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THE DIRECTIONAL SPECTRUM OF OCEAN WAVES:
AN EXPERIMENTAL INVESTIGATION OF CERTAIN
PREDICTIONS OF THE MILES-PHILLIPS THEORY OF
WAVE GENERATION

ABSTRACT

The directional spectrum of wind-driven surface waves has been measured under conditions of limited fetch, in order to check the predictions of Miles-Phillips theory of wave generation. The expression obtained for the directional spectrum in this theory involves the spectrum of the atmospheric pressure fluctuations, but it is possible to obtain theoretical estimates for the major features of the directional spectrum without knowledge of the pressures. Specifically it is possible to predict the frequency at which the power spectrum should peak, and for the higher frequencies the range of azimuth over which high spectral values should be observed; for the lower frequencies the theory indicates a bimodal distribution in azimuth (Phillips' resonance waves), and gives the angle of travel relative to the wind as a function of frequency.

The results of the measurements are in fairly good agreement with the theoretical predictions for the higher frequencies. The asymmetry of the fetch results in the prediction that the waves will travel at an angle to the wind which varies with frequency, and this was observed. The range of azimuth over which the spectral density is high is also close to the theoretical prediction. For the low frequencies the bimodal distribution was not observed: the waves were found to have a single predominant direction of travel at each frequency. However this direction conformed closely to that of one of the two wave trains predicted by Phillips, and its variation with frequency was also that given by the theory. It is suggested that the peculiarities of the experimental site may be responsible for the absence of the second wave train, since the observed effects would be difficult to account for on any basis other than that of Phillips' theory.

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by

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

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DOCTOR OF PHILOSOPHY

in the Department

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**We accept this thesis as conforming to the
required standard**

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October, 1965

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Department of PHYSICS

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ABSTRACT

The directional spectrum of wind-driven surface waves has been measured under conditions of well-defined fetch in Burrard Inlet, which separates the cities of Vancouver and West Vancouver in British Columbia, Canada, and extends seaward to Howe Sound and Strait of Georgia.

This investigation was undertaken to check the predictions of two recent theories concerning the mechanism by which the wind acts on the water surface to produce waves. These theories, one due to Phillips and the other to Miles, were first advanced in 1957, and have since been combined into a unified theory by the latter author.

The unified theory yields an expression for the directional spectrum of the waves in terms of wind speed and direction, the duration of the wind (or alternatively the fetch available), and the two-dimensional spectrum of the turbulent pressure fluctuations in the wind. The fetch is the distance from the shore to the point of observation. The pressure spectrum has never been measured; it appears, indeed, that even single point pressure measurement is a matter of considerable difficulty. However, it is possible even without knowledge of the pressures, to obtain theoretical estimates of the major features of the directional spectrum. Thus, given the wind speed and direction, and the fetch as a function of azimuth, it is possible to predict the frequency at which the power

spectrum should peak; and for this and higher frequencies to predict the range of azimuth over which high spectral values should be observed. Further, when the fetch is small, so that the waves are not fully developed, there is a range of frequencies below that of the spectral peak in which only Phillips' resonance waves should be present. For these, the theory indicates a bimodal distribution in azimuth, and it also predicts the angle of travel relative to the wind as a function of frequency.

The results of the measurements made in the present investigation are in fairly good agreement with the theoretical predictions for the higher frequencies, from the frequency of the peak of the power spectrum up (referred to here as the Miles' regime; the lower frequencies will be referred to as the Phillips' regime). The asymmetrical fetch results in the prediction that these waves will travel at an angle to the wind which varies with frequency, and this was observed. The range of azimuth over which the spectral density is high is also close to the theoretical expectation.

In the Phillips' regime, the bimodal distribution was not observed; the waves were found to have a single predominant direction of travel at each frequency. However, this direction conformed closely to that of one of the two wave trains predicted by Phillips, and its variation with frequency was also that given by the theory. It is suggested

that peculiarities of the experimental site may be responsible for the absence of the second wave train, since the observed effects would be difficult to account for on any basis other than that of Phillips' theory.

In order to demonstrate the physical reality of resonance waves in an absolutely decisive way it would be necessary to measure a spectrum exhibiting the characteristic bimodal form. Further investigations at a more suitable site are planned by the Institute of Oceanography, University of British Columbia, (IOUBC) to determine whether this second wave train is actually present.

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A.W.R. Gilchrist

Section 1: INTRODUCTION

That wind gives rise to waves has been known from ancient times; it is indeed a matter of the most casual observation to anyone who has lived near a lake. Until very recently however, there has been no adequate theory of the mechanism by which the flow of one fluid (the air) over another (the water) produces the observed undulatory motion of the interface. In a review article published in 1956 Ursell remarked that an understanding of the mechanism required three elements: a correctly derived theory, a body of relevant measurements, and a demonstration of agreement between the two; he added that at that time all three were lacking.

Stimulated by Ursell's review, two papers appeared in the following year (Phillips 1957, Miles 1957) which provided two distinct theories of the mechanism. These theories, however, were complementary rather than competitive; Phillips' theory accounted for the initial stages of the transition of the interface from a plane to a wavy surface, while that of Miles described the subsequent growth of the waves. The two theories were combined by Miles (1960).

Phillips' theory is based on the fact that large scale fluid motions, such as that of the atmosphere, are

almost invariably turbulent--i.e. the wind is not a smooth flow, but is gusty. Associated with the fluctuating motion are fluctuations in pressure, and it is these fluctuating pressures, acting normally on the water surface, which produce the initial deformation. Phillips' model therefore considers a pressure pattern which travels over the water surface at a convection velocity U , and varies as it travels much more slowly than does the pressure at a fixed point on the surface as the pattern crosses it. The Fourier components of this pressure pattern generate small forced oscillations in the surface. If the pattern contains components whose wavenumber and frequency coincide with those of free surface waves, the response is resonant to these components, and the corresponding surface waves are preferentially energised. The condition for this, that the waves remain stationary relative to the pressure pattern, may be expressed as $U \cos \theta = c$, where $c = c(\lambda)$ is the free wave speed, λ is the wavelength, and θ is the angle between the direction of propagation of the waves and that of convection of the pressure pattern. In the subsequent motion the slow random variation of the pressure pattern must be taken into account. The wave which is stationary relative to the pattern continues to be acted on by a component of the correct wavenumber, but of varying phase and amplitude. Under these conditions the amplitude of the wave increases in proportion to the square root of the time,

rather than linearly as in the initial period; the process is analogous in this respect to diffusion and the random walk.

In the Miles' theory turbulence does not play an explicit role; the essential idea is the interaction between the airflow and the surface wave. It differs critically from the earlier instability theory of Kelvin (1871) and the "sheltering" theory of Jeffries (1925) in that the boundary condition of zero relative motion of the fluids at the interface is taken into account. This leads to a vertical gradient of wind velocity, and incidentally makes it at once obvious that any "sheltering" due to flow separation, if it occurs, would be on the windward side of the wave and so would provide a damping rather than a growth mechanism. Miles postulates a logarithmic wind velocity profile, which is predicted by turbulence theory for a neutrally stable atmosphere, and agrees moderately well with observations for this case. The model is one for the growth of pre-existing waves, and does not explain the initial formation of the waves from a plane surface. In the solution for the perturbation pressure, Miles finds a component in quadrature with the surface elevation, and of the correct sign to transfer energy from the air to the water; this component is zero in the irrotational fluid model. The power spectral density of the surface elevation is found to vary exponentially with the time, rather than linearly as in Phillips' process.

The processes considered in the two theories are evidently complementary, that of Phillips accounting for the creation and growth of the waves during the early part of their history, and that treated by Miles taking over when the development has proceeded sufficiently far. The theories were combined by Miles (1960), who succeeded in deriving a single analytical expression for the spectral density of the surface elevation, as a function of the vector wave number and the time, which takes into account the effects of both of the physical processes described above.

It now appears that the first of Ursell's three requirements has been supplied. However, in a recently published book, Kinsman (1965) remarks that the other two requirements are still substantially lacking. This is certainly true. A full verification of the theory would require the measurement of the two-dimensional spectrum of pressure fluctuations in the atmosphere, simultaneously with that of the surface elevation. The former has never been measured, and appears to present formidable difficulties. However, it transpires that the major predictions of the theory can be investigated using measurements of the two-dimensional (directional) spectrum of the surface elevation alone. But even this partial check presents difficulties. The directional spectrum has been measured in the ocean only three or four times; in fact if we exclude experiments aimed at finding the direction of swell (long isolated wave trains),

only two sets of observations have been obtained (Longuet-Higgins et al, 1963; Cote et al, 1960). In addition to the general difficulties of deep-water experimentation, these investigators were faced with the problems of developing new techniques, of determining the wind strength and direction over areas of oceanic size, and of isolating the locally-generated wave field from other waves travelling into the area. In spite of this they obtained results which may be said to confirm the theory in some respects, or at least are not inconsistent with it. The particularly interesting question of the existence of resonance waves was still open, although both of the investigations referred to provided some evidence which can be interpreted as supporting the theory; these interpretations, however, are not generally agreed upon.

This, then, formed the background to the present work, which was aimed at verifying the Miles-Phillips theory. It would obviously be impossible for a single investigator to undertake deep-water measurements such as those referred to above, both of which were supported by large national organisations. However, it was felt that by working in a small coastal inlet, not only would the magnitude of the project be reduced to manageable proportions, but some of the uncertainties in the earlier work could be minimised. Further, the fetch limitation over most of the azimuth should provide a more discriminating test of the

theory. While in the prior investigations resonance waves had to be sought as residual traces under conditions where their characteristic features were swamped through the action of the Miles' mechanism, it was anticipated that in the present case waves of measurable amplitude should exist over a range of frequencies in which the Miles' mechanism had not become operative; these should be pure resonance waves. A study of these frequencies appeared to offer the possibility of the first positive demonstration of the existence of resonance waves.

Section 2: THEORY

- 2.1 The Phillips-Miles Spectrum: Method for Predicting Characteristics of Spectrum.
- 2.2 Predicted Characteristics of Spectrum for the Conditions of the Investigation.
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 - 2.2.2. The Growth Rate Parameter.
 - 2.2.3 The Power Spectrum.
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Section 2: THEORY

2.1 The Phillips-Miles Spectrum: Method for Predicting Characteristics of Spectrum

The end result of the unified Phillips-Miles theory is an expression for the directional spectrum of the surface displacement, which describes the effects of the two physical processes outlined in Section 1. This expression as given by Phillips (1961) is:

$$\phi(\underline{k}, t) \sim \frac{f(t, m)}{2\rho^2 c^2} \int_0^\infty \pi(\underline{k}, \tau) \cos \omega \tau \, d\tau. \quad 2.1$$

where $\phi(\underline{k}, t)$ is the directional spectrum,

$\underline{k}$	vector wave number,
t	time from the onset of the wind,
ρ	water density,
c	wave phase speed,
ω	wave angular frequency,
$\pi(\underline{k}, \tau)$	time-lagged spectrum of the turbulent atmospheric pressure fluctuations, i.e. the spatial Fourier transform of the covariance between pressures at points of the surface separated by a distance $\underline{r}$ and time τ ,

$$f(t, m) = (e^{2mt} - 1)(2m)^{-1}$$

$$= t \left\{ 1 + mt + O[(mt)^2] \right\}$$

$$m = \frac{1}{2} \beta k c$$

and β is the fractional increase in wave energy per radian; its evaluation requires the solution of the Orr-Sommerfeld equation. A solution has been obtained by Conte and Miles (1959) for the case of a logarithmic wind velocity profile.

Although no measurements of the pressure spectrum have been made, some indirect evidence exists which suggests that $\frac{1}{c^2} \int_0^\infty \Pi(k, \tau) \cos \omega \tau d\tau$ is a unimodal function, approaching zero for both large and small wavenumbers, and with a fairly broad maximum in the range of wavenumbers around $k = 10^{-2} \text{ cm}^{-1}$ (Phillips, 1957). This evidence therefore suggests that the unknown part of the expression for $\phi(k, t)$ may be assumed to vary only slowly over the range of wavenumbers of interest in the present investigation. The directional spectrum is then dominated by the exponential factor $f(t, m)$.

Considering the series expansion for $f(t, m)$, it is apparent that for $mt \ll 1$, $\phi(k, t)$ increases linearly with the wind duration t , as is characteristic of Phillips' mechanism; the expression for $\phi(k, t)$ then in fact reduces to that derived by Phillips. For fixed k (or m) the rate of growth increases with increase in t and eventually exhibits the exponential variation typical of Miles' process. The

transition from one mechanism to the other may be assumed to occur when $mt \sim 1$, i.e. at a wind duration:

$$t_T(k) = m^{-1} = \left(\frac{1}{2} \beta k c\right)^{-1} \quad 2.2$$

Now in the present investigation the waves are limited by fetch rather than by the duration of the wind. The relation between fetch and duration is $F = c_g t$, where c_g is the group velocity of the waves (Phillips, 1958). The above expression for the wind duration necessary to produce transition can therefore be written in terms of fetch: $F_T(k) = (\beta k)^{-1}$; if the fetch is measured in wavelengths we obtain:

$$\left\{ F_T(k) / \lambda_T \right\} = (2\pi \beta)^{-1} \quad 2.3$$

which is a function of $\frac{c}{U}$ and α , the angle between the directions of the wind and of wave propagation; the quantity U is the wind speed. In order to apply this result the numerical values of β calculated by Conte and Miles must be used.

Experimental wave power spectra are found to exhibit a characteristic variation with frequency: from low levels at low frequencies the density rises very rapidly to a maximum, and then falls off more slowly with further increase in frequency. Up to the stage at which the spectrum is no longer limited by fetch, i.e. until the sea is fully

developed, it is found that the rapid rise in the spectrum occurs at lower frequencies as the fetch is increased. Considering the spectral density as a function of fetch for a fixed frequency, as the fetch increases the density at first increases slowly, and then rises rapidly to a much higher level; with further increase in fetch this higher level is maintained. But this is the behaviour we would expect to occur in the transition from the Phillips' to the Miles' mechanism, if we realise that the exponential growth of the latter must ultimately be limited by non-linear processes not taken into account in the theory. This suggests a correspondence between the theoretical transition frequency and the observable frequency of the rapid rise of the spectrum. A quantitative comparison has been made by Phillips (1961) using the experimental data of Burling (1955), Kinsman (1960), and Pierson (1960). The comparison was actually made in terms of the transition fetch rather than the transition frequency for a given fetch, and a considerable amount of scatter was found in the data, although the observed fetches were certainly in order of magnitude agreement with the theoretical transition fetch. As will be seen in Section 2.2 the transition fetch-frequency relation is such that the transition frequencies are much less variable. The preceding argument, including the derivation of the theoretical transition fetch, is due to Phillips (1961).

The above ideas may be applied to the directional spectrum also, since the transition fetch is a function of the vector wave number. The indicated correspondence between the theoretical transition and the sharp rise of the power spectrum enables us in this case to predict ranges of azimuth for a given frequency over which high spectral values should be found; these are the ranges in which the Miles mechanism has become effective. At any frequency above that of the peak of the power spectrum (the Miles regime) this mechanism may be expected to dominate the directional spectrum, although the effect of energy transfers between wave modes (Phillips, 1960) may produce a tendency towards isotropy --i.e. towards broadening the directional spectrum--at the higher frequencies.

For the low frequencies, such that transition does not occur at any azimuth, only Phillips' mechanism can be operative. Only forced waves and resonance waves should then be present, and the latter should dominate the directional spectrum. The theoretical prediction is particularly simple in this case: at any frequency there should exist two wave trains making angles of $\pm \cos^{-1} \frac{c}{U}$ with the wind direction. We do not of course expect to observe such sharply defined trains, if only because any physically realisable filter has a finite bandwidth. Neither do we expect sharply defined trains to exist in the

water; the normal fluctuations in wind speed and direction would ensure some angular spread, and so too would the finite correlation times of the pattern of atmospheric pressure fluctuations. The actual prediction is that the spectral density will be high in two more or less narrow ranges of azimuth centred on the theoretical resonance angles.

If experimental conditions are such that $\frac{c}{U}$ is close to unity at the peak of the power spectrum the sea is said to be fully developed. Since the Miles' mechanism is very inefficient at frequencies such that $\frac{c}{U} \sim 1$, it might be thought that the waves at these frequencies would be mainly resonance waves. However, the resonance angle is then zero, so that the resonance waves travel with the wind, as also do Miles waves in the open sea. Thus, the two cannot be distinguished by their directional properties. At higher frequencies such that $\frac{c}{U} < 1$ resonance waves may be identified by their characteristic angles of travel, but in the fully developed sea the Miles' mechanism has taken over at these frequencies, and so there is little hope of identifying them. However, under appropriate conditions of limited fetch and wind speed, which are easily derivable from the theoretical expression for the transition fetch, the power spectrum peaks at small values of $\frac{c}{U}$, and a decisive identification of the resonance waves should be possible. The present investigation was designed to achieve this.

Quantitative characteristics of the directional spectrum predicted by the theory for the conditions of this investigation are presented in Section 2.2.

2.2 Predicted Characteristics of the Spectrum for the Conditions of the Investigation

2.2.1 Experimental Conditions:

- (a) Fetch: The distance from the shore to the point of observation is called the fetch. This is shown in Fig. 1, which is a map of Burrard Inlet showing the coastline and the location of the IOUBC oceanographic research station, where the measurements for this project were made. Fig. 2 is derived from Fig. 1 and presents the fetch as a function of azimuth.
- (b) Wind: For the set of measurements analysed in detail the mean wind direction was 102° true. This direction is indicated by an arrow in Fig. 1, and is the zero of the scale marked θ_{WIND} in Fig. 2. The wind speed, measured at a height of 2 metres above the water surface was 5.8 m/sec. The measurement of wind speed and direction is discussed in Section 3.
- (c) Array Orientation: The array of wave probes was arranged in a line running roughly north and south, actually $175^\circ - 355^\circ$ true. In Fig. 2 the scale marked θ_{ARRAY} represents angles measured from an

origin corresponding to 175° true; the sense of the angle θ_{ARRAY} follows the mathematical convention, which is opposite to the convention used for compass bearing.

2.2.2 The Growth Rate Parameter: It was pointed out in Section 2.1 that the evaluation of ξ , the fractional increase in wave energy per cycle, requires a knowledge of the wind profile. The values calculated by Conte and Miles apply to a logarithmic profile, which is the theoretical form for a turbulent boundary layer in an unstratified fluid. Measurements made over water (e.g. Hamblin, 1965) occasionally give profiles of this form, but more often do not, even under neutral conditions.

According to Miles' theory the value of ξ is proportional to the profile curvature, so that even if the profile is close to logarithmic it is possible for the curvature to be significantly different. Benjamin (1959) has shown that different but not unreasonable profiles can result in variations of the order of 50% in the theoretical transition fetch. In his comparison of the measured transition fetch with theoretical predictions, Phillips found a scatter of this order of magnitude among the observations. This was presumably due, in part at least, to differences in the profile curvature. The theoretical fetch calculated by Phillips was based on a logarithmic profile:

$$U(z) = U_1 \log_e \frac{z}{z_0} \quad 2.4$$

in which the parameters U_1 and z_0 were given values typical of measured logarithmic profiles at the wind speed concerned (usually measured at a standard height).

Because of the great difficulty of measuring profile curvature, and the virtual uselessness for this purpose of ordinary profile measurements, the theoretical predictions in this section are based on a logarithmic profile with parameters that are typical of such a profile for the measured wind speed. Only order-of-magnitude agreement can be expected with these predictions, but this is in any case inherent in the analysis, which starts from the order-of-magnitude relations that transition will occur when $mt \sim 1$. It is possible that the variation with frequency of the quantities concerned may be predicted more reliably than their absolute magnitude.

In the following estimates the profile parameters used are $U_1 = 0.6$ m/sec and $z_0 = 0.1$ mm. These values are derived from Hamblin's work and are similar to those obtained by other investigators. Hamblin does not consider the z_0 estimate to be at all reliable, but fortunately the dependence of β on z_0 is small: see Miles (1960; Fig. 2).

2.2.3 The Power Spectrum: On the basis of Phillips'

interpretation of the Miles-Phillips theory, we can predict that the power spectrum will exhibit low densities at the

low frequencies, and that there will be a rapid rise in the spectrum at the theoretical transition frequency. At higher frequencies we expect that the spectral density will be limited by processes not accounted for in this theory, which imposes no limit to the exponential growth. Phillips (1958) has shown by dimensional arguments that the high frequency end of the spectrum must fall off according to f^{-5} ; however his analysis does not indicate the frequency at which the f^{-5} variation begins. Burling (1955) found that his measurements indicated a -5.5 exponent, over a range of frequencies that extended from the peak of the spectrum up to the ripple frequencies. Kinsman (1960) found that his spectra, measured at longer fetches, were described by a -4.5 power law, and he believes that his measurements were accurate enough to distinguish between the -4.5 and the -5 law. If we regard the evidence as confirming the -5 law within the indicated limits of agreement, we may predict that the spectral density will fall off as f^{-5} , approximately, at the higher frequencies. On the basis of empirical evidence alone we expect that the -5 law will hold from the frequency of the spectral peak up. We could also make an empirical estimate of the magnitude of the spectral density from the results of Burling and Kinsman; however, a theoretical estimate would require, for the low frequencies, a knowledge of the pressure spectrum, and for the high frequencies a more detailed theory of the non-linear

processes than at present exists.

The frequency of the rapid rise in the power spectrum was estimated using the formula $F_T = \lambda_T / 2\pi\delta$. δ was obtained from Miles (1960, Equ. 5.9 and Fig. 2) using the parameters $U_1 = 0.6$ m/sec. and $z_0 = 0.1$ mm from Section 2.2.2. The actual upwind fetch available is

TABLE 2.1: TRANSITION FETCH VS. FREQUENCY

f cps.	c m/s.	λ m	$\frac{c}{U_1}$	$\left(\frac{U_1}{c}\right)^2$	δ	F_T m.
.4	3.91	9.76	6.7	.022	7.4×10^{-5}	21000
.5	3.12	6.75	5.3	.035	13.5×10^{-5}	7400
.6	2.60	4.34	4.5	.050	20.6×10^{-5}	3400

6300 m. This intersects the $F_T \times f$ curve at $f = 0.52 \sim$ which is the predicted frequency of the rapid rise.

Corresponding to a 50% uncertainty in F_T , the uncertainty in the predicted frequency is $\pm 0.05 \sim$.

2.2.4 The Directional Spectrum:

(a) The Low Frequencies: Below the peak of the spectrum we anticipate resonance waves at azimuths measured from the wind direction of $\pm \cos^{-1} \frac{c}{U}$, where c is the wave velocity and U is the convection velocity of the component

of the pressure pattern having the corresponding wave number. Following Lighthill (1962), the latter is identified with the wind speed at a height of 0.25λ above the surface. The wind speed was actually measured at a fixed height, 2 m above the surface, but since the variation with height is small at heights of 0.25λ for the relatively long waves under consideration, estimates based on the logarithmic profile parameters used in the calculation of the transition frequency are considered sufficiently accurate. The resonance angles are given in Table 2.2 for the actual frequencies at which the data analysis was carried out.

