A THEORETICAL AND EXPERIMENTAL INVESTIGATION OF SINUSOIDAL AND RELAXATION OSCILLATIONS IN THERMISTOR - CAPACITOR SYSTEMS

by

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ABSTRACT

It is known that a thermistor displays inductive reactance and negative resistance at low frequencies when biased with a current greater than turnover current. Thus when shunted by a capacitance in this condition sustained oscillations are possible. These oscillations range from a sinusoidal small - amplitude character, when the system is just above the threshold for oscillation, to a strongly relaxational type when a large shunt capacitance is employed.

The investigation described in this thesis involved a study of the dynamic properties of these oscillations and their relation to the static properties of the thermistor. The relevant thermistor parameters which were measured included the dependence of resistance on temperature, the thermal conductance and the thermal time constant. For the determination of the latter, special very low frequency techniques were developed. Thermistors of high resistance and low thermal time constant were selected in order to avoid the necessity for very large capacitances in the study of relaxation oscillations.

The oscillations were studied over a wide range of capacitances covering the transition from sinusoidal to relaxation type. The voltage extrema and period were measured as functions of capacitance, voltage supply resistance and operating point. Also, in order to elucidate the dynamic processes involved, the transient phenomena produced by abrupt pertubations were investigated.

In parallel with the experimental program, an investigation was made of the features of the differential equations describing the behaviour of the system, especially for the limit cycles corresponding to relaxation oscillations. The asymptotic form of the cycles were derived for the case where the circuital time constant greatly exceeds the thermal time constant of the thermistor. The complicated nature of the equations precluded a solution in a closed form and approximational methods were found to be necessary. However, in the case of sinusoidal oscillations of small amplitude, the period can be exactly expressed in terms of the system parameters (Burgess, Nov. 1955) and this result was confirmed experimentally.

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PART 1

INTRODUCTION

The thermistor is a two terminal device possessing a high negative temperature coefficient of resistance and a large thermal inertia. Burgess (Oct. 1955) has shown that under certain conditions of d - c bias, the thermistor will exhibit an incremental admittance consisting of a negative conductance and negative susceptance. In a later paper, Burgess (Nov. 1955) also has shown that if a positive susceptance of equal magnitude, in the form of a capacitance, be connected in parallel to the thermistor, small-amplitude sinusoidal oscillations will be sustained if the d - c source conductance is only slightly less in magnitude than the negative incremental conductance of the thermistor.

If the magnitude of the parallel capacitance is increased, the oscillations become relaxational and do not appear to bear treatment in terms of the Van der Pol equation. This paper presents the results of a theoretical and experimental investigation of these relaxation oscillations.

Preliminary to the investigation, existing theory of the thermistor and some general remarks on non linear vibration theory are given. In Part 2 the form of the static current - voltage characteristic is deduced and the results of measurements of its parameters for certain thermistors are quoted.

Part 3 concerns small - amplitude time varying phenomena in the thermistor. It essentially reproduces the two papers of Burgess and relates how the small - amplitude a - c theory was used to measure the thermal time constant of the thermistor.

Part 4 gives some definitions and theorems of non-linear vibration theory which are employed in the investigation. Part 5 presents the investigation of relaxation - type oscillations over a wide range of the value of the parallel capacitance.

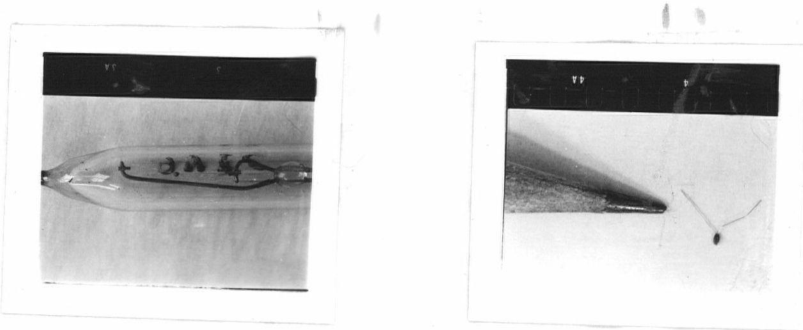
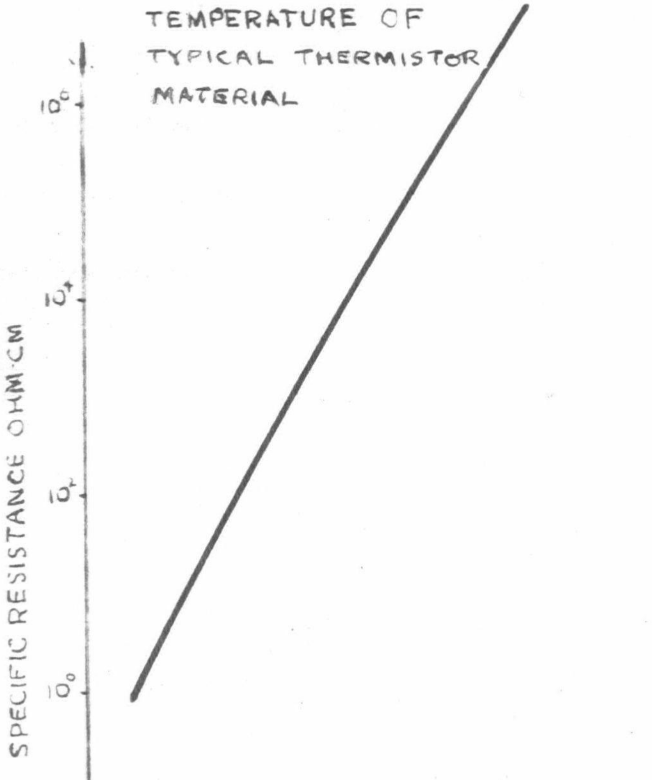


FIGURE 2.1 PHOTOGRAPHS OF BEAD-TYPE THERMISTORS

FIGURE 2.2 DEPENDENCE OF SPECIFIC RESISTANCE UPON INVERSE TEMPERATURE OF TYPICAL THERMISTOR MATERIAL



PART 2

STATIC PROPERTIES OF THERMISTORS

1. Physical Properties

Thermistors are thermally sensitive resistors whose variation in resistance is characterized by a high negative temperature coefficient. They are generally made by heating compressed powders of semiconducting materials to a temperature at which they sinter into a compact mass and then firing them on metal - powder contacts. Semiconductors are substances whose electrical resistivity at or near room temperature is much more than that of typical conductors and much less than that of typical insulators. In general, around room temperature these materials when pure, have negative temperature coefficients of resistivity of about 5% per degree centigrade compared to platinum and copper which are typical conductors and have positive temperature coefficients of about 0.4% per degree centigrade.

Figure 2.2 is a graph of the logarithm of specific resistance versus reciprocal absolute temperature of a typical thermistor material. This graph indicates that the logarithm of the thermistor resistance R varies approximately linearly as the reciprocal of the absolute temperature of the thermistor T. Thus for a given thermistor one may write

$$\log R \propto \frac{1}{T},$$

or

$$\ln R = \frac{b}{T} + \text{constant},$$

or finally

$$R = R_{\infty} e^{\frac{b}{T}} \tag{2.1}$$

where,

R = thermistor resistance

T = absolute temperature of thermistor

b = constant equal to the slope of ln R - vs - $\frac{1}{T}$ graph

e = naperian base

R_{∞} = value of R where projected curve crosses the line $\frac{1}{T} = 0$.

It is apparent that b is a characteristic of a given material since a plot of log R versus $\frac{1}{T}$ has the same slope as a plot of the logarithm of specific resistance versus $\frac{1}{T}$. The dimension of b is temperature and it is generally specified in degrees kelvin or degrees centigrade. It plays the role of activation energy in (2.1) similar to the work function in the equation for thermionic emission. It is apparent that R_{∞} is dependent in general upon the type of material and the physical construction such as the size of the element and area of the contacts of a given thermistor.

If the dependence of ln R upon $\frac{1}{T}$ be carefully examined, it may be found (Becker et al 1947) that the slope increases as the temperature increases, thus a more precise expression may be

$$R = \frac{A}{T^c} e^{\frac{d}{T}}, \tag{2.2}$$

where A, c and d are empirical constants. The constant c is generally a small positive or negative number at zero. Equation (2.1) will be used for the model in this paper.

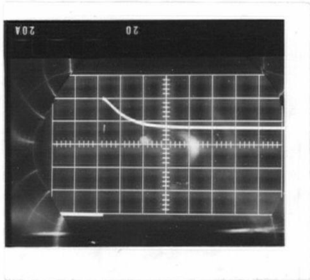


FIGURE 2.3 APPLICATION OF CURRENT TO THERMISTOR

TIME-HORIZONTAL
VOLTAGE-VERTICAL

2. Static Voltage - Current Characteristic

(2.a) General

It is convenient here to introduce the variable Θ which is defined as the excess of the temperature T of the element above the ambient temperature Ta, or

$$\Theta = T - Ta \qquad (2.3)$$

If a current I is passed through a thermistor and Ta is held constant, then after sufficient time the voltage at the terminals of the thermistor V will reach a steady value (see Figure 2.3). If the current is increased with Ta held constant then the applied electric power will increase causing an increase in the excess temperature Θ of the thermistor. Thus a series of points (V, I) may be obtained which define a curve called the static characteristic.

If a current I is passed through the thermistor and the electric power is prevented from increasing the temperature above the ambient Ta, then since $T = Ta + \Theta$ and $\Theta = 0$, the relation

$$\frac{V}{I} = R \quad e^{\frac{b}{Ta}}$$

is valid for all the values of V and I. The points (V, I) again define a characteristic curve. This curve is called the isothermal characteristic or simply isothermal. It is apparent that in the V - I plane an isothermal is a straight line the slope of which depends upon Ta. Thus, the thermistor may be considered a linear circuit element device if its temperature is held constant. In practice the isothermal condition may be observed by increasing the cooling apparatus or by applying the power in pulses.

(2.b) Definition of Thermal Conductivity

It will now be assumed that the dissipated power in the steady state is a function of the excess temperature Θ. The balance of the heat supplied and the rate of cooling gives

$$VI = f(T - Ta) = f(\Theta) . \tag{2.4}$$

The term on the right $f(\Theta)$ is the rate at which heat is lost from the thermistor. The form of $f(\Theta)$ has been given (Bollman and Kreer, 1950) as the sum of a thermal conductance term and a radiation - loss term, resulting in,

$$f(\Theta) = k\Theta + k_R\left[(Ta + \Theta)^4 - Ta^4\right], \tag{2.5}$$

where k is the thermal conductance of the thermistor and k_R is the radiation coefficient of the thermistor. The dimensions of k are power per unit temperature and the units are generally watts per degree centigrade. Over most measurable temperature ranges either the second term of (2.5) is generally negligible with respect to the first term or at least the dominant factor of the second term is $k_R\, Ta^3\Theta$. In the model in this paper the relation

$$VI = k\Theta, \tag{2.6}$$

expressing Newton's law of cooling will be assumed. Combining equations (2.1) and (2.6), one obtains the following static characteristic

$$\frac{V}{I} = R_{\infty} \exp \frac{b}{Ta + \frac{VI}{k}} . \tag{2.7}$$

(2.c) Properties of the Static Characteristic

There are several interesting properties which can be obtained from (2.7). To facilitate this, several new functions will be defined.

The <u>temperature coefficient of resistance</u> μ will be defined by the relation

$$\mu = -\frac{1}{R}\left(\frac{dR}{d\theta}\right)_{Ta}, \qquad (2.8)$$

which upon calculation from (2.7) gives

$$\mu = \frac{b}{T^2} = \frac{b}{(Ta+\theta)^2}\,. \qquad (2.9)$$

The <u>thermal function</u> ϕ will be defined by

$$\phi = \frac{b\theta}{(Ta+\theta)^2} \qquad (2.10)$$

and is a dimensionless quantity which can never exceed $\frac{b}{4Ta}$. It is noted that since $VI = k\theta$ both μ and ϕ may be written as functions of or as functions of VI.

The slope r of the static characteristic is

$$r = \frac{dV}{dI} = \frac{1-\phi}{1+\phi}R = \frac{T^2 - b(T-Ta)}{T^2 \quad b(T-Ta)}R \qquad (2.11)$$

and is the ac resistance at zero frequency for any operating point. The first part of (2.11) may be rearranged so that

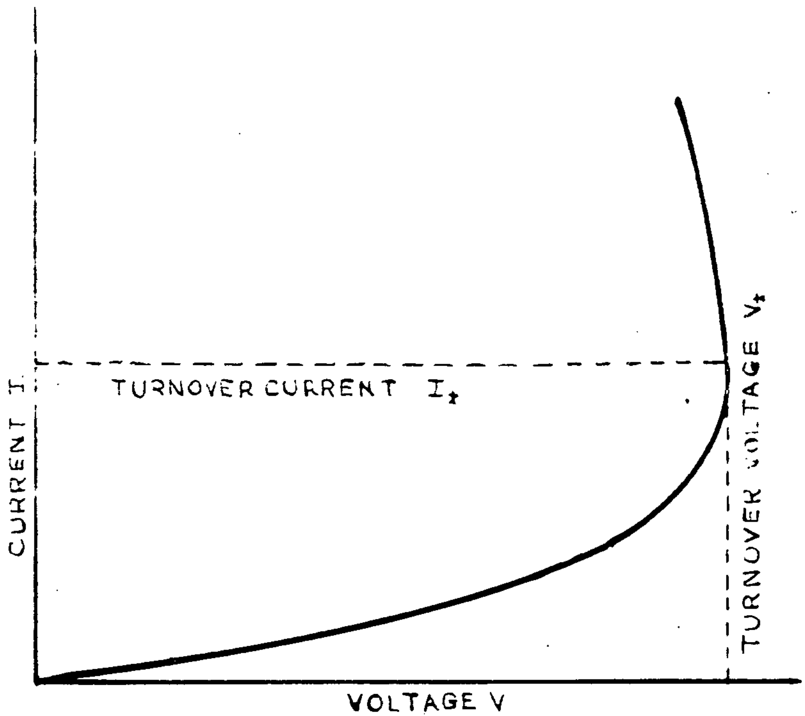
$$\phi = \frac{R-r}{R+r}\,. \qquad (2.12)$$

At the origin of the V - I plane, $V = I = \phi = 0$ and

$$r = R_{\infty}\, e^{\frac{b}{Ta}} = Ra$$

the zero-current resistance of the thermistor at the ambient temperature Ta. Thus the static characteristic at the origin is tangent to the isothermal corresponding to $T = Ta$. As I is increased ϕ becomes larger until finally it may equal unity. At the point where $\phi = 1$, $r = 0$ and $T = T_t$, the

FIGURE 24
TURNOVER OF STATIC CHARACTERISTIC



phenomena of turnover is observed since this corresponds to the voltage maximum in the static characteristic (see Figure 2.4). The value $T = T_t$ is the thermistor temperature at turnover

$$T_t = \frac{b - (b^2 - 4b\,Ta)^{\frac{1}{2}}}{2}. \tag{2.13}$$

It is noted that turnover will only occur if $b > 4\,Ta$. If the ambient temperature is $300\,^{\circ}K$ and b has a typical value for thermistor materials of $4000\,^{\circ}K$, turnover will always occur. If $b \gg 4\,Ta$, then

$$T_t = Ta\left(1 + \frac{Ta}{b}\right)$$

is valid. The voltage minimum at $Ta + b$ is of mathematical interest only since this temperature is well beyond the melting point of materials used in thermistor production. The turnover condition $T_t - Ta = \frac{Ta^2}{b} \simeq 20\,^{\circ}C$ is physically realizeable and is of great interest in many properties and applications of thermistors.

