The University of British Columbia

FACULTY OF GRADUATE STUDIES

PROGRAMME OF THE

FINAL ORAL EXAMINATION

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

of

THOMAS WILLIAM DONNELLY

B.Sc.(Hons.) University of British Columbia, 1964

WEDNESDAY, APRIL 12, 1967 AT 3:30 P.M.

IN ROOM 304, HENNINGS BUILDING

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TWO-BODY CALCULATIONS FOR THE DIRECT RADIATIVE REACTIONS
\[ \text{D(p,}\gamma\text{He}^3, \text{He}^3(\gamma,p) \] AND \[ \text{0}^{16}(p,\gamma)\text{F}^{17} \]

ABSTRACT

The direct radiative capture reactions \( \text{D(p,}\gamma\text{He}^3 \) and \( \text{0}^{16}(p,\gamma)\text{F}^{17} \), both of which are of interest in astrophysical processes, have been studied theoretically using a simple two-body direct radiative capture model in order to estimate the cross sections at low energies. In addition, the time inverse of the first reaction, namely the photodisintegration of \( \text{He}^3 \), has been studied for high excitation energies in \( \text{He}^3 \) by applying the reciprocity relations to the direct capture theory. The calculations involve taking matrix elements of the particle-radiation interaction Hamiltonian between bound and continuum states and using first-order perturbation theory to obtain the cross sections. Bound state wave functions are generated in simple potentials involving square-well and Saxon-Woods forms with appropriate Coulomb barriers and with one free parameter which is adjusted to fit the binding energy. The potential parameters for the continuum state wave functions are adjusted to fit available scattering data.

For the reaction \( \text{0}^{16}(p,\gamma)\text{F}^{17} \) the cross sections for transitions to both the ground and first excited states are in good agreement with the somewhat limited experimental data from 150 keV to 2.5 MeV and the astrophysical S-factors are shown to be energy dependent even at energies below 100 keV. The photodisintegration cross section for the reaction \( \text{He}^3(\gamma,p)\text{D} \) is well fitted in the neighbourhood of the peak at around 11 MeV as well as at lower energies. The \( \text{D(p,}\gamma\text{He}^3 \) direct capture cross sections in the energy range around 1 MeV.
are shown to be sensitive to admixtures of \(^2\)S-state of mixed symmetry and of \(^4\)D-state in the ground state of \(^3\)He, which is predominantly symmetric \(^2\)S. The same model including the \(^2\)S-state of mixed symmetry leads to a capture cross section for thermal neutrons by deuterons in good agreement with the experimental value.

AWARDS

1965-67 National Research Council of Canada, Studentship
1967- National Research Council of Canada, Post-doctoral Fellowship
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TWO-BODY CALCULATIONS FOR
THE DIRECT RADIATIVE REACTIONS

\[ D(p, \gamma)He^3, He^3(\gamma, p)D \text{ AND } ^{16}O(p, \gamma)^{17}F \]

by

THOMAS WILLIAM DONNELLY
B.Sc., University of British Columbia, 1964

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department of

PHYSICS

We accept this thesis as conforming to the
required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April, 1967

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Date April 13, 1967
ABSTRACT

The direct radiative capture reactions \( D(p,\gamma)He^3 \) and \( ^{16}O(p,\gamma)^{17}F \), both of which are of interest in astrophysical processes, have been studied theoretically using a simple two-body direct radiative capture model in order to estimate the cross sections at low energies. In addition, the time inverse of the first reaction, namely the photodisintegration of \( He^3 \), has been studied for high excitation energies in \( He^3 \) by applying the reciprocity relations to the direct capture theory. The calculations involve taking matrix elements of the particle-radiation interaction Hamiltonian between bound and continuum states and using first-order perturbation theory to obtain the cross sections. Bound state wave functions are generated in simple potentials involving square-well and Saxon-Woods forms with appropriate Coulomb barriers and with one free parameter which is adjusted to fit the binding energy. The potential parameters for the continuum state wave functions are adjusted to fit available scattering data.

For the reaction \( ^{16}O(p,\gamma)^{17}F \) the cross sections for transitions to both the ground and first excited states are in good agreement with the somewhat limited experimental data from 150 KeV to 2.5 MeV and the astrophysical S-factors are shown to be energy dependent even at energies below 100 KeV. The photodisintegration cross section for the reaction \( He^3(\gamma,p)D \) is well fitted in the neighbourhood of the peak at around 11
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ACKNOWLEDGEMENTS

It is a pleasure to express my sincere gratitude to my supervisor, Prof. G. M. Griffiths and to Dr. G. M. Bailey, under whose guidance and supervision this work was carried out.

I am also indebted to Prof. E. W. Vogt and to Dr. M. McMillan for helpful discussions on various aspects of this work, particularly those sections involving the three-body problem. Furthermore, I should like to thank A. G. Fowler of the University of British Columbia Computing Centre for valuable assistance involving computer programming.

Finally, I should like to thank the National Research Council of Canada for financial assistance through N.R.C. Scholarships over the course of my graduate studies.
CHAPTER 1

INTRODUCTION

The present work is a study of the direct radiative processes $D(p, \gamma)He^3$, $He^3(\gamma, p)D$ and $o^{16}(p, \gamma)^{17}$ on the basis of a simple two-body model. These direct one-step processes provide a means for extracting details of the bound states. Since these reactions proceed directly between continuum and bound states via interactions with the electromagnetic radiation field and since the continuum states are known from scattering, comparisons of theoretical and experimental cross sections can provide information about the bound states. The direct radiative processes have this feature in common with other direct reactions (for example, stripping), but have the advantage that the transitions are induced by the well-known electromagnetic field rather than the less well-known nuclear force. In addition, the weakness of electromagnetic forces (relative to nuclear forces) allows first-order perturbation theory to be used (with some confidence) in computing transition probabilities.

Much interest is presently being directed towards studies of the three-nucleon system as a means for testing theories of nuclear forces. The facts that the bound three-body states ($T$ and $He^3$) have both singlet and triplet two-nucleon spin configurations and are more tightly bound than the deuteron indicate that a study of these systems could reveal some pro-
properties of the nucleon-nucleon force to which the deuteron is insensitive, for example, short range components of the force or components sensitive to singlet spin configurations. The present work is a study in which the complicated three-nucleon problem is approximated by a deuteron plus proton, ignoring the neutron-proton structure of the deuteron. Some features of the problem are of course neglected in this approximation, in particular those involving the Pauli Exclusion Principle for the two protons in He$^3$ or two neutrons in T. A complete treatment of the three-body system involves a detailed classification of the three-nucleon states according to their permutation symmetry in spatial, spin and isospin coordinates. As will be shown, the approximate two-body calculations lead directly to results in very good agreement with experiment for some radiative transitions, while for others, it is necessary to incorporate three-body features into the two-body model in an empirical way.

One of the motivations for investigating the reaction D(p,γ)He$^3$ was provided by astrophysics. In the hydrogen-burning stage of smaller main sequence stars, the main energy supply comes from a series of reactions known as the p-p chain (Burbidge et al., 1957) in which the reaction D(p,γ)He$^3$ is the second step:

\[
\begin{align*}
\text{p} + \text{p} & \rightarrow \text{D} + \beta^+ + \gamma \\
\text{p} + \text{He}^3 & \rightarrow \text{He}^4 + \gamma \\
\text{He}^3 + \text{He}^3 & \rightarrow \text{He}^4 + 2\text{p}
\end{align*}
\]

\[
\begin{align*}
\text{p} + \text{p} & \rightarrow \text{D} + \beta^+ + \gamma \\
\text{p} + \text{He}^3 & \rightarrow \text{He}^4 + \gamma \\
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\end{align*}
\]

\[
\begin{align*}
\text{He}^3 & \rightarrow \text{He}^4 + 2\gamma + 2\beta^+ + 2\gamma
\end{align*}
\]

\[
\begin{align*}
\text{He}^3 & \rightarrow \text{He}^4 + 2\gamma + 2\beta^+ + 2\gamma
\end{align*}
\]
In the smaller stars where the p-p chain predominates over the C-N cycle, the interior temperatures fall in the range of $5 \times 10^6$ to $15 \times 10^6$ K, corresponding to mean thermal energies of about 1 keV for the centre of this temperature range ($10^6$ K). Consequently the reaction rates of interest are in the energy range from 1 to 10 keV. For a star on the main sequence the rate at which energy is released is controlled by the very small cross section for the first step in the chain which depends on the weak $\beta$ interaction. The reaction $D(p,\gamma)He^3$, which depends on the much stronger electromagnetic interaction, is so fast in comparison that it has little effect on the rate of energy release. However in the early stages of condensation towards the main sequence, when the temperature is rising, the reaction $D(p,\gamma)He^3$ operating on the primordial deuterium in the interstellar gas will be one of the first to occur and consequently may affect the rate of condensation. Furthermore, as Cameron (1962) has pointed out, the reaction $D(p,\gamma)He^3$ competes with the reaction $D(d,n)He^3$ which has a much larger cross section but a smaller probability due to the small deuterium concentration in the interstellar gas. This competition then affects the number of neutrons available from the initial deuterium. The number of neutrons in turn is of interest because of the possibility that they could affect isotope ratios among the heavy elements formed by neutron capture during the early stages of condensation of main sequence stars.

In the present calculations the photodisintegration
reaction $\text{He}^3(\gamma, p)D$ is also studied using the same direct capture model including the reciprocity relation. The photodisintegration cross sections are calculated at relatively high excitation energies of $\text{He}^3$ where one perhaps would not expect this approximate model to give as good agreement with experiment as at low energies (where the cross sections are largely model-independent).

The $^{16}\text{O}(p, \gamma)^{17}\text{F}$ direct capture reaction was also considered on the basis of this simple two-body model. Due to the tightly bound $^{16}\text{O}$ core, the $^{17}\text{F}$ system should be well described by a single-particle model in both bound and continuum states. This reaction then provides a good means for testing the direct capture model and for studying the bound states of $^{17}\text{F}$. As for the three-nucleon system, this reaction is also of astrophysical interest, in particular in the C-N cycle:

\[
\begin{align*}
^{12}\text{C}(p, \gamma)^{13}_L(\beta^+ \nu) & \rightarrow ^{13}\text{C}(p, \gamma)^{14}_H(p, \gamma)^{15}_L(\beta^+ \nu)N^{15}(p, \alpha)^{12}_H \\
^{15}_L(p, \gamma)^{16}_H(p, \gamma)^{17}_L(\beta^+ \nu) & \rightarrow ^{17}_L(p, \alpha)^{14}_H
\end{align*}
\]

The reaction $N^{15}(p, \gamma)^{16}_H$ competes with the reaction which returns $^{12}\text{C}$ to the beginning of the C-N cycle, namely $N^{15}(p, \alpha)^{12}_H$. The competition results in a leakage of C-N catalyst from the cycle. However the loss is returned at a rate controlled by the slowest reaction in the subcycle shown above, namely $O^{16}(p, \gamma)^{17}_L$. Hebbard (1960) made a detailed analysis
of the reaction $\text{N}^{15}(p,\gamma)\text{O}^{16}$ and showed that for low energies there was constructive interference between the tails of the two $1^-$ resonances at 338 and 1010 keV, so that the cross section, and hence the leakage rate of catalyst from the C-N cycle, was considerably larger than previously estimated on the basis of the properties of the 338 keV resonance alone. Most of the material leaked into the side cycle is in the form of $\text{O}^{16}$ and the ratio of carbon to oxygen or nitrogen to oxygen is directly affected by the $\text{O}^{16}(p,\gamma)\text{F}^{17}$ cross section.
2.1 Introduction and Model

The properties of the three-nucleon systems T and $^3\text{He}$ are of some interest from the viewpoint of assessing the character of internucleon forces. These nuclei are more tightly bound than the bound state of two nucleons, the deuteron, and thus should be more sensitive to shorter range components of the forces. Also, they contain pairs of nucleons in both triplet and singlet spin states, compared to the deuteron which has only a triplet spin state of the two particles, and thus their properties depend on more components of the possible types of internucleon forces.

In principle, if one inserts into the three-nucleon system the two-body forces between each pair of nucleons, as obtained from a study of the deuteron and of neutron-proton and proton-proton scattering, one should be able to calculate the properties of the three-nucleon systems and compare the results with experiment. Such a comparison would provide a critical test of the two-body data or, on the other hand, could give evidence for specifically three-body forces which have been suggested by some forms of meson theory. A realistic attempt to carry out a program based on this suggestion is plagued by numerous difficulties, not the least of which are the facts that even in classi-
cal mechanics there is no closed form of solution to the dynamical behaviour of three interacting particles, and second, that the internucleon potential is very complicated, containing non-central terms of several possible kinds.

Because of the complexity, in much of nuclear physics simple models are often introduced, guided by theoretical understanding but containing arbitrary parameters which can be evaluated by experiment. These models then help to condense large amounts of experimental data into a few significant parameters which may become the input data for a further elaboration of the model. Such models, if successful, can then be used to predict results in ranges of some of the parameters outside those measured directly, which may be empirically useful or may suggest new measurements to the experimentalist. In the present work on the mass three system a very simplified model is introduced, with some theoretical guidance, and adjusted to fit experimental data. This gives in turn approximate values for several parameters of theoretical interest and indicates how further experimental work could improve the data on these parameters, as well as predicting some cross sections in the energy range of astrophysical interest, where laboratory measurements would be extremely difficult if not impossible.

In order to incorporate the Pauli Exclusion Principle into the description of the three-nucleon system, it is convenient to classify the states in terms of the basic symmetries of the
permutation group for three things. This group has three irreducibile representations: a one-dimensional representation completely symmetric under interchange of all pairs of particles, a one-dimensional representation anti-symmetric under interchange of all pairs and a two-dimensional representation of mixed symmetry, symmetric for some permutations and neither symmetric nor antisymmetric for other permutations. If spatial wave functions of the three-nucleon states are produced with symmetric, antisymmetric and mixed permutation symmetry, then total wave functions which are antisymmetric under all permutations (Pauli Principle) can be written as products of spatial and spin-isospin parts of specific symmetry. On the basis of kinetic energy arguments, one would expect states of highest spatial symmetry to have the lowest energy and correspondingly the highest probability in the bound state wave functions. In general the state which is completely anti-symmetric in spatial coordinates can be neglected.

In what follows, two kinds of radiative processes will be considered, photodisintegration and direct radiative capture, the two being related by reciprocity. Early calculations on the photodisintegration of three-body nuclei were done by Burhop and Massey (1948) and Gunn and Irving (1951) who considered only electric dipole transitions from symmetric $^2S$ bound states to p-waves in the continuum, the latter treated as plane waves. Gunn and Irving compared results when Gaussian and exponential forms were used in the bound state wave functions and showed that
the energy and value of the maximum in the cross section was sensitive to the scale size parameter as well as to the form of the radial wave function. In 1950 Verde included a symmetry classification of the bound states and made rough estimates of the photodisintegration cross section on the basis of a Gaussian form for the radial wave functions. As might be expected, these forms, in having cut off the long range parts of the wave function and consequently making it too compact, lead to energies for the maximum in the cross section considerably higher than have actually been found by experiment.

Observations of the electric quadrupole moment of the deuteron indicated the need for including tensor forces in the two-body interaction. If tensor forces are introduced into the description of the ground states of He and T, the orbital angular momentum (L) and total spin (S) are no longer good quantum numbers. The total angular momentum (J=1/2), the total isospin (T=1/2) and the parity (+) however remain good quantum numbers. The ground state wave functions can then contain the following components (Sachs, 1955): 2S\text{1/2}, 2P\text{1/2}, 4P\text{1/2} and 4D\text{3/2}. A more detailed group theoretical symmetry classification of the three-body bound states has been given by Derrick and Blatt (1958). Of the ten possible parts of the total wave function, Derrick (1960) has shown that the amplitudes of the 2P\text{1/2} and 4P\text{1/2} components are negligible, leaving the 2S\text{1/2} and 4D\text{3/2} components, with the former making the major contribution. The individual space and spin-isospin parts of the 2S\text{1/2} wave function can be
decomposed into symmetric, antisymmetric and mixed-symmetry parts. Derrick (1960) has shown that the $2S_\frac{1}{2}$-state can have a mixed-symmetry component due to the difference between the central triplet even and singlet even two-body forces and that the amplitude of this component, with respect to the major symmetric contribution, is small due to the approximate equality of the central triplet even and singlet even forces. In addition, the $4D_\frac{1}{2}$-state, which is coupled to the $2S_\frac{1}{2}$-state by the tensor force, can be decomposed into components of given symmetry. As the spin is $3/2$, with the spin functions then being completely symmetric, and the isospin is $\frac{1}{2}$, with the isospin functions then being of mixed symmetry, the spin-isospin and consequently the spatial functions must be of mixed symmetry. In fact, the $4D_\frac{1}{2}$ can be decomposed into three pairs of mixed-symmetry states.

In the present work, a two-body approximation to the three-nucleon system is used, so that many of the three-body aspects of the problem are neglected. The system is treated, both in bound and continuum states, as an odd nucleon interacting with a deuteron.