TABLE 2.2: RESONANCE ANGLES VS. FREQUENCY

f c.p.s.	λ m.	$.25\lambda$ m.	$U(.25\lambda)$ m.p.s.	c m.p.s.	$\frac{c}{U}$	$\cos^{-1} \frac{c}{U}$
.261	23.0	5.75	6.45	5.97	.925	22°
.284	19.4	4.85	6.35	5.48	.867	30°
.310	16.3	4.1	6.25	5.05	.807	36°
.338	13.7	3.4	6.15	4.63	.752	41°
.368	11.5	2.9	6.06	4.24	.703	45°
.401	9.7	2.4	5.95	3.81	.641	50°
.437	8.2	2.05	5.82	3.56	.608	53°
.476	6.9	1.7	5.70	3.27	.575	55°

(b) **The Higher Frequencies:** In calculating the frequency of the rapid rise in the power spectrum we considered only the upwind fetch, and by implication, waves travelling with the wind. Actually F_T is a function of vector wave number, and the expression for β involves the angle α between the direction of wave propagation and that of the wind; α was taken to be zero in the earlier calculation. Using the vector form of F_T we can compute the transition fetch for waves travelling in any direction, and by comparison with the fetch actually available in that direction, we can estimate for waves of any frequency the azimuths over which transition should have occurred according to the theory. The theoretical fetch required for transition is calculated in Table 2.3 as a function of azimuth for the frequencies at which the measurements were analysed. Curves of "fetch required" are plotted in Fig. 3 along with the curve of "fetch available", replotted from Fig. 2. For a fixed frequency the azimuths between the points of intersection of the two curves give the range over which high spectral values are to be expected. If θ_1 and θ_2 are the points of intersection, we define the mean

TABLE 2. 3: TRANSITION FETCH VS. FREQUENCY AND AZIMUTH

(Fetch in metres)

FREQ. C.P.S.	AZIMUTH							
	0°	10°	20°	30°	40°	50°	60°	70°
.500	7400	7600	8500	11000	16000	82000	∞	∞
.520	6100	6300	7100	8800	13000	44000	∞	∞
.566	4300	4400	4900	6000	8600	20000	∞	∞
.618	3000	3100	3400	4200	4800	12000	∞	∞
.674	2100	2100	2400	2800	3900	7500	40000	∞
.735	1500	1500	1700	2000	2800	4700	18000	∞
.802	1000	1100	1200	1400	1800	3000	9800	∞
.875	730	750	830	970	1300	2000	5300	∞
.954	520	540	590	690	890	1400	3300	∞
1.04	370	380	420	500	640	960	2100	∞
1.135	260	270	300	350	450	660	1400	13000
1.24	180	190	210	250	320	470	890	6000

direction of travel to be $\bar{\theta} = \frac{1}{2}(\theta_1 + \theta_2)$ and the angular width of the spectrum as $\Delta\theta = (\theta_1 - \theta_2)$. Values of $\bar{\theta}$ and $\Delta\theta$ are plotted against frequency in Figs. 4 and 5, and these curves constitute the theoretical prediction for the directional spectrum at the higher frequencies.

Values of θ_1 and θ_2 obtained from Fig. 3 are given in Table 2.4, along with the derived quantities $\bar{\theta}$ and $\Delta\theta$. Angles are in array coordinates.

TABLE 2.4: MEAN WAVE DIRECTION AND ANGULAR SPREAD

f	.500	.520	.566	.618	.674	.735	.802	.875	.954	1.04	1.135	1.24
θ_1	80	91	107	119	123	127	131	134	136	138	141	143
θ_2	78	72	69	66	59	55	50	42	35	27	22	18
$\bar{\theta}$	79	82	88	93	91	91	91	88	86	83	82	81
$\Delta\theta$	2	19	38	53	64	72	81	92	101	111	119	125

2.3 Methods of Deducing the Directional Spectrum from

Array Data

2.3.1 General: The directional properties of a field of radiant energy can be determined by the use of an array of two or more detectors, each of which individually is incapable of distinguishing directions. It is evident that the ensemble of signals from such an array contains

directional information, in the phase relations between signals from pairs of detectors at a known separation. The extraction of this information must therefore depend in principle on the complex correlations between the signals. The actual techniques used fall into two quite distinct categories, namely:

- (a) Direct techniques, which may involve:
 - (i) Physical rotation of the array, or
 - (ii) Time or phase-lagging of the signals
- and (b) Correlation techniques, involving:
 - (i) Fourier transform of the cross spectrum of the signals, or
 - (ii) Fourier-Bessel analysis of the cross spectrum.

In the direct techniques the correlations do not appear explicitly. In method (a)(i) the signals are summed and the average power of the output signal is measured. It can be shown (Barber, 1958) that this output is a weighted integral of the entire spectrum, depending on all the waves present whatever their direction. The weighting function is characteristic of the array, and for a well designed array is such that the output power depends almost entirely on waves travelling in a direction normal to the array. Hence by rotating the array it is possible to examine the distribution of signal power in azimuth. In the alternative

method (a)(ii), instead of rotating the array physically, time or phase lags are introduced between the signals from the various detectors before summing, which makes the array sensitive to waves travelling in directions other than along the normal. This alternative is preferable when the frequency of the waves is low, so that the averaging times necessary are long, especially when the phenomenon being examined may be considered to be statistically stationary only over periods not much longer than the averaging time. The direct techniques have the great advantage of requiring a minimum of computation, but, at least when analogue methods are used and frequencies are very low, the introduction of long, accurate, and variable lags may present a problem. It was for this reason that the correlation techniques were adopted in the present investigation.

2.3.2 The Fourier Transform Method: The following analysis draws on a series of papers by Barber (1957, 58, 59, 63) and Munk, Miller, Snodgrass and Barber (1963).

For a wave system composed of many complex wave trains we may write:

$$\eta = \sum_{r=1}^{\infty} a_r \exp 2\pi i (l_r x + m_r y + f_r t + \alpha_r) \quad 2.5$$

noting that the sum of two such waves differing only in the sign of the exponent is real. η represents the displacement of the water surface from its mean level; a_r is the (real)

amplitude of the r^{th} component wavetrain; l_r and m_r are the components of wavenumber in the OX and OY directions, where the wave number k is the reciprocal of the wavelength. We may form the following conjugate product:

$$\overline{\eta^*(x, y, t) \eta(x+X, y+Y, t+T)} = \sum_{r=1}^{\infty} a_r^2 \exp 2\pi i (l_r X + m_r Y + f_r T) \quad 2.6$$

in which X, Y, T are constants, and the bar denotes an average taken over long intervals of space and time.

We may suppose that in any small range of l, m, f there are very many component waves, and that the sum of the powers a_r^2 in such a range is proportional to the size of the range and to a density $E(l, m, f)$. Then we have:

$$\overline{\eta^* \eta} = \rho(X, Y, T) = \iiint_{-\infty}^{\infty} E(l, m, f) \exp 2\pi i (lX + mY + fT) dl dm df \quad 2.7$$

$E(l, m, f)$ is the density of the directional spectrum, and this relation shows that it is the Fourier transform of the time-and space-lagged correlogram $\rho(X, Y, T)$.

For real waves the spectral densities at (l, m, f) and $(-l, -m, -f)$ are equal and the above expression becomes:

$$\rho(X, Y, T) = \iiint_{-\infty}^{\infty} E(l, m, f) \cos 2\pi (lX + mY + fT) dl dm df. \quad 2.8$$

This may be written:

$$\rho(X, Y, T) = \int_{-\infty}^{\infty} \cos 2\pi fT df \left\{ \iint_{-\infty}^{\infty} E(l, m, f) \cos 2\pi (lX + mY) dl dm \right\} - \quad 2.9$$

$$\int_{-\infty}^{\infty} \sin 2\pi fT df \left\{ \iint_{-\infty}^{\infty} E(l, m, f) \sin 2\pi (lX + mY) dl dm \right\}.$$

Hence the cosine Fourier transform of ρ as a function of T is:

$$C(X,Y,f) = \iint_{-\infty}^{\infty} E(l,m,f) \cos 2\pi(lX+mY) dl dm \quad - \text{ the co-spectrum.} \quad 2.10$$

and its sine transform is:

$$Q(X,Y,f) = - \iint_{-\infty}^{\infty} E(l,m,f) \sin 2\pi(lX+mY) dl dm \quad - \text{ the quad-spectrum.} \quad 2.11$$

Combining these results we have:

$$C - iQ = \iint_{-\infty}^{\infty} E(l,m,f) \exp 2\pi i(lX+mY) dl dm \quad 2.12$$

For a given frequency $C - iQ$ may be regarded as a complex spatial correlation whose transform $E(l,m,f)$ is the directional spectrum for that frequency. Inverting the last relation gives $E(l,m,f)$:

$$E(l,m,f) = \iint_{-\infty}^{\infty} \{ C(X,Y,f) - iQ(X,Y,f) \} \exp \{ -2\pi i(lX+mY) \} dX dY \quad . \quad 2.13$$

Thus, to obtain the directional spectrum we must first obtain the co- and quad-spectra, either by computing the correlogram $\rho(X,Y,T)$ and then finding its temporal Fourier transform, or by equivalent analogue techniques. The operation described the last equation must then be performed. It is usual in evaluating $\rho(X,Y,T)$ to assume ergodicity, so that the time and space averages involved may be replaced by a simple time average; the validity of this assumption is open to question.

In the present case, choosing the line of the array as the x-axis, the function ρ is known only for 35 values of X , i.e. $X = \pm nD$, $n = 0, 1, 2, \dots, 17$, where D is the unit space of the array, and the corresponding value of Y is zero. For these values,

$$\begin{aligned} \rho(X, 0, f) &= \int_{-\infty}^{\infty} \exp 2\pi i l X \left\{ \int_{-\infty}^{\infty} E(l, m, f) dm \right\} dl \\ &= \int_{-\infty}^{\infty} E(l, f) \exp 2\pi i l X dl. \end{aligned} \quad 2.14$$

Inverting this result,

$$\begin{aligned} E(l, f) &= \int_{-\infty}^{\infty} \rho(X, f) \exp(-2\pi i l X) dX \\ &= \int_{-\infty}^{\infty} \left\{ C(X, f) \cos 2\pi l X - Q(X, f) \sin 2\pi l X \right\} dX \end{aligned} \quad 2.15$$

The last line follows either by the symmetry properties of C and Q , or more immediately, by the fact that E must be real. Since C and Q are not known as continuous functions of X , we cannot determine the true spectrum E , but only an approximation E' obtained by some sort of approximate integration from the 35 values of C and Q at equally spaced intervals in X . Using the simplest integration technique, the "histogram" method, and making the assumption that C and Q are zero outside the range of the measurements (together equivalent to the use of a rectangular lag window), we obtain:

$$E'(\ell, f) = D \left[C(0, f) + 2 \sum_{n=1}^{17} C(nD, f) \cos 2\pi n \ell D - 2 \sum_{n=1}^{17} Q(nD, f) \sin 2\pi n \ell D \right] \quad 2.16$$

The method using the rectangular lag window is only one of a more general class of approximate integration techniques, which comprises those using arbitrary functions of the lag X to weight the measured values of C and Q . In some circumstances functions other than the rectangular "boxcar" function may be preferable, as discussed in Appendix 1. This infinite class of methods does not exhaust the possibilities, since all such methods in effect assume that C and Q are zero beyond the range of the measurements. This form of the extrapolated functions is only one possibility; to be quite general the extrapolation may be made arbitrarily. The choice of an integration technique may therefore be made from a doubly infinite class. We may note in passing that although the subject lends itself to mathematical study, extrapolations other than the zero extrapolation do not appear to have received attention in the literature, despite the fact that in some cases the latter clearly gives a poor representation of the correlogram.

The effect of the widely-used cosine ("Hanning") lag window is discussed in Appendix 1, but for the present we shall restrict our attention to the rectangular lag window, which is chosen because it has several desirable properties (see Appendix 1). We shall make one modification

to equ. 2.16. This equation gives no guarantee that the computed spectrum $\mathcal{E}'$ will be correctly normalised, i.e. that its integral with respect to $\mathcal{A}$ will be equal to the power spectral density $C(o, f)$ at the frequency concerned. Accordingly we write:

$$\mathcal{E}'(l, f) = K \left[C(o, f) + 2 \sum_{n=1}^{17} C(nD, f) \cos 2\pi n l D - 2 \sum_{n=1}^{17} Q(nD, f) \sin 2\pi n l D \right] \quad 2.17$$

which differs from equ. 2.16 in that the factor D has been replaced by a constant K , to be determined so as to normalise $\mathcal{E}'$.

The normalisation of $\mathcal{E}'$ involves some matters of theoretical interest which are discussed in Appendix 4.

2.3.3 The Fourier-Bessel Method: The Fourier-Bessel method starts from the relations, derived in the previous subsection:

$$C(X_n Y_n f) = \iint_{-\infty}^{\infty} E(l, m, f) \cos 2\pi (l X_n + m Y_n) dl dm \quad 2.10$$

$$Q(X_n Y_n f) = - \iint_{-\infty}^{\infty} E(l, m, f) \sin 2\pi (l X_n + m Y_n) dl dm \quad 2.11$$

Making the transformations:

$$l = k \cos \theta \quad m = k \sin \theta$$

$$X_n = D_n \cos \theta_n \quad Y_n = D_n \sin \theta_n$$

where D_n is the separation of the n^{th} pair of detectors and θ_n the direction of the line joining them, we obtain:

$$C_n(f) = \int_0^{2\pi} \int_0^\infty E_1(k, \theta, f) \cos \{ 2\pi k D_n \cos(\theta - \theta_n) \} k dk d\theta \quad 2.18$$

Now we expect that at a given f all the energy is concentrated at the value of k given by the dispersion relation, so that the integral with respect to k has a delta-function behaviour, and we may therefore write:

$$C_n(f) = \int_0^{2\pi} \mathcal{E}(\theta, f) \cos \{ z_n \cos(\theta - \theta_n) \} d\theta, \quad \text{where } z_n = 2\pi k D_n$$

and similarly for Q ,

$$Q_n(f) = - \int_0^{2\pi} \mathcal{E}(\theta, f) \sin \{ z_n \cos(\theta - \theta_n) \} d\theta.$$

Combining these results we have:

$$C_n(f) - i Q_n(f) = \int_0^{2\pi} \mathcal{E}(\theta, f) \exp i \{ z_n \cos(\theta - \theta_n) \} d\theta \quad 2.19$$

The density $\mathcal{E}(\theta, f)$ may be expanded in a Fourier series in θ :

$$\mathcal{E}(\theta, f) = \sum_{r=0}^{\infty} (A_r \cos r\theta + B_r \sin r\theta) \quad 2.20$$

After substituting this series for $\mathcal{E}(\theta, f)$ in the above integral, and making the transformation $\bar{\theta} = \theta - \theta_n$ the terms in the integral are found to have the form of the integral representation of the Bessel function:

$$J_r(z) = \frac{1}{2\pi i} \int_0^{2\pi} \cos r\phi \cdot \exp(iz \cos \phi) d\phi$$

Hence we obtain the principal result of this sub-section:

$$C_n(f) - i Q_n(f) = 2\pi \sum_{r=0}^{\infty} i^r (A_r \cos r\theta_n + B_r \sin r\theta_n) \cdot J_r(2\pi k D_n). \quad 2.21$$

The directional spectrum $\mathcal{E}(\theta)$ for a given frequency may thus be found from the measured co- and quad-spectra if we can evaluate the coefficients A_r , B_r in this relation. For each n , that is for each separation presented by the array, we have one equation. If the number of separations is N , we have N equations, but the number of the unknowns A_r , B_r is infinite. However, for a fixed z the value of $J_r(z)$ decreases rapidly with r after a certain value of r , so that the coefficients of the higher A_r and B_r approach zero. We might therefore suppose that if only N unknowns are retained we could invert the resulting $N \times N$ matrix and obtain approximate values for the first N coefficients of $\mathcal{E}(\theta)$. The problem is actually more difficult than this, since, at least in the present investigation, the matrix obtained in this way was always singular. The above discussion however, illustrates the principles of the method. The technique by which the Fourier coefficients were actually determined is treated in Appendix 3.

The analysis above is based on the treatments given by Barber (1959, 63) and Munk, Miller, Snodgrass and Barber (1963). The result quoted in the last paper (equ. 7.13) is in error.

2.4 Principles of Design of Direction-Finding Arrays

2.4.1 Introduction: In designing a directional array it is necessary to know the resolving power as a function of the unit spacing of the individual detectors and the overall span of the array. The resolving power in this connection is a measure of the angular discrimination of which the array is capable. Just as in optics a monochromatic source viewed through a grating of finite aperture is seen not as a single spectral line but as a diffraction pattern, so a single wave train of sharply defined direction is seen by an array as a diffraction pattern; the observed wave energy is spread over all azimuths. The analogy is between the time frequency in the optical case and the wavenumbers or space frequencies in the case of the directional array. In order to assess the ability of an array to distinguish between wave trains travelling in different directions we must therefore determine its diffraction pattern. The analysis up to equ. 2.24 is due to Barber (1963).

2.4.2 The Diffraction Pattern of an Array: In Section 2.3.2 we obtained the following equation for the spatial correlogram $\{C(X,Y,f) - iQ(X,Y,f)\}$, which we shall denote here by $\rho(X,Y,f)$:

$$\rho(X,Y,f) = \iint_{-\infty}^{\infty} E(l,m,f) \exp 2\pi i (lX + mY) dl dm \quad 2.12$$

In this equation $E(l,m,f)$ is the true directional spectrum, and ρ is a continuous function of the separations (X,Y) . However, with a finite number of detectors in the array, we can determine ρ only for the discrete set of values (X,Y) given by the separations of detector pairs in the array. Thus, we actually measure the function ρg , where g is a function defined in separation space, having the value zero everywhere except on the set of points representing the separations presented by the array; at each such point the function has a delta-function behaviour. From the above equation the Fourier transform of ρ is E . If we denote the transform of g by G , we may write the transform of ρg :

$$\mathcal{E}(l,m,f) = \iint_{-\infty}^{\infty} E(l',m',f) G(l-l', m-m') dl' dm' ; \quad 2.22$$

this result follows by the convolution theorem. Thus, our limited knowledge of ρ implies that we cannot determine the true directional spectrum E , but only the approximation $\mathcal{E}$, which is a weighted integral of the true spectrum over all wavenumber space. Each element of E is spread out over the (l,m) plane according to the pattern of G , and the measured spectrum is the sum of all these "smeared out" elements. The pattern of G is characteristic of the array and determines its resolving power. Evidently if G were a delta-function there would be no spreading, and $\mathcal{E}$ would be identical with E . If G were a function with a high narrow

peak at the origin and low values elsewhere, $\hat{E}$ would be a good approximation to E , while if G has a wide spread, the approximation would be poor.

Now G was defined to be the Fourier transform of $q(X,Y)$, which consists of a set of delta-functions at the separations (X_i, Y_i) , so that we may write:

$$G(l,m) = \iint_{-\infty}^{\infty} q(X,Y) \exp -2\pi i (lX + mY) dX dY \quad 2.23$$

The transform of a delta-function at the origin is:

$$\iint_{-\infty}^{\infty} \delta(X,Y) \exp -2\pi i (lX + mY) dX dY, = 1$$

That for a delta-function at (X_i, Y_i) is:

$$\iint_{-\infty}^{\infty} \delta(X-X_i, Y-Y_i) \exp -2\pi i (lX + mY) dX dY, = \exp -2\pi i (lX_i + mY_i)$$

But if an array presents the separation (X_i, Y_i) it must also present $(-X_i, -Y_i)$, which has the transform:

$$\iint_{-\infty}^{\infty} \delta(X+X_i, Y+Y_i) \exp -2\pi i (lX + mY) dX dY, = \exp 2\pi i (lX_i + mY_i)$$

The last two transforms combine to give:

$$2 \cos 2\pi (lX_i + mY_i)$$

Hence we may write:

$$G(l,m) = 1 + 2 \sum_i \cos 2\pi (lX_i + mY_i) \quad 2.24$$

where the summation is over all the positive separations

presented by the array. Thus, G can be evaluated for any array from a knowledge of its geometry.

2.4.3 Linear Arrays: If all the detectors are arranged in a straight line, the array is said to be one-dimensional, or linear. A linear array can measure only one component of wave number, but the directional spectrum can be deduced from such measurements provided two conditions are met. First, it must be possible to infer the wavelength and hence the total scalar wave number from a knowledge of the frequency. If this is so, the component of wave number in a direction perpendicular to the line of the array can be determined, apart from sign, from measurements of frequency and the component of wave number along the line of the array. The second condition is that the directions of travel of all waves present in the wave field must be on one side of the line of the array. If this is so, the ambiguity of sign of the second wave number component is removed.

For free gravity waves, the linear theory yields a dispersion relation which gives a unique value of the wavelength for each frequency. This has been verified experimentally for swell, and is usually assumed to be valid for waves in the generating area also. The observed peakedness of the waveform in the latter case suggests however, that the simple dispersion relation cannot hold exactly.

In the present investigation it was decided to use a linear array. As regards the conditions discussed above, this decision necessitated the usual assumption for the dispersion relation. It also involved the assumption that, with the array approximately perpendicular to the wind direction, no significant energy is associated with waves travelling upwind, so that the second condition is satisfied. This is believed to be a very reasonable assumption, at least under the conditions of the experiment; its validity could be investigated by the use of a two-dimensional array.

The advantages of the linear array include the fact that the mounting structure for the detectors can be physically simpler. More important, if the total number of detectors and the unit spacing are fixed, the span of the array, and hence the resolution, can be increased. The principal limitation to the number of detectors, apart from cost, is the fact that the computational effort involved increases roughly as the square of the number. It was arbitrarily decided in the present case to use seven detectors.

2.4.4 Optimum Linear Arrays: Barber (1958) defines an optimum array to be one which gives an approximation to the true directional spectrum having the smallest possible mean square error. This optimum is relative to other arrays with diffraction patterns depending on the same highest

harmonic. He shows that an optimum linear array is one in which the separations of detector pairs include all integral multiples of the unit separation D , up to the maximum span of the array. Thus with a given number of detectors we wish to find the arrangement giving all the separations τD for $\tau = 0, 1, 2, \dots, N$, and such that N is as large as possible.

Barber defines an " n " system as one in which the central intervals are n units long, the last interval is $(n-1)$ units, and the first $(n-2)$ intervals are each one unit long. If there are K detectors, the overall span of the array is N units where $N = n(K-n+2) - 3$. The maximum value of N is given by $n = \frac{1}{2}(K+1)$ or $n = \frac{1}{2}(K+3)$ when K is odd, and by $n = \frac{1}{2}(K+2)$ when K is even. Then

$$N = \frac{1}{4} K^2 + K - 2\frac{1}{4}, \quad K \text{ odd},$$

$$N = \frac{1}{4} K^2 + K - 2, \quad K \text{ even}.$$

With $K = 7$, the systems with $n = 4$ and $n = 5$ both give the maximum span of 17 units. The intervals between adjacent detectors are 1-1-4-4-4-3 for $n = 4$, and 1-1-1-5-5-4 for $n = 5$. Both the "4" and "5" systems were used in the course of this investigation, the choice between them being decided by convenience for the particular detector mounting arrangements employed.

2.4.5 Diffraction Pattern of the 7-Detector Array: Applying the general result given by equ 2.24 to the seven-detector optimum linear array, the function G becomes:

$$G(\ell) = 1 + 2 \sum_{r=1}^{17} \cos 2\pi r \ell D \quad , \quad = \left\{ \frac{\sin 35\pi \ell D}{\sin \pi \ell D} \right\} \quad 2.25$$

The form of this function is shown in Fig. 6, from which it is seen that even with only seven detectors the diffraction pattern is quite sharp, approximating the ideal delta function reasonably well. It remains to apply equ 2.25 to determine a value of D suitable for the study of a particular range of wave numbers. We should like to find some simple numerical measure of the array performance, which is fully described by the function $G(\ell)$.