The following relations are valid for $T_t = Ta + \frac{Ta^2}{b}$

$$VI = (VI)_t \simeq \frac{k}{b}\,Ta^2$$

$$R = R_t \simeq Ra\,e^{-1}.$$

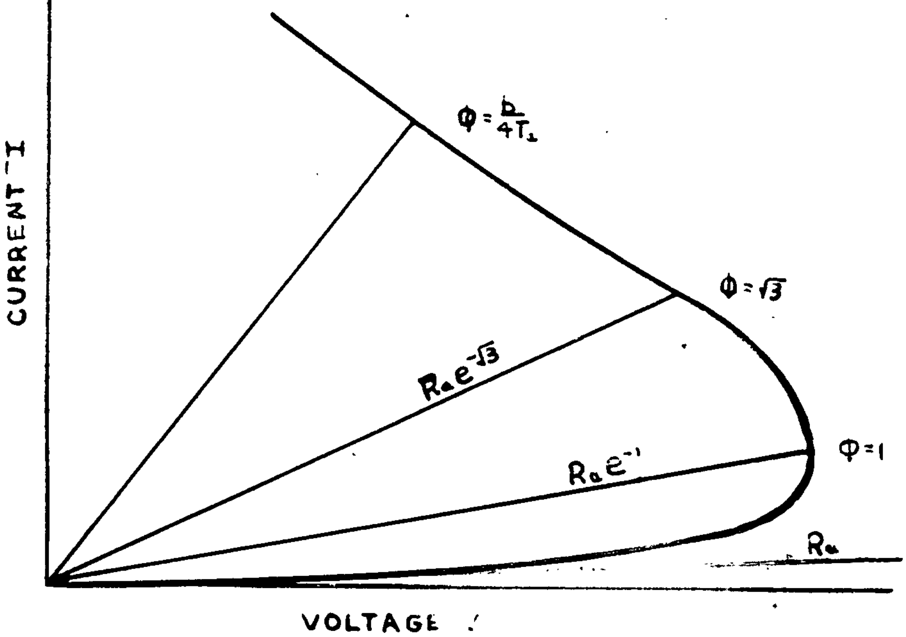
The point of inflexion of the static characteristic where the slope r reaches its maximum negative value is at the point where

$$\frac{d}{dT}\left(\frac{dV}{dI}\right) = 0 \tag{2.14}$$

Putting $x = \frac{T - Ta}{Ta}$ and $c = \frac{b}{Ta}$ and $T = T_i$ at the point $\frac{dr}{dT} = 0$, it is found

FIGURE 2.5 STATIC CHARACTERISTIC FOR IDEAL THERMISTOR FOLLOWING

$$\frac{V}{I} = R_\infty \exp \frac{b}{T_a + \Theta}$$



that

$$(x_i + 1)^3 \ (x_i - 3) + c^2 x_i^2 = 0 ,$$

If $x_i \ll \frac{3}{8}$ then

$$c \simeq \frac{\sqrt{3}}{x_i} + \frac{4}{\sqrt{3}} ,$$

and

$$\Theta_i = \frac{3\,Ta^2}{\sqrt{3}b - 4\,Ta}$$

resulting in

$$R_t \simeq R_i\, e^{\sqrt{3} - 1} = 2.16\, R_i$$

or

$$R_i \simeq Ra\, e^{-\sqrt{3}} = .18 Ra$$

and also

$$I_i \simeq I_t \left(e^{\sqrt{3} - 1}\right)^{\frac{1}{2}} = 1.93\, I_t .$$

These relations occuring for an ideal thermistor following

$$\frac{V}{I} = R_\infty \quad \exp \frac{b}{Ta + \frac{VI}{k}} \tag{2.7}$$

under the condition Ta $\gg$ b may be summarized in the following table and in Figure 2.5.

Summary of the Properties of Ideal Thermistor Characteristics

Θ	R	r	ϕ	Comment
$\ll$ Ta	$Ra(1+\phi)$	Ra	$\frac{b\,\Theta}{Ta^2}$	Origin
$\frac{Ta^2}{b}$	$Ra\,e^{-1}$	0	1	Turnover
$\sqrt{3}\,\frac{Ta^2}{b}$	$Ra\,e^{-\sqrt{3}}$	$-Ra\,\frac{e^{-\sqrt{3}}}{(2+\sqrt{3})}$	$\sqrt{3}$	Inflexion
Ta	$Ra\,e^{\frac{b}{4Ta}}$	$-Ra\,e^{\frac{b}{4\,Ta}}$	$\frac{b}{4\,Ta}$	Maximum

FIGURE 2.6 DEPENDENCE OF RESISTANCE UPON INVERSE TEMPERATURE FOR A VECO TYPE 65A1 THERMISTOR

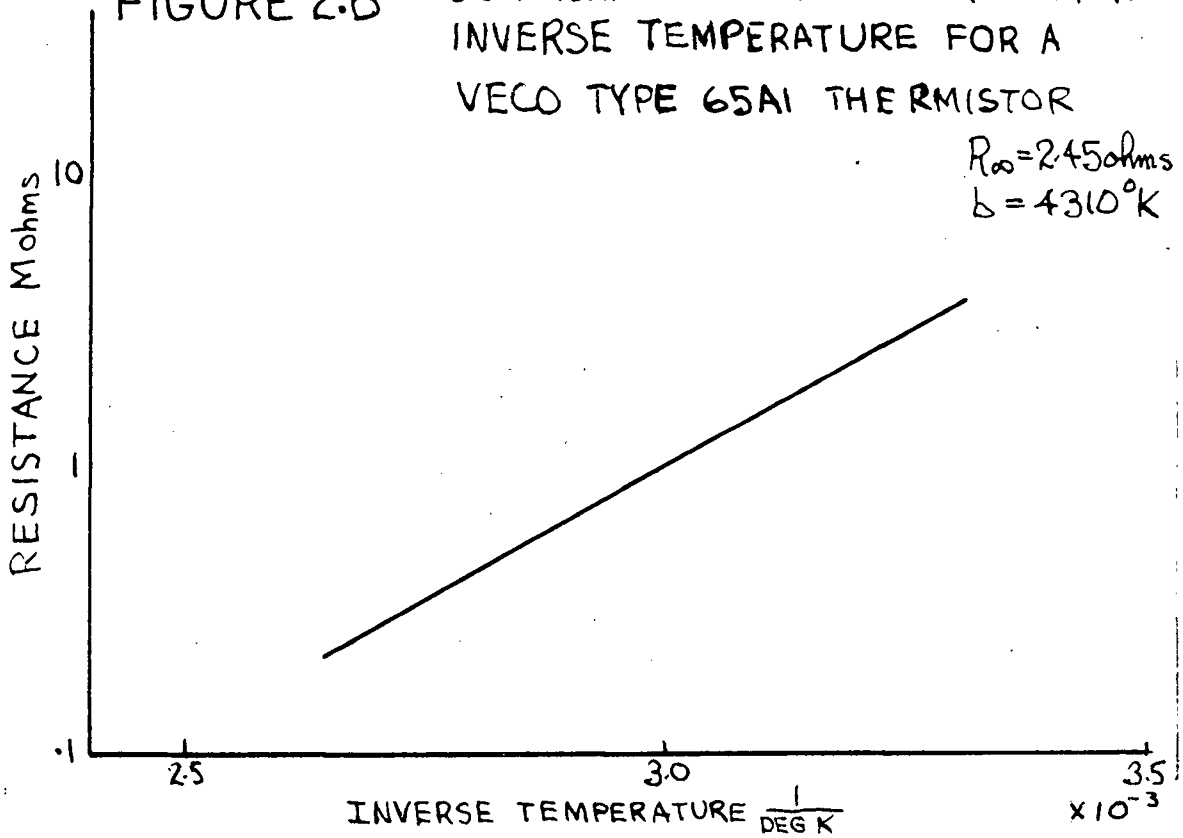
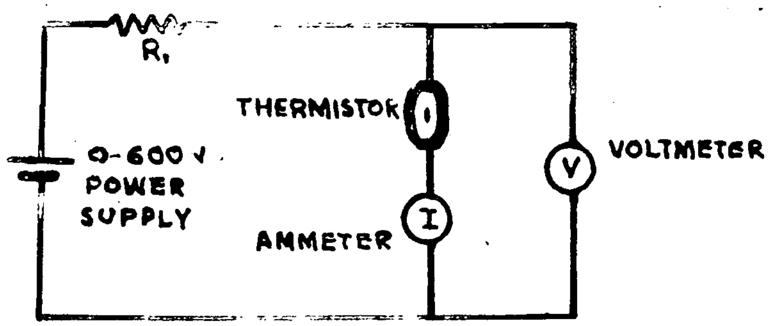


FIGURE 2.7

CIRCUIT USED FOR STATIC MEASUREMENTS



3. Experimental

Data were obtained for the temperature variation of resistance of several different thermistors. The thermistor under measurement was placed in a bath of hot transformer oil in a vacuum flask. Simultaneous readings of resistance R from a Wheatstone bridge connected to the thermistor and of the temperature T from a thermistor immersed in the oil were taken while the oil cooled to room temperature. The Wheatstone bridge was operated so as to pass negligible power into the thermistor and the thermometer was immersed so that its bulb was at the same depth in the oil as the thermistor bead. Figure 2.6 shows the graph of log R vs T^{-1} from a set of readings. Over the temperature range measured no variation from a straight line was observed. Thus, the equation

$$R = R_{\infty} e^{\frac{b}{T}} \tag{2.1}$$

is adequate.

Measurements of the static characteristic were made using the circuit shown in Figure 2.7. In the V - I plane the equation

$$E = IR_1 + V$$

gives the load line. Its intersection with the static characteristic is the operating point (V_o, I_o). R_1 was composed of a variable resistance and a fixed safety resistance. E was a variable - voltage power supply. Figure 2.8 shows a graph of the static characteristic for a Victory Engineering Corporation (VECO) type 65A1 thermistor in air. Values of R were calculated at various points on the curve and corresponding values of Θ were calculated from the log R - versus - T^{-1} graph. Values of ϕ were calculated from the values of Θ by the relation

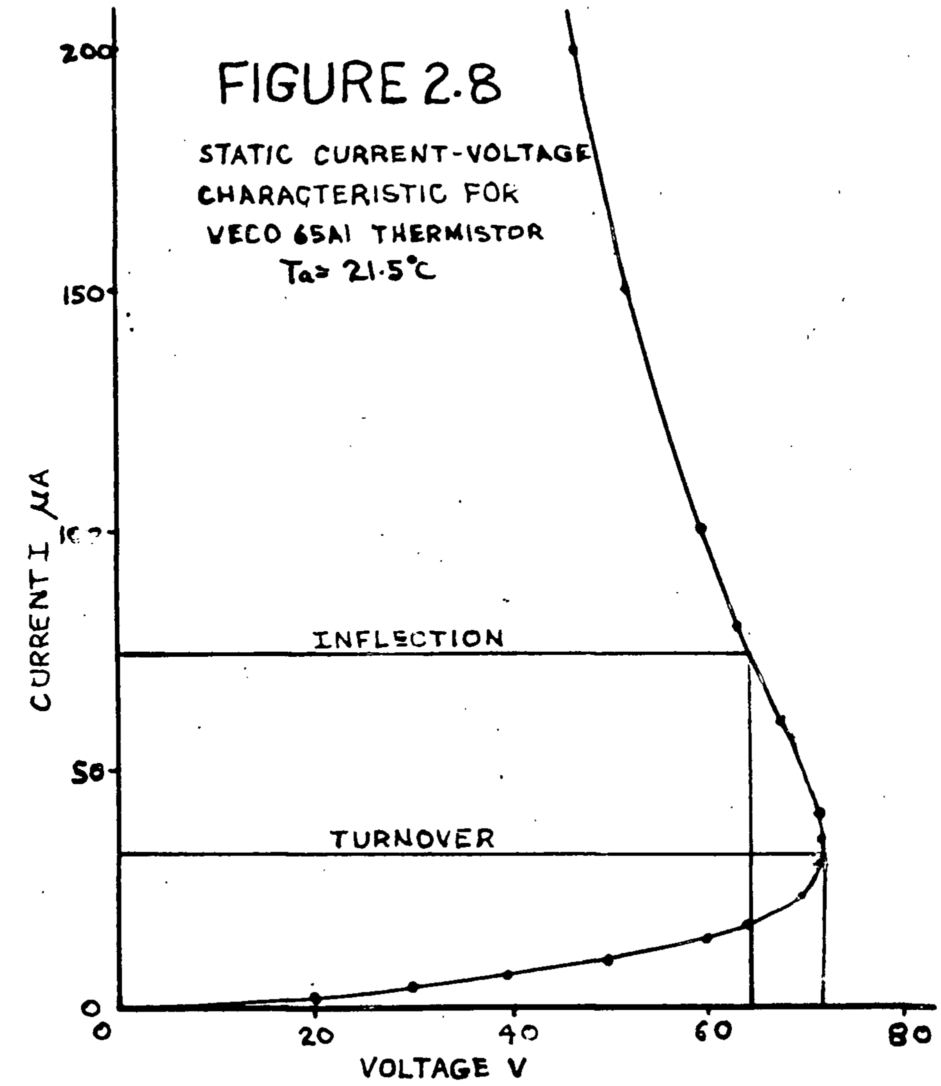


FIGURE 2.8
STATIC CURRENT-VOLTAGE
CHARACTERISTIC FOR
VECO 65A1 THERMISTOR
Ta= 21.5°C
INFLECTION
TURNOVER
CURRENT I μA
200
150
100
50
0
0
20
40
60
80
VOLTAGE V

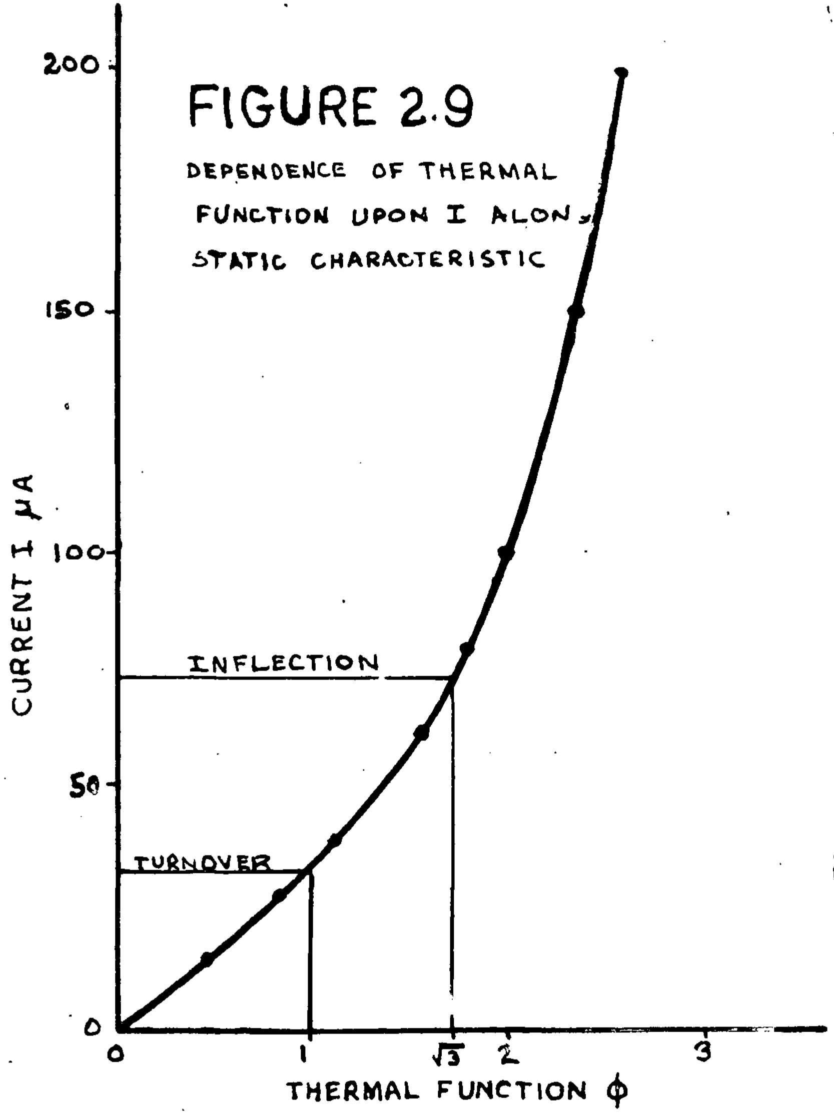
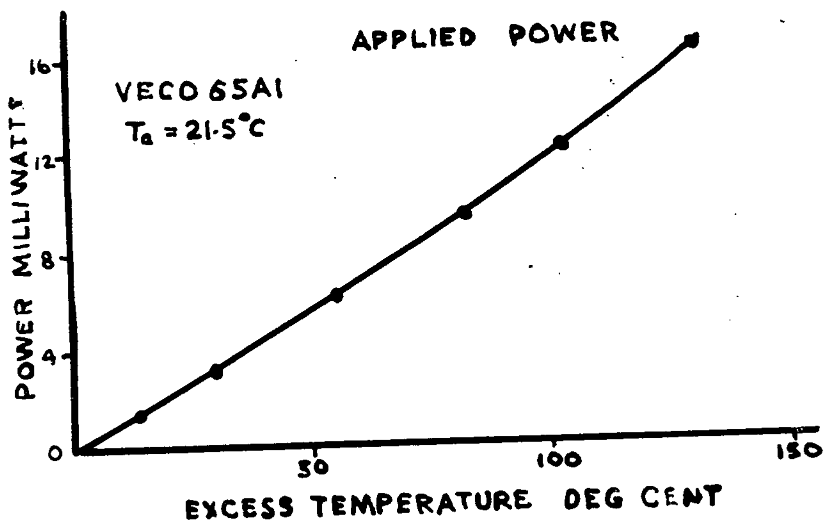


FIGURE 2.9
DEPENDENCE OF THERMAL
FUNCTION UPON I ALONG
STATIC CHARACTERISTIC
INFLECTION
TURNOVER
CURRENT I μA
200
150
100
50
0
0
1
√3
2
3
THERMAL FUNCTION φ

FIGURE 2.10 DEPENDENCE OF EXCESS TEMPERATURE UPON APPLIED POWER



$$\phi = \frac{b}{(Ta + |\Theta|)^2} \tag{2.10}$$

Figure 2.9 shows the dependence of ϕ upon the thermistor current I.

From the values of (V, I) at various points along the static curve, values of the power VI were calculated and a graph was made of the power - excess temperature relation along the static curve (see Figure 2.10). For the equation

$$VI = k|\Theta| \tag{2.6}$$

to be valid, this curve must be straight line. Deviations from a straight line might be due to:

1. Uneven heating of the thermistor bead resulting in the non-applicability of the log R vs T^{-1} relation measured with $T = Ta$ for $T \neq Ta$.
2. Heat losses due to radiation at high temperatures of the bead necessitating consideration of second or higher order terms in $|\Theta|$ of the relation

$$VI = k|\Theta| + k_R \left[(Ta + |\Theta|)^4 - Ta^4 \right] \tag{2.5}$$

3. Heating of the surroundings of the bead to such an extent that changes in the effective ambient temperature are not proportional to the excess temperature.
4. Heating of the surroundings of the bead resulting in a change of the thermal conductivity k.

