COMPOSITION OF MASSIVE FROM MASSLESS BOSONS

by

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ABSTRACT

Massive bosons are constructed from generalized biparticle states of massless scalar particles ("zerons") and of massless vector particles (photons). The self-coupling of the zeron field σ_o is described by the Lagrangian density $g_o\sigma_o^4$. The self-coupling of the photon field is described by the duality invariant Misner-Wheeler Lagrangian density. Doublet mass spectra are generated if the coupling constants take the appropriate signs. Each such spectrum is fitted to that of an experimentally observed meson doublet having the correct spin and parity. The coupling constants and the cut-offs that are introduced are thereby determined uniquely. The coupling constant and the cut-off λ in the zeron model assume the values $g_o = -9.50$ and $\lambda = 515$ n.u. In the photon model, both scalar and pseudoscalar particles are constructed. The respective cut-offs and coupling constants are $\lambda_{(+)} = 729$ n.u., $g_{(+)} = 1.67 \times 10^{-5}$ n.u. and $\lambda_{(-)} = 574$ n.u., $g_{(-)} = 3.34 \times 10^{-5}$ n.u.

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INTRODUCTION

The procedure of using the self-coupling of massless fields to construct massive bosons has been shown possible in the case of fermions for the V-A interaction (Kaempffer 1970) and for the renormalizable Nambu interaction (Esch 1971). Similar calculations are reported in this thesis for the case of massless bosons.

It is shown in section 1 that there are no kinematical objections to models of this sort. An outline of the necessary formalism employed in the quantization of the free zeron field and of the free electromagnetic field, using the radiation gauge, is presented in section 2. Interaction Lagrangians that are chosen for the main calculation are introduced and motivated in section 3. After introducing the notion of generalized biparticle states for zerons and photons in the center-of-momentum and determining their spin-parity assignments in section 4, the calculations of the mass spectra of the composite particles are presented in section 5 for the zeron model and in section 6 for the more involved photon model.

1. KINEMATICAL CONSIDERATIONS

Since the total proper mass of a system of free, zero mass particles is, in general, nonzero (Terletskii 1968), there is no basic kinematical objection to a model of massive particles that are considered to be composites of massless ones. Consider N free particles having proper masses $m_1, m_2, \ldots, m_N$, with respective energies E_i and momenta $\underline{p}_i$

[1.1] $$E_i = \gamma_i m_i;\ \underline{p}_i = \gamma_i m_i \underline{v}_i = E_i \underline{v}_i,\ i=1, 2, \ldots, N,$$

where $\gamma_i = (1-v_i^2)^{-\frac{1}{2}}$ and $\underline{v}_i$ is the velocity of the i^{th} particle. The sum of the proper masses of such individual particles,

[1.2] $$\sum_i m_i = \sum_i (E_i^2 - p_i^2)^{\frac{1}{2}} = \sum_i \gamma_i^{-1} E_i,$$

is a scalar under Lorentz transformations. It is, in general, different from the scalar

[1.3] $$m_s = (E^2 - p^2)^{\frac{1}{2}}$$

formed with the total energy $E = \sum_i E_i$ and the total momentum $\underline{p} = \sum_i \underline{p}_i$ of an isolated system S consisting of N such particles. The scalar m_s may be regarded as the total proper mass of the system because in the center of momentum frame $\underline{p}=0$ it reduces to

[1.4] $$m_s = \sum_i E_i = E.$$

Substituting $E_i = \gamma_i m_i$ from [1.1] into [1.4] reveals the non-additivity property of the proper masses, since now

[1.5] $$m_s = \sum_i \gamma_i m_i \geq \sum_i m_i .$$

The proper masses are additive only in the special case of identical velocities, $\underline{v}_i = \underline{v}$, $\gamma_i = \gamma$, enabling one to write

[1.6] $$m_s = \sum_i E_i (1-v^2)^{\frac{1}{2}} = \sum_i m_i \gamma_i (1-v^2)^{\frac{1}{2}} = \sum_i m_i .$$

For the special case of systems of massless particles, [1.3] can be written in a convenient form. For systems of one, two and three massless particles respectively,

[1.7a] $$m_s^2 = 0,$$

[1.7b] $$m_s^2 = 2E_1E_2(1-\underline{v}_1 \cdot \underline{v}_2)$$

and

[1.7c] $$m_s^2 = 2E_1E_2(1-\underline{v}_1 \cdot \underline{v}_2)+2E_1E_3(1-\underline{v}_1 \cdot \underline{v}_3)+2E_2E_3(1-\underline{v}_2 \cdot \underline{v}_3),$$

where each $|\underline{v}_i| = 1$. It is apparent that only in the case when all particles are moving in the same direction is the total proper mass of the system additive, that is, equal to zero. An ordinary light beam, for example, that exhibits any divergence at all has a nonzero proper mass. Only an infinite plane wave, unattainable in practice, would have zero proper mass.

2. SUMMARY OF FORMALISM

Free massless particles of spin 0, called "zerons", are described by the scalar field operator (Lurié 1968)

$$[2.1] \quad \sigma_0(\underline{x},t) = \frac{1}{\sqrt{V}}\sum_{\underline{k}}\sqrt{\frac{1}{2k}}[a(\underline{k})e^{-ikx} + b^{\dagger}(\underline{k})e^{ikx}]$$

and its hermitean adjoint

$$[2.2] \quad \sigma_0^{\dagger}(\underline{x},t) = \frac{1}{\sqrt{V}}\sum_{\underline{k}}\sqrt{\frac{1}{2k}}[a^{\dagger}(\underline{k})e^{ikx} + b(\underline{k})e^{-ikx}],$$

where $kx \equiv -\underline{k}\cdot\underline{x} + |\underline{k}|t$ and $|\underline{k}| \equiv k$. The creation and annihilation operators $a^{\dagger}$, $b^{\dagger}$, a and b satisfy the commutation relations

$$[2.3] \quad [a(\underline{k}), a^{\dagger}(\underline{k}')] = [b(\underline{k}), b^{\dagger}(\underline{k}')] = \delta_{\underline{kk}'},$$

$$\text{all other } [\quad] = 0.$$

The field operators satisfy the wave equation

$$[2.4] \quad \Box\sigma_0(\underline{x},t) = \Box\sigma_0^{\dagger}(\underline{x},t) = 0.$$

Unless the field is hermitean, $\sigma_0 = \sigma_0^{\dagger}$, which amounts to having $a(\underline{k})=b(\underline{k})$, this description can accommodate the presence of some dichotomic attribute of the zerons, such as electric charge or hypercharge, if required. The freedom to put $a(\underline{k})=b(\underline{k})$ at any stage will be retained.

The free-field Lagrangian density L_0 that leads to [2.4] via the action principle, [2.5] $\delta\int_{\Omega} L_0 d^4x = 0$, Ω is an arbitrary 4-volume,

is given by (Roman 1969)

[2.6] $L_0 = \partial_\mu \sigma_0^\dagger \partial^\mu \sigma_0$, summation over $\mu = 0,1,2,3$.

Variation with respect to σ_0 gives the field equations for $\sigma_0^\dagger$ and conversely.

The canonical momentum is constructed from L_0 by the standard prescription

[2.7] $\pi_0 = \dfrac{\partial L_0}{\partial \dot{\sigma}_0} = \dot{\sigma}_0$

and the free Hamiltonian density H_0 is formed by

[2.8] $H_0 = \pi_0 \dot{\sigma}_0 - L_0 = \frac{1}{2}[\pi_0^2 + \underline{\nabla}\sigma_0 \cdot \underline{\nabla}\sigma_0]$.

H_0 is positive-definite as required. The equal-time commutation relations are

[2.9] $[\sigma_0(x), \sigma_0(x')] = [\pi_0(x), \pi_0(x')] = 0,$

$[\sigma_0(x), \pi_0(x')]_{x_0 = x_0'} = i\delta(\underline{x} - \underline{x}')$, $x \equiv (\underline{x}, x_0)$.

Free massless particles of spin 1, photons, may be described by a 3-vector operator (Lurié 1968)

[2.10] $\underline{\overset{(0)}{A}}(\underline{x},t) = \dfrac{1}{\sqrt{V}} \sum_{\underline{k}} \sum_{\lambda=1}^{2} \sqrt{\dfrac{1}{2k}}\, \underline{\epsilon}(\underline{k},\lambda)[a(\underline{k},\lambda)e^{-ikx}$

$+ a^\dagger(\underline{k},\lambda)e^{ikx}]$,

where $a^\dagger(\underline{k},\lambda)$ and $a(\underline{k},\lambda)$ are creation and annihilation operators respectively of free photons of momentum $\underline{k}$ having linear polarizations labelled by λ. $\underline{A}^{(o)}(\underline{x},t)$ satisfies the wave equation

$$[2.11] \qquad \Box \underline{A}^{(o)}(\underline{x},t) = 0$$

and the transversality condition, in the radiation gauge,

$$[2.12] \qquad \underline{\nabla}\cdot\underline{A}^{(o)}(\underline{x},t) = 0.$$

Classically, [2.12] and [2.11] are equivalent to Maxwell's free-field equations.

The commutation relations obeyed by the operators $a(\underline{k},\lambda)$ and $a^{+}(\underline{k},\lambda)$ are

$$[2.13] \qquad [a(\underline{k},\lambda), a^{+}(\underline{k}',\lambda')] = \delta_{\underline{k}\underline{k}'}\delta_{\lambda\lambda'}.$$

In this gauge the electric field operator is

$$[2.14] \qquad -\dot{\underline{A}}^{(o)} = \underline{E}^{(o)} = \frac{i}{\sqrt{V}}\sum_{\underline{k}}\sum_{\lambda=1}^{2}\sqrt{\frac{k}{2}}\,\underline{\epsilon}(\underline{k},\lambda)[a(\underline{k},\lambda)e^{-ikx} - a^{+}(\underline{k},\lambda)e^{ikx}]$$

and the magnetic field operator is

$$[2.15] \qquad \underline{\nabla}\times\underline{A}^{(o)} = \underline{B}^{(o)} = \frac{i}{\sqrt{V}}\sum_{\underline{k}}\sum_{\lambda=1}^{2}\sqrt{\frac{k}{2}}\,(-1)^{3-\lambda}\,\underline{\epsilon}(\underline{k},3-\lambda)[a(\underline{k},\lambda)e^{-ikx} - a^{+}(\underline{k},\lambda)e^{ikx}],$$

where use has been made of the relations

[2.16] $\underline{k} \times \underline{\epsilon}(\underline{k},1) = k\, \underline{\epsilon}(\underline{k},2)$; $\underline{k} \times \underline{\epsilon}(\underline{k},2) = -k\, \underline{\epsilon}(\underline{k},1)$.

Other properties of the polarization vectors that will be used later are

[2.17] $\underline{\epsilon}(-\underline{k},1) = -\underline{\epsilon}(\underline{k},1)$

[2.18] $\underline{\epsilon}(-\underline{k},2) = \underline{\epsilon}(\underline{k},2)$

[2.19] $\underline{\epsilon}(-\underline{k},\lambda)\cdot\underline{\epsilon}(\underline{k},\lambda') = (-1)^{\lambda}\,\delta_{\lambda\lambda'}$.