This kind of model has been used in direct radiative capture calculations at low energies by Christy and Duck (1961) and Tombrello and Parker (1963) and, in particular, for the three-nucleon problem by Lal (1961) and Griffiths et al. (1963).

In the present work the p-d interaction is represented by a square-well potential in the nuclear interior, with two para-
meters, the nuclear radius $R$ and the well depth $V$, plus a Coulomb potential outside the nucleus, that is,

$$V(r) = \begin{cases} -V & r < R \\ \frac{e^2}{r} & r > R \end{cases}$$

(2.1)

The same form is used for the potential both for the continuum states of the $p$-$d$ system and for the bound state of $\text{He}^3$, the latter being regarded as the bound state of the $p$-$d$ system in which the deuteron is treated as an inert core, that is, as a single particle. In making this approximation, effects of the exclusion principle involving the free proton and the proton in the deuteron are neglected.

Using this simple potential, the Schrodinger equation is solved numerically to obtain the bound and continuum wave functions of the $p$-$d$ system. The potential parameters, the radius and depth, are adjusted to produce fits, where possible, to the experimental binding energy and scattering phase shifts. Cross sections then follow by evaluating the matrix elements of the multipole operators for the radiation field between continuum and bound states, as summarized in Appendix A. These radial integrals are again computed numerically with integrations which include contributions from the nuclear interior using the exact forms of the multipole operators, rather than the forms obtained when the long wavelength approximation is invoked (Appendix A).

At low energies the major contributions to the direct cap-
ture cross section arise from outside the nucleus and so should be relatively independent of the details of the model used for the nuclear interior. In fact good agreement with low energy direct capture data is obtained using this model. Furthermore, calculations using this model have now been extended to much higher energies where one would not necessarily expect to find good agreement with experiment. Comparison with recent photodisintegration data in fact indicates very satisfactory agreement with the model calculations, even in the neighbourhood of the peak in the photodisintegration cross section. With this knowledge that the direct capture and photodisintegration cross sections can be reproduced using this model, additional calculations have been performed to predict the relative importance of some of the transitions which are weaker than the El transitions (constituting the major contributions to the cross sections). These transitions show up as interferences with the El transitions or as contributions to the cross sections at $0^\circ$ where the dominant $\sin^2\theta$ El cross section is zero.

As a check on the parameters used in the present calculations for $D(p,\gamma)He$ and $He^3(\gamma,p)D$, the mirror reactions $D(n,\gamma)T$ and $T(\gamma,n)D$ were considered and the results compared with experiment. The following is a brief review of the experimental data on these four reactions. The cross sections for the direct capture reactions can be related to the photodisintegration cross sections by the reciprocity relations, so that these processes give essentially the same information about the nuclear wave
The reaction $D(p,\gamma)He^3$ was first observed by Curran and Strothers (1939). Ten years later Fowler et al. (1949) showed that the angular distribution was nearly pure $\sin^2\theta$ and that the cross section was non-resonant, indicating a direct capture process. In 1952 Wilkinson showed that at $90^\circ$ the gamma-rays were plane polarized with the electric vector in the reaction plane, proving that the capture resulted in El radiation from a continuum p-wave to the ground s-state of $He^3$. Furthermore, Wilkinson noted that the pure $\sin^2\theta$ angular distribution indicated that there was very little spin-orbit interaction mixing in $\Delta m = \pm 1$ transitions which would contribute at $0^\circ$. Griffiths and Warren (1955) measured the cross section between .5 and 2.0 MeV with particular emphasis on the region around $0^\circ$ with the object of determining the amount contributed by the spin-orbit interaction. However, the energy dependence of the yield at $0^\circ$ was different from that at $90^\circ$ and suggested that the $0^\circ$ yield was due to s-wave capture compared to the p-wave energy dependence of the $90^\circ$ yield. These measurements were repeated later (Griffiths et al. 1962) between .275 and 1.75 MeV with a much higher accuracy using both heavy ice and gas targets, confirming that the $0^\circ$ yield followed an energy dependence characteristic of s-wave capture based on relatively crude penetrability arguments. Further confirmation came from measurements (Griffiths et al., 1963) in the energy range from 24 to 48 keV. On the basis of these measurements, astrophysical S-factors were
calculated and extrapolated to the 1 keV region using a crude direct radiative capture model including both p- and s-wave capture. Recently Wölfli et al. (1966) have measured the capture cross section between 2 and 12 MeV including angular distributions for several energies up to 5.25 MeV.

It is of some interest to compare the D(p,γ)He₃ cross section with the cross section for the mirror reaction D(n,γ)T, which has been measured only for thermal neutrons, first by an indirect method by Sargent et al. (1947) and later by a more direct method by Kaplan et al. (1952), who obtained a cross section of .57 ± .01 mb for 2200 m/sec neutrons. More recent measurements by Jurney and Motz (1964) and by Trail and Raboy (1964) with less precision are in agreement with the Kaplan et al. result.

The inverse photodisintegration reactions He₃(γ,p)D and T(γ,n)D have been extensively studied since 1963 as a result of the availability of T and He₃. Berman et al. (1963) studied the photodisintegration of He₃ between 8.5 and 22 MeV using bremsstrahlung from the Illinois betatron and detected the charged particles with solid-state counters at 90° to the beam. Warren et al. (1963) measured the total cross section in the low energy range using discrete energy gamma-rays produced by nuclear reactions and detected the particles in a gridded ionization chamber. The observed photodisintegration cross sections measured at energies which overlapped the direct capture cross sections.
were in good agreement with the latter taking into account the reciprocity relations between them. Further measurements of both total and 90° differential cross sections were made in this same period by Gorbunov and Varfolomeev (1963) using 170 MeV bremsstrahlung and observing the particles in a cloud chamber. These results were later extended by Fetisov et al. (1965) and compared with a number of theoretical predictions. The 90° differential cross section has also been measured in the energy range 8.5 to 46 MeV by Stewart et al. (1965) using the Yale electron linear accelerator to produce bremsstrahlung, observing the charged reaction products with a quadrupole triplet magnetic focussing system.

Work on the photodisintegration of T has been more limited. Bösch et al. (1965) have measured the T(γ,n)D cross section using discrete gamma-rays produced by (n,γ) capture in various targets located in a nuclear reactor and detecting the neutrons with BF₃ counters surrounded by water moderators placed at various positions around the T target. Kosiek et al. (1966) using 32.5 MeV bremsstrahlung from a betatron measured the outgoing deuterons in T(γ,n)D in a counter telescope and determined the photodisintegration cross section for gamma-rays of energies from 17 to 31 MeV.

Most of the experimental results discussed here will be compared with theoretical calculations in the following sections.
2.2 Bound States

The ground state of $\text{He}^3$ can be regarded as a combination of $^2S_\frac{1}{2}$ and $^4D_\frac{3}{2}$ states, the latter being included due to tensor forces. If a tensor potential is introduced in a manner completely analogous to the introduction of such a potential in the derivation of the Rarita-Schwinger equations for the deuteron (e.g., Blatt and Weisskopf, 1952) coupled differential equations of the following form obtain

\[ -\frac{\hbar^2}{2\mu} \frac{d^2 u_0(r)}{dr^2} + (V_1(r) - E) u_0(r) = -A \frac{\sqrt{P_D}}{\sqrt{1-P_D}} V_T(r) u_2(r) \]  
\[ -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{\ell^2}{r^2} \right) u_2(r) + (V_2(r) - E) u_2(r) = -B \frac{\sqrt{1-P_D}}{\sqrt{P_D}} V_T(r) u_0(r) \]

where

\( u_0(r) = \) $^2S$-component of the radial wave function,

\( u_2(r) = \) $^4D$-component of the radial wave function,

both normalized to unity, and

\( P_D = \) $^4D$-component probability, so that the total radial wave function is

\[ \mu(r) = \sqrt{1-P_D} u_0(r) + \sqrt{P_D} u_2(r) \]

\( \mu = \) reduced mass of the proton-deuteron system,

\( E = \) binding energy of the $\text{He}^3$ ground state,

\( V_1(r), V_2(r) = \) potentials, including spin-dependence

\( (V_2(r) \) also contains the tensor potential

\[ V_T(r) \]

and \( A, B = \) constants determined by details of the theory ( \( \sim 1 \)).
For \( r > R_{s} \), if all potentials are taken to be square wells, these equations decouple, leaving the equations for pure S- and D- states. For \( r < R_{s} \), assuming the forms of the interior solutions to be the same (by analogy with the deuteron case and knowing the \(^4\)D-component probability to be small, in which case such an assumption should hold at least approximately), these equations become

\[
\begin{align*}
\frac{-\hbar^2}{2\mu} \frac{d^2u_{0}(r)}{dr^2} + (V_{1}'(r) - E) u_{0}(r) &= 0 \\
\frac{-\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{2}{r^2} \right) u_{2}(r) + (V_{2}'(r) - E) u_{2}(r) &= 0
\end{align*}
\]

where

\[
V_{1}'(r) = V_{1}(r) + A \sqrt{\frac{P_{D}}{1 - P_{D}}} V_{T}(r)
\]

and

\[
V_{2}'(r) = V_{2}(r) + B \sqrt{\frac{1 - P_{D}}{P_{D}}} V_{T}(r)
\]

These equations then describe pure S- and D-wave functions with "effective potentials" \( V_{1}'(r) \) and \( V_{2}'(r) \). As the \(^4\)D-component probability is known to be small (and hence the term \( \sqrt{\frac{1 - P_{D}}{P_{D}}} \) large) if the tensor potential depth is taken to be of the same order of magnitude as the central potential depth (\( \approx 10-20 \) MeV) with the same range, then the "effective potential" \( V_{2}'(r) \) will be greater in depth (\( \approx 50-100 \) MeV).

The wave functions computed in this way are not true solutions of the coupled equations in the nuclear interior but are considered to be adequate for providing an estimate of the
effect of a small $^4D$-component of the direct capture cross section. Furthermore, as will be demonstrated subsequently, the major contributions to the matrix elements involving the $^4D$-state occur in the nuclear surface and beyond, where the wave functions are approximately correct.

Initial calculations of direct capture and photodisintegration cross sections were performed using three values of the nuclear radius, 2.26, 3.19 and 4.35 fm. For all subsequent calculations, the value 3.19 fm was used, both for bound and continuum states. This was the only radius of the three which produced agreement both with the low energy $D(p, \gamma)He^3$ $E1(2^P-2^S)$ cross section measured at 24 keV by Griffiths et al. (1963) and with the higher energy photodisintegration data. With this choice of radius, the bound state well depths were fixed by requiring that the proton and deuteron forming the ground state of $He^3$ be bound by 5.49 MeV. This required well depths of 19.54 and 70.14 MeV for the $^2S$- and $^4D$-components respectively.

The ground state wave function can be written

$$\Phi_M = \sum L \sum S a^S_L \frac{\psi^S_L(r)}{r} \sum \frac{C(LSJ; M-\beta, \beta)}{r^\beta} \gamma^M_{\beta}(\theta, \varphi) \chi^\beta_S$$  \hspace{1cm} (2.8)

where the notation is that used in Appendix A, with the amplitudes

$$a^S_L = \begin{cases} \sqrt{1-P_3} & L=0, S=1/2 \\ \sqrt{P_3} & L=2, S=3/2 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (2.9)
The radial wave functions, $U_L^S(r)$, are shown in Fig. 2.1 for the parameters discussed above, both being normalized to unity.
Figure 2.1: Normalized radial wave functions for $\text{He}^3$. 

$U(r)$ (fm$^{-1/2}$) 

$^2S$-component 
$^4D$-component 

$r$ (fm) 

0 1 2 3 $R$ 4 5 6 7 8 9 10
2.3 Continuum States

The continuum states of the proton-deuterons system can be expanded in a series of spherical waves of different channel spin $A$ and orbital angular momentum $l$ as

$$
\psi_m = \sum_l \sum_A \sqrt{4\pi (2l+1)} i^l e^{i\varphi^A_l} R^A_l(r) Y^A_l(\theta, \phi) \chi^m
$$

where the notation is that used in Appendix A, with the phase shifts $\varphi^A_l$ containing both Coulomb and nuclear phase shifts. The nuclear phase shifts can be obtained from the computed solutions to the radial Schrödinger equation for a given potential. The continuum state well depths for each partial wave were determined (again for $R=3.19$ fm) by requiring that they produce phase shifts in accord where possible with those of Christian and Gammel (1953) obtained from an analysis of p-d elastic scattering data. The various partial waves considered are discussed below.

(1) $^4S$-waves

The $^4S$ p-d phase shifts of Christian and Gammel are shown in Fig. 2.2 along with the phase shifts which result when a well depth of 13.0 MeV is employed in the present model. As the fit is excellent, this value of 13.0 MeV was retained in the capture calculations for the $^4S$-wave. An equally good fit was obtained to the n-d $^4S$-wave phase shift data of Aaron et al. (1965) for a well depth which slowly increases with increasing energy from 13 to 14 MeV; one might expect the p-d well depth to be less than the n-d depth as the Coulomb barrier has not been included.
Figure 2.2: $^4S$ p-d phase shift

$4\delta_0$ (radians)

\[ -E_{cm} \text{ (MeV)} \]

- PRESENT THEORY ($V=13\text{ MeV}$)
- CHRISTIAN and GAMMEL (1953)
in the nuclear interior in the former case. Similar equivalences between p-d and n-d phase shifts and the well depths necessary to produce these phase shifts were found for all partial waves considered.

(2) $^2P$-waves

The $^2P$ phase shifts of Christian and Gammel could not be reproduced with the two-body model for any values of the potential parameters which would, at the same time, fit the experimental El($^2P$-$^2S$) cross section for the D(p,γ)He$^3$ and He$^3$(γ,p)D reactions. It was found that this El cross section, for both reactions, could be fitted when a continuum well depth of 1.0 MeV was used. The $^2P$ phase shifts which result when this well depth is used are small and positive, whereas the Christian and Gammel phase shifts are small and negative (Fig.2.3). Potentials adjusted to fit the direct capture and photodisintegration data using the other radii which were considered produced even greater discrepancies with the Christian and Gammel phase shifts. A potential depth of 1.0 MeV for the $^2P$-wave has been used in subsequent calculations in spite of the fact that it does not reproduce the Christian and Gammel phase shifts.

Since the $^2P$-wave phase shifts were poorly reproduced with the present model, whereas the $^4S$ phase shifts were accurately fitted with a constant well depth of 13.0 MeV, a check was made on the sensitivity of the p-d elastic scattering cross section to variations of the $^2P$ phase shift. The scattering cross sec-
Figure 2.3: $^2\text{P}$ p-d phase shift

PRESENT MODEL

(V = 1.0 MeV)

CHRISTIAN and GAMMEL (1953)
tion was calculated using the exact expression in terms of phase shifts (Christian and Gammel, 1953) employing the Christian and Gammel phase shifts (for \( \ell = 0 \) to 4) and compared with the scattering data of Seth et al. (1963) taken at 30 angles for \( E_{CM} = 1.69 \) MeV. For this comparison the \( ^2P \) phase shift was varied from the Christian and Gammel value to the value given by the present model for a 1.0 MeV well depth, keeping all other phase shifts at the Christian and Gammel values. The \( \chi^2 \) test indicated that the differences in the quality of fit were small. The values of \( \chi^2 \), defined as

\[
\chi^2 = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{d\sigma_{\text{calc}}(\theta_i) - d\sigma_{\text{expt}}(\theta_i)}{d\sigma_{\text{expt}}(\theta_i)} \right)^2
\]

varied from \( 6.4 \times 10^{-2} \) for the Christian and Gammel value to \( 7.7 \times 10^{-2} \) for the value produced by the model. It is clear that the scattering cross section is not very sensitive to variations in the \( ^2P \) phase shift from small negative to small positive values.

The small \( ^2P \) phase shift, corresponding to the small well depth required to fit the direct capture and photodisintegration El cross sections, suggests that there is little interaction between the proton and deuteron in the \( ^2P \) continuum state. This is consistent with the fact that several calculations of three-body photodisintegration (e.g., Bösch et al., 1965), which have neglected the p-d interaction in the final state, have given good
agreement with experiment. As mentioned above, similar results were found in the case of $^2P$ n-d continuum waves.

(3) $^2D$, $^4P$, and $^4F$-waves

The remaining partial waves considered in the direct capture and photodisintegration calculations were calculated by allowing the well depths to vary with energy, keeping the radius fixed at 3.19 fm, to reproduce the phase shifts given by Christian and Gammel.

All well depths used in calculating wave functions for use in subsequent direct capture and photodisintegration computations are shown in Fig. 2.4. For calculations at energies below those at which the experimental phase shifts were known some extrapolations were required. In the case of $^2D$- and $^4F$-waves graphical extrapolations were used to obtain low energy phase shifts and corresponding well depths necessary to produce these phase shifts. However for the $^4P$-wave, an analytic formula for the phase shift as a function of energy was taken from effective range theory (Kalikstein et al., 1963). The scattering length and effective range parameters in this theory were adjusted to fit the Christian and Gammel $^4P$ phase shifts and the phase shifts were then extrapolated to low energies. A similar procedure was followed for the $^4S$-wave in fitting the phase shifts which result when a well depth of 13.0 MeV is used, giving values of the $^4S$ scattering length and effective range of 12.4 fm and 2.1
Figure 2.4: Well depths used for $p + d$ continuum wave functions
fm respectively, in good agreement with the corresponding experimental values, 12.5 ± 1.0 fm and 1.99 ± .07 fm (Christian and Gammel, 1953).