It was noted that VI - vs - $|\Theta|$ curve had an increasing slope with increasing $|\Theta|$. Items 2 and 4 could have caused this.

Values of b, R_∞ and average slopes of the VI vs $|\Theta|$ curves for several thermistors were:

Thermistor	R ohms	b^oK	k $\frac{\text{milliwatts}}{^oK}$	Surroundings
Servotherm 1317	3.15	4160	1.1	Still air
VECO 65A3	3.11	4320	0.13	Still air
VECO 65A1	2.45	4310	.0094	Vacuum - as supplied by manufacturer
			0.12	Still air - with bulb opened to admit atmosphere

It was noted that the manufacturer's specifications give a value of 0.1 milliwatts per degree centigrade for k for both VECO type 65A3 and 65A1. This agrees with the measured value for the type 65A3 in still air. However, in the case of the type 65A1 which is in a sealed evacuated glass bulb there is a discrepancy unless the glass bulb is opened and the bead comes in contact with still air.

PART 3

SMALL AMPLITUDE TIME VARYING PHENOMENA IN THERMISTORS

1. Definition of Thermal Time Constant

If the non-steady-state condition is considered, it is evident that the balance of power can no longer be represented by the equation

$$VI = k\Theta$$

but rather the input electrical power must equal the sum of the dissipated power plus the rate at which thermal energy is being supplied to the thermistor. If the rate at which heat is lost is determined only by the instantaneous excess temperature Θ, the balance condition may be written

$$VI = k\Theta + H\frac{d\Theta}{dt} \qquad (3.1)$$

where H is the heat capacity of the thermistor at temperature $Ta + \Theta$. Several cases where the rate of heat loss is not dependent only upon Θ are considered by Burgess (Oct 1955). However, in the development here, the validity of (3.1) and the independence of H upon Θ will be assumed.

If a power $V_0 I_0 = k\Theta_0$ has been applied to the thermistor for such a time that all transients have died out and then the thermistor is open circuited, the excess temperature will be given by

$$\Theta = \Theta_0 \exp\left(-\frac{k}{H}t\right) \qquad (3.2)$$

The quantity $\frac{H}{k}$ is called the <u>thermal time constant of the thermistor</u> and will be denoted by τ . Equation (3.1) may be rewritten as

$$VI = k\left(\Theta + \tau \frac{d\Theta}{dt}\right). \tag{3.3}$$

2. Small Signal Differential Equation

The quantities v, i and θ will be defined by

$$\left.\begin{aligned} v &= V - V_o \\ i &= I - I_o \\ \theta &= \Theta - \Theta_o \end{aligned}\right) \tag{3.4}$$

where V_o, I_o and Θ_o are the values of V, I and Θ at some operating point. The quantities ϕ_o, μ_o and R_o are the respective values of ϕ , μ and R at the operating point so that

$$\left.\begin{aligned} \frac{V_o}{I_o} &= R_o = R_\infty \exp \frac{b}{Ta+\Theta_o} \\ V_o I_o &= k\Theta_o \\ \phi_o = \mu_o\Theta_o &= \frac{b\,\Theta_o}{(Ta+\Theta_o)^2} \end{aligned}\right) \tag{3.5}$$

If the first equation of (3.5) is expanded in a Taylor's series in θ , it is seen that

$$\frac{V_o + v}{I_o + i} = R_o + \theta\frac{\partial R}{\partial \theta} + \frac{\theta^2}{2}\frac{\partial^2 R}{\partial \theta^2} \cdots$$

If the condition

$$vi \ll V_o I_o$$

holds, only the first order terms of the expansion need be considered. Solving thus for θ , it is found that

$$\theta = \frac{1}{\mu_o}\left(\frac{v}{V_o} - \frac{i}{I_o}\right) \tag{3.6}$$

and

$$\frac{d\theta}{dt} = \frac{1}{\mu_o}\left(\frac{1}{V_o}\frac{dv}{dt} - \frac{1}{I_o}\frac{di}{dt}\right). \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.3) and expanding VI in terms of v and i add keeping only first order terms it is found that

$$V_o I_o + v I_o + i V_o + \cdots = k\left[\Theta_o + \frac{1}{\mu_o}\left(\frac{v}{V_o} - \frac{i}{I_o}\right) + \frac{\tau}{\mu_o}\left(\frac{1}{V_o}\frac{dv}{dt} - \frac{1}{I_o}\frac{di}{dt}\right)\right].$$

or since $V_o I_o = k\Theta_o$ and $\phi_o = \mu_o \dfrac{V_o I_o}{k}$,

$$\frac{v}{V_o}(1+\phi_o) + \frac{\tau}{V_o}\frac{dv}{dt} = \frac{i}{I_o}(1-\phi_o) + \frac{\tau}{I_o}\frac{di}{dt}. \tag{3.8}$$

This is the general differential equation governing small changes in V and I.

3. Response to Sinusoidal Input

If $v = V_1 e^{jwt}$ and $i = I_1 e^{jwt}$ where V_1 and I_1 may be complex and where $w = 2\pi \times$ frequency f and if $Z = \dfrac{V_1}{I_1}$, it is found from (3.8) that

$$Z = R_o\left[\frac{1-\phi_o+jwt}{1+\phi_o+jwt}\right] \tag{3.9}$$

is the relation for the small - signal driving - point impedance of the thermistor at any operating defined by R_o and ϕ_o.

If

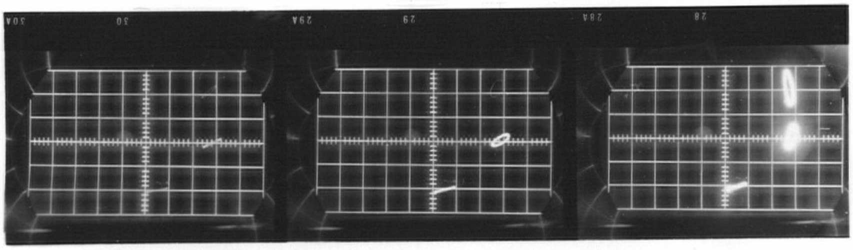
$$Z = R(w) + jX(w)$$

it is found that

$$\frac{R(w)}{R_o} = \frac{1-\phi_o^{\,2} + w^2\tau^2}{(1+\phi_o)^2 + w^2\tau^2} \tag{3.10}$$

and

$$\frac{X(w)}{R_o} = \frac{2\phi_o w\tau}{(1+\phi_o)^2 + w^2\tau^2}. \tag{3.11}$$



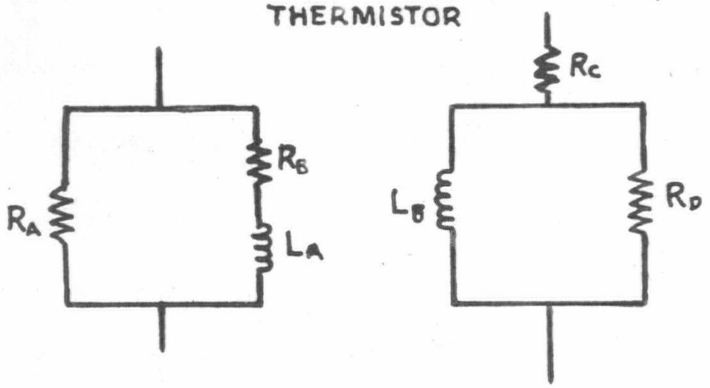
f=2.5cps f=.5cps f=.1cps

T_a=23°C HORIZONTAL 1DIV=10V VERTICAL 1DIV=10μA

FIGURE 3.1 CURRENT VOLTAGE LOCI WITH APPLIED SINUSOID

FIGURE 3.2

SMALL AMPLITUDE AC EQUIVALENT CIRCUITS OF THERMISTOR



$$R_A = R_o \qquad R_C = \frac{1-\Phi_o}{1+\Phi_o} R_o$$

$$R_B = R_o \frac{1-\Phi_o}{2\Phi_o} \qquad R_D = \frac{2\Phi_o}{1+\Phi_o} R_o$$

$$L_A = \frac{\tau R_o}{2\Phi_o} \qquad L_B = \frac{2\tau\Phi_o}{1+\Phi_o} R_o$$

If $w\tau >> 1$ it is clear that $R(w) = R_o$ and $X(w) = 0$. This is the condition that the applied power is varying so rapidly that the thermal inertia of the thermistor causes the excess temperature to remain essentially constant over a whole cycle at the value

$$\Theta_o = \frac{V_o I_o}{k}$$

if

$$\overline{vi} << V_o I_o .$$

The impedance then corresponds to R_o, the reciprocal of the slope of the isothermal passing through the operating point. If $w\tau << 1$ it is clear that

$$R(w) = R_o \frac{(1 - \phi_o)}{(1 + \phi_o)} = r$$

and $X(w) = 0$. This is the condition that the applied power VI is varying so slowly that the temperature essentially reaches its equilibrium value at each point of the cycle, thus the current - voltage locus follows the static characteristic and the impedance is equal to the slope of the static characteristic at the operating point. At medium frequencies where w is neither much larger nor much smaller than $\frac{1}{\tau}$, the current will lag the applied voltage and the current - voltage locus becomes an ellipse. Photographs of current - voltage loci at different operating points and frequencies are shown in Figure 3.1.

Several equivalent circuits are immediately apparent from the form of the impedance function and are shown in Figure 3.2.

Regarding the impedance equation

$$\frac{Z}{R_o} = \frac{1-\phi_o^2 + w^2\tau^2}{1+\phi_o^2 + w^2\tau^2} + j\frac{2\phi_o w\tau}{(1+\phi_o)^2 + w^2\tau^2},$$

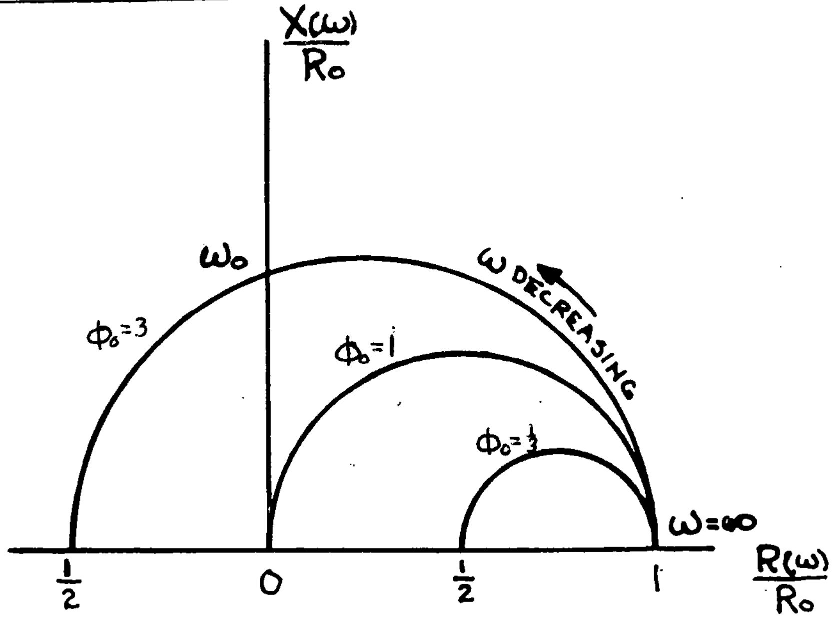
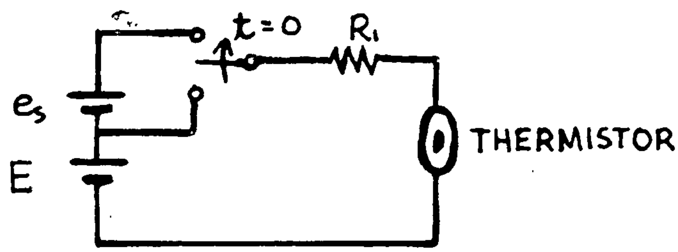


FIGURE 3.3 LOCI OF

$$\frac{1-\Phi_o+j\omega\tau}{1+\Phi_o+j\omega\tau}$$

FIGURE 3.4

CIRCUIT FOR STEP INPUT



it is seen that its representation in the $\frac{Z}{R_o}$ plane is a series of semicircles with ϕ_o as a parameter (see Figure 3.3). The infinite frequency value of $\frac{Z}{R_o}$ is real and has the magnitude unity. The zero frequency value is $\frac{r_o}{R_o}$ and is also real. If $\phi_o > 1$ and hence $r_o < 0$; i.e. the thermistor is biased beyond turnover, it is noted that R (w) becomes negative for all frequencies below some critical frequency

$$f_o = \frac{w_o}{2\pi} = \frac{(\phi_o^2 - 1)^{\frac{1}{2}}}{2\pi\tau}. \qquad (3.12)$$

At this frequency f_o the real part of the impedance is zero and the thermistor behaves like a pure, positive reactance given by

$$Z(w_o) = jR_o \left(\frac{\phi_o - 1}{\phi_o + 1}\right)^{\frac{1}{2}} = j(-r_o R_o)^{\frac{1}{2}}. \qquad (3.13)$$

4. Response to Step Input

The response of the thermistor to a small step function of voltage will now be considered. Since linear theory will be used, the value of the change of voltage e_s applied to the thermistor and load resistor R_1 in series must be small enough that

$$e_s \ll V_o$$

holds. The circuit will be of the form shown in Figure 3.4. The relation

$$e_s = iR_1 + v \qquad (3.14)$$

will hold for this circuit. Combining this with the general equation

$$\tau\frac{dv}{dt} + v(1 + \phi_o) = R_o\left[\tau\frac{di}{dt} + i(1 - \phi_o)\right] \qquad (3.8)$$

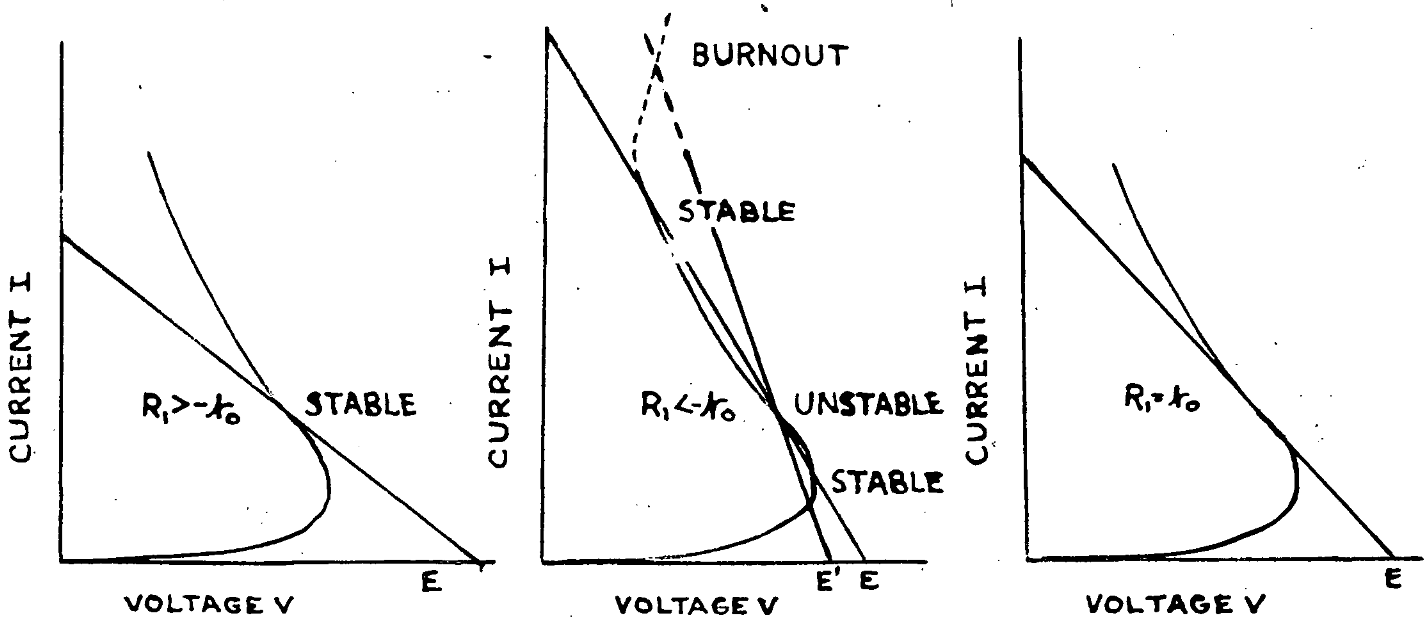


FIGURE 3.5 INTERSECTIONS OF LOAD LINE AND STATIC CHARACTERISTIC

a first order differential equation in i is obtained which may be easily integrated to give for the pertubation of the current

$$i = \frac{e_s(1+\phi_0)}{R_0(1-\phi_0)+R_1(1+\phi_0)}$$

$$\text{constant} \exp - \left[\frac{R_0(1-\phi_0)+R_1(1+\phi_0)}{R_0+R_1}\right]\frac{t}{\tau}$$

If τ' is defined by

$$\tau' = \tau\frac{(R_0+R_1)}{R_0(1-\phi_0)+R_1(1+\phi_0)}$$

and the initial condition

$$i = \frac{e_s}{R_1+R_0}$$

at $t = 0$ is applied, the equation for the current i becomes

$$i = \frac{e_s}{R_0(1-\phi_0)+R_1(1+\phi_0)}\left(1+\phi_0 \quad \frac{-2\phi_0 R_0}{R_0+R_1}e^{-\frac{t}{\tau'}}\right). \tag{3.15}$$

It is interesting to note that if

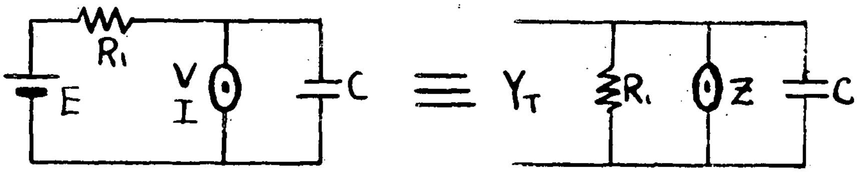
$$R_1 < -R_0\frac{(1-\phi_0)}{(1+\phi_0)} = -r_0$$

then

and the exponent in (3.15) becomes positive so that the circuit becomes unstable. If the thermistor is biased at a point on the static characteristic where $R_1 < -r_0$, the load line will cross the static characteristic in at least one other place (see Figure 3.5) and any pertubation e_s will cause the system to jump to one of the other intersections of the load line and the static characteristics. For certain values of E and R_1, the upper intersection exists only theoretically since the hypothetical value of the corresponding temperature is above the melting point

FIGURE 3.6

CIRCUIT FOR OSCILLATIONS



ACTUAL CIRCUIT

AC EQUIVALENT

of the thermistor material. If the thermistor tends to jump to such a point it of course burns out. If $R_1 = -r_o$, the denominator of (3.15) vanishes indicating that under this condition the linear terms are insufficient and that higher order terms must be considered in the expansion of V, I and Θ.