Since the states of linear polarization are not eigenstates of the spin, one introduces states of transverse circular polarization by rotating the reference base of the polarization vectors. The corresponding creation operators for circularly polarized photons take the form

[2.20] $a^{+}(\underline{k}) = 2^{-\frac{1}{2}}[a^{+}(\underline{k},1)+ia^{+}(\underline{k},2)]$

for right-handed polarization and

[2.21] $b^{+}(\underline{k}) = 2^{-\frac{1}{2}}[a^{+}(\underline{k},1)-ia^{+}(\underline{k},2)]$

for left-handed polarization.

The total photon occupation number operator is

[2.22] $N(\underline{k}) = a^{+}(\underline{k})a(\underline{k})+b^{+}(\underline{k})b(\underline{k})$

$\qquad = a^{+}(\underline{k},1)a(\underline{k},1)+a^{+}(\underline{k},2)a(\underline{k},2)$.

A satisfactory Lagrangian density for the free, Maxwell field is (Bjorken, Drell 1965)

$$[2.23] \quad L_o = \tfrac{1}{2}(E^{(o)2}-B^{(o)2}) = \tfrac{1}{2}[\dot{A}^{(o)2}-(\underline{\nabla}\times\underline{A}^{(o)})^2].$$

From the canonical 3-momentum,

$$[2.24] \quad \underline{\pi}_o = \frac{\partial L_o}{\partial \dot{\underline{A}}^{(o)}} = \dot{\underline{A}}^{(o)} = -\underline{E}^{(o)},$$

the (positive-definite) Hamiltonian density may be constructed:

$$[2.25] \quad H_o = \underline{\pi}_o\cdot\dot{\underline{A}}^{(o)}-L_o = \tfrac{1}{2}(E^{(o)2}+B^{(o)2}).$$

A difficulty now appears. The usual form of the equal-time canonical commutation relations, namely

$$[2.26] \quad [A_i^{(o)}(x),\pi_{oj}(x')] = i\delta_{ij}\delta(\underline{x}-\underline{x}'), \quad (i,j)=(1,2,3)$$

is inconsistent with the transversality constraints $\underline{\nabla}\cdot\underline{A}^{(o)} = \underline{\nabla}\cdot\underline{E}^{(o)} = 0$ in force here, as can readily be seen by taking the divergence of [2.26]. For then, at equal times,

$$[2.27] \quad [\underline{\nabla}\cdot\underline{A}^{(o)}(x),\pi_{oj}(x')] = i\delta_j\delta(\underline{x}-\underline{x}') \neq 0.$$

The correct canonical commutation relations are given instead by (Lurié 1968)

$$[2.28] \quad [A_i^{(o)}(x),\pi_{oj}(x')]_{t=t'} = i\left(\delta_{ij}-\frac{\partial i\partial j}{\nabla 2}\right)\delta(\underline{x}-\underline{x}'),$$

where the symbol ∇^{-2} represents the operation

[2.29] $$\frac{1}{\nabla^2}\underline{V}(\underline{x}) = \int D(\underline{x}-\underline{x}')\underline{V}(\underline{x}')d\underline{x}',$$

where

[2.30] $$D(\underline{x}-\underline{x}) = \frac{1}{4\pi|\underline{x}-\underline{x}'|}\left[\delta(t+|\underline{x}-\underline{x}'|)-\delta(t-|\underline{x}-\underline{x}'|)\right].$$

3. SELF-COUPLING OF MASSLESS FIELDS

The interaction Lagrangian densities L_I that are chosen for this work to describe the self-coupling of massless fields exhibit as many of the symmetries of the corresponding free-field equations as possible.

The wave equation [2.4] for the hermitean zeron field is conformally invariant and so a conformally invariant interaction Lagrangian density is employed to describe the zeron-zeron coupling. $L_I = g_o\sigma_o^4$ is the only choice that exhibits scale invariance because it is the only choice that provides a dimensionless coupling constant. Scale invariance, in this case, implies full conformal invariance (Carruthers 1971).

The full Lagrangian density $L=L_o+L_I$ is thus

[3.1] $$L = -\tfrac{1}{2}(\underline{\nabla}\sigma_o\cdot\underline{\nabla}\sigma_o - \dot{\sigma}_o^2) + g_o\sigma_o^4$$

and the corresponding Hamiltonian density is given by

[3.2] $$H = \frac{\partial L}{\partial\dot{\sigma}_o}\dot{\sigma}_o - L = \tfrac{1}{2}(\underline{\nabla}\sigma_o\cdot\underline{\nabla}\sigma_o + \dot{\sigma}_o^2) - g_o\sigma_o^4.$$

Since L_I is a nonderivative coupling, σ_o may be expanded in free creation and annihilation operators according to [2.1] with $a(\underline{k})=b(\underline{k})$. Normal ordering of the operator products is denoted by the usual double dot notation.

The free part becomes, after integration over all 3-space,

$$[3.3] \quad \mathcal{H}_0 = \sum_{\underline{k}} k a^{+}(\underline{k}) a(\underline{k}).$$

A truncated version of the interaction Hamiltonian is employed by dropping all terms that do not contain an equal number of creation and annihilation operators. These particle nonconserving terms will constitute a perturbation. The truncated Hamiltonian density is

$$[3.4] \quad H_I = : \frac{-g_0}{4V^2} \sum_{\underline{k}} \sum_{\underline{k}'} \sum_{\underline{k}''} \sum_{\underline{k}'''} (kk'k''k''')^{-\frac{1}{2}}$$

$$[a^{\dagger} a'^{\dagger} a'' a''' e^{-i(\underline{k}+\underline{k}'-\underline{k}''-\underline{k}''')\cdot\underline{x}} + a^{\dagger} a' a''^{\dagger} a'''$$

$$e^{-i(\underline{k}-\underline{k}'+\underline{k}''-\underline{k}''')\cdot\underline{x}} + a a'^{\dagger} a''^{\dagger} a''' e^{-i(-\underline{k}+\underline{k}'+\underline{k}''-\underline{k}''')\cdot\underline{x}}$$

$$+ a^{\dagger} a' a'' a'''^{\dagger} e^{i(\underline{k}-\underline{k}'-\underline{k}''+\underline{k}''')\cdot\underline{x}} + a a'^{\dagger} a'' a'''^{\dagger}$$

$$e^{-i(-\underline{k}+\underline{k}'-\underline{k}''+\underline{k}''')\cdot\underline{x}} + a a' a''^{\dagger} a'''^{\dagger} e^{i(\underline{k}+\underline{k}'-\underline{k}''-\underline{k}''')\cdot\underline{x}}];$$

where $a'' \equiv a(\underline{k}'')e^{-i|\underline{k}''|t}$, etc. Integrating over all 3-space,

$$[3.5] \quad \mathcal{H}_I = \int :H_I: d\underline{x} = : \frac{-g_0}{4V}[\delta(\underline{k}+\underline{k}'-\underline{k}''-\underline{k}''')(a^{\dagger} a'^{\dagger} a'' a'''$$

$$+ a a' a''^{\dagger} a'''^{\dagger}) + \delta(\underline{k}-\underline{k}'+\underline{k}''-\underline{k}''')(a^{\dagger} a' a''^{\dagger} a'''$$

$$+ a a'^{\dagger} a'' a'''^{\dagger}) + \delta(\underline{k}-\underline{k}'-\underline{k}''+\underline{k}''')(a a'^{\dagger} a''^{\dagger} a'''$$

$$+ a^{\dagger} a' a'' a'''^{\dagger})]:.$$

In addition to being Lorentz invariant, the free Maxwell equations still hold if the fields $\underline{E}^{(o)}$ and $\underline{B}^{(o)}$, called a "duality pair", undergo the so-called "duality rotation" through an arbitrary angle α, (Misner, Wheeler 1957)

$$[3.6] \quad \begin{pmatrix} \underline{E}^{(o)\prime} \\ \underline{B}^{(o)\prime} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \underline{E}^{(o)} \\ \underline{B}^{(o)} \end{pmatrix}.$$

The interaction Lagrangian density chosen in this work to describe a self-coupling of photons is the simplest, non-trivial, duality invariant and Lorentz invariant expression in operators $\underline{E}$ and $\underline{B}$, namely

$$[3.7] \quad L_I = \tfrac{1}{4}g[(B^2-E^2)^2+4(B\cdot E)^2].$$

The duality invariance of [3.7] follows immediately from the fact that $\underline{B}\cdot\underline{E}$ and $\frac{1}{2}(E^2-B^2)$ form a duality pair.

Since this choice amounts to the introduction of a derivative coupling, a complication arises in the canonical formalism. The canonical 3-momentum for the total Lagrangian density $L=L_o+L_I$ is

$$[3.8] \quad \underline{\pi} = \frac{\partial L_o}{\partial \dot{\underline{A}}} + \frac{\partial L_I}{\partial \dot{\underline{A}}} = \dot{\underline{A}}+g\dot{\underline{A}}\dot{A}^2+2g\underline{B}(\underline{B}\cdot\dot{\underline{A}})-g\dot{\underline{A}}B^2, \quad \underline{B}=\underline{\nabla}\times\underline{A}.$$

This equation cannot be solved for $\dot{\underline{A}}$ in closed form. Therefore, $\underline{A}$ is expanded in powers of g according to

[3.9] $\underline{A} = \sum_{n=0} g^n \underline{A}^{(n)}$,

where $\dot{\underline{A}}^{(o)} = \underline{\Pi}_o = \partial L_o / \partial \dot{\underline{A}}^{(o)}$ is the time derivative of the free field. To first order in g, [3.8] becomes

[3.10] $\underline{\Pi}^{(1)} = \underline{\Pi}_o + g[\dot{\underline{A}}^{(1)} + \underline{\Pi}_o \Pi_o^2 + 2\underline{B}^{(o)}(\underline{B}^{(o)} \cdot \underline{\Pi}_o) - B^{(o)2}\underline{\Pi}_o]$.

