containing x that is T-isomorphic to a T-representation space, the equivariant multiplicity considered in the last section can be extended to one for Y and it is the same as that considered by Rossmann. In particular, the equivariant multiplicity is an invariant of $A_k^T(X)$, and is defined even when $x \notin X$.

Let X be an equidimensional T-subscheme of an ambient smooth T-variety Y. If $x \in X$ is a fixed point of T in X, we define the equivariant multiplicity $mult_T(x, X, Y)$ of x in X relative to Y:

Definition: Let $N_xY = Spec(\oplus_{i\geq 0} m_x^i/m_x^{i+1}) = Spec(R)$, where m_x is the maximal ideal defining x in Y. Since Y is nonsingular, m_x is generated by weight vectors $x_1, \ldots, x_n$. So, $N_x(Y) = Spec(k[x_1, \ldots, x_n])$. Let $C_x(X)$ be the cone to x in X. $\mathcal{O}_{C_x(X)}$ is a ring with T-action and is a quotient ring of R. So, $\mathcal{O}_{C_x(X)}$ is a R, T module. Let

$$mult_T(x, X, Y) = \begin{cases} mult_T(\mathcal{O}_{C_x(X)}, R) & \text{if } x \in X \\ 0 & \text{if } x \notin X. \end{cases} \tag{5.41}$$

Note that $\mathcal{O}_{C_x(X)}$ is homogeneous in the usual sense, so $mult_T(x, X, Y)$ is defined even when $\lambda_i = 0$ for some i.

Let U be an open affine T-subset of Y such that U is T-isomorphic to a representation space F. We consider

$$X\,|_U \subset U \subset \mathbf{P}(F \oplus 1), \tag{5.42}$$

where 1 has weight 0. For notational purposes, we assume the element in $k[x_0, x_1, \ldots, x_n]$ corresponding to 1 is x_0.

Proposition 5.2.1 *If $x \in Y$ is a T fixed point and there is an open affine T-subset of Y containing x that is T-isomorphic to a T-representation space, then*

$$mult_{T,\mathbf{P}(F\otimes 1)}(x, X) = mult_T(x, X, Y). \tag{5.43}$$

Proof: Let $x \in X$. Since $U \simeq F$ $N_x(Y) \simeq F$ and we can close $N_x(Y)$ and $C_x(X)$ in $\mathbf{P}(F \oplus 1)$.

$$mult_T(x, X, Y) = mult_T(\mathcal{O}_{C_x(X)}(U), \mathcal{O}_{N_x(Y)}(U)) \tag{5.44}$$

$$= mult_{T,\mathbf{P}(F\oplus 1)}(x, C_x(X)) \tag{5.45}$$

$$= mult_{T,\mathbf{P}(F\oplus 1)}(x, X). \tag{5.46}$$

□

Since the right-hand sides have been defined when $x \notin X$ and are 0, defining $mult_T(x, X, Y) = 0$ in this case is consistent.

For completeness we list all the properties of $mult_T(x, X, Y)$.

Proposition 5.2.2 *Suppose Y is a smooth T-variety, $x \in Y^T$ and there is an affine open T-subset U of Y with $x \in U$ such that U is T-isomorphic to a T representation space with weights λ_i.*

1. If X is an equidimensional T-scheme with components V_i of geometric multiplicity m_i, then

$$mult_T(x, X, Y) = mult_T(x, [X], Y) = \sum m_i mult_T(x, V_i, Y). \tag{5.47}$$

2. If $H|_U$ is a T-subvariety defined by a single function of weight $-\lambda_{H|_U}$ and H intersects X in codimension 1, then

$$mult_T(x, X \cap H, Y) = \lambda_{H|_U} mult_T(x, X, Y). \tag{5.48}$$

In particular, if α is T-rationally equivalent to zero, then $mult_T(x,\alpha,Y) = 0$ and $mult_T(x,\alpha,Y)$ is an invariant of $\alpha \in A_k^T(X)$.

3. With notation as in 1 and 2, we have the following Bezout theorem:

$$\lambda_H mult_T(x,X,Y) = mult_T(x,X\cap H,Y). \tag{5.49}$$

4.

$$mult_T(x,X,Y) = \frac{1}{N!}\sum_{\lambda\in\Lambda(T)} a_\lambda \lambda^N, \tag{5.50}$$

where $N = \operatorname{codim}(X,Y)$, $a_\lambda \in \mathbf{Z}$, all but a finite number of the a_λ are zero, $a_\lambda \neq 0$ implies that λ is a sum of the λ_i, and $mult_T(x,X,Y)$ is a polynomial with coefficients in $\mathbf{Z}$ in the λ_i.

5. Let $\varphi : T' \to T$ be a morphism of tori, Y be a T and a T' invariant variety. Suppose

$$\begin{tikzcd} T'\times Y \arrow[r, "\varphi\times 1"] \arrow[dr] & T\times Y \arrow[d] \\ & Y \end{tikzcd} \tag{5.51}$$

commutes. If X is a T-subvariety of Y and the weights of T' in the open set containing x are λ_i', then

$$\left(\prod \lambda_i \circ d\varphi\right) mult_{T'}(x,X,Y) = \left(\prod \lambda_i'\right) mult_T(x,X,Y)\circ d\varphi. \tag{5.52}$$

6. Let E be a T-vector bundle over Y, $x \in Y^T$, let E be trivial over U, and let E have weights $\mu_1,\dots,\mu_k$. If $p : E \to Y$ is the projection, and i is the zero section embedding of Y in E then

$$mult_T(i(x),p^*(X),E) = mult_T(x,X,Y). \tag{5.53}$$

7. With notation as in 6,

$$mult_T(i(x),i(X),E) = (\prod \mu_i) mult(x,X,Y). \tag{5.54}$$

8. If X is 0-dimensional with x as its only component, then

$$mult_T(x, X, Y) = (\prod \lambda_i)m, \tag{5.55}$$

where m is the geometric multiplicity of x in X.

Proof: These are all consequences of the results of the last section. □

Chapter 6

Intersections of T-line bundles

In order to do intersection theory, we want to be able to define the Chern classes of a T-line bundle. We do this by associating a T-Cartier divisor to the pair consisting of a T-line bundle L and an equivariant meromorphic section of L. To this T-Cartier divisor, we associate a T-Weil divisor. This allows us to define the intersection of a T-line bundle with a T-subvariety V of X provided an equivariant section exists on V. As in chapter 3, our results are by and large the equivariant versions of Fulton's [6]. The main problem in generalizing Fulton's work to the equivariant case is, in fact, that equivariant sections need not necessarily exist. We also consider how equivariant multiplicities behave with respect to intersections.

6.1 T-Cartier Divisors

Definition: A T-Cartier divisor D on a T-variety X is a collection of pairs (U_α, f_α) where U_α is an open T-subset of X and $f_\alpha \in R(U_\alpha)^*$ is a weight vector of weight $-\lambda_\alpha$ such that on $U_\alpha \cap U_\beta$, $f_\alpha f_\beta^{-1}$ is invertible in $\mathcal{O}_{U_\alpha \cap U_\beta}$. The support of a T-Cartier divisor is the set of points in X where the local equations are not invertible. We label this by $|D|$. We will also label the collection of components of $|D|$ by $|D|$. We say a T-Cartier divisor is effective if $f_\alpha \in \mathcal{O}_{U_\alpha}$ for all α.

As with usual Cartier divisors, we have a group structure:

1. If D and D' are two T-Cartier divisors represented locally by (U_α, f_α) and (U_α, f'_α), then $D + D'$ is represented by $(U_\alpha, f_\alpha f'_\alpha)$, and has support $|D| \cup |D'|$.
2. The identity element is represented by $(U_\alpha, 1)$ for all α.
3. If D is represented locally by (U_α, f_α), then $-D$ is represented locally by $(U_\alpha, f_\alpha^{-1})$.

Definition: If $f \in R(X)^*$ is a weight vector we label the associated T-Cartier divisor by $\text{div}f$.

We say two T-Cartier divisors D and D' are T-linearly equivalent if there is a weight vector $f \in R(X)^*$ of weight 0 such that

$$D = D' + \text{div}f. \tag{6.1}$$

So, locally, if D and D' are represented by (U_α, f_α) and (U_α, f'_α), then $f_\alpha = f'_\alpha f$.

A T-Cartier divisor defines a T-line bundle $\mathcal{O}_D$. Let F_α be the 1-dimensional representation space generated by $1/f_\alpha$. Let L be the 1-dimensional $\mathcal{O}_{U_\alpha}$-sheaf defined over U_α by $\mathcal{O}_{U_\alpha} \otimes F_\alpha$. We set

$$\mathcal{O}_D = Spec(Symm(L^\vee)). \tag{6.2}$$

For notational purposes, we write $\mathcal{O}_D = Spec(\mathcal{O}_{U_\alpha} \otimes k[x_\alpha])$. Note that if f_α has weight $-\lambda_\alpha$ then $\mathcal{O}_D$ has weight λ_α, and x_α has weight $-\lambda_\alpha$.

The transition functions $g_{\alpha\beta} : \mathcal{O}_{U_\alpha \cap U_\beta \times F} \to \mathcal{O}_{U_\alpha \cap U_\beta \times F'}$ are defined by the function $g_{\alpha\beta} = f_\alpha / f_\beta \in \mathcal{O}^*_{U_\alpha \cap U_\beta}$.

There is a section s_D defined locally over the U_α by the $f_\alpha \in R(U_\alpha)^*$. Since x_α has the same weight as f_α, this is an equivariant section.

Note that if D and D' are T-linearly equivalent, then $\mathcal{O}_D$ and $\mathcal{O}_{D'}$ are T-isomorphic.

Some of the group properties of T-Cartier divisors pass to the associated T-line bundles:

1. If D and D' are two T-Cartier divisors, then $\mathcal{O}_{D+D'} = \mathcal{O}_D \otimes \mathcal{O}_{D'}$, $s_{D+D'} = s_D \otimes s_{D'}$, and $\text{supp}(s_{D+D'}) = |D| \cup |D'|$.

2. If D is a T-Cartier divisor, then $\mathcal{O}_{-D}$ is generated locally by the f_α, $s_{-D} = {s_D}^{-1}$, and $\text{supp}(s_{-D}) = \text{supp}(s_D)$.

Of course, if $f \in R(X)^*$ is a weight vector of weight 0, then the T-line bundle associated to $\text{div}f$ is the trivial one with trivial T-action, and the associated equivariant section is defined by f.

We associate a T-Weil divisor to D in the usual fashion. Over U_α we set

$$[D] = \sum_V \mathrm{ord}_V f_\alpha [V] \tag{6.3}$$

in $Z^T_{n-1}(U_\alpha)$ where the sum is over all codimension one subvarieties of U_α. Since f_α is a weight vector, the V for which $\mathrm{ord} f_\alpha$ is non-zero are T-subvarieties.

Since $\mathrm{ord}_V f$ is well defined up to units, this extends to a global definition.

With respect to equivariant multiplicity, the results of the previous chapters give:

Proposition 6.1.1 *Suppose that X is a T-subvariety of a smooth T-variety Y, D is a T-Cartier divisor on X, $x \in Y^T$, and there is an open affine T-subset $U \subset Y$ containing x that is T-isomorphic to a T representation space. If $x \in U_\alpha$ and f_α has weight $-\lambda_\alpha$ then,*

$$mult_T(x, [D], Y) = \lambda_\alpha mult_T(x, X, Y). \tag{6.4}$$

Proof: The result is a local result. Let (U_α, f_α) represent D, where $x \in U_\alpha$. To be able to use Proposition 5.2.2 part 2 we need the T-Cartier divisor defined on an open T-subset that is T-isomorphic to a T-representation space. Since U_α is not necessarily of this form, we consider a T-Cartier divisor D' that is defined on U and that restricts to D on U_α.

Let $U' = U \cap U_\alpha$. The closure of U' in $U \cap X$ is $U \cap X$, so $R(U')^* \simeq R(U \cap X)^*$. Let $\bar{f}_\alpha$ be the weight vector in $R(U \cap X)^*$ associated to f_α. Since the ideal defining X in U is prime, there is a weight vector $\bar{f} \in R(U)^*$ that restricts to $\bar{f}_\alpha$ in $R(U \cap X)^*$. The numerator and the denominator of $\bar{f}$ can be chosen to be weight vectors, and they act as a non-zero divisors on $\mathcal{O}_X(U \cap X)$. So,

$$mult_T(x, \mathrm{div}\bar{f}, Y) = \lambda_\alpha mult_T(x, X, Y). \tag{6.5}$$

On $U \cap X$, $[\mathrm{div}\bar{f}] - [\mathrm{div} f_\alpha]$ lies in $U \cap X - U'$, which does not contain x. So, the equivariant multiplicity of x in the two cycles is the same. So,

$$mult_T(x, [D], Y) = \lambda_\alpha mult_T(x, X, Y). \tag{6.6}$$

□

Remark: If D is T-linearly equivalent to D', then

$$mult_T(x, [D], Y) = mult_T(x, [D'], Y). \tag{6.7}$$

So, the equivariant multiplicity of the T-Weil divisor associated to a T-Cartier divisor is an invariant with respect to T-linear equivalence. If $D = D' + [\text{div} f]$ where f has weight $-\lambda$, then

$$mult_T(x, [D], Y) = \lambda mult_T(x, [D'], Y). \tag{6.8}$$

6.2 T-pseudo Divisors

Definition: A T-pseudo divisor D is a triple (L, Z, s) where Z is a closed T-subset of X, L is a T-line bundle and s is an equivariant section of L invertible over $X - Z$. We will sometimes write Z as $|D|$. A T-pseudo divisor (L', Z', s') is T-equivalent to (L, Z, s) if $Z = Z'$, and there is a T-isomorphism of T-line bundles $\varphi : L \to L'$ such that $\varphi(s) = s'$.