2.4.6 Measures of Array Performance: The similarity noted above between directional arrays and optical gratings suggested that a useful measure could be obtained by the use of Rayleigh's criterion. Accordingly, we define the resolution as the angle between the directions of travel of two wave trains which can just be resolved according to this criterion, assuming that the directions of travel are close to the array normal. This means that the central maximum of the diffraction pattern of one wave train coincides with the first minimum of that of the other. The first minimum is given by the first non-zero solution of $G'(\ell) = 0$, which yields the equation:

$$35 \tan \pi l D = \tan 35 \pi l D \quad 2.26$$

The required solution is $\pi l_1 D = 0.129$, or $l_1 = \frac{.0411}{D}$.
 The central maximum is at $l_0 = 0$. Hence, two wave trains can just be resolved if their components of wave number along the line of the array differ by $\Delta l = \frac{.0411}{D}$, or alternatively if their directions of travel differ by the angle:

$$\Delta \theta = \frac{.0411 \lambda}{D} \quad 2.27$$

In any investigation the range of wavelengths λ that is of interest is known, and equ (7) therefore enables us to estimate the value of the unit spacing D required to obtain the desired resolution. This is done in Section 3.6.1 for the array used.

Another criterion considered was the angular bandwidth, defined by analogy with electrical filters as the angular span between the "half-power" points of the central maximum. This is related to the Rayleigh resolution angle $\Delta \theta$, the bandwidth being $0.84 \Delta \theta$. The bandwidth criterion appears to be more appropriate when the spectra are continuous, since the Rayleigh criterion is based on the requirements for distinguishing discrete spectral lines. However, in view of the near-equality of the resulting measures of performance, it clearly does not matter which is chosen.

2.4.7 Persistence of the Correlation: We have not yet taken into account the fact that at some separation the spatial correlation becomes so small that it cannot be distinguished from the noise background. It is usually considered that for wind waves the correlation along a line transverse to the mean direction of propagation "runs out" in about $1\frac{1}{2}$ wavelengths, so that there is no advantage in extending the array to spans greater than this. However, this is an empirical limit only, and presumably the correlation could be measured to greater separations by suitable signal-processing techniques; the ultimate limit would then be imposed by the time duration over which the wave state remains stationary.

Section 3: EXPERIMENT

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Section 3: EXPERIMENT

3.1 Objective

The overall aim of the investigation was stated in Section 1. In summary, it is to compare some theoretically predicted characteristics of the directional spectrum of wind waves in the sea with those found experimentally. These characteristics, and the predictions concerning them, are discussed in detail in Sections 2.1 and 2.2. In order to make the proposed comparison it is necessary to measure the directional wave spectrum under conditions where the sea is not fully developed--this term was defined in Section 2.1--which requires a limited fetch. The objective of the experimental part of this investigation was to obtain the data from which the spectrum could be computed.

3.2 Outline of the Experiment

The experiment was conducted at the IOUBC research station in Burrard Inlet. At this location the fetch is well defined over the entire azimuth, except to the west. The site would therefore be suitable when the wind is from the east, and it was known that winds from this direction occur frequently. Apart from this consideration the principal factors governing the choice of site were logistic: the Burrard Inlet station was

already established, it is close to the IOUBC laboratory and workshop, a radio communication link is installed, power is available, the station is manned 24 hours daily during the operating season, and instruments for making some of the measurements required--such as wind speed and direction--are installed and routinely operated. The site is described in further detail in Section 3.4.

A linear array of seven wave detectors was used to obtain the data for the directional spectrum. The reasons for this choice were presented in Section 2.4, where the principles of the design of directional arrays were discussed; the detailed design is described in Section 3.6. The seven detectors were carried on a frame which was originally mounted on the sea-bed, but at a later stage the frame was suspended from a cantilever structure attached to the pilings of the IOUBC tower.

The wave detectors were of the capacitance type in which the sea water formed the outer plate of a concentric cylinder condenser; the inner conductor consisted of a brass rod, and the dielectric was of teflon tubing, sealed at the bottom. An electrical connection was made to the brass rod at the top, and another connection to the metal frame on which the detectors were mounted provided a sea ground. As the water level changes the capacitance of the instrument changes accordingly. This variable condenser was used as a frequency-determining element in a Wien bridge oscillator,

so that a F.M. signal was obtained whose frequency deviation depended on the change in the water level from its mean value, i.e. on the instantaneous wave height. The signals were transmitted by coaxial cables to a multi-channel tape recorder, where they were recorded in continuous form for subsequent analysis. Multiplexing was used in the early stages of the experiment, but this was later abandoned in favour of single channel recording. Further discussions of the capacitance wave detector, the signal generator, and the recording arrangements will be found in Sections 3.5 and 3.7.

3.3 Design of the Experiment

The objectives of the experiment called for measurements in a partially developed sea. A desirable target appeared to be a value of $\frac{c}{U}$ of about 0.5 at the peak of the power spectrum, although the exact value is not critical. On the one hand, low values should permit the observation of resonance waves travelling at large angles relative to the wind, but wave amplitudes are then small, and the accuracy of measurement is reduced. High values give greater amplitudes, but isolated resonance waves can then be observed only in a smaller range of frequencies, in which they travel closer to the wind, so that their positive identification becomes more difficult. It was necessary therefore, to estimate a range of wind speeds in which the

sea state would be suitable for our purposes. As already remarked, at the Burrard Inlet site the most favourable wind direction for the proposed measurements was from the east, and the upwind fetch is then approximately 7000 metres. By the method described in Section 2.2 the frequency of the waves at the peak of the power spectrum was estimated for winds of various speeds, and the corresponding values of $\frac{c}{U}$ calculated. These estimates indicated that the desired sea state would exist with wind speeds of 4 to 8 m/sec. Accordingly, it was planned that measurements should be taken with easterly winds of 4 to 8 m/sec.

The choice of winds from the east was based on the fact that the fetch is not well defined to the westward. In this direction there are several islands, and the maximum fetch is limited by the eastern coast of Vancouver Island, which runs so that the fetch can vary by as much as 60 kilometres over a few degrees of azimuth; however, the fetch is well defined over more than 100° on either side of east. It appears then that the fetch restricts the choice of wind direction only within generous limits. There are, however, two further considerations that had to be borne in mind. The first was the relative frequency with which winds of various directions occur; obviously it would not be practical to plan an experiment which required conditions which are present only 0.1% of the time. It was known that winds more northerly than a compass bearing of 070° are exceedingly

rare, while those more southerly than about 130° are not common, except for light southerly land breezes. Winds in the range $070^\circ - 130^\circ$ are frequently present, often for several days at a time; for practical purposes, then, we were restricted to this range. Thus, fetch and wind frequency considerations agreed in suggesting a mean wind direction for planning purposes, of about 090° or 100° . This in turn suggested that the line of the array should be laid roughly north and south, since we expect the waves to travel more or less in the wind direction, although with considerable angular spread, and the resolution of the array is best for waves travelling along the array normal. But as soon as we fix the line of the array we introduce the second restriction on the wind directions that we can use: we wish to keep the mean direction of the waves within a small angle of the array normal in order to ensure that there is little energy associated with directions close to the line of the array, where the resolution is poor. Hence, we would wish to take measurements only with winds which are within a small angle of the direction adopted for planning. In practice this consideration did not limit the choice, since the range of directions of frequently occurring winds is itself small, and we could exploit any wind of the desired strength from a generally easterly bearing.

We have just stated that the array orientation should be chosen so that the array normal is parallel to the mean direction of wave propagation, and we implied that this coincides with the wind direction. Now according to the predictions of Section 2.2, this implication is not correct under the conditions of the experiment, for which the predicted mean direction of the waves is considerably to the north of the wind direction. For reasons of practical convenience the array was actually arranged with its normal at an intermediate angle between the expected directions of the wind and of wave travel.

From the procedure by which the desired wind speed was estimated we can also estimate the frequency of the peak of the power spectrum, and assuming we can measure the directional spectrum for frequencies an octave on either side of this, we can derive the range of wavelengths for which the array should be designed. For a wind speed of 6 m/s, in the centre of the chosen range, the frequency of the spectral peak was estimated to be about 0.5 cycle per second, so that the frequency range to be considered is 0.25 to 1.0 cycle per second. The corresponding range of wavelength is 1 to 25 metres. The design of the array is dealt with in Section 3.6.

In the design of the wave detectors it is necessary to know the range of wave heights expected under the conditions of the experiment. There is no purely theoretical

way of making this estimate, but it can be done empirically, for example by using the expression given by Burling (1955) for the power spectrum: $\phi(f) = 7.5 f^{-5.5} \text{ cm}^2\text{-sec}$. If we assume that this holds for frequencies from the estimated frequency of the peak up, and further that the spectrum is zero for lower frequencies, the expression may be integrated to yield a value for the mean square of the surface displacement. A factor of about three applied to the square root of this value gives an order-of-magnitude estimate of the maximum excursion. In the planning stages of this experiment, wave height estimates were actually obtained from very rough visual observations at the site, from which it appeared that 0.5 metre was a reasonable figure for the trough-to-crest height. A further discussion of the wave detectors is given in Section 3.5.

It would be very desirable in an experiment of this kind to choose a site such that the waves of all frequencies to be studied were effectively in deep water, which implies that the water depth should be everywhere more than half the wavelength of the longest waves. This was satisfied at the IOUBC site over most of the fetch, but the station itself was built on a sand bank, and even at high spring tides there is only 4 metres of water over the sand. Thus, for frequencies below that of the spectral peak the waves would be affected by the bottom. However, the

most serious effect--refraction--at these low frequencies could be corrected for, in the case of waves coming from directions north of the wind direction, as a result of the rather simple bottom topography (see Section 3.4). The effects of the bottom are discussed further in Section 6. This disadvantage of the site was accepted in view of its many advantages.

Another apparent disadvantage of the IOUBC site is the fact that strong currents--up to a metre per second--occur on the flood and ebb tides. Apart from any interaction effects, if measurements are made in a current its speed and direction must be accurately known in order to correct the apparent wave number components. However, as we have seen in the last paragraph, the measurements must in any case be taken with the greatest possible depth of water, i.e. at or near high tide, and the currents are then small or zero.

3.4 The Experimental Site

Fig. 1 presents a map of the site, showing the location of the IOUBC Research Station and the neighbouring coast line. The fetch has been measured from this map and is presented as a function of azimuth in Fig. 2. The marked asymmetry will be observed; the fetch varies from about 7000 metres for azimuths to the north of east, to about 1000 metres for directions south of east. The effect of this asymmetry

on the directional distribution of the higher frequency waves has been estimated in Section 2.2; the effect on the lower frequencies will be discussed in Section 6.

Waves of all frequencies of interest in this investigation may be considered to be effectively in deep water over the whole inlet, except for the shaded part which is a sandbank. The dashed line in Fig. 1 denotes the edge of the bank where the depth rapidly increases to several tens of metres. The part of the bank to the north of an east-west line drawn through a point about 100 metres south of the IOUBC station is almost perfectly flat; it barely dries at low spring tides and at the corresponding high tides is covered to a depth of about 4 metres. To the south of this line the bottom shoals gradually towards the shore, on the average, but has a few shallow depressions and elevations. The fact that the bank is dry at the lowest tides is convenient for erecting mounting structures for instruments, and for laying cables.

The IOUBC station is a platform about five metres square mounted on piles about seven metres above the sand, with a hut to house the equipment and personnel occupying the northern half of the platform. A boat landing stage and mooring arrangements are provided at the south end of the platform. Two masts for mounting instruments are located about 40 metres to the north of the platform; the height of the instruments on these masts above the water can be

adjusted. The instruments are connected by cable to their associated circuitry in the hut, which also contains the recording equipment. Fig. 11 is a photograph showing the IOUBC station.

3.5 The Wave Detector

3.5.1 Description: A brief description of the wave detector was given in Section 3.2. The inner conductor of the probe element consisted of a brass rod, 0.25 in. (0.63 cm.) diameter, 3 metres long in the version used with the bottom-mounted frame, and 2 metres long in that used with the suspended frame. An electrical connection was made to the inner conductor at the top of the probe. The insulation was of teflon tubing, 0.268 in. $\pm$.010 in. inside diameter, and 0.015 in. nominal wall thickness. The teflon tubing was fitted over the brass rod and sealed at the lower end of the probe, while at the upper end the electrical connector was mechanically supported by a standard cable sleeve drilled out to give a tight press fit over the teflon. The dielectric constant of the teflon is 2.0, so that the calculated capacitance of the probe was 6 pF per centimetre.

The oscillator circuit is shown in Fig. 7, and the amplifier circuit in Fig. 8. The output of the oscillator is taken between the positive feedback point and the sea ground. The amplifier is designed to have a high input impedance to minimise the loading on the oscillator, and a low output impedance, since the signal is fed to a long output cable. The arrangement of grounds is such that the shield of the output cable is at sea ground potential. The circuits were turned on and off by a double-pole double-throw relay, one per oscillator-amplifier unit; the relays were activated remotely from the hut using two relay cables, one per group of three or four oscillators.

In the original arrangement the detectors were dual units consisting of two parallel probes 10 cms apart. These were connected as the two bridge condensers, with the common point being at sea ground potential. Using the theoretical expression for the frequency of the Wien bridge oscillator, $f = \frac{1}{2\pi} (R_1 R_2 C_1 C_2)^{-\frac{1}{2}}$, it can be shown that the dual probe detector has better sensitivity-linearity characteristics than the single probe type. This result assumes that the bridge is balanced and that there are no phase shifts through the amplifier and feedback circuits, neither of which conditions holds in an actual oscillator. It was found that the frequency was considerably more sensitive

to capacitance changes in the series arm than in the parallel arm, and that in fact the performance of a single probe detector using the series arm of the bridge was almost as good as that of the dual unit. Since it required only half as much labour for construction, installation, and maintenance, the single probe detector was used in the later stages of the experiment.

3.5.2 Sensitivity and Linearity: We may write the theoretical frequency of oscillation for a single probe detector in the form $f = kC^{-\frac{1}{2}}$, where C is the total capacitance in the series arm of the bridge. The sensitivity may be defined in terms of the frequency deviation per cm. change in the elevation of the water surface, i.e. $s = -\frac{df}{dz} = -\frac{df}{dC} \cdot \frac{dC}{dz}$. But the theoretical value of $\frac{df}{dC}$ is $-\frac{1}{2}kC^{-\frac{3}{2}}$, or $-\frac{f}{2C}$, and $\frac{dC}{dz}$ is the capacitance of the probe per cm, which was calculated to be 6 pF/cm . The theoretical sensitivity is therefore $s = \frac{3f}{C} \sim / \text{cm}$. However, the sensitivity expressed in these terms is not in itself very useful. The detector signal will eventually be demodulated to give a voltage whose amplitude represents the instantaneous elevation of the water surface, and the true measure of sensitivity is the signal to noise ratio of this analogue voltage. This ratio cannot be determined a priori, but assuming the tape noise and demodulation noise to be a significant part of the entire system noise,

a useful lower bound for the system noise level can be obtained by a simple laboratory experiment.

The proposed tape speed for recording was $1\frac{7}{8}$ ips, the lowest available in the recorder used, since the higher speed ranges did not appear to offer any net advantage. At this speed the direct record system has a response that is flat up to 6 kc. A laboratory signal generator was set at a frequency in the middle of this range, and its output recorded on the tape. The taped signal was then played back through the Ampex demodulator which it was intended to use with the wave data signals, and then through a half-octave filter set at a centre frequency of $2\sim$, a typical wave frequency. The voltage of the filter output was measured with a RMS voltmeter. Since the input signal was not modulated, the entire output was noise. The measured value was about 1 mV , which was taken as indicating the order of magnitude of the noise voltage to be expected in the system.

Now the frequency deviation corresponding to the maximum linear half-range of the Ampex demodulator, when set up for a tape speed of $1\frac{7}{8}$ ips, is $1350\sim$. This deviation gives an output voltage of 1.414 V, so that a deviation of $1\sim$ corresponds closely to 1 mV . Hence, the estimate of system noise corresponds to a frequency deviation of the order of $1\sim$. With this information we are in a position to make use of our theoretical sensitivity

estimate.

We first obtain an upper bound for s . In Section 3.3 we estimated the maximum wave amplitudes expected at about $\pm 20 \text{ cm}$. Assuming that we use the full dynamic range of the demodulator, $\pm 1350 \sim$, it would appear that we could consider values of s up to $67 \sim / \text{cm}$. However, this would allow nothing for changes in the mean water level. If we wish to be able to record data when the mean level is within $\pm 50 \text{ cm}$ of some fixed value, the total change that we should be able to accommodate is $\pm 70 \text{ cm}$. This limits s to a maximum value of about $20 \sim / \text{cm}$. It should be noted however, that the restriction due to changes in mean level does not apply to an array such as that described in Section 3.6.5.

Now we shall find when we discuss linearity in the next paragraph that there are severe disadvantages in using any greater sensitivity than is necessary. What is our minimum requirement? Although the predicted wave amplitude of $\pm 20 \text{ cm}$ is relatively large, this represents the sum of all the component frequencies present, and the filtered components will become small at some frequencies. With an f^{-5} variation for the squared amplitudes, we expect the component an octave away from the spectral peak to have an amplitude of a couple of centimetres, and at two octaves away it should be in the millimetre range; these estimates of course imply a filter bandwidth, but we are

interested here only in orders of magnitude. To allow for the fact that our noise estimate is a lower bound, it was decided that the minimum detectable change of surface elevation should be of the order of a millimetre, or more precisely that the signal to noise ratio should be unity for waves of this amplitude. Hence a sensitivity of $1 \sim / \text{mm}$ or $10 \sim / \text{cm}$ is required.

The oscillator centre frequency was set at $4200 \sim$, towards the upper end of the linear demodulation range. Using the estimated minimum value of sensitivity of $10 \sim / \text{cm}$, we can compute the values of the capacitance C and the constant k in the formula $f = kC^{-\frac{1}{2}}$. Then by the use of the calculated probe capacitance per cm, we can express the oscillator frequency as a function of the surface elevation z :

$$f = 6.08 \times 10^4 (z + 210 \text{ cm})^{-\frac{1}{2}} \text{ cm}^{\frac{1}{2}} \text{ sec}^{-1}$$

This result exhibits the non-linear relation between the frequency deviation ($f - 4200$) and the surface elevation. If z varies between ± 20 cms the frequency varies between $4009 \sim$ and $4411 \sim$, so that the frequency deviation is $-191 \sim$ to $+211 \sim$. Applying the sensitivity $S = 10 \sim / \text{cm}$, we find that a surface displacement of ± 20 cms would appear from the analogue signal to be a displacement of $+19.1$ cms to -21.1 cms. This of course applies to the maximum anticipated surface displacement; the linearity improves rapidly as the displacement is reduced. It can readily be seen that high

linearity can only be obtained with this type of detector at the cost of increasing the loading capacitance, and hence of poor sensitivity and vice versa. This statement is strictly true only if the oscillator centre frequency is fixed. If this frequency is increased, the loading capacitance can be increased while maintaining a given sensitivity, and hence the linearity can be improved. The centre frequency is actually determined only to the extent that this frequency, plus or minus the maximum expected deviation, must be within the linear range of the demodulator. However, the value quoted above, 4200 ~, was selected with this in mind, and no further gains in this direction were possible. Although the indicated non-linearity was undesirably large, the associated sensitivity represented our minimum requirement; some compromise had to be made, and the value $S = 10 \sim / \text{cm}$ was adopted for design purposes.

It is worth remarking that although the relation between sensitivity and linearity expressed by the equations above is a fundamental limitation of the type of detector used in this investigation, this limitation can be overcome by the use of other circuit arrangements. Sensitive detectors of high linearity are now being designed at the IOUBC for use in the study of non-linear effects in the waves themselves.

3.5.3 Calibration and Stability: When the detectors had been built in accordance with the design criteria described,

it was necessary to check their performance by calibration, and to examine their short-period frequency stability and temperature stability. There is no completely satisfactory method for obtaining a dynamic calibration of a wave detector, and after unsuccessful attempts using standing waves in a tank, it was decided to rely on a static calibration. This was done first by varying the depth of probe immersion in a tank, and measuring the corresponding oscillator frequency; later an in-situ calibration was obtained, exploiting the rising tide to give varying depths of probe immersion.

Fig. 9 presents a measured calibration curve obtained in this way. A rough check of the equality of the static and dynamic sensitivities was obtained by comparing the visually observed height of a train of waves and that calculated from the demodulated output of the detector, using the static calibration. These were found to agree to the accuracy with which the wave height could be estimated visually.

The temperature stability was measured by placing the oscillator in a box containing a heater and a thermometer. The oscillator frequency was found to increase by $2 \sim$ per degree C. This was considered very satisfactory, since temperature changes occur much too slowly to affect the signal in the range of frequencies of interest, and the observed effect could influence the sensitivity only to a negligible extent. The short term stability was examined using a Hewlett-Packard counter. Over 20-sec intervals

(10 secs counting and 10 secs display) the frequency was found to be constant within one tenth of a cycle. Over 2-sec intervals the counter reads integral numbers of cycles only, and it was found that the unit figure occasionally changed by one unit. This was probably a rounding error, and it was concluded that for short periods the frequency was stable to better than one cycle per second. The quantity of operational significance, however, is the noise in the entire system, including that due to short term instability in the oscillators, pick-up in the leads, tape noise, and noise introduced in demodulation. This was obtained by making a "noise recording" with the probes not immersed, and measuring the spectrum of the signal thus obtained. The noise spectrum was subtracted from the measured wave spectrum to give the corrected spectrum. This procedure involves the assumptions that the noise is not correlated with the waves, and that its spectrum does not vary with time, both of which appear to be very reasonable.

3.5.4 Probe Fouling: We now turn to a serious problem encountered in the operation of the wave detectors. It is usually considered that teflon is not wetted by water, but under some conditions it was in fact wetted; the water in contact with the teflon, instead of instantaneously following the wave profile, adhered to the teflon in a thin film, which could be observed by the eye. When the water was clean, as judged visually and by the presence of small

capillary waves, the film could not be seen, or else it broke and drained almost instantaneously. At other times the film could easily be seen extending up the probe as far as the highest previous wave crest; it did not break, and drained very slowly. It was found that under these conditions the oscillator signal was not modulated, so that the waves were not recorded.

There were several clues to the cause of this effect. In the first place wave detectors employing capacitance probes similar to those used in this investigation had previously been used successfully at the same location. However, these had almost invariably been used under west wind conditions, while in the present case the detectors were used with east winds; it was assumed that the difference was due to contaminants being carried out of Vancouver harbour. Also it had been observed that visually clean water seemed to be associated with the occurrence of rain and of wave breaking, and the probes worked well under these conditions. The facts were consistent with the supposition that a surface film on the sea, probably more in the nature of a tenuous dispersion of organic molecules than a uniform monolayer, was responsible for the malfunctioning of the detectors. On this basis the action of rain or of wave breaking could be explained as the effect of driving the surface film into solution. This hypothesis was suggested by Dr. R.W. Stewart of IOUBC.

It was found by Dr. R.W. Burling of IOUBC that a similar wave detector using a blocking oscillator in place of the Wien bridge type would continue to modulate well in conditions where the Wien bridge type failed completely, even using the same capacitance probe. This suggested that the effect might be due to differences in the polarisation of the dielectric. Accordingly, the DC polarisation, which amounted to only a few tenths of a volt in the Wien bridge device, was removed completely by inserting a large capacitance in series with the probe. This had no apparent effect. A full investigation would require the application of a range of DC bias voltages of both polarities, but this was not done. The other apparent differences between the two oscillators were the size of the AC voltage swing across the probe, which was an order of magnitude greater with the blocking oscillator, and the waveform--that of the blocking oscillator containing very much higher frequency components, for which the film resistance would presumably be greater as a result of skin effect.

It was not necessary for the purposes of this investigation to elucidate the physical and chemical mechanisms of this fouling phenomenon, and the problem is still unsolved. Since rain frequently accompanies the weather conditions required, it was decided to record data

only when it was raining, when the probes invariably worked well.