The full Lagrangian density, also to first order, is

[3.11] $$L^{(1)} = \tfrac{1}{2}(E^{(o)2} - B^{(o)2}) + \tfrac{1}{4}g\left[(E^{(o)2} - B^{(o)2})^2 + 4(\underline{B}^{(o)} \cdot \underline{E}^{(o)})^2\right] - g\underline{E}^{(o)} \cdot \dot{\underline{A}}^{(1)} - g\underline{B}^{(o)} \cdot (\underline{\nabla} \times \underline{A}^{(1)})$$

and the corresponding Hamiltonian density takes the form

[3.12] $$H^{(1)} = (\underline{\Pi} \cdot \dot{\underline{A}})^{(1)} - L^{(1)} - \tfrac{1}{2}(E^{(o)2} + B^{(o)2}) + \tfrac{1}{4}g[(E^{(o)2} - B^{(o)2})(3E^{(o)2} + B^{(o)2}) + 4(\underline{B}^{(o)} \cdot \underline{E}^{(o)})^2] - g\underline{E}^{(o)} \cdot \dot{\underline{A}}^{(1)} + g\underline{B}^{(o)} \cdot (\underline{\nabla} \times \underline{A}^{(1)}).$$

It is possible to express the last two terms of [3.12] in terms of the free fields $\underline{E}^{(o)}$ and $\underline{B}^{(o)}$. In covariant notation, the Euler-Lagrange equation for the full Lagrangian density [3.11] is (see Appendix A for more details)

[3.13] $$\partial_\alpha \left[\frac{\partial L^{(1)}}{\partial(\partial_\alpha A_\beta)} \right] = -\partial_\alpha f^{(1)\alpha\beta} - \tfrac{1}{2}\partial_\alpha [f^{(o)}_{\mu\nu} f^{(o)\mu\nu} f^{(o)\alpha\beta} - 4f^{(o)\nu\alpha} f^{(o)\beta\mu} f^{(o)}_{\mu\nu}] = 0,$$

where $f^{(n)\alpha\beta} \equiv \partial^{[\alpha} A^{(n)\beta]}$. It is important to note that $A^0 \neq 0$ in the radiation gauge for interacting fields. [3.13] is of the form of Maxwell's inhomogeneous equations with a conserved 4-current given by

$$[3.14] \quad j^{(0)\beta} = -\tfrac{1}{2}\partial_\alpha\left[f^{(0)}_{\mu\nu} f^{(0)\mu\nu} f^{(0)\alpha\beta} - 4 f^{(0)\nu\alpha} f^{(0)\beta\mu} f^{(0)}_{\mu\nu}\right].$$

By inspection of [3.13],

$$[3.15] \quad f^{(1)\alpha\beta} = -\tfrac{1}{2}\left[f^{(0)}_{\mu\nu} f^{(0)\mu\nu} f^{(0)\alpha\beta} - 4 f^{(0)\nu\alpha} f^{(0)\beta\mu} f^{(0)}_{\mu\nu} + h^{\alpha\beta}\right],$$

where $\partial_\alpha h^{\alpha\beta} = 0$, $h^{\alpha\beta}$ to be determined. Hence, $\underline{\nabla} \times \underline{A}^{(1)}$ and $\dot{\underline{A}}^{(1)}$, which follow from $f^{(1)\alpha\beta}$, are expressible in terms of free fields. The last two terms of [3.12] become

$$[3.16] \quad -g\underline{E}^{(0)}\cdot\dot{\underline{A}}^{(1)} + g\underline{B}^{(0)}\cdot(\underline{\nabla}\times\underline{A}^{(1)}) = -g(E^{(0)4} - B^{(0)4} + \underline{\nabla}A^{0(1)}\cdot\underline{E}^{(0)})$$

so that

$$[3.17] \quad \mathcal{H}^{(1)} = \int :H^{(1)}:d\underline{x} = :\int \Big[\tfrac{1}{2}(E^{(0)2}+B^{(0)2})$$
$$+ \tfrac{1}{4}g\left\{(B^{(0)2}-E^{(0)2})(3B^{(0)2}+E^{(0)2}) + 4(\underline{B}^{(0)}\cdot\underline{E}^{(0)})^2\right\}\Big]d\underline{x}:,$$

where $\underline{\nabla}A^{0(1)}\cdot\underline{E}^{(0)}$ has vanished upon integration.

At this stage it is legitimate to expand the Hamiltonian in terms of the free-field operators given in [2.14] and [2.15]. Retaining only particle conserving terms, the field products that occur in the interaction part are given by

$$[3.18] \quad :E^{(0)4}: = :\frac{1}{V^2}\sum_{\underline{k},\lambda}\sum_{\underline{k}',\lambda'}\sum_{\underline{k}'',\lambda''}\sum_{\underline{k}''',\lambda'''}\sqrt{\frac{kk'k''k'''}{16}}$$
$$\underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(\underline{k}',\lambda')\,\underline{\epsilon}(\underline{k}'',\lambda'')\cdot\underline{\epsilon}(\underline{k}''',\lambda''')$$

$$[aa'a''^{\dagger}a'''^{\dagger}e^{i(\underline{k}+\underline{k}'-\underline{k}''-\underline{k}''')\cdot\underline{x}} + a^{\dagger}a'a''a'''^{\dagger}$$

$$e^{i(-\underline{k}+\underline{k}'+\underline{k}''-\underline{k}''')\cdot\underline{x}} + aa'^{\dagger}a''^{\dagger}a'''e^{i(\underline{k}-\underline{k}'-\underline{k}''+\underline{k}''')\cdot\underline{x}}$$

$$+ a^{\dagger}a'^{\dagger}a''a'''e^{i(-\underline{k}-\underline{k}'+\underline{k}''+\underline{k}''')\cdot\underline{x}} + aa'^{\dagger}a''a'''^{\dagger}$$

$$e^{i(\underline{k}-\underline{k}'+\underline{k}''-\underline{k}''')\cdot\underline{x}} + a^{\dagger}a'a''^{\dagger}a'''e^{i(-\underline{k}+\underline{k}'-\underline{k}''+\underline{k}''')\cdot\underline{x}}]:,$$

[3.19] $$:B^{(0)4}: = :\frac{1}{V^2}\sum_{\underline{k},\lambda}\sum_{\underline{k}',\lambda'}\sum_{\underline{k}'',\lambda''}\sum_{\underline{k}''',\lambda'''}\sqrt{\frac{kk'k''k'''}{16}}$$

$$(-1)^{-\lambda-\lambda'-\lambda''-\lambda'''}\underline{\epsilon}(\underline{k},3-\lambda)\cdot\underline{\epsilon}(\underline{k}',3-\lambda')$$

$$\underline{\epsilon}(\underline{k}'',3-\lambda'')\cdot\underline{\epsilon}(\underline{k}''',3-\lambda''')[\ldots]:,$$

where [...] is the same set of operators as in [3.18],

[3.20] $$:B^{(0)2}E^{(0)2}: = :\frac{1}{V^2}\sum_{\underline{k},\lambda}\sum_{\underline{k}',\lambda'}\sum_{\underline{k}'',\lambda''}\sum_{\underline{k}''',\lambda'''}\sqrt{\frac{kk'k''k'''}{16}}$$

$$(-1)^{-\lambda-\lambda'}\underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(\underline{k}',\lambda')$$

$$\underline{\epsilon}(\underline{k}'',3-\lambda'')\cdot\underline{\epsilon}(\underline{k}''',3-\lambda''')[\ldots]:,$$

[3.21] $$:E^{(0)2}B^{(0)2}: = :\frac{1}{V^2}\sum_{\underline{k},\lambda}\sum_{\underline{k}',\lambda'}\sum_{\underline{k}'',\lambda''}\sum_{\underline{k}''',\lambda'''}\sqrt{\frac{kk'k''k'''}{16}}$$

$$(-1)^{-\lambda-\lambda'}\underline{\epsilon}(\underline{k},3-\lambda)\cdot\underline{\epsilon}(\underline{k}',3-\lambda')$$

$$\underline{\epsilon}(\underline{k}'',\lambda'')\cdot\underline{\epsilon}(\underline{k}''',\lambda''')[\ldots]:$$

and

[3.22] $$:(\underline{E}^{(0)}\cdot\underline{B}^{(0)}+\underline{B}^{(0)}\cdot\underline{E}^{(0)})^2: = :\frac{1}{V^2}\sum_{\underline{k},\lambda}\sum_{\underline{k}',\lambda'}\sum_{\underline{k}'',\lambda''}\sum_{\underline{k}''',\lambda'''}\sqrt{\frac{kk'k''k'''}{16}}$$

$$[\underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(\underline{k}',3-\lambda')(-1)^{3-\lambda'}$$
$$+\ \underline{\epsilon}(\underline{k},3-\lambda)\cdot\underline{\epsilon}(\underline{k}',\lambda')(-1)^{3-\lambda}]$$
$$[\underline{\epsilon}(\underline{k}'',\lambda'')\cdot\underline{\epsilon}(\underline{k}''',3-\lambda''')(-1)^{3-\lambda'''}$$
$$+\ \underline{\epsilon}(\underline{k}'',3-\lambda'')\cdot\underline{\epsilon}(\underline{k}''',\lambda''')(-1)^{3-\lambda''}][\ldots]:,$$

where $a'' = a(\underline{k}'',\lambda'')e^{-i|\underline{k}''|t}$, etc.

The free Hamiltonian is

$$[3.23]\quad \mathcal{H}_o = :\tfrac{1}{2}\int(E^{(o)2}+B^{(o)2})d\underline{x}: = \sum_{\underline{k},\lambda} ka^{\dagger}(\underline{k},\lambda)a(\underline{k},\lambda).$$

4. BIPARTICLE STATES

Under the operation of parity P, the hermitean zeron field transforms according to

[4.1] $P\sigma_0(x)P^{-1} = s\sigma_0(-\underline{x},t)$,

where s=1 for scalar σ_0 and s=-1 for pseudoscalar σ_0. Using [2.1] with $a(\underline{k})=b(\underline{k})$, it follows that

[4.2] $Pa^\dagger(\underline{k})P^{-1} = sa^\dagger(-\underline{k})$.

Consider now the biparticle state $|B(\underline{k})\rangle = |1_{\underline{k}},1_{-\underline{k}}\rangle$ consisting of two zerons having equal and opposite momenta $\underline{k}$ and $-\underline{k}$. Thus, under P,

[4.3] $Pa^\dagger(k)a^\dagger(-\underline{k})|0\rangle \equiv P|1_{\underline{k}},1_{-\underline{k}}\rangle$

$$=[Pa^\dagger(\underline{k})P^{-1}][Pa^\dagger(-\underline{k})P^{-1}][P|0\rangle] = s^2|1_{\underline{k}},1_{-\underline{k}}\rangle ,$$

where $P|0\rangle=|0\rangle$ by assumption. The two-zeron state $|B(\underline{k})\rangle$ is thus an eigenstate of parity with eigenvalue +1.

Consider states of two photons having momenta $\underline{k}$ and $-\underline{k}$. Eigenstates of the parity operator are sought. With circularly polarized photons it is possible to construct the following four helicity states:

[4.4] $|1_{\underline{k},R},1_{-\underline{k},R}\rangle = |RR\rangle = a^\dagger(\underline{k})a^\dagger(-\underline{k})|0\rangle$,

[4.5] $|1_{\underline{k},R},1_{-\underline{k},L}\rangle = |RL\rangle = a^\dagger(\underline{k})b^\dagger(-\underline{k})|0\rangle$,

$$[4.6] \quad |1_{\underline{k},L}, 1_{-\underline{k},R}\rangle \equiv |LR\rangle = b^{\dagger}(\underline{k})a^{\dagger}(-\underline{k})|0\rangle,$$

$$[4.7] \quad |1_{\underline{k},L}, 1_{-\underline{k},L}\rangle \equiv |LL\rangle = b^{\dagger}(\underline{k})b^{\dagger}(-\underline{k})|0\rangle.$$

Under P, the mixed states go into themselves, $|RL\rangle \to |RL\rangle$, $|LR\rangle \to |LR\rangle$ and $|RR\rangle \to |LL\rangle$ and $|LL\rangle \to |RR\rangle$. The triplet of states $|RL\rangle$, $|LR\rangle$ and $|RR\rangle + |LL\rangle$ are therefore eigenstates of the parity operator with even parity and the singlet state $|RR\rangle - |LL\rangle$ is an eigenstate with odd parity.

A two-photon state $|RR\rangle + |LL\rangle$ has therefore the appropriate quantum numbers of a massive, neutral scalar particle in the center-of-momentum; the state $|RR\rangle - |LL\rangle$ has the appropriate quantum numbers of a massive, neutral pseudoscalar particle. Either of the states $|RL\rangle$ or $|LR\rangle$ has the quantum numbers of a massive, neutral, spin 2, even parity particle.