As a consistency check, the n-d $^4S$ scattering length was also calculated on the basis of the present model (using the $^4S$ well depth of 13.0 MeV fixed by fitting the scattering phase shift data at low energies) and found to give reasonable agreement with the value of Aaron et al. (1965) (6.1 fm versus 6.38 ± .06 fm respectively).

Radial wave functions are shown in Figs. 2.5 and 2.6 for several partial waves and energies.
Figure 2.5: Radial wave functions for $p + d$
$^{4S}$ continuum state
Figure 2.6: Radial wave functions for $p + d$

$^2_p$ continuum state
2.4. Radial Integrals

The matrix elements of the electromagnetic radiation operators between continuum and bound states involve radial integrals of the form (Appendix B),

\[ \int_{0}^{\infty} u^*_L(r) \mathcal{O}(r) R^\alpha(r) \, dr \]

where \( u^*_L \) and \( R^\alpha \) are the bound and continuum radial wave functions and \( \mathcal{O}(r) \) is the radial part of one of the three multipole operators considered in Appendix A. The radial integrals (and the wave functions themselves) were calculated numerically using several computer programs, including the program ABACUS 2*, with the IBM 7040 computer of the University of British Columbia. The program ABACUS 2, obtained from the Brookhaven National Laboratory (Auerbach, 1962) was modified basically in three ways: (1) the integration routines were improved so that the radial integrations could be carried out to many times the nuclear radius; (2) the routine for calculation of Coulomb functions was extended so that calculations could be performed at very low energies; and (3) a routine to produce Whittaker functions was included.

Integrations were carried out between 0 and 44 fm in all cases, the integrand being negligible at the latter limit when compared to its value at the nuclear radius, even in the extreme case of a laboratory energy of 24 keV where the ratio of the value of the integrand at 44 fm to its value at 3.19 fm
was less than $10^{-4}$. Some typical electric dipole radial integrands are shown in Fig. 2.7. It is apparent that the interior contributions become progressively more important with increasing energy — for example the interior contributions in the dominant $E1(^2S-^2P)$ transition in the photodisintegration reaction amount to 4%, 11% and 43% of the total at energies of 0.016, 4.0 and 15.0 MeV respectively. A comparison of calculations with the long wavelength approximation and with the exact multipole operators is shown in the figure for an energy of 30 MeV (the dotted and solid curves respectively), indicating the significance of using the exact operators at high energies. The exact operators were in fact employed at all energies.

The radial integrands for transitions involving the $^4D$-component of the $\text{He}^3$ ground state are very similar to those shown in Fig. 2.7, although peaking more in the nuclear surface. For example, the contributions to the radial integrals for the $E1(^4P-^4D)$ transition in direct capture arising from the interior region ($r<R$) is 32% of the total at 1.5 MeV. Most of this comes from the region around $r=R$, where the interior function is joined smoothly to the exterior function. The contributions contained inside 0.8R (containing half the nuclear volume) are only 13%, indicating that the results are not very sensitive to the assumptions concerning the nuclear interior region.
Figure 2.7: $D(p,\gamma)He^3 El$ normalized radial integrands for several centre-of-mass energies (--- exact multipole operator; --- long wavelength approximation)
2.5 Cross Sections -- Formulae

The direct capture transitions between continuum and bound states considered are

\begin{align*}
(1) & \quad E_1(\text{^2P}^{-}\text{^2S}) \\
(2) & \quad E_2(\text{^2D}^{-}\text{^2S}) \\
(3) & \quad M_1(\text{^4S}^{-}\text{^2S}) \\
(4) & \quad E_1(\text{^4P}^{-}\text{^4D}) \\
(5) & \quad E_1(\text{^4F}^{-}\text{^4D}) \\
(6) & \quad E_2(\text{^4S}^{-}\text{^4D}) \\
\end{align*}

and interferences

\begin{align*}
(7) & \quad \frac{E_1}{E_2} \left(\frac{\text{^2P}^{-}\text{^2S}}{\text{^2D}^{-}\text{^2S}}\right) \\
(8) & \quad \frac{E_1}{E_1} \left(\frac{\text{^4P}^{-}\text{^4D}}{\text{^4F}^{-}\text{^4D}}\right) \\
(9) & \quad \frac{E_1}{E_2} \left(\frac{\text{^4P}^{-}\text{^4D}}{\text{^4S}^{-}\text{^4D}}\right) \\
(10) & \quad \frac{E_1}{E_2} \left(\frac{\text{^4F}^{-}\text{^4D}}{\text{^4S}^{-}\text{^4D}}\right) \\
(11) & \quad \frac{E_1}{M_1} \left(\frac{\text{^4P}^{-}\text{^4D}}{\text{^4S}^{-}\text{^2S}}\right) \\
(12) & \quad \frac{E_1}{M_1} \left(\frac{\text{^4F}^{-}\text{^4D}}{\text{^4S}^{-}\text{^2S}}\right) \\
(13) & \quad \frac{E_2}{M_1} \left(\frac{\text{^4S}^{-}\text{^4D}}{\text{^4S}^{-}\text{^2S}}\right). \\
\end{align*}

These transitions are illustrated schematically in Fig. 2.8.

From Appendix B, the differential direct capture cross sections in the centre-of-mass system are given by:

\begin{align*}
(1) & \quad E_1 \left(\text{^2P}^{-}\text{^2S}\right) \\
& \quad \left(\frac{d\sigma}{d\Omega}\right)_1 = 6W_c^2 (1-P_D) \left(\frac{11}{2\pi}\right)^2 \sin^2 \theta_Y \\
& \quad (2.13)
\end{align*}

where the notation is that used in Appendix B, with $P_D$ being the
Figure 2.8: D(p,γ)He\(^3\) transition scheme
4D-component probability, \( \Theta_Y \) specifying the gamma-ray direction and \( (k) I_{L^S_{LS}} \) being the radial integral defined in Appendix B for a transition of character \( k \) (E1, E2 or M1) from a continuum state with orbital angular momentum \( \ell \) and channel spin \( \Delta \) to a bound state with corresponding quantum numbers \( L \) and \( S \).

(2) \( E2 \) \((^2D-^2S)\)

\[
\frac{d\sigma}{d\Omega} = \frac{15}{2} \mathcal{W} c_z^2 (1 - P_D) \left( I_0 \right)^{2\frac{1}{2}} \sin^2 \Theta_Y \tag{2.14}
\]

(3) \( M1 \) \((^4S-^2S)\)

\[
\frac{d\sigma}{d\Omega} = \frac{32\pi}{9} \mathcal{W} c_z^2 (1 - P_D) \left( 2\mu_P - \mu_D \right)^2 \left( I_0 \right)^{2\frac{1}{2}} \tag{2.15}
\]

where \( \mu_P, \mu_D \) are the proton and deuteron magnetic moments.

The appearance of these magnetic moments comes from Appendix B (B.18) since

\[
\mathcal{M}_{\frac{3}{2}, \frac{1}{2}} = -\frac{1}{4\mathcal{W}} \left( 2\mu_P - \mu_D \right) \tag{2.16}
\]

(4) \( E1 \) \((^4P-^4D)\)

\[
\frac{d\sigma}{d\Omega} = \frac{4\pi}{25} \mathcal{W} c_z^2 P_D \left( I_2 \right)^{1\frac{1}{2}} \left( 1 - \frac{5}{7} \cos^2 \Theta_Y \right) \tag{2.17}
\]

(5) \( E1 \) \((^4F-^4D)\)

\[
\frac{d\sigma}{d\Omega} = \frac{22}{25} \mathcal{W} c_z^2 P_D \left( I_2 \right)^{3\frac{1}{2}} \left( 1 - \frac{1}{2} \cos^2 \Theta_Y \right) \tag{2.18}
\]

(6) \( E2 \) \((^4S-^4D)\)

\[
\frac{d\sigma}{d\Omega} = \frac{4\pi}{5} \mathcal{W} c_z^2 P_D \left( I_2 \right)^{2\frac{1}{2}} \tag{2.19}
\]
(7) \( \frac{d\sigma}{d\Omega} (E1/E2, ^2\text{P}_1-^2\text{D}_1/^-^2\text{S}) \)
\[
\left( \frac{d\sigma}{d\Omega} \right)_7 = 6 \sqrt{5} w c_1 c_2 (1 - P_D) \left( ^{1/2}_{1/2} I_0 \right)^{1/2} \left( ^{1/2}_{1/2} I_0 \right)^{1/2} \times \cos (\varphi_1^2 - \varphi_2^2) \sin \theta \gamma \sin 2 \theta \gamma
\]
where \( \varphi_n^\lambda \) is the combined Coulomb and nuclear phase shift for quantum numbers \( \lambda \) and \( \mu \).

(8) \( \frac{d\sigma}{d\Omega} (E1/E1, ^4\text{P}_1-^4\text{F}_1/^-^4\text{D}_1) \)
\[
\left( \frac{d\sigma}{d\Omega} \right)_8 = \frac{36}{25} w c_1^2 P_D \left( ^{3/2}_{1/2} I_2 \right)^{3/2} \left( ^{3/2}_{1/2} I_2 \right)^{3/2} \times \cos (\varphi_1^1 - \varphi_2^1) (3 \cos^2 \theta \gamma - 1)
\]

(9) \( \frac{d\sigma}{d\Omega} (E1/E2, ^4\text{P}_1-^4\text{S}_0/^-^4\text{D}_1) \)
\[
\left( \frac{d\sigma}{d\Omega} \right)_9 = -\frac{24}{25} w c_1 c_2 P_D \left( ^{3/2}_{1/2} I_2 \right)^{3/2} \left( ^{3/2}_{1/2} I_2 \right)^{3/2} \times \cos (\varphi_1^3 - \varphi_2^3) \cos \theta \gamma
\]

(10) \( \frac{d\sigma}{d\Omega} (E1/E2, ^4\text{F}_1-^4\text{D}_1/^-^4\text{S}_0) \)
\[
\left( \frac{d\sigma}{d\Omega} \right)_{10} = -\frac{24}{25} w c_1 c_2 P_D \left( ^{3/2}_{1/2} I_2 \right)^{3/2} \left( ^{3/2}_{1/2} I_2 \right)^{3/2} \times \cos (\varphi_3^0 - \varphi_5^0) \cos \theta \gamma
\]

The following interference terms are zero (Appendix B):

(11) \( E1/M1 (^4\text{P}_1-^4\text{D}_1/^-^4\text{S}_0) \), (12) \( E1/M1 (^4\text{F}_1-^4\text{D}_1/^-^4\text{S}_0) \) and

(13) \( E2/M1 (^4\text{S}_0-^4\text{D}_1/^-^4\text{S}_0) \).

The total cross sections for direct capture, obtained by integrating the above over all solid angles are given below. The interference terms give no contribution to the total cross sections.
The direct capture and photodisintegration cross sections are obtained (sections 2.6 and 2.7) by calculating the radial integrals (section 2.4) using the bound and continuum wave functions (sections 2.2 and 2.3) for well parameters determined by fitting the experimental binding energies and scattering data and substituting in the above expressions for direct capture and in similar expressions, obtained using detailed balance, for photodisintegration.
2.6 Cross Sections -- Results for \(D(p, \gamma)\text{He}^3\)

Calculations of the \(D(p, \gamma)\text{He}^3\) direct capture cross section were obtained for centre-of-mass energies between 16 keV and 4 MeV, including the transitions \(M1\left(^4S-^2S\right)\), \(E1\left(^2P-^2S, ^4P-^4D, ^4P-^4D\right)\), \(E2\left(^2D-^2S, ^4S-^4D\right)\) and interference terms.

The present model does not incorporate the symmetry features of the three-nucleon system, however, an attempt was made to introduce some of these features in the case of the \(M1\left(^4S-^2S\right)\) transition. In particular, a three-body classification (Eichmann, 1963) shows that the \(^4S\) continuum state, consisting of a proton and a deuteron with spins aligned, belongs to a mixed-symmetry representation (of the spatial or spin-isospin functions). As the magnetic dipole operator (in the long wavelength approximation and at least approximately in the exact multipole operator case) is symmetric and in fact unity, the \(M1\) matrix element is essentially an overlap between continuum and bound states. Thus this transition can proceed only between the \(^4S\)-wave and the mixed-symmetry component in the \(^2S\) ground state of \(\text{He}^3\). This component has small amplitude relative to the main symmetric \(^2S\)-component and arises from the small difference between the central triplet even and singlet even two-body forces (Derrick, 1960). In addition, as Eichmann has pointed out, there is an \(M1\left(^2S-^2S\right)\) transition, since the orthogonality condition for wave functions of different energy applies only to the total wave functions, and not to the symmetric (or mixed-symmetry) components separately. The overall \(M1\) transition is
therefore inhibited by selection rules which are a direct consequence of the Pauli Exclusion Principle. For this reason, a scaling factor was included in the $^4S-^2S$ cross section calculated on the basis of the present two-body model where no Pauli Principle was considered. The M1 cross section, calculated for a $^2S$ wave function with $\sqrt{1-P_D}$ normalization, was reduced by a factor 10 to produce agreement with the isotropic part of the $D(p,\gamma)\text{He}^3$ cross section at very low energy (16 keV in the centre-of-mass system (Griffiths et al., 1963)) where the cross section is essentially all magnetic dipole in character. We are unable to attribute any absolute significance to this factor of 10, since the wave functions are only approximate and there are contributions from both the symmetric $^2S$ and mixed-symmetry $^2S$-components of the $\text{He}^3$ ground state.

The resulting cross sections are shown in Fig. 2.9.

When the $E2(^4S-^4D)$ term and the $E1/E2(^4P-^4D/^4S-^4D$ and $^4P-^4D/^4S-^4D$) interferences are neglected, being considerably smaller than the other terms considered, the differential cross section in the centre-of-mass system can be written

$$\frac{d\sigma}{d\Omega} = \alpha + b \sin^2 \theta \gamma (1 + \beta \cos \theta \gamma + \gamma \cos^2 \theta \gamma) \quad (2.30)$$

where

$$\alpha = \sqrt{\frac{32}{9}} \pi c_3^2 \left(1-P_D\right) \left(2\mu_p - \mu_D\right)^2 \left(\begin{array}{l} 1 3 \hfill \\
\end{array}\right) \left(\begin{array}{l} \frac{3}{2} 3 \hfill \\
\end{array}\right)^2$$

$$+ \frac{3\sqrt{2}}{25} c_1^2 P_D \left[ \left(\begin{array}{l} 1 3 \hfill \\
\end{array}\right) \left(\begin{array}{l} \frac{3}{2} 3 \hfill \\
\end{array}\right)^2 + \left(\begin{array}{l} 1 3 \hfill \\
\end{array}\right) \left(\begin{array}{l} \frac{3}{2} 3 \hfill \\
\end{array}\right)^2 \right.$$
Figure 2.9: $D(p,\gamma)He^3$ cross sections

($1\% \ ^4D$-state; $1/10$ in $M1(^4S\rightarrow^2S)$)
In Fig. 2.10, the isotropic part of the cross section \( (\frac{4\pi a}{\rho}) \) at low energies is shown for various \( ^4D \)-state probabilities with experimental data for comparison. Above 100 keV the cross section is very sensitive to the amount of \( ^4D \)-state included, a little under 1% probability appearing most suitable, although, due to the approximations made, the actual value is at most an indication of the true amount. Evidently the main contributions to the isotropic cross section, in addition to the \( M1(^4S-^2S) \) transition, arise from the \( E1(^4P-^4D) \) term and its interference with the smaller \( E1(^4P-^4D) \) transition.

The ratio \( a/b \) is shown in Fig. 2.11 for various \( ^4D \)-component probabilities. Significant improvement is again noted for about 1% \( ^4D \)-state.

The differential cross section can also be expanded in a Legendre polynomial series as

\[
\frac{d\sigma}{d\Omega} = \sum_{i=0}^{4} \alpha_i \, P_i (\cos \theta) \]  

(2.35)

The coefficients \( \alpha_i, i=0,4 \) are given by expressions similar to

\[
b = (\theta \, w^{2/3} \, c_{1/2} \, (\theta - P_D) \, (\frac{1}{4} I_{01}^{1/2} \, I_{12}^{1/2})^2
+ \frac{P_D}{25} \bigg \{ (\frac{1}{4} I_{12}^{1/2})^2 + 18 \cos (\omega_{12}^{1/2} - \omega_{12}^{1/2}) (\frac{1}{4} I_{12}^{1/2})^2
- 18 \cos (\omega_{12}^{1/2} - \omega_{12}^{1/2}) \bigg \} \bigg \}
\]

(2.32)

\[
b \gamma = 30 \, w \, c_{2/3} \, (1 - P_D) \, (\frac{1}{4} I_{01}^{1/2} \, I_{12}^{1/2})^2
\]

(2.33)

\[
+ 2 \cos (\phi_1^{1/2} - \phi_2^{1/2}) (\frac{1}{4} I_{12}^{1/2} \, I_{13}^{1/2} \, I_{14}^{1/2} \, I_{15}^{1/2} \, I_{16}^{1/2}) \bigg \}
\]

(2.31)
Figure 2.10: $4\pi a$ for $D(p,\gamma)He^3$ shown for several $^4D$-component probabilities.
Figure 2.11: a/b for D(p,γ)He³ showing curves for several 4D probabilities (dashed) and for calculations (Bösch et al., 1965) with Gunn-Irving wave functions (solid; χ⁻¹ = 2.5, δ = 0.07)
those for \(a, b, \beta, \gamma\) above; the coefficients are given as functions of energy in Table 2.1, where here all terms neglected above have been included.