3.6 The Wave Detector Array

3.6.1 Design of the Array: The principles of array design were discussed in Section 2.4, and in Sections 2.4.3 and 2.4.4 reasons were given for the selection of a linear array of seven detectors, spaced at intervals in the ratio of 1-1-4-4-4-3 or 1-1-1-5-5-4. It remains to select the size of the unit space.

In Section 2.4.6 it was shown that an array of this type may be regarded as an azimuth filter with a half-power bandwidth $\delta\theta = \frac{.035\lambda}{D}$ radians, $= \frac{2.0\lambda}{D}$ degrees, where D is the length of the unit space. While as we have seen, the true power density associated with any azimuth appears spread over the entire azimuth circle in the measured spectrum, the spreading is mainly confined to an angle $\psi = \frac{1}{2}\delta\theta$ on either side of the true azimuth. From a slightly different point of view, we may regard the measured spectrum as a running average over azimuth of the true spectrum, where the average is effectively over the angle ψ on either side of the azimuth concerned.

We have seen in Section 3.3 that the peak of the power spectrum under the planned experimental conditions was expected to occur at a frequency of about $0.5\sim$, corresponding to a wavelength of about six metres, and that the range of wavelengths of interest could be assumed to be from 1 to 25 metres. Table 3.1 presents the resolving power of the array in terms of the one-sided angle of spread ψ , for these wavelengths and for various values of the unit space D .

TABLE 3.1: ANGLE OF SPREAD ψ

$\begin{array}{c} \lambda \\ D \end{array} \begin{array}{c} m. \\ m. \end{array}$	1	2	4	6	10	15	25
1.0	-	2°	4°	6°	10°	15°	25°
0.67	-	3°	6°	9°	15°	22°	37°
0.5	2°	4°	8°	12°	20°	30°	50°
0.2	5°	10°	20°	30°	50°	-	-

The omissions in the bottom right corner reflect the fact that the resolution is becoming unacceptably poor. The criterion is not precise; with values of ψ as high as 90° or even somewhat greater, some information such as a rough

estimate of the mean direction, could be obtained, but at $\psi = 180^\circ$ even a single wave train would appear to be spread almost uniformly over the azimuth circle. In the present state of knowledge we must suppose that some information is better than none, and accordingly $\psi = 50^\circ$ was taken to be the limit of usefulness, poor though this performance is. Higher values of D , up to 2 metres, might be considered; with $D = 2$ metres, the overall span would be 34-metres, or about $1\frac{1}{2}$ times the wavelength of the longest waves considered. According to Section 2.4.7, this is the largest spacing that could usefully be employed. However, for a reason about to be discussed, an array with $D = 2$ metres could not be used to study wavelengths of less than 4 metres, and this was considered a more serious restriction than the loss of resolution at the longest wavelengths.

We now turn to the short wavelength limit referred to, which also accounts for the omissions in the top left of the table. By virtue of the assumed dispersion relation, the scalar wave number k is fixed when the frequency is known, and the maximum possible value of ℓ , the component of wave number along the line of the array, is equal to k . Now if $\lambda < 2D$, $k > \frac{1}{2D}$, and so $|\ell|$ may exceed $\frac{1}{2D}$. But from Section 2.4 we know that the apparent spectrum $\mathcal{E}(\ell)$ is given by the convolution of the true spectrum $E(\ell)$ with the function $G(\ell)$, where $G(\ell) = \left\{ \frac{\sin 35\pi\ell D}{\sin \pi\ell D} \right\}$, which has the period 2π in the argument $(2\pi\ell D)$. The graph of $G(\ell)$

shown in Fig. 6 is drawn for $0 \leq 2\pi lD \leq \pi$, and so represents a half period only; the rest of the function may be constructed knowing its period, and the fact that it is an even function. Hence $G(l)$ actually approximates an infinite Dirac comb, rather than the ideal single delta-function at the origin. However, as long as $|l| \leq \frac{1}{2D}$, only a single period of the function, or less, comes into play, and all is well. As soon as $|l|$ exceeds $\frac{1}{2D}$, which can occur if $\lambda < 2D$, it becomes possible for two or more "spikes" of $G(l)$ to take effect. If the true spectrum

$E(l)$ consists of a single line at $l = l_1$, the apparent spectrum $\mathcal{E}(l) = \int_{-\infty}^{\infty} \delta(l-l_1) G(l-l') dl' = G(l-l_1)$, which has peaks at $2\pi(l-l_1)D = 2\pi n$, $n=0, \pm 1, \pm 2$, etc.; the number of values of l for which the associated value of k could represent real waves is limited by the fact that $l \neq k$. However, if $\lambda < 2D$, we see that spurious lines can appear, and the total energy may then be divided between the possible values of wave number component, or direction, in an arbitrary way. It is to avoid the ambiguities thus arising that arrays are used only to study wavelengths such that $\lambda \geq 2D$; that is to say, the unit space must not be greater than half the wavelength of the shortest waves which it is desired to study.

Returning now to the table of values of ψ , we see that with $D = 1$ metre the resolution is excellent for wavelengths below that of the spectral peak ($\lambda = 6m$), and

fairly good to the longest wavelengths of interest, according to our rather imprecise criterion. However, with this unit space, we cannot study waves of length less than 2 metres at all. With $D = 0.67$ metre the resolution is fair for the long wavelengths, and the short wavelength cutoff is 1.3 metres. If we choose $D = 0.5$ metre, the resolution is still good to wavelengths beyond that of the spectral peak, but becomes poor at the longest wavelengths; the cutoff is now at only 1 metre. In this range the choice is largely a matter of the relative importance attached to the upper and lower limits of wavelength. Actually all of the tabulated values of D were used at various times in this investigation, the last one for a special purpose in which the wavelengths of interest were around 2 metres. As for the relative importance of long and short waves, it was felt that the identification of resonance waves was of special significance, so that the initial choice was made in favour of long waves, and the unit space was set at 1 metre. For reasons to be discussed, this was reduced in successive installations, first, to 0.67 metre and finally to 0.5 metre.

3.6.2 Description of First Array: Fig. 10 is a photograph of the first wave detector array used. It was taken at low tide, so that the sand is uncovered. The detectors, which were of the dual probe type, were arranged with relative spacings of 1-1-1-5-5-4, and with a unit space of 1 metre,

so that the overall span between the end detectors was 17 metres. The first group of four detectors, with neighbouring pairs 1 metre apart, was mounted on a frame consisting of two 4 in. (10 cm) aluminum pipe uprights, 5 m. high, and two horizontal cross members of $1\frac{1}{2}$ in. (3.7 cm) square aluminum tubing, 5 m. long, at heights of 2 m. and 5 m. above the bottom. The cross-members were bolted to lugs welded to the uprights, and welded lugs were also provided to attach the guy wires by which the frame was braced. The probes were attached to the top rail of the frame by Amphenol elbow fittings, and secured at the bottom rail by locally made clamps. The 4 oscillator units were housed in a wooden box carried on the top rail, which passed through a false bottom in the box. The cables from the probes to the oscillators were routed internally through the top rail, and passed into the wooden box via BNC chassis mount connectors on the rail, inside the false bottom. The only external wiring was that connecting the array with the IOUBC tower, i.e. the relay and signal cables; these passed into the false bottom of the electronics box, and thence into the box proper, through rubber stoppers. Disconnects were provided for all cables inside the false bottom, and the electronics box could be removed from the frame without disturbing any cable runs. The oscillator-amplifier units with their relays and batteries were housed in individual

plug-in aluminum boxes, with all external electrical connections made through 8-pin Jones plugs. A lid with a deep flange, secured by snap clips, was provided on the electronics box so that an unserviceable oscillator could be removed without removing the electronics box. This rather complex arrangement was devised to ensure weather-tightness and ease of maintenance.

The fifth detector was separately mounted on the main instrument mast of the IOUBC research station, and the electrical and other arrangements were similar to those already described. The sixth and seventh were mounted on a frame identical to that carrying the first four detectors, except of course that this frame carried only two detectors, at a separation of 4 metres. The two frames were arranged to be in line with the detector on the main mast, and placed so that the separations between the fourth and fifth, and between the fifth and sixth detectors were each five metres.

3.6.3 Operational Problems: The methods adopted to deal with most of the difficulties that had been foreseen were on the whole successful. Thus, rain and salt spray, for example, offered no problem. One difficulty that had been foreseen, however, turned out to be very much more serious than had been anticipated. This was damage caused by logs and other floating debris. It had been realised that a large log, weighing many tons, and propelled by wind and current at speeds of up to two knots, could completely wreck one of

the frames, or even the entire array. Based on previous IOUBC experience at the site, however, the probability of such a strike was considered to be small, even though the array was evidently a more efficient "log-catcher" than a single guyed mast. With the benefit of hindsight, the soundness of this judgment may perhaps be questioned, but the most important factor affecting the outcome could hardly have been foreseen. By the time the array had been constructed and installed the early summer run-off was in spate. It proved to be abnormally heavy, and the levels of the Fraser and all other rivers in the area were higher than they had been for years. As a result, debris which had laid high and dry on the river banks for several seasons was carried away, and the number of logs, pieces of timber, trees, and branches of trees floating in Burrard Inlet was greater by an order of magnitude than it had been in the previous few years. The number of these articles which struck the array was also many times more than the most pessimistic expectation. The extent of the resulting damage varied: a light strike might bend and twist the probes, tear the teflon, knock one of the frames out of line, or break the elbow connectors fastening the probes to the top rail; a heavy strike could bend and twist the frame members, tear up the guy-anchors, collapse the frame and immerse the electronics in the sea. All of these events occurred, except that the electronics boxes had been removed

prior to the first major disaster; as a result of experience it had been decided that the electronics would be safer if located remotely in the IOUBC tower; the penalty of reduced sensitivity due to the excess cable capacitance was regarded as the lesser of two evils. Minor damage occurred on average about once every two days, and it was found that when the equipment was serviceable the weather was not, and vice versa. Persistence was eventually rewarded, to the extent that both periods of serviceability coincided for a couple of days. It was then that the probe fouling problem was encountered. This has already been discussed in Section 3.5.4, and the means by which it was circumvented, if not solved, has been described. At this stage, following a final disastrous log strike, the first array was dismantled.

3.6.4 Description of Second Array: After a period spent in investigating the fouling problem, a second array was designed and built. Since it was by this time late in the summer the run-off was much reduced, and the amount of debris in the Inlet was much less. For this reason, and because there seemed to be no easy alternative, the second array was also mounted on the bottom. In order not to endanger the main mast all seven probes were mounted on the two five-metre frames, which were moved to a new location away from the mast. The dual probe detectors were replaced by a single probe type, largely to reduce the labour of manufacture and maintenance. The fragile elbow connectors

and the internal wiring arrangements were superseded for obvious reasons by a less elegant but more serviceable arrangement, using external cables. A relative probe spacing of 1-1-4-4-4-3 was adopted since it permitted a slightly greater unit spacing than the 1-1-1-5-5-4 arrangement when the restrictions associated with mounting all seven probes on the two frames were taken into account. The unit space was 0.67 metre. In other respects this design was generally similar to the first. The installation was demolished by a log strike before any data could be obtained.

3.6.5 Description of Third Array: It was finally realised that a bottom-mounted array was not feasible under the conditions existing in Burrard Inlet, and a design for a third array was conceived, in which the over-riding consideration was immunity from damage by floating debris. This array was constructed and operated successfully. Fig. 11 is a photograph of the installation.

The design objectives were mostly the same as for the earlier arrays, but certain additional requirements were specified in the light of experience. The most important were:

- (a) the probes should be in the water only when recording
- (b) the electronics should be indoors
- (c) there should be no underwater cable runs

- (d) all installation, adjustment, inspection, and removal should be possible from the platform
- (e) probe cables should be as short as possible
- (f) the depth of probe immersion should be adjustable
- (g) it should be possible to set the probe spacings accurately, and to maintain this setting
- (h) the mounting structure should offer the minimum possible obstruction at any azimuth, but no obstruction at all could be tolerated on the east side of the array.

The first three of these requirements were specified to provide protection against damage by debris; the second and third are not essential, but were considered to be an easier alternative to providing complete immersion-proofing. The fourth requirement amounted to a necessity, since this array was built during the winter months. The fifth and sixth are consequences of the second, and have the aim of reducing unwanted loading capacitance in the oscillator circuit. The need for the seventh is obvious, but was emphasised by the difficulty encountered in maintaining the probe alignment and spacing with the bottom-mounted arrays. The final requirement is associated with the aim of the experiment, to measure waves travelling from the east; for other azimuths the purpose was to avoid the possibility of wave reflection.

The solution to the problem posed by this specification is illustrated in Fig. 11. It involved the use of a probe-mounting frame suspended from a large cantilever structure which was bolted to the heavy beams supporting the platform of the IOUBC tower. The beams of the cantilever projected about 8 metres on the east side of the tower, and carried steel tracks on their lower inside faces with roller hangers to support the probe frame. Ropes passing over pulleys at the outboard end of the tracks allowed the frame to be rolled out to the end of the cantilever when recording data. When not in use, the frame could be pulled in, raised by means of pulley blocks attached to the hangers, and swung up completely out of the water, with the bottom of the frame resting on a specially constructed walkway. A pair of bracing ropes ran from the bottom of the frame to the walkway, and another pair ran out to pulleys anchored on the bottom, about 6 metres beyond the end of the cantilever, and back to the walkway.

The top member of the probe frame was an aluminum I-beam 10 m. long with 3 in. (7.5 cm) flanges and 4 in. (10 cm) web. There were four uprights of $1\frac{1}{2}$ in. (3.7 cm) square aluminum tubing, and the bottom member, also 10 m. long, was of the same material. The structure was bolted together using aluminum fishplates and angle brackets. This frame was considered to be the largest that could be handled in the proposed installation, and experience

confirmed this view. The length limitation obliged us to reduce the unit spacing of the array to 0.5 metre. Even so, the frame overhung the platform by about a quarter of its length at each end, and the frame suspension had to be designed to permit sliding the frame lengthwise in its supports, in order to get access to the end probes for installation and maintenance. The probe connectors at the top end were housed in wooden boxes to provide weather-proofing, and both the probe and the probe cable passed through rubber stoppers in the bottom of the box. The top of the box had an overhanging flange, by which it was bolted to the lower flange of the I-beam. The probes were secured to the lower member of the frame by electrical tape.

This array was immediately successful, and the results presented in Section 5 were derived from measurements made with it. It was found that the large physical size of the frame made handling rather awkward, but it was manageable. Perhaps the greatest inconvenience was the poor accessibility of the end probes, but at least they could be reached. During the whole period that the array was in use, no mechanical damage of any kind was suffered.

3.6.6 Description of Fourth Array: A fourth array was constructed in the following year for the purpose of examining waves with a spectral peak wavelength of two metres, so that water depth effects should be minimised.

This array followed the general design of the third, but the unit space was reduced to 0.2 metre. Because of the smaller anticipated wave height, the sensitivity loss due to the capacitance of the long probe cables could not be tolerated, and the electronics box was mounted on the top member of the frame. Provision was also made for swivelling the frame and locking it at angles from -90° to $+90^{\circ}$, in 15° steps, so that winds from directions other than the east could be exploited. The exact weather condition sought did not occur, and the measurements made were not analysed.

3.7 Measurement Procedure

The conditions required to warrant a data run were quite restrictive. As we have seen, we have specified a range of wind speeds and wind directions, and we need a high tide with zero current, within about 10 cm/sec. The wind must be as steady as possible in both speed and direction, and must have been so for a considerable time before the start of the run; a minimum value for this time is given by the longest fetch divided by the group velocity of the shortest waves to be studied. Not all high tides could be used, because at the highest tides a deep flat-sided structural member of the IOUBC tower became immersed, and it was feared that this would cause wave reflection.

Then the waves present must be pure wind waves; the principal trouble here was that Burrard Inlet is the entrance to Vancouver harbour, so that there is a lot of ship traffic, with associated ship waves. Swell from the west did not appear to be a problem in the weather conditions of interest; certainly none was visible, and with steady gradient winds none should be expected, since Juan de Fuca Straits are effectively blocked by islands and any swell entering from the Pacific should be quickly dissipated. Finally the fouling problem added the requirement that it should be raining; this, however, was more usual than not, with east winds in the winter months.

It might well be asked whether all of these conditions are ever met simultaneously. In fact, it was found that they were not infrequent, except for the absence of ship waves. One run, however, was made in ideal circumstances, with no ships in sight during the entire run. The best section of this run, of 32 minutes duration, was analysed in detail. The choice was governed primarily by the zero current condition.

At the start of a run the probes were first carefully cleaned with moistened paper towels. The frame was then lowered into the water, run out to the end of the cantilever, and the bracing ropes tightened. The oscillators were then started and switched on to separate direct record channels in the Ampex 14-channel tape recorder. Another

direct record channel was used for voice announcements, which included the date, purpose of the run, time checks, comments on the wave state, wind speed and direction, tide level, current, etc. It was found necessary to be extremely careful in monitoring, and in addition to displaying the oscillator output signal on the scope, where the varying frequency could be directly observed, the demodulated signals from the tape recorder were continuously displayed on a two-channel Sanborn paper recorder. The signals were switched from time to time to keep a check on the performance of all seven detectors.

A Savonius rotor type meter was used at first to measure the current speed and direction. However, since the currents of interest were at or near zero speed, another technique was adopted which was found to give a more accurate indication of this condition. A piece of paper towel was thrown into the water, and when this had become thoroughly sodden, and was lying below the surface, its passage past the pilings of the tower was timed. When the current speed was nearly zero, it was difficult to distinguish any mean motion among the turbulent and orbital motions, but velocities of the order of 1-10 cm/sec could easily be detected.

The wind speed was measured by a Casella 3-cup anemometer, which was mounted on the south end of the top member of the probe frame, with the cups half a metre above

the top of the frame, and 2 metres above the water surface. The anemometer was thus well removed from any obstruction to the airflow. The turning shaft opened and closed a pair of electrical contacts, and the number of contacts was counted by an electro-mechanical counter. Differences in the counter readings over a period of one minute give the wind speed, using the calibration chart supplied.

Wind direction was obtained from a Casella vane of the distant-reading type. This was fixed to an upright on the tower about 4 metres above the platform. It was aligned so that the instrument north corresponded to the bearing of the Point Atkinson lighthouse; the true reading could then be obtained by subtracting 18° from the indicated bearing. A smaller vane, aligned along the line of the array was mounted beside the anemometer and the probe frame. The difference in the readings, which should of course be the angle between the line of the array and the bearing of the lighthouse, was found to be constant, as accurately as readings could be made.

Section 4: DATA ANALYSIS

4.1 Outline of Data Analysis Procedure.

Section 4: DATA ANALYSIS

4.1 Outline of Data Analysis Procedure

It will be recalled from Section 3.2 that the wave detector outputs were recorded on a multi-channel magnetic tape in the form of F.M. signals. The method by which these seven simultaneous records were subsequently analysed to obtain the directional spectrum will now be outlined.

The first stage in the analysis was to play back the signals in pairs through demodulators, thus producing analogue signals representing the wave height to a known scale. The playback was carried out at several times the recording speed, which of course increased the signal frequencies by a factor equal to the speed ratio. The maximum voltage swing of the analogue signals was of the order of one volt peak to peak. The simultaneous pair of the demodulated signals was then passed through matched narrow-band filters to isolate a particular frequency band component. The filters used provided for amplification, and the gains were set to maintain a one volt RMS level in the output signals. The signals from the filters were next passed to the input of an analogue computer. The dynamic range of the computer was 100 volts, and most of this range had to be utilised in order to obtain the greatest possible

computational accuracy. The first computer operation was therefore the preamplification of both signals to give a gain of about 70. The signals were then passed to the multipliers and phase shifter. For input signals $X(t)$ and $Y(t)$, these stages produced the signals $X^2(t)$, $Y^2(t)$, $XY(t)$, and $X\tilde{Y}(t)$, where the tilda denotes a phase advance of $\frac{\pi}{2}$ in the signal to which it applies. These signals were passed to four integrators, which produced $\int_0^T X^2(t) dt = \overline{X^2(t)} \cdot T$, etc. where T is the integrating time. The integrated square and product signals were recorded on Varian paper recorders. The final output of the analogue computer thus yielded $\overline{X^2}$, $\overline{Y^2}$, $\overline{XY}$, and $\overline{X\tilde{Y}}$, where $X(t)$ and $Y(t)$ are the scaled values of one frequency component of the analogue outputs from two of the seven wave detectors. At each pass of the tape we therefore developed one point of the cospectrum $C(X,Y,f)$ and quad-spectrum $Q(X,Y,f)$ surfaces, namely, that corresponding to the separation given by the particular pair of detectors from which the signals $X(t)$ and $Y(t)$ were obtained, and to a frequency equal to the scaled value of the centre frequency to which the filters were set. It may be noted that $C(0,0,f)$ is the power spectral density.

The derivation of the power spectrum from the quantities given by the analogue computer called for no further data processing, except for simple scaling and

correction of the calculated spectrum for the effects of system noise.

The directional spectrum was derived, for all frequencies analysed, using the Fourier-Bessel method, with the values of the co- and quad-spectra obtained from the analogue computation. The principles of the method have been presented in Section 2.3.3, and the practical technique for obtaining the coefficients is given in Appendix 3. The numerical work was carried out by an IBM 7040 digital computer. Smoothing of the calculated directional spectrum was effected using the Cesaro technique described in Appendix 1.

The results obtained by the previous method were checked, for a few frequencies only, by computing the spatial Fourier transform of the cross-correlogram given by $C-iQ$. As was shown in Section 2.3.2, this operation also gives the directional spectrum. The numerical values of C and Q were those given by the analogue computer, and a Friden desk calculator was used for the computation. For one frequency only, the directional spectrum thus obtained was recalculated using the cosine lag window smoothing technique ("Hanning") described in Appendix 1.

Section 5: RESULTS

5.1 Summary of Results.

Section 5: RESULTS

5.1 Summary of Results

The results of the investigation are presented in Figures 14-41 inclusive. Fig. 14 gives the power spectrum on a linear plot. This has been corrected for system noise as described in Section 3.5.3, and covers three octaves in frequency. The same data are presented in Fig. 15, using a log-log plot. In Fig. 16 the measured values of the cospectrum C, and quadrature spectrum Q, are shown for one of the middle frequencies, just below the peak of the power spectrum. Fig. 17 gives the coherence R and phase ϕ , defined by:

$$R(X) e^{i\phi(X)} = \left\{ \frac{C(X) - i Q(X)}{C(0)} \right\}$$

where X is the spatial separation of the pair of detectors involved; the data apply to the same frequency as Fig. 16. Fig. 18 is an example of the directional spectrum $E(l,f)$ computed by the Fourier transform method from the data of Fig. 16. This spectrum is replotted in Fig. 19 with azimuth as the abscissa. The transformation from $E(l,f)$ to $E(\theta,f)$ was carried out according to the method of Appendix 5. A smoothed version of the spectrum of Fig. 18

is presented in Fig. 20, which was computed using the cosine lag window as described in Appendix 1, and the corresponding spectrum $E(\theta, f)$ is given in Fig. 21. The main body of the results is given in Figs. 22-41, and comprises the directional spectra $E(\theta, f)$ computed by the Fourier-Bessel method. The Cesaro smoothing technique, described in Appendix 1, was used, and the Fourier-Bessel equations were solved by the method given in Appendix 3.