A more general biparticle state may be constructed by superposition:

$$[4.8] \quad |B\rangle = \sum_{\underline{k}} f(\underline{k})|B(\underline{k})\rangle,$$

where $f(\underline{k})$ is an arbitrary amplitude function. One finds here a direct analog to the two-electron state, Cooper pair, formalism used in the BCS theory of superconductivity (Cooper 1956).

The general state $|B\rangle$ may have non-zero orbital angular momentum depending on the form of $f(\underline{k})$. For example, $f(\underline{k})$ may itself be expanded in the usual set of spherical harmonics, which serve as eigenfunctions of the orbital angular momentum operator L^2. In the spherically symmetric case in which f is a function only of $|\underline{k}|$, the orbital angular momentum of $|B\rangle$ is zero.

The state $|B\rangle$ has the same parity assignment as that of $|B(\underline{k})\rangle$ iff $f(\underline{k})$ is even and has the opposite parity assignment iff $f(\underline{k})$ is odd.

5. CALCULATION OF THE MASS SPECTRUM OF COMPOSITE PARTICLES AS AN EIGENVALUE PROBLEM I: THE ZERON MODEL

Operating the full Hamiltonian $\mathcal{H}$ given by [3.3] and [3.5] on the two-zeron state $|B(\underline{q})\rangle = |1_{\underline{q}}, 1_{-\underline{q}}\rangle$, it is found that

[5.1] $$\mathcal{H}|B(\underline{q})\rangle = 2q|B(\underline{q})\rangle - \frac{3g_0}{V}\sum_{\underline{k}} \frac{1}{kq}|B(\underline{k})\rangle ,$$

where use has been made of identities of the form

[5.2] $$\delta(\underline{k}+\underline{k}'-\underline{k}''-\underline{k}''')\, a^{\dagger} a'^{\dagger} a'' a''' |B(\underline{q})\rangle$$
$$\equiv [\delta(\underline{k}'''-\underline{q})\delta(\underline{k}''+\underline{q}) + \delta(\underline{k}'''+\underline{q})\delta(\underline{k}''-\underline{q})]\,|B(\underline{k})\rangle .$$

Thus $|B(\underline{q})\rangle$ is not an eigenstate of the Hamiltonian. However, by introducing the superposition given in [4.8], namely

[5.3] $$|B\rangle = \sum_{\underline{q}} f(\underline{q})\,|B(\underline{q})\rangle ,$$

where the unknown amplitude function $f(\underline{q})$ is to be determined, then [5.1] becomes

[5.4] $$\mathcal{H}|B\rangle = \sum_{\underline{q}} [2qf(\underline{q})|B(\underline{q})\rangle - \frac{3g_0}{V}\sum_{\underline{k}} \frac{1}{kq} f(\underline{q})|B(\underline{k})\rangle]$$

or, interchanging $\underline{k}$ and $\underline{q}$ in the second summation,

[5.5] $$\mathcal{H}|B\rangle = \sum_{\underline{q}} [2qf(\underline{q}) - \frac{3g_0}{V}\sum_{\underline{k}} \frac{1}{kq} f(\underline{k})]\,|B(\underline{q})\rangle .$$

It follows that $|B\rangle$ is an eigenstate of the Hamiltonian with energy eigenvalue E provided that $f(\underline{q})$ satisfies

[5.6] $$f(\underline{q}) = \frac{3g_o}{(2q-E)q} \frac{1}{V} \sum_{\underline{k}} \frac{f(\underline{k})}{k}$$

or, in integral form,

[5.7] $$f(\underline{q}) = \frac{3g_o}{(2q-E)(2\pi)^3 q} \int \frac{f(\underline{k})}{k} d\underline{k}.$$

The solution of this integral equation is, by inspection,

[5.8] $f(\underline{k}) = f(-\underline{k}) = A/(E-2k)k$, A is a constant.

The spherical symmetry implies that $|B\rangle$, like $|B(\underline{q})\rangle$, is a state of even parity. Substituting [5.8] into [5.7] yields the eigenvalue condition

[5.9] $$\int \frac{d\underline{k}}{k^2(E-2k)} = -\frac{(2\pi)^3}{3g_o}, \quad g_o \neq 0.$$

Thus, in polar coordinates,

[5.10] $$\int_0^\pi \int_0^{2\pi} \int_0^\lambda \frac{\sin\theta d\theta d\phi dk}{E-2k} = 4\pi \int_0^\lambda [P \frac{1}{(E-2k)} - i\pi\delta(E-2k)]\, dk = \frac{(2\pi)^3}{-3g_o},$$

where a cut-off λ has been introduced in the k-integration and P denotes the principal value. Integrating,

[5.11] $\frac{1}{2}\ln(1-2\lambda/E) + i\pi = 2\pi^2/3g_o.$

If the assumption is made that the states are quasistationary, then discrete, complex energy eigenvalues $E = \alpha + i\beta$ will be obtained, where β corresponds to the width of the energy level α. Substituting $E = \alpha + i\beta$ in [5.11] and separating into real and imaginary parts gives

$$[5.12] \qquad \tfrac{1}{4}\ln\left[\left(1-\frac{2\lambda\alpha}{\alpha^2+\beta^2}\right)^2 + \frac{4\lambda^2\beta^2}{(\alpha^2+\beta^2)^2}\right] = \frac{2\pi^2}{3g_0}$$

and

$$[5.13] \qquad \tfrac{1}{2}\tan^{-1}\left(\frac{2\lambda\beta}{\alpha^2+\beta^2-2\lambda\alpha}\right) + n\pi = 0$$

From [5.13], $\beta = 0$ for all n, so that [5.12] reduces to

$$[5.14] \qquad \tfrac{1}{2}\ln|1-x| = \frac{2\pi^2}{3g_0}, \quad x \equiv 2\lambda/\alpha.$$

Equation [5.14] is displayed graphically in figure 1.

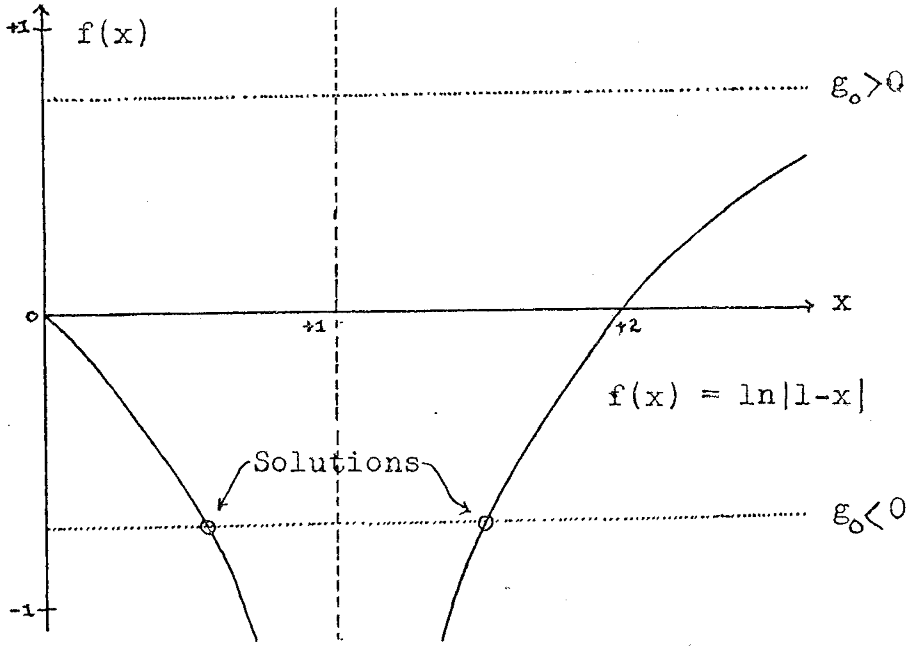

Figure 1: Graph of equation [5.14]

The graph reveals that only one value of x will satisfy [5.14] if $g_0 > 0$ but that two solutions exist if $g_0 < 0$. In this case,

[5.15] $\ln(1-x_1) = \ln(x_2-1)$, $x_i = 2\Lambda/\alpha_i$, $i=1,2$.

The only particles so far experimentally observed with the spin-parity assignment of 0^+, as would be required in this case, is the $\eta_{0^+}(1373\text{ n.u.})-\eta_{0^+}(2080)$ doublet (Particle Data Group 1970). If α_1 and α_2 are fitted to these particles respectively, then g_0 and Λ can be determined uniquely. Introducing $R \equiv \alpha_2/\alpha_1$, [5.15] becomes

[5.16] $x_1 = R(2-R)$.

With $R \approx 3/2$ and hence $x_1 \approx 3/4$ for the chosen doublet, the cut-off Λ takes the value

[5.17] $\Lambda = x_1\alpha_1/2 \approx 515$ n.u.

and the coupling constant becomes

[5.18] $g_0 = \dfrac{-4\pi^2}{3\ln 4} \approx -9.5$.

6. CALCULATION OF THE MASS SPECTRUM OF COMPOSITE PARTICLES AS AN EIGENVALUE PROBLEM II: THE PHOTON MODEL

If the interaction Hamiltonian [3.17], expressed in terms of the expansions [3.18]-[3.22], is operated on either the even parity state $|B(\underline{q})\rangle_+ = |RR\rangle + |LL\rangle$ or the odd parity state $|B(\underline{q})\rangle_- = |RR\rangle - |LL\rangle$, the result for both, after integration over all space, is [see Appendix B]

$$[6.1] \quad \int :H_I|B(\underline{q})\rangle_\pm : d\underline{x} = \frac{g}{2V}\sum_{\underline{k}} kq\{4 + [\underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(\underline{q},1)]^2 + [\underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(\underline{q},2)]^2 + [\underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(\underline{q},1)]^2 + [\underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(\underline{q},2)]^2\}|B(\underline{k})\rangle_\pm .$$

Using the representation,

$$[6.2] \quad \underline{\epsilon}(\underline{k},1) = (1-\hat{k}_3^2)^{-\frac{1}{2}}(\hat{k}_2,-\hat{k}_1,0), \quad \hat{\underline{k}} = \underline{k}/k;$$

$$\underline{\epsilon}(\underline{k},2) = (1-\hat{k}_3^2)^{-\frac{1}{2}}(\hat{k}_1\hat{k}_3,\hat{k}_2\hat{k}_3,\hat{k}_3^2-1),$$

then [6.1], together with the free part, can be expressed wholly in terms of $\underline{k}$ and $\underline{q}$ as [see Appendix C]

$$[6.3] \quad \mathcal{H}|B(\underline{q})\rangle_\pm = 2q|B(\underline{q})\rangle_\pm + \frac{g}{2V}\sum_{\underline{k}} kq[5 + (\hat{\underline{k}}\cdot\hat{\underline{q}})^2]|B(\underline{k})\rangle_\pm .$$

Introducing, as in [5.7], the superposition

$$[6.4] \quad |B\rangle_\pm = \sum_{\underline{q}} f(\underline{q})|B(\underline{q})\rangle_\pm ,$$

with the amplitude $f(\underline{q})$ to be determined, then

[6.5] $$\mathcal{H}|B\rangle_{\pm} = \sum_{\underline{q}} \{2qf(\underline{q}) + \frac{g}{2V}\sum_{\underline{k}} f(\underline{k})kq[5 + (\hat{\underline{k}}\cdot\hat{\underline{q}})^2]\}|B(\underline{q})\rangle_{\pm} .$$