A similar equation to (3.15) involving the change of voltage v may be developed and is

$$v = \frac{e_s \quad R_o}{R_1 (1+\phi_o) + R_o (1-\phi_o)} \left(1 - \phi_o + \frac{2\,\phi_o R_1 e^{-\frac{t}{\tau'}}}{R_o + R_1}\right). \tag{3.16}$$

5. Small Amplitude Sinusoidal Oscillations

The circuit to be considered is shown in Figure 3.6. The development will follow that of Burgess (Nov 1955). The thermistor will be assumed to be biased beyond turnover and the amplitude of oscillation will be assumed to be small so that the linear equations will be valid. The admittance of the circuit is of the form

$$Y_T = \frac{1}{R_1} + jwC + \frac{1}{R_o}\left(\frac{1+\phi_o + jw\tau}{1-\phi_o + jw\tau}\right) \tag{3.17}$$

or if

$$Y_T = G_T(w) + j\,B_T(w),$$

then

$$G_T(w) = \frac{1}{R_1} + \frac{1}{R_o}\left(\frac{1+\phi_o^2 + w^2\tau^2}{(1-\phi_o)^2 + w^2\tau^2}\right)$$

$$B_T(w) = wC - \frac{1}{R_o}\left(\frac{2w\,\tau\,\phi_o}{(1-\phi_o)^2 + w^2\tau^2}\right).$$

The conditions for oscillations to occur are

$$\left.\begin{aligned} G_T(w) &\leq 0 \\ B_T(w) &= 0 \end{aligned}\right) \tag{3.18}$$

For this development to be valid the oscillations must be small so that non-linearities are small. Thus the first relation of (3.18) should approach the equality as closely as possible. Applying equations (3.18) to the admittance in (3.17) it is seen that

$$C \geq C_{min} = \frac{\tau}{\phi_o - 1}\left(\frac{1}{R_1} + \frac{1}{R_o}\right) \qquad (3.19)$$

and

$$w = \frac{1}{\tau}\left[\frac{2\tau\phi_o}{C\,R_o} - (1 - \phi_o)^2\right]^{\frac{1}{2}}. \qquad (3.20)$$

An upper limit of oscillation frequency corresponding to w_{max} occurs when the d - c supply is a perfect constant - current source; i.e., $\frac{1}{R_o} = 0$, and C_{min} has its lowest possible value

$$C_{min} = \frac{\tau}{(\phi_o - 1)}\,\frac{1}{R_o}$$

and

$$w_{max} = \frac{(\phi_o{}^2 - 1)^{\frac{1}{2}}}{\tau}$$

It is noted that w_{max} corresponds to the w_o defined in (3.12).

6. Experimental

Early in the programme of research, attempts were made to verify the semi-circular locus of the impedance function and to measure the thermal time constant of the thermistor flakes in several Servotherm type 1317 bolometer units. Several differant types of bridges were set up but found to be unsuccessful because of the difficulties associated with the high impedance levels and high voltages involved in these thermistors. Finally, a bridged - T

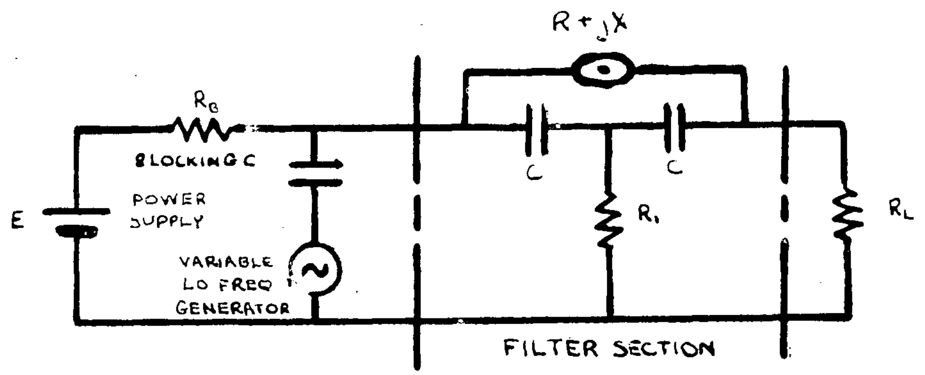
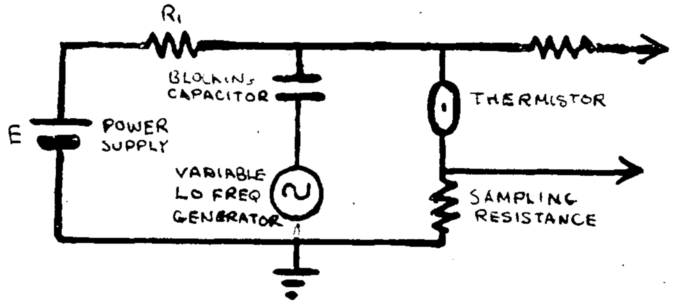
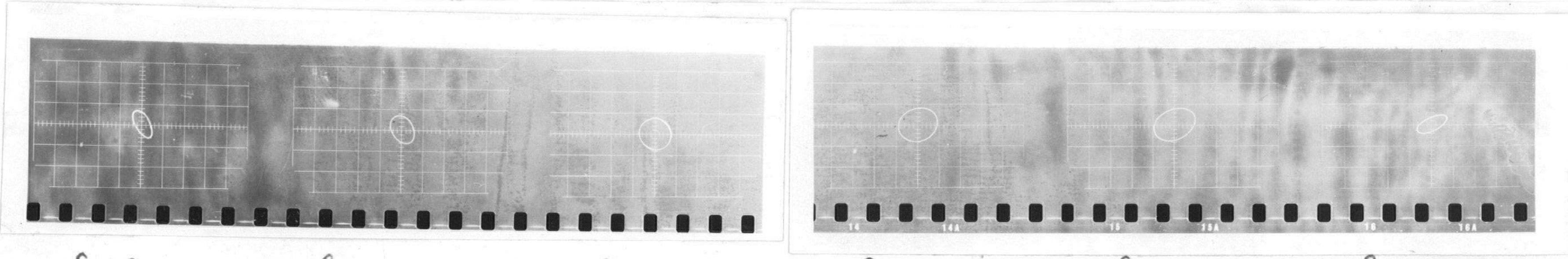


FIGURE 3.7 BRIDGED T CIRCUIT FOR MEASURING IMPEDANCE

FIGURE 3.8
CIRCUIT FOR OBSERVING CURRENT-VOLTAGE LOCI





$f=0.2$ cps $f=0.3$ cps $f=0.4$ cps $f=0.5$ cps $f=0.6$ cps $f=1.0$ cps

FIGURE 3.9 PHOTOGRAPHS OF CURRENT-VOLTAGE LOCI TO SHOW EFFECT OF CHANGE OF FREQUENCY OF APPLIED SIGNAL WHEN THERMISTOR IS BIASED BEYOND TURNOVER

$I_o = 100\mu A$

$V_o = 60V$

$T_a = 23°C$

VOLTAGE - VERTICAL

CURRENT - HORIZONTAL

filter of the type shown in Figure 3.7 was chosen. In this bridge the resistors R_B, R_L and the supply voltage E determine the operating point (V_o, I_o). None of E, R_B and R_L need be varied during a - c measurements.

Measurements of the impedance were made over a range of frequency from 5 to 100 cps. At several different operating points the measured impedance was essentially the value of the reciprocal of the slope of the isothermal through the operating point; i.e., $Z(w) \simeq R_o$, $X = 0$. This led to the conclusion that manufacturer's value of 0.01 sec for τ was much smaller than the actual which must have been at least 0.1 sec.

Later in the research programme the thermal time constant was measured of a VECO type 65A1 (with its glass tube opened), using two different methods. The first method consisted of applying a variable-frequency, small-amplitude, a - c signal to the thermistor biased beyond turnover and observing the voltage - current locus on an oscilloscope as the frequency of the generator was varied (see Figure 3.8). The loci were ellipses whose axe's' direction varied with frequency (see Figure 3.9). At the frequency where the axes were vertical and horizontal the thermistor behaved like a pure reactance and the relation

$$f_o = \frac{1}{2\pi\tau} \left(\phi_o^2 - 1 \right)^{\frac{1}{2}} \qquad (3.12)$$

or

$$\tau = \frac{1}{2\pi f_o} \left(\phi_o^2 - 1 \right)$$

was known to hold. The value of τ at different operating points were

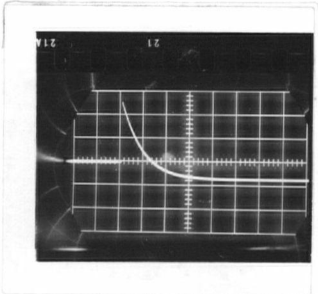


FIGURE 3.10 RESPONSE TO STEP INPUT WHEN THERMISTOR IS BIASED BEYOND TURNOVER

I_o (μAmps)	τ (sec)
90	.52
100	.61
125	.56
150	.63
200	.63
250	.62
300	.64

The other method consisted of biasing the thermistor beyond turnover and applying a small and instantaneous pertubation e_s to the supply voltage (see Figure 3.4). The change of voltage v across the thermistor follows

$$v = \frac{e_s R_o}{R_1(1+\phi_o)+R_o(1-\phi_o)} \left(1-\phi_o+\frac{2\phi_o R_1}{R_o+R_1} e^{-\frac{t}{\tau'}}\right) \tag{3.16}$$

At time t = 0, v will jump from v = 0 to

$$v = \frac{e_s R_o}{R_1+R_o}$$

As t increases, v decreases (see Figure 3.10) to a negative value if $\phi_o > 1$. Then the condition v = 0 will occur at t = t_o given by

$$1-\phi_o+\frac{2\phi_o R_1}{R_o+R_1} e^{-\frac{t_o}{\tau'}} = 0,$$

or

$$\tau = \left[\frac{R_1(1+\phi_o)+R_o(1-\phi_o)}{R_1+R_o}\right] \frac{t_o}{\ln \frac{2\phi_o R_1}{(R_o+R_1)(\phi_o-1)}} \tag{3.21}$$

Using (3.21) the value of τ was calculated at several operating points

I_0 (μAmps)	τ (sec)
50	.67
100	.64
200	.58

Averaging the values of τ measured by both these methods gives τ = .61 sec compared to the manufacturer's value of τ = 1 sec.

PART 4

THEORY OF NON LINEAR OSCILLATIONS

1. General

In this section, the various methods of examination of systems which can be described by two first - order differential equations of the form

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned}\right) \qquad (4.1)$$

will be discussed. This is an autonomous dynamical system since P and Q are not explicit functions of time. It is noted that the general second order differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt}\, f\left(x, \frac{dx}{dt}\right) + g\left(x, \frac{dx}{dt}\right) = 0 \qquad (4.2)$$

can be transformed to (4.1) thus

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = f(x, y) - g(x, y) \,.$$

2. Phase Planes

Suppose a solution of (4.1) exists of the form

$$x = x(t)$$
$$y = y(t)$$

Then since this is a dynamical system with two degrees of freedom and since the state of the system at time t is fixed by the values (x, y), these two quantities may be considered as coordinates of a phase plane. To each state of the system there corresponds a point M (x, y) which is known as a representative point. As t varies M will describe a curve called a trajectory or integral curve which is a description of the history of the system. A complete trajectory, of course, represents the history of system throughout all time. The totality of all paths of M represents all possible histories of the system, any one of which is determined by a single point. Thus with the possible exception of the intersection of the curves P = 0 and Q = 0, only one path may pass through any one representative point. The velocity of the point M along a trajectory is the phase velocity of the system. It is a plane vector with P and Q its components in the y and x directions. Its direction is given by

$$\text{arc tan } \frac{P(x, y)}{Q(x, y)} = \text{arc tan } \frac{dy}{dx}$$

at every point where P and Q do not vanish simultaneously. The locus of points where $\frac{dy}{dx}$ is constant is called an isocline. The points where P and Q vanish simultaneously are called singular points.

3. Singular Points

3.a General

Since singular points are at the intersection of the curves P = 0 and Q = 0, all velocities of the system vanish and the system is in a rest or equilibrium position there. The nature of a singular point reveals much qualitative information regarding the solution of the differential equation, thus some general theorems regarding singular points will be quoted.

Poincaré (1892) has shown that the differential equation

$$\frac{dy}{dx} = \frac{ax + by + P_2(x,y)}{cx + dy + Q_2(x, y)}$$

in which $\Delta = ad - bc \neq 0$ and in which P_2 and Q_2 approach zero like $x^2 + y^2$ has its only singularity at the origin and the behaviour at the origin is similar to that of the linear differential equation

$$\frac{dy}{dx} = \frac{ax + by}{cx + dy} \qquad (4.3)$$

If a differential equation is of the form

$$\frac{dy^1}{dx^1} = \frac{A + ax^1 + by^1}{B + cx^1 + dy^1}, \qquad (4.4)$$

the only singularity is at the point (x_o, y_o) where

$$x_o = \frac{\begin{vmatrix} -A & b \\ -B & d \end{vmatrix}}{\Delta} \qquad y_o = \frac{\begin{vmatrix} a & -A \\ c & -B \end{vmatrix}}{\Delta}$$

and

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

If the transformations

$$x = x^1 - x_o$$

$$y = y^1 - y_o$$

are made (4.4) is reduced to the form of (4.3). Since Poincare (1892) has shown that the nature of a singularity is preserved in undergoing linear transformations, there is no loss in generality in considering (4.3) instead of (4.4).

The general solution of (4.3) is

$$(x - \alpha_1 y)^{\lambda_1} (x - \alpha_2 y)^{\lambda_2} = \text{constant} \qquad (4.5)$$

where

$$\lambda_{1,2} = \frac{b + c \pm \sqrt{(b-c)^2 - 4\,ad}}{2} \tag{4.6}$$

and

$$\alpha_{1,2} = b - c \pm \sqrt{(b-c)^2 + 4\,ad} \tag{4.7}$$

3.b Special Cases of $\frac{dy}{dx} = \frac{ax + by}{cx + dy}$

It can be shown (Stoker 1950 Chap. III) that any form of (4.3) can be reduced to certain special cases by linear transformations of x and y. These special cases and the names given to their singularities are listed in the following table.

	Condition	Differential Equation	Solution	Type of Singularity at Origin
1.	$a = d = 0$ $bc > 0$ $m = \frac{b}{c}$	$\frac{dy}{dx} = m\,\frac{y}{x}$	$y = y_o\left(\frac{x}{x_o}\right)^m$	node
2.	$a = d = 0$ $bc < 0$ $\frac{b}{c} = -n$	$\frac{dy}{dx} = -n\,\frac{y}{x}$	$yx^n = y_o x_o^{\,n}$	saddle
3.	$b = c = 0$ $ad < 0$ $\frac{a}{d} = -p^2$	$\frac{dy}{dx} = -p^2\,\frac{x}{y}$	$y^2 + p^2x^2 = y_o + p^2x_o^2$	center
4.	$a\ d = 0$ $b - c = 0$ $b = qa$ $q > 0$	$\frac{dr}{d\theta} = qr$ If $x = r\cos\theta$ $y = r\sin\theta$	$r = r_o \exp q(\theta - \theta_o)$	spiral
5.	$a = b = c$ $d = 0$	$\frac{dy}{dn} = \frac{x+y}{y}$	$y = x\left(\frac{y_o}{x_o} + \ln\left\lvert\frac{x}{x_o}\right\rvert\right)$	node

3.c Stability and Classifications of Singular Points

A singular point is stable if there exists some neighbourhood around the singularity inside of which all representative points approach the singularity as t increases. A singular point is unstable if there exists some neighbourhood around the singularity inside of which all representative tend to leave the neighbourhood as t increases. Using these criteria for stability, the following classifictions can be made of the types of singularities of (4.3).