The direct capture calculations for the M1 transition were repeated with the \(^4S\) phase shift reduced to zero in order to check the effect of this phase shift on the magnetic dipole transition. The resulting cross section was found to be 10 times larger than that calculated with the Christian and Gammel phase shifts, in agreement with results of other workers (Barucchi et al. 1965; Erdas et al., 1966). Neglecting the \(^2P\) phase shifts in the predominant El\((^2P - ^2S)\) transition has little effect on the cross section as this phase shift is small. It is for this reason that a number of earlier calculations (eg. Bösch et al. 1965) of the main El\((^2S - ^2P)\) contribution to the photodisintegration cross section, which did not include final-state interactions in the \(^2P\) continuum states, were able to provide good fits to the experimental cross sections.
Table 2.1: Angular Distribution Coefficients

(with 1% $^4D$, 1/10 in M1($^4S-^2S$))

<table>
<thead>
<tr>
<th>$E_{CM}$ (MeV)</th>
<th>$\alpha_1/\alpha_0$</th>
<th>$\alpha_2/\alpha_0$</th>
<th>$\alpha_3/\alpha_0$</th>
<th>$\alpha_4/\alpha_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>0.260</td>
<td>-0.970</td>
<td>-0.260</td>
<td>-0.0165</td>
</tr>
<tr>
<td>3.48</td>
<td>0.242</td>
<td>-0.971</td>
<td>-0.242</td>
<td>-0.0143</td>
</tr>
<tr>
<td>2.48</td>
<td>0.209</td>
<td>-0.972</td>
<td>-0.208</td>
<td>-0.0106</td>
</tr>
<tr>
<td>1.66</td>
<td>0.179</td>
<td>-0.970</td>
<td>-0.179</td>
<td>-0.00781</td>
</tr>
<tr>
<td>0.993</td>
<td>0.147</td>
<td>-0.965</td>
<td>-0.147</td>
<td>-0.00535</td>
</tr>
<tr>
<td>0.740</td>
<td>0.131</td>
<td>-0.961</td>
<td>-0.131</td>
<td>-0.00424</td>
</tr>
<tr>
<td>0.533</td>
<td>0.118</td>
<td>-0.954</td>
<td>-0.118</td>
<td>-0.00345</td>
</tr>
<tr>
<td>0.390</td>
<td>0.114</td>
<td>-0.945</td>
<td>-0.114</td>
<td>-0.00330</td>
</tr>
</tbody>
</table>

where $\alpha_1 \approx \alpha_3$; they differ in the 4th figure due to contributions from the E1/E2 interference terms involving the $^4D$-state.
2.7 Cross Sections -- Results for $\text{He}^3(\gamma, p)^D$

As a result of a great deal of new experimental information on the photodisintegration of $\text{He}^3$ which has become available since 1963, it is possible to assess the validity of the present theory at higher excitation energies in $\text{He}^3$. The principle of detailed balance gives the following relationship between the direct captures $D(p, \gamma)\text{He}^3$ cross section and the inverse photodisintegration $\text{He}^3(\gamma, p)^D$ cross section,

$$d\sigma_{\text{photo}} = \frac{3}{2} \left( \frac{k}{\xi} \right)^2 d\sigma_{\text{cap}}$$

(2.36)

where $k$ and $\xi$ are the particle and radiation wavenumbers respectively and where the factor $3/2$ comes from the statistical weighting: $3/2 = 1/2 (2s_p + 1)(2s_D + 1)/(2s_{\text{He}^3} + 1)$, where $S_p$, $S_D$ and $S_{\text{He}^3}$ are the spins of the particles $p$, $D$ and $\text{He}^3$ respectively and where the factor $1/2$ arises from the two polarization states of the gamma-ray.

Calculations of the $\text{He}^3(\gamma, p)^D$ cross sections were carried out for energies from the gamma-ray threshold to 45 MeV. Only the $E1(2s-2p)$, $E2(2s-2D)$ terms and the interference between them have been included, since the other transitions, which were used in the $D(p, \gamma)\text{He}^3$ calculation at lower energies, give negligible contributions compared to the above transitions at the higher energies considered here.

The photodisintegration cross sections are shown in Figs.
Total cross section for He$^3(\gamma,p)D$ shown for the long wavelength approximation (A), for the exact multipole operators (B) and for calculations (Bösch et al., 1965) with Gunn-Irving wave functions (C). The direct capture data of Griffiths et al. (1962) has been converted to photodisintegration using detailed balance.
Figure 2.13: $\text{He}^3(\gamma,p)D$ differential cross section at 90°
2.12 and 2.13 where they are compared with recent experimental data. In these figures, curves A and B correspond to calculations with the long wavelength approximation and with the exact forms for the multipole operators respectively. The cross sections which result in both cases are very similar, even though the radial integrands may be somewhat different (see section 2.4). A number of workers (eg. Berman et al., 1963; Bösch et al., 1965) have previously compared photodisintegration data with calculations based on the Gunn and Irving (1951) wave function, which has the following modified exponential form

\[ \psi = A \frac{e^{-\mu \left( \sum r_{ij}^2 \right)^{1/2}}}{\left( \sum r_{ij} \right)^{1/2}} \quad (2.37) \]

where \( \mu \) is a size parameter, A a normalization factor and \( r_{ij} \) the interparticle distances. This form, originally introduced empirically, was found to give a good fit to the binding energies of T and He\(^3\). The photodisintegration cross section was obtained by evaluating matrix elements of the El operator between this ground state wave function and a final state consisting of a plane wave proton and a simple exponential deuteron function of the form

\[ \psi_d = B \frac{e^{-\beta r}}{r} \quad (2.38) \]
where \( B = \text{normalization factor}, \)

\[
\beta = \frac{\sqrt{M W_D}}{A}
\]

\( M = \text{nucleon mass}, \)

and \( W_D = \text{deuteron binding energy} = 2.226 \text{ MeV}. \)

Bösch et al. considered in addition matrix elements for E2 and M1 transitions and their total cross section is shown in Fig. 2.12 (curve C) for a size parameter \( \mu = 2.5 \text{ fm} \) and a \( ^2S \) mixed-symmetry component amplitude of 0.07. The M1 transition makes only a small contribution to the total cross section.

The agreement between these three-body calculations, the experimental results and the present calculations is in general reasonable considering the large errors on the experimental data above 2 MeV. In particular, there is good agreement with the University of British Columbia ionization chamber results of Warren et al. (1963) and MacDonald (1964) which overlap the direct capture data of Griffiths et al. (1962).

The angular distribution is illustrated in Fig. 2.14 for 15 MeV at which energy experimental and other theoretical results are available. Although the E2(\(^2S-^2D\)) contribution (curve B) is small, when taken with the large E1 (\(^2S-^2P\)) term (curve A), the contribution from the E1/E2 interference is appreciable and leads to a marked asymmetry in the angular distribution (curve C). The experimental data are not very satisfactory but do show the same general asymmetry. The dotted curves in Fig. 2.14 show results of a calculation by Eichmann
Figure 2.14: Angular distribution for $\text{He}^3(\gamma,p)D$ at $E_{cm}=15$ MeV showing E1 only (A), E2 only (B) and total, including E1/E2 interference, (C).
(1963) based on three-body wave functions incorporating the Pauli Exclusion Principle and taking into account final-state interactions for the continuum waves. There is marked agreement between the results of present model and those of Eichmann, especially considering that no arbitrary normalizations are used in these calculations. The parameters which give the asymmetry, $\beta$ and $\gamma$ (see section 2.6) are given as functions of energy in Table 2.2.

**Table 2.2: $\text{He}^3(\gamma,p)\text{D}$ Asymmetry Parameters $\beta$ and $\gamma$**

<table>
<thead>
<tr>
<th>$E_\gamma - E_B$ (MeV)</th>
<th>$\beta$ (P)</th>
<th>$\beta$ (B)</th>
<th>$\beta$ (C)</th>
<th>$\gamma$ (P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.49</td>
<td>0.38</td>
<td></td>
<td>0.058</td>
</tr>
<tr>
<td>13</td>
<td>0.76</td>
<td>0.76</td>
<td>0.8</td>
<td>0.15</td>
</tr>
<tr>
<td>20</td>
<td>0.95</td>
<td>1.1</td>
<td></td>
<td>0.23</td>
</tr>
<tr>
<td>30</td>
<td>1.2</td>
<td>1.5</td>
<td></td>
<td>0.36</td>
</tr>
<tr>
<td>40</td>
<td>1.3</td>
<td>1.9</td>
<td></td>
<td>0.48</td>
</tr>
</tbody>
</table>

where $E_\gamma$ is the gamma-ray energy, $E_B$ is the $\text{He}^3$ binding energy (5.49 MeV) and where (P) refers to present calculations, (B) to work of Bösch et al. (1965) and (C) to the measurement of Cranberg (1958).
2.8 Summary

On the basis of the comparisons made in the previous two sections, it would appear that the simple two-body model predicts results in better agreement with experimental data, over a wider energy range, than one had any a priori reason to expect, particularly at higher energies. The agreement at low energies is not particularly surprising since the main contributions to the radial matrix elements come from regions outside the nuclear surface and so should be relatively model-independent. Due to the approximations made, one cannot expect this model to predict accurately the amplitudes of the $^4D$ and mixed-symmetry components of the bound state wave functions. However, the contributions due to these states should be reasonably predicted as functions of energy and angle with the empirical normalizations used here.

As a check on the parameters used in the $D(p, \gamma)He^3$ calculations, the $D(n, \gamma)T$ cross section, including only the $E1(^2P-^2S)$ and $M1(^4S-^2S)$ transitions, was computed from thermal energies to 4.0 MeV. The same factor $1/10$ was used in the $M1$ transition as for the proton capture reaction. The well depths used were 20.5 MeV for the $^2S$ bound state of the triton (to give the binding energy of 6.26 MeV), 13.0 MeV for the $^4S$-wave (to reproduce the $^4S$ phase shifts of Aaron et al. (1965)), and 1.0 MeV for the $^2P$-wave (to agree with the value used in the p-d case).
The El and Ml cross sections for the reactions D(p, γ)He^3 and D(n, γ)T are shown in Fig. 2.15. The D(p, γ)He^3 cross sections are in good agreement with the experimental data of Griffiths et al. (1962, 1963) and Wölfli et al. (1966) except for the Ml cross section at higher energies (> 1 MeV). With the inclusion of contributions due to the \(^4\text{D}\)-state, with 1% probability, the isotropic part of the cross section \((\pi \text{a})\) is in reasonable agreement with the data of Wölfli et al. and shows the importance of including the \(^4\text{D}\)-state.

There are no direct measurements for the reaction D(n, γ)T except at thermal energies. However, a comparison is made with the recent photodisintegration measurements of Kosiek et al. (1966) using the reciprocity relation to convert their data to direct capture cross sections, again with results in reasonable agreement with theory. The calculated D(n, γ)T Ml cross section is shown up to 1.0 MeV only and contains only contributions from the Ml matrix elements (i.e. no \(^4\text{D}\)-component has been introduced). Calculations for both D(n, γ)T and D(p, γ)He^3 reactions have been carried to low energies, in the former case to compare with the measured thermal neutron capture cross section and in the latter to produce the astrophysical S-factor at low energies.

The D(n, γ)T Ml cross section was calculated for thermal neutrons (2200 m/sec) using the same potential parameters referred to above. A cross section of .60 mb was found, in good agreement with the latest experimental value of .60 ± .05 mb.
Figure 2.15: $D(p,\gamma)He^3$ and $D(n,\gamma)T$ cross sections compared with experimental data (the data of Kosiek et al. (1966) has been converted from $T(\gamma,n)D$ results by using detailed balance).
given by Jurney and Motz (1964) and not in quite as good agreement with the earlier cross section given with a smaller error by Kaplan et al. (1952), namely $0.57 \pm 0.01$ mb. This is rather good agreement considering that the calculated value is essentially based on the $D(p, \gamma)He^{3}$ cross section measured at an energy $10^{6}$ times higher.

Since the reaction $D(p, \gamma)He^{3}$ has some astrophysical significance (Cameron, 1962), the low energy cross sections are shown in Fig. 2.16 in terms of the astrophysical S-factor defined as

$$S = \sigma E e^{2\pi \eta}$$

(2.39)

where $\sigma$ is the cross section, $E$ is the energy in the centre-of-mass system and $\eta$ is the Coulomb factor. These show a more complicated energy dependence than the linear relations that are often assumed and which resulted from the very approximate extrapolations made by Griffiths et al. (1963). The p-wave cross section remains greater than the s-wave cross section down to about 2.5 keV, below which it is smaller. The s-wave S-factor rises slowly with decreasing energy, reaching a constant value of $0.144eV\cdot bn$ at about 1 keV as shown in the inset to Fig. 2.16. Note the expanded ordinate scale on the inset.
Figure 2.16: Astrophysical S-factor for $D(p, \gamma)He^3$ showing s-wave and p-wave capture separately.
CHAPTER 3

THE DIRECT CAPTURE REACTION $^{16}_0(p,\gamma)^{17}F$

3.1 Introduction and Models

The direct capture reaction $^{16}_0(p,\gamma)^{17}F$ has been studied to a lesser extent than the reactions $D(p,\gamma)^3He$ and $^3He(\gamma,p)^3D$ considered in the preceding chapter. The reaction $^{16}_0(p,\gamma)^{17}F$ was first measured by DuBridge et al., (1938) by observation of the positron decay of $^{17}F$ following capture. The positron yield was measured as a function of energy from 1.1 to 4.1 MeV by Laubenstein et al. (1951) and found to increase almost linearly with energy up to 3.75 MeV except for sharp resonances at 2.66 and 3.47 MeV. Subsequent measurements were made using the same method by Hester et al. (1958) in the energy range 140 to 170 keV and by Tanner (1959) for energies between 275 and 616 keV.

The gamma-rays were observed directly by Warren et al. (1954) at energies between 0.9 and 2.1 MeV and the yield was used to estimate the absolute cross section for $^{16}_0(p,\gamma)^{17}F$. Subsequently similar more accurate measurements were undertaken by Robertson (1957) and Riley (1958). More recently Domingo (1965) has measured the yields of the gamma rays in the energy range 2.56 to 2.75 MeV spanning the $1/2^-$ resonance at 2.66 MeV.

Measurements of the slowly rising capture cross section sug-
gested that the non-resonant yield was likely due to the direct radiative capture process. This was confirmed by preliminary theoretical studies made by Griffiths (1958) and Nash (1959). Subsequently, more detailed theoretical studies of \( ^{16}P(p,\gamma)^{17}F \) direct capture have been made by Christy and Duck (1961), Lal (1961) and Griffiths et al. (1962a).

The present work is a theoretical study of the reaction \( ^{16}P(p,\gamma)^{17}F \) on the basis of a single-particle direct radiative capture model. The single-particle model is used for both the bound and continuum states of \( ^{17}F \). Due to the tightly bound \( ^{16}O \) core this approximation is valid at low excitation energies in \( ^{17}F \) (below the \( 1/2^- \) resonance at 2.66 MeV) and in fact the \( 5/2^+ \) ground and \( 1/2^+ \) excited bound states of \( ^{17}F \), with binding energies of 0.5978 and 0.1025 MeV respectively, are well described by the single-particle model. Because of the small binding energies of these states (and the consequently large spatial extension of the wave functions) the direct capture cross sections are strongly extranuclear in character. A recent measurement of Alburger (1966) for the energy separation of the bound states has been used in the present calculations (\( 495.33\pm10 \) keV). The level scheme for levels in \( ^{17}F \) is given in Fig. 3.1.

Two models are used in the present work to describe the \( p-^{16}P \) interaction:
Figure 3.1: $^{17}$F level scheme.
Model I

In this case the p-0$^{16}$ interaction is represented by a square-well plus Coulomb potential

$$V_I(r) = \begin{cases} 
-V & r \leq R \\
\frac{8e^2}{r} & r > R 
\end{cases} \quad (3.1)$$

with radius R and well depth V.

The radius parameter R was fixed at 4.8 fm in accord with prior calculations by Lai (1961). This is an unusually large radius, however it is understandable in terms of model II below which uses a diffuse-edged potential with a smaller and more reasonable radius. In order to get approximately the same wave function with a square well one must choose a large radius parameter to produce a comparable reduction in the Coulomb barrier that results automatically from the diffuse-edged potential. This has been discussed in some detail by Vogt et al. (1965).