This equation is of the form $\mathcal{H}|B\rangle_{\pm} = E|B\rangle_{\pm}$ provided

[6.6] $$f(\underline{q}) = \frac{g}{2V(E-2q)} \sum_{\underline{k}} kqf(\underline{k})[5 + (\hat{\underline{k}}\cdot\hat{\underline{q}})^2]$$

Putting $(\hat{\underline{k}}\cdot\hat{\underline{q}})^2 = \cos^2\psi$, where ψ is the angle between $\underline{k}$ and $\underline{q}$, then [6.6] becomes, in integral form,

[6.7] $$f(\underline{q}) = \frac{gq}{2(2\pi)^3(E-2q)} \int k^3 f(\underline{k})\sin\theta(5+\cos^2\psi)\,d\theta\, d\phi\, dk.$$

If $(\cos\alpha, \cos\beta, \cos\gamma)$ and $(\cos\alpha', \cos\beta', \cos\gamma')$ are the direction cosines of $\underline{k}$ and $\underline{q}$ respectively, then

[6.8] $$\cos\psi = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

These direction cosines are defined in terms of θ and ϕ by

[6.9] $$\cos\alpha = \sin\theta\cos\phi;\ \cos\beta = \sin\theta\sin\phi;\ \cos\gamma = \cos\theta$$

and similarly for the primed angles. Performing the integration in [6.7] over the angular coordinates,

[6.10] $$\begin{aligned}&\int_0^{\pi}\int_0^{2\pi} \sin\theta(5+\cos^2\psi)\,d\theta\, d\phi \\ &= 5(4\pi) + \frac{4\pi}{3}(\cos^2\theta'\cos^2\phi' + \sin^2\theta'\sin^2\phi' + \cos^2\theta') \\ &= 5(4\pi) + \frac{4\pi}{3}(\cos^2\alpha' + \cos^2\beta' + \cos^2\gamma') = 16(4\pi/3),\end{aligned}$$

the integral equation reduces to

$$[6.11] \quad f(\underline{q}) = \frac{2gq}{3\pi^2(E-2q)} \int_0^{\lambda} k^3 f(\underline{k})\,dk,$$

where, as in section 5, λ is a cut-off parameter to be determined. The solution of [6.11] is of the form

$$[6.12] \quad f(\underline{k}) = f(-\underline{k}) = Aq/(E-2k); \ A \text{ is a constant,}$$

which when substituted into [6.11] yields the eigen-value condition

$$[6.13] \quad \int_0^{\lambda} \frac{k^4\,dk}{E-2k} = \frac{3\pi^2}{2g}, \ g \neq 0.$$

Performing the integration as before,

$$[6.14] \quad \lambda^4 + \tfrac{1}{2}E^3\lambda + \tfrac{1}{2}E^2\lambda^2 + \tfrac{2}{3}E\lambda^3 + \tfrac{1}{4}E^4 \ln\left(1-\frac{2\lambda}{E}\right) + \tfrac{1}{2}i\pi E^4 = -\frac{12\pi^2}{g}.$$

Separating [6.14] into real and imaginary parts,

$$[6.15] \quad \tfrac{1}{2}\lambda\left[\beta(\alpha^2-\beta^2)+2\alpha^2\beta\right] + \alpha\beta\lambda^2 + \tfrac{2}{3}\beta\lambda^3 + \tfrac{1}{2}\alpha\beta(\alpha^2-\beta^2)\ln\left[\left(1-\frac{2\lambda\alpha}{\alpha^2+\beta^2}\right)^2 + \frac{4\lambda^2\beta^2}{(\alpha^2+\beta^2)^2}\right] + \tfrac{1}{4}\left[(\alpha^2-\beta^2)^2-4\alpha^2\beta^2\right]\left[\tan^{-1}\frac{2\lambda\beta}{\alpha^2+\beta^2-2\lambda\alpha} + 2\pi n_1\right] + \tfrac{1}{2}\pi\left[(\alpha^2-\beta^2)^2-4\alpha^2\beta^2\right] = 0$$

and

$$[6.16] \quad \lambda^4 + \tfrac{1}{2}\alpha\lambda\left[(\alpha^2-\beta^2)-2\beta^2\right] + \tfrac{1}{2}\lambda^2(\alpha^2-\beta^2) + \tfrac{2}{3}\alpha\lambda^3 + \tfrac{1}{8}\left[(\alpha^2-\beta^2)-4\alpha^2\beta^2\right]\ln\left[\left(1-\frac{2\lambda\alpha}{\alpha^2+\beta^2}\right)^2 + \frac{4\lambda^2\beta^2}{(\alpha^2+\beta^2)^2}\right] - \alpha\beta(\alpha^2-\beta^2)\tan^{-1}\left(\frac{2\lambda\beta}{\alpha^2+\beta^2-2\lambda\alpha}\right) - 2\alpha\beta(\alpha^2-\beta^2)n_1\pi - 2\pi\alpha\beta(\alpha^2-\beta^2) = -\frac{48\pi^2}{4g}; \ n_1 \text{ is an integer.}$$

Assuming that the state is stable with respect to the interaction considered here so that $\beta = 0$, then [6.15] and [6.16] become

$$[6.17] \quad \lambda^4+\tfrac{2}{3}\alpha\lambda^3+\tfrac{1}{2}\alpha^2\lambda^2+\tfrac{1}{2}\alpha^3\lambda+\tfrac{1}{4}\alpha^4 \ln\left|1-\frac{2\lambda}{\alpha}\right| = -\frac{48\pi^2}{4g}$$

and

$$[6.18] \quad \alpha^4(n_1+n_2+2) = 0, \; n_2 \text{ is an integer.}$$

If $\alpha^4 > 0$, then $n \equiv n_1+n_2+2 = 0$ and corresponds to a groundstate; for higher n, $\beta \neq 0$ and the state decays. Introducing $x = 2\lambda/\alpha$ in [6.17],

$$[6.19] \quad \tfrac{1}{4}x^4+\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+x+\ln|1-x| = -\frac{48\pi^2}{g\alpha^4}.$$

This equation is displayed graphically in figure 2.

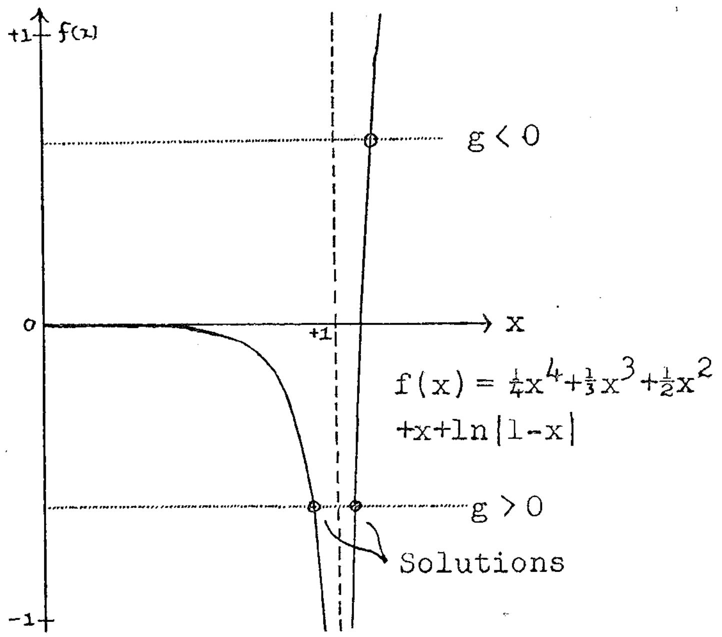

Figure 2: Graph of equation [6.19]

The graph reveals that in this case it is possible to accommodate a doublet spectrum if $g > 0$. Putting α_1 and α_2 as the two possible mass values, then from [6.17]

$$[6.20] \quad \tfrac{1}{3}x_1^3(R-1) + \tfrac{1}{2}x_1^2(R^2-1) + x_1(R^3-1) + R^4\ln(1-x_1/R) - \ln(x_1-1) = 0,$$

where $R = \alpha_2/\alpha_1$ and $x_1 = 2\lambda/\alpha_1$.

For the even parity state $|B(\underline{q})\rangle_+ = |RR\rangle + |LL\rangle$, the fit is made to the η_0+ doublet as in the zeron model. Equation [6.20] becomes

$$[6.21] \quad \tfrac{8}{3}x_1^3 + 10x_1^2 + 38x_1 + 81\ln(1-\tfrac{2}{3}x_1) - 16\ln(x_1-1) = 0.$$

The solution to three significant figures is $x_1 = 1.06$. Hence the cut-off is

$$[6.22] \quad \lambda_{(+)} = x_1\alpha_1/2 \simeq 729 \text{ n.u.} = 372 \text{ Mev}$$

and the corresponding value for the coupling constant is

$$[6.23] \quad g \approx 2.80 \times 10^{-10} \text{ n.u. (units of length}^4).$$

Putting $g^2 = g_{(+)}$ for standard units,

$$[6.24] \quad g_{(+)} \approx 1.67 \times 10^{-5} \text{ n.u.} \approx 7.8 \times 10^{-43} \text{ erg-cm}^3.$$

For the odd parity state $|B(\underline{q})\rangle_- = |RR\rangle - |LL\rangle$, it is possible to make a fit to a number of particles.

The most natural choice is the η_{0^-}(1078 n.u.)-η_{0^-}(1870 n.u.) doublet. The cut-off and the coupling constant take the values

[6.25] $\quad \lambda_{(-)} \fallingdotseq 574$ n.u. $\fallingdotseq 293$ Mev

and

[6.26] $\quad g_{(-)} \fallingdotseq 3.34 \times 10^{-5}$ n.u.

7. DISCUSSION OF RESULTS

The interactions used here are of the "direct" particle interaction type. That is, no intermediate boson mechanism is supposed. It is appropriate therefore to compare the zeron interaction with the direct, weak, 4-fermion interaction, even though the coupling constant in the zeron model was found to be of the order of 10, a number usually associated with the strong interaction. In order to compare interactions characterized by coupling constants of different dimensions, a dimensionless measure M is sought for each interaction. It is possible to construct such M here since there exist natural intrinsic lengths in the theory, namely the reciprocals of the invariant cut-offs. The correct combination of the coupling constants and their associated cut-offs that give dimensionless numbers are listed in table I for each model, including the neutrino models of Kaempffer (1970) and Esch (1971). The appropriate combination is of the form $M = g\lambda^2$ for each model shown except for the zeron model, which has a coupling already dimensionless.

Table I: Comparison of Models

Model	Interaction Lagrangian	J^P of the Composite*	Momentum Cut-off (n.u.)	Coupling Constant $\times 10^{-5}$ (n.u.)	M
Zeron ($\sigma_o - \sigma_o$)	$g_o \sigma_o^4$	0^+	515	9.50 $\times 10^5$	9.50
Photon ($\gamma - \gamma$)	Misner-Wheeler	0^+	729	1.67	8.87
		0^-	574	3.34	10.9
Neutrino ($\nu_e - \nu_e$)	V-A	0^-	598	5.20	18.6
Neutrino ($\nu_\mu - \nu_e$)	Nambu	0^+	739	4.77	26.0

* Each 0^+ corresponds to the η_{o^+} doublet and each 0^- corresponds to the η_{o^-} doublet.