I	$D > 0$	(A) Node if $\Delta < 0$	(i) Stable if $b+c < 0$ (ii) Unstable if $b+c > 0$
		(B) Saddle if $\Delta > 0$	
II	$D < 0$	(A) Center if $b+c = 0$	
		(B) Spiral if $b+c \neq 0$	(i) Stable if $b+c < 0$ (ii) Unstable if $b+c > 0$
III	$D = 0$	(A) Node	(i) Stable if $b+c < 0$ (ii) Unstable if $b+c > 0$

where $D = (b - c)^2 + 4\,ad$; $\Delta = ad - bc$

3.d Examples of Singular Points

It is instructive to note that the singularity of the equation for the linear harmonic oscillator exhibits all the possible types of singularities, except a saddle, if negative damping is allowed. The equation for such a system are

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + s^2x = 0$$

and

$$\frac{dy}{dt} = -s^2x - 2ry$$

$$\frac{dx}{dt} = y$$

which are of the form of (4.3) if

$$a = -s^2 \qquad b = -2r$$

$$c = 0 \qquad d = 1 .$$

It is seen that

$$b+c = 2r$$

$$\Delta = -s^2$$

$$D = 4r^2 - s^2 .$$

The following table illustrates the nature of the singularities for different values of r and s.

Singularity	Condition	Damping	Solution
Stable Node	$r \geq s$	Positive, greater or equal to critical	$c_1 e^{(-r-\sqrt{r^2-s^2})t} + c_2 e^{(-r+\sqrt{r^2-s^2})t}$ or $(c_1 t + c_2) e^{-rt}$
Stable Spiral Point	$0 > r > s$	Positive less than critical	$e^{-rt} c_1 \cos(\sqrt{s^2-r^2} + c_2) t$
Center	$r = 0$	None	$c_1 \cos(st + c_2)$
Unstable Spiral Point	$0 > -r > s$	Negative, less than critical	$e^{rt} c_1 \cos(\sqrt{s^2-r^2} + c_2) t$
Unstable Node	$-r \geq s$	Negative, greater or equal to critical	$c_1 e^{(r-\sqrt{r^2-s^2})t} + c_2 e^{(r+\sqrt{r^2-s^2})t}$ or $(c_1 t + c_2) e^{rt}$

An example of a saddle point is the singularity of the equation governing the behaviour of a simple, rigid pendulum near its uppermost position. This equation is

$$\frac{d^2 x}{dt^2} - q^2 \sin x = 0$$

or

$$\frac{d^2 x}{dt^2} - q^2 x = 0$$

keeping only linear terms. It is apparent

$$\Delta = -q^2 < 0$$

and

$$D = 4q^2 > 0$$

which are the conditions for a saddle.

4. Oscillations

4.a General

A closed trajectory in a phase plane corresponds to a periodic phenomena in the system represented by the phase plane. This statement follows from the fact that a representative point on the closed curve returns to the same position after some time T; i.e.,

$$x\,(t + T) = x\,(t)$$

$$y\,(t + T) = y(t)$$

The symbol T will be used to denote period in the sequel and will not be confused with T the absolute temperature. The period T of an oscillatory process may in principle be calculated from the line integral

$$T = \oint \frac{dx}{P} = \oint \frac{dy}{Q}$$

where the path of integration is to be taken around the closed trajectory in the direction of increasing time.

It can be shown (Stoker 1950 Chap. III) that inside a closed trajectory the number of saddle points must be one less than the sum of the number of nodes, centers and spiral points. Thus it follows that inside a closed trajectory there must exist one node, center or spiral point.

If all the representative points in a region of a phase plane tend to a single closed trajectory free of singular points as t increases that trajectory is called a limit cycle.

4.b Relaxation Oscillations

In most oscillatory systems of (4.1), P and Q involve parameters of the system which may be varied; e.g., the factor ϵ in the Van der Pol equation

$$\frac{dy}{dt} = \epsilon\left(y - \frac{y^3}{3}\right) - x$$

$$\frac{dx}{dt} = y$$

For certain values of the parameters; e.g., $\epsilon \ll 1$, the oscillations are very nearly sinusoidal. As the parameters change; e.g., ϵ is increased, the oscillations may ultimately become characterized by two distinct epochs; e.g., $\epsilon > 10$, one in which energy is stored up and one in which energy is discharged nearly instantaneously. Oscillations of this nature are called relaxation oscillations.

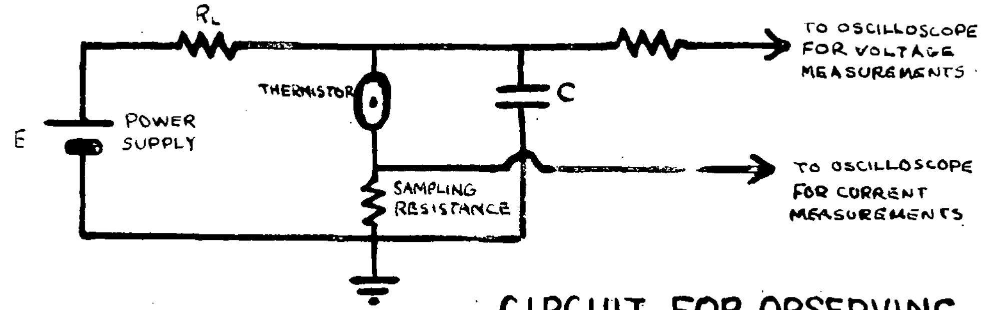


FIGURE 5.1

CIRCUIT FOR OBSERVING OSCILLATIONS

PART 5

STUDY OF OSCILLATIONS IN THERMISTORS

1. Experimental Techniques

The basic circuit employed was that of Figure 5.1 which has been shown in 5 of Part 3 to oscillate if E and R_1 are such that the thermistor is biased beyond turnover and C is greater than some minimum value.

Great difficulty was experienced in obtaining thermistors both sufficiently robust to withstand the extremely high temperatures during large - amplitude oscillations and having sufficiently low value of the ratio $\frac{\tau}{R_o}$ so that

$$C_{min} = \frac{\tau}{(\phi_o - 1)}\left(\frac{1}{R_o} + \frac{1}{R_1}\right) \qquad (3.19)$$

was easily obtainable using oil and paper condensers. Also difficulties were met because of the apparent errors in the manufacturer's specifications of τ and k in several thermistors (see 3 of Part 2 and 6 of Part 3). The final choice of a thermistor type for experimental investigation was made by biasing a number of thermistors beyond turnover and increasing the parallel capacitance in each case until sustained oscillations were observed and then choosing the one requiring the least capacitance. The type chosen was a VECO type 65A3 which was later substituted by a VECO type 65A1 which has, as explained in 3 of Part 1, after opening its glass tube, similar electrical and thermal properties to the 65A3, but is somewhat better protected from mechanical damage. The measured values of its parameters were

$k = 0.13 \text{ m W deg}^{-1}$

$\tau = 0.61 \text{ sec}$

$R = 3.45 \text{ ohms}$

$b = 4310^{\circ}K$

Oscillations were observed on an oscilloscope in the V - I plane and in the time domain, i.e., V (t) and I (t). A current sampling resistor of 1 Kohm was inserted in series with the thermistor for current measurements and a large resistor was used in series with the oscilloscope for voltage measurements. The effect of the 1 Kohm resistance was found to be negligible except at very large instantaneous currents occuring when the thermistor experienced its peak of temperature rise. An arbitrary current maximum of 1.5 mA, corresponding to $|\Theta| \simeq 150^{\circ}C$ on the static curve, was set on the current during oscillations to avoid risk of burn - out. The effect of loading of the shunt resistance of the oscilloscope circuit for voltage measurements was calculated and the appropriate adjustments to R_1 and E were made to preserve a known operating point.

2. Limited Applicability of Linear Theory of Oscillations

In Part 3 an equation was derived relating the frequency f of small - amplitude sinusoidal oscillations to the parallel capacitance C,

$$2\pi f = w = \frac{1}{\tau}\left[\frac{2\tau\phi_o}{CR_o} - (1 - \phi_o)^2\right]^{\frac{1}{2}} \qquad (3.20)$$

when

$$C \geq C_{min} = \frac{\tau}{(\phi_o - 1)}\left(\frac{1}{R_o} + \frac{1}{R_1}\right). \qquad (3.19)$$

If the period of oscillation $T = \frac{1}{f}$,

$$T = \frac{2\pi\tau}{\left[\frac{2\phi_o\tau}{CR_o} - (1 - \phi_o)^2\right]^{\frac{1}{2}}} \qquad (5.1)$$

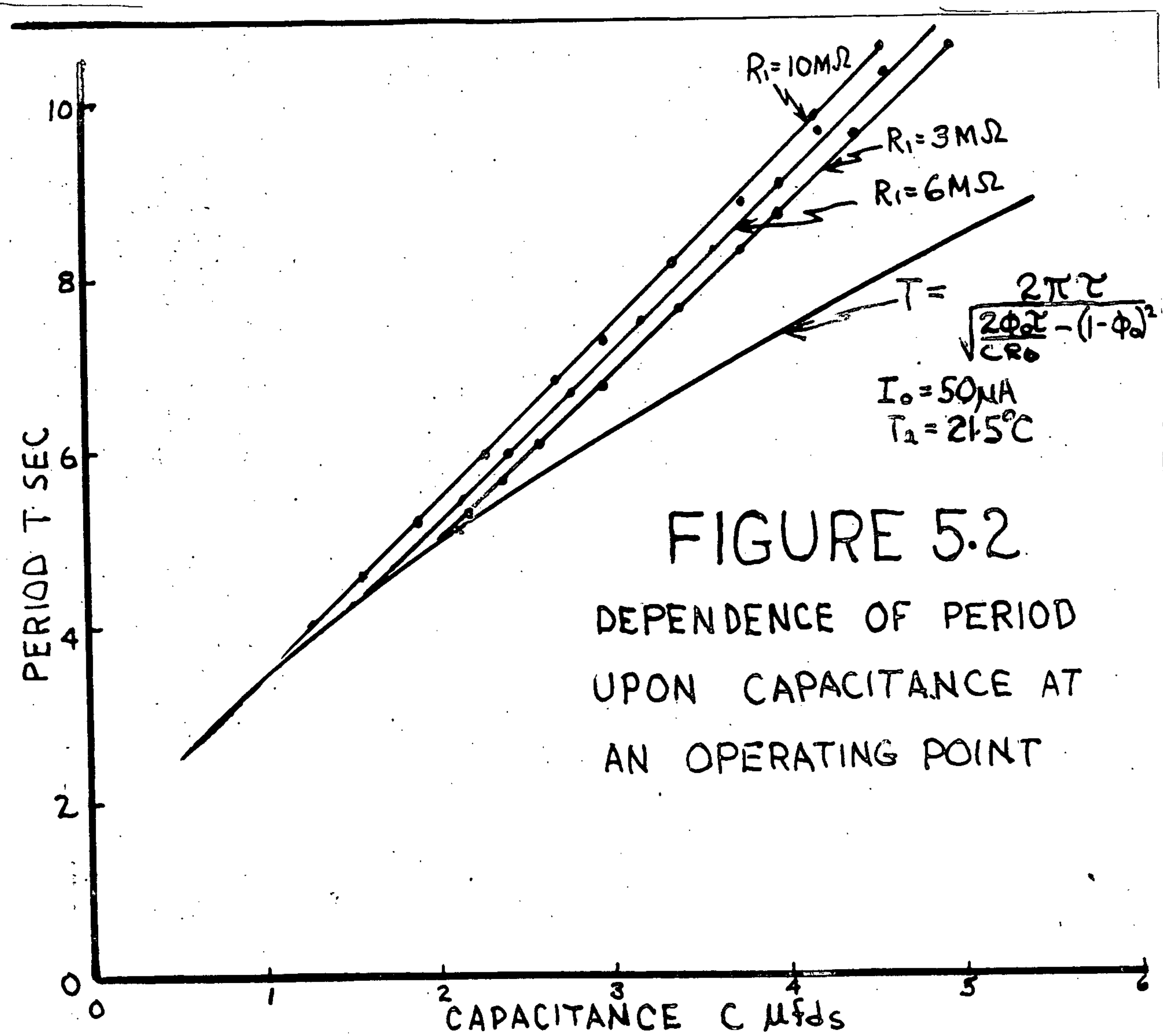


FIGURE 5.2

DEPENDENCE OF PERIOD UPON CAPACITANCE AT AN OPERATING POINT

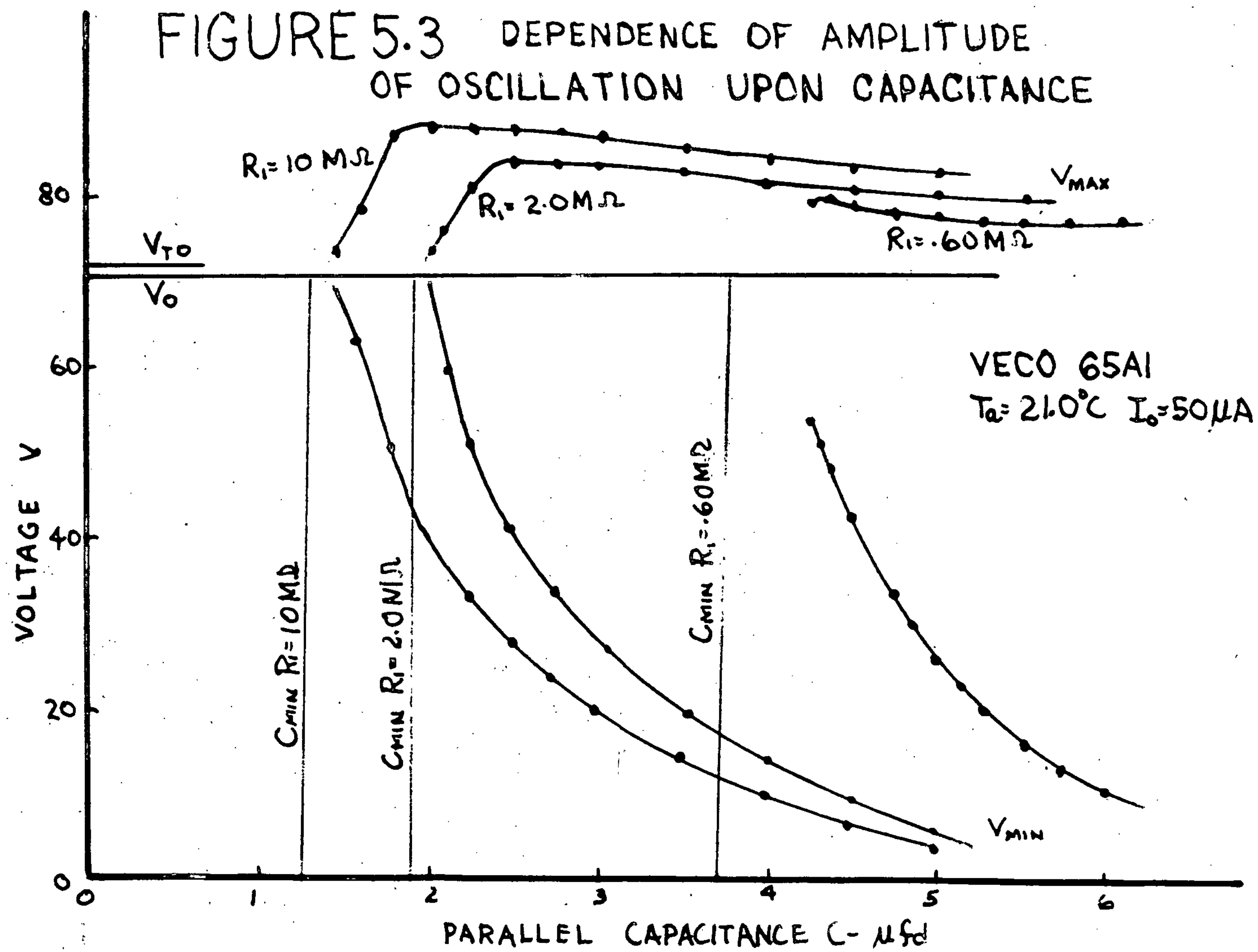


FIGURE 5.3 DEPENDENCE OF AMPLITUDE OF OSCILLATION UPON CAPACITANCE
R_1 = 10 MΩ
R_1 = 2.0 MΩ
R_1 = .60 MΩ
V_{MAX}
V_{TO}
V_0
V_{MIN}
C_{MIN} R_1 = 10 MΩ
C_{MIN} R_1 = 2.0 MΩ
C_{MIN} R_1 = .60 MΩ
VECO 65A1
T_a = 21.0°C I_0 = 50 μA
VOLTAGE V
PARALLEL CAPACITANCE C – μfd
0
20
40
60
80
1
2
3
4
5
6

Data were taken to show the dependence of T upon C at a fixed operating point for several different values of R_1 and was compared to (5.1) (see Figure 5.2). The curves nearly coincided for $C \simeq C_{min}$; i.e., where the amplitude of oscillation was small. Naturally a marked divergence was noted as the value of C was increased since (3.20) and (3.19) are valid only for infitesimal oscillations; i.e., there is only one value of C and f for a given operating point and R_1.

Data were also taken to show the relation of the amplitude of the voltage waveform and the value of C. The difference between the voltage maximum V_{max} and the voltage minimum V_{min} indicates that for only small increases of C beyond C_{min} the voltage amplitude becomes very large and the voltage variation is very unsymmetrical relative to the value of the voltage at the operating point V_o (see Figure 5.3).