Section 6: DISCUSSION OF RESULTS

6.1 The Power Spectrum.

6.2 The Directional Spectrum.

6.2.1 The Miles Regime.

6.2.2 The Phillips Regime.

Section 6: DISCUSSION OF RESULTS

6.1 The Power Spectrum

The power spectrum of the water surface displacement is presented in Fig. 14. For the purposes of this investigation, the feature of greatest interest in the power spectrum is the frequency at which it peaks, for which a theoretical estimate was obtained in Section 2.2.3. This estimate was based on a model in which the spectrum rises very rapidly to a peak, from the low levels characteristic of the low frequencies, and really applies to the frequency of this rapid rise rather than the peak. The rise of the measured spectrum is not so abrupt, although this is probably due in part to the finite bandwidth of the filters. Allowing for this effect, we may take the measured values as indicating a rapid rise at $f = 0.48 \sim$, and the peak frequency is $0.52 \sim$. The predicted value for the frequency of the rapid rise was $f = 0.52 \pm .05 \sim$, which agrees well with the measurements. The agreement is actually somewhat better than this, for the prediction was based on the upwind fetch, whereas from Fig. 3 we see that transition should first occur, according to the theory, for waves travelling about 6° off the wind, and the transition frequency is more accurately estimated at $f = 0.50 \sim$. Although the central interest in this investigation is in the directional spectrum

the theoretical predictions for the latter, at least in the Miles regime, depend upon an extension of the argument used in predicting the peak of the power spectrum. The excellent agreement obtained in this case therefore increases our confidence in the predictions for the directional spectrum.

As a matter of secondary interest we may determine the best-fitting power law, $\phi(f) = A f^{\sim}$, for the high frequency part of the power spectrum. The spectrum is replotted in log-log form in Fig. 15, from which we estimate the values of the parameters to be $A = 7.9 \text{ cm}^2 \text{ sec.}$, $\sim = -4.2$. These values agree reasonably well with earlier measurements.

6.2 The Directional Spectrum

The characteristics of the directional spectrum are discussed separately for the Miles regime (frequencies above that of the peak of the power spectrum) and for the Phillips regime (frequencies below that of the peak of the power spectrum).

6.2.1 The Miles Regime: The principal characteristics of the directional spectrum to be expected on the basis of the theory were presented in Section 2.2.4. For the Miles regime the predictions gave the variation with frequency of the mean direction of wave travel, and of the angular spread of the waves. It will be recalled that these quantities were defined in terms of the angles Θ_1 and Θ_2 , the

azimuths between which high spectral values were predicted. In order to compare the theoretical predictions with the results of measurement, we identify θ_1 and θ_2 with the azimuths of the half-power points of the measured spectra, i.e. with the azimuths where the spectral density has half its peak value for the frequency concerned. In Table 6.1 these values, together with the mean direction of travel $\bar{\theta} = \frac{1}{2}(\theta_1 + \theta_2)$, and the angular width of the spectrum $\Delta\theta = (\theta_2 - \theta_1)$, are tabulated using array coordinates.

TABLE 6.1: EXPERIMENTAL VALUES OF $\bar{\theta}$ AND $\Delta\theta$

f	.500	.520	.566	.618	.674	.735	.802	.875	.954	1.04	1.14	1.24
θ_1	86	82	77	78	74	52	47	41	30	37	12	10
θ_2	142	136	141	142	142	146	147	145	148	147	138	130
$\bar{\theta}$	114	109	109	110	108	99	97	93	89	92	75	70
$\Delta\theta$	56	54	64	64	68	94	100	104	118	110	126	120

In Figs. 42 and 43, the experimental values of $\bar{\theta}$ and $\Delta\theta$ are compared graphically with the theoretical predictions. A scale of wind coordinates is given in Fig. 42 to emphasise the difference between the mean direction of travel of the waves, and the wind direction.

The first conclusion to be drawn from Fig. 42 is that the direction of wave travel is indeed displaced from the wind direction in the sense predicted by the theory. Further, the amount of the displacement is in order of magnitude agreement with the predictions. Although the definitions adopted for the angles θ_1 and θ_2 are somewhat arbitrary, it is easily seen by inspection of the directional spectra that the value of $\bar{\theta}$ is affected only slightly by any reasonable definitions of these angles.

The quantitative agreement of the theoretical and observed mean directions is good over the middle frequencies of this range. The results for the highest three frequencies are probably becoming unreliable, since the wave energy is very small. The experimental points are scattered on both sides of the theoretical curve, so that we may consider the results as confirming the theory for the high frequencies also.

The discrepancy between the measurements and the predictions is greatest at the lower end of the frequency range, and it increases with decreasing frequency. The following argument is offered as a possible explanation. Although at these frequencies the Miles mechanism is strongly operative, it may not have obliterated all traces of the resonance waves. But there is reason to suppose that the resonance wave approaching from north of the array normal has much greater energy than that from the south; this will

be explained in Section 6.2.2. Effectively, therefore, the resonance waves are displaced from the wind direction in the same sense as Miles waves, and at the frequencies concerned they are displaced to a greater extent. The resultant effect would be an increase in the net displacement, as observed. This possibility is supported by the fact that the angle θ_1 is in fairly good agreement with the predicted value, and the discrepancy in $\bar{\theta}$ arises mainly because θ_2 is greater than the theory indicates. The effect suggested is not amenable to treatment by the existing theories.

Another effect arises from the variability of the wind direction, and the irregularity of the fetch. The wind direction was observed to vary by $\pm 10^\circ$ about the mean direction. Normally, with a symmetrical fetch, variation of wind direction would not be expected to influence the mean direction of wave travel. The effect in the present case is most easily understood by an examination of Fig. 3, considering the result of a translation in azimuth of the U-shaped curves relative to the geometric fetch curve. For a variation of $\pm 10^\circ$ the maximum displacement is 2° , at a frequency of $0.566 \sim$. This effect is small, but it has the sense required to account for the discrepancy, and it occurs in the frequency range where the discrepancy is greatest.

The angular spread results are shown in Fig. 43, from which it is seen that the observed values agree fairly well with the theory both in order of magnitude and frequency trend. In this case the magnitude does depend considerably on the definitions adopted for θ_1 and θ_2 , and exact agreement is hardly to be expected. The greatest discrepancy is again at the low end of the frequency range. The greater observed spread at these frequencies is consistent with the argument that residual resonance waves are present. The effect of wind direction variability would of course be to increase the observed angular spread; while the observed spread angles do lie above the theoretical curve no definite conclusions should be drawn from this fact in view of the arbitrary definitions of θ_1 and θ_2 .

6.2.2 The Phillips Regime: According to the theoretical considerations of Section 2.1 the wave energy at the low frequencies should be concentrated around two azimuths equally displaced from the wind direction. The displacement is given by Phillips' resonance angle, $\pm \cos^{-1} \frac{c}{U}$, the values of which are tabulated for each frequency analysed, in Section 2.2.4.

The directional spectra for these frequencies, shown in Section 5, do not exhibit the bimodal characteristic that would be expected according to a straightforward application of the theory. We shall return to this point

later, but first let us consider the mean direction of travel of the waves actually observed. The azimuth of the maximum spectral density is probably the most appropriate measure in this case, although the choice is not critical. The experimental values are given in Table 6.2 together with the theoretical angles for resonance waves from Section 2.2.4; to facilitate comparison, the experimental angles are quoted in wind coordinates.

TABLE 6.2: EXPERIMENTAL VALUES OF $\bar{\theta}$

f	.261	.284	.310	.338	.368	.401	.437	.476	.500	.520
$\bar{\theta}$ Expt.	41	45	46	45	44	48	49	48	40	32
$\cos^{-1} \frac{C}{U}$	22°	30°	36°	41°	45°	50°	53°	55°		

Agreement would not be expected for the highest two frequencies; since as can be seen from the power spectrum the Miles mechanism is acting in these cases; the highest frequency in fact represents the peak of the power spectrum. But what can be said as regards the rest of the range? The order of magnitude is certainly correct, but the frequency trend of the experimental results--if indeed a trend exists--does not agree at all well.

However, we have so far ignored the fact that waves of the frequencies involved are not effectively deep-water waves for the last part of their travel, as they traverse the sandbank on which the IOUBC experimental station is located, so that the waves will be refracted. The sandbank is shown shaded in Fig. 1, and the bottom topography is described in Section 3.4. A larger scale map of the bank is shown in Fig. 44, from which it may be seen that the edge of the bank can be represented quite well by two straight lines. For all directions of travel under consideration the waves will cross the edge of the bank where the latter bears 13° south of east. The bottom contours are such that this line may be considered to divide the fetch into a deep water region north and east of the line, and a shallow water region of uniform depth from the line to the measuring site. The depth in the shallow region at the time of the measurements was 3 metres. In this simple situation it is possible to estimate the refraction effect quite accurately.

The problem is to find the direction of travel of the incident wave over most of the fetch, where the water is deep. The geometry of the idealised model is given in Fig. 45. If the wave phase speed in deep water is c_0 , and in a depth of 3 metres is c , the simple refraction formula gives:

$$\frac{\cos \phi_0}{\cos \phi} = \frac{c_0}{c} = \frac{\lambda_0}{\lambda} = \coth \left(2\pi \frac{h}{\lambda} \right) \quad 6.1$$

where h is the water depth, and λ the wavelength in the shallow region; the suffix zero indicates deep water quantities, and the angles ϕ and ϕ_0 are defined by Fig. 45. To avoid iteration, let $u = \frac{2\pi h}{\lambda}$, and $A = \frac{4\pi^2 h f^2}{g}$; then $\frac{\cos \phi_0}{\cos \phi} = \frac{u}{A}$ where u is the solution of $A = u \tanh u$, which is easily found graphically. From Fig. 45, $\phi = \bar{\theta} + 1$, and hence $\phi_0 = \cos^{-1} \left\{ \frac{u}{A} \cos(\bar{\theta} + 1) \right\}$. The angle we wish to find is $\bar{\theta}_0$, the angle between the wind direction and the direction of travel of the wave in deep water. From the figure this is seen to be:

$$\bar{\theta}_0 = \phi_0 - \phi + \bar{\theta}$$

$$= \phi_0 - 1$$

$$= \left[\cos^{-1} \left\{ \frac{u}{A} \cos(\bar{\theta} + 1) \right\} - 1 \right] \quad 6.2$$

The mean directions of travel, corrected for refraction effects according to equ 6.2 are given in Table 6.3.

TABLE 6.3: CORRECTED VALUES $\bar{\theta}_0$

f	.261	.284	.310	.338	.368	.401	.437	.476	.500	.520
$\frac{u}{A}$	1.268	1.202	1.142	1.094	1.060	1.032	1.013	1.013	1.000	1.000
$\bar{\theta}$	41	45	46	45	44	48	49	48	40	32
$\bar{\theta}_0$	18	$32\frac{1}{2}$	38	$39\frac{1}{2}$	$40\frac{1}{2}$	$46\frac{1}{2}$	48	48°	40	32
$\cos^{-1} \frac{C}{U}$	22°	30°	36°	41°	45°	50°	53°	55°		

As expected, the effect of refraction is small for the higher frequencies in this range, but becomes appreciable at the lower frequencies. The net effect is to give much better agreement between the experimental results and the theoretical resonance angle, particularly in regard to the frequency trend. The comparison is shown graphically in Fig. 46. The agreement is such that one can scarcely avoid the conclusion that the observed waves represent one of the two trains predicted in Phillips' theory, especially as it would be difficult to account for the results on any other basis. However, if this is so, what has happened to the other wave train?

The first part of the answer to this question is to be found from an examination of Fig. 2. The observed waves approach the array from directions to the north of the wind direction, where the available fetch is about 8 km. Waves coming from directions equally inclined to the wind, but to the south of the wind direction, have a fetch of only about 1 km. Now from Section 2.1 the spectral density of waves in the Phillips regime increases linearly with the wind duration, or equivalently, with the fetch. Hence, we would expect the second spectral peak to be much lower than the first, of the order of 12%, on the basis of the asymmetry of the fetch. But the waves corresponding to this hypothetical second peak travel in shallow water over their entire fetch, and moreover, as described in Section 3.4,

the depth decreases in an irregular way towards the shore. As a result the waves experience continuous variation in wavelength and speed of travel, and in direction of travel also, because of refraction. But the resonant response of Phillips' mechanism depends upon the waves remaining in step with the forcing pressure pattern. If for some wave-number component of the pattern and some free wave the necessary condition $U \cos \theta = c$ holds at a given instant, and if θ , c , and λ then change arbitrarily, as dictated by the bottom topography, the condition will no longer be satisfied and so the wave cannot continue to grow resonantly. Actually if the fetch were divided into regions of approximately constant depth, we might expect that in each region resonance waves of a given frequency would start to grow, each with a different direction of travel. Each wave would cease to grow as it passed out of the region in which it was formed, but it would continue to suffer refraction. The total energy of this frequency observed at the measuring site would be the total energy of the waves from each region, each with its own refracted direction of travel, plus the energy of the forced waves. Since the theory predicts a linear rate of energy growth, the integral over azimuth of the energy of the waves coming from south of the wind direction might be expected to be of the order of 12% of that for waves from the north, as estimated on the basis of the fetch. Because of refraction and the

different directions of travel as generated, the observed energy would be spread out in azimuth, and especially for the lower frequencies would tend to concentrate near the array normal, which is roughly parallel to the contour lines, and so would be lost in the tail of the first spectral peak. In other words this argument suggests that no second peak should be discernable under the conditions of the experiment.

If this argument is correct it disposes of the principal obstacle to the identification of the observed waves as one of the trains predicted by Phillips, and the good agreement in magnitude and frequency trend, and especially the sudden jump in the off-wind angle below the frequency of the spectral peak, constitutes strong evidence for the validity of Phillips' theory. It would, however, be very desirable to repeat the experiment under conditions of symmetrical fetch and in deep water, since observation of the bimodal spectrum would be the most convincing proof of the theory. Another investigator at IOUBC plans to make such measurements in the near future.

The experimental and theoretical values of $\bar{\theta}$ for both the Miles and Phillips regimes are brought together in Fig. 47, primarily to illustrate the jump in the off-wind angle at the frequency where the dominant mechanism changes. The jump is a notable feature of the experimental results, but it occurs less abruptly than a simple application of the theory would predict. Evidently as already suggested in

Section 6.2.1 there is a band of frequencies centred on the theoretical transition frequency (0.50~) in which the effects of both mechanisms may be seen.

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APPENDICES

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APPENDIX 1

Smoothing Techniques

(a) Lag and Spectral Windows:

One method for deriving the one-dimensional spectrum from a continuous experimental record involves the computation of the autocovariance, followed by the Fourier transformation of the latter. It is usual, especially when the length of the record is short, to weight the autocovariance before transforming, using a weighting function with a value of unity at zero lag, and falling to zero for the largest lag. The weighting function is often called a "lag window", by an obvious analogy. The effect on the computed spectrum may be calculated--the resulting spectral window is the Fourier transform of the lag window. With discrete data it is usual to apply the spectral window after the transformation rather than the lag window before this step; the two processes are equivalent. Qualitatively the effect is to smooth the spectrum; the spectral density estimate for any frequency is a running average over frequency of the unsmoothed spectrum. Since the autocovariance is known only up to some finite maximum lag, it is obvious that some lag window must be used; the omission of any explicit weighting process is equivalent to adopting a rectangular window. The reason for smoothing is that fine

detail in the spectrum is likely to be peculiar to the particular sample of the phenomenon represented by the experimental record, whereas the broad features that remain after smoothing should represent the phenomenon rather than the sample.

In the Fourier transform method for obtaining the directional spectrum (see Section 2.3.2) the final step is the Fourier transformation of the function $C-iQ$. This is exactly analogous to the above process for the power spectrum, and the same case might be made for smoothing in the same way. However, there are practical differences, as pointed out by Barber (1963a). In studying ocean waves, reasonable statistical stability of the sample is fairly readily obtained by making the averaging time long enough. The "expensive" commodity is the limited number of space lags that can be obtained; the dimensions of the array fix the span of the lag window, and the number of detectors used determines the number of lags available within this span. It might seem best, with a limited number of well-determined lags, to use a rectangular lag window rather than downgrading good data by tapering the correlogram. The title of his paper: "A Plea for the Rectangular Lag Window" indicates Barber's opinion. However, the spectral window corresponding to this lag window has many negative lobes, so that unless the true spectrum is fairly flat, the

estimated spectrum may exhibit regions of negative power, which of course are not realistic. It can be shown that the rectangular lag window leads to an estimate of the spectrum with the smallest possible error in the least squares sense. However, this is not necessarily the best estimate, since we have more information than is contained in the measured record: namely, the fact that the power is not negative. With this in mind, we might reasonably decide to use the rectangular lag window, provided that it leads to a spectrum which does not exhibit any significant region of negative power. If negative power does appear, then some tapering, sufficient to remove the negative regions, should be used.

The Fourier transform method was actually used only for a few frequencies, to check its agreement with the Fourier-Bessel method. In one of these cases ($f = 0.437 \sim$) smoothing was applied in order to observe the effect. The window used was the "Hanning", or cosine lag window (Blackman and Tukey; 1959), which greatly attenuates the side lobes in the corresponding spectral window, compared to the rectangular window. The Hanning weighting function is:

$$W(X) = \frac{1}{2} \left[1 + \cos \left(\pi \frac{X}{X_{\max}} \right) \right] = \cos^2 \left(\frac{\pi}{2} \frac{X}{X_{\max}} \right)$$

which multiplies the values of $C(X)$ and $Q(X)$ for the frequency concerned. These are known for $X = nD$, $n = 0, 1, 2, \dots, 17$, and $X_{\max} = 17D$. Hence, we may

write the weighting function:

$$W(n) = \cos^2\left(\frac{\pi}{2} \frac{n}{17}\right)$$

The modified co- and quad- variances: $C'(nD) = C(nD) \cdot W(n)$,
 $Q'(nD) = Q(nD) \cdot W(n)$ were then transformed to obtain the smoothed spectrum.

As expected, the spectrum obtained was similar to the unsmoothed spectrum, but with a lower and wider peak. The peak occurred within 2° of the same azimuth. One unexpected feature was that the integral of the spectrum over azimuth was about 12% lower than for the unsmoothed spectrum. This discrepancy is explained in Appendix 4, where an investigation of the normalisation of the spectrum is presented.

(b) Cesaro Smoothing

In the Fourier-Bessel method (Section 2.3.3) the coefficients of the Fourier series for $\mathcal{E}(\theta)$ are determined by a technique which gives estimates of decreasing accuracy as the order of the coefficient increases (Appendix 3). An overestimate of one of the higher order coefficients can evidently lead to negative estimates of the power over some range of azimuth. This suggests a smoothing of the coefficients.

Even if the coefficients could be determined exactly, the fact that we can obtain only a finite number of them involves the risk of negative power estimates. The

partial sum of a Fourier series for a function $f(x)$ may be written:

$$S_{2n+1}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') \left[\frac{\sin(2n+1)\left(\frac{x'-x}{2}\right)}{\sin\left(\frac{x'-x}{2}\right)} \right] dx' = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') \cdot q(x'-x) \cdot dx' \quad A1.1$$

Since $q(x)$ assumes negative values, the convolution may be negative for some range of x even if $f(x)$ is a non-negative function. However, it is known (Hobson, 27) that this partial sum is the best estimate for $f(x)$ that can be made, based only on a knowledge of the first $(2n+1)$ coefficients of its Fourier expansion.

Longuet-Higgins (1955) makes passing reference to the fact that the partial Cesaro sums of the Fourier series for a non-negative function are always positive, which inspired the present investigation of the application of Cesaro sums to spectral smoothing. The Cesaro sum is defined by $S_r = \left(\frac{S_1 + S_2 + \dots + S_r}{r} \right)$, where S_i is the i^{th} partial Fourier sum. Hobson shows that the Cesaro sum for a function $f(x)$ may be expressed as follows:

$$S_{2n+1}(x) = \frac{1}{2\pi(n+1)} \int_{-\pi}^{\pi} f(x') \left[\frac{\sin(n+1)\left(\frac{x'-x}{2}\right)}{\sin\left(\frac{x'-x}{2}\right)} \right]^2 dx' = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') h(x'-x) \cdot dx' \quad A1.2$$

Evidently $h(x)$ is never negative. The sequence of Cesaro sums has the further desirable property that its limit is $f(x)$.

Each of the functions $q(x)$ and $h(x)$ may be expressed as a finite sum of cosines of multiple angles:

$$q(x) = 1 + 2 \sum_{r=1}^n \cos rx, \quad h(x) = 1 + 2 \sum_{r=1}^n \left(\frac{n-r+1}{n+1} \right) \cos rx \quad \text{A1.3a,b}$$

If $f(x)$ is represented by its Fourier expansion and $h(x)$ by the finite sum above, the convolution for $S_{2n+1}(x)$ is easily evaluated, and the resulting expression is found to have the form of a partial Fourier sum in which the coefficients of the r^{th} harmonic terms are equal to those in the original series multiplied by the factor $\left(\frac{n-r+1}{n+1} \right)$. The Cesaro partial sum, therefore, provides a smoothing which satisfies the requirements implied in the first two paragraphs.

Cesaro smoothing is the best possible in the sense that if we know the exact values of first $(2n+1)$ coefficients in the Fourier expansion of any spectrum $\xi(\theta)$, it "smears out" the spectrum less than any other smoothing which can guarantee that no negative power estimates will arise. In practice we are not concerned with all possible spectra, and we only have estimates for the coefficients. Using the criterion that we should smooth no more than we have to, we conclude that in practice there is nothing unique about the Cesaro method: with a fairly smooth spectrum and good estimates we might smooth less, or even use the "minimum error" Fourier partial sum. If the estimates of the coefficients are poor we might smooth more heavily; with $n=2$, Longuet-Higgins (1963) uses a convolution function $\frac{2}{3} \cos^4 \frac{x}{2} = 1 + \frac{4}{3} \cos x + \frac{1}{3} \cos 2x$, which obviously has the

same non-negative property as $h(x)$, but has a slightly lower and wider peak, so that it gives a rather heavier smoothing.

Smoothing by tapering the correlogram is applicable to the Fourier-Bessel method also, although it was not tried. The technique discussed here seems to be preferable, because of the decreasing accuracy with which the higher order coefficients can be determined.

APPENDIX 2

Equivalence of Two Definitions for the Co- and Quad-Spectra

Introduction: The quantities C and Q, first introduced in the theory of Section 2.3, were defined by the relations:

$$C_{r,s}(f) = \int_{-\infty}^{\infty} \rho_{r,s}(\tau) \cos 2\pi f\tau \cdot d\tau \quad A2.1$$

$$Q_{r,s}(f) = \int_{-\infty}^{\infty} \rho_{r,s}(\tau) \sin 2\pi f\tau \cdot d\tau \quad A2.2$$

$$\rho_{r,s}(\tau) = \overline{\eta_r(t) \cdot \eta_s(t+\tau)} \quad A2.3$$

A change has been made in the notation so that the separation (X,Y) does not appear in the argument of the functions; this is indicated instead by the subscripts (r,s) denoting the r^{th} and s^{th} detectors, which have the separation $(X_{r,s}, Y_{r,s})$. C and Q were identified as the co- and quad-spectra, and the definitions in fact coincide with the process by which these functions are computed when using digital techniques.

In an actual digital computation the cross-correlation $\rho(\tau)$ is replaced by an approximate truncated function, since obviously $\rho(\tau)$ cannot be determined for infinite time lags from measurements of finite duration, nor, strictly speaking, can it be determined exactly even

for finite lags. If $\rho(\tau)$ is computed only for lags that are small relative to the length of the records, the computed values should be good approximations to the true values, and the use of the approximate truncated function may be regarded as equivalent to exchanging the infinite limits of integration for finite ones.

The analogue process actually used to determine C and Q does not follow the definitions, so that it is necessary to demonstrate the equivalence of the functions obtained to those defined in the theory. The following argument is intended as a heuristic demonstration of this equivalence.