We have some of the group properties we'd expect:
If $D = (L, Z, s)$, and $D' = (L', Z', s')$ are two T-pseudo divisors on X, then

1. $D + D' = (L \otimes L', Z \cup Z', s \otimes s')$
2. $-D = (L^{-1}, Z, s^{-1})$.

A T-Cartier divisor D determines a T-pseudo divisor $(\mathcal{O}_D, |D|, s_D)$. We say D T-represents (L, Z, s) if $|D| \subset Z$ and there is a T-isomorphism of T-line bundles $\varphi : \mathcal{O}_D \to L$ such that $\varphi(s_D) = s$ on $X - Z$. Note that if $Z = X$, two T-Cartier divisors T-represent the same T-pseudo divisor if and only if they are T-linearly equivalent.

Note that if (L, Z, s) and (L', Z', s') are T-represented by D and D', then

1. $(L \otimes L', Z \cup Z', s \otimes s')$ is T-represented by $D + D'$
2. (L^{-1}, Z, s^{-1}) is T-represented by $-D$.

Theorem 6.2.1 *Every T-pseudo divisor is T-represented by a T-Cartier divisor that is*

1. *unique if* $Z \neq X$
2. *unique up to T-linear equivalence if* $Z = X$.

Proof: The proof is essentially contained in Fulton [6] Lemma 2.2. For an open cover $\{U_\alpha\}$ such that $L|_{U_\alpha}$ is trivial, choose an α_0 and set $f_\alpha = g_{\alpha\alpha_0}$ for all α. $g_{\alpha\alpha_0} \in R(U_\alpha \cap U_{\alpha_0})^* = R(U_\alpha)^*$ is a weight vector, so f_α is a weight vector in $R(U_\alpha)^*$. Also, $f_\alpha f_\beta^{-1} = g_{\alpha\beta}$ is invertible on $U_\alpha \cap U_\beta$. This shows the existence of a representing (but not T-representing) T-Cartier divisor D for $Z = X$.

To get T-representation, we assume for the moment that $Z = \mathrm{supp}(s)$. Since $s_\alpha = g_{\alpha\beta} s_\beta$, $s_\alpha / f_\alpha = s_\beta / f_\beta$, and there is a $r \in R(X)^*$ such that $r = s_\alpha / f_\alpha \in \mathcal{O}_X(U_\alpha)^*$ for all α. We set $D' = D + \mathrm{div} r$. We then have $s_{D'} = s$. Since $s_{D'}$ is an equivariant section, D' T-represents (L, Z, s).

For uniqueness, if D and D' both T-represent (L, Z, s) and have local equations f_α and f'_α, then $f_\alpha / f'_\alpha = f \in R(U_\alpha)^*$. Also, $f_\alpha / f'_\alpha = f_\beta / f'_\beta$. So, $f \in R(X)^*$, and f is a weight vector of weight 0. If $Z = X$, D and D' are T-linearly equivalent. If $Z \neq X$, then $s_D = s_{D'}$ off Z, and we have $f_\alpha = f'_\alpha$ on $U_\alpha - Z \cap U_\alpha$. So, $D = D'$. □

Definition: If D is a T-pseudo divisor on an n-dimensional T-variety X, we define its T-Weil class $[D] \in A^T_{n-1}(|D|)$ as the Weil class of a T-representing T-Cartier divisor of D.

If $f : X' \to X$ is a morphism, $f(X') \not\subset \mathrm{supp} s$, and D is a T-pseudo divisor on X, then the pullback $f^*(D) = (f^*(L), f^{-1}(Z), f^*(s))$ is a T-pseudo divisor on X'.

The pullback property illustrates the main problem with T bundles. We can only guarantee the existence of an equivariant section on X' if $f(X') \not\subset \mathrm{supp} s$.

Definition: Let $D = (L, Z, s)$ be a T-pseudo divisor on X and let $V \subset X$ be a T-subvariety of X. We say D is V-admissible if L has an equivariant section over V. If $\alpha \in Z_k^T(X)$, we say that D is α-admissible if for every component V of α L is V-admissible. If L has an equivariant section over every T-subvariety of dimension k of X, we will say that D is k-admissible.

Proposition 6.2.2 *Suppose that $X \subset Y$ is a T-subscheme of a smooth T-variety Y, $D = (L, Z, s)$ is a T-pseudo divisor on X, $x \in Y^T$, and there is a open affine T-subset $U \subset Y$ containing x which is T-isomorphic to a representation space. Then, if L has weight λ_x over x,*

$$mult_T(x, [D], Y) = \lambda_x mult_T(x, X, Y). \tag{6.9}$$

Proof: This is a consequence of Proposition 6.1.1. □

Remark: If D and D' are T-equivalent T-pseudo divisors, then

$$mult_T(x, [D], Y) = mult_T(x, [D'], Y). \tag{6.10}$$

If $D = D' + \mathrm{div} f$ for f of weight $-\lambda$, then

$$mult_T(x, [D], Y) = mult_T(x, [D'], Y) + \lambda mult_T(x, X, Y). \tag{6.11}$$

Example: We show that the pullback of a T-pseudo divisor is not necessarily defined.

Let $X = \mathbf{A}^3$ and let $T = G_m \times G_m$, be the action with weights

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \tag{6.12}$$

Let $D = (L, X, s)$ be the T-pseudo divisor where L is the trivial line bundle with weight $(1,3)$ and s is the section defined by the function $x_2x_3 \in R(X)^*$. The equivariant sections of L are defined by the functions $ax_1x_2^3 - bx_2x_3$ for $a,\ b \in k$. So, on the subvariety W defined by the ideal (x_2), L has no equivariant section and if $i : W \to X$ is the injection, then $i^*(L, X, s)$ does not exist. Note, however, that since

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{6.13}$$

has solution $(-1/2, 3/2)$, $L^{\otimes 2}$ does have an equivariant section, and $i^*(L^{\otimes 2}, X, s^{\otimes 2})$ does exist.

6.3 Intersecting T-pseudo Divisors

Definition: Let D be a T-pseudo divisor on X and V a k-dimensional T-subvariety of X such that D is V-admissible. If $j : V \to X$ is the inclusion of V in X, we define

$$D \cdot [V] = [j^* D] \tag{6.14}$$

in $A^T_{k-1}(|D| \cap V)$.

Note that if $f : X' \to X$ is a T-morphism of T-schemes, V is a T-subvariety of X' and D is $f(V)$-admissible, then $\mathcal{O}_D$ has an equivariant section defined on $f(V)$ and so f^*D is V-admissible.

We extend the definition linearly to cycles:

Definition: If $\alpha = \sum n_V [V] \in Z^T_k(X)$ and D is an α-admissible T-pseudo divisor on X, then

$$D \cdot \alpha = \sum n_V D \cdot [V] \tag{6.15}$$

in $A^T_{k-1}(|D| \cap |\alpha|)$.

Proposition 6.3.1 *1. Let α, $\alpha' \in Z^T_k(X)$. If D is an α and an α'-admissible T-pseudo divisor, then*

$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha' \tag{6.16}$$

in $A^T_{k-1}(|D| \cap (|\alpha| \cup |\alpha'|))$.

2. If $\alpha \in Z^T_k(X)$ and D, D' are two α-admissible T-pseudo divisors, then

$$(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha \tag{6.17}$$

in $A^T_{k-1}((|D| \cup |D'|) \cap |\alpha|)$.

3. Let $f : X' \to X$ be a proper T-morphism, $\alpha \in Z^T_k(X')$, D a $f_*(\alpha)$-admissible T-pseudo divisor on X and $g : f^{-1}(|D|) \cap |\alpha| \to |D| \cap f\,|\alpha|$ the induced morphism. Then

$$g_*((f^*D) \cdot \alpha) = D \cdot f_*\alpha \tag{6.18}$$

in $A^T_{k-1}(D \cap f(|\alpha|))$.

4. Let $f : X' \to X$ *be a flat T-morphism of relative dimension* n, $\alpha \in Z^T_k(X)$, D *an* α*-admissible T-pseudo divisor on* X *and* $g : |f^*D| \cap |f^*(\alpha)| \to |D| \cap |\alpha|$ *the induced morphism. Then*

$$(f^*D) \cdot f^*(\alpha) = g^*(D \cdot \alpha) \tag{6.19}$$

in $A^T_{k+n-1}(f^{-1}(|D|) \cap |\alpha|)$.

Proof: The proof is contained in Fulton [6]. For 3, note that since D is $f_*\alpha$-admissible, $\mathcal{O}_D$ has an equivariant section over every component of $f_*\alpha$, and f^*D is α-admissible.

For 4, recall that locally over U, any T-rational function is a ratio of elements of $\mathcal{O}_U$. So, any T-Cartier divisor is locally a difference of effective T-Cartier divisors.

$$[f^{-1}(Z)] = f^*[Z] \tag{6.20}$$

then gives the result. □

Proposition 6.3.2 *Suppose that* $X \subset Y$ *is a T-subscheme of a smooth T-variety* Y, $\alpha \in Z^T_k(X)$, $D = (L, Z, s)$ *is an* α*-admissible T-pseudo divisor on* X, $x \in Y^T$ *and there is an open affine T-subset* U *of* Y *containing* x *that is T-isomorphic to a representation space. Suppose that* L *has weight* λ *over* x*. Then*

$$mult_T(x, D \cdot \alpha, Y) = \lambda mult_T(x, \alpha, Y). \tag{6.21}$$

Proof: This is a consequence of Proposition 6.2.2. □

Example: For flat pullback the admissibility conditions on X are necessary.

Let $X = \mathbf{A}^2$, let $T = G_m \times G_m$ and let X have the T-action defined by the weights $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $D = (L, X, s)$ be the T-pseudo divisor where L is the trivial T-line bundle with weight $(0, 1)$ and s is defined by $x_2 \in \mathcal{O}_{\mathbf{A}^2}$. Let $X' = \mathbf{A}^3$ with weights $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Let $f : \mathbf{A}^3 \to \mathbf{A}^2$ be the projection onto the first and second coordinates. L has no equivariant section on the variety

V defined by (x_2) in $\mathbf{A}^2$, but f^*L has an equivariant section defined on f^*V. So flat pullback does require the admissibility condition on X.

Example: Additivity of T-pseudo divisors requires the admissibility conditions.

Let $X = \mathbf{A}^3$, let $T = G_m \times G_m$ and let X have the T-action defined by the weights $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \end{smallmatrix}\right)$, let $D = (L, X, s)$ be the T-pseudo divisor where L is the trivial T-line bundle with weight $(1,0)$ and s is defined by the function x_3, and let $D' = (L', X, s')$ be the T-pseudo divisor where L' is the trivial T-line bundle with weight $(0,2)$ and s' is defined by the function x_1/x_3. Let V be the T-subvariety of X defined by the ideal (x_3). Then neither D nor D' are V-admissible, but $D + D'$ is V-admissible.

6.4 Commutativity

Theorem 6.4.1 *Let D and D' be two T-Cartier divisors on an n-dimensional T-variety X. Suppose D is $|D'|$-admissible, and D' is $|D|$-admissible. Then*

$$D \cdot [D'] = D' \cdot [D]. \tag{6.22}$$

Proof: We proceed by cases. If D and D' are effective and intersect properly, the theorem is local, and is purely algebraic and is contained in Fulton [6], Theorem 2.4, case 1. The proof is a rewriting of Fulton [6], Theorem 2.4.

For the other cases we consider the blow-up $\widetilde{X}$ of X along certain subschemes. We pull back D and D' to $\widetilde{X}$ and intersect there. Since we have problems with the existence of equivariant sections, the only difference between what we do and what Fulton does is that we have to check for the existence of equivariant sections on the subvarieties of $\widetilde{X}$ that map to the codimension 1 subvarieties of X that we are interested in. Checking for the existence of equivariant sections and getting around their non-existence makes up the bulk of this section.

If D and D' are effective, let

$$\epsilon(D, D') = \max\{\mathrm{ord}_V(D)\mathrm{ord}_V(D') : \mathrm{codim}(V, X) = 1\} \tag{6.23}$$

where the max is over all codimension 1 T-subvarieties of X. Note that $\epsilon(D, D') = 0$ precisely when D and D' intersect properly.

Let $D \cap D'$ be the intersection scheme of D and D'. If the local equations for D and D' are a and a', then $D \cap D'$ is defined locally over U by the ideal $I = (a, a')$. For convenience, we also use $I = (s, t)$. Let $\pi : \widetilde{X} \to X$ be the blow-up of X along $D \cap D'$. Locally, $\widetilde{X} = Proj(\oplus_{n=0}^{\infty} I^n) = Proj(\mathcal{O}_X \oplus (\oplus_{n=1}^{\infty}(s,t)^n))$.