These two results point out the very limited range of C in which the linear theory is applicable and also the inadequacy of applying pertubation methods to the linear theory.

3. Phase Planes for Thermistor Oscillations

3.a. General

The general problem of oscillation consists of the simultaneous solution of the two thermistor equations

$$\frac{V}{I} = R_\infty \exp \frac{b}{T_a + \Theta} = R(\Theta) = \frac{1}{G(\Theta)} \qquad (2.1)$$

$$P = VI = k\left(\Theta + \tau \frac{d\Theta}{dt}\right) \qquad (3.3)$$

and the circuital relation

$$E = V + IR_1 + CR_1 \frac{dV}{dt} . \tag{5.2}$$

It is seen that there are many pairs of variables for which phase planes exist. Among them are those listed in the following table.

Variables	Static Characteristic	Load Line	Comments
I, V	$\frac{V}{I} = R_{\infty} \exp \frac{b}{Ta + \frac{VI}{k}}$	$E = IR_1 + V$	Variables easily measured. Load line is a straight line.
θ, V	$V^2 = k\theta R(\theta)$	$E = V\left[1 + R_1 G(\theta)\right]$	Simplest form of differential equation
P, θ	$P = k\theta$	$E = \left[R(\theta)\, P\right]^{\frac{1}{2}} \times \left[1 + R_1 G(\theta)\right]$	Static characteristic is straight line. Differential equation is complicated.
θ, $\frac{d\theta}{dt}$	$\frac{d\theta}{dt} = 0$	$E = \left[k(\theta + \tau \frac{d\theta}{dt}) R(\theta)\right]^{\frac{1}{2}} \times \left[1 + R_1 G(\theta)\right]$	Static characteristic is one of axes. Differential equation is complicated.

3.b. The V - I Phase Plane

The first to be investigated was the V - I plane. Eliminating all other variables the system may be reduced to the following first order differential equation

$$\frac{dI}{dV} = \frac{I}{V} + \frac{CR_1\, bI\left[VI - k\theta(I, V)\right]}{\left[Ta + \theta(I, V)\right]^2 k\left[E - V - IR_1\right]} \tag{5.3}$$

where

$$\theta(I, V) = \frac{b}{\ln \frac{V}{IR_{\infty}}} - Ta.$$

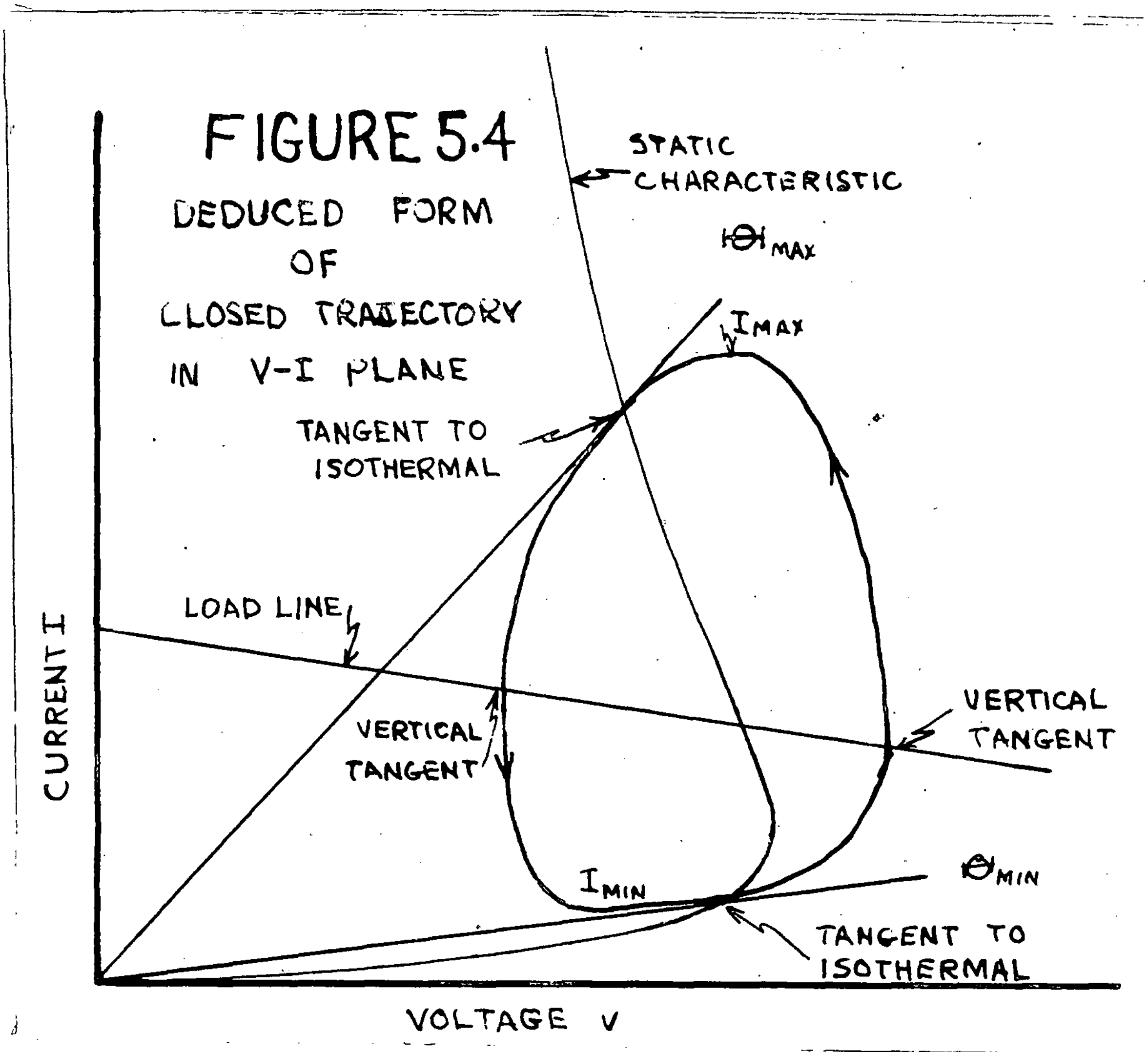


FIGURE 5.4
DEDUCED FORM OF CLOSED TRAJECTORY IN V-I PLANE

Oscillations correspond to closed trajectories around the singular point (V_o, I_o) at the intersection of $E = IR_1 + V$ and $VI = k\Theta(I, V)$ where $\frac{dV}{dt} = \frac{dI}{dt} = 0$. The trajectories of (5.3) have vertical tangents on the locus of $\frac{dV}{dt} = 0$, thus the load line $E - IR_1 - V = 0$ is the isocline for $\frac{dI}{dV} = \infty$. The trajectories are tangent to the isothermal corresponding to maximum and minimum temperatures where $\frac{dI}{dV} = \frac{I}{V}$ or $VI = k\Theta(I, V)$ which are the points where the trajectories cross the static characteristic. The isocline for $\frac{dI}{dV} = 0$ corresponding to current maximum and minimum is not easily soluble. It is to the right of the static characteristic above the load line and to the left of the static characteristic below the load line (see Figure 5.4). Since $\frac{dV}{dt} > 0$ below the load line and $\frac{dV}{dt} < 0$ above the load line it is apparent that the motion of any representative point is in a counterclockwise direction around the operating point (V_o, I_o).

The nature of the singular point (V_o, I_o) was next examined. The transformations

$$v = V - V_o$$

$$i = I - I_o$$

were made and $\frac{dv}{dt}$ and $\frac{di}{dt}$ were solved with second and higher order terms in v and i discarded to give the pair of linear equations

$$\frac{dv}{dt} = -\frac{v}{CR_1} - \frac{i}{C}$$

$$\frac{di}{dt} = v\left(\frac{\phi_o + 1}{\tau} - \frac{1}{CR_1}\right)\frac{1}{R_o} + i\left(\frac{\phi_o - 1}{\tau} - \frac{1}{CR_o}\right)$$

The quantities D, Δ and $b + c$ are

$$D = (b - c)^2 + 4\,ad = \left(\frac{\Phi_o - 1}{\tau} - \frac{1}{C}\left(\frac{1}{R} - \frac{1}{R_1}\right)\right)^2 - 4\left(\frac{\Phi_o + 1}{\tau} - \frac{1}{CR_1}\right)\frac{1}{CR_o}$$

$$\Delta = ad - bc = \frac{1}{\tau C}\left(\frac{\Phi_o - 1}{R_1} - \frac{\Phi_o + 1}{R_o}\right).$$

$$b + c = \frac{\Phi_o - 1}{\tau} - \frac{1}{C}\left(\frac{1}{R_o} + \frac{1}{R_1}\right).$$

A condition of great interest is that for the operating point to be a center. For $b + c = 0$ one has the condition that the system is just oscillating with infitesimal amplitude. For $b + c = 0$, D is less than zero and

$$C = \frac{\tau}{\Phi_o - 1}\left(\frac{1}{R_o} + \frac{1}{R_1}\right) \tag{5.4}$$

This value of C is identical to the value C_{min} in (3.19). For values of greater than C_{min}, the singularity is an unstable spiral point provided $D < 0$ and for values of C smaller than C_{min}, the singularity is a stable spiral point also provided that $D < 0$. The conditions at a typical operating point are

$\Phi_o = 2.0$

$R_o = 0.60$ Mohms

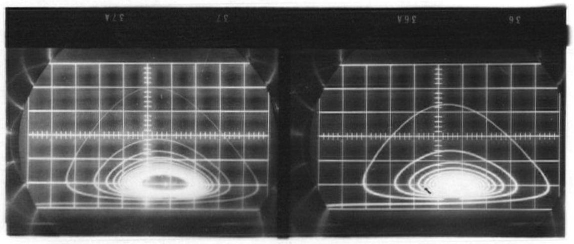
$R_1 = 4.0$ Mohms

$\tau = 0.61$ sec

$r_o = -0.20$ Mohms

for which the singularity is

Unstable Node	$C > 13.8$ µfd
Unstable Spiral	$13.8 \geq C > 1.17$ µfd
Center	$C = 1.17$ µfd



$C = 1.5 \mu fd > C_{MIN}$ $\quad$ $C = 1.2 \mu fd < C_{MIN}$

VERTICAL 1DIV = 10V
HORIZONTAL 1DIV = 20μA

$C_{MIN} = 1.25 \mu fd$

FIGURE 5.5 PHOTOGRAPHS OF I-V PLANE SHOWING STABLE AND UNSTABLE SPIRAL

Stable Spiral	$.104 < C \leq 1.17$ ufd
Stable Node	$C < .104$ ufd

Figure 5.5 shows some photographs illustrating the nature of the singularity in the V - I plane for several values of C.

3.c. The V - Θ Plane

The differential equations in the V - Θ plane are

$$\left.\begin{aligned} \frac{d\Theta}{dt} &= \frac{1}{\tau}\left(\frac{V^2 G(\Theta)}{k} - \Theta\right) \\ \frac{dV}{dt} &= \frac{1}{CR_1}\left(E - V(1+R_1 G(\Theta))\right) \end{aligned}\right\} \qquad (5.5)$$

where

$$G(\Theta) = G_\infty \exp \frac{-b}{Ta+\Theta}$$

and their first order approximations valid near the operating point are

$$\frac{d\theta}{dt} = \frac{\phi_o - 1}{\tau}\theta + \frac{2}{\tau}\frac{\Theta_o}{V_o} v$$

$$\frac{dv}{dt} = \frac{-\phi_o V_o G_o}{C\,\Theta_o}\theta - \frac{(1+R_1 G_o)}{R_1} v$$

which have the same values of D, Δ and b+c as the equations for v and i since to first order they are linear transformations from the equations in v and i.

The equations of (5.4) may be rewritten as the single first-order differential equation

$$\frac{d\Theta}{dV} = \frac{CR_1}{\tau}\,\frac{V^2 G(\Theta)\frac{1}{k} - \Theta}{E - V(1+R_1 G(\Theta))} \qquad (5.6)$$

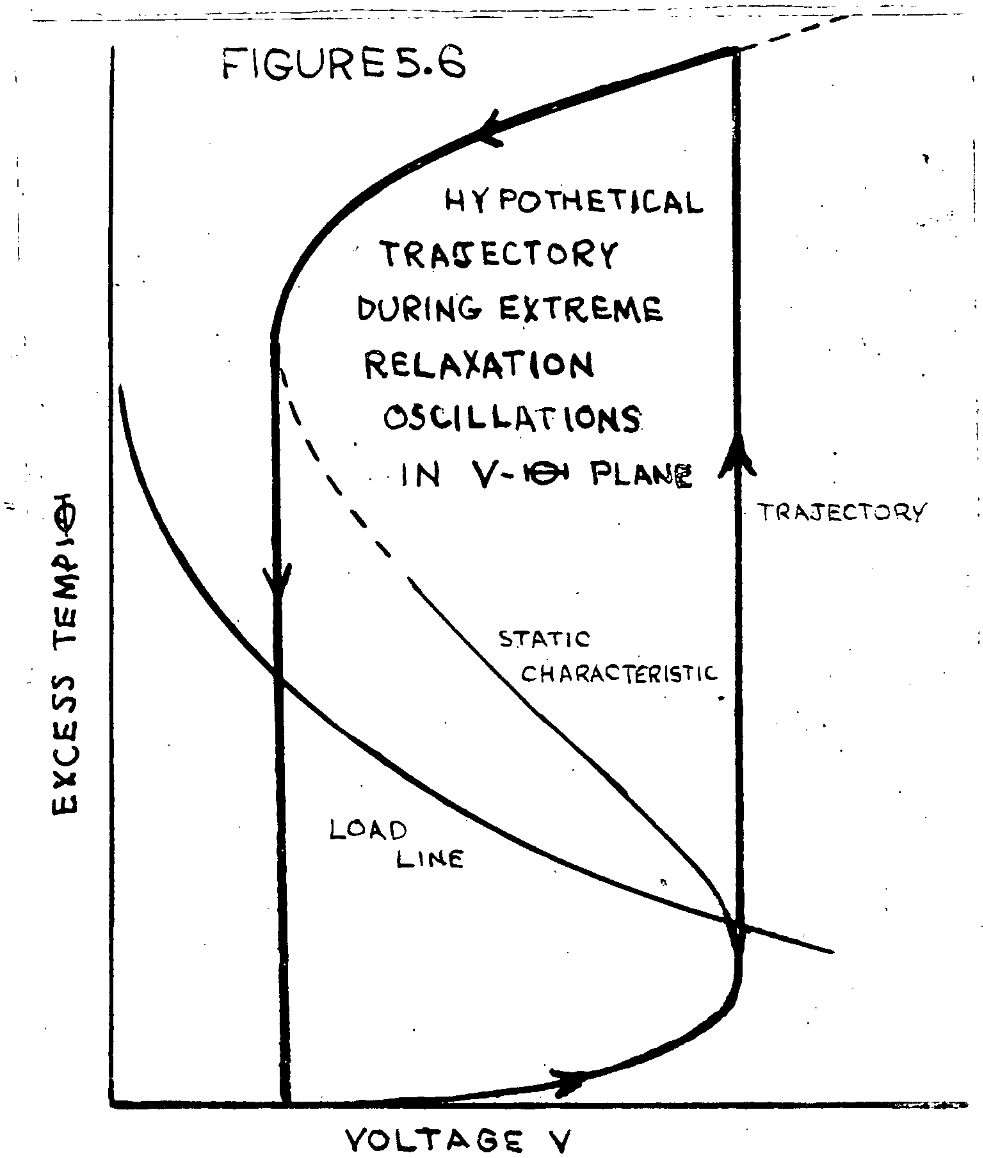


FIGURE 5.6
HYPOTHETICAL TRAJECTORY DURING EXTREME RELAXATION OSCILLATIONS IN V-Θ PLANE
TRAJECTORY
STATIC CHARACTERISTIC
LOAD LINE
EXCESS TEMP. Θ
VOLTAGE V

It is noted that this equation (5.6) is much simpler than (5.3). It is apparent that the isocline for $\frac{d|\Theta|}{dV} = 0$ is the static characteristic and the isocline for $\frac{d|\Theta|}{dV} = \infty$ is the load line.

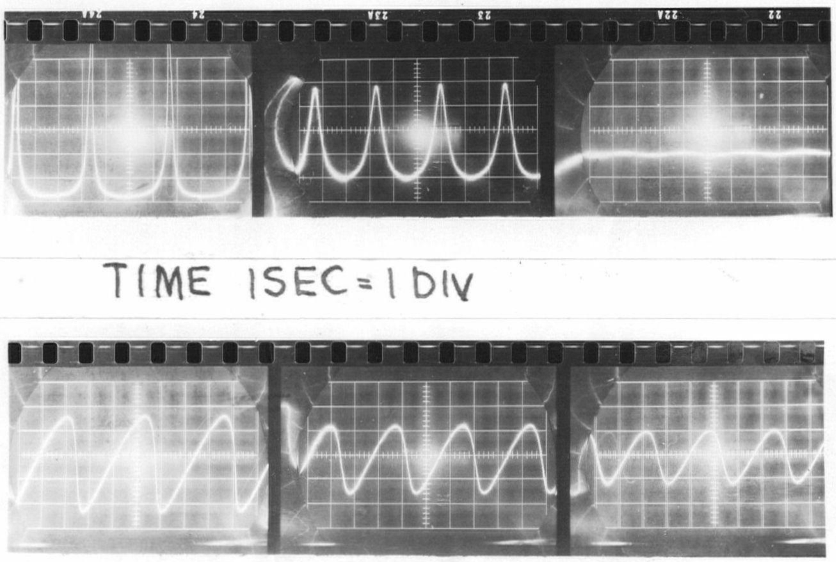
Also, it is seen that if $C \to 0$, then $\frac{d|\Theta|}{dV} \to 0$ everywhere in the plane except at the load line. Thus in the limit as C becomes zero one would expect all representative points to jump horizontally to the load line and then to follow the load line to the operating point. If $C \to \infty$, the case for extreme relaxation oscillations, $\frac{d|\Theta|}{dV} \to \infty$ everywhere in the plane except at the static characteristic. Thus one would expect all representative points to jump vertically to the static characteristic and once there, to follow it. In particular, a representative point starting from the origin would be expected to follow the static curve to the turnover point. At this point it would tend to jump to the hypothetical branch of the ideal static characteristic and follow it until the voltage minimum was reached and then jump to the lower branch of the static characteristic (see Figure 5.6). Since a trajectory such as this is not physically realizable as the upper branch of the ideal static characteristic is at a hypothetical temperature greater than $b \simeq 4000^{\circ}K$ it was not possible to set up extreme relaxation oscillations in the thermistor.