The Analogue Process: The signals $\eta_r(t)$ and $\eta_s(t)$ from the r^{th} and s^{th} wave detectors are passed through a matched pair of filters to isolate components of a given frequency. The filtered signals are then multiplied and time averaged to yield the (analogue) co-spectrum. The filtered signal η_s is then given a phase advance of $\frac{\pi}{2}$, and the resulting signal multiplied by the unmodified filtered signal η_r . The time-averaged product signal is the (analogue) quad-spectrum. Using a tilde ($\sim$) to denote a phase advance of $\frac{\pi}{2}$, δf to denote the filter bandwidth, and a bar to denote the time average, we may indicate the process as follows:

$$\text{analogue } C_{r,s}(f) \cdot \delta f = \overline{\eta_r(t; f) \cdot \eta_s(t; f)}$$

$$\text{analogue } Q_{\tau,s}(f) \cdot \delta f = \overline{\eta_{\tau}(t;f) \cdot \tilde{\eta}_s(t;f)} \quad \text{A2.5}$$

Equivalence of Cospectra: It is not at all obvious that the analogue and digital processes yield the same function. A rigorous proof of this equivalence depends upon a generalisation of the Wiener-Khintchine theorem. The possibility of an elementary demonstration was suggested by the interpretation of the Fourier transformation as a filtering process. This in turn suggested the decomposition of $\rho(\tau)$ using a Fourier series representation. It is legitimate to replace the integral by the series since, as we have seen, the truncation of $\rho(\tau)$ in the digital process is equivalent to the use of finite limits. Thus, we may represent $\rho(\tau)$ in the interval $-T < \tau < T$ as follows:

$$\rho(\tau) = \sum_n \left\{ a_n \cos\left(\frac{2\pi n\tau}{T}\right) + b_n \sin\left(\frac{2\pi n\tau}{T}\right) \right\} \quad \text{A2.6}$$

The coefficients are given by Fourier's theorem:

$$a_n = \frac{1}{T} \int_{-T}^T \rho(\tau) \cos\left(\frac{2\pi n\tau}{T}\right) d\tau \quad \text{A2.7}$$

$$b_n = \frac{1}{T} \int_{-T}^T \rho(\tau) \sin\left(\frac{2\pi n\tau}{T}\right) d\tau \quad \text{A2.8}$$

In these formulas, $\frac{1}{T}$ is the constant frequency difference between the sinusoids, which we denote by Δf , and $\frac{n}{T}$ is the frequency f of the n^{th} sinusoid. Comparing equs A2.1

and A2.7, and making use of the definition A2.4 and the expansion A2.6, we obtain:

$$\text{digital } C_{r,s}(f) \cdot \Delta f = a_n = \rho_f(0) = \overline{\eta_r(t; f) \cdot \eta_s(t+0; f)} = \text{analogue } C_{r,s}(f) \cdot \Delta f \quad \text{A2.9}$$

where $\rho_f(0)$ is the frequency $f = \frac{n}{T}$ component of $\rho(\tau)$ evaluated at zero lag ($\tau=0$). The third equality in A2.9 follows provided the frequency f component of ρ is derived from the components of η_r, η_s of the same frequency. This might not appear at first sight to be so, since in the one case the filtering is done before multiplying and averaging, and in the other case after. To investigate this point we write:

$$\eta_r(t) = \sum_n A_n \cos\left(\frac{2\pi n t}{T} + \varepsilon_n\right) \quad \text{A2.10}$$

Then, using dashes to refer to η_s ,

$$\begin{aligned} \rho_{r,s}(\tau) &= \sum_n \sum_m A_n A'_m \left\{ \cos\left(\frac{2\pi n t}{T} + \varepsilon_n\right) \cdot \cos\left(\frac{2\pi m(t+\tau)}{T} + \varepsilon'_m\right) \right\}_{\text{averaged}} \\ &= \sum_n \sum_m \frac{1}{2} A_n A'_m \left\{ \cos\left(\frac{2\pi((m+n)t + m\tau)}{T} + \varepsilon_n + \varepsilon'_m\right) + \cos\left(\frac{2\pi((n-m)t - m\tau)}{T} + \varepsilon_n - \varepsilon'_m\right) \right\}_{\text{averaged}} \\ &= \sum_n \frac{1}{2} A_n A'_n \cos\left(\frac{2\pi n \tau}{T} + \varepsilon'_n - \varepsilon_n\right) \quad \text{A2.11} \end{aligned}$$

Thus, the frequency f component of ρ arises from the components of η_r, η_s of the same frequency. The equivalence asserted in equ A2.9 is therefore established.

Equivalence of Quadrature Spectra: In order to show that the two definitions of the quad-spectrum are equivalent, we define a new function:

$$\tilde{\rho}_{r,s}(\tau) = \overline{\eta_r(t) \cdot \tilde{\eta}_s(t+\tau)} \quad \text{A2.12}$$

where $\tilde{\eta}_s(t+\tau)$ denotes the time-lagged unfiltered signal $\eta_s(t+\tau)$ with every frequency component advanced in phase by $\frac{\pi}{2}$. The introduction of $\tilde{\rho}(\tau)$ is simply a mathematical device; it is not necessary to produce an electrical signal of this form. Exactly as before we find:

$$\tilde{\rho}_{r,s}(\tau) = \sum_n \frac{1}{2} A_n A'_n \cos\left(\frac{2\pi n \tau}{T} + \epsilon'_n - \epsilon_n + \frac{\pi}{2}\right) \quad \text{A2.13}$$

Comparing equs A2.11 and A2.13 we see that the component of $\tilde{\rho}_{r,s}$ of frequency f is the same as that of $\rho_{r,s}$ except that its phase is advanced by $\frac{\pi}{2}$. It follows that if we replace cos by -sin, and sin by cos in A2.6, we have a Fourier series for $\tilde{\rho}_{r,s}(\tau)$:

$$\tilde{\rho}_{r,s}(\tau) = \sum_n \left\{ -a_n \sin\left(\frac{2\pi n \tau}{T}\right) + b_n \cos\left(\frac{2\pi n \tau}{T}\right) \right\} \quad \text{A2.14}$$

The coefficients are given by A2.7 and A2.8 as before.

Comparing equs A2.2 and A2.8, and making use of the definition A2.5 and the expansion A2.14, we obtain:

$$\text{digital } Q_{r,s}(f) \cdot \Delta f = b_n = \tilde{\rho}_f(0) = \overline{\eta_r(t; f) \cdot \tilde{\eta}_s(t+0; f)} = \text{analogue } Q_{r,s}(f) \cdot \Delta f \quad \text{A2.15}$$

where $\tilde{\rho}_f(0)$ is the frequency $f = \frac{n}{T}$ component of $\tilde{\rho}_{\tau,s}(\tau)$ evaluated at zero lag ($\tau=0$). This establishes the equivalence of the two definitions of the quad-spectrum.

The Analogue Process in Practice: We now consider the extent to which the actual analogue process produces the functions defined in eqs A2.4 and A2.5. Our concern is primarily with the operations of filtering and phase-shifting, since multiplication and averaging can be carried out with considerable accuracy.

We normally think of the filtered signal as consisting of a single frequency, but we now wish to examine it in closer detail. The original signal is of finite duration, say T_0 , and it may accordingly be decomposed into a set of sinusoidal components of constant amplitude, with definite phase relations, and with frequencies which are integral multiples of $\frac{1}{T_0}$. We are here considering one specific signal, and not the ensemble of which it is a sample. For fairly large values of T_0 even a narrow-band filter will pass a large number of these components.

It is known that real filters always introduce distortion in both amplitude and phase. The former arises from the fact that the ideal rectangular filter characteristic is not physically realisable. Its effect is to produce an output which is a weighted average of the amplitudes in the passband. This non-uniform weighting is not serious if the spectrum of the original signal is fairly flat in the

neighbourhood of the centre frequency of the filter, that is if it changes by only a small fraction over the filter bandwidth. It is possible to modify a signal with a rapidly varying spectrum so that this condition is satisfied; the process is called pre-whitening.

Phase distortion, which increases as the frequency departs from the centre frequency, appears to be more serious. The frequency components present in the filtered signals have phase shifts due to distortion, which vary with frequency, as well as the phase shift deliberately introduced in the components of one of the signals, in the quad case. It might be thought that the complicated interaction of these components in the subsequent multiplication and averaging could scarcely yield the simple average over the passband of the quantities defined in equs A2.4 and A2.5. However, the orthogonality of the components ensures that the cross-frequency terms contribute nothing to the mean product, and since the time-averaged product of two components of the same frequency is independent of a constant phase shift, we see that the phase distortion introduced by the filters does not affect the result, provided it is the same for each signal.

Hence, to the extent that the filters are matched, and the phase-advancing device produces accurate quadrature without amplitude distortion over the frequency range of the filter passband--a condition that can be approximated

reasonably well if the filter bandwidth is narrow--we may indeed conclude that the analogue process yields the average value of the functions defined in A2.4 and A2.5, where the average applies to the components with frequencies lying within the filter passband.

The above argument applies to a single pair of signals; however, our aim is really to draw conclusions about the ensemble obtained from a large number of realisations of the experiment, conducted under identical conditions. Since the frequency difference between the components discussed above is very small, we may consider that their amplitudes and phases are representative of the centre frequency. It may plausibly be argued, then, that the band-averaged quantities measured for a single pair of signals represent the ensemble-average of these quantities measured at the centre frequency. Thus, we may regard band-averaging not as something we are forced to do by the physical limitations of the equipment, but as a desirable technique by which we can obtain stable estimates of ensemble averages from a single sample.

APPENDIX 3

Method for Determining the Coefficients of the Fourier-Bessel Series

(a) Introduction

In Section 2.3.3 the Fourier-Bessel method of analysis was found to lead to the set of equations:

$$C_n(f) - iQ_n(f) = 2\pi \sum_{r=0}^{\infty} i^r (A_r \cos r\theta_n + B_r \sin r\theta_n) \cdot J_r(2\pi k D_n). \quad 2.21$$

There is one such equation for each separation of detector pairs presented by the array. The problem is to determine the A's and B's, which are the Fourier coefficients of $\xi(\theta)$. Because of a property of Bessel functions, the factors multiplying the higher order A's and B's rapidly approach zero, and it appeared reasonable, having N equations, to truncate the infinite sum so as to leave only N unknowns, which could then in principle be determined. However, it turned out that the matrix of the resulting set of equations was ill-conditioned, so that a solution could not be obtained in this way.

The obvious next step was to reduce the number of the unknowns, while retaining the full number N of the equations, and attempt a least squares solution of the resulting set. For ease of discussion we shall change the

nomenclature at this point and refer to the A's and B's as the parameters, and the factors multiplying them as the coefficients. It was found that this least squares technique did indeed enable a solution to be obtained, but the estimates of the parameters were unstable; that is to say, the values obtained for a particular parameter did not converge as the number of parameters retained in the set was increased from $N-m$ to $N-m+1$, etc. Pencil and paper calculations--the main body of the calculations was carried out using a digital computer--showed that the problem was still the same: the small values of the higher coefficients enabled the corresponding parameters to become large.

At this stage it was realised that the values obtained for the $(N-m)$ parameters were imposing conditions on the spatial correlogram $\{C_n - iQ_n\}$ for values of n beyond those for which we had measured information, and that the instability was the result, in effect, of transforming an unphysical correlogram. The dangers of extrapolating a correlogram are well known, and so too is the remedy: a lag window must be used. This was done by adjoining to the set of N equations another set, corresponding to separations greater than those presented by the array and imposing the condition that the correlogram is zero for these separations. The least squares technique applied to this augmented set of equations was found to give satisfactory convergence of the estimates of the parameters,

and the resulting directional spectra agreed well with those computed by the Fourier transform method.

(b) The Regression Analysis

We denote the n C's or Q's by $y_1, y_2, \dots, y_n$, where n is equal to the number N of measured values plus the number of the additional equations with zero left hand sides. The expected ("true") values of the y_i are related to p parameters ($p < n$) : $x_1, x_2, \dots, x_p$ by the linear equations:

$$y_i = \sum_{j=1}^p a_{ij} x_j \quad A3.1$$

where the coefficients a_{ij} are known. It will be assumed that the $n \times p$ matrix of coefficients is of rank p .

In the least squares technique we seek as estimates of the parameters the values $x_1^*, x_2^*, \dots, x_p^*$ which minimise:

$$W = \sum_{i=1}^n \left\{ y_i - \sum_{j=1}^p a_{ij} x_j^* \right\}^2 \quad A3.2$$

It follows that:

$$\sum_{i=1}^n a_{ik} \left\{ y_i - \sum_{j=1}^p a_{ij} x_j^* \right\} = 0 \quad A3.3$$

for $k = 1, 2, \dots, p$. Thus, the estimates are given by a set of linear equations, usually called the normal equations.

It is convenient for machine computation and for further development to express these equations in matrix

form. The following analysis is due to Plackett (1960), but the steps in his rather brief derivations have been filled in and explained.

$$E y = A x \quad A3.4$$

$$W = [y - Ax]'[y - Ax] \quad A3.5$$

$$A'y = Cx^* \quad , \quad \text{where } C = A'A \quad A3.6$$

The solution to the normal equations A3.6 can be written:

$$x^* = \bar{C}^{-1} A' y \quad A3.7$$

since $\bar{C}^{-1}$ exists under the assumption that the rank of A is p. The solution shows that the estimates x_i^* are linear combinations of the observations y_i . We may note also that the estimates are unbiased, i.e. $E x_i^* = x_i$; for

$$\begin{aligned} E x^* &= E \bar{C}^{-1} A' y \\ &= \bar{C}^{-1} A' E y \\ &= \bar{C}^{-1} A' A x \\ E x^* &= x \end{aligned} \quad A3.8$$

where the last line follows by definition of C.

(c) Variance of the Estimates

We have reason to suppose, from the previous discussion, that the reliability of the estimates of the parameters x_i^* will decrease after some value of i . An estimate of the reliability is given by the variance. Since the equations are linear, the variance of the estimates can be obtained in terms of the variance and covariance of the observations.

The covariance matrix of the y_i is defined by the relation:

$$[\text{Cov } y]_{ij} = \text{Cov}(y_i, y_j) = E\{(y_i - E y_i)(y_j - E y_j)\} \quad A3.9$$

We may write:

$$\text{Cov } y = \sigma^2 V \quad A3.10$$

We assume that $V = I$, the identity matrix, which implies that the variances of all the y_i are equal ($= \sigma^2$), and the covariances are zero. Similarly we have:

$$[\text{Cov } x^*]_{ij} = E\{(C'A'y - E C'A'y)_i \cdot (C'A'y - E C'A'y)_j\}$$

$$[\text{Cov } x^*] = E\{[C'A'y - E C'A'y][C'A'y - E C'A'y]'\}$$

$$= E\{[C'A'(y - E y)][C'A'(y - E y)]'\}$$

$$= E\{C'A'[y - E y][y - E y]'(C'A')'\}$$

$$\begin{aligned}
[\text{Cov } x^*] &= \bar{C}'A' E\{[y - \bar{C}x][y - \bar{C}x]'\} (\bar{C}'A')' \\
&= \bar{C}'A' [\text{Cov } y] (\bar{C}'A')' \\
&= \bar{C}'A' \sigma^2 I (\bar{C}'A')' \\
&= \sigma^2 \bar{C}'A'A\bar{C}^{-1}
\end{aligned}$$

$$[\text{Cov } x^*] = \sigma^2 \bar{C}^{-1} \quad \text{A3.12}$$

The second line above follows since $\bar{C}^{-1}$ is a symmetric matrix. The main diagonal elements of this matrix multiplied by the factor σ^2 are the variances of the estimates. It remains to find σ^2 , the variance of the observations.

Variance of the Observations

It is evident that σ^2 is closely related to the minimum value of W , the sum of squares of the deviations of the observations y_i from their fitted values given by the least squares analysis. Denoting this minimum value of W by R , we have:

$$R = [y - Ax^*]' [y - Ax^*] \quad \text{A3.13}$$

$$\text{By lemma 1 below, } R = [y - Ax]' [I - A\bar{C}'A'] [y - Ax] \quad \text{A3.14}$$

It is easily seen that the second matrix on the right of equ A3.14 is symmetric, so that R is a quadratic form. For

any quadratic form $z' B z$ we have by lemma 2:

$$E(z' B z) = \text{trace} [B \text{Cov } z] + E z' B E z \quad A3.15$$

$$\text{Hence } E R = \text{trace} [I - A \bar{C}' A'] [\text{Cov } (y - A x)] \quad A3.16$$

$$\begin{aligned} \text{But } \text{trace} [I - A \bar{C}' A'] &= \text{trace } I - \text{trace } A \bar{C}' A' \\ &= n - \text{trace } \bar{C}' A' A = (n-p). \end{aligned}$$

The second line above follows since the trace of a product matrix is independent of the order of multiplication.

Also

$$[\text{Cov } (y - A x)] = [\text{Cov } y] = \sigma^2 I$$

$$\text{Hence } E R = (n-p) \sigma^2 \quad A3.17$$

It follows that an unbiased estimate for σ^2 is given by $\left\{ \frac{R}{n-p} \right\}$.

To find R , set $x = 0$ in the identity stated in equ A3.21, and note that the first term on the right of this equation has the form of R .

$$R = y' y - x^{*'} A' A x^*$$

$$R = y' y - y' A x^* \quad A3.18$$

Combining equations A3.17 and A3.18 we obtain σ^2 in terms of the observations y_i , the parameter estimates x_i^* and the known coefficients a_{ij} :

$$\sigma^2 = \left(\frac{1}{n-p} \right) (y'y - y'A x^*) \quad \text{A3.19}$$

Equations A3.12 and A3.19 enable the variances of the estimates to be computed. The computed values were used as a guide in deciding the point at which to truncate the Fourier series for the directional spectrum $\xi(\theta)$. This is discussed further in part (d) of this appendix.

Lemma 1

$$(y - Ax) = (y - Ax^*) + A(x^* - x) \quad - \text{an identity} \quad \text{A3.20}$$

$$\text{Hence } (y - Ax)'(y - Ax) = (y - Ax^*)'(y - Ax^*) + (x^* - x)'A'A(x^* - x) \quad \text{A3.21}$$

The product $(x^* - x)'A'(y - Ax^*)$ vanishes by definition of x^* .

In the second term on the right in equ A3.21 substitute

$x^* = C'A'y$; it becomes:

$$(x^* - x)'A'A(x^* - x) = (y - Ax)'AC'A'(y - Ax).$$

Note that the first term on the right in equ A3.21 has the form of R.

$$\text{Hence } R = (y - Ax)'[I - AC'A'](y - Ax). \quad \text{A3.14}$$

Lemma 2

A quadratic form $z'Bz$ is defined by $\sum_r \sum_s b_{rs} z_r z_s$, given that B is symmetric. Hence

$$\begin{aligned}
\mathcal{E}(z' B z) &= \mathcal{E} \sum_r \sum_s b_{r,s} z_r z_s \\
&= \sum_r \sum_s b_{r,s} \mathcal{E}(z_r z_s)
\end{aligned}$$
A3.23

By definition,

$$\begin{aligned}
\text{Cov}(z_r, z_s) &= \mathcal{E}\{(z_r - \mathcal{E}z_r)(z_s - \mathcal{E}z_s)\} \\
&= \mathcal{E}(z_r z_s) - \mathcal{E}(z_r) \mathcal{E}(z_s)
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{E}(z' B z) &= \sum_r \sum_s b_{r,s} \{ \text{Cov}(z_r, z_s) + \mathcal{E}(z_r) \mathcal{E}(z_s) \} \\
&= \text{trace}[B][\text{Cov}z] + \mathcal{E}z' \cdot B \cdot \mathcal{E}z
\end{aligned}$$
A3.15

To obtain the first term on the right in equ A3.15, let

$$[M] = [B][\text{Cov}z]$$

Then

$$\begin{aligned}
\text{trace}[M] &= \sum_s [M]_{ss} \\
&= \sum_s \left\{ \sum_r b_{s,r} [\text{Cov}z]_{r,s} \right\} \\
&= \sum_r \sum_s b_{r,s} \text{Cov}(z_r, z_s), \text{ by symmetry of } B.
\end{aligned}$$

(d) Truncation of the $\mathcal{E}(\theta)$ Series:

In principle, the Fourier-Bessel method enables the coefficients of the first 17 harmonics of the Fourier series for $\mathcal{E}(\theta, f)$ to be determined. In practice, as discussed earlier, the number of coefficients that can be

found is less than this. The computation of the coefficients was actually carried out retaining 3,4, ..., 13 harmonics in the series, so that 11 solutions were obtained for each frequency. It therefore became necessary to establish criteria to decide which of these solutions should be adopted, or in other words to select the point at which the Fourier series should be truncated. The following criteria were chosen:

- (1) The theoretical resolution of the array determines an upper bound to the order of the terms that should be retained. If the array can distinguish, at best, two waves travelling in directions which differ by θ_1 , then spectral structure finer than θ_1 cannot be significant, and terms in the series giving more than one cycle in θ_1 may be ignored. But the term $A_n \cos n\theta$ corresponds to one cycle in $\left(\frac{360}{n}\right)$ degrees, and so the highest significant order is given by the nearest integer to $\left(\frac{360}{\theta_1}\right)$, where θ_1 is in degrees. From Section 2.4 θ_1 is given by $\sin^{-1}\left(\frac{l_1}{k}\right)$, where l_1 is the first non-zero solution of:

$$(2N+1) \tan \pi l_1 D = \tan (2N+1) \pi l_1 D$$

and N is the maximum number of lags presented by the array, 17, or such smaller number as was actually used if the correlation ran out in fewer lags. Since $n = 13$ is the highest order that it was possible to

compute, this value of n was used if the above calculation gave greater values, as was the case for all frequencies above $f = .520 \sim$. The values of n for the lower frequencies are as follows:

f	.261	.284	.310	.338	.368	.401	.437	.476	.520
n	4	5	5	6	7	8	10	11	13

- (2) A lower bound to the order n is provided by the requirement that the estimates of the coefficients of all the terms retained should be stable. If the estimates of the coefficient A_n obtained from computations retaining 3, 4, . . . , 13 terms of the series are arranged in order of increasing number of terms retained, it is found that for low orders n the difference between adjacent estimates is small, while for higher orders the estimates stabilise after the first or second member of the sequence, except as noted below.
- (3) The requirement that the estimates be stable also provides an upper bound to the order n , at least for the low frequencies. In these cases as the number of terms retained is increased, a point is

reached at which the estimates of the higher order coefficients begin to diverge. When such divergence occurred, it was almost always at orders n greater than the highest significant order, according to criterion (1). In the worst case it occurred at an order one less than the highest significant order.