Let $E = \pi^{-1}(D \cap D')$ be the exceptional divisor. Let U_s and U_t be the open T-subsets of $\pi^{-1}U$ obtained by inverting s and t. Then, on U_t, $\pi^* D$ is defined by $a = \frac{s}{t}a'$, $\pi^* D'$ is defined by a', and E is defined by a'. On U_s, $\pi^* D$ is defined by a, $\pi^* D'$ is defined by $a' = \frac{t}{s}a$, and E is defined by a. So, if on U_t we define C by $\frac{s}{t}$ and C' by 1, and on U_s we define C by 1 and C' by $\frac{t}{s}$, then $\pi^* D = E + C$, $\pi^* D' = E + C'$, and $\pi^* D$, $\pi^* D'$, E, C and C' are effective T-Cartier divisors on $\widetilde{X}$.

Lemma 6.4.2 *With notation as above,*

1. $|C| \cap |C'| = \emptyset$
2. *If* $\epsilon(D, D') > 0$, *then* $\epsilon(C, E)$, $\epsilon(C', E)$ *are strictly less than* $\epsilon(D, D')$.

Proof: These claims are local in nature, so we assume that $X = Spec(A)$. Our explicit description of C and C' shows that $|C| \cap |C'| = \emptyset$. For part 2, consider the map

$$\begin{aligned} A[S,T] &\to \oplus_{n=0}^{\infty} I^n \\ S &\mapsto s \\ T &\mapsto t \end{aligned} \tag{6.24}$$

where S has weight λ_s and T has weight λ_t. This map is equivariant, defines an injection φ and

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & X \times \mathbf{P}^1 \arrow[d, "p"] \\ & X \end{tikzcd} \tag{6.25}$$

commutes.

We can also consider s and t as T-sections. Consider the pull back of $\mathcal{O}(1)$ from $X \times \mathbf{P}^1$ to $\widetilde{X}$. Let s and t be the sections on $\widetilde{X}$ defined by the pullbacks of the sections associated to S and T. We represent the associated functions by s and t as well. Then $|C|$ is the zero-scheme $Z(s)$ of s, and $|C'|$ is $Z(t)$.

Since $Z(S)$ and $Z(T)$ are mapped isomorphically to subschemes of X by p, $Z(s)$ and $Z(t)$ are also mapped isomorphically to subschemes of X by π. So, if $\widetilde{V}$ is a codimension 1 T-subvariety of $\widetilde{X}$ contained in $|C|$ or in $|C'|$, then $V = \pi(\widetilde{V})$ is of codimension 1 in X. Also, $[D] = \pi_*[E+C]$, so that,

$$\mathrm{ord}_V(D) \geq \mathrm{ord}_{\widetilde{V}}(E) + \mathrm{ord}_{\widetilde{V}}(C). \tag{6.26}$$

Repeating this for D' and $E + C'$, we get

$$\mathrm{ord}_V(D') \geq \mathrm{ord}_{\widetilde{V}}(E) + \mathrm{ord}_{\widetilde{V}}(C'). \tag{6.27}$$

Now, suppose $0 < \epsilon(D, D')$, and $\widetilde{V}$ is such that $\mathrm{ord}_{\widetilde{V}}(E)\mathrm{ord}_{\widetilde{V}}(C) = \epsilon(C, E)$. Then,

$$\mathrm{ord}_V(D)\mathrm{ord}_V(D') \geq (\mathrm{ord}_{\widetilde{V}}(E) + \mathrm{ord}_{\widetilde{V}}(C))(\mathrm{ord}_{\widetilde{V}}(E) + \mathrm{ord}_{\widetilde{V}}(C')) \tag{6.28}$$

$$\geq \mathrm{ord}_{\widetilde{V}}(E)^2 + \epsilon(C, E). \tag{6.29}$$

Since $\mathrm{ord}_{\widetilde{V}}(E)^2 > 0$, $\epsilon(C, E) < \epsilon(D, D')$. □

For the final technique we (temporarily) redefine equivariant intersection. If D is V-admissible, then the intersection is defined as before. If D is not V-admissible, we set $D \cdot V = 0$.

Because of this redefinition, we have to show proper pushforward still works for our $\widetilde{V}$ and that equivariant sections exist on the appropriate varieties.

Lemma 6.4.3 *Let D and D' be T-Cartier divisors on a T-scheme X, D be $|D'|$-admissible, $\pi : \widetilde{X} \to X$ be a proper T-morphism, $\widetilde{V} \subset |\pi^*D'|$ be a T-subvariety of X' of codimension 1, and let $V = \pi(\widetilde{V})$. Then,*

$$\pi_*(\pi^*D \cdot [\widetilde{V}]) = D \cdot [V] \tag{6.30}$$

in $A^T_{n-2}(X)$.

Proof: If V is of codimension 1, then $V \subset |D'|$ and $\mathcal{O}_D$ has an equivariant section on V. So,

$$\pi_*(\pi^*D \cdot [\widetilde{V}]) = D \cdot [V]. \tag{6.31}$$

in $A^T_{n-2}(X)$.

If V is of codimension greater than 1, then

$$\pi_*(\pi^*D \cdot [\widetilde{V}]) = 0 = D \cdot [V], \tag{6.32}$$

in $A^T_{n-2}(X)$, independent of whether a section exists or not. □

Now, we need to know whether $\mathcal{O}_E$, $\mathcal{O}_C$, $\mathcal{O}_{C'}$ have equivariant sections over the appropriate varieties.

Lemma 6.4.4 *Suppose $\widetilde{V}$ is a T-subvariety of $\widetilde{X}$ of codimension 1 and V is of codimension 1 in X. If $\widetilde{V} \subset |\pi^*D|$, then $\mathcal{O}_E$ and $\mathcal{O}_{C'}$ are $\widetilde{V}$-admissible. If $\widetilde{V} \subset |\pi^*D'|$, then $\mathcal{O}_E$ and $\mathcal{O}_C$ are $\widetilde{V}$-admissible.*

Proof: We check the result using our explicit descriptions of $\mathcal{O}_E$, $\mathcal{O}_C$ and $\mathcal{O}_{C'}$.

Suppose $\widetilde{V} \subset |C| - |E| \cap |C|$. Locally $\mathcal{O}_{C'}$ has section 1 over $\widetilde{V}$ and $\mathcal{O}_E$ has section a' over $\widetilde{V}$ which is invertible over $\widetilde{V}$. Similarly, if $\widetilde{V} \subset |C'| - |E| \cap |C'|$, then $\mathcal{O}_C$ has section 1 over $\widetilde{V}$ and $\mathcal{O}_E$ has section a over $\widetilde{V}$ which is also invertible on $\widetilde{V}$.

If $\widetilde{V} \subset |C| \cap |E|$, then V is of codimension 1 in X and is contained in $|D| \cap |D'|$. So $\mathcal{O}_D$ and $\mathcal{O}_{D'}$ have equivariant sections on V, $\mathcal{O}_{C'}$ has section 1 on $\widetilde{V}$, and since

$$\pi^* D' = E + C', \tag{6.33}$$

$\mathcal{O}_E$ has an equivariant section over $\widetilde{V}$. Similarly, since $\pi^* D = E + C$, $\mathcal{O}_C$ has an equivariant section on $\widetilde{V}$. For $\widetilde{V} \subset |C'| \cap |E|$, we have $\mathcal{O}_C$ with section 1 on $\widetilde{V}$, and $\mathcal{O}_E$ and $\mathcal{O}_{C'}$ have equivariant sections defined on $\widetilde{V}$ as well.

Finally, let $\widetilde{V} \subset |E| - (|C| \cup |C'|) \cap |E|$ and let V be of codimension 1 in X. $\mathcal{O}_D$ has an equivariant section on V, and so $\mathcal{O}_{\pi^* D}$ has an equivariant section on $\widetilde{V}$. $\mathcal{O}_C$ has an invertible T-section over $\widetilde{V}$ and so $\mathcal{O}_E$ has an equivariant section on $\widetilde{V}$. Similarly, $\mathcal{O}_{C'}$ has an invertible equivariant section over $\widetilde{V}$. □

case 2

Suppose that D and D' are both effective. We show that for the redefined intersection $D \cdot [D'] = D' \cdot [D]$ by induction on $\epsilon(D, D')$. For $\epsilon(D, D') = 0$ we have already shown the result. Suppose the result is true for all effective T-Cartier divisors B and B' with $\epsilon(B, B') < \epsilon(D, D')$. For $\pi : \widetilde{X} \to X$ a proper bi-rational T-morphism, let $\pi^* D = E \pm C$, and $\pi^* D' = E \pm C'$ where E, C and C' are effective T-Cartier divisors. Then,

$$D \cdot [D'] = \pi_*(\pi^* D \cdot [\pi^* D']) \tag{6.34}$$

$$= \pi_*((E \pm C) \cdot [E \pm C']) \tag{6.35}$$

$$= \pi_*(E \cdot [E] \pm E \cdot [C'] \pm C \cdot [E] \pm C \cdot [C']) \tag{6.36}$$

$$= \pi_*(E \cdot [E] \pm C' \cdot [E] \pm E \cdot [C] \pm C \cdot [C']) \tag{6.37}$$

$$= D' \cdot [D] \tag{6.38}$$

in $A^T_{n-2}(X)$. Note that we use bi-rationality to get $\pi_*[\pi^* D] = [D]$.

Since D is $|D'|$-admissible and D' is $|D|$-admissible, the usual equivariant intersection and the redefined intersection agree. So, for the usual intersection,

$$D \cdot [D'] = D' \cdot [D] \tag{6.39}$$

in $A^T_{n-2}(X)$.

case 3.

Suppose one of D or D' is effective, say D'. Let D be T-represented on U by $d = a/b$ and D' by a' for $a, b, a' \in \mathcal{O}_U = A$. Let J be the sheaf of denominators of D. So, on U,

$$J = \{c \in A : cd \in A\}. \tag{6.40}$$

Let I be the sheaf of numerators, *i.e.*

$$I = dJ. \tag{6.41}$$

I and J are both T-sheaves. To show that J is a T-sheaf, note that if $c \in J$,

$$\chi_d(t)(t \cdot c)d = t \cdot (cd) \in A. \tag{6.42}$$

So, $t \cdot c \in J$. Since d is a weight vector, $I = dJ$ is a T-sheaf as well.

Let K be the sheaf generated by both I and J. We blow-up X along the subscheme defined by K. On U this ideal is generated by (a, b). As before we label this by (s, t). Let $\widetilde{U} = Proj(\oplus_{n=0}^{\infty} K^n)) = Proj(A \oplus (\oplus_{n=1}^{\infty}(s,t)))$. On U_t we have $a = (s/t)b$, and on U_s, $b = (t/s)a$.

We consider the map $\pi : \widetilde{X} \to X \times \mathbf{P}^1$ induced by the local map on functions,

$$\begin{aligned} k[S,T] &\to \oplus_{n=0}^{\infty}(s,t)^n \\ S &\mapsto s \\ T &\mapsto t. \end{aligned} \tag{6.43}$$

As before, we can consider s and t as T-sections. We consider the pullback of $\mathcal{O}(1)$ from $\mathbf{P}^1$ to $\widetilde{X}$. Let s and t be the T-rational functions associated to the sections on $\widetilde{X}$ defined by the pullbacks from $X \times \mathbf{P}^1$ of the sections associated to S and T. We label these T-sections by s and t as well. Then $C = Z(s)$, and $C' = Z(t)$. These are mapped isomorphically by π to X, so if $\widetilde{V}$ is a codimension 1 T-subvariety of $\widetilde{X}$ contained in $|C|$ or in $|C'|$, then V is of codimension

1 in X. Now, on U_t, C is defined by $\frac{s}{t}$ and C' is defined by 1. On U_s, C is defined by 1 and C' is defined by $\frac{t}{s}$. So, on U_t, π^*D is defined by $(s/t)b \cdot b^{-1}$ and on U_s by $a((t/s)a)^{-1}$. We have $C \cap C' = \emptyset$, and

$$\pi^*D = C - C'. \tag{6.44}$$

We again check for the existence of equivariant sections.

Lemma 6.4.5 *Let D' be any T-Cartier divisor on X. If D is a $|D'|$-admissible T-Cartier divisor on X and $\widetilde{V} \subset |\pi^*D'|$ is of codimension 1, then $\mathcal{O}_{\pi^*D}$, $\mathcal{O}_C$ and $\mathcal{O}_{C'}$ are $\widetilde{V}$-admissible. If D' is $|D|$-admissible, $\widetilde{V} \subset |\pi^*D|$ and V is of codimension 1 in X, then $\mathcal{O}_{\pi^*D'}$ is $\widetilde{V}$-admissible.*

Proof: Suppose $\widetilde{V} \subset |\pi^*D'|$. If $\widetilde{V} \subset |\pi^*D'| - |C| \cup |C'|$, then $\mathcal{O}_C$ and $\mathcal{O}_{C'}$ have invertible equivariant sections defined on $\widetilde{V}$.

If $\widetilde{V} \subset |\pi^*D'| \cap |C|$, then V is of codimension 1 in X and $\mathcal{O}_{\pi^*D}$ has an equivariant section on $\widetilde{V}$. $\mathcal{O}_{C'}$ has section 1 on $\widetilde{V}$, so $\mathcal{O}_C$ has an equivariant section on $\widetilde{V}$.