For values of $C < \infty$ the trajectories would be expected to have a value of $|\Theta|$ somewhat lower than the value of $|\Theta|$ corresponding to the static characteristic for a given V.

CURRENT
50 μA = 1 DIV

TIME 1SEC = 1 DIV

VOLTAGE
20V = 1 DIV



C = 2.0 μfd C = 1.5 μfd C = 1.4 μfd

FIGURE 5.7 PHOTOGRAPHS OF VOLTAGE & CURRENT AS FUNCTIONS OF TIME DURING OSCILLATION

4. Division of the Period of Oscillations into Epochs

4.a. General

Figure 5.7 shows photographs of thermistor voltage V and current I as functions of time during oscillation. It is seen that as the value of C increases, in both the current and voltage functions two distinct epochs may be recognized. These were given the names charging and discharging epochs corresponding to the time regions in which $\frac{dV}{dt}$ is respectively positive and negative.

4.b. Charging Epoch

For the purposes of analysis the charging epoch T_c is defined as that portion of the period for which the closed trajectory is below the load line.

If $\frac{CR_1}{\tau}$ is sufficiently large it has been shown that $\frac{d\Theta}{dt}$ is very small compared with $\frac{\Theta}{\tau}$ over the charging portion of the cycle. Thus the equation for the balance of power

$$\frac{V^2 G(\Theta)}{k} = \Theta + \tau \frac{d\Theta}{dt}$$

becomes approximated by the equation for the static characteristic

$$\Theta = \frac{V^2 G(\Theta)}{k}$$

which may in principle be solved for Θ and substituted in the equation for $\frac{dV}{dt}$ giving for T_c

$$T_c = CR_1 \int_{V_{min}}^{V_{max}} \frac{dV}{E - V(1 + R_1 G(\Theta))} \qquad (5.7)$$

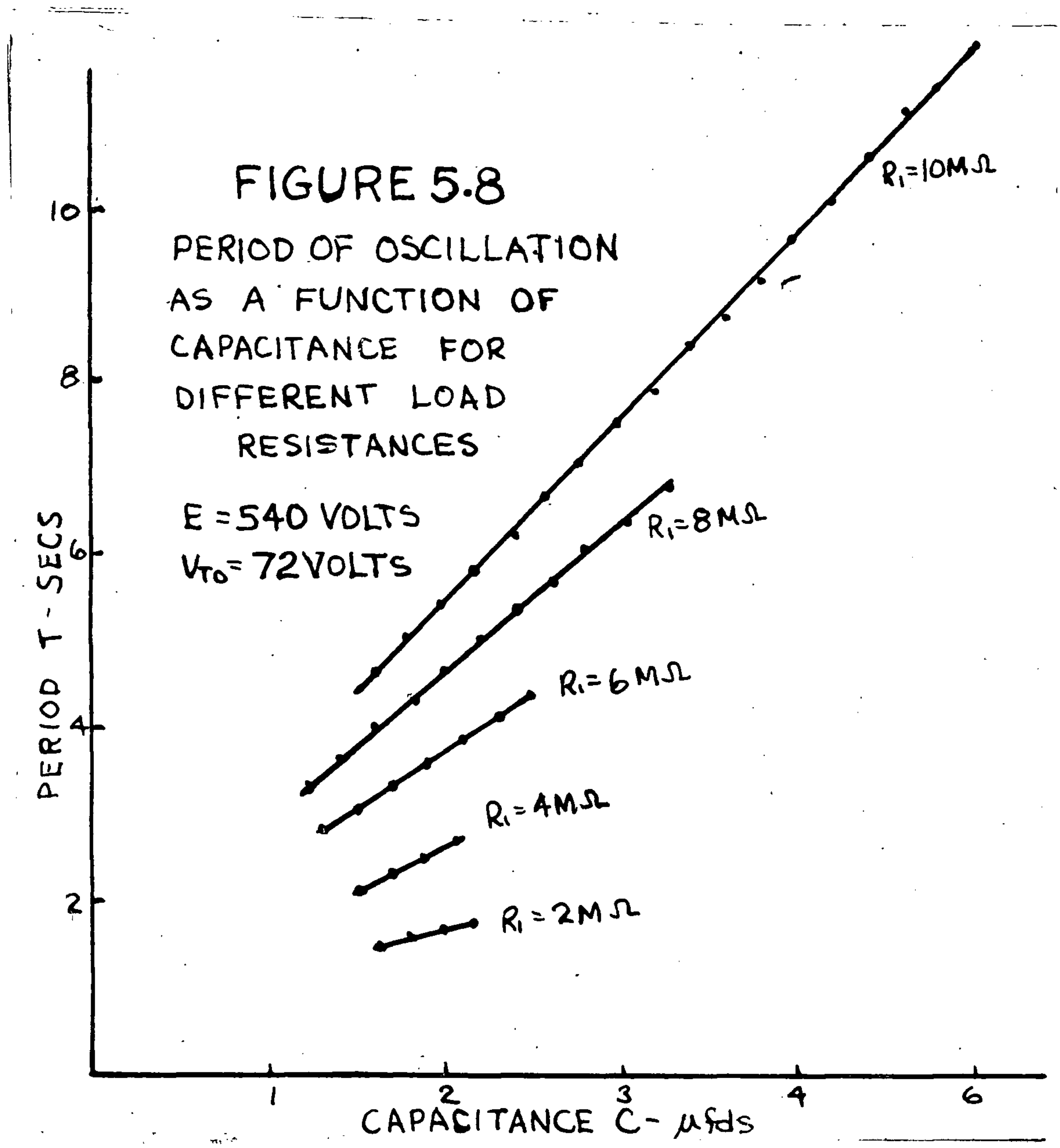


FIGURE 5.8
PERIOD OF OSCILLATION AS A FUNCTION OF CAPACITANCE FOR DIFFERENT LOAD RESISTANCES
E = 540 VOLTS
V_{TO} = 72 VOLTS
R_1 = 10 MΩ
R_1 = 8 MΩ
R_1 = 6 MΩ
R_1 = 4 MΩ
R_1 = 2 MΩ
PERIOD T - SECS
CAPACITANCE C - μfds
10
8
6
4
2
1
2
3
4
6

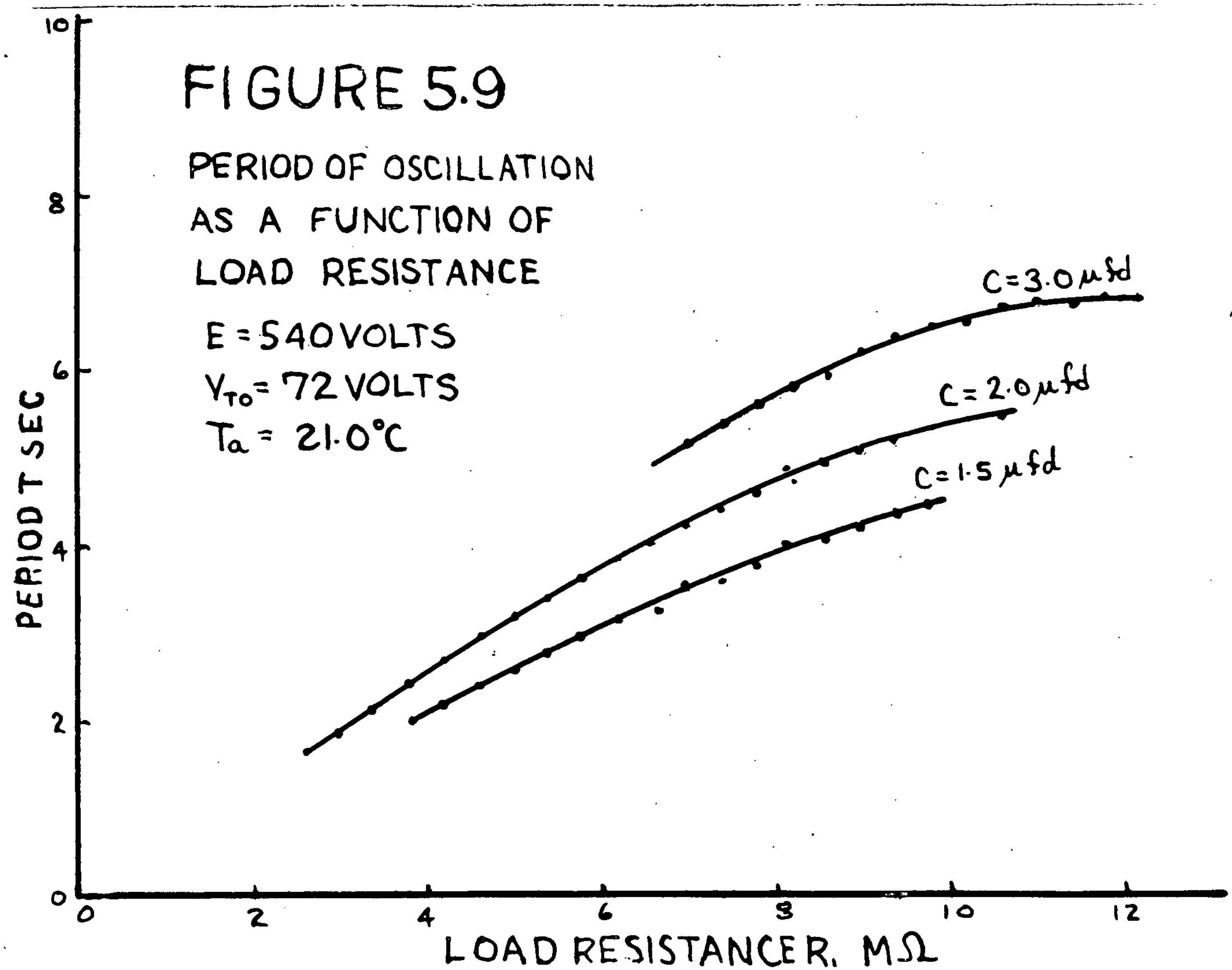


FIGURE 5.9
PERIOD OF OSCILLATION
AS A FUNCTION OF
LOAD RESISTANCE
E = 540 VOLTS
V_{TO} = 72 VOLTS
T_a = 21.0°C
C = 3.0 μfd
C = 2.0 μfd
C = 1.5 μfd
PERIOD T SEC
LOAD RESISTANCE R, MΩ
0
2
4
6
8
10
12

where V_{max} and V_{min} are the maximum and minimum values of V over a cycle. A lower bound on T_c is given by the inequality

$$T_c > CR_1 \ln \frac{E - V_{min}}{E - V_{max}}$$

or for sufficiently large values of C since V_{min} becomes small (see Figure 5.3)

$$T_c > CR_1 \quad \frac{V_{max} - V_{min}}{E}$$

If the discharging epoch can be assumed small enough, T_c should be a first order approximation to the period T. The form of (5.7) indicates that the period of oscillation should be proportional to the parallel capacitance C. This was true over a wide range of the value of C and the slope of the T versus C line increases with increasing R_1. Figure 5.8 shows the period of oscillation T as a function of capacitance C. Figure 5.9 shows the period of oscillation T as a function of load resistance R_1 for several different values of C. The relation between R_1 and T is linear for values of R_1 such that the operating point is well above turnover. The form of these functional relations between T, C and T, R_1 suggest the following empirical relation

$$T = a_0 + a_1 C + a_2 R_1 + a_3 CR_1$$

would be valid over a range of operating points.

4.c. Discharging Epoch

It was decided that a study of the discharging of a condenser through a thermistor would possibly yield information regarding the discharging epoch of the oscillation.

For the circuit of a capacitor discharging into a thermistor the following equations hold

$$\frac{I}{V} = G_{\infty} \exp \frac{-b}{T_a + \Theta}$$

$$P = IV = k\left(\Theta + \tau \frac{d\Theta}{dt}\right)$$

$$C \frac{dV}{dt} + I = 0$$

If T_{DIS} is the duration of the discharging epoch,

$$T_{DIS} = C \int_{V_{min}}^{V_{max}} \frac{dV}{VG(\Theta)} \qquad (5.8)$$

and

$$\frac{C}{G(\Theta_{max})} \ln \frac{V_{max}}{V_{min}} < T_{DIS} < \frac{C}{G(0)} \ln \frac{V_{max}}{V_{min}} .$$

Several phase planes were considered for regarding this problem among which were Θ - V and the Θ - $\frac{d\Theta}{dt}$ planes.

The differential equations were

$$\frac{d\Theta}{dV} = -\frac{C}{\tau}\left[\frac{V}{k} - \frac{\Theta R}{V}\right] \qquad (5.9)$$

where

$$R = \frac{V}{I} = R_{\infty} \exp \frac{b}{T_a + \Theta}$$

and

$$\frac{dx}{d\Theta} = \frac{(\Theta + \tau x)(\mu x - 2\frac{G}{C}) - x}{x\tau} \qquad (5.10)$$