- (4) The calculated probable error of the estimates for a given coefficient A_n was found to decrease initially as the number of terms retained in the computation was increased, but after some point it began to increase. The increase was always greatest for the higher order coefficients. The error became large, of the order of A_n , for values of n that would be rejected by criterion (3), but the reversal in trend of the error could be detected before the divergence, and provided a more sensitive indication of the reliability of the estimates. The criterion adopted was that the series should be truncated at the n^{th} term if the retention of more than n terms leads to an increase in the probable error of the estimates.
- (5) A fifth criterion is based on the fact that the correlation runs out in about $1\frac{1}{2}$ wavelengths. This affects the number of terms that may be retained at the high frequency end of the spectrum. For $\lambda = 2\pi D$,

D being the unit space of the array, the correlation runs out in a distance $l_2^1 m D$, and if M is the nearest integer to $l_2^1 m$, the measurements give only M pieces of information, i.e.: the values of the correlation for the first M separations. Using the representation:

$$\frac{1}{\pi} C(nD) = \sum_{r=0}^{\infty} (-1)^r A_{2r} J_{2r} \left(2\pi n \frac{D}{\lambda} \right)$$

the measurements can be fitted by determining M coefficients, and more than M are not meaningful, although at the higher frequencies more can be determined. We therefore find A_{2r} for $0 \leq r \leq M-1$ and take A_N , where $N = 2(M-1)$, to be the highest significant coefficient. The limiting order N according to this criterion is:

f	.735	.802	.875	.954	1.04	1.14	1.24
N	> 13	12	10	8	6	4	4

APPENDIX 4

Normalisation of the Computed Spectrum

In Section 2.3.2 the following approximation to the directional spectrum was obtained:

$$\mathcal{E}'(\ell, f) = \kappa \left[C(0, f) + 2 \sum_{n=1}^{17} C(nD, f) \cos 2\pi n \ell D - 2 \sum_{n=1}^{17} Q(nD, f) \sin 2\pi n \ell D \right] \quad 2.17$$

where the factor K is to be determined empirically, such that the integral of $\mathcal{E}'$ agrees with the power spectral density at frequency f . The use of the rectangular lag window gives $K = D$, the unit space of the array, but with this value the spectrum $\mathcal{E}'$ is not in general correctly normalised. Suppose in a particular case the empirically determined value of K is, say, αD , $\alpha \neq 1$. We may ask if the difference between α and unity is inherent in the approximation involved in equ 2.17, or if it is the result of errors in the values of C and Q , or in the graphical integration. It will be found in general that the integral of $\mathcal{E}'$ changes if a different lag window is used, and again we may ask why this is so. In this section the normalisation of the spectrum will be investigated theoretically for the case of the rectangular lag window; the extension necessary with different lag windows will become obvious.

From equ 2.14 we obtain:

$$C(o, f) = \int_{-\infty}^{\infty} E(l, f) dl \quad A4.1$$

$$C(nD, f) = \int_{-\infty}^{\infty} E(l, f) \cos 2\pi n l D dl \quad A4.2$$

$$Q(nD, f) = - \int_{-\infty}^{\infty} E(l, f) \sin 2\pi n l D dl \quad A4.3$$

Substituting these expressions, equ 2.17 becomes:

$$\begin{aligned} E'(l, f) &= K \left[\int_{-\infty}^{\infty} E(l', f) dl' + 2 \sum_{n=1}^{17} \left\{ \int_{-\infty}^{\infty} E(l', f) \cos 2\pi n l' D dl' \right\} \cos 2\pi n l D + \right. \\ &\quad \left. 2 \sum_{n=1}^{17} \left\{ \int_{-\infty}^{\infty} E(l', f) \sin 2\pi n l' D dl' \right\} \sin 2\pi n l D \right] \\ &= K \left[\int_{-\infty}^{\infty} E(l', f) \left\{ 1 + 2 \sum_{n=1}^{17} \cos 2\pi n (l-l') D \right\} dl' \right] \\ &= K \int_{-\infty}^{\infty} E(l', f) \left\{ \frac{\sin 35\pi (l-l') D}{\sin \pi (l-l') D} \right\} dl' \\ E'(l, f) &= K \int_{-\infty}^{\infty} E(l', f) G(l-l') dl' \quad A.4.4 \end{aligned}$$

where:

$$G(l) = \left\{ \frac{\sin 35\pi l D}{\sin \pi l D} \right\} \quad A4.5$$

Apart from the factor K, which was not explicitly introduced before, this expression is identical to that derived by a different method in Section 2.4.5, when discussing the resolving power of an array. K represents the coefficient or strength of the delta functions, assumed previously to be of unit strength, in the analysis of Section 2.4.

The problem is to find the "best" value for K . In order to define this value we make the assumption that in the true spectrum $\mathcal{E}(\ell, f)$ no energy is associated with values of $|\ell|$ exceeding k , the scalar wavenumber determined from f according to the dispersion relation of linear theory. Then equ A4.4 becomes:

$$\mathcal{E}'(\ell, f) = \kappa \int_{-k}^k \mathcal{E}(\ell', f) G(\ell - \ell') d\ell', \quad |\ell| \leq k \quad \text{A4.6}$$

in which the limits of the integral are now finite. The condition $|\ell| \leq k$ has been imposed since equ A4.6 gives non-zero values of $\mathcal{E}'$ for $|\ell| > k$ and these cannot be significant in view of the assumption that the true spectrum $\mathcal{E}$ is then zero. Accordingly we add to the definition A4.6 the requirement:

$$\mathcal{E}'(\ell, f) = 0, \quad |\ell| > k \quad \text{A4.6a}$$

We now define the "best" value for K to be that for which the integral of the approximate directional spectrum has its proper value, i.e. that given by:

$$\int_{-k}^k \mathcal{E}'(\ell, f) d\ell = \int_{-k}^k \mathcal{E}(\ell, f) d\ell \quad \text{A4.7}$$

or in explicit form:

$$K = \left[\frac{\int_{-k}^k \mathcal{E}(\ell, f) d\ell}{\int_{-k}^k \int_{-k}^k \mathcal{E}(\ell', f) G(\ell - \ell') d\ell' d\ell} \right] \quad \text{A4.7a}$$

In this equation k and G are known, but $\hat{\epsilon}$ is not. As already noted, an empirical estimate of K can readily be obtained, since if the assumption we have just made is valid, the numerator in equ A4.7a is equal to $C(o.f)$, and the denominator is simply the right hand side of equ 2.17 with $K = 1$. Equ A4.7a can also be used to calculate the theoretical value of K if the form of the true spectrum $\hat{\epsilon}$ is prescribed. It will be shown in the following analysis that there is one wavenumber for which K is independent of $\hat{\epsilon}$, and the corresponding value of K and wavenumber k will be determined. For other wavenumbers and for simple forms of the spectrum the magnitude of K can be predicted qualitatively; the method could actually be used to obtain the numerical value of K in these cases, but it is easier to use equ A4.7a.

The explicit introduction of the array weighting function G in equ A4.7a enables us to investigate the effect of other lag windows on the integral of the spectrum. For such windows equ A4.7a still gives the correct normalisation if G is replaced by the appropriate function, namely, the convolution of G with the Fourier transform of W , where $W(X)$ is the weighting function applied to the space-lagged correlogram. On this understanding equ A4.7a and the following analysis apply to the general case.

Suppose that the true spectrum $\hat{\epsilon}$ is non-zero and constant with respect to λ over the range $|\lambda| \leq k$, and

zero for $|\ell| > k$, and let G be an arbitrary even function of ℓ . We shall later generalize $\mathcal{E}$ and specialise G . Let I_1 and I_2 be defined as follows:

$$I_1 = \int_{-k}^k \mathcal{E}(\ell, f) d\ell \quad A4.8$$

$$I_2 = \kappa \int_{-k}^k \mathcal{E}(\ell', f) G(\ell - \ell') d\ell', \quad |\ell| \leq k \quad A4.9a$$

$$= 0, \quad |\ell| > k \quad A4.9b$$

We note that I_1 and I_2 are of the form of the right hand sides of equs A4.7 and A4.6 respectively; according to A4.6 I_2 is therefore the approximate spectral density $\mathcal{E}'$ corresponding to a constant true spectrum $\mathcal{E}$, as seen by an array whose weighting function is G . Let us replace each of the integrals I_1 and I_2 by one of their approximating sums, using identical partitions of the ℓ -axis for each; denote these by Σ_1 and Σ_2 .

Consider the element $\mathcal{E}(\ell_n, f) \cdot \Delta \ell_n$ in Σ_1 ; the corresponding element in Σ_2 contributes an amount $\kappa \mathcal{E}(\ell_n, f) G(\ell - \ell_n) \cdot \Delta \ell_n$ to Σ_2 , and this element of "area" is distributed over the range $|\ell| \leq k$ in proportion to the ordinates of the curve $G(\ell)$, translated along the ℓ -axis so that its origin coincides with ℓ_n .

We note that the part of the curve of $G(\ell)$ that is effective for this element, i.e. such that the ordinates of G multiply non-zero values of $\mathcal{E}$, is that on the

interval $-(k+l_n) \leq l \leq (k-l_n)$. Since l_n ranges over $|l_n| \leq k$, we see that for any element in Σ_1 , the corresponding element in Σ_2 is obtained using a part of the curve $G(l)$ of length $2k$, from somewhere in the interval $|l| \leq 2k$. So long as k is fixed, the form of for $|l| > k$ is immaterial.

Let us return to the elements of Σ_1 and Σ_2 . The element $\mathcal{E}(l_n f) \cdot \Delta l_n$ of Σ_1 is an element of the true spectrum (not a density), and the corresponding element in Σ_2 may be interpreted as the amount contributed by the element of Σ_1 to the density of the approximate spectrum $\mathcal{E}'$. This amount is $K \mathcal{E}(l_n f) G(l-l_n) \cdot \Delta l_n$, and it is spread out over the whole range $|l| \leq k$.

Now if the function G and wavenumber k are such that the integral over $|l| \leq k$ of the contribution to $\mathcal{E}'$ is proportional to $\mathcal{E}(l_n f) \cdot \Delta l_n$ for every l_n in $|l_n| \leq k$, we may evidently choose K to make the factor of proportionality unity. But these conditions imply that $\int_{-k}^k G(l-l_n) \cdot dl$ is constant for all l_n in $|l_n| \leq k$, i.e. that $\int_{-(k+l_n)}^{(k-l_n)} G(l) dl$ is constant; from the latter form we see that the integral of $G(l)$ over every interval of length $2k$ in the range $|l| \leq 2k$ must be constant. In other words, we must have $G(l) = G(l+2k)$ for the wavenumber k concerned.

If this condition is satisfied, and K has the value specified, then the sum of the integrated contributions to $\mathcal{E}'$ from all the elements of the true spectrum is equal

to the sum of all the latter elements, i.e. to $\sum_n$, or to $\int_{-k}^k \mathcal{E}(l, f) dl$, in the limit as the largest partition interval Δl_n tends to zero. The sum of all the integrated contributions to $\mathcal{E}'$ is:

$$S_2 = \sum_n \int \kappa \mathcal{E}(l_n, f) G(l - l_n) \Delta l_n dl \quad \text{A4.10}$$

in which the limits of integration differ by $2k$, but are different for every n . But under the assumed condition the value of the integral is the same for every n , a constant. We may therefore invert the order of integration and summation and choose any limits in the range $|l| \leq 2k$ which differ by $2k$. Hence

$$\begin{aligned} S_2 &= \int_{-k}^k \sum_n \kappa \mathcal{E}(l_n, f) G(l - l_n) \Delta l_n dl \\ &= \int_{-k}^k \Sigma_2 dl \\ &= \int_{-k}^k I_2 dl, \quad \text{in the limit} \\ &= \int_{-k}^k \mathcal{E}'(l, f) dl, \quad \text{using Eqn A4.6} \end{aligned} \quad \text{A4.11}$$

We have therefore proved that subject to the given condition and with K as specified:

$$\int_{-k}^k \mathcal{E}'(l, f) dl = \int_{-k}^k \mathcal{E}(l, f) dl \quad \text{A4.12}$$

But equ. A4.12 is identical to equ A4.7, from which equ A4.7a follows. What has been proved is therefore that if the true spectrum $\mathcal{E}$ is constant, and if G and k satisfy:

$$G(\ell) = G(\ell + 2k) \quad \text{A4.13}$$

then the value of K given by equ A4.7a is identical with that given by:

$$K \int_{-k}^k G(\ell) d\ell = 1 \quad \text{A4.14}$$

But the assumption $\mathcal{E} = \text{constant}$ was introduced only to fix ideas; we have not used it. The argument stands unchanged if any or every element of Σ_1 is factored arbitrarily. In other words the result A4.14 is valid, subject to the condition A4.13, for any form of spectrum $\mathcal{E}$. Moreover, although K was defined in A4.7a in terms of $\mathcal{E}$, it is actually independent of $\mathcal{E}$ in this case, as may be seen from equ A4.14.

If we now specialise the arbitrary even function G , choosing the weighting function A4.5 of the array used in the present investigation, can the condition A4.13 be satisfied for any k within the range to which this array is restricted (see Section 3.6.1)? It is found that A4.13 can be satisfied, but only at one wavenumber: the limiting value itself, $k = \frac{1}{2D}$. The value of K is then, according to A4.14, given by:

$$K \int_{-\frac{1}{2D}}^{\frac{1}{2D}} \left\{ \frac{\sin 35\pi l D}{\sin \pi l D} \right\} dl = 1$$

Making the transformation $\phi = 2\pi l D$, this reduces to:

$$\frac{K}{D} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{\sin \frac{35\phi}{2}}{\sin \frac{\phi}{2}} \right\} d\phi \right] = 1$$

The quantity in square brackets has the value unity, and therefore $K = D$ is the value which correctly normalises $\mathcal{E}(l, f)$ for this value of k , regardless of the form of the true spectrum.

For other wavenumbers the value of K depends on the true spectrum as well as on the properties of the array. However, the above analysis can be extended to give an order of magnitude estimate of K for these wavenumbers and a typical wind wave form of the spectrum.

We define the function G_1 , for arbitrary k , as follows:

$$G_1(l) = G(l), \quad |l| \leq k \quad \text{A4.15a}$$

$$G_1(l) = G_1(l + 2k) \quad \text{A4.15b}$$

The function G_1 satisfies the condition A4.13 and if this were the actual weighting function of the array, the value of K given by $K \int_{-k}^k G_1(l) dl = 1$ would normalise $\mathcal{E}'$ correctly for any spectrum $\mathcal{E}$. This value is readily calculable; let us denote it by $K_1 = \beta D$. The variation

of β with k can be inferred from Fig. 6. When $k = \frac{1}{2D}$, $\beta = 1$; as k is reduced the value $\int_{-k}^k G_1(l) dl = \int_{-k}^k G(l) dl$ at first increases, and β decreases, attaining a local minimum when k has the value of l corresponding to the last zero of $G(l)$ in Fig. 6. β then increases, reaching the value 1 again when k is just slightly greater than the value of l at the next maximum of $G(l)$; it continues to increase to a local maximum value at the next zero of $G(l)$, and so on. β thus oscillates about the value 1, with swings of increasing amplitude as k decreases, until k attains the value l_0 , the first zero of $G(l)$, where β has a local minimum. As k is reduced further β increases, attains the value 1, and continues to increase without limit.

However, since G rather than G_1 is the true weighting function of the array, the value $K_1 = \beta D$ must be corrected to account for the fact that the "tails" of G and G_1 --the part of the curves in the range $k < |l| \leq 2k$ --are different. This can be done only for prescribed forms of the spectrum. The true value of K is:

$$K = \beta D \cdot \left\{ \frac{\int_{-k}^k \mathcal{E}'_1(l) dl}{\int_{-k}^k \mathcal{E}'(l) dl} \right\} = \alpha D$$

where $\mathcal{E}'_1$ is the apparent spectrum as seen by an array having a weighting function G_1 and $\mathcal{E}'$ that corresponding to the function G . For a given wavenumber and a simple true spectrum $\mathcal{E}$, for example a constant, a rough estimate of the

correction factor can readily be obtained using sketch graphs of G and G_1 . If k is greater than about $2\ell_0$ it is found that $E'(\ell)$ and $E_1'(\ell)$ are similar except for values of ℓ close to k . If $E(\ell)$ is concentrated around $\ell = 0$, and falls off toward zero for $|\ell|$ close to k , as is typical of wind waves whose mean direction of travel is close to the array normal, the correction factor is close to unity. But if the wavenumber is greater than ℓ_0 , we have seen that β is also of order unity, and hence the value of K will be fairly close to D . On the other hand if $k \ll \ell_0$, while the correction factor is very close to unity, the value of β is then large, and the normalisation constant K is many times greater than D . Of course if an array were used to study such wavenumbers its resolution would be extremely poor, and so we may say that for practical purposes K will always be approximately equal to D . Those directional spectra presented in Section 5 which were obtained by the Fourier transform method were computed using $K = D$, and the fact that the normalisation is not exactly correct was ignored.

APPENDIX 5

Relations between Various Definitions of the Wave Spectrum

The directional spectrum may be expressed as a function of the wavenumber components $E_1(k_1, k_2)$, or of frequency and azimuth $E_2(f, \theta)$, or of several other pairs of arguments. The one-dimensional spectrum may be expressed in terms of scalar wavenumber $E_3(k)$, or of frequency $E_4(f)$, as well as in other ways. If the usual dispersion relation is assumed to hold, a spectrum given in one of these forms can be obtained in any of the other forms, except of course that E_1 or E_2 cannot be obtained from a knowledge of E_3 or E_4 , since the former contain more information than the latter. Of the eight possible relations we shall derive three.

The contribution to the variance of the surface displacement from a region R of wavenumber space is:

$$I = \iint_R E_1(k_1, k_2) dk_1 dk_2 \quad A5.1$$

If R' is the corresponding region in frequency-angle space,

$$I = \iint_{R'} E_2(f, \theta) df d\theta \quad A5.2$$

and the mapping is given by:

$$4\pi f^2 = qk, \quad k_1 = k \cos \theta, \quad k_2 = k \sin \theta \quad A5.3$$

We therefore have by equ A5.1:

$$I = \iint_{R'} E_1 \left(\frac{4\pi^2 f^2}{q} \cos \theta, \frac{4\pi^2 f^2}{q} \sin \theta \right) |J| df d\theta \quad A5.4$$

where

$$J = \frac{\partial(k_1, k_2)}{\partial(f, \theta)} = \frac{2k^2}{f} = \frac{32\pi^2 f^3}{q^2} \quad A5.5$$

Since the regions are arbitrary, the integrands in A5.2 and A5.4 must be equal. Hence

$$E_2(f, \theta) = \frac{32\pi^2 f^3}{q^2} E_1 \left(\frac{4\pi^2 f^2}{q} \cos \theta, \frac{4\pi^2 f^2}{q} \sin \theta \right) \quad A5.6$$

so that if E_1 is known we can find E_2 .

The one-dimensional frequency spectrum can be found from E_2 , or from E_1 by the use of equ A5.6, by integrating with respect to θ :

$$E_4(f) = \int_0^{2\pi} E_2(f, \theta) d\theta \quad A5.7$$

Applying the method used to derive equ A5.6 to E_3 and E_4 , we obtain:

$$E_4(f) = \frac{8\pi^2 f}{q} E_3 \left(\frac{4\pi^2 f^2}{q} \right) \quad A5.8$$

The spectrum can be obtained in terms of other variables in a similar way.

APPENDIX 6

The Wien Bridge Oscillator

Although the Wien bridge oscillator is a standard circuit its treatment in the literature is often inadequate. For example the frequency of oscillation is frequently derived by considering the balance condition of the Wien bridge, and although this gives the correct result it does not provide much physical insight. In at least one case where a physical explanation is offered (Malmstadt et al, 1963) the account is incorrect.

The action of the Wien bridge oscillator may be described using the following block diagram:

The amplifier is normally a two-stage tube or transistor type giving two reversals of phase and, ideally, no other phase shifts. There are two feedback paths, the network $R_1 C_1 R_2 C_2$ giving positive feedback, and $R_3 R_4$ giving

negative feedback. Evidently the circuit cannot oscillate at any frequency where the net feedback is negative. With respect to the signal applied between the points A and B, the two networks may be regarded as voltage dividers. The voltage at D is a constant fraction of the applied voltage, and is always in phase with it regardless of the frequencies present in the signal. The conditions for oscillation are that for one of the signal frequencies the voltage at C must be in phase with the component of this frequency applied between A and B, and that the fraction of this component voltage appearing across CB must exceed that across DB. As we shall see, the phase condition alone determines the frequency, but the amplitude condition can be arranged to reinforce the frequency selectivity by causing attenuation of all but one frequency. This action might be expected from the following considerations: for very low frequencies the impedance in the arm AC is very large, because of the series condenser C_1 , and so the fraction of the applied voltage appearing across CB is very small; for very high frequencies this fraction is again very small because R_2 is effectively shorted by C_2 ; we therefore infer that the voltage across CB is a maximum for some intermediate frequency. If R_3 and R_4 are selected so that the net feedback is just sufficiently positive to maintain oscillation at this intermediate frequency, evidently the network will attenuate all other frequencies. But, as we

shall see, this intermediate frequency is identical to that giving zero phase shift at the point C, which is the first condition for oscillation stated above.

To determine the frequency for zero phase shift at the point C, let z_1 be the impedance of the arm AC and z_2 that of the arm CB. Then the voltage across CB is $\left(\frac{z_2}{z_1 + z_2}\right)$ times the voltage applied across AB. The condition is that $\left(\frac{z_2}{z_1 + z_2}\right)$ be real, or more simply that $\frac{z_1}{z_2}$ be real. But $\frac{z_1}{z_2} = \left(R_1 + \frac{1}{j\omega C_1}\right)\left(\frac{1}{R_2} + j\omega C_2\right)$, and the imaginary part of this is $\left(\omega R_1 C_2 - \frac{1}{\omega R_2 C_1}\right)$, which is zero for the radian frequency given by:

$$\omega^2 = R_1 R_2 C_1 C_2$$

With the same notation, the frequency for the maximum voltage across CB is that for which $\left|\frac{z_2}{z_1 + z_2}\right|$ is a maximum, or $\left|1 + \frac{z_1}{z_2}\right|$ a minimum. This yields the condition that $\left|\left(1 + \frac{R_1}{R_2} + \frac{C_2}{C_1}\right) + j\left(\omega R_1 C_2 - \frac{1}{\omega R_2 C_1}\right)\right|$ should be a minimum, which is obviously when the imaginary part of the expression between the modulus bars is zero, and this occurs at the same frequency as that found for the zero phase shift condition.

With the correct choice of R_3 and R_4 the two conditions therefore reinforce, and sinusoidal oscillations of very stable frequency can be obtained. The conditions on R_3 and R_4 are not exactly the same as the conditions for bridge balance, since the latter would give a net

feedback of zero; R_4 must be slightly smaller than the balance value to give a net positive feedback. Provided this is so the exact values of R_3 and R_4 are not critical, and in fact the oscillator will operate if both are removed, and the points B and D connected. From the argument above, we should not expect the frequency stability to be so good in this case.

APPENDIX 7

Techniques of Analogue Computation

(a) Introduction

The methods used in the reduction of the data have been described in outline in Section 4.1, and for those operations that were carried out digitally, the methods used have been described in detail in Appendices 1 and 3. The latter operations required the co- and quadrature spectra as their input data, and these were obtained by analogue computation, employing techniques that will be described in this Appendix.

Starting from the recorded signals, the analogue system may be considered to include the processes of signal demodulation and frequency filtering, as well as those processes carried out on the analogue computer proper. The following account will cover the entire system.

The IOUBC analogue computer comprises three Philbrick K7-A10 manifolds, each of ten operational amplifiers, and R-300 and R-600 regulated power supplies. A result required in most of the following paragraphs is the input-output relation for an operational amplifier. As is shown in most accounts of the subject, for example Seddon (1960), this relation is:

$$e_o = -K \frac{z_2}{z_1} e_i$$

where K is a factor very close to unity in almost all cases; its value for the general case is given by Seddon; z_1 is the impedance of the computing elements connected between the input and the summing point, and z_2 is that of the elements between the summing point and the output.

Fig. 12 presents a block diagram showing the signal path through the analogue computation system, the separate blocks denoting only the operation performed. The methods by which the operations were performed are described in detail below.

(b) Signal Demodulation

The frequency-modulated wave detector signals from the tape were demodulated using Ampex F.M. Reproduce units specially designed for use with the Ampex CP-100 tape recorder. These units operate on a somewhat unusual principle: in essence they count the zero crossings in the input F.M. signal and produce an analogue output proportional to the count, with a suitable DC offset so that the output is zero when the frequency of the input signal is equal to the design centre frequency. It is apparent that this technique is inherently more accurate than that commonly used in F.M. radio practice.