Similarly, if $\widetilde{V} \subset |\pi^*D'| \cap |C'|$, then V is of codimension 1 in X, $\mathcal{O}_{\pi^*D}$ has an equivariant section defined on $\widetilde{V}$, $\mathcal{O}_C$ has section 1 and $\mathcal{O}_{C'}$ has an equivariant section on $\widetilde{V}$.

Suppose that $\widetilde{V} \subset |\pi^*D|$. If V is of codimension 1 in X, then $\mathcal{O}_{D'}$ has an equivariant section over V and $\mathcal{O}_{\pi^*D'}$ has an equivariant section over $\widetilde{V}$. □

Note that if E is the canonical divisor on $\widetilde{X}$ then $|\pi^*D|$ contains E. However, since $\pi^*D = C - C' + E - E$, for the redefined equivariant intersection whether E has an equivariant section over any $\widetilde{V}$ in $\widetilde{X}$ is immaterial.

Applying case 2, and in particular equations 6.34 to 6.38, since C, C' and D are effective and $\pi : \widetilde{X} \to X$ is bi-rational, for the redefined intersection,

$$D \cdot [D'] = \pi_*(\pi^*D \cdot [\pi^*D']) = \pi_*(\pi^*D' \cdot [\pi^*D]) = D' \cdot [D] \tag{6.45}$$

in $A^T_{n-2}(X)$.

Again, since D is $|D'|$-admissible and D' is $|D|$-admissible, the usual equivariant intersection and the redefined intersection agree, so with the usual intersection,

$$D \cdot [D'] = D' \cdot [D] \tag{6.46}$$

in $A_{n-2}^T(X)$.

For the final case where D and D' are arbitrary, we blow-up along the T-subscheme defined by the sheaf of denominators and numerators of D as above to get

$$\pi^* D = C - C' \tag{6.47}$$

on $\widetilde{X}$ where C and C' effective and $|C| \cap |C'| = \emptyset$. Applying case 3 then gives

$$D \cdot [D'] = \pi_*((C + C') \cdot [\pi^* D']) = \pi_*(\pi^* D' \cdot [C + C']) = D' \cdot [D] \tag{6.48}$$

for the redefined intersection. Again, as above, this implies commutativity for the usual equivariant intersection. □

Corollary 6.4.6 *Suppose D is a k and $k+1$-admisssible T-pseudo divisor on X and suppose that for all k and $k+1$-dimensional T-subvarieties V of X there is a weight vector $f \in R(V)^*$ of weight 0. If $\alpha \in Z_k^T(X)$ is T-rationally equivalent to 0, then $D \cdot \alpha = 0$ in $A_{k-1}^T(X)$.*

Proof: Let V be a $k+1$-dimensional T-subvariety, and let $f \in R(V)^*$ be a weight vector of weight 0. Then,

$$D \cdot [\mathrm{div} f] = \mathrm{div} f \cdot [D] = 0. \tag{6.49}$$

□

Example: The condition on V is necessary.

We use a previous example. Let $X = \mathbf{A}^3$, with torus $G_m \times G_m$ and weights

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \tag{6.50}$$

Let L be the T-line bundle with weight $(1, 3)$. The equivariant sections of L are defined by functions $ax_2x_3 - bx_1x_2^3$ for $a, b \in k$. The T-rational functions of weight 0 are of the form $a(x_1x_2^2/x_3)^n$ for $a \in k$ and $n \in \mathbf{Z}$. Over the subvariety defined by (x_3) L has an equivariant section, but over the cycle defined by $(x_1x_2^2)$ which is T-rationally equivalent to it, no equivariant section exists. So, we do need the second condition in the corollary.

Because of the problems with rationality, we define:

Definition: X is k-nice if for every k-dimensional T-subvariety V of X there is a weight vector $f \in R(X)^*$ of weight 0. Equivalently, the trivial T-line bundle of weight 0 has an equivariant section on every k-dimensional T-subvariety of X.

Remark: Niceness is not as strong a condition as admissibilty. Any T-variety has an open subset U over which algebraic quotient $U//T$ exists. If $k > \dim(X-U)$, since $\mathcal{O}_{U//T} \simeq \mathcal{O}_X(U)^T$, X is k-nice.

6.5 Intersection with T-line bundles

Definition: Let L be a T-line bundle on a T-scheme X. We will say L is k-admissible if L has an equivariant section over every k-dimensional T-subvariety of X.

Definition: Let L be a T-line bundle on X. If V is a k-dimensional T-subvariety of X and L is k-admissible, then L is T-represented by a T-pseudo divisor D and we define

$$c_1(L) \cap V = D \cdot [V] \tag{6.51}$$

in $A_{k-1}^T(V)$. We extend this definition linearly to $\alpha \in Z_k^T(X)$.

Proposition 6.5.1 *1. Let X be k and $k+1$-nice, $\alpha \in Z_k^T(X)$, and let L be a k and a $k+1$-admisssible T-line bundle on X. If α is T-rationally equivalent to 0, then*

$$c_1(L) \cap \alpha = 0 \tag{6.52}$$

in $A_{k-1}^T(X)$.

2. Let X be k and $k+1$-nice, $\alpha \in A_k^T(X)$, and let L and L' be k and $k+1$-admisssible T-line bundles on X. Then,

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha \tag{6.53}$$

in $A^T_{k-1}(X)$.

3. Let $f : X' \to X$ *be a proper* T*-morphism,* X *and* X' *be* k *and* $k+1$*-nice, and let* L *be a* k *and* $k+1$*-admisssible* T*-line bundle on* Y. *If* $\alpha \in A^T_k(X')$*, then*

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha) \tag{6.54}$$

in $A^T_{k-1}(X)$.

4. Let $f : X' \to X$ *be a flat* T*-morphism of relative dimension* n*,* X *be* k *and* $k+1$*-nice,* X' *be* $k+n$ *and* $k+n+1$*-nice and let* L *be a* k *and* $k+1$*-admisssible* T*-line bundle on* X. *If* $\alpha \in A^T_k(X)$*, then*

$$f^*(c_1(L) \cap \alpha) = c_1(f^*L) \cap f^*\alpha \tag{6.55}$$

in $A^T_{k+n-1}(X')$.

5. Let X *be* $k-1$*,* k *and* $k+1$*-nice, and let* L *and* L' *be* $k-1$*,* k *and* $k+1$*-admisssible* T*-line bundles on* X. *If* $\alpha \in A^T_k(X)$*, then*

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha), \tag{6.56}$$

in $A^T_{k-2}(X)$.

Proof: We know the results hold if $\alpha \in Z^T_k(X)$. Part 1 follows from Corollary 6.4.6. The remainder follow from part 1, and Proposition 6.3.1. □

Proposition 6.5.2 *Suppose that* $X \subset Y$ *is a* k *and* $k+1$*-nice* T*-subscheme of a smooth* T*-variety* Y*,* $\alpha \in A^T_k(X)$*,* L *is a* k*-admissible* T*-line bundle on* X*,* $x \in Y^T$ *and there is an open affine* T*-subset of* Y *containing* x *that is* T*-isomorphic to a* T*-representation space. If* L *has weight* λ *over* x*, then*

$$mult_T(x, c_1(L) \cap \alpha, Y) = \lambda mult_T(x, \alpha, Y). \tag{6.57}$$

Proof: This results from Proposition 6.3.2, and that $mult_T(x, \alpha, Y)$ is an invariant of $A^T_k(X)$. □

6.6 Notes

We could have considered different forms of equivariant intersection.

If (L, Z, s) is a T-pseudo divisor, we could demand that $|D| \subset Z$, $\varphi : \mathcal{O}_D \to L$ be a weight isomorphism (but not necessarily a T-isomorphism), and $\varphi(s_D) = s$ off Z. Note that since the transition functions are equivariant, if φ is a weight morphism of weight λ on one U_α, then it is a weight morphism of weight λ on all the U_α. This would lead to a different form of equivariant intersection for which a section always exists, but for which most of the equivariant properties are lost. In future, we will say such a T-Cartier divisor represents (but not T-represents) the T-pseudo divisor (L, Z, s). Note that for normal projective varieties, Theorem 4.7.3 shows that the strongest equivalence relation we can use with this form of intersection is rational equivalence.

If $Z = X$, D and D' represent the same T-pseudo divisor if

$$D = D' + \mathrm{div} f \tag{6.58}$$

where $f \in R(X)^*$ is a weight vector.

For this form of intersection, all the results of Proposition 6.5.1 hold without the admissibility conditions, but instead of getting a cycle in $A_k^T(X)$, we get a cycle in $A_k(X)$. The equivariant multiplicity is also dependent on the choice of representing T-Cartier divisor. If (L, Z, s) is represented by the T-Cartier divisor D, L has weight λ_x in the fibre over $x \in X^T$ and D has weight $\lambda_x + \lambda$ over x, then

$$mult_T(x, D \cdot \alpha, Y) = (\lambda_x + \lambda) mult_T(x, \alpha, Y). \tag{6.59}$$

There is another form of equivariant intersection, which we call $*$ intersection:

Definition: For D a T-pseudo divisor on X, V a k-dimensional T-subvariety of X, we set

$$D * \alpha = \begin{cases} \frac{1}{n}(nD) \cdot V & \text{if } L^{\otimes n} \text{ is } V\text{-admissible} \\ 0 & \text{if no } n \text{ exists such that } L^{\otimes n} \text{ is } V\text{-admissible} \end{cases} \tag{6.60}$$

in $A^T_{k-1}(X) \otimes \mathbf{Q}$.

If L is a T-line bundle, we can replace our usual equivariant intersection with $*$ intersection. We write this form of intersection as $c_1(L) * \alpha$. This form of intersection is well-defined.

With respect to $*$ intersection, the only property of Theorem 6.5.1 to hold without the admissibility condition is proper push forward. If $p : X' \to X$ is a proper T-morphism, $V \subset X'$ is a k-dimensional T-subvariety and $f \in R(U)^*$ is a weight vector where U is an open affine T-subset of V, then $p_*([\mathrm{div} p^*(f)]) = d[\mathrm{div} N^{R(X')}_{R(X)}(f)]$, where $d = [R(X') : R(X)]$. Since $N^{R(X')}_{R(X)}(f)$ has weight $d\lambda_f$, if $(p^*L)^{\otimes n}$ has an equivariant section over V for some n, then $L^{\otimes dn}$ has an equivariant section over $p(V)$. So, pushforward holds for $*$ intersection without the admissibility condition.

The main reason for considering $*$ intersection is that if $T = G_m$, a section is always defined for some tensor power of L. This is just an application of the Chinese Remainder Theorem.

For $*$ intersection, if X is k and $k+1$-nice then

$$mult_T(x, c_1(L) * [V], Y) = \begin{cases} \lambda mult_T(x, [V], Y) & \text{if } L^{\otimes n} \text{ has an equivariant section over} \\ & V \text{ for some } n \\ 0 & \text{if } L^{\otimes n} \text{ has no equivariant section over} \\ & V \text{ for any } n. \end{cases} \tag{6.61}$$

Chapter 7

Intersections with T-Vector Bundles

We define the intersection of T-vector bundles with T-cycles in a T-variety X. In doing this, we follow Fulton [6]. First we define the T-Segre classes and then we define the T-Chern classes in terms of the T-Segre classes. As in the previous chapter, we have admissibility and niceness conditions. For T-vector bundles these are a lot more restrictive than they were for T-line bundles.

7.1 T-Segre classes

Definition: Suppose that E is a T-vector bundle over an n-dimensional T-scheme X of rank $e+1$. We say that E is a k^+-admissible T-vector bundle if $\mathcal{O}(1)$ is a k to $(n+e)$-admissible T-line bundle on $\mathbf{P}(E)$. We will say that $\mathbf{P}(E)$ is k^+-nice if $\mathbf{P}(E)$ is k to $(n+e)$-nice.

In future, we shall call the admissibilty and niceness conditions just admissibility conditions.

Definition: Let E be a $(k-i+1)^+$-admissible T-vector bundle, $\mathbf{P}(E)$ be $(k-i+1)^+$-nice and let $p : \mathbf{P}(E) \to X$ be the projection. For $\alpha \in A_k^T(X)$ we define the T-Segre class as

$$s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*\alpha) \tag{7.1}$$

in $A_k^T(X)$.

We start with a basic fact:

Lemma 7.1.1 *Suppose $f : X' \to X$ is a flat T-morphism of relative dimension n and L is a $(k-i+1)^+$-admissible T-line bundle on X. Then f^*L is a $(k+n-i+1)^+$-admissible T-line bundle on X'.*

Proof: Note that if $V \in Z_l^T(X')$ for $l \geq (k+n-i+1)$, then $\dim f(V) \geq l-i+1$. So L has an equivariant section over $f(V)$ and f^*L has an equivariant section over V. □

Corollary 7.1.2 *Suppose that $f : X' \to X$ is a flat T-morphism of relative dimension n, E is a $(k-i+1)^+$-admissible T-vector bundle over X and $\mathbf{P}(E)$ is $(k-i+1)^+$-nice . Then f^*E is $(k+n-i+1)^+$-admissible on X' and $\mathbf{P}(f^*E)$ is $(k+n-i+1)^+$-nice.*

Proof: Note that $\mathcal{O}_{\mathbf{P}(f^*E)}(1) = f^*(\mathcal{O}_E(1))$. The lemma applied to $\mathcal{O}_E(1)$ and the trivial T-line bundle on $\mathbf{P}(E)$ with trivial T-action then show the admissibility and niceness conditions respectively. □

Proposition 7.1.3 *1. Let E be $(k-i+1)^+$-admissible, let $\mathbf{P}(E)$ be $(k-i+1)^+$-nice and let $\alpha \in A_k^T(X)$. Then,*

a. $s_i(E) \cap \alpha = 0$ if $i < 0$

b. $s_0(E) \cap \alpha = \alpha$.