The values of M reveal that the interactions strengths of all the models are of the same order of magnitude. (It should be noted that the values of M reflect the choice of the numerical factors such as $\frac{1}{4}$ and $\sqrt{2}$ etc. in the interaction Lagrangians.) These interaction strengths may be compared with that of the weak interaction provided one knows the intrinsic length λ_w associated with the weak interaction. Then $M_{weak} = 3 \times 10^{-12} \lambda_w^2$.

The γ-γ interaction remains one of the fundamental unmeasured quantities in particle physics. However, the measurement of the total γ-γ cross-section may be within the scope of current techniques (Stodolsky 1971). In the low energy region ($\ll$ 1 Mev), the probable upper limit to the cross-section, based on q.e.d. calculations, is of the order $\bar{\sigma} \sim 2.5 \times 10^{-29} \mathrm{cm}^2$ (Kunszt et. al. 1970). It is possible, in this low energy limit, to simulate photon-photon scattering phenomena by an effective non-linear interaction Lagrangian density of the form

$$[7.1] \qquad L_I^! = \frac{-2\alpha^2}{45\,m^4}\left[(E^2-B^2)^2 + 7(\underline{B}\cdot\underline{E})^2\right] ,$$

where α is the fine structure constant $e^2/4\pi^2$ and m is electron rest mass. The formal similarity of $L_I^!$ to the Misner-Wheeler Lagrangian density

$$[7.2] \qquad L_I = \tfrac{1}{4}g\left[(E^2-B^2)^2 + 4(\underline{B}\cdot\underline{E})^2\right]$$

enables one to make a crude order of magnitude estimate of the energy domain for which the g of the photon model is compatible with the aforementioned cross-section. Using (adapted from Jauch and Rohrlich 1955),

$$[7.3] \qquad \bar{\sigma} \approx 10\,G^2\omega^6 ,$$

where G may correspond to either $-2\alpha^2/45\,m^4$ of [7.1] or to the $\frac{1}{4}g$ of [7.2]. Then

[7.4] $$\omega^3 \lesssim 10^{-15} g^{-1}$$
$$\Rightarrow \omega \lesssim 10^{-2}\ \text{Mev.}$$

If 10^{-2} Mev $\lesssim \omega \lesssim 1$ Mev, then this direct photon interaction will lead to cross-sections higher than those predicted solely on the basis of the virtual positronium exchange.

The photon model distinguishes itself from the other models by the fact that the canonical formalism was carried out using potentials, the derivatives of which are the fields. The interaction Lagrangian may be expressed in a "4-field" form similar to the others by introducing the combinations

[7.5] $$\underline{F} \equiv \underline{E} + i\underline{B} \text{ and } \underline{F}^{\dagger} \equiv \underline{E} - i\underline{B}.$$

For then, [7.2] can be written in the now manifestly duality invariant form

[7.6] $$L_I = \tfrac{1}{4}g(\underline{F}^{\dagger}\cdot\underline{F}^{\dagger})(\underline{F}\cdot\underline{F}),$$

where variation is now taken with respect to $\underline{F}$ and $\underline{F}^{\dagger}$. This model is not equivalent to the photon model presented in this work. Difficulties encountered with it have not been overcome.

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APPENDIX A: ON EQUATION [3.12]

In terms of the usual antisymmetric, second rank tensor,

[A.1] $f^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} = \partial^{[\mu} A^{\nu]}$,

then (Jauch and Rohrlich, 1955)

[A.2] $(B^2 - E^2) \equiv I_1 = \frac{1}{2} f_{\mu\nu} f^{\mu\nu}$,

[A.3] $\underline{B} \cdot \underline{E} \equiv I_2 = \frac{1}{4} \epsilon_{\mu\nu\lambda\sigma} f^{\mu\nu} f^{\lambda\sigma}$,

[A.4] $I_3 \equiv \frac{1}{2} I_1^2 + I_2^2 = \frac{1}{4} f_{\mu\nu} f^{\nu\lambda} f_{\lambda\sigma} f^{\sigma\mu}$,

where $\epsilon_{\mu\nu\lambda\sigma}$ is the totally antisymmetric tensor for which $\epsilon_{1234} = -1$, so that the interaction Lagrangian density [3.7] can be written as

[A.5] $L_I = -\frac{1}{4} g \left[\frac{1}{4} (f_{\mu\nu} f^{\mu\nu})^2 - f_{\mu\nu} f^{\nu\lambda} f_{\lambda\sigma} f^{\sigma\mu} \right]$.

Since

[A.6] $$\frac{\partial}{\partial(\partial_\alpha A_\beta)} (f_{\mu\nu} f^{\nu\lambda} f_{\lambda\sigma} f^{\sigma\mu}) = 8 f^{\nu\alpha} f^{\beta\mu} f_{\mu\nu}$$

and

[A.7] $$\frac{\partial}{\partial(\partial_\alpha A_\beta)} (f_{\mu\nu} f^{\mu\nu}) = 4 f^{\alpha\beta} \Rightarrow \frac{\partial}{\partial(\partial_\alpha A_\beta)} (f_{\mu\nu} f^{\mu\nu})^2 = 8 f_{\mu\nu} f^{\mu\nu} f^{\alpha\beta},$$

then the canonical momentum tensor for the total Lagrangian density $L = L_0 + L_I$ is

[A.8] $\Pi^{\alpha\beta} \equiv \dfrac{\partial L}{\partial(\partial_\alpha A_\beta)} = -f^{\alpha\beta} - \frac{1}{4}g[2f_{\mu\nu}f^{\mu\nu}f^{\alpha\beta} - 8f^{\nu\alpha}f^{\beta\mu}f_{\mu\nu}]$.

The Euler-Lagrange equation

[A.9] $\partial_\alpha \Pi^{\alpha\beta} = 0$

thus becomes

[A.10] $-\partial_\alpha\partial^\alpha A^\beta + \partial_\alpha(\partial^\beta A^\alpha) = \partial_\alpha S^{\alpha\beta}$,

where

[A.11] $S^{\alpha\beta} = \frac{1}{2}g[f_{\mu\nu}f^{\mu\nu}f^{\alpha\beta} - 4f^{\nu\alpha}f^{\beta\mu}f_{\mu\nu}]$.

To zeroth order in g, [A.10] becomes

[A.12] $\partial_\alpha f^{(0)\alpha\beta} = 0$,

which are free Maxwell equations as required. To first order in g,

[A.13] $\partial_\alpha f^{(1)\alpha\beta} = -\frac{1}{2}\partial_\alpha[f^{(0)}_{\mu\nu}f^{(0)\mu\nu}f^{(0)\alpha\beta} - 4f^{(0)\nu\alpha}f^{(0)\beta\mu}f^{(0)}_{\mu\nu}]$.

Equation [A.13] is of the form of Maxwell's inhomogeneous equations with a conserved 4-current given by

[A.14] $j^{(0)\beta} = -\frac{1}{2}\partial_\alpha[f^{(0)}_{\mu\nu}f^{(0)\mu\nu}f^{(0)\alpha\beta} - 4f^{(0)\nu\alpha}f^{(0)\beta\mu}f^{(0)}_{\mu\nu}]$.

The divergence condition $\partial_\beta j^{(0)\beta} = 0$ is satisfied due to the antisymmetry of $f^{(0)\alpha\beta}$. Furthermore, $f^{(1)\alpha\beta}$ satisfies the remaining two (homogeneous) Maxwell equations expressed as

[A.15] $\partial_\lambda f^{(1)}_{\mu\nu} + \partial_\mu f^{(1)}_{\nu\lambda} + \partial_\nu f^{(1)}_{\lambda\mu} = 0$.

By inspection of [A.13],

$$[A.16] \quad f^{(1)\alpha\beta} = -\tfrac{1}{2}f^{(0)}_{\mu\nu}f^{(0)\mu\nu}f^{(0)\alpha\beta} + 2f^{(0)\nu\alpha}f^{(0)\beta\mu}f^{(0)}_{\mu\nu} + h^{\alpha\beta},$$

where $h^{\alpha\beta}$ is an antisymmetric tensor to be determined subject to the condition $\partial_\alpha h^{\alpha\beta} = 0$. Although the dimensions of $h^{\alpha\beta}$ are those of E^3 or B^3, substituting [A.16] into [A.15] reveals that $h^{\alpha\beta}$ satisfies also the homogeneous Maxwell equations. The only non-trivial solution is $h^{\alpha\beta} = cf^{(0)\alpha\beta}$, where c is a constant having the dimensions of 1/g. In order that the interaction part consist only of terms quadrilinear in the free fields, the simplifying and consistent choice of the trivial solution $h^{\alpha\beta} = 0$ is adopted.

Since

$$[A.17] \quad \partial_j A^{(1)}_k - \partial_k A^{(1)}_j = -(B^{(0)2} - E^{(0)2})f^{(0)}_{jk} + 2f^{(0)}_{\nu j}f^{(0)}_{k\mu}f^{(0)\mu\nu}; \; j,k = 1,2,3,$$

then

$$[A.18] \quad (\underline{\nabla} \times \underline{A}^{(1)})_1 = -(B^{(0)2} - E^{(0)2})B^{(0)}_1 + 2B^{(0)}_1 B^{(0)2} - 2B^{(0)}_1(E^{(0)2}_2 + E^{(0)2}_3) + 2E^{(0)}_1(E^{(0)}_2 B^{(0)}_2 + E^{(0)}_3 B^{(0)}_3)$$

and similarly for $(\underline{\nabla} \times \underline{A}^{(1)})_2$ and $(\underline{\nabla} \times \underline{A}^{(1)})_3$. Also,

$$[A.19] \quad \partial_0 A^{(1)}_j - \partial_j A^{(1)}_0 = -(B^{(0)2} - E^{(0)2})f^{(0)}_{0j} + 2f^{(0)}_{\nu 0}f^{(0)}_{j\mu}f^{(0)\mu\nu},$$

so that

$$[A.20] \quad \dot{A}_1^{(1)} - \frac{\partial A_0^{(1)}}{\partial x} = (B^{(0)2} - E^{(0)2})E_1^{(0)} + 2E_1^{(0)}E^{(0)2} - 2E_1^{(0)}(B_2^{(0)2} + B_3^{(0)2})$$
$$+ 2B_1^{(0)}(E_2^{(0)}B_2^{(0)} + E_3^{(0)}B_3^{(0)})$$

and similarly for $\dot{A}_2^{(1)} - \partial A_0^{(1)}/\partial y$ and $\dot{A}_3^{(1)} - \partial A_0^{(1)}/\partial z$.