The well depth V was adjusted in the case of bound states of F$^{17}$ to fit their binding energies. For the continuum wave functions Lai used a potential depth which was the average of the potential depths necessary to produce the correct binding energies for the 1$d_{5/2}$ and 2$s_{1/2}$ bound states of F$^{17}$. These continuum functions gave a poor fit to elastic scattering data. In the present work the continuum wave functions have been generated in potentials whose parameters were adjusted to fit the 0$^{16}(p,p)0^{16}$ scattering data of Eppling (1952) over a range of
angles (90.4° to 168°) and energies (1.18 MeV to 1.88 MeV). More possible radiative transitions are used in these calculations and, in addition, the solutions to the Schrödinger equation are computed with increased accuracy, particularly in regards the evaluation of Coulomb functions at low energy.

Model II

In order to assess the accuracy of Model I, with square-well potentials, calculations were performed using a diffuse-edged Saxon-Woods potential with a spin-orbit term and with a Coulomb potential corresponding to a uniformly charged sphere to represent the p-0\(^{16}\) interaction in both bound and continuum states:

\[ V_{\text{II}}(r) = V_{\text{SW}}(r) + V_{\text{SO}}(r) + V_{\text{Coul}}(r) \quad (3.2) \]

where

\[ V_{\text{SW}}(r) = \frac{-V_0}{1 + \exp\left(\frac{r-R}{a}\right)} \quad (3.3) \]

\[ V_{\text{SO}}(r) = -V_s \left(\frac{\hbar}{m_p c}\right)^2 \frac{1}{r} \left| \frac{d}{dr} \left( \frac{V_{\text{SW}}(r)}{V_0} \right) \right| \frac{l(l+1)}{2} \quad (3.4) \]

\[ V_{\text{Coul}}(r) = \begin{cases} \frac{Ze^2}{r} \left(3 - \frac{r^2}{R^2}\right) & r \leq R \\ \frac{Ze^2}{aR} & r > R \end{cases} \quad (3.5) \]
and where

\[ R = \text{nuclear radius parameter}, \]
\[ a = \text{diffuseness}, \]
\[ V_0 = \text{central well depth}, \]
\[ V_s = \text{spin-orbit well depth}, \]
\[ Z_e = \text{charge of target (} = 8e \text{ for } ^{16}_0 \text{)}, \]
\[ m_\pi = \text{pion mass}, \]
\[ J. = \text{orbital angular momentum}, \]
\[ \mathcal{J} = \text{spin angular momentum}. \]

No imaginary term was included in the potential as there is no resonant absorption in the region considered here and the absorption due to the electromagnetic interaction is small, corresponding to a small perturbation of the wave functions.

The radius \( R \) was fixed at 3.33 fm (corresponding to \( r_o = 1.32 \text{ fm} \) in \( R = r_o A^{1/3} \)) following Satchler's fit to \( ^{16}_0(n,n) ^{16}_0 \) scattering data. This leaves three parameters, the well depths \( V_0 \) and \( V_s \) and the diffuseness \( a \). These parameters were then adjusted to fit binding energies and elastic scattering data as for Model I. The effects on the \( ^{16}_0(p,\gamma)^{17}_F \) cross section of varying \( V_s \) and \( a \) were investigated.

The potentials of Models I and II are compared in Fig.3.2 for typical values of the parameters (Cf. the lists on pp 69 and 71). The spin-orbit term in Model II, being proportional to the derivative of the Saxon-Woods potential, peaks in the nuclear surface and can be attractive or repulsive depending on the sign of \( J \cdot \mathcal{J} \).
Figure 3.2: Potentials for models I and II with typical well parameters.
3.2 Elastic Scattering and Continuum States

Using the potentials of Models I and II, the $^16(p,p)^16$ elastic scattering differential cross sections were computed by numerically solving the Schrodinger equation for partial waves up to $l = 6$. The scattering data of Eppling (1952) at eight angles and at four energies (1.18, 1.41, 1.65 and 1.88 MeV in the centre-of-mass system) were fitted by varying the potential parameters to minimize $\chi^2$ for a best fit to all 32 experimental points.

Model I

For Model I the well depth $V$ was varied to obtain a minimum $\chi^2$ fit to the experimental data, resulting in a well depth of $V=24.09$ MeV (for the radius $R=4.8$ fm). The resulting theoretical fit to Eppling's data is shown in Figs. 3.3 (a-d). This fit has a $\chi^2$ (as defined in section 2.3) of $5.8 \times 10^{-3}$ corresponding to an rms deviation of 7.6%.

Model II

A similar procedure was adopted for Model II, i.e. varying the central well depth $V_o$, for fixed values of the remaining parameters, to obtain a minimum $\chi^2$ fit to the scattering data. This search was repeated for several values of $V_s$ and $a$, minimizing $\chi^2$ by varying $V_o$ for each pair $V_s$, $a$. It was found that the quality of fit was relatively insensitive to variations of $V_s$ and $a$; for a range of $5$ MeV $< V_s < 10$ MeV and $0.50$ fm $< a < 0.65$ fm there was less than 10% change in $\chi^2$. 
Figure 3.3: $^0_{16}(p,p)^0_{16}$ elastic scattering fit to data of Eppling (1952).
Consequently, the values $V_s = 6.0 \text{ MeV}$, $a = 0.55 \text{ fm}$ and hence $V_0 = 49.85 \text{ MeV}$ for $R = 3.33 \text{ fm}$ were chosen as being representative of values in this range. Subsequently the sensitivity of the direct capture cross section to variations in $V_s$ and $a$ was investigated.

The resulting theoretical fit to the experimental data using Model II is so nearly the same as that for Model I, shown in Figs. 3.3 (a-d), that the differences would neither show in the figures nor have statistical significance.

The potential parameters, determined by fitting scattering data, were then used to compute the continuum wave functions involved in the direct capture calculations. The continuum wave functions are given by:

Model I

$$\Psi_m = \sum_{\lambda} \sqrt{4\pi} (2\lambda+1) \; i^{\lambda} e^{i\varphi_{\lambda}} \frac{R_{\lambda}^{\varphi}(r)}{\sqrt{r}} \; Y_{\lambda}^0(\theta, \varphi) \; \chi_{\lambda}^m$$  \hspace{1cm} (3.6)

where the notation is that of Appendix A, with $\lambda = 1/2$ and $V^{\lambda}_{\lambda}(r) = V_{\lambda}(r)$.

Model II

$$\Psi_m = \sum_{\lambda} \sum_{\varphi} \sum_{\chi} C(\lambda \frac{1}{2} j; 0, m) C(\lambda \frac{1}{2} j; m-\alpha, \alpha) \times \sqrt{4\pi} (2\lambda+1) \; i^{\lambda} e^{i\varphi_{\lambda}} \frac{R_{\lambda}^{\varphi}(r)}{\sqrt{r}} \; Y_{\lambda}^{m-\alpha}(\theta, \varphi) \; \chi_{\lambda}^\varphi$$  \hspace{1cm} (3.7)

where the notation is again that of Appendix A, with $V_{\lambda j}(r) = V_{\varphi}(r)$. 
The radial wave functions involved, \( R_j^r(r) \) and \( R_j(r) \), were computed numerically as solutions to the radial Schrödinger equation with potentials \( V_I(r) \) and \( V_{II}(r) \) respectively. The values of the parameters used in the potentials for the continuum states are tabulated below:

Model I:  
\[ R = 4.8 \text{ fm} \]  
\[ V = 24.09 \text{ MeV} \]

Model II:  
\[ R = 3.33 \text{ fm} \]  
\[ a = 0.55 \text{ fm} \]  
\[ V_0 = 49.85 \text{ MeV} \]  
\[ V_g = 6.0 \text{ MeV} \]
3.3 Bound States

The wave functions for the \(1d_{5/2}\) and \(2s_{1/2}\) bound states of \(^{17}F\) are obtained by solving the Schrödinger equation with the potentials \(V_I(r)\) or \(V_{II}(r)\). The bound states wave functions are given by

Model I

\[
\Phi_M = \frac{\psi_{\frac{5}{2}}(r)}{r} \sum_{\ell} C(2 \frac{1}{2}, \ell, M-\beta, \beta) Y_{2}^{M-\beta}(\theta, \varphi) \chi_{\frac{5}{2}}^{\beta} \tag{3.8}
\]

for the \(1d_{5/2}\) state, and

\[
\Phi'_M = \frac{\psi_{\frac{1}{2}}(r)}{r} Y_{0}^{0}(\theta, \varphi) \chi_{\frac{1}{2}}^{M} \tag{3.9}
\]

for the \(2s_{1/2}\) state, and,

Model II

\[
\Phi_M = \frac{\psi_{\frac{5}{2}}(r)}{r} \sum_{\ell} C(2 \frac{1}{2}, \ell, M-\beta, \beta) Y_{2}^{M-\beta}(\theta, \varphi) \chi_{\frac{5}{2}}^{\beta} \tag{3.10}
\]

for the \(1d_{5/2}\) state, and

\[
\Phi'_M = \frac{\psi_{\frac{1}{2}}(r)}{r} Y_{0}^{0}(\theta, \varphi) \chi_{\frac{1}{2}}^{M} \tag{3.11}
\]

for the \(2s_{1/2}\) state, where again the notation of Appendix A has been used with the radial wave functions satisfying the radial Schrödinger equation employing the appropriate potentials.

In both Models I and II all parameters except the well depths \(V\) and \(V_o\) were retained from the fits to the elastic scattering data, the depths \(V\) and \(V_o\) being adjusted to give the binding energies of the \(1d_{5/2}\) and \(2s_{1/2}\) states (0.5978 MeV and
This resulted in well depths of \( V = 21.12 \text{ MeV} \) and \( 23.94 \text{ MeV} \) and \( V_0 = 47.38 \text{ MeV} \) and \( 50.00 \text{ MeV} \) for the \( 1d_{5/2} \) and \( 2s_{1/2} \) states respectively. The normalized radial wave functions are shown in Fig. 3.4.

The \( 2s_{1/2} \) wave functions for the two models are seen to be very similar beyond about 5 fm, although they are significantly different for distances less than this value. In fact the strengths (\( VR^2 \) and \( V_0R^2 \)) of the two potentials in the \( 2s_{1/2} \) case are nearly equal. However for the \( 1d_{5/2} \) case, the wave functions are significantly different at all distances. The effect of the spin-orbit potential, which was zero for the \( 2s_{1/2} \)-state (since \( J \) is zero for an s-state), is to cause the Model II wave function to be shifted inwards with respect to the Model I wave function. As will be demonstrated in what follows, the calculated direct capture cross sections are found to differ when these two models are used to produce the bound and continuum wave functions.

The bound state well depths used are tabulated below:

Model I: \[ V = 21.12 \text{ MeV} \]

\[ = 23.94 \text{ MeV} \]

(\( 1d_{5/2} \) state)

(\( 2s_{1/2} \) state)

Model II: \[ V_0 = 47.38 \text{ MeV} \]

\[ = 50.00 \text{ MeV} \]

(\( 1d_{5/2} \) state)

(\( 2s_{1/2} \) state).

All the other parameters are the same as for the continuum states (Cf. p. 69).
Figure 3.4: $^{17}$F bound state normalized radial wave functions
3.4 Cross Sections -- Formulae

Model I

The Transitions considered are

(1) \( E_1 \ (p - 2s_{1/2}) \)
(2) \( E_2 \ (d - 2s_{1/2}) \)
(3) \( E_1 \ (p - 1d_{5/2}) \)
(4) \( E_1 \ (f - 1d_{5/2}) \)
(5) \( E_2 \ (s - 1d_{5/2}) \)
(6) \( E_2 \ (d - 1d_{5/2}) \),

and interferences,

(7) \( E_1/E_2 \ (p - 2s_{1/2} / d - 2s_{1/2}) \)
(8) \( E_1/E_1 \ (p - 1d_{5/2} / f - 1d_{5/2}) \)
(9) \( E_1/E_2 \ (p - 1d_{5/2} / s - 1d_{5/2}) \)
(10) \( E_1/E_2 \ (p - 1d_{5/2} / d - 1d_{5/2}) \)
(11) \( E_1/E_2 \ (f - 1d_{5/2} / s - 1d_{5/2}) \)
(12) \( E_1/E_2 \ (f - 1d_{5/2} / d - 1d_{5/2}) \)
(13) \( E_2/E_2 \ (s - 1d_{5/2} / d - 1d_{5/2}) \).

These transitions are illustrated schematically in Fig. 3.5.

From Appendix B (with the notation used therein) the differential direct capture cross sections in the centre-of-mass system are given by

(1) \( E_1 \ (p-2s_{1/2}) \)

\[
\left( \frac{d \sigma}{d \Omega} \right)_1 = 6 \, w^2 \, c^2 \left( \, \gamma \, I_0^{1/2} \right)^2 \, (1 - x^2) \]  \hspace{1cm} (3.12)
Figure 3.5: $^0{}^{16}(p, \gamma)^{17}F$ transition scheme
(no spin-orbit splitting)
where \( x = \cos \theta \) and \( I_{L/2}^{L} \) is the radial integral for a transition of character \( k (E_1 \text{ or } E_2) \) from continuum wave \((l, s)\) to bound state \((L, S)\), where here \( L = S = \frac{1}{2} \).

(2) \( E_2 (d-2s_{1/2}) \)

\[
\left( \frac{d\sigma}{dn} \right)_2 = 30 \, \omega \, c^2 \, \left( I_0^{3/2} \right)^2 \, x^2 \, (1-x^2)
\]

(3) \( E_1 (p-1d_{5/2}) \)

\[
\left( \frac{d\sigma}{dn} \right)_3 = \frac{12}{25} \, \omega \, c^2 \, \left( I_2^{1/2} \right)^2 \, (1-\frac{1}{2} \, x^2)
\]

(4) \( E_1 (f-1d_{5/2}) \)

\[
\left( \frac{d\sigma}{dn} \right)_4 = \frac{216}{25} \, \omega \, c^2 \, \left( I_2^{3/2} \right)^2 \, (1-\frac{1}{2} \, x^2)
\]

(5) \( E_2 (s-1d_{5/2}) \)

\[
\left( \frac{d\sigma}{dn} \right)_5 = \frac{12}{5} \, \omega \, c^2 \, \left( I_2^{0/2} \right)^2
\]

(6) \( E_2 (d-1d_{5/2}) \)

\[
\left( \frac{d\sigma}{dn} \right)_6 = \frac{30}{49} \, \omega \, c^2 \, \left( I_2^{3/2} \right)^2 \, (5+9x^2-12x^4)
\]

(7) \( E_1/E_2 (p-2s_{1/2}/d-2s_{1/2}) \)

\[
\left( \frac{d\sigma}{dn} \right)_7 = 12 \sqrt{5} \, \omega \, c \, c_2 \, \left( I_0^{1/2} \right) \left( I_2^{1/2} \right) \cos (\psi_k - \psi'_k) \, x \, (1-x^2)
\]

where \( \psi_k^{\alpha} \) is the total phase shift for quantum numbers \( \ell \) and \( \alpha \).
(8) $E_1/E_1 \ (p-1d_{5/2}/f-1d_{5/2})$

$$\left( \frac{d\sigma}{d\Omega} \right)_8 = \frac{108}{25} \ \omega \ c_1^2 \ I_2^{-1/2} \ I_2^{0\ 1/2} \ \cos (\varphi_3^{1/2} - \varphi_3^{1/2}) \ (3x^2 - 1) \quad (3.19)$$

(9) $E_1/E_2 \ (p-1d_{5/2}/s-1d_{5/2})$

$$\left( \frac{d\sigma}{d\Omega} \right)_9 = -\frac{72}{25} \ \omega \ c_1 \ c_2 \ I_2^{-1/2} \ I_2^{0\ 1/2} \ \cos (\varphi_3^{1/2} - \varphi_0^{1/2}) \ \times \quad (3.20)$$

(10) $E_1/E_2 \ (p-1d_{5/2}/d-1d_{5/2})$

$$\left( \frac{d\sigma}{d\Omega} \right)_{10} = \frac{72}{25} \ \omega \ c_1 \ c_2 \ I_2^{-1/2} \ I_2^{0\ 1/2} \ \cos (\varphi_3^{1/2} - \varphi_0^{1/2}) \ \times \ (1 - x^2) \quad (3.21)$$

(11) $E_1/E_2 \ (f-1d_{5/2}/s-1d_{5/2})$

$$\left( \frac{d\sigma}{d\Omega} \right)_{11} = \frac{36}{25} \ \omega \ c_1 \ c_2 \ I_2^{-1/2} \ I_2^{0\ 1/2} \ \cos (\varphi_3^{1/2} - \varphi_0^{1/2}) \ \times \ (3 - 5x^2) \quad (3.22)$$