2. Let F be a rank $f+1$ T-vector bundle on X, E and F be $(k-i-j+1)^+$-admissible T-vector bundles, $\mathbf{P}(E)$ and $\mathbf{P}(F)$ be $(k-i-j+1)^+$-nice and $\alpha \in A_k^T(X)$. Then

$$s_i(E) \cap [s_j(F) \cap \alpha] = s_j(F) \cap [s_i(E) \cap \alpha] \tag{7.2}$$

in $A_{k-i-j}^T(X)$.

3. Let $f : X' \to X$ be a proper T-morphism, E be a $(k-i+1)^+$-admissible T-vector bundle over X. Suppose that f^*E is a $(k-i+1)^+$-admissible T-vector bundle over X' and $\mathbf{P}(E)$ and $\mathbf{P}(f^*E)$ are both be $(k-i+1)^+$-nice. For $\alpha \in A_k^T(X')$,

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*(\alpha) \tag{7.3}$$

in $A^T_{k-i}(X)$.

4. Let $f : X' \to X$ *be a flat T-morphism of relative dimension* n, E *be a* $(k-i+1)^+$*-admissible T-vector bundle over* X, *and* $\mathbf{P}(E)$ *be* $(k-i+1)^+$*-nice. For* $\alpha \in A^T_k(X')$,

$$f^*(s_i(E) \cap \alpha) = s_i(f^*E) \cap f^*\alpha \tag{7.4}$$

in $A^T_{k+n-i}(X')$.

5. If $\alpha \in A^T_k(X')$, E *is a* k *and* $(k+1)$*-admissible T-line bundle on* X, *and* $\mathbf{P}(E)$ *is* k *and* $(k+1)$*-nice, then*

$$s_1(E) \cap \alpha = -c_1(E) \cap \alpha \tag{7.5}$$

in $A^T_{k-1}(X)$.

Proof: The proof is contained in Fulton [6]. The only part that is different is part 1.*b*. For 3, we need the admissibility conditions on X' since if V is a j-dimensional T-subvariety of X' such that $\dim p(V) < k-i+1$ the admissibility conditions on X do not guarantee an equivariant section on V.

To prove 1, by 3 we can restrict to the case where $\alpha = [V]$ and $V = X$. We have,

$$s_0(E) \cap [V] = p_*(c_1(\mathcal{O}_E(1))^e \cap p^*[V]) = m[V] \tag{7.6}$$

in $A^T_k(V)$ for some $m \in \mathbf{Z}$. We need to show that $m = 1$. Since this is a local result we can assume that E is trivial over V.

Since the map $j : A^T_k(V) \to A_k(V)$ sending a T-cycle in $A^T_k(V)$ to its rational equivalence class in $A_k(V)$ is a surjection and since $A^T_k(V) = A_k(V) = \mathbf{Z}$ are both generated by $[V]$, we have $j[V] = \pm[V]$ in $A_k(V)$. Since the only difference between equivariant intersection and usual intersection is that we demand that the sections be equivariant, if $\alpha \in A^T_l(V)$, we have $j(s_i(E) \cap \alpha) = s_i(E) \cap \alpha$ in $A_{l-i}(V)$ provided the left-hand side is defined. In [6] Proposition 3.1 Fulton shows that $s_0(E) \cap [V] = [V]$ in $A_k(X)$. So, $s_0(E) \cap [V] = [V]$ in $A^T_k(V)$, and $m = 1$. $\square$

The proof of 1.*b* has the following corollary:

Corollary 7.1.4 *If E is $(k+1)^+$-admissible and $(k+1)^+$-nice, then*

$$p^* : A_k^T(X) \to A_{k+e}^T(\mathbf{P}(E)) \tag{7.7}$$

is a split monomorphism.

Proof: The inverse map is given by $p_*(c_1(\mathcal{O}(1))^e \cap -)$. □

Proposition 7.1.5 *Suppose that X is a T-subscheme of a smooth n-dimensional T-variety Y, $x \in Y^T$ and there is an affine open T-subset U' of Y containing x that is T-isomorphic to a T-representation space. Suppose that E is a $(k-i+1)^+$-admissible T-vector bundle on X such that the weights $\mu_0, \ldots, \mu_e$ of E in the fibre over x are distinct and $\mathbf{P}(E)$ is $(k-i+1)^+$-nice. If $\alpha \in A_k^T(X)$, then,*

$$mult_T(x, s_i(E) \cap \alpha, Y) = \sum_{j=0}^{e} \frac{(\mu_j)^{e+i}}{\prod_{l \neq j}(\mu_l - \mu_j)} mult_T(x, \alpha, Y). \tag{7.8}$$

Proof: As in Proposition 6.1.1, we need the open T-subset containing x over which E is trivial to be isomorphic to a T-representation space. Since this is not necessarily the case, we consider a different T-vector bundle that is of this form. Let U be a T-subset of U' containing x over which E is trivial. We extend E to U' to get a trivial T-vector bundle E' over U' that restricts to E on U. Let $p' : \mathbf{P}(E') = U' \times \mathbf{P}(F) \to U'$. We need to show that

$$mult_T(x, s_i(E) \cap \alpha, Y) = mult_T(x, s_i(E') \cap \alpha, Y). \tag{7.9}$$

We first show that

$$mult_T(x, p_*(c_1(\mathcal{O}_E(1)) \cap \alpha), Y) = mult_T(x, p'_*(c_1(\mathcal{O}_{E'}(1)) \cap \alpha'), Y) \tag{7.10}$$

where $\alpha \in Z_k^T(\mathbf{P}(E))$ and $\alpha' \in Z_k^T(\mathbf{P}(E'))$ is the cycle related to α. We can restrict to the case where α is a T-subvariety of X. First, suppose that $V \subset \mathbf{P}(E)|_U$ is a k-dimensional T-subvariety. We restrict V to $\mathbf{P}(E)|_U$ and close it in $\mathbf{P}(E')$ to get a k-dimensional T-subvariety $V' \subset \mathbf{P}(E')$ that restricts to V in $\mathbf{P}(E)|_U$. Note that $p_*[V] = p'_*[V']$. A T-section s of $\mathcal{O}_E(1)$

determines a weight vector in $R(V)^*$. Since $R(V^*) = R(V')^*$, this in turn determines a T-section of $\mathcal{O}_{E'}(1)$ over V' which restricts to s on V. Let D and D' be the associated T-pseudo divisors on V and V'. $D' \cdot [V']$ and the closure of $D \cdot [V]$ in $\mathbf{P}(E')$ agree, except possibly on $p'^{-1}(U - U')$. However, subvarieties of $p'^{-1}(U - U')$ map to subvarieties of $U - U'$. Since $x \notin U - U'$, these subvarieties have equivariant multiplicity 0. So,

$$mult_T(x, p_*(c_1(\mathcal{O}_E(1)) \cap \alpha), Y) = mult_T(x, p'_*(c_1(\mathcal{O}_{E'}(1)) \cap \alpha'), Y). \tag{7.11}$$

If $\alpha \in A_k^T(X)$, this shows that

$$mult_T(x, p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap p^*\alpha), Y) = mult_T(x, p'_*(c_1(\mathcal{O}_{E'}(1))^{e+i} \cap p'^*\alpha), Y) \tag{7.12}$$

and

$$mult_T(x, s_i(E) \cap \alpha, Y) = mult_T(x, s_i(E') \cap \alpha, Y). \tag{7.13}$$

We now need to show that

$$mult_T(x, p'_*[V'], Y) = \sum_{j=0}^{e} \frac{1}{\prod_{l \neq j}(\mu_l - \mu_j)} mult_T(x \times P_j, [V'], \mathbf{P}(E')). \tag{7.14}$$

for $[V'] \in Z_k^T(\mathbf{P}(E'))$.

Any T-subvariety of $\mathbf{P}(E')$ is T-rationally equivalent to a cycle whose components are of the form $W \times F_{(i_1,\ldots,i_q)}$ where W is a $(k-q)$-dimensional T-subvariety of X and of the form $W \times P_i$ where W is a k-dimensional T-subvariety of X. In the first case,

$$mult_T(x \times P_i, W \times F_{(i_1,\ldots,i_q)}, \mathbf{P}(E')) = mult_T(x, W, Y) mult_T(P_i, F_{(i_1,\ldots,i_q)} \mathbf{P}(F)). \tag{7.15}$$

Proposition 5.1.10 shows that

$$\sum_{j=0}^{e} \frac{1}{\prod_{l \neq j}(\mu_l - \mu_j)} mult_T(P_j, F_{(i_1,\ldots,i_q)}, \mathbf{P}(F)) = 0. \tag{7.16}$$

Since $p'_*(W \times F_{(i_1,\ldots,i_q)}) = 0$,

$$mult_T(x, p'_*(W \times F_{(i_1,\ldots,i_q)}), Y) = \sum_{j=0}^{e} \frac{1}{\prod_{l \neq j}(\mu_l - \mu_j)} mult_T(x/timesP_j, W \times F_{(i_1,\ldots,i_q)}, \mathbf{P}(F)) = 0. \tag{7.17}$$

In the second case, if $j \neq i$, then $mult_T(x \times P_j, W \times P_i, \mathbf{P}(E')) = 0$. Since $p'_*(W \times P_i) = W$, Proposition 5.2.2 part 7 shows that

$$\sum_{j=0}^{e} \frac{1}{\prod_{l \neq j}(\mu_l - \mu_j)} mult_T(x \times P_j, W \times P_i, \mathbf{P}(E')) = mult_T(x, p'_*(W \times P_i), Y). \tag{7.18}$$

So, for any $\alpha \in A_k^T(\mathbf{P}(E'))$,

$$mult_T(x, p_*\alpha, Y) = \sum_{j=0}^{e} \frac{1}{\prod_{l \neq j}(\mu_l - \mu_j)} mult_T(x \times P_j, \alpha, \mathbf{P}(E')). \tag{7.19}$$

Now,

$$\begin{aligned} mult_T(x, s_i(E) \cap \alpha, Y) &= mult_T(x, s_i(E') \cap \alpha, Y) \\ &= mult_T(x, p_*(c_1(\mathcal{O}_{E'}(1)) \cap p^*\alpha), Y) \\ &= \sum_{j=0}^{e} \frac{1}{\prod_{l \neq j}(\mu_l - \mu_j)} mult_T(x \times P_j, c_1(\mathcal{O}_{E'}(1))^{e+i} \cap \alpha, \mathbf{P}(E')) \\ &= \sum_{j=0}^{e} \frac{\mu_j^{e+i}}{\prod_{l \neq j}(\mu_l - \mu_j)} mult_T(x \times P_j, p^*\alpha, \mathbf{P}(E')) \\ &= \sum_{j=0}^{e} \frac{\mu_j^{e+i}}{\prod_{l \neq j}(\mu_l - \mu_j)} mult_T(x, \alpha, Y) \end{aligned} \tag{7.20}$$

□

Remark: Let X be a T-subscheme of a smooth T-variety Y. If we consider the quotient group $A_k^T(X)/\sim$ where $\alpha \sim 0$ if $mult_T(x, \alpha, Y) = 0$ for all $x \in X^T$, the proposition holds independent of niceness conditions. The reason is that if $mult_T(x, \alpha, Y) = 0$ for all $x \in X^T$, then $mult_T(x, c_1(L) \cap \alpha, Y) = 0$ for any α-admissible T-line bundle L.

The following is the T-Segre class analogue of a characteristic number formula that has been proved under various conditions by Iversen and Nielsen [9] and Brion [3]. While all of these require some use of the Riemann-Roch theorem, our result is purely combinatorial.

Proposition 7.1.6 *Suppose $X \subset \mathbf{P}^n$ is a T-subscheme where $\mathbf{P}^n$ has isolated fixed points and weights $\lambda_0, \ldots, \lambda_n$. If $x = P_j \in X^T$, let $\lambda_x = \lambda_j$. Let E be a T-vector bundle on X such that* *the weights $\mu_{x0}, \ldots, \mu_{xe}$ of E in the fibre over $x \in X^T$ are distinct. Let $\alpha \in A_k^T(X)$. If P is an isobaric polynonial of degree k in $x_0, \ldots, x_e$ where x_i has degree i, then*

$$m = \sum_{x \in X^T} \frac{1}{\prod_{\substack{y \in X^T \\ y \neq x}} (\lambda_y - \lambda_x)} P(\tau_0(x), \ldots, \tau_e(x)) mult_T(x, \alpha, \mathbf{P}^n), \tag{7.21}$$

where

$$\tau_i(x) = \sum_{j=0}^{e} \frac{\mu_{xj}^{e+i}}{\prod_{l \neq j} (\mu_{xl} - \mu_{xj})}, \tag{7.22}$$

and m is the geometric multiplicity of $P(s_1(E), \ldots, s_e(E)) \cap \alpha$.