The terms $-g\underline{E}^{(0)}\cdot\underline{\dot{A}}^{(1)} + g\underline{B}^{(0)}\cdot(\underline{\nabla}\times\underline{A}^{(1)})$ can be expressed in terms of $\underline{E}^{(0)}$ and $\underline{B}^{(0)}$ as

$$[A.21] \quad -g\underline{E}^{(0)}\cdot\underline{\dot{A}}^{(1)} + g\underline{B}^{(0)}\cdot(\underline{\nabla}\times\underline{A}^{(1)}) = -g(E^{(0)4} - B^{(0)4} + \underline{\nabla}A_0^{(1)}\cdot\underline{E}^{(0)}).$$

Since $\underline{\nabla}A_0^{(1)}\cdot\underline{E}^{(0)}$ vanishes upon integration, the longitudinal part of $\underline{E}^{(1)}$, which could not be suppressed since $j^{(0)\beta} \neq 0$, does not contribute to the Hamiltonian. The correct Hamiltonian to first order in g is

$$[A.22] \quad \mathcal{H}^{(1)} = :\int\Big\{\tfrac{1}{2}(E^{(0)2} + B^{(0)2}) + \tfrac{1}{4}g\big[(B^{(0)2} - E^{(0)2})(3B^{(0)2} + E^{(0)2})$$
$$+ 4(\underline{B}^{(0)}\cdot\underline{E}^{(0)})^2\big]\Big\}d\underline{x}:$$

Higher order calculations are possible in principle. If $f_{\mu\nu}^{(0)} + gf_{\mu\nu}^{(1)} + g^2f_{\mu\nu}^{(2)}$ is substituted for $f_{\mu\nu}$ in [A.10], then $\partial_\alpha f^{(2)\alpha\beta}$ is expressible in terms of a current $j^{(1)\beta}$ containing terms involving $f^{(0)\alpha\beta}$ and $f^{(1)\alpha\beta}$. The divergence of such $j^{(n)\beta}$ always vanishes as required.

APPENDIX B: ON EQUATION [6.1]

Expressing the operators $a(\underline{k},\lambda)$ in terms of $a(\underline{k})$ and $b(\underline{k})$, the expansion [6.18], integrated over all 3-space, is

[B.1] $$\int :E^{(o)4}: d\underline{x} = :\frac{1}{V} \sum_{\underline{k},\lambda} \sum_{\underline{k}',\lambda'} \sum_{\underline{k}'',\lambda''} \sum_{\underline{k}''',\lambda'''} \sqrt{\frac{kk'k''k'''}{16}}$$

$$\tfrac{1}{4}(-1)^{\lambda+\lambda'+\lambda''+\lambda'''}\, i^{-(\lambda+\lambda'+\lambda''+\lambda''')}\, \underline{\epsilon}(\underline{k},\lambda)\cdot \underline{\epsilon}(\underline{k}',\lambda')$$

$$\underline{\epsilon}(\underline{k}'',\lambda'')\cdot \underline{\epsilon}(\underline{k}''',\lambda''') \Big\{ (-1)^{\lambda+\lambda'} \big[b''^{\dagger} b'''^{\dagger} b b' + (-1)^{\lambda+\lambda'}$$

$$b''^{\dagger} b'''^{\dagger} a a' + (-1)^{\lambda''+\lambda'''} a''^{\dagger} a'''^{\dagger} b b' + (-1)^{\lambda+\lambda'+\lambda''+\lambda'''}$$

$$a''^{\dagger} a'''^{\dagger} a a' \big] \delta(\underline{k}+\underline{k}'-\underline{k}''-\underline{k}''') + (-1)^{\lambda''+\lambda'''} \big[b^{\dagger} b'^{\dagger} b'' b'''$$

$$+ (-1)^{\lambda''+\lambda'''} b^{\dagger} b'^{\dagger} a'' a''' + (-1)^{\lambda+\lambda'+\lambda''+\lambda'''} a^{\dagger} a'^{\dagger} a'' a'''$$

$$+ (-1)^{\lambda+\lambda'} a^{\dagger} a'^{\dagger} b'' b''' \big] \delta(-\underline{k}-\underline{k}'+\underline{k}''+\underline{k}''') + (-1)^{\lambda'+\lambda''}$$

$$\big[b^{\dagger} b'''^{\dagger} b' b'' + (-1)^{\lambda'+\lambda''} b^{\dagger} b'''^{\dagger} a' a'' + (-1)^{\lambda+\lambda'''}$$

$$a'''^{\dagger} a^{\dagger} b' b'' + (-1)^{\lambda+\lambda'+\lambda''+\lambda'''} a^{\dagger} a'''^{\dagger} a' a'' \big] \delta(-\underline{k}+\underline{k}'+\underline{k}''-\underline{k}''')$$

$$(-1)^{\lambda+\lambda'''} \big[b'^{\dagger} b''^{\dagger} b b''' + (-1)^{\lambda+\lambda'''} b'^{\dagger} b''^{\dagger} a a''' + (-1)^{\lambda'+\lambda''}$$

$$a'^{\dagger} a''^{\dagger} b b''' + (-1)^{\lambda+\lambda'+\lambda''+\lambda'''} a'^{\dagger} a''^{\dagger} a a''' \big] \delta(\underline{k}-\underline{k}'-\underline{k}''+\underline{k}''')$$

$$+ (-1)^{\lambda+\lambda''} \big[b'^{\dagger} b'''^{\dagger} b b'' + (-1)^{\lambda+\lambda''} b'^{\dagger} b'''^{\dagger} a a''$$

$$+ (-1)^{\lambda'+\lambda'''} a'^{\dagger} a'''^{\dagger} b b'' + (-1)^{\lambda+\lambda'+\lambda''+\lambda'''} a'^{\dagger} a'''^{\dagger} a a'' \big]$$

$$\delta(-\underline{k}+\underline{k}'-\underline{k}''+\underline{k}''') + (-1)^{\lambda'+\lambda'''} \big[b^{\dagger} b''^{\dagger} b' b''' + (-1)^{\lambda'+\lambda'''}$$

$$b^{\dagger} b''^{\dagger} a' a''' + (-1)^{\lambda+\lambda''} a^{\dagger} a''^{\dagger} b' b''' + (-1)^{\lambda+\lambda'+\lambda''+\lambda'''}$$

$$a^{\dagger} a'^{\dagger} a a''' \big] \delta(\underline{k}-\underline{k}'+\underline{k}''-\underline{k}''') \Big\} :,$$

where terms that lead to $|RL\rangle$ and $|LR\rangle$ states, or to zero, when applied to $|LL\rangle$ or $|RR\rangle$ states are dropped. Applying the above expression on $|LL\rangle$ and $|RR\rangle$ states leads to conditions on the $\underline{k}$'s, permitting a separation of the δ-functions. For example, in the first term, $\underline{k} = \pm \underline{q}$, $\underline{k}' = \mp \underline{q}$ so that $\underline{k}'' = -\underline{k}'''$. The term $(kk'k''k''')^{\frac{1}{2}}$ becomes k^2q^2 in each case. Performing this operation,

$$[B.2] \quad \int :E^{(o)4}: |RR\rangle d\underline{x} = \int :E^{(o)4}: |LL\rangle d\underline{x} = \frac{1}{V} \sum_{\underline{k}} \sum_{\lambda\lambda'\lambda''\lambda'''} \frac{kq}{16}$$

$$\Big\{ \big[2\underline{\epsilon}(\underline{q},\lambda)\cdot\underline{\epsilon}(-\underline{q},\lambda)\underline{\epsilon}(\underline{k},\lambda'')\cdot\underline{\epsilon}(-\underline{k},\lambda'')(-1)^{\lambda+\lambda''}$$
$$(|LL\rangle + |RR\rangle)\big] + \big[2\underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(-\underline{k},\lambda)\underline{\epsilon}(\underline{q},\lambda'')\cdot\underline{\epsilon}(-\underline{q},\lambda'')$$
$$(-1)^{\lambda+\lambda''}(|LL\rangle + |RR\rangle)\big] + \big[\underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(-\underline{q},\lambda')$$
$$\underline{\epsilon}(\underline{q},\lambda'')\cdot\underline{\epsilon}(-\underline{k},\lambda''') + \underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(\underline{q},\lambda')\underline{\epsilon}(-\underline{q},\lambda'')\cdot\underline{\epsilon}(-\underline{k},\lambda''')\big]$$
$$(-1)^{\lambda+\lambda'''}\, i^{-(\lambda+\lambda'+\lambda''+\lambda''')}\big[(-1)^{\lambda'+\lambda''}|LL\rangle +$$
$$(-1)^{\lambda+\lambda'+\lambda''+\lambda'''}|RR\rangle\big] + \big[\underline{\epsilon}(-\underline{q},\lambda)\cdot\underline{\epsilon}(\underline{k},\lambda')\underline{\epsilon}(-\underline{k},\lambda'')\cdot$$
$$\underline{\epsilon}(\underline{q},\lambda''') + \underline{\epsilon}(\underline{q},\lambda)\cdot\underline{\epsilon}(\underline{k},\lambda')\underline{\epsilon}(-\underline{k},\lambda'')\cdot\underline{\epsilon}(-\underline{q},\lambda''')\big]$$
$$(-1)^{\lambda'+\lambda''}\, i^{-(\lambda+\lambda'+\lambda''+\lambda''')}\big[(-1)^{\lambda+\lambda'''}|LL\rangle +$$
$$(-1)^{\lambda+\lambda'+\lambda''+\lambda'''}|RR\rangle\big] + \big[\underline{\epsilon}(-\underline{q},\lambda)\cdot\underline{\epsilon}(\underline{k},\lambda')\underline{\epsilon}(\underline{q},\lambda'')\cdot$$
$$\underline{\epsilon}(-\underline{k},\lambda''') + \underline{\epsilon}(\underline{q},\lambda)\cdot\underline{\epsilon}(\underline{k},\lambda')\underline{\epsilon}(-\underline{q},\lambda'')\cdot\underline{\epsilon}(-\underline{k},\lambda''')\big]$$
$$(-1)^{\lambda'+\lambda'''}\, i^{-(\lambda+\lambda'+\lambda''+\lambda''')}\big[(-1)^{\lambda+\lambda''}|LL\rangle +$$
$$(-1)^{\lambda+\lambda'+\lambda''+\lambda'''}|RR\rangle\big] + \big[\underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(\underline{q},\lambda')\underline{\epsilon}(-\underline{k},\lambda'')\cdot$$
$$\underline{\epsilon}(-\underline{q},\lambda''') + \underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(-\underline{q},\lambda')\underline{\epsilon}(-\underline{k},\lambda'')\cdot\underline{\epsilon}(\underline{q},\lambda''')\big]$$
$$(-1)^{\lambda+\lambda''}\, i^{-(\lambda+\lambda'+\lambda''+\lambda''')}\big[(-1)^{\lambda'+\lambda'''}|LL\rangle +$$
$$(-1)^{\lambda+\lambda'+\lambda''+\lambda'''}|RR\rangle\big]\Big\}$$

$$= \frac{1}{V}\sum_{\underline{k}} kq(|LL\rangle + |RR\rangle) + \frac{4}{V}\sum_{\underline{k}}\sum_{\text{all}\,\lambda} \frac{kq}{16}\Big\{\underline{\epsilon}(\underline{k},\lambda)\cdot$$

$$\underline{\epsilon}(-\underline{q},1)\underline{\epsilon}(\underline{q},1)\cdot\underline{\epsilon}(-\underline{k},\lambda''')\big[|LL\rangle + (-1)^{\lambda+\lambda'''}|RR\rangle\big]$$

$$+ i^{-\lambda-\lambda'''}\,\underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(-\underline{q},2)\underline{\epsilon}(\underline{q},2)\cdot\underline{\epsilon}(-\underline{k},\lambda''')\big[|LL\rangle$$

$$+ (-1)^{\lambda+\lambda'''}|RR\rangle\big]\Big\} + \frac{1}{8V}\sum_{\underline{k}}\sum_{\text{all}\,\lambda} kq\Big\{(-1)^{\lambda'+\lambda'''}\,i^{-\lambda'-\lambda'''}$$