(12) $E_1/E_2 \ (f-1d_{5/2}/d-1d_{5/2})$

$$\left( \frac{d\sigma}{d\Omega} \right)_{12} = -\frac{72}{25} \ \omega \ c_1 \ c_2 \ I_2^{-1/2} \ I_2^{0\ 1/2} \ \cos (\varphi_3^{1/2} - \varphi_0^{1/2}) \ \times \ (3 - 4x^2) \quad (3.23)$$

(13) $E_2/E_2 \ (s-1d_{5/2}/d-1d_{5/2})$

$$\left( \frac{d\sigma}{d\Omega} \right)_{13} = -\frac{12}{7} \ \omega \ c_2^2 \ I_2^{0\ 1/2} \ I_2^{1\ 1/2} \ \cos (\varphi_3^{1/2} - \varphi_0^{1/2}) \ (3x^2 - 1) \quad (3.24)$$

The corresponding total cross sections are given by

(1) $E_1(p-2s_{1/2})$

$$\sigma_1 = 16\pi \ \omega \ c_1^2 \ I_2^{0\ 1/2} \ I_2^{0\ 1/2} \quad (3.25)$$
Model II

Due to the complexity of the situation when spin-orbit forces are introduced and the lack of very accurate experimental direct capture differential cross sections, only the total cross sections are considered on the basis of Model II (Appendix C).

\[ \sigma_2 = 16\pi r \frac{c_2}{2} \left( \frac{1}{2} I_{2s_1/2}^2 \right)^2 \]  
\[ \sigma_3 = \frac{96}{5} \pi r c_1^2 \left( \frac{1}{2} I_{1s_1/2}^2 \right)^2 \]  
\[ \sigma_4 = \frac{144}{5} \pi r c_1^2 \left( \frac{1}{2} I_{3s_1/2}^2 \right)^2 \]  
\[ \sigma_5 = \frac{144}{5} \pi r c_1^2 \left( \frac{1}{2} I_{-\frac{5}{2}}^2 \right)^2 \]  
\[ \sigma_6 = \frac{96}{7} \pi r c_2^2 \left( \frac{1}{2} I_{-\frac{7}{2}}^2 \right)^2 \]
(5) \( E_{1}(p_{3/2} \rightarrow ld_{5/2}) \)
(6) \( E_{1}(f_{5/2} \rightarrow ld_{5/2}) \)
(7) \( E_{1}(f_{7/2} \rightarrow ld_{5/2}) \)
(8) \( E_{2}(s_{1/2} \rightarrow ld_{5/2}) \)
(9) \( E_{2}(d_{3/2} \rightarrow ld_{5/2}) \)
(10) \( E_{2}(d_{5/2} \rightarrow ld_{5/2}) \)
(11) \( M_{1}(d_{3/2} \rightarrow ld_{5/2}) \)

These transitions are shown schematically in Fig. 3.6 and are given explicitly below using the notation of Appendix C, where \( I_{L, J}^{(k)} \) is the radial integral for a transition of character \( k \) from a continuum state with quantum numbers \( L \) and \( j \) to a bound state with quantum numbers \( L \) and \( J \):

(1) \( E_{1}(p_{1/2} \rightarrow 2s_{1/2}) \)
\[
\sigma_{1} = \frac{16\pi}{3} \omega c_{1}^{2} \left( I_{1/2, 0}^{(1)} \right)^{2}
\]
(2) \( E_{1}(p_{3/2} \rightarrow 2s_{1/2}) \)
\[
\sigma_{2} = \frac{32\pi}{3} \omega c_{1}^{2} \left( I_{1/2, 0}^{(1)} \right)^{2}
\]
(3) \( E_{2}(d_{3/2} \rightarrow 2s_{1/2}) \)
\[
\sigma_{3} = \frac{32\pi}{5} \omega c_{1}^{2} \left( I_{2, 0}^{(2)} \right)^{2}
\]
(4) \( E_{2}(d_{5/2} \rightarrow 2s_{1/2}) \)
\[
\sigma_{4} = \frac{48\pi}{5} \omega c_{1}^{2} \left( I_{2, 0}^{(2)} \right)^{2}
\]
Figure 3.6: $^{16}\text{O}(p,\gamma)^{17}\text{F}$ transition scheme (with spin-orbit splitting)
(5) \( \text{El}(p_{3/2} - ld_{5/2}) \)
\[ \sigma_5 = \frac{96}{5} \pi \ w \ c_1^2 \left( I^{(1)}_{3/2} ; 2 \frac{3}{2} \right)^2 \]  
(3.35)

(6) \( \text{El}(f_{5/2} - ld_{5/2}) \)
\[ \sigma_6 = \frac{48}{35} \pi \ w \ c_1^2 \left( I^{(1)}_{3/2} ; 2 \frac{3}{2} \right)^2 \]  
(3.36)

(7) \( \text{El}(f_{7/2} - ld_{5/2}) \)
\[ \sigma_7 = \frac{192}{7} \pi \ w \ c_1^2 \left( I^{(1)}_{3/2} ; 2 \frac{3}{2} \right)^2 \]  
(3.37)

(8) \( \text{E2}(s_{1/2} - ld_{5/2}) \)
\[ \sigma_8 = \frac{48}{5} \pi \ w \ c_2^2 \left( I^{(2)}_{0} ; 2 \frac{1}{2} \right)^2 \]  
(3.38)

(9) \( \text{E2}(d_{3/2} - ld_{5/2}) \)
\[ \sigma_9 = \frac{96}{35} \pi \ w \ c_2^2 \left( I^{(2)}_{3/2} ; 2 \frac{3}{2} \right)^2 \]  
(3.39)

(10) \( \text{E2}(d_{5/2} - ld_{5/2}) \)
\[ \sigma_{10} = \frac{384}{35} \pi \ w \ c_2^2 \left( I^{(2)}_{3} ; 2 \frac{3}{2} \right)^2 \]  
(3.40)

(11) \( \text{M1}(d_{3/2} - ld_{5/2}) \)
\[ \sigma_{11} = \frac{128}{5} \pi^2 \ w \ c_3^2 \left( 4m^2 + c_4^2 \right) \left( I^{(3)}_{3} ; 2 \frac{3}{2} \right)^2 \]  
(3.41)

Where in Model II, with spin-orbit forces introduced, the \( \text{M1}(d_{3/2} - ld_{5/2}) \) transition is allowed as the \( d_{3/2} \) -wave and \( ld_{5/2} \)-state are no longer orthogonal.
3.5 Cross Sections -- Results

The $^9_0 \text{p} \to F^{17}$ cross sections can be obtained upon evaluating the necessary radial integrals involving the bound and continuum states with the appropriate multipole operators and inserting these in the expressions given in Section 3.4. Throughout these calculations the exact multipole operators were used (Appendix A), although little error arises from use of the long wavelength approximation. The radial integrands obtained are similar to those for the $D(p,\gamma)\text{He}^3$ reaction, except that, due to the small binding energies of the $F^{17}$ levels and the resulting long exponential tails to the bound state wave functions, the radial integrands for $^9_0 \text{p} \to F^{17}$ have their maxima at large distances. These maxima occur typically at several times the nuclear radius — for example, for the $E1(p-2s_{1/2})$ transition at 2.5, 1.0 and 0.15 MeV the peaks in the radial integrand occur at 10, 15 and 30 fm respectively. For this reason the integrations involved were carried out to 100-200 fm, depending on the energies considered. Furthermore, contributions to the radial integrals from the nuclear interior were included although amounting to only a small fraction of the total integrals. For example, even at 2.5 MeV, the interior contributions to the $E1(p-1d_{5/2})$ matrix element account for only 10% of the total, so that, in the case of $^9_0 \text{p} \to F^{17}$ at low energies, the cross sections are indeed extranuclear in origin.

The total cross sections are shown in Figs. 3.7 and 3.8, where they are compared with experimental data. The agreement
Figure 3.7: $^{16}\text{(p,\gamma)}^{17}$ cross sections.
Figure 3.8: $^{16}(p, \gamma)^{17}$ cross sections
with experiment is good, particularly so when one considers that no free parameter has been used in these calculations to fit the direct capture data. The differences between the results using Models I and II are not large, but certainly significant, particularly at the higher energies considered where details of the bound state wave functions in the nuclear surface and interior have an appreciable effect on the cross sections. The cross sections for transitions to the $1d_{5/2}$-state are lower when calculated on the basis of Model II than when calculated for Model I. This difference is due principally to the fact that the Model II bound state wave functions are smaller in magnitude than those for Model I in the exterior regions where the radial integrand has its maximum. However, for transitions to the $2s_{1/2}$-state at low energies, a similar effect is not observed since the two models have nearly identical exterior bound state wave functions. At higher energies, differences between the $2s_{1/2}$ bound state wave functions in the nuclear surface and interior become more important and in fact result in higher cross sections for transitions to the $2s_{1/2}$-state when Model II is used than when Model I is used.

The cross sections calculated on the basis of models I and II at $E_{CM} = 1.0$ MeV are given in Table 3.1 to illustrate the relative importance of the various transitions considered. Clearly the E2 and M1 contributions to the total cross section are small in comparison with the E1 cross sections. The E1 transitions from f-waves to the $1d_{5/2}$-state are significant,
however, even in comparison with transitions involving p-waves, amounting to about 12% of the total cross section to the $1d_{5/2}$ state at 1.0 MeV and even more at higher energies (35% at 2.0 MeV). Results of calculations with the f-waves omitted are illustrated in Fig. 3.7.

The E2 transitions do have a small effect on the differential capture cross sections in that they interfere with the large E1 transitions. The differential cross section for transitions to the $2s_{1/2}$ excited state is modified from the pure \( \sin^2 \theta \) distribution of the \((p-2s_{1/2})\) term by the E1/E2 \( (p-2s_{1/2}/d-2s_{1/2}) \) interference, producing an asymmetry and causing the differential cross section to peak at forward angles (87° at 1.0 MeV, 83° at 2.0 MeV in the centre-of-mass system). In a similar way; E1/E1 and E1/E2 interferences contribute to the differential cross section for transitions to the $1d_{5/2}$ ground state. The E1\((p-1d_{5/2})\), the E1\((f-1d_{5/2})\) and the E1/E1 interference between them contribute most to the differential cross section. In Fig. 3.9 these differential cross sections are shown at an energy of \( E_{CM} = 1.0 \) MeV.

Calculations were also performed using Model II at an energy of 1.0 MeV with different values of the spin-orbit well depth \( V_s \) and diffuseness \( a \) in the potential for the continuum wave functions. The capture cross sections were found to be relatively insensitive to variations of these parameters in the range considered in fitting elastic scattering data (section
Figure 3.9: $^{16}(p, \gamma)^{17}$ angular distribution at $E_{cm} = 1.0$ MeV
In fact, the direct capture cross sections varied only by about 1% for changes of 1 MeV and 0.1 fm in $V_5$ and $\alpha$, respectively, about their values used above.

**Table 3.1: $^{16}\text{O}(p,\gamma)^{17}\text{F}$ Cross Sections at $E_{CM} = 1.0$ MeV**

**Model I**

<table>
<thead>
<tr>
<th>Transition</th>
<th>Value (microbarns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E1(p-2s_{1/2})$</td>
<td>1.454</td>
</tr>
<tr>
<td>$E2(d-2s_{1/2})$</td>
<td>0.00153</td>
</tr>
<tr>
<td>$E1(p-1d_{5/2})$</td>
<td>0.379</td>
</tr>
<tr>
<td>$E1(f-1d_{5/2})$</td>
<td>0.0450</td>
</tr>
<tr>
<td>$E2(s-1d_{5/2})$</td>
<td>0.000241</td>
</tr>
<tr>
<td>$E2(d-1d_{5/2})$</td>
<td>0.000242</td>
</tr>
</tbody>
</table>

**Model II**

<table>
<thead>
<tr>
<th>Transition</th>
<th>Value (microbarns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E1(\pi_{1/2}-2s_{1/2})$</td>
<td>0.480</td>
</tr>
<tr>
<td>$E1(\pi_{3/2}-2s_{1/2})$</td>
<td>0.983</td>
</tr>
<tr>
<td>$E2(d_{3/2}-2s_{1/2})$</td>
<td>0.000595</td>
</tr>
<tr>
<td>$E2(d_{5/2}-2s_{1/2})$</td>
<td>0.000866</td>
</tr>
<tr>
<td>$E1(\pi_{3/2}-1d_{5/2})$</td>
<td>0.239</td>
</tr>
<tr>
<td>$E1(f_{5/2}-1d_{5/2})$</td>
<td>0.000981</td>
</tr>
<tr>
<td>$E1(f_{7/2}-1d_{5/2})$</td>
<td>0.0200</td>
</tr>
<tr>
<td>$E2(s_{1/2}-1d_{5/2})$</td>
<td>0.0000842</td>
</tr>
<tr>
<td>$E2(d_{3/2}-1d_{5/2})$</td>
<td>0.0000284</td>
</tr>
<tr>
<td>$E2(d_{5/2}-1d_{5/2})$</td>
<td>0.0000953</td>
</tr>
<tr>
<td>$M1(d_{3/2}-1d_{5/2})$</td>
<td>$1.50 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
3.6 Summary

The present calculations, employing either a simple square-well plus Coulomb potential or a more elaborate Saxon-Woods plus spin-orbit plus Coulomb potential, show that good agreement with experimental $^{16}(p,\gamma)^{17}$F cross sections can be obtained using the direct capture model. With well parameters fixed entirely by fitting $^{16}(p,p)^{16}$ elastic scattering data and by fitting the binding energies of the $5/2^+$ and $1/2^+$ bound states of $^{17}$F, agreement with experimental direct capture data is obtained without further parameter variation. The results of calculations on the basis of models I and II are found to differ, largely because of the different bound state wave functions which are obtained. Model II is found to reproduce the experimental cross sections better than Model I, however, because of the large errors involved in the experimental data, it is difficult to assess the significance of the differences in the results obtained using the two models.

Since the $^{16}(p,\gamma)^{17}$F reaction is of astrophysical interest, the results of the present calculations are given in Fig. 3.10 as S-factors (defined in section 2.8) and compared with experiment. The S-factor, obtained using either Model I or Model II, rises with decreasing energy from 3.7 keV-bn at 2.0 MeV to 8.5 keV-bn (for Model I) or 7.8 keV-bn (for Model II) at 50 keV. At low energies the calculated S-factor is somewhat smaller than previously calculated by Lal (1961) or Domingo (1965), yet is in good agreement with experiment.
Figure 3.10: $^{16}\text{O}(p,\gamma)^{17}$ astrophysical S-factor shown for transitions to the s- and d-states, as well as for the total.
With the advent of more accurate experimental data, a more detailed analysis and parameter variation would be warranted; however, with the current status of the $^0_{16}(p,\gamma)^{17}_F$ data, this is not justified at present.
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APPENDIX A

DIRECT CAPTURE THEORY

This appendix contains a brief summary of the theory used to describe the direct radiative capture reaction. It essentially follows the works of Christy and Duck (1961) and Tombrello and Phillips (1961). The notation is largely that of Parker (1963).

The direct radiative capture reaction \( A(x, \gamma)B \) representing the capture of particle \( x \) by a target \( A \) to form a bound state \( B \) of \( x \) and \( A \) with the emission of radiation can be described by the Hamiltonian \( \mathcal{H} \),

\[
\mathcal{H} = \mathcal{H}^0 + \mathcal{H}_{\text{int}} = \mathcal{H}_p + \mathcal{H}_\gamma + \mathcal{H}_{\text{int}} \quad \text{(A.1)}
\]

\( \mathcal{H}_p \) contains the kinetic energy of the particles plus the potential energy of their nuclear and Coulomb interactions while \( \mathcal{H}_\gamma \) contains the energy of the free electromagnetic field. \( \mathcal{H}_{\text{int}} \) then represents the usual Hamiltonian for the interaction of the particles and electromagnetic field, which we can treat as a perturbation on the Hamiltonian since the electromagnetic interaction is very much weaker than the nuclear interaction.

The eigenstates of \( \mathcal{H}^0 \) are of the form

\[
|\psi\rangle = |\psi\rangle |\nu\rangle \quad \text{(A.2)}
\]
where \( |\psi\rangle \) is an eigenstate of \( \mathcal{H}_p \), a state of the \( x + A \), continuum or \( B \), bound systems, and \( |n\rangle \) is an eigenstate of \( \mathcal{H}_\gamma \), a state of \( n \) photons of given energy and polarization; this is \( |0\rangle \), the vacuum state in the initial state and \( |1\rangle \), the state of a free photon of energy \( \hbar \omega \) and circular polarization \( P \) in the final state.

First-order time-dependent perturbation theory for the perturbation of \( \mathcal{H}^0 \) by \( \mathcal{H}_{\text{int}} \) is employed in the form of Fermi's Golden Rule. That this approximation is valid is evidenced by the small cross sections involved in direct capture reactions, of the order of microbarns compared to cross sections of the order of barns in the case of elastic scattering.