Proof: The result is purely combinatorial. Let V be a k-dimensional T-subvariety of $\mathbf{P}(E)$. Let the fixed points of $\mathbf{P}^e$ be P_j and the open T-subset of $\mathbf{P}^e$ defined by inverting x_j be U_j. The fixed points of $\mathbf{P}(E)$ over $x \in X^T$ are then $x \times P_j$. We prove the proposition by first summing over the $x \times P_j$ where x is fixed and then summing over the fixed points of $\mathbf{P}^n$.

As in the previous proposition, let E be trivial over U. We can extend E to U_i to obtain a trivial T-vector bundle E_i over U_i.

Let $[V] \in A_k^T(\mathbf{P}(E))$. Since $\mathcal{O}_E(1)$ is not necessarily k^+-admissible, we can only guarantee the existence of a weight section s of weight ν, say. As in the previous proposition, we can extend $s\,|_{U_i}$ to a weight section s_i of $\mathcal{O}_{E_i}(1)$. Let D be the representing T-Cartier divisor defined by s and D_i, that defined by s_i. As in the previous proposition, if we identify the variety in $\mathbf{P}(E_i)$ associated to V with V,

$$\begin{aligned} &\sum_{j=0}^{e} \frac{1}{\prod_{l \neq j} (\mu_{xl} - \mu_{xj})}\; mult_T(x \times P_j, D \cdot [V], \mathbf{P}(E)) = \\ &\qquad \sum_{j=0}^{e} \frac{1}{\prod_{l \neq j} (\mu_{xl} - \mu_{xj})} mult_T(x \times P_j, D_j \cdot [V], \mathbf{P}(E_i)) = \\ &\qquad \sum_{j=0}^{e} \frac{\mu_{xj} + \nu}{\prod_{l \neq j} (\mu_{xl} - \mu_{xj})} mult_T(x \times P_j, [V], \mathbf{P}(E_i)). \end{aligned} \tag{7.23}$$

If $\dim V > 0$, either $p_*[V] = 0$ or $\dim p_*[V] > 0$. In either case,

$$\sum_{x \in X^T} \frac{1}{\prod_{\substack{y \in X^T \\ y \neq x}} (\lambda_y - \lambda_x)} mult_T(x, p_*[V], \mathbf{P}^n) = 0. \tag{7.24}$$

Since

$$mult_T(x, p_*[V], \mathbf{P}^n) = \sum_{j=0}^{e} \frac{1}{\prod_{l \neq j}(\mu_{xl} - \mu_{xj})} mult_T(x \times P_j, [V], \mathbf{P}(E_i)), \tag{7.25}$$

we have,

$$\begin{aligned} \sum_{x \in X^T} \frac{1}{\prod_{\substack{y \in X^T \\ y \neq x}} (\lambda_y - \lambda_x)} \sum_{j=0}^{e} \frac{\mu_{xj} + \nu}{\prod_{l \neq j}(\mu_{xl} - \mu_{xj})} mult_T(x \times P_j, [V], \mathbf{P}(E_i)) = \\ \sum_{x \in X^T} \frac{1}{\prod_{\substack{y \in X^T \\ y \neq x}} (\lambda_y - \lambda_x)} \sum_{j=0}^{e} \frac{\mu_{xj}}{\prod_{l \neq j}(\mu_{xl} - \mu_{xj})} mult_T(x \times P_j, [V], \mathbf{P}(E_i)). \end{aligned} \tag{7.26}$$

Recalling that for $\alpha \in A_k^T(X)$,

$$mult_T(x \times P_j, p^*\alpha, \mathbf{P}(E_i)) = mult_T(x, \alpha, \mathbf{P}^n) \tag{7.27}$$

and using the above we find,

$$\begin{aligned} \sum_{x \in X^T} \frac{1}{\prod_{\substack{y \in X^T \\ y \neq x}} (\lambda_y - \lambda_x)} mult_T(x, s_i(E) \cap \alpha, \mathbf{P}^n) = \\ \sum_{x \in X^T} \frac{1}{\prod_{\substack{y \in X^T \\ y \neq x}} (\lambda_y - \lambda_x)} \sum_{j=0}^{e} \frac{\mu_{xj}^{e+i}}{\prod_{l \neq j}(\mu_{xl} - \mu_{xj})} mult_T(x, \alpha, \mathbf{P}^n). \end{aligned} \tag{7.28}$$

So,

$$\begin{aligned} \sum_{x \in X^T} \frac{1}{\prod_{\substack{y \in X^T \\ y \neq x}} (\lambda_y - \lambda_x)} mult_T(x, P(s_0(E), \ldots, s_e(E)) \cap \alpha, X) = \\ \sum_{x \in X^T} \frac{1}{\prod_{\substack{y \in X^T \\ y \neq x}} (\lambda_y - \lambda_x)} mult_T(x, P(\tau_0(x), \ldots, \tau_e(x)), \mathbf{P}^n). \end{aligned} \tag{7.29}$$

Recalling that if $\alpha \in A_0^T(X)$,

$$m = \sum_{x \in X} \frac{1}{\prod_{\substack{y \in X^T \\ y \neq x}} (\lambda_y - \lambda_x)} mult_T(x, \alpha, X), \tag{7.30}$$

we get the result. □

While the sums in the proposition may look formidable, note that $\prod_{y \neq x}(\lambda_y - \lambda_x)$ is just the

Van Dermonde determinant and that the numerator of the fraction

$$\prod_{j=0}^{e} \frac{\mu_{xj}^{e+i}}{\prod_{k\neq j}(\mu_{xk} - \mu_{xj})} \tag{7.31}$$

is the determinant of the matrix

$$VDM(\mu_{xj}, i) = \det \begin{pmatrix} 1 & \mu_{x0} & \mu_{x0}^2 & \cdots & \mu_{x0}^{e-1} & \mu_{x0}^{e+i} \\ 1 & \mu_{x1} & \mu_{x1}^2 & \cdots & \mu_{x1}^{e-1} & \mu_{x1}^{e+i} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \mu_{xe} & \mu_{xe}^2 & \cdots & \mu_{xe}^{e-1} & \mu_{xe}^{e+i} \end{pmatrix} \tag{7.32}$$

7.2 T-Chern Classes

We define the T-Chern classes in terms of the T-Segre classes and we show their basic properties. The bulk of this section involves showing that the splitting construction works in the equivariant setting.

We imitate the construction of the T-Chern classes in Fulton. If E is $(k-j+1)^+$-admissible and $\mathbf{P}(E)$ is $(k-j+1)^+$-nice, we define the polynomial

$$s_t(E) = \sum_{i=0}^{j} s_i(E)t^i. \tag{7.33}$$

We define the T-Chern series to be,

$$c_t(E) = \sum_{i=0}^{\infty} c_i(E)t^i, \tag{7.34}$$

where $c_t(E) = s_t(E)^{-1}$. The first j T-Chern classes are the first j terms of the series.

Explicitly,

$$c_0(E) = 1 \tag{7.35}$$

$$c_1(E) = -s_1(E) \tag{7.36}$$

$$c_2(E) = -s_1(E)c_1(E) - s_2(E) \tag{7.37}$$

$$\vdots$$

$$c_j(E) = -s_1(E)c_{j-1}(E) - s_2(E)c_{j-2}(E) - \cdots - s_j(E). \tag{7.38}$$

Theorem 7.2.1 *1. Suppose that E is a $(k-i+1)^+$-admissible T-vector bundle on a T-scheme X such that $\mathbf{P}(E)$ is $(k-i+1)^+$-nice. If $i > \mathrm{rank}E$, then*

$$c_i(E) = 0. \tag{7.39}$$

2. Let F be a rank $f+1$ T-vector bundle on X, E and F be $(k-i-j+1)^+$-admissible T-vector bundles, $\mathbf{P}(E)$ and $\mathbf{P}(F)$ be $(k-i-j+1)^+$-nice and $\alpha \in A_k^T(X)$. Then

$$c_i(E) \cap [c_j(F) \cap \alpha] = c_j(F) \cap [c_i(E) \cap \alpha] \tag{7.40}$$

in $A_{k-i-j}^T(X)$.

3. Let $f : X' \to X$ be a proper T-morphism, E a $(k-i+1)^+$-admissible T-vector bundle on X, f^*E be a $(k-i+1)^+$-admissible T-vector bundle on X' and let $\mathbf{P}(E)$ and $\mathbf{P}(f^*E)$ be $(k-i+1)^+$-nice. If $\alpha \in A_k^T(X)$, then

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha \tag{7.41}$$

in $A_{k-i}^T(X)$.

4. Let $f : X' \to X$ be a flat T-morphism, E a $(k-i+1)^+$-admissible T-vector bundle on X and $\mathbf{P}(E)$ and be $(k-i+1)^+$-nice. If $\alpha \in A_k^T(X)$, then

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha) \tag{7.42}$$

in $A_{k+n-i}^T(X)$.

5. Let $\alpha \in A_k^T(X)$. If E is of rank $r = e+1$,

$$0 \to E' \to E \to E'' \to 0 \tag{7.43}$$

is an exact sequence of $(k-r)^+$-admissible T-vector bundles and $\mathbf{P}(E')$, $\mathbf{P}(E)$ and $\mathbf{P}(E'')$ are $(k-r+1)^+$-nice, then

$$c_t(E) = c_t(E')c_t(E''). \tag{7.44}$$

6. If E is a $[X]$-admissible T-line bundle on X, D a T-pseudo divisor on X with $\mathcal{O}_D \simeq E$, then

$$c_1(E) \cap X = [\mathcal{O}_D]. \tag{7.45}$$

Proof: 2, 3, 4 and 6 follow from Proposition 7.1.3. We show 1 and 5 by using the splitting construction.

Splitting Construction

Given a $(k-r+1)^+$-admissible T-vector bundle E of rank $r = e+1$ on X such that $\mathbf{P}(E)$ is $(k-r+1)^+$-nice, we construct a space $f : X' \to X$ such that f is a flat T-morphism of relative dimension n, say, $f^* : A_k^T(X) \to A_{k+n}^T(X')$ is injective, f^*E is $(k+n-r+1)^+$-admissible, and f^*E has a filtration,

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = f^*E \tag{7.46}$$

where the quotient bundles

$$M_i/M_{i-1} = L_i \tag{7.47}$$

are $(k+n-r+1)^+$-admissible T-line bundles.

The construction is the usual splitting construction. Consider

$$\begin{array}{ccc} p_0^*E & \longrightarrow & E \\ \Big\downarrow & & \Big\downarrow \\ \mathbf{P}(E) & \xrightarrow[p_0]{} & X. \end{array} \tag{7.48}$$

p_0^*E has a T-subbundle $\mathcal{O}_E(-1)$. Let E^1 be the quotient T-vector bundle $p_0^*E/\mathcal{O}(-1)$ on $\mathbf{P}(E)$. E^1 is of rank $r-1$. If we set $X^1 = \mathbf{P}(E)$, we can repeat the construction to get $X^2 = \mathbf{P}(E^1)$, and $E^2 = p_1^*E^1/\mathcal{O}_{E^1}(-1)$. Continuing this, we arrive at $X' = X^r$. Let f be the composition of all the p_i. Since each p_i is flat, f is flat, and since $p_i^* : A_k^T(X^i) \to A_{k+e-i}^T(\mathbf{P}(E^i))$ injective by Corollary 7.1.4, $f^* : A_k^T(X') \to A_k^T(X)$ injective. This gives us the diagram:

$$
\begin{tikzcd}
E^r & & & & \\
\vdots & & & & \\
q_2^*E^2 \arrow[r] & \cdots \arrow[r] & E^2 & & \\
q_1^*E^1 \arrow[r] \arrow[u] & \cdots \arrow[r] & p_1^*E^1 \arrow[r] \arrow[u] & E^1 & \\
q_0^*E \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & p_1^*p_0^*E \arrow[r] \arrow[u] \arrow[d] & p_0^*E \arrow[r] \arrow[u] \arrow[d] & E \arrow[d] \\
X^r \arrow[r, "p_r"'] & \cdots \arrow[r] & \mathbf{P}(E^1) \arrow[r, "p_1"'] & \mathbf{P}(E) \arrow[r, "p_0"'] & X,
\end{tikzcd}
\tag{7.49}
$$

where $q_i = p_r \circ p_{r-1} \circ \cdots \circ p_i$.

Let M_i be the kernel of the map $q_0^*E \to q_i^*E^i$. We have $M_{i-1} \subset M_i$ and

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = f^*E. \tag{7.50}$$

Finally, since the E^i are T-vector bundles of rank $r-i$, the M_i are T-vector bundles of rank i.

We still need to show admissibility.