$$\big[\underline{\epsilon}(-\underline{q},2)\cdot\underline{\epsilon}(\underline{k},\lambda')\underline{\epsilon}(\underline{q},2)\cdot\underline{\epsilon}(-\underline{k},\lambda''') - \underline{\epsilon}(-\underline{q},1)\cdot\underline{\epsilon}(\underline{k},\lambda')$$

$$\underline{\epsilon}(\underline{q},1)\cdot\underline{\epsilon}(-\underline{k},\lambda''')\big]\big[|LL\rangle + (-1)^{\lambda'+\lambda'''}|RR\rangle\Big\} + \frac{1}{8V}\sum_{\underline{k}}\sum_{\text{all}\,\lambda}$$

$$kq\Big\{(-1)^{\lambda+\lambda''}i^{-\lambda-\lambda''}\big[\underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(\underline{q},2)\underline{\epsilon}(-\underline{k},\lambda'')\cdot\underline{\epsilon}(-\underline{q},2)$$

$$- \underline{\epsilon}(\underline{k},\lambda)\cdot\underline{\epsilon}(\underline{q},1)\underline{\epsilon}(-\underline{k},\lambda'')\cdot\underline{\epsilon}(-\underline{q},1)\big]\big[|LL\rangle + (-1)^{\lambda+\lambda''}|RR\rangle\big]\Big\}$$

$$= \frac{1}{V}\sum_{\underline{k}} kq(|LL\rangle + |RR\rangle) + \frac{1}{2V}\sum_{\underline{k}}\sum_{\text{all}\,\lambda} kq\Big\{-\epsilon_{(1)}(|LL\rangle$$

$$+ |RR\rangle) + i(|LL\rangle - |RR\rangle)\epsilon_{(2)} + i(|LL\rangle - |RR\rangle)\epsilon_{(3)}$$

$$+ (|LL\rangle + |RR\rangle)\epsilon_{(4)}\Big\},$$

where

[B.3] $$\epsilon_{(1)} \equiv \underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(-\underline{q},2)\underline{\epsilon}(\underline{q},2)\cdot\underline{\epsilon}(-\underline{k},1) - \underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(-\underline{q},1)\underline{\epsilon}(\underline{q},1)\cdot\underline{\epsilon}(-\underline{k},1)$$

[B.4] $$\epsilon_{(2)} \equiv \underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(-\underline{q},2)\underline{\epsilon}(\underline{q},2)\cdot\underline{\epsilon}(-\underline{k},1) - \underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(-\underline{q},1)\underline{\epsilon}(\underline{q},1)\cdot\underline{\epsilon}(-\underline{k},1)$$

[B.5] $$\epsilon_{(3)} \equiv \underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(-\underline{q},2)\underline{\epsilon}(\underline{q},2)\cdot\underline{\epsilon}(-\underline{k},2) - \underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(-\underline{q},1)\underline{\epsilon}(\underline{q},1)\cdot\underline{\epsilon}(-\underline{k},2)$$

[B.6] $$\epsilon_{(4)} \equiv \underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(-\underline{q},2)\underline{\epsilon}(\underline{q},2)\cdot\underline{\epsilon}(-\underline{k},2) - \underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(-\underline{q},1)\underline{\epsilon}(\underline{q},1)\cdot\underline{\epsilon}(-\underline{k},2).$$

Therefore,

[B.7] $$\int :E^{(o)4}: |RR\rangle d\underline{x} = \int :E^{(o)4}: |LL\rangle d\underline{x} = \frac{1}{V}\sum_{\underline{k}} kq(|LL\rangle + |RR\rangle) + \frac{1}{2V}\sum_{\underline{k}} kq\Big[(-\epsilon_{(1)} + i\epsilon_{(2)} + i\epsilon_{(3)} + \epsilon_{(4)})|LL\rangle + (-\epsilon_{(1)} - i\epsilon_{(2)} - i\epsilon_{(3)} + \epsilon_{(4)})|RR\rangle\Big].$$

A similar calculation for $B^{(o)4}$ gives

[B.8] $$\int :B^{(o)4}: |RR\rangle d\underline{x} = \int :B^{(o)4}: |LL\rangle d\underline{x} = \frac{1}{V}\sum_{\underline{k}} kq(|LL\rangle + |RR\rangle) + \frac{1}{2V}\sum_{\underline{k}} kq\Big[(-\epsilon_{(1)} - i\epsilon_{(2)} - i\epsilon_{(3)} + \epsilon_{(4)})|LL\rangle + (-\epsilon_{(1)} + i\epsilon_{(2)} + i\epsilon_{(3)} + \epsilon_{(4)})|RR\rangle\Big].$$

In the calculation for $E^{(o)2}B^{(o)2}$, different results are obtained when the operation is on $|RR\rangle$ from those obtained when the operation is on $|LL\rangle$. For this case, the results are:

[B.9] $$\int :E^{(o)2}B^{(o)2}: |RR\rangle = -\frac{1}{V}\sum_{\underline{k}} kq(|LL\rangle + |RR\rangle) + \frac{1}{2V}\sum_{\underline{k}} kq\Big[(\epsilon_{(1)} + i\epsilon_{(2)} + i\epsilon_{(3)} - \epsilon_{(4)})|LL\rangle + (-\epsilon_{(1)} + i\epsilon_{(2)} + i\epsilon_{(3)} + \epsilon_{(4)})|RR\rangle\Big]$$

and

[B.10] $$\int :E^{(0)2}B^{(0)2}: |LL\rangle d\underline{x} = -\frac{1}{V}\sum_{\underline{k}} kq(\,|LL\rangle + |RR\rangle\,)$$

$$-\frac{1}{2V}\sum_{\underline{k}} kq\left[(\,\epsilon_{(1)} + i\epsilon_{(2)} + i\epsilon_{(3)} - \epsilon_{(4)}\,)|LL\rangle\right.$$
$$\left. + (-\epsilon_{(1)} + i\epsilon_{(2)} + i\epsilon_{(3)} - \epsilon_{(4)}\,)|RR\rangle\right].$$

For the last product, $(\underline{B}^{(0)}\cdot\underline{E}^{(0)})^2$, it is found that

[B.11] $$\int :(\underline{B}^{(0)}\cdot\underline{E}^{(0)})^2: |RR\rangle d\underline{x} = -\int :(\underline{B}^{(0)}\cdot\underline{E}^{(0)})^2 |LL\rangle d\underline{x}$$
$$= \frac{1}{V}\sum_{\underline{k}} kq(\,|RR\rangle - |LL\rangle\,) + \frac{1}{2V}\sum_{\underline{k}} kq\left[(\,\epsilon_{(1)} + i\epsilon_{(2)}\right.$$
$$\left. + i\epsilon_{(3)} - \epsilon_{(4)})|LL\rangle + (-\epsilon_{(1)} + i\epsilon_{(2)} + i\epsilon_{(3)} - \epsilon_{(4)}\,)|RR\rangle\right].$$

It follows that if $\mathcal{H}$ operates on the even parity state $|LL\rangle + |RR\rangle$, then no contribution is obtained from the pseudoscalar part $(\underline{B}^{(0)}\cdot\underline{E}^{(0)})^2$ of the interaction. On the other hand, if $\mathcal{H}$ operates on the odd parity state $|LL\rangle - |RR\rangle$, no contribution is obtained from the $E^{(0)4}$ and $B^{(0)4}$ terms. The cross-term $E^{(0)2}B^{(0)2}$ gives contribution in both cases. Gathering the results together, one obtains:

[B.12] $$\int :H_I:(\,|RR\rangle \pm |LL\rangle\,)d\underline{x} = \mathcal{H}_I|B(\underline{q})\rangle_\pm$$
$$= \frac{2g}{V}\sum_{\underline{k}} kq\,|B(\underline{q})\rangle_\pm + \frac{g}{2V}\sum_{\underline{k}} kq(\,\epsilon_{(4)} - \epsilon_{(1)}\,)|B(\underline{q})\rangle_\pm,$$

where $$\epsilon_{(4)} - \epsilon_{(1)} = [\underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(\underline{q},1)]^2 + [\underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(\underline{q},1)]^2$$
$$+ [\underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(\underline{q},2)]^2 + [\underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(\underline{q},2)]^2$$
$$\equiv P(\underline{k},\underline{q}).$$

It is the duality invariance of the interaction Lagrangian that accounts for the fact that $\mathcal{H}_I |B(\underline{q})\rangle_+ = \mathcal{H}_I |B(\underline{q})\rangle_-$. Equation [6.1] follows immediately from [B.12].

APPENDIX C: ON EQUATION [6.3]

In equation [6.1], consider the terms

$$[C.1] \quad P(\underline{k},\underline{q}) \equiv [\underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(\underline{q},1)]^2 + [\underline{\epsilon}(\underline{k},1)\cdot\underline{\epsilon}(\underline{q},2)]^2 + [\underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(\underline{q},1)]^2 + [\underline{\epsilon}(\underline{k},2)\cdot\underline{\epsilon}(\underline{q},2)]^2$$

Using the representation [6.2],

$$\begin{aligned}[C.2] \quad P(\underline{k},\underline{q}) = \frac{1}{\Delta}\Big\{&(q_1^2q_3^2k_2^2 + k_1^2q_2^2q_3^2 - 2q_1q_3^2q_2k_1k_2)k^2 \\ &+ (k_1^2k_3^2q_2^2 + k_2^2k_3^2q_1^2 - 2k_1k_2k_3^2q_1q_2)q^2 \\ &+ (k_1^2q_1^2 + 2k_1k_2q_1q_2 + k_2^2q_2^2)k^2q^2 + k_1^2k_3^2q_1^2q_3^2 \\ &+ 2k_1k_2k_3^2q_1q_2q_3^2 + 2(k_3^2-k^2)(q_3^2-q^2)k_1k_3q_1q_3 \\ &+ k_2^2k_3^3q_2^2q_3^2 + 2(k_3^2-k^2)(q_3^2-q^2)k_2k_3q_2q_3 \\ &+ (k_3^2-k^2)^2(q_3^2-q^2)^2\Big\},\end{aligned}$$

where $\Delta = k^2q^2(k_3^2-k^2)(q_3^2-q^2)$. Since

$$[C.3] \quad (k_1q_1+k_2q_2)^2 = (k_3^2-k^2)(q_3^2-q^2) - (k_1q_2-k_2q_1)^2$$

then

$$\begin{aligned}[C.4] \quad P(\underline{k},\underline{q}) &= \frac{1}{\Delta}\Big\{k^2q_3^2(k_1q_2-k_2q_1)^2 + q^2k_3^2(k_1q_2-k_2q_1)^2 \\ &\quad + k^2q^2(k_1q_1+q_2k_2)^2 + k_3^2q_3^2(k_1q_1+k_2q_2)^2 \\ &\quad + 2k_3q_3(k_3^2-k^2)(q_3^2-q^2)(k_1q_1+k_2q_2) \\ &\quad + (k_3^2-k^2)^2(q_3^2-q^2)^2\Big\} \\ &= \frac{1}{\Delta}\Big\{(k_3^2-k^2)(q_3^2-q^2)\big[(\underline{k}\cdot\underline{q})^2 + k^2q^2\big]\Big\} \\ &= 1 + \hat{\underline{k}}\cdot\hat{\underline{q}}.\end{aligned}$$