The differential cross section can then be written

\[
\frac{d\sigma(\theta)}{d\Omega} = \frac{2\pi}{k_{\gamma}^2} \left( \frac{d\nu}{d\omega} \right) \sum_{m, M} \frac{1}{(2I_1 + 1)(2I_2 + 1)} \left| \langle \Phi_M | \mathcal{H}_{\text{int}} | \Psi_m \rangle \right|^2
\]

where \( \nu = \) speed of incident particle \( x \),
\( I_1, I_2 = \) spins of particles \( x \) and \( A \),
\( P = \) circular polarization of photon,
\( n(E) = \) density of states function,
\( |\psi_m\rangle = \) initial state, with magnetic quantum number \( m \),
\( |\Phi_M\rangle = \) final state, with magnetic quantum number \( M \),

and where, for the initial and final state wave functions used in calculating these matrix elements, two cases will be considered. Firstly, the case of two particles of arbitrary spins with no spin-orbit forces will be considered, in which case the
wave functions can be written:

(1) for the initial state,

\[ \Psi_m = \sum_l \sum_\Delta \sqrt{4\pi (2l + 1)} i^l \Phi_\Delta^\ell(r) \frac{R_\Delta^\ell(r)}{kr} Y_\ell^m(\theta, \phi) \chi_m \]  

where the sum extends over all values of \( l \) and \( \Delta \), the orbital angular momentum and channel spin respectively, and,

\( (r, \theta, \phi) = \) relative coordinates of the particles,

\( \Phi_\Delta^\ell = \Phi_\ell^\Delta - \Phi_\ell^\Delta \)

\( \Phi_\ell^\Delta = \) Coulomb phase shift,

\( \Phi_\ell^\Delta = \) nuclear phase shift for quantum numbers \( l, \Delta \),

\( k = \) particle wavenumber,

\( Y_\ell^m(\theta, \phi) = \) spherical harmonic,

\[ \chi_m = \sum_\alpha \phi(I_1, I_2, \Delta; \alpha, m-\alpha) \xi_{I_1}^\alpha \xi_{I_2}^{m-\alpha} \]  

\( \phi_{I_1}^\alpha, \phi_{I_2}^{m-\alpha} = \) spin functions of the two particles of spins \( I_1 \) and \( I_2 \) \( (\Delta = I_1 + I_2) \),

\( \phi(I_1, I_2, \Delta; \alpha, \beta) = \) Clebsch-Gordan coefficient, using the notation of Rose (1957),

and \( R_\Delta^\ell(r) = \) radial wave function for the initial state, satisfying

\[ R_\Delta^\ell''(r) + \left( k^2 - \frac{\Delta^2}{r^2} - \frac{\ell(\ell + 1)}{r^2} \right) R_\Delta^\ell(r) = 0 \]

and normalized asymptotically to unit flux, where
\[ k^2 = \frac{2m_1 m_2}{(m_1 + m_2)} E \]

\[ m_1, m_2 = \text{masses of the two particles,} \]

\[ E = \text{energy in the centre-of-mass system,} \]

\[ W_L^2(r) = \frac{2m_1 m_2}{(m_1 + m_2)} V_L^2(r) \]

\[ V_L^2(r) = \text{nuclear potential for quantum numbers} \]

\[ l \text{ and } s \text{ plus Coulomb potential;} \]

(ii) for the final state,

\[ \Phi_M = \sum L \sum S a_L^S \mathcal{U}_L^S(r) \sum C(LS; M-\beta, \beta) Y_L^{-\beta}(\theta, \phi) \chi_S^\beta \]

(A.7)

where the sums extend over all values of L and S, the orbital and channel spin angular momenta respectively, to allow for various terms of different L and S in a bound wave function of total angular momentum J, with amplitudes \(a_L^S\) (eg. the ground states of He\(^3\) and T, which may be regarded as \(^2S_{\frac{1}{2}}\) plus \(^4D_{\frac{3}{2}}\) states, having \(a_L^S = \sqrt{1-P_D} \delta_{L0} \delta_{S \frac{1}{2}} + \sqrt{P_D} \delta_{L2} \delta_{S \frac{3}{2}}\), where \(P_D\) is the \(^4D\)-state probability), and where \(\mathcal{U}_L^S(r)\) is the radial wave function for quantum numbers L and S, satisfying

\[ \mathcal{U}_L^S''(r) + \left( \varepsilon - W_L^S(r) - \frac{L(L+1)}{r^2} \right) \mathcal{U}_L^S(r) = 0 \]

(A.8)

with normalization condition

\[ \int_0^\infty |\mathcal{U}_L^S(r)|^2 \, dr = 1 \]

(A.9)

where \(\varepsilon = -\frac{2m_1 m_2}{(m_1 + m_2)} E \)
and \[ E_B = \text{binding energy of the bound } x + A \text{ system,} \]
i.e. for B.

All states are treated as having L and S as good quantum numbers, with the total final state being a combination of such states with amplitudes \( a^S_L \). This case will be considered in detail in Appendix B.

Secondly, the system of two particles in which one has spins \( 0 \) and \( \frac{1}{2} \) and in which spin-orbit forces are included will be considered. The wave functions in this case are,

(i) for the initial state,

\[
\Psi_m = \sum_l \sum_j \sum_\alpha C(l \frac{1}{2} j; \alpha m) C(l \frac{1}{2} j; m-\alpha, \alpha) \\
\times \sqrt{4\pi(2l+1)} \ i^l e^{i\theta_j} R_{l,j}(r) Y_{l-m}^\alpha(\theta, \phi) \chi_\frac{l}{2}^\alpha
\]

where the notation is similar to that above, with sums over all values of \( l \) and \( \alpha \) and of \( j = l \pm \frac{1}{2} \), the total angular momentum of the partial wave, with phase shifts and radial wave functions now being specified by \( l \) and \( j \) rather than \( l \) and \( \Delta \) (\( \Delta = \frac{1}{2} \) here);

(ii) for the final state,

\[
\Phi_M = \frac{\mathcal{U}_{-\frac{1}{2} J}(r)}{r} \sum_\beta C(L \frac{1}{2} J; M-\beta, \beta) Y_{\frac{1}{2}}^{M-\beta}(\theta, \phi) \chi_\frac{1}{2}^\beta
\]

where in this case the total bound state is regarded for simplicity as having \( L, S \) (\( = \frac{1}{2} \)) and \( J \) as good quantum numbers.
The radial wave functions for these states satisfy equations similar to the ones above. This case will be considered in detail in Appendix G.

The interaction Hamiltonian is given by

$$H_{\text{int}}^P = -\frac{i}{e} \mathbf{J} \cdot \mathbf{A}^P$$  \hspace{1cm} (A.12)

where $\mathbf{J}$ is the nuclear-charge current vector and $\mathbf{A}^P$ that part of the vector potential for the electromagnetic field which describes the creation of a photon of circular polarization $P$. With normalization to energy $\hbar \omega$ in volume $V$, the vector potential becomes

$$\mathbf{A}^P = -\frac{e}{i \omega} \left( \frac{2 \pi \hbar \omega}{V} \right)^{1/2} \mathbf{\alpha}_1^P e^{-i \mathbf{x} \cdot \mathbf{r}}$$  \hspace{1cm} (A.13)

where $\mathbf{\alpha}_1^P$ is a spherical unit vector, projection $P$, and $\mathbf{x} = \omega/c$ is the radiation wavenumber.

$\mathbf{\alpha}_1^P e^{-i \mathbf{x} \cdot \mathbf{r}}$ may be expanded in magnetic and electric multipoles ($m$ and $e$) of multipolarity $\ell$ as

$$\mathbf{\alpha}_1^P e^{-i \mathbf{x} \cdot \mathbf{r}} = \sum_{\ell=1}^{\infty} \sum_{\ell=1}^{\ell} \sqrt{\frac{2}{(2\ell+1)}} (-i)^{\ell} \mathbf{D}_{\ell}^{(\ell)\ell} \left( \theta_\gamma, \phi_\gamma, 0 \right) \left\{ \mathbf{A}_\ell^P (m) - i P \mathbf{A}_\ell^P (e) \right\}$$  \hspace{1cm} (A.14)

where $\mathbf{D}_{\ell}^{(\ell)\ell} \left( \theta_\gamma, \phi_\gamma, 0 \right)$ is an element of the rotation matrix and $(\theta_\gamma, \phi_\gamma)$ are the polar angles of the $\gamma$-ray. Furthermore, using Siegert's Theorem and invoking the long wavelength approximation ($\mathbf{x} \mathbf{r} = \rho \ll 1$), one obtains
\[ \mathbf{A} \cdot \mathbf{A}^{\star} (e) \xrightarrow{\rho \ll 1} -i e c \sqrt{\frac{x+1}{x}} \frac{p^x}{(2x+1)!!} Y_{x}^{m*} \quad (A.15) \]

and similarly for magnetic multipoles,

\[ \mathbf{A} \cdot \mathbf{A}^{\star} (m) \xrightarrow{\rho \ll 1} -i c \sqrt{\frac{x+1}{x}} \frac{e^h}{m_p c} \frac{1}{2} \frac{1}{x+1} \left\{ - \frac{1}{x+1} \left( L + \frac{1}{2} \mu_S S \right) \right\} \cdot (\text{grad} \ r) Y_{x}^{m*} \quad (A.16) \]

where \( m_p \) is the proton mass, \( \mu_S \) is the magnetic moment in nuclear magnetons, and \( L \) and \( S \) are the orbital and spin angular momentum operators respectively. For magnetic dipole radiation this becomes

\[ \mathbf{A} \cdot \mathbf{A}^{\star} (m) \xrightarrow{\rho \ll 1} (-)^m \left\{ - \frac{1}{x+1} \left( L + \frac{1}{2} \mu_S S \right) \right\} \cdot \mathbf{r} Y_{x}^{m*} \quad (A.17) \]

As the density of states is given by

\[ \frac{d\Omega}{dE} = \frac{\gamma^2 V}{8\pi^3 \hbar c} \quad (A.18) \]

the differential cross section can be written

\[ \frac{d\sigma}{d\Omega} (\theta) = \frac{e^2 \gamma^3}{\hbar \nu} \frac{1}{(2\alpha_1+1)(2\alpha_2+1)} \sum_{m_1, m_2} \left| \langle \Phi | H_{\text{int}} P' | \Psi_m \rangle \right|^2 \quad (A.19) \]

where \( H_{\text{int}}' \) is \( H_{\text{int}} \) with the factor \( i e c \sqrt{\frac{2\pi \hbar}{\nu}} \) taken outside, and is given by (considering only \( E_1, E_2, M_1 \))

\[ H_{\text{int}}' = \sum_{m} \left\{ -i \sqrt{\frac{4\pi}{3}} P \mathcal{D}_{mp}^{(1)} (\theta, \phi, 0) Y_{1}^{m*} + \sqrt{\frac{\pi}{15}} P \times \mathcal{D}_{mp}^{(2)} (\theta, \phi, 0) r^2 Y_{2}^{m*} \right\} \]
Furthermore, as each term contains a sum over all nucleons in the system, we have

\[
(\mathbf{L} + \mu_5 \mathbf{S}) \rightarrow \sum_j \left( m_p \frac{\mathbf{z}_j}{m_j} \mathbf{L}_j + g_j \mathbf{S}_j \right)
\]

\[
= m_p \left( \frac{m_1 m_2}{m_1 + m_2} \right) \left( \frac{\mathbf{z}_1}{m_1} + \frac{\mathbf{z}_2}{m_2} \right) \mathbf{L} + g_1 \mathbf{S}_1 + g_2 \mathbf{S}_2 \quad (A.21)
\]

for two particles, where \( g_j \) are the gyromagnetic ratios and \( \mathbf{s}_j \) the spin operators (\( \sigma = 2 \mathbf{S} \) for spin \( \frac{3}{2} \), \( \sigma = \mathbf{S} \) for spin 1, etc.)

\[
r \mathbf{Y}_1^{m*} \rightarrow \sum_j \mathbf{z}_j r_j \mathbf{Y}_1^{m*}(\theta_j, \phi_j) \quad (A.22)
\]

becoming

\[
\left( \frac{m_1 m_2}{m_1 + m_2} \right) \left( \frac{\mathbf{z}_1}{m_1} - \frac{\mathbf{z}_2}{m_2} \right) r \mathbf{Y}_1^{m*}(\theta, \phi) \quad (A.23)
\]

for electric dipole, and similarly

\[
\sum_j \mathbf{z}_j r_j^2 \mathbf{Y}_2^{m*}(\theta_j, \phi_j) = \left( \frac{m_1 m_2}{m_1 + m_2} \right)^2 \left( \frac{\mathbf{z}_1}{m_1} + \frac{\mathbf{z}_2}{m_2} \right) r^2 \mathbf{Y}_2^{m*}(\theta, \phi) \quad (A.24)
\]

for electric quadrupole, where

\[
z_j = \text{charge of the } j^{th} \text{ particle in units of } e,
\]

\[
m_j = \text{mass of the } j^{th} \text{ particle},
\]

\((r_j, \theta_j, \phi_j) = \text{coordinates of the } j^{th} \text{ particle}, \text{ and} \)

\[(r, \theta, \phi) = \text{relative coordinates of the two particles}.\]

In fact, if the long wavelength approximation is not used, but rather the correct expressions are employed, then the following replacements are made in the radial parts of the multipole operators:
(1) for $E_1$, $r$ is replaced by
\[ O_{E1}(r) = \frac{3}{\rho^3} \left\{ (\rho^2 - 2) \sin \rho + 2 \rho \cos \rho \right\} r \]  
(A.25)
(2) for $E_2$, $r^2$ is replaced by
\[ O_{E2}(r) = \frac{15}{\rho^5} \left\{ (5\rho^2 - 12) \sin \rho + (12 - \rho^2) \rho \cos \rho \right\} r^2 \]  
(A.26)
and (3) for $M_1$, $\rho$ is replaced by
\[ O_{M1}(r) = \frac{1}{2\rho} (\sin \rho + \rho \cos \rho) \]  
(A.27)
The ratios $O_{E1}/r$, $O_{E2}/r^2$ and $O_{M1}$ are shown as functions of $\rho$ in Fig.A.1.

Then, with the definitions
\[ C_1 = -\sqrt{\frac{4\pi}{3}} \left( \frac{m_1 m_2}{m_1 + m_2} \right) \left( \frac{E_1}{m_1} - \frac{E_2}{m_2} \right) \]  
(A.28)
\[ C_2 = -\sqrt{\frac{\pi}{15}} J \left( \frac{m_1 m_2}{m_1 + m_2} \right)^2 \left( \frac{E_1}{m_1^2} + \frac{E_2}{m_2^2} \right) \]  
(A.29)
\[ C_3 = \frac{2mp}{c} \]  
(A.30)
\[ C_4 = mp \left( \frac{m_1 m_2}{m_1 + m_2} \right) \left( \frac{E_1^2}{m_1^2} + \frac{E_2^2}{m_2^2} \right) \]  
(A.31)
The interaction Hamiltonian can be written
\[ H_{int}^I = \sum \left\{ iC_1 P D^{(1)*}_{M \rho} (\varphi_y, \theta_y, \phi) O_{E1}(r) Y_1^*(\theta, \phi) + C_2 P D^{(2)*}_{M \rho} (\varphi_y, \theta_y, \phi) O_{E2}(r) Y_2^*(\theta, \phi) + (-)^m C_3 D^{(1)*}_{M \rho} (\varphi_y, \theta_y, \phi) O_{M1}(r) \left[ g \frac{1}{2} \right] + C_4 \right\} X_{1-M} \]  
(A.32)
and with
\[ w = \frac{e^2 c^3}{h \nu} \frac{1}{(2\mathcal{I}_1 + 1)(2\mathcal{I}_2 + 1)} \]  
(A.33)
the differential direct capture cross section is given by

\[
\frac{d\sigma}{d\Omega} = \mathcal{W} \sum_{m', m, \rho = \pm 1} |<\Phi_{m'} | \mathcal{H}_{\text{int}}^\rho | \Phi_m>|^2
\]  

(A.34)
Figure A.1: Exact multipole operators, radial forms.
APPENDIX B

DIRECT CAPTURE INVOLVING
PARTICLES OF ARBITRARY SPINS
WITH NO SPIN-ORBIT POTENTIAL

General expressions for calculating the capture cross section have been given in Appendix A, these including wave functions for the continuum states (A.4), the bound states (A.7), the interaction Hamiltonian (A.32) and a general expression for the cross section (A.34). Here we develop results for the special case where the nuclear potential contains spin-dependent terms, which are different for different channel spins, but does not contain spin-orbit interactions which split states of different \( j \).