where

$$x = \frac{d\Theta}{dt}$$

$$\mu = \frac{b}{(T_a + \Theta)^2}$$

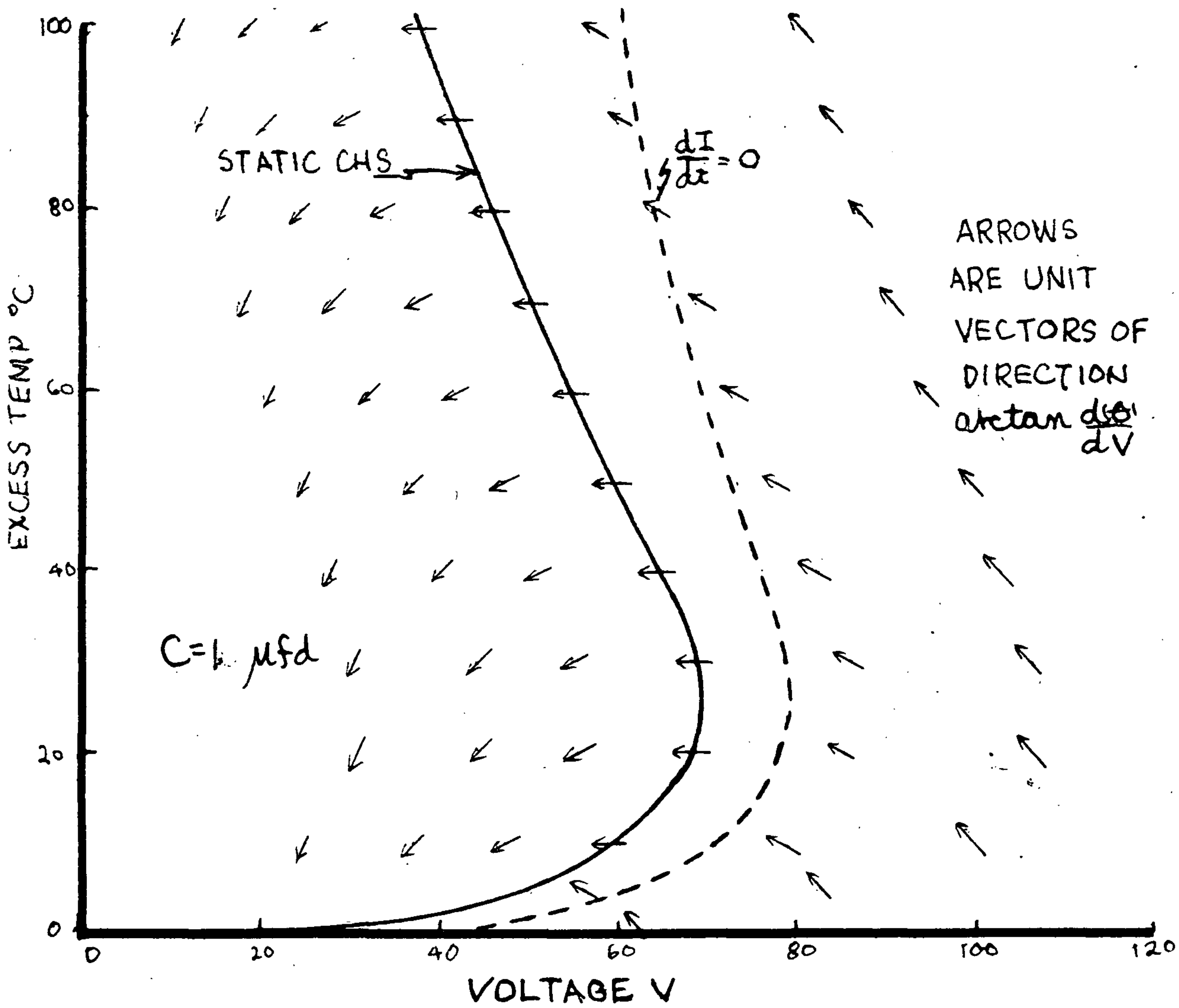
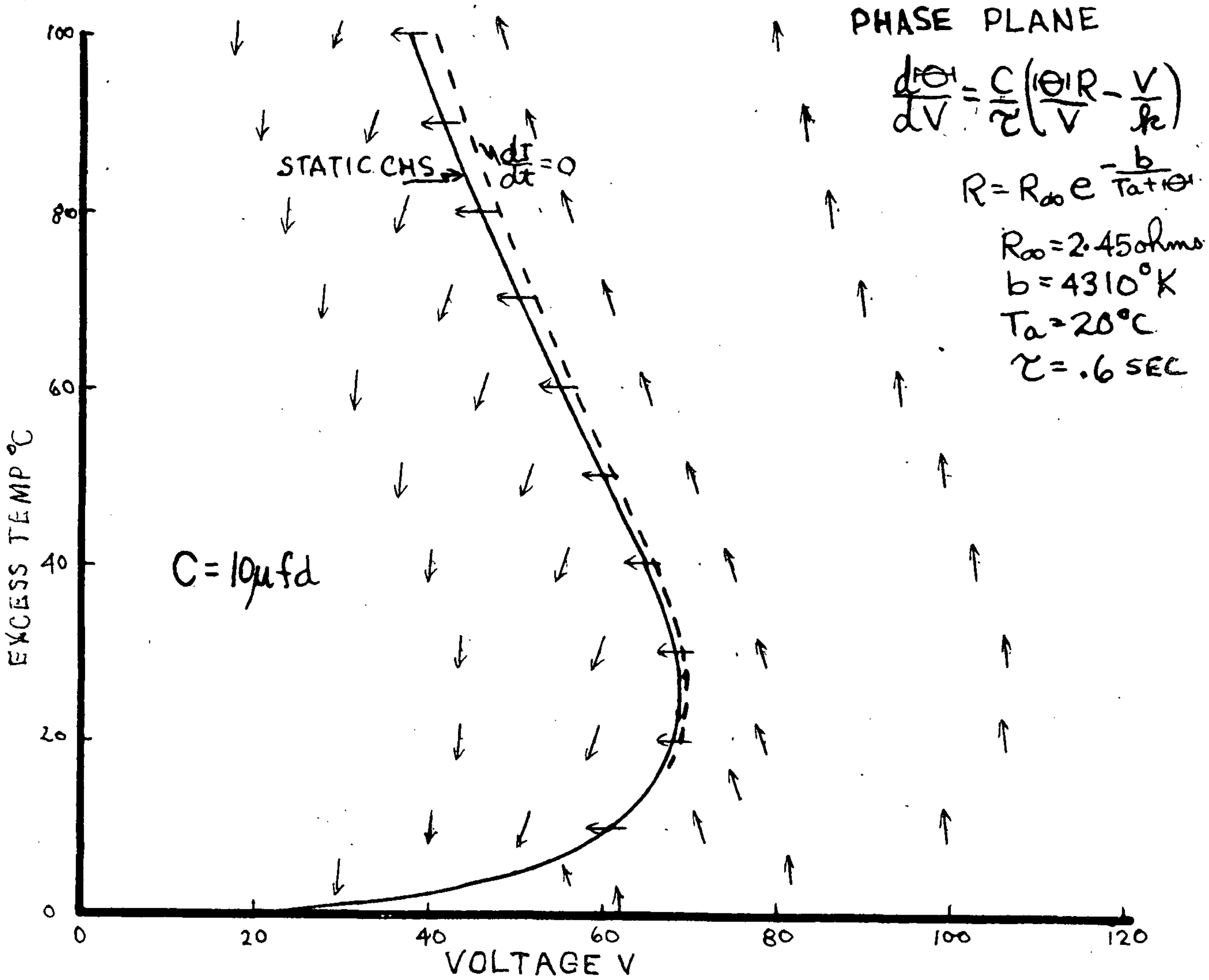


FIGURE 5.10 DRAWINGS OF VOLTAGE-EXCESS TEMPERATURE PHASE PLANE



and

$$G = G_{\infty} \exp \frac{-b}{Ta + \theta} .$$

The θ-V plane was first considered. The origin is a stable node for all real values of the parameters C, τ and k. Isoclines corresponding to slope $\beta = \frac{d\theta}{dV}$ are given by

$$V = -\frac{k\tau}{2C}\beta \pm \frac{1}{2}\sqrt{k^2 \frac{\tau^2\beta^2}{C^2} + 4k\theta R}$$

In particular for $\beta = 0$

$$V = (k\theta R)^{\frac{1}{2}}$$

which is the static characteristic. For the locus of points where the current I is maximum,

$$\frac{dI}{dt} = 0$$

which gives

$$\frac{dV}{d\theta} = -\mu V = \frac{bV}{(Ta+\theta)^2}$$

and

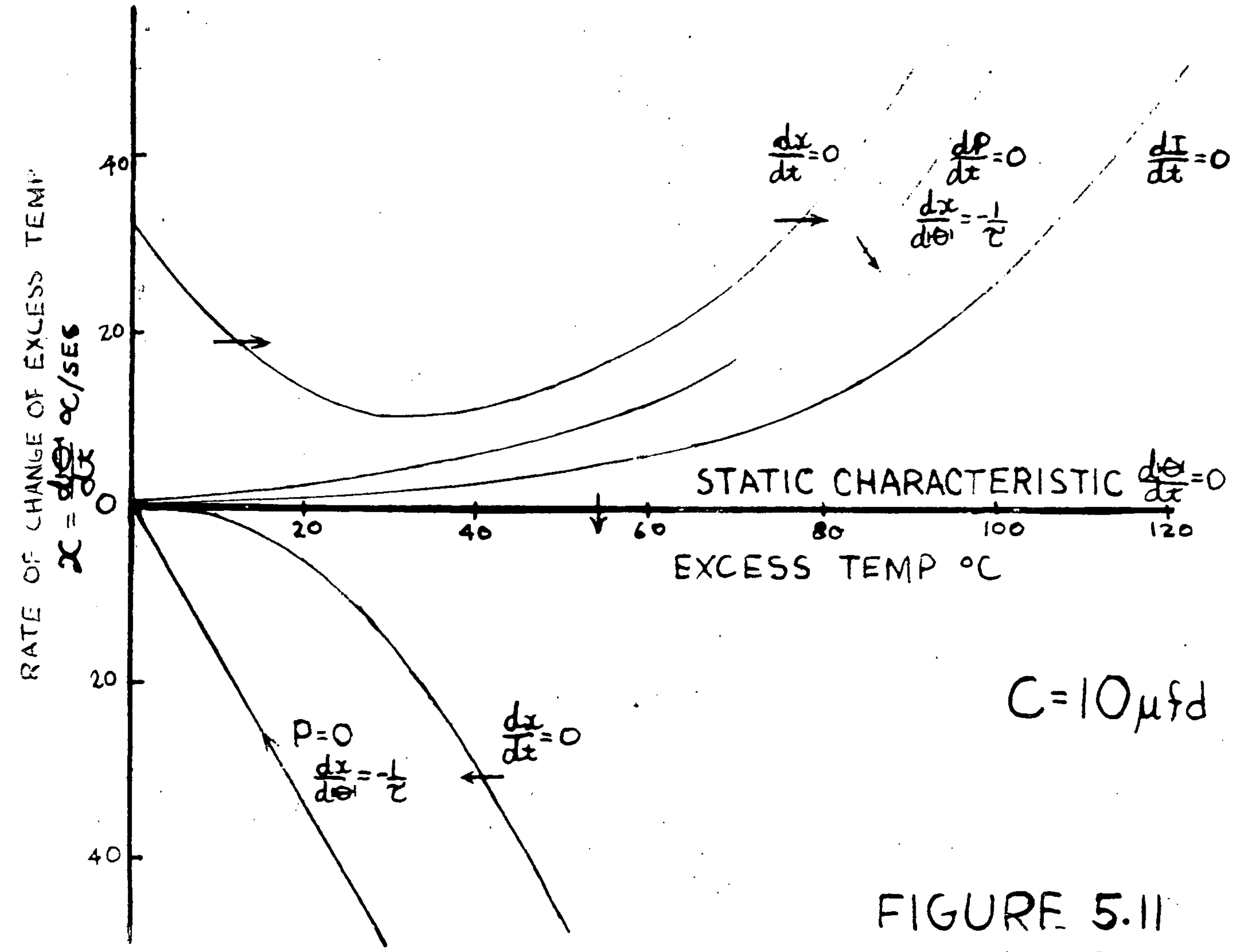
$$V = \sqrt{k(\theta R + \frac{\tau}{C}\frac{(Ta+\theta)^2)}{b}} ,$$

For sufficiently large values of C and sufficiently small values θ this locus very nearly coincides with the static characteristic. For large values of θ this locus is given by

$$V \simeq \left(\frac{k\tau}{bC}\right)^{\frac{1}{2}} (Ta + \theta) .$$

Figure 5.10 is a drawing of the V -θ plane for two different values of C with several isoclines and the locus for $\frac{dI}{dt} = 0$ drawn in.

Regarding the differential equation in the θ , $x = \frac{d\theta}{dt}$ plane (5.9) it is seen that the static characteristic is given by $x = 0$,



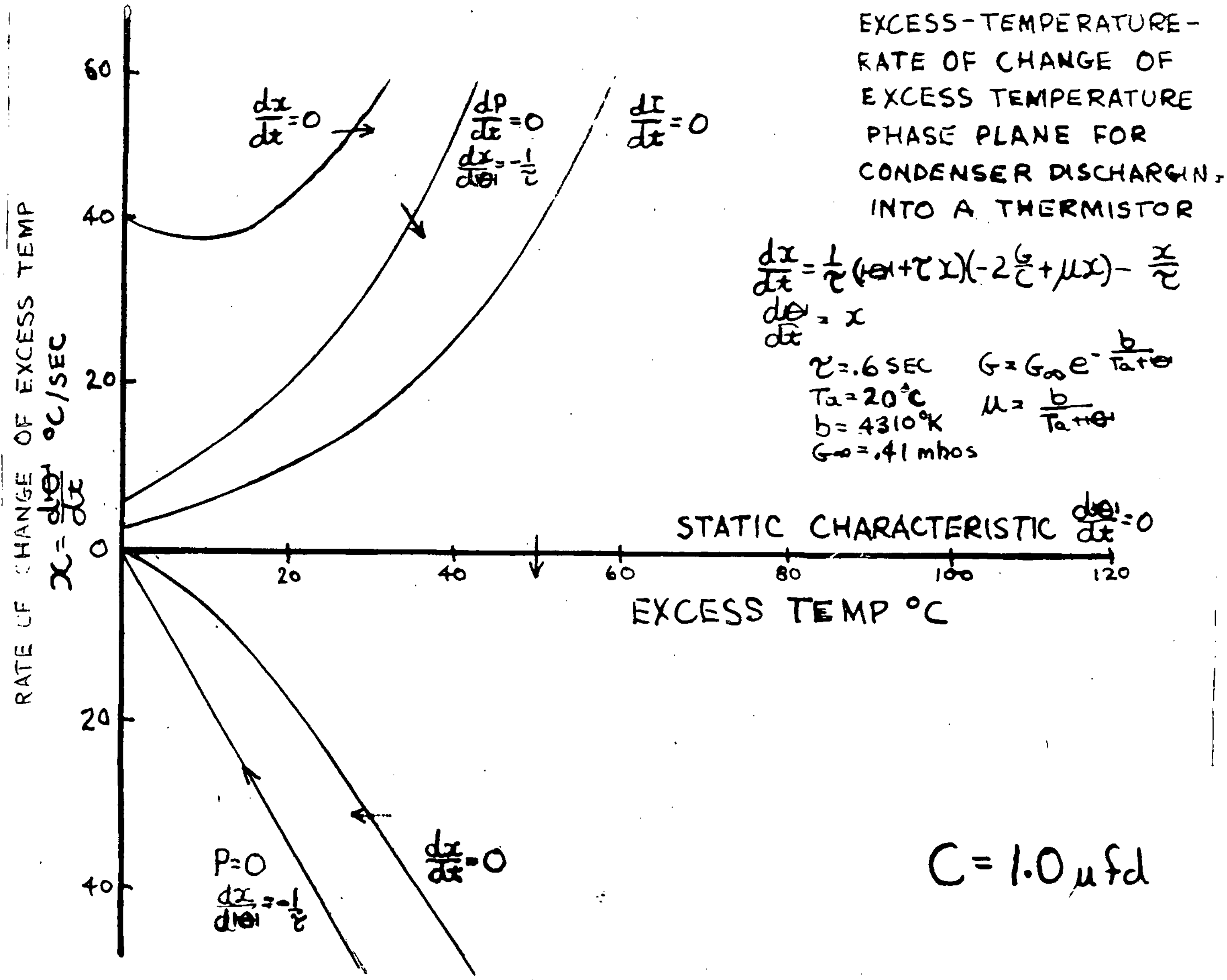


FIGURE 5.11

DRAWING OF EXCESS-TEMPERATURE-RATE OF CHANGE OF EXCESS TEMPERATURE PHASE PLANE FOR CONDENSER DISCHARGING INTO A THERMISTOR

$$\frac{dx}{dt} = \frac{1}{\tau}(\Theta+\tau x)\left(-2\frac{G}{C}+\mu x\right) - \frac{x}{\tau}$$

$$\frac{d\Theta}{dt} = x$$

$\tau = .6$ SEC $\quad G = G_\infty e^{-\frac{b}{T_a+\Theta}}$

$T_a = 20°C \quad \mu = \frac{b}{T_a+\Theta}$

$b = 4310°K$

$G_\infty = .41$ mhos

which is the isocline for $\frac{dx}{d\theta} = \infty$. The isocline for $\frac{dx}{d\theta} = 0$,

or for the maximum $\frac{d\theta}{dt}$ is given by the equation

$$x^2 \tau\mu + x\left(\mu\theta + 1 - 2\frac{G}{C}\tau\right) - 2\theta\frac{G}{C} = 0$$

The isoclines with slope $\frac{dx}{d\theta} = -\frac{1}{\tau}$ are the locus where P is a maximum in which case

$$x = \frac{2G\,(Ta + \theta)^2}{C\,b}$$

and the locus where $P = 0$ in which case

$$x = -\tau\theta$$

The locus for maximum current I where $\frac{dI}{dt} = 0$ is

$$x = \frac{G}{\mu C}$$

which is not an isocline. The trajectories along this locus have a slope given by

$$\frac{dx}{d\theta} = \frac{-\mu(\theta + \tau x) - 1}{\tau}$$

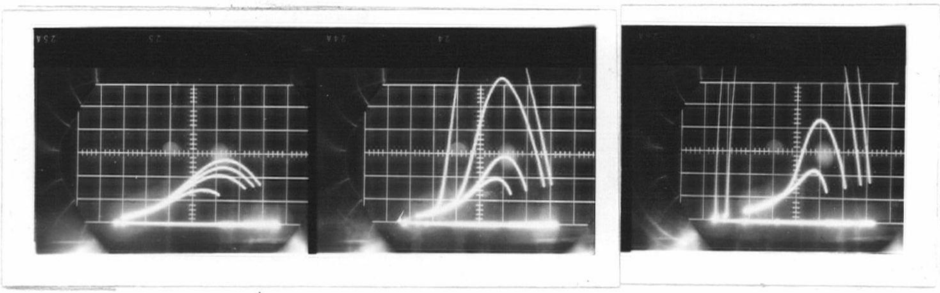
Figure 5.11 shows drawings of the θ, $x = \frac{d\theta}{dt}$ plane for values of $C = 1$ and $C = 10$ ufd.

It is seen that trajectories starting above a certain minimum value of x have in succession

maximum of $\frac{d\theta}{dt}$ if $GaV_i^2 > (VI)_t$

maximum of P if $\frac{CV_i^2}{2} > (VI)_t\,\tau$

maximum of I if $\frac{CV_i^2}{2} > \frac{(VI)_t}{2}\,\tau$



C=3.0μfd   C=4.0μfd   C=5.0μfd

VERTICAL 1DIV=10μA   HORIZONTAL 1DIV=1.5V

FIGURE 5.12 PHOTOGRAPHS OF CURRENT-VOLTAGE PLANE DURING CONDENSER DISCHARGE THROUGH THERMISTOR maximum of $|\Theta|$

minimum of $\frac{d|\Theta|}{dt}$

where V_i is the initial voltage of the condenser.

Attempts were made to approximate the discharge by several different functions such as

$$|\Theta| = \beta t e^{-\gamma t}$$

and

$$|\Theta| = \frac{\gamma t^n}{\alpha + t^m} .$$

These functions were found to be greatly inadequate due to their lack of sufficient number of adjustable parameters preventing an adequate approximation to the solution of the differential equation for $|\Theta|$:

$$\frac{d}{dt}\left[\ln \left(\frac{\frac{|\Theta|}{\tau} + \frac{d|\Theta|}{dt}}{G(|\Theta|)} \right) \right] + \frac{2\,G(|\Theta|)}{C} = 0$$

Figure 5.12 shows photos of the V- I plane during discharging of a condenser.

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