Lemma 7.2.2 *If E is a $(k-i+1)^+$-admissible T-vector bundle of rank e and $\mathbf{P}(E)$ is j^+-nice for any $j > 0$, then any rank $e-1$ T-subbundle E' and any quotient T-line bundle L of E are $(k-i+1)^+$-admissible and j^+-nice.*

Proof: we have the commuting diagram

$$\begin{array}{ccc} \mathcal{O}_{E'}(1) & \longrightarrow & \mathcal{O}_E(1) \\ \downarrow & & \downarrow \\ \mathbf{P}(E') & \longrightarrow & \mathbf{P}(E) \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X'. \end{array} \tag{7.51}$$

$\mathcal{O}_{E'}(1)$ is the restriction of $\mathcal{O}_E(1)$ to $\mathbf{P}(E')$, so, since E is $(k-i+1)^+$-admissible, and any T-subvariety of $\mathbf{P}(E')$ is a T-subvariety of $\mathbf{P}(E)$, E' is $(k-i+1)^+$-admissible.

Suppose that $\mathcal{O}_L$ is locally generated over $\mathcal{O}_U$ by x. Since $E \to L$, there is a morphism on the structure sheaves $\varphi : \mathcal{O}_U \otimes k[x] \to \mathcal{O}_E$. Let $\varphi(1 \otimes x) = y$. This induces a morphism $\mathbf{P}(E) \to \mathbf{P}(L)$, and we also get a morphism $\mathcal{O}_E(-1) \to \mathcal{O}_L(-1) \simeq L$. Since $\mathcal{O}_E(-1)$ has an equivariant section over any T-subvariety of X of dimension greater than $(k-i+1)$, composing with the morphism above gives a T-section of L. So, L is $(k-i+1)^+$-admissible.

To show the niceness conditions, note that if $\mathbf{P}(E)$ is j^+-nice, then the trivial T-line bundle with trivial T-action over any T-subvariety of $\mathbf{P}(E)$ of dimension greater than j has a T-section. If E' is a T-subbundle of E, then any T-subvariety of $\mathbf{P}(E')$ is contained in $\mathbf{P}(E)$, so any T-subvariety of $\mathbf{P}(E')$ of dimension greater than j has a T-section of the trivial T-line bundle with trivial T-action. Similarly, if L is a quotient T-line bundle, then $\mathbf{P}(L) = X$. Since X can be embedded is $\mathbf{P}(E)$, any T-subvariety of X of dimension greater than j has a T-section of the trivial T-line bundle with trivial T-action defined on it. □

For future reference, we would like to know the weights of the L_i. We find them inductively. Let $F_{(j_0,\ldots,j_i)}$ be the T-representation space with basis $X_{j_0}, \ldots, X_{j_i}$. We let $U^i_{j_i}$ be the open subset of $\mathbf{P}(F_{(j_0,\ldots,j_i)})$ obtained by inverting x_{j_i}. Locally, $\mathbf{P}(E^i)$ is of the form $U \times U^0_{j_0} \times \cdots \times U^i_{j_i}$, and locally E^i has weights $\{\lambda_l : l \neq j_i, \text{ for all } i\}$. Pulling this back to X', and taking kernels, we find M_i has weights $\lambda_{j_0}, \ldots, \lambda_{j_i}$ over $U \times U^0_{j_0} \times \cdots \times U^i_{j_i}$. So, the quotient bundle $M_i/M_{i-1} = L_i$ has weight λ_i over $U \times U^0_{j_0} \times \cdots \times U^i_{j_i}$.

Suppose $i \geq r$. We want to show that

$$c_t(E) = \prod_{j=1}^{r}(1 + c_1(L_j)t). \tag{7.52}$$

Lemma 7.2.3 *Suppose E is filtered as above, E is $(k-i+1)^+$-admissible for $i \geq r$ and $\mathbf{P}(E)$ is $(k-i+1)^+$-nice. Let s be an equivariant section of E with support Z. Then, for $\alpha \in A^T_k(X)$, there exists a cycle $\beta \in |Z|$ such that*

$$\prod_{j=1}^{r} c_1(L_j) \cap \alpha = \beta. \tag{7.53}$$

In particular, if s is trivial, i.e. $Z = \emptyset$, then $\prod c_1(L_j) \cap \alpha = 0$.

Proof: The proof is as in Fulton [6]. We proceed by induction. If $r = 1$, since E is a k^+-admissible T-line bundle, E is T-represented by the T-pseudo divisor (E, Z, s) where $Z = \text{supp}(s)$. So, $c_1(E) \cap \alpha \subset Z$. Suppose that we have the result for T-vector bundles of rank $r-1$. Consider the exact sequence,

$$0 \to M_{r-1} \to E \xrightarrow{\varphi} L_r \to 0. \tag{7.54}$$

Let $\bar{s} = \varphi(s)$. Then, $\bar{s}$ is a (possibly 0) T-section of L_r. Since E is $(k-i+1)^+$-admissible, L is as well, and we set

$$D_r = \begin{cases} (L_r, Z, \bar{s}) & \text{if } \bar{s} \neq 0 \\ (L_r, Z, s') & \text{if } \bar{s} = 0, \end{cases} \tag{7.55}$$

where s' is an equivariant section of L_r over X. If $j : Z \to X$ is the inclusion,

$$c_1(L_r) \cap \alpha = j_*(D_r \cdot \alpha). \tag{7.56}$$

Since M_{r-1} is a rank $r-1$ T-subbundle of M_r, M_{r-1} has a T-section induced by s with support Z.

$$\prod_{j=0}^{r} c_1(L_j) \cap \alpha = (\prod_{j=0}^{r-1} c_1(L_j)) \cap j_*(D_r \cdot \alpha) = \beta \tag{7.57}$$

for $\beta \in |Z|$. $\square$

To complete the splitting, let E be a $(k-i+1)^+$-admissible T-vector bundle for $i \geq r$ over X which is filtered as above. Consider $p : \mathbf{P}(E) \to X$. $\mathcal{O}_E(-1)$ is a T-subbundle of p^*E and $\mathcal{O}(-1) \otimes \mathcal{O}(1)$ is a trivial T-line subbundle of weight 0 of $p^*E \otimes \mathcal{O}(1)$. Since $\mathcal{O}(-1) \otimes \mathcal{O}(1)$ is $(k-i+1)^+$-admissible it has an equivariant section over any T-subvariety of $\mathbf{P}(E)$ of dimension greater than $k-i+1$. So we have a trivial T-section of $p^*E \otimes \mathcal{O}(1)$. $p^*E \otimes \mathcal{O}(1)$ has a filtration by T-subbundles with T-line bundle quotients $p^*L_j \otimes \mathcal{O}(1)$. So,

$$\prod c_1(p^*L_j \otimes \mathcal{O}(1)) = 0. \tag{7.58}$$

Let $\xi = c_1(\mathcal{O}(1))$, σ_k (resp. $\tilde{\sigma}_k$) be the k^{th} symmetric function in the $c_1(L_j)$ (resp. $c_1(p^*L_j)$). Since p^*L_j, and $\mathcal{O}(1)$ are $(k-i+1)^+$-admissible,

$$\prod c_1(p^*L_j \otimes \mathcal{O}(1)) = \xi^r + \tilde{\sigma}_1 \xi^{r-1} + \ldots + \tilde{\sigma}_r = 0. \tag{7.59}$$

Multiplying by ξ^{l-1} for $1 \leq l \leq i+1$, and recalling that $r = e+1$,

$$\xi^{e+l} + \tilde{\sigma}_1 \xi^{e+l-1} + \ldots + \tilde{\sigma}_r \xi^{l-1} = 0. \tag{7.60}$$

So,

$$p_*(c_1(\mathcal{O}(1))^{e+l} \cap p^*\alpha) + p_*(\tilde{\sigma}_1 c_1(\mathcal{O}(1))^{e+l-1} \cap \alpha) + \ldots + p_*(\tilde{\sigma}_r c_1(\mathcal{O}(1))^{l-1} \cap p^*\alpha) = 0 \tag{7.61}$$

$$s_l(E) \cap \alpha + \sigma_1 s_{l-1}(E) \cap \alpha + \ldots + \sigma_r s_{l-r-1}(E) \cap \alpha = 0. \tag{7.62}$$

So,

$$(1 + \sigma_1 t + \ldots + \sigma_r t^r) s_t(E) = 1, \tag{7.63}$$

and

$$c_t(E) = \prod_{i=1}^{r} (1 + c_1(L_i) t). \tag{7.64}$$

We now show 1a and 5.

Injectivity of f^*, and

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha) \tag{7.65}$$

now imply 1a. For 5, we find $f : X' \to X$ such that f^*E' and f^*E'' split, with $(k-r+1)^+$-admissible T-line bundle quotients L'_j and L''_k. f^*E then has an induced filtration with quotients L'_j, and L''_k. So,

$$c_t(E) = c_t(E')c_t(E''). \tag{7.66}$$

□

The splitting construction also yields,

Proposition 7.2.4 *Suppose that $X \subset Y$ is a T-subscheme of a smooth T-variety Y, $x \in Y^T$ and there is an open affine T-subset U of Y containing x which is T-isomorphic to a T-representation space. Let E be a $(k-i+1)^+$-admissible T-vector bundle on X with weights $\mu_0, \ldots, \mu_e$, and let $\mathbf{P}(E)$ is $(k-i+1)^+$-nice. If $\alpha \in A_k^T(X)$, then*

$$mult_T(x, c_i(E) \cap \alpha, Y) = \sigma_i(\mu_0, \ldots, \mu_e) mult_T(x, \alpha, X). \tag{7.67}$$

Proof: We note that in the spliting construction, X' is locally of the form $U \times F$ where U is an open T-subset of X and F is a T-representation space. If $x \times 0 \in U \times F$, then

$$mult_T(x, c_i(E) \cap \alpha, Y) = mult_T(x \times 0, c_i(f^*(E)) \cap f^*\alpha, U \times F). \tag{7.68}$$

Since $c_i(f^*E) \cap f^*\alpha = \prod_{i=1}^r (c_1(L_i) \cap f^*\alpha)$, and L_i has weight λ_i over x,

$$mult_T(x \times 0, c_i(f^*E) \cap f^*\alpha, X') = \sigma_i(\mu_0, \ldots, \mu_e) mult_T(x \times 0, f^*\alpha, X') \tag{7.69}$$

$$= mult_T(x, c_i(E) \cap \alpha, X). \tag{7.70}$$

□

Corollary 7.2.5 *If X, Y and E satisfy the conditions of the proposition,*

$$mult_T(x, s_i(E) \cap \alpha, X) = \tau_i(\mu_0, \ldots, \mu_e) mult(x, \alpha, X) \tag{7.71}$$

where $\tau_i(\mu_0, \ldots, \mu_e)$ is the sum of all monomials of degree i in the μ_j's.

Proof: This results from the relations between the T-Segre and T-Chern classes. These relations are precisely the ones between the elementary symmetric functions and the τ_i. □

We also get the following seemingly purely algebraic fact:

Corollary 7.2.6 $VDM(\mu_j, i) = \tau_i(\mu_0, \ldots, \mu_e) \prod_{j \neq k}(\mu_j - \mu_k)$.

Oddly enough, the determinant $VDM(\mu_{jl}, i)$ does not seem to have been calculated before.

If we omit the admissibility conditions, since we can not determine the equivariant multiplicity of the intersection except when the final cycle is of dimension 0, we only get the characteristic number formula mentioned before:

Proposition 7.2.7 *Let $X \subset \mathbf{P}^m$ be an n-dimensional T-variety. Suppose that $\mathbf{P}^m$ has isolated fixed points and weights $\lambda_0, \ldots, \lambda_m$. Let E be a T-vector bundle over $\mathbf{P}^m$ with weights $\{\mu_{i0}, \ldots, \mu_{ie}\}$ over $P_i \in \mathbf{P}^m$. Let $\alpha \in A_k^T(X)$. If $P(x_0, \ldots, x_e)$ is an isobaric polynomial of degree n, where x_i has degree i, then*

$$\int_X P(c_1(E), \ldots, c_e(E) \cap \alpha) = \sum_{i=0}^{n} \frac{1}{\prod_{j \neq i}(\lambda_j - \lambda_i)} P(\sigma_{i1}, \ldots, \sigma_{ie}) mult_T(P_i, \alpha, \mathbf{P}^n), \tag{7.72}$$

where σ_{ik} are the k^{th} symmetric functions in the μ_{ij}.

Proof: This is merely a consequence of the result concerning T-Segre classes. If we replace the c_i by its expression in terms of T-Segre classes, we get an isobaric polynomial as above, but in the T-Segre classes. Proposition 7.1.6 and the expression of the T-Segre classes in tems of the T-Chern classes then yield the result. □

7.3 Notes

If we consider $*$ intersection (*i.e.* we have $c_1(\mathcal{O}(1))^{e+i} * p^*\alpha$ in the definition of intersection) then all the properties of Proposition 7.1.3 and Proposition 7.2.1 hold, and the only ones to hold without admissibility conditions are Proposition 7.1.3 part $1a$ and the push forward properties for both the Segre and Chern classes. The various properties concerning multiplicities hold as well, provided we do not omit the admissibility conditions.

Chapter 8

Applications and Relations

We consider how $A_k^T(X)$ is related to the other equivariant objects. The objects we are interested in are $A^k(X//T)$ where $X//T$ is the algebraic quotient, vector bundles on $X//T$ and $K_T(X)$. The main result we show is that if X is a complex variety with free T-action and $\dim T = r$, then $A_k^T(X) \otimes \mathbf{Q} = A_{k+r}(X//T) \otimes \mathbf{Q}$. In the third section we consider $A_k^G(X)$ for G a reductive algebraic group.