The circuit comprises six main sections--limiter, frequency doubler, multivibrator, ramp generator, blocking

oscillator, and output driver. The limiter amplifies and clips the input signal. The doubler doubles the fundamental frequency of the limiter output, and passes its output to a network comprising a differentiating circuit and diodes, which passes the negative pulses associated with the negative-going swings of the double frequency wave, but blocks the positive pulses. The negative pulses are used to set the multivibrator, which is of the bistable type, but they cannot reset it. Resetting is effected by a pulse from the blocking oscillator, which occurs at a fixed time interval after the setting pulse. To obtain this fixed time delay, the blocking oscillator is triggered by a ramp generator which is itself triggered by the multivibrator setting pulse. The action of the ramp generator upon receiving its triggering pulse is to initiate the discharge of a condenser through a resistor; when the condenser potential reaches a predetermined value, it triggers the blocking oscillator. The waveform of the multivibrator output is therefore approximately rectangular, with a negative-going swing occurring simultaneously with the zero-crossings, in either direction, of the original signal, and with a positive-going swing occurring at a fixed time interval after the preceding negative swing. This output is applied to the output driver which provides amplification and voltage limiting, thus producing an accurate rectangular wave of constant amplitude. The output of the driver stage is fed

into an integrating network which produces an analogue output corresponding to the modulation of the original signal.

The Ampex F.M. Reproduce unit can accommodate frequency deviations of $\pm 40\%$ of the design centre frequency, and is linear to 0.1% over this range. It is supplied with plug-in adaptors, comprising the timing condenser for the ramp generator and the integrating network, which change the centre frequency for each tape speed of the CP-100 recorder, and maintain the output level of 1 volt RMS for full deviation. If a record is played back at a higher speed than the recording speed, as it was in the analogue computation described in Section 4.1, the voltages are therefore correctly reproduced, but the time integrals of squares and products of the signals are reduced by the ratio of the tape speeds because of the compression of the time scale.

(c) Filters: The filters used were two General Radio type 1554-A Sound and Vibration Analysers. Four frequency ranges are provided, the lowest of which goes down to $2.5\sim$. There are three bandwidth settings: all-pass, $1/3$ octave, and narrow-band; the instruments were used with the narrow-band setting in which the response is designed to be 3db down at $\pm 4\%$ of the centre frequency. The output can be amplified or attenuated in 10db steps, using the control provided for this purpose, but since the output meter was not used in this application, the calibration control was available to give continuously variable gain control. The instrument is

battery powered to avoid the need for elaborate power supply regulation.

Electrically the 1554-A consists of two RC-type tuned amplifiers in cascade, with an untuned input amplifier and attenuator network, and an untuned output amplifier. For the narrow-band mode, the two RC amplifiers are tuned to the same frequency, and for the 1/3-octave mode the tuning is staggered; on the all-pass mode the RC networks are replaced by resistances to cut out the frequency selectivity.

In the present application the two filters had to be set and matched more accurately than was possible using the frequency dial and meter provided in the instruments. This was done in the following way. A General Radio low frequency signal generator was set to the precise centre frequency required, using a Hewlett-Packard counter. This signal was fed to the two filters, which were set to the narrow-band mode. The frequency controls were then adjusted for maximum response, as read on the output meter. The filter outputs were next applied to the horizontal and vertical deflection plates of a Tectronics dual beam 'scope, to produce a Lissajous figure, which was normally a narrow ellipse with its major axis at 45° to the grid lines on the scope. The frequency control of one of the filters was then adjusted very carefully until the ellipse collapsed into a straight line at 45° . With the scope amplifiers set to give sufficient gain, this was a highly precise method of setting the filter outputs to exact phase agreement. The amount of

movement of the frequency control required was always exceedingly small, and the meter needle did not move from its previous maximum response indication. It was found that the matching depended slightly on the gain settings used, and the operation was therefore performed after the gains had been determined and set. To determine the appropriate gain, the filters were first adjusted for maximum response at the frequency desired, as described above, and one of the taped wave detector signals was passed through each filter and the analogue computer preamplifiers. The gain was then set so that the highest peak voltage at the preamplifier output was slightly less than 100V, the dynamic range of the analogue computer.

Since the filter bandwidth was a constant percentage of the centre frequency rather than a constant frequency interval, the wave frequencies selected for analysis were such that adjacent pairs of frequencies were in a constant ratio. The ratio adopted was $\sqrt[3]{2}$, or 1.089, at which the upper half-power frequency for one setting just overlapped the lower half-power frequency for the next higher setting. The shortest wavelength at which the array could be used (see Section 3.6.1) determined the highest frequency in absolute measure, and this together with the frequency ratio chosen determined all the wave frequencies to be analysed. Since the analogue computation was carried out at the highest available tape speeds, 30 or 60 ins. per sec., i.e. 16 or 32 times the recording speed, the filter centre frequencies were set at 16 or 32 times

the wave frequencies as thus calculated.

In order to check the intrinsic matching of the two filters, their power transfer characteristics were measured for each frequency and gain setting used. A typical pair of characteristics is shown in Fig. 13. It was found that the actual bandwidths were somewhat greater than the nominal values, averaging 10.5% of the centre frequency rather than 8%, and the difference in bandwidths of the two filters was about 0.5%. With the filters set to phase coincidence at the centre frequency, the difference in phase angle was checked at the half-power points, and found to be of the order of one degree.

(d) Preamplification: The need for preamplification was discussed in Section 4.1. In principle a gain of 70 can be achieved by using resistors as the elements z_1 and z_2 , with $R_2 = 70 R_1$. It was found, however, that the filter output contained a small DC component, due to leakage across the output condenser. The standard method of dealing with this situation would be to introduce an opposing DC voltage at the input. However, in the present case the magnitude of the DC depended on the signal level. The solution adopted was to design a preamplifier which had a flat response over the range of filter centre frequencies of interest, but with zero DC gain. The circuit was as shown below.

The triangle is the standard symbol for an operational amplifier, and S.P. stands for the summing point. Applying the input-output formula of Part(a), we have:

$$-\frac{e_o}{e_i} = \left[\frac{\omega R_2 C_1 (\omega R_1 C_1 + \omega R_2 C_2)}{(1 + \omega^2 R_1^2 C_1^2)(1 + \omega^2 R_2^2 C_2^2)} \right] \cdot \left[1 + j \left\{ \frac{1 - \omega R_1 C_1 \cdot \omega R_2 C_2}{\omega R_1 C_1 + \omega R_2 C_2} \right\} \right]$$

If $(\omega R_1 C_1)^2 \ll 1$, $(\omega R_2 C_2)^2 \gg 1$, this simplifies to:

$$-\frac{e_o}{e_i} = \frac{R_2}{R_1} \left[1 + j \left\{ \frac{1}{\omega R_1 C_1} - \omega R_2 C_2 \right\} \right]$$

If we choose the values $R_1 = 47 \text{ k}\Omega$, $R_2 = 3.3 \text{ M}\Omega$, $C_1 = 10^{-5}$ farad, $C_2 = 10^{-10}$ farad, these conditions are amply satisfied for the frequency range of interest,

$60 < \omega < 100$, and the coefficient of j is zero at the centre of this range and small elsewhere. The calculated gain is thus $\frac{R_2}{R_1} = 70.2$. Since it is the same for both signals, the phase reversal does not matter, and neither does the phase shift at off-centre frequencies provided the two units are matched. Matching is easily achieved, since the coefficient of j is small over the design range.

Two preamplifiers were built to this design. The gains were found to be independent of frequency within the accuracy of measurement, but differed slightly between the two units because of differences in nominal and actual component values. With identical inputs there was no measurable phase discrepancy between the outputs.

(e) Phase-Shifting: The first technique used to produce the 90° phase shift needed to obtain the quad-spectrum (see Section 4.1) was to connect an operational amplifier as a simple differentiating circuit. This was effective, but since the gain increased with frequency, and the signal frequencies were less than the mains frequency, there was a great deal of $60 \sim$ noise in the output. To overcome this a circuit of the same type as that used for the preamplifier was adopted. The gain of such a circuit initially increases with frequency--the differentiating regime, then becomes constant--the amplifying regime, as exploited in the pre-amplifier design, and finally falls off as f^{-1} , the integrating regime. While there is practically no phase shift in the second regime, in the first and third the phase shift tends to 90° as the frequency tends to zero and ∞ , respectively. Evidently the falling characteristic of the third regime would give the noise attenuation required, and so the device was designed to operate in this regime. In this case it is convenient to write the input-output relation in the form:

$$-\frac{e_o}{e_i} = \left[\frac{\omega R_2 C_1 (\omega^2 R_1 R_2 C_1 C_2 - 1)}{(1 + \omega^2 R_1^2 C_1^2)(1 + \omega^2 R_2^2 C_2^2)} \right] \left[\left\{ \frac{\omega R_1 C_1 + \omega R_2 C_2}{\omega^2 R_1 R_2 C_1 C_2 - 1} \right\} - j \right]$$

Denoting $\omega R_1 C_1$ by a , $\omega R_2 C_2$ by b , we wish to make $\left| \frac{a+b}{ab-1} \right| \ll 1$. This evidently requires $a, b \gg 1$, and so the condition becomes $ab \gg a+b$. The maximum product with the minimum sum is obtained when $a = b$. If we specify further that the gain is to be unity, we find that while a, b are large, $\omega R_1 C_1$ must be very large. This condition had to be imposed because of a shortage of amplifiers at the time the unit was designed. With feasible component values, more accurate quadrature could be obtained if this condition were relaxed, and a variable gain amplifier added to correct to unit gain. For $a = b \gg 1$, we obtain the approximate result:

$$-\frac{e_o}{e_i} = \frac{\omega R_1 C_1}{a^2} \left[\frac{2}{a} - j \right]$$

so that the gain is $\left(\frac{\omega R_1 C_1}{a^2} \right) = \left(\frac{1}{\omega} \frac{R_2}{R_1^2} \frac{1}{C_1} \right)$, and the phase error is $\frac{2}{a} = \left(\frac{2}{\omega R_1 C_1} \right)$. With $R_2 = 50 \text{ M}\Omega$, $C_1 = 10^{-5}$ farad,

$C_2 = 5.2 \times 10^{-8}$ farad, and R_1 a $500 \text{ K}\Omega$ potentiometer, the phase error is 0.5° at the mid-range frequency $\omega = 80$, and R_1 set to give a gain of unity. As was mentioned, this error could have been made completely negligible by adding an amplifier. The device also introduces another error: because of the gain-frequency relation, frequencies within the filter passband, but above or below the centre frequency, are amplified by factors less than or greater

than unity, respectively. This error is common to all analogue phase shifters, but is small in the present case because of the narrow filter bandwidth used. With this design, the calculated amplitude of 60 ~ noise is reduced by a factor of 25 at the mid-range frequency as compared to a phase shifter of the differentiating type.

The phase-shifter constructed to this design was found to be very satisfactory in performance. Its quadrature accuracy was checked using identical inputs and measuring their co- and quad-variance. The latter should ideally be zero for identical inputs; it was actually about 1% of the covariance, corresponding to a phase error in close agreement with the calculated value. The zero offsets, to be described later, could be used to reduce this error. Whereas with the differentiating shifter it had been difficult to set the gain accurately because of the noise present, even when using a low-pass filter, with this device noise presented no problems.

(f) Multiplication: The multipliers used were the Donner 3732 P type, which operate on the so-called quarter-square principle. If the two inputs are denoted by X and Y the device simulates electrically the algebraic operations on the left side of the equation:

$$\left[\frac{1}{2}(X+Y)\right]^2 - \left[\frac{1}{2}(X-Y)\right]^2 = XY$$

The two inputs are first inverted, producing $-X$ and $-Y$, and using two amplifiers. X and Y are then applied through matched resistors to a diode, and $-X$ and $-Y$ are similarly applied to an identical diode, and the diode outputs joined. Since obviously either $(X + Y)$ is positive or $((-X) + (-Y))$ is positive, one or other, but not both, diodes will pass current at all times except when X and Y are both zero. The current leaving the junction of the diode outputs is therefore proportional to $|X+Y|$. This is applied to one of two squaring networks, to be described, giving the output current $K(X+Y)^2$. In a similar way the pairs X and $-Y$, $-X$ and Y are applied through resistors to reversed diodes, so that the current leaving the junction of the diode outputs represents $-|X-Y|$. This is applied to the other squaring network, which is such that the negative sign is maintained, and the output current is $-K(X-Y)^2$. Note that no operational amplifiers have been used up to this stage, apart from the two used at the start for phase inversion. The outputs from the two squaring networks are now summed using a third amplifier and giving the output voltage $-4KXY$, where the minus sign results from the usual inversion of phase of input and output of an operational amplifier. The scale factor $4K = 0.01$, so that if the amplitudes of X and Y are simultaneously equal to 100V, the maximum permissible signal amplitude, the amplitude of the output will also be 100V.

The squaring networks are diode function generators, which approximate the square curve by nine straight line segments. (It is principally to obtain a good approximation at this stage that it is necessary to use most of the dynamic range of the computer.) The two networks are identical except for the polarity of the diode connections and bias voltages, one being designed to operate only with positive inputs, and the other with negative inputs. Each network consists of eight diodes, each with a series resistor, and a voltage divider resistance chain to bias the diodes at successively higher levels in 10V steps. For the positive network each diode starts to conduct when the applied positive voltage exceeds its negative bias; the current through it is then equal to the ratio of the excess positive voltage to the resistance of the series resistor. The output current is the sum of the currents through the conducting diodes, plus that through a ninth resistor which has no diode in series with it.

A mode switch is provided by which the two squaring circuits may be used independently, to produce outputs of $-.01X^2$ and $+.01Y^2$. In this case a fourth operational amplifier is required; although the output currents from the squaring networks represent X^2 and Y^2 to some scale, the third and fourth amplifiers are required to provide the appropriately scaled voltage while isolating

each squaring network from the input impedance of the following stage.

(g) Integration: The integrators used were the Philbrick U-2J type, which provides the necessary computing elements mounted in a shielded box, designed to plug in to a Philbrick operational amplifier. For integration the input element is a resistance and the output element a condenser. The U-2J has several convenient features. Four input terminals, each with an input resistor connected to the summing point, are provided, so that the integral of the sum of up to four signals can be obtained. There are five output condensers with capacitances in the ratio of powers of ten; these can be selected by a switch on the front of the box. This switch also provides for manual resetting of the integrator, by shorting the output condenser. The component values are such that the U-2J can perform the operation:

$$e_o = -10^m \int_0^T (e_1 + e_2 + e_3 + e_4) dt, \quad \text{for } m = 0, 1, 2, 3, \text{ or } 4.$$

In addition a relay operated switch is provided by which an applied 6V DC signal of the correct polarity shorts the output condenser. By taking an appropriate fraction of the integrator output voltage, and changing its polarity if necessary, this switch can be used to short the output condenser automatically when its potential reaches a pre-determined limit. This method of resetting the integrators is applicable when integrating a signal such as $X^2(t)$,

which is always of the same polarity. For product signals which can have either polarity, the polarity of the output condenser potential may also be of either sign, and if automatic resetting is desired a double-sided flyback is necessary. The method adopted for achieving automatic double-sided resetting was suggested by Mr. S.D. Smith of IOUBC and is illustrated below. It was found to work well, but requires an extra amplifier for phase inversion.

When no inputs were applied to the preamplifiers, it was found that the output from the multipliers, although very small, was not zero. The integrated signals showed a small steady drift in one direction, evidently due to noise pickup. This was cancelled out at the start of every run by applying a suitable small DC voltage to one of the four integrator input terminals. The voltage was obtained from the variable tap of a 10-turn dial-type potentiometer of

100 K Ω , which was connected between two 1-megohm fixed resistors, the ends of the chain being connected to the positive and negative power supplies of the computer. The offset was adjusted until the integrator output showed no drift, which could be determined very accurately.

(h) Recorders: The integrator outputs were recorded on Varian paper recorders. The slowest paper speed, 1 in. per min., was used, and the voltage scale was set so that the full width of the paper represented 100V. For the squared signals, the zero was chosen to be at the appropriate edge of the paper, depending on the output polarity, and the fly-backs were arranged to reset the integrators when the output condenser potential reached 100V. The Varian chart trace was therefore of roughly sawtooth form, since after slowly advancing across the chart to the edge representing 100V, the pens, following the integrator output potential, returned very rapidly to the zero edge of the chart before starting another slow advance. The value of the integral for a given integration time could be read from the chart to a known scale, by counting the number of completed "saw-teeth" and adding the final fractional part. In the case of the mixed products the procedure was similar, but since the integrator output could be of either sign, the centre line of the chart was chosen as the zero. With the full width of the paper still representing 100V, the integrators were arranged to reset at $\pm 50V$, so that one complete sawtooth

from the centre to the edge represented 50V. The sense of the deflection corresponding to positive values of the product could be determined either experimentally using input signals with a known product, or by counting the number of phase reversals in the computer network and observing the sense of the deflection for a positive voltage applied directly to the recorder.

(i) Calibration: The complete system from the filter input to the recorder was calibrated by applying to the two filter inputs identical sinusoidal signals of known amplitude and of a frequency equal to the centre frequency of the filters, and integrating for a fixed time. The integrals read from the charts gave the calibration for three of the four outputs. It did not give the calibration for the quad-variance output, since with identical inputs this was zero. In order to calibrate the quad channel, two inputs of the same frequency and amplitude as before, but with a phase difference of 120° , were used in the same way. These signals were obtained from a low-frequency three-phase signal generator. An alternative method for calibrating the quad channel is to bypass the phase shifter, using identical inputs, and measure the gain of the shifter separately; the latter was always set to unity before each run. In practice the gains of each stage were separately measured as a check on the calibration.

The method of using the calibration results to obtain the spectral density is illustrated below for the simplest case of the power spectrum. The cross-spectral densities involve only a trivial extension.

Suppose the frequency-selection and amplification functions of the filter are carried out in separate stages, the former occurring first. Denote the instantaneous signal output of this imaginary pure filter by $v(t)$, and the gain of the filter amplifier by G_1 . If G_2 is the gain of the preamplifier the overall gain G is approximately $G_1 G_2$; this result is exact if the multiplier, integrator, and recorder calibration factors are precisely unity. The recorded data signal voltage is then $.01 G^2 \overline{v^2(t)} T$, where the .01 factor is built into the multiplier, and T is the time duration of the integration; we denote this recorded voltage by $\overline{X^2}$. For any signal we may therefore write $\overline{v^2} = \frac{k}{T} \overline{X^2}$. Now the sinusoidal calibration signal is $v_c(t) = \frac{1}{2} V_c \sin \omega t$, where V_c is its measured peak-to-peak voltage, and the corresponding recorded voltage is $\overline{X_c^2} = \frac{1}{8} \times .01 G^2 V_c^2 T$. Hence $k = \frac{1}{8} V_c^2 T / \overline{X_c^2}$, and for the data signal $v(t)$ we may write:

$$\overline{v^2} = \left[\frac{V_c^2}{8} \frac{T}{T} \frac{\overline{X^2}}{\overline{X_c^2}} \right]$$

If the data signal is played back at α times the recording speed, the measurement made with filter centre frequency f applies to a centre frequency of $\frac{1}{\alpha} f$ in the original

wave data, and if the filter power bandwidth is βf_0 ,
the corresponding data bandwidth is $\frac{\beta}{\alpha} f_0$. Hence, for
waves of frequency $\frac{1}{\alpha} f_0$ the spectral density is given by:

$$\left(\frac{\overline{v^2}}{\frac{\beta}{\alpha} f_0} \right) = \left[\frac{\alpha}{\beta} \frac{1}{f_0} \frac{V_c^2}{8} \frac{T_c}{T} \frac{\overline{X^2}}{X_c^2} \right] \text{ volt}^2\text{-sec}$$

Using the detector sensitivity s expressed in volts (of
the demodulated signal) per cm. of surface displacement,
the spectral density may be written:

$$\Phi \left(\frac{f_0}{\alpha} \right) = \left[\frac{\alpha}{\beta} \frac{1}{f_0} \frac{1}{s^2} \frac{V_c^2}{8} \frac{T_c}{T} \frac{\overline{X^2}}{X_c^2} \right] \text{ cm}^2\text{-sec}$$

where every term occurring is a measured quantity.

FROM CANADIAN HYDROGRAPHIC SERVICE CHART NO. 3586.

FIG 1 : MAP OF EXPERIMENTAL SITE.

FIG 2: FETCH AVAILABLE AT EXPERIMENTAL SITE

FIG 4: PREDICTED MEAN DIRECTION OF PROPAGATION - MILES REGIME

FIG 5: PREDICTED ANGULAR WIDTH OF SPECTRUM - MILL'S REGIME

FIG 6: DIFFRACTION PATTERN FOR 7-DETECTOR ARRAY.

TRANSISTOR TYPE 2N1305.

FIG 7 : WIEN BRIDGE OSCILLATOR CIRCUIT.

FIG 8 : OUTPUT AMPLIFIER CIRCUIT

FIG 9 : WAVE DETECTOR CALIBRATION CURVE.

FIG. 10: PHOTOGRAPH OF FIRST ARRAY INSTALLATION

FIG. 11: PHOTOGRAPH OF THIRD ARRAY INSTALLATION

FIG 12 : ANALOGUE COMPUTER SIGNAL FLOW DIAGRAM

FIG 15: WAVE POWER SPECTRUM - LOG LOG PLOT

FIG 16: CO- AND QUADRATURE SPECTRA : $f = 0.437 \sim$

FIG 19: DIRECTIONAL SPECTRUM - FOURIER TRANSFORM METHOD $f = 0.437 \sim$

FIG 20: SMOOTHED DIRECTIONAL SPECTRUM $E(l)$: $f = 0.437$

FIG 21 : SMOOTHED DIRECTIONAL SPECTRUM - FOURIER TRANSFORM METHOD

FIG 22: DIRECTIONAL SPECTRUM $f = 0.261 \sim$

FIG 23: DIRECTIONAL SPECTRUM $f = 0.284$

FIG 24: DIRECTIONAL SPECTRUM $f=0.310\sim$

FIG 25 : DIRECTIONAL SPECTRUM $f = 0.338 \sim$

FIG 26: DIRECTIONAL SPECTRUM $f = 0.368 \sim$

FIG 27: DIRECTIONAL SPECTRUM $f=0.401$

FIG 28: DIRECTIONAL SPECTRUM $f=0.437 \sim$

FIG 29: DIRECTIONAL SPECTRUM $f = 0.476 \sim$

FIG 30: DIRECTIONAL SPECTRUM $f = 0.500$

FIG 31: DIRECTIONAL SPECTRUM $f = 0.520 \sim$

FIG 32 : DIRECTIONAL SPECTRUM $f = 0.566 \sim$

FIG 34: DIRECTIONAL SPECTRUM $\lambda = 0.674 \sim$

FIG 35: DIRECTIONAL SPECTRUM $f = 0.735 \sim$

FIG 36: DIRECTIONAL SPECTRUM $f = 0.802$

FIG 37: DIRECTIONAL SPECTRUM $f=0.875\sim$

FIG 38: DIRECTIONAL SPECTRUM $f = 0.954 \sim$

FIG 39: DIRECTIONAL SPECTRUM $f = 1.04 \sim$

FIG 40: DIRECTIONAL SPECTRUM $f=1135$ m

FIG 41: DIRECTIONAL SPECTRUM $f = 1.24 \sim$

FIG 44 : MAP OF SPANISH BANKS

FIG 45 : WAVE REFRACTION DIAGRAM

FIG 46: GOODNESS OF FIT DIAGRAM

FIG 47: COMPARISON OF THEORETICAL AND EXPERIMENTAL DIRECTIONS OF PROPAGATION.