8.1 $A(X//T)$

Suppose $\dim T = r$. Let X be a complex T-variety with free T-action, and suppose that X can be covered by open affine T-subsets. We show that $A_*(X//T) \otimes \mathbf{Q} = A_*^T(X) \otimes \mathbf{Q}$. If $f : X \to X//T$ is the quotient morphism, we show that if E is a bundle on $X//T$, then f^*E is an r-admissible T-vector bundle. We show that in some cases, multiplicity is an invariant of $A_*(X//T)$.

We start with a result of Vistoli [19].

Proposition 8.1.1 *Let $f : X \to X//T = Y$ be the quotient variety, Y' be the normalization of Y and let X' be the normalization of X. There exists a normal variety Y'' such that if X'' is the normalization of $Y'' \times_Y X$, then*

$$\begin{array}{ccccc} X'' & \xrightarrow{q'} & X' & \xrightarrow{q} & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow[p']{} & Y' & \xrightarrow[p]{} & Y \end{array} \tag{8.1}$$

commutes. Furthermore, there are finite groups F' and F acting on Y'' such that $Y''//F' = Y'$ and $Y''//F = Y$.

Proof: The proof is in Vistoli [19], Lemma 4. In fact, the result he proves is more general since it concerns reductive algebraic groups. □

Definition: Let G be a group acting on X' and let $p : X' \to X = X'//G$ be the quotient. Let W be a subvariety of X and let e_W be the order of the inertia group of a general point of $p^{-1}(W)$. We define $p^* : Z_k^T(X) \to Z_k^T(X')$ by,

$$p^*[W] = e_W/e_Y[p^{-1}(W)]. \tag{8.2}$$

Lemma 8.1.2 *Suppose G is finite. Then the set of cycles T-rationally equivalent to 0 in $Z_k^T(X') \otimes \mathbf{Q}$ are generated by cycles of the form*

$$\sum_{g\in G}[\mathrm{div}(g^{-1}\cdot r)] \tag{8.3}$$

where V is a $k+1$-dimensional T-subvariety of X' and $r \in R(V)^*$ is a weight vector of weight 0. In particular, $A_k^T(X') \otimes \mathbf{Q} = A_k^T(X) \otimes \mathbf{Q}$.

Proof: Suppose that α is G stable and T-rationally equivalent to 0. There exists a collection of $k+1$-dimensional T-subvarieties $V_1, \ldots, V_n$ of X' and T-weight vectors $f_i \in R(V_i)^*$ of weight 0 such that $\alpha = \sum_{i=1}^n \mathrm{div} f_i$. Averaging over G we have,

$$(\sum_{g\in G} g)\cdot\alpha = \sum_{i=1}^n \sum_{g\in G}[\mathrm{div}(g^{-1}\cdot f_i)] = |G|\,\alpha. \tag{8.4}$$

So, after tensoring with $\mathbf{Q}$, cycles of this form generate the set of G stable cycles that are T-rationally equivalent to 0 in $Z_k^T(X') \otimes \mathbf{Q}$. □

Theorem 8.1.3 *1. f^* passes to T-rational equivalence*

2. $f^* : A_k(X//T) \otimes \mathbf{Q} \to A_{k+r}^T(X) \otimes \mathbf{Q}$ is an isomorphism.

Proof: Vistoli shows [19] Lemmas 2 and 3, that $f'^*q'^*$, $f''^*p'^* : Z_k(Y') \otimes \mathbf{Q} \to Z_k^T(X'') \otimes \mathbf{Q}$ are defined and are equal. We know that $f''^* : A_k(Y'') \otimes \mathbf{Q} \simeq A_{k+r}^T(X'') \otimes \mathbf{Q}$ from Proposition 4.8.2. Since p'^* and q'^* are also isomorphisms on the $A_k^T(-) \otimes \mathbf{Q}$, f'^* passes to T-rational equivalence and is an isomorphism as well.

Vistoli [19] also shows in the proof of Theorem 1, that f^*p_*, $q_*f'^* : Z_k(Y') \to Z_{k+r}^T(X)$ are defined and are equal. As we have just seen, f'^* passes to T-rational equivalence and is an isomorphism after tensoring with $\mathbf{Q}$. Since p_* and q_* are also isomorphisms after tensoring with $\mathbf{Q}$, $f^* : Z_k(Y) \otimes \mathbf{Q} \to Z_{k+r}^T(X) \otimes \mathbf{Q}$ passes to T-rational equivalence and is an isomorphism on the Chow groups.. □

The theorem allows us to calculate the Chow groups for quotients of affine spaces fairly easily.

Corollary 8.1.4 *Let V be a T-representation space, and let V^{ss} be the open set of V over which the T-action is free. Then,*

$$f^* : A_k(V^{ss}//T) \otimes \mathbf{Q} \simeq A_{k+r}^T(V^{ss}) \otimes \mathbf{Q}. \tag{8.5}$$

Note that this is essentially the same result that Ellingsrud and Stromme [5] get. They also show that if $V^{ss}//T$ is non-singular then $A_k^T(V^{ss})$ is a free group and

$$f^* : A_k(V^{ss}//T) \simeq A_{k+r}^T(V^{ss}) \otimes \mathbf{Q}. \tag{8.6}$$

Example: The morphism f^* is not necessarily surjective (even if defined) if we do not tensor with $\mathbf{Q}$.

Let $X = \mathbf{A}^2$ with the G_m-action with weights $\lambda_x = 1$ and $\lambda_y = 2$. $X//T \simeq \mathbf{P}^1$, but $A_1^T(X) = \mathbf{Z} \oplus \mathbf{Z}/(2,-1)$. $f^* : X//T \to X$ sends the generator of $A_0(X//T)$ to $(0,1) \in A_1^T(X)$. Since $(0,1) \sim (2,0)$, $(1,0)$ has no pre-image and f^* is not surjective.

Example: The map $f_* : Z_*^T(X) \to Z(X//T)$ induced by $f_*[V] = [f(V)]$ does not induce a map on the equivariant Chow groups.

We use the previous example. Let P and Q be the points of $X//T$ defined by the semi-invariant ideals (x) and (y). Since x^2/y is T-rationally equivalent to 0, if $f_* : A_k^T(X) \to A_{k-r}(X//T)$ were a morphism, then $f_*(\mathrm{div}(x^2/y)) = 2[P] - [Q]$ would be rationally equivalent to 0. However, since $X//T \simeq \mathbf{P}^1$, this is not the case. So, $f_* : A_k^T(X) \to A_{k-r}(X//T)$ is not a morphism.

We relate the vector bundles on $X//T$ to the T-vector bundles on X. We start with a theorem whose proof is due to F. Knop [11].

Theorem 8.1.5 *If the action of T on the complex variety X is free, then every T-vector bundle is isomorphic (not necessarily T-isomorphic) to the pull back of a vector bundle on $X//T$. If the weights of the T-vector bundle are all 0, then the isomorphism is a T-isomorphism.*

Proof: The proof of the first statement is in Kraft [11]. All we have to note is that the pull back bundles really are T-bundles. Let $p : X \to X//T$. If E is a vector bundle over $X//T$, then locally over some open $U \subset X//T$, E is trivial. $p^{-1}U$ is an open T-subset of X over which p^*E is trivial. For the second statement, note that E has the trivial T-action. So, the weights of E over U are all 0, and those of p^*E over $p^{-1}(U)$ are also 0. □

We consider equivariant multiplicities and quotient varieties.

Let Y be a smooth T-variety such that for $x \in Y^T$ there exists an open T-subset U containing x that is T-isomorphic to a T-representation space. Let X be a T-subvariety of Y. Let A be the free group generated by the $mult_T(x, [V], Y)$ for all k-dimensional T-subvarieties V of X. Let B be the free group generated by all the $mult_T(x, [V], Y)$ for all k-dimensional T-subvarieties of X contained in $X - X^{ss}$. Then $mult_T(x, \alpha, Y)$ is an invariant of $\alpha \in A_{k-r}(X^{ss})$ in A/B where we identify $mult_T(x, \alpha, Y)$ with its class in A/B. This follows from the exact sequence

$$A_k^T(X - X^{ss}) \to A_k^T(X) \to A_k^T(X^{ss}) \to 0. \tag{8.7}$$

Remark: We would like to understand the relationship between $A(X//T)$ and $A_k^T(X)$ better. We would like to know if in Theorem 8.1.3 the morphism is an injection without tensoring with $\mathbf{Q}$. In many examples, we do find that it is an injection.

As far as intersections go, if E is a vector bundle on $X//T$, we would like to know how $s_i(E) \cap -$ and $c_i(E) \cap -$ behave when pulled back to X.

8.2 $K_T(X)$

Since $A_k^T(X)$ involves T-cycles, we would expect some relationship between $A_k^T(X)$ and $K_T(X)$. This is in fact the case, but the relation is not quite the one we would expect.

First of all, the obvious morphism $\varphi : Z_k^T(X) \to K^T(X)$ does exist and is an injection. If M is a T, $\mathcal{O}_X$ module, then M has a composition series with quotients locally T-isomorphic to $\mathcal{O}_X/P_i \cdot v_i$. Let V_i be the variety associated to R/P_i. Then $\sum_i (-1)^i R/P_i$ is in the image of φ, but $\sum_i (-1)^i R/P_i \cdot v_i$ need not be. So, φ is not an isomorphism.

Repeating S.G.A. VI, we do get a map that respects T-rational equivalence. So,

$$A_k^T(X) \to K^T(X) \qquad mod \;\; K^T(X)^{k+1} \tag{8.8}$$

exists. On resolving, we get

$$A_k^T(X) \otimes \mathbf{Q} \to K_T(X) \otimes \mathbf{Q}. \tag{8.9}$$

Remark: As in usual K-theory, we would expect the map to have an inverse at least on $\mathrm{Im}(A_*^T(X) \otimes \mathbf{Q})$. The proposed inverse would be the Chern class. However, since we have admissibility problems, the Chern class does not necessarily exist, and the inverse need not exist either.

We also have other problems. Since $A_*^T(X)$ does not necessarily have a product structure, the localization result we would expect from Nielsen's [16] result does not hold.

8.3 $A_k^G(X)$

We consider a definition of $A_k^G(X)$ for G a reductive algebraic group and X a G-scheme. If G is a reductive group with maximal torus T, we can define a G/T-action on $Z_k^T(X)$ as follows: let $g \in G$ be a representative for $g' \in G/T$. For $[V] \in Z_k^T(X)$ we set $g' \cdot V = g \cdot V$. Since V is a T-variety, this action is independent of the choice of representative. This action extends to one on $A_k^T(X)$.

Definition: If G is reductive with maximal torus T we say $\alpha \in Z_k^T(X)^{G/T} = Z_k(X)^G$ is G-rationally equivalent to 0 if there is a collection pairs $\{(V_1, f_1), \ldots, (V_n, f_n)\}$ where V_i is a T-variety, $f_i \in R(V_i)^*$ is a weight vector of weight 0 and the collection is G-stable. By this we mean that the pair $(g \cdot V_i, g^{-1} \cdot f_i)$ is in the collection for every $g \in G$ and every $1 \leq i \leq n$.

Using Lemma 2 of Vistoli [19], if G/T is finite, we have

$$A_k^G(X) \otimes \mathbf{Q} = A_k^T(X)^{G/T} \otimes \mathbf{Q} = A_{k-r}(X//T)^{G/T} \otimes \mathbf{Q} = A_{k-r}(X//G) \otimes \mathbf{Q}. \tag{8.10}$$

where $r = \dim T$, and provided $X//G$ exists.

For this definition of the equivariant Chow groups most of the properties Fulton considers in chapter 1 hold. For the alternate definition of G-rational equivalence we require that the collection of pairs $\{(V_i, f_i)\}$ be G-stable where V_i is a T-subvariety of $X \times \mathbf{P}^1$ where $\mathbf{P}^1$ has the trivial G-action and $f_i : V_i \to \mathbf{P}^1$ is the T-morphism induced by the projection onto $\mathbf{P}^1$ and is dominant. As above, by G-stable we mean that the pair $(g \cdot V_i, f_i \circ g^{-1})$ is in the collection for every $g \in G$ and for every i. We also have the exact sequence of section 4.6, provided we require that the subscheme X be G stable. We also have flat pullback. What we do not have, though, is the moving result of section 4.4, nor the change of groups results of section 4.7, nor the affine bundle results of section 4.8. The reason for this is that the part involving G/T is not always satisfied.

This should give a fairly good idea of why we did not consider G-rational equivalence. The definition is too hard to work with, and not all the properties we want necessarily exist.

We consider another possible definition.

Example: As with $A_k^T(X)$, we could demand that the subvarieties generating $A_k^G(X)$ be G invariant. This however, is too restrictive. Consider $X = \mathbf{A}^2$ with the $G = \mathbf{Z}/2\mathbf{Z}$-action given by interchanging X_1 and X_2. For this definition of equivariant Chow groups, let B be the free group on the elements of the field k. Then $A_0^G(X) = B/2B$ and is generated by the cycles of the form $(X_1 - a, X_2 - a) \in \mathbf{A}^2$, where $a \in k$.

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