The matrix elements are decomposed to show explicitly the individual contributions from \( E1, E2, \) or \( M1 \) transitions (of multipolarity \( \mathcal{L} \), parity \( \pi \) and magnetic quantum number \( \mu \) ) from states in the continuum with orbital quantum number \( l \), channel spin \( \mathcal{J} \) and magnetic quantum number \( m \) to bound states described by corresponding quantum numbers \( L, S \) and \( M \), with total angular momentum quantum number \( J \). A general term is illustrated schematically in Fig.B.1.
Matrix Elements

The following matrix elements are defined for the El, E2, M1 ("spin-flip") and Ml ("orbital") parts respectively of the total matrix element, but with factors $P$ and $D^{(k)}_{mp}$ removed (Cf. A.32 and A.34),

\[ Q^{(1)} \mu \mu m = < \Phi_M \mid i c_1 O_{El} Y_1^{\mu*} \mid \Psi_m > \] (8.1)

\[ Q^{(2)} \mu \mu m = < \Phi_M \mid c_2 O_{E2} Y_2^{\mu*} \mid \Psi_m > \] (8.2)

\[ Q^{(3)} \mu \mu m = < \Phi_M \mid (-)^{\mu} c_3 O_{M1} (g_1 s_{L-M} + g_2 s_{2-M}) \mid \Psi_m > \] (8.3)

\[ Q^{(4)} \mu \mu m = < \Phi_M \mid (-)^{\mu} c_3 O_{M1} (c_4 L_{-M}) \mid \Psi_m > \] (8.4)

These matrix elements are in turn considered individually:

1. Electric dipole

\[ Q^{(1)} \mu \mu m = < \Phi_M \mid i c_1 O_{El} Y_1^{\mu*} \mid \Psi_m > \]
\[ = \sum_{L} \sum_{S} \sum_{C} a_L^S \mathcal{O}_{E1} \mathcal{R}_{\ell}^m Y_L^{m-\beta} \chi_S^{\beta} \chi_{L}^{\ell} \mathcal{C}(L_SJ; M-\beta, \beta) Y_{L}^{-\ell} \chi_{L}^{\ell} \chi_{L}^{\ell} \mathcal{C}(L_SJ; M-\beta, \beta) \]

\[ \sum_{L} \sum_{S} \frac{4\pi (2L+1)}{r} e^{i\phi^{\beta}_S} \frac{R_{\ell}^m}{kr} \mathcal{C}(L_SJ; M-\beta, \beta) \]

\[ \times \int_0^\infty r^2 dr \frac{u_S^{\beta S}}{R_{\ell}^m} \mathcal{O}_{E1} \frac{R_{\ell}^m}{kr} \int d\Omega Y_L^{m-\beta} Y_L^{m-\ell} Y_L^{m-\ell} \mathcal{C}(L_SJ; M-\beta, \beta) \]

and, letting

\[ ^{(i)} I_{L_S} = \int_0^\infty r^2 dr \frac{u_S^{\beta S}}{R_{\ell}^m} \mathcal{O}_{E1} \frac{R_{\ell}^m}{kr} \]

\[ ^{(i)} J_{L_S} = c_l a_L^S \mathcal{C}(L_SJ; M-\beta, \beta) \]

using Gaunt's formula to give

\[ \int d\Omega Y_L^{m-\beta} Y_L^{m-\ell} Y_L^{m-\ell} = \frac{\sqrt{3(2L+1)}}{4\pi} C(L12; 00) C(L12; M-\beta, \mu, \ell) \]

\[ \sqrt{4\pi (2L+1)} \]

and, using

\[ < \chi_{L}^{\beta} | \chi_{L}^{m} > = \delta_{\alpha \beta} \delta_{\beta m} \]

we obtain

\[ Q^{(i)}_{\mu m} = \sum_{L} \sum_{S} \sum_{C} \sqrt{3(2L+1)} \mathcal{C}(L_SJ; M-\mu, m) \]

\[ \times C(L12; 00) C(L12; M-\mu, m) ^{(i)} J_{L_S}^{\mu m} \]

where the M has been suppressed due to the relation M = m - \mu.

(2) Electric quadrupole

Similarly, letting
\[ (2^*) I_{LS} = \int_0^\infty r^2 dr \, \frac{u_i^*}{r} O_E \frac{R^2}{kr} \tag{B.10} \]

\[ (2^*) J_{LS} = c_a a^L_i \, \mathcal{I}_{\psi^2} \, I_{LS} \tag{B.11} \]

we obtain

\[ Q_{\mu m}^{(2^*)} = \sum_L \sum_S \sum_L \sqrt{5 (2L+1)} \, \delta_{\mu, m-M} \, C(LSj; -\mu, m) \times C(LZl; 00) \, C(LZl; -\mu, \mu) \, (2^*) J_{LS} , \, \Delta = S \tag{B.12} \]

(3) Magnetic dipole, "spin-flip"

Letting

\[ (3) I_{LS} = \int_0^\infty r^2 dr \, \frac{u_i^*}{r} O_{m} \frac{R^2}{kr} \tag{B.13} \]

and

\[ (3) J_{LS} = c_a a^L_i \, \mathcal{I}_{\psi^2} \, (3) I_{LS} \tag{B.14} \]

and using the orthogonality relation for the spherical harmonics, we have

\[ Q_{\mu m}^{(3)} = \sum_L \sum_S \sum_{\Delta} \sqrt{4 \pi (2L+1)} \, (-)^\Delta \, C(LSj; \mu, M) \times (3) J_{LS} \, \langle \chi_s^M | (g_1 s_1 - \mu + g_2 s_2 - \mu) | \chi_o^M \rangle , \, l=L \tag{B.15} \]

where now that part of the matrix element involving the spin functions must be evaluated. The channel spin functions are written in terms of the spin functions for the individual particles (A.5) and the relation (Rose, 1957)

\[ S_{-\mu} | \xi_i^\Delta > = (-)^\mu \sqrt{I(I+1)} \, C(I \mu ; -\mu, \mu) \, | \xi_i^\Delta > \tag{B.16} \]

is used, to give

\[ \langle \chi_s^M | (g_1 s_1 - \mu + g_2 s_2 - \mu) | \chi_o^M \rangle = \delta_{\mu, m-M} \sqrt{2\Delta+1} \, C(1 \Delta S; -\mu, m) \, M_{\Delta S} \tag{B.17} \]
where

\[ M_{\Delta S} = g_1 \sqrt{I_1(I_1+1)}(2I_1+1)(-)^{I_1-I_1-\Delta} W(\Delta S I_1 I_1 ; I_1 I_2) \]

\[ + g_2 \sqrt{I_2(I_2+1)}(2I_2+1)(-)^{I_1-I_2-\Delta} W(\Delta S I_2 I_2 ; I_1 I_1) \]

and where \( W(abcd;ef) \) are Racah coefficients (using Rose's notation). Finally, this matrix element becomes

\[ Q^{(3)}_{\mu m} = \sum \sum \sum \sqrt{4\pi(2L+1)}(-)^{\mu} C(LSJ;0,m-\mu) \delta_{\mu,m-M} \]

\[ \times \sqrt{2L+1} C(1\Delta S;-m,m) M_{\Delta S} ^{(3)} J_{LS}^{L} , \Delta = S \] (B.19)

(4) Magnetic dipole, "orbital"

Using a relation similar to (B.16) for the orbital angular momentum operator and spherical harmonics, and using orthogonality relations for the spin functions and spherical harmonics, we obtain

\[ Q^{(4)}_{\mu m} = \sum \sum \sum \mu (-)^{\mu+1} \sqrt{2\pi L(L+1)(2L+1)} C_{L} \delta_{\mu,m-M} \]

\[ \times C(LSJ; -m,m) ^{(4)} J_{LS}^{L} , \Delta = S, L = L \] (B.20)

If the notation

(1) \[ q_{\Delta \Delta; LS} \]

\[ = \sqrt{3(2L+1)} C(LSJ; -m,m) C(LLL; 00) \]

\[ \times C(LLL; -m,m) ^{(1)} J_{LS}^{L} \delta_{\mu,m-M} \delta_{\Delta S} \] (B.21)

(2) \[ q_{\Delta \Delta; LS} \]

\[ = \sqrt{5(2L+1)} C(LSJ; -m,m) C(LLL; 00) \]

\[ \times C(LLL; -m,m) ^{(2)} J_{LS}^{L} \delta_{\mu,m-M} \delta_{\Delta S} \] (B.22)

(3) \[ q_{\Delta \Delta; LS} \]

\[ = (-)^{\mu} \sqrt{4\pi(2L+1)} C(LSJ; 0,m-\mu) \sqrt{2L+1} \]

\[ \times C(1\Delta S; -m,m) M_{\Delta S} ^{(3)} J_{LS}^{L} \delta_{\mu,m-M} \delta_{\Delta L} \] (B.23)

(4) \[ q_{\Delta \Delta; LS} \]

\[ = C_{L} \mu (-)^{\mu+1} \sqrt{2\pi L(L+1)(2L+1)} C(LSJ; -m,m) \]

\[ \times ^{(4)} J_{LS}^{L} \delta_{\mu,m-M} \delta_{\Delta L} \delta_{\Delta S} \] (B.24)
is introduced as a convenience in what follows, then

\[ Q^{(k)}_{\mu \nu} = \sum_{L \leq l} \sum_{s} \sum_{s'} \sum_{s''} (k) q_{\mu \nu}^{L; l; s} \theta_{s', s} \quad , \quad k = 1, 4 \]  \hspace{1cm} (B.25)

**Cross Section**

Using (A.32-34) with (B.1-4), the cross section can be written

\[
\frac{1}{\mathcal{N}} \frac{d \sigma}{d \Omega} = \sum_{m, m'} \left| \langle \Phi_m | \mathcal{H}_{int} | \Psi_m \rangle \right|^2 \\
= \sum_{|m|, m'} \left[ \right. P \Delta_{\mu P}^{(1)} Q_{\mu \mu}^{(1)} + P \Delta_{\mu P}^{(2)} Q_{\mu \mu}^{(2)} + \Delta_{\mu P}^{(3)} Q_{\mu \mu}^{(3)} + \Delta_{\mu P}^{(4)} Q_{\mu \mu}^{(4)} \left. \right] ^2 \\
= \sum_{|m|, m'} \left\{ \right. 1 \left| \Delta_{\mu P}^{(1)} \right|^2 \left| Q_{\mu \mu}^{(1)} \right|^2 + 1 \left| \Delta_{\mu P}^{(2)} \right|^2 \left| Q_{\mu \mu}^{(2)} \right|^2 + 1 \left| \Delta_{\mu P}^{(3)} \right|^2 \left| Q_{\mu \mu}^{(3)} \right|^2 + \left. 1 \left| \Delta_{\mu P}^{(4)} \right|^2 \left| Q_{\mu \mu}^{(4)} \right|^2 \right\} \\
+ 2 \text{Re} \left\{ \right. \Delta_{\mu P}^{(1)} \Delta_{\mu P}^{(3)} Q_{\mu \mu}^{(1)} Q_{\mu \mu}^{(3)} + \Delta_{\mu P}^{(1)} \Delta_{\mu P}^{(4)} Q_{\mu \mu}^{(1)} Q_{\mu \mu}^{(4)} + \Delta_{\mu P}^{(2)} \Delta_{\mu P}^{(3)} Q_{\mu \mu}^{(2)} Q_{\mu \mu}^{(3)} + \Delta_{\mu P}^{(2)} \Delta_{\mu P}^{(4)} Q_{\mu \mu}^{(2)} Q_{\mu \mu}^{(4)} + \Delta_{\mu P}^{(3)} \Delta_{\mu P}^{(4)} Q_{\mu \mu}^{(3)} Q_{\mu \mu}^{(4)} \left. \right\} \hspace{1cm} (B.26)

It is convenient to define the following purely angular-dependent functions

\[ \Delta_{\mu}^{(1)} = \sum_P \left| \Delta_{\mu P}^{(1)} \right|^2 \] \hspace{1cm} (B.27)

\[ \Delta_{\mu}^{(2)} = \sum_P \left| \Delta_{\mu P}^{(2)} \right|^2 \] \hspace{1cm} (B.28)

\[ \Delta_{\mu}^{(1,2)} = \sum_P \Delta_{\mu P}^{(1)} \Delta_{\mu P}^{(2)*} \] \hspace{1cm} (B.29)
\[ \Delta^{(1,2)} = \sum_{\mu} \left| D_{\mu \mu}^{(1)} \right|^2 \] (B.30)

\[ \Delta^{(2,2)} = \sum_{\mu} D_{\mu \mu}^{(1)} D_{\mu \mu}^{(2)*} \] (B.31)

It is possible to evaluate these $\Delta$'s without explicitly taking elements of the rotation matrices, $D_{\mu \mu}^{(x)}$, by using the Clebsch-Gordan series (Rose, 1957),

\[ D_{\mu_1 \mu_2}^{(j_1)} D_{\mu_3 \mu_2}^{(j_2)} = \sum_{j} C(j_1 j_2 j; \mu_1 \mu_2) C(j_1 j_2 j; \mu_3 \mu_2) D_{\mu_1 \mu_2}^{(j)} \] along with

\[ D_{\mu \mu}^{(x)*} = \left( -1 \right)^{\mu_1} D_{-\mu_1 -\mu}^{(x)} \quad (\mu = \pm 1) \] (B.32)

and

\[ D_{00}^{(x)} (\psi, \theta, \phi) = P_{\alpha} (x), \quad x = \cos \theta \] (B.33)

where $P_{\alpha} (x)$ is a Legendre polynomial, to give

\[ D_{\mu \mu}^{(x)} D_{-\mu_1 -\mu}^{(x)*} = \left( -1 \right)^{\mu_1} \sum_{\alpha} C(\alpha \alpha'; \mu_1 \mu) C(\alpha \alpha'; \mu_2 \mu) P_{\alpha} (x) \] (B.34)

The evaluation of the $\Delta$'s is now straightforward and gives

\[ \Delta^{(1)} = \begin{cases} \frac{2}{3} (1 - P_2 (x)) & \mu = 0 \\ \frac{2}{3} (1 + \frac{1}{2} P_2 (x)) & \mu = \pm 1 \end{cases} \] (B.35)

\[ \Delta^{(2)} = \begin{cases} \frac{2}{35} (7 + 5 P_2 (x) - 12 P_4 (x)) & \mu = 0 \\ \frac{1}{35} (14 + 5 P_2 (x) + 16 P_4 (x)) & \mu = \pm 1 \\ \frac{2}{35} (7 - 5 P_2 (x) - 2 P_4 (x)) & \mu = \pm 2 \end{cases} \] (B.36)

\[ \Delta^{(1,2)} = \begin{cases} \frac{2}{35} (P_1 (x) - P_3 (x)) & \mu = 0 \\ \frac{1}{5} (3 P_1 (x) + 2 P_3 (x)) & \mu = \pm 1 \end{cases} \] (B.37)
\[ \Delta^{(1,3)} = \mu \mathcal{P}_1(x), \quad \mu = 0, \pm 1 \]  
\[ \Delta^{(2,3)} = \mu \mathcal{P}_2(x), \quad \mu = 0, \pm 1 \]  

The \( Q \)'s (B.1-4) are now expanded in terms of the \( q \)'s (B.21-24) and inserted in (B.26), giving terms of the general type

\[ Q^{(k)}_{mm} Q^{(k')*}_{mm} = \sum_{\lambda, L_S \lambda', L_{S'}} \langle \lambda \lambda' \rangle \langle \lambda' \lambda \rangle \langle L_S \lambda' \rangle \langle L_{S'} \lambda \rangle \delta_{\lambda \lambda'} \]  

where several of these sums drop out due to the Kronecker deltas in (B.21-24). Furthermore, waves of different channel spin in the continuum are assumed to be combined incoherently, thus terms in (B.41) will be zero unless \( \lambda = \lambda' \).

With the definition

\[ \mathcal{J}^{(k,k')}_{\mu}(\lambda \lambda'; L_S L_{S'}) = \sum_{m} \langle \lambda \lambda' \rangle \langle \lambda' \lambda \rangle \langle L_S \lambda' \rangle \langle L_{S'} \lambda \rangle \delta_{\lambda \lambda'} \]  

representing interference between transitions (which may be the same) \( (\lambda \lambda') \rightarrow (L S) \) of character \( k \) (E1, E2, etc.) and \( (\lambda' \lambda') \rightarrow (L' S') \) of character \( k' \), we have for the cross section

\[ \frac{1}{W} \frac{d\sigma}{d\Omega} = \sum_{\lambda \lambda'} \sum_{\mu, L_S L_{S'}} \left\{ \Delta^{(1)}_{\mu} \mathcal{J}^{(1)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(2)}_{\mu} \mathcal{J}^{(2)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(3)}_{\mu} \mathcal{J}^{(3)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(4)}_{\mu} \mathcal{J}^{(4)}_{\mu}(\lambda \lambda'; L_S L_{S'}) \right\} + 2 \operatorname{Re} \left\{ \Delta^{(1)}_{\mu} \mathcal{J}^{(2)}_{\mu}(\lambda \lambda'; L_S L_{S'}) \right\} + \Delta^{(1)}_{\mu} \mathcal{J}^{(3)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(1)}_{\mu} \mathcal{J}^{(4)}_{\mu}(\lambda \lambda'; L_S L_{S'}) \]

+ \Delta^{(2)}_{\mu} \mathcal{J}^{(3)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(2)}_{\mu} \mathcal{J}^{(4)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(3)}_{\mu} \mathcal{J}^{(1)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(3)}_{\mu} \mathcal{J}^{(2)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(3)}_{\mu} \mathcal{J}^{(4)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(4)}_{\mu} \mathcal{J}^{(1)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(4)}_{\mu} \mathcal{J}^{(2)}_{\mu}(\lambda \lambda'; L_S L_{S'}) + \Delta^{(4)}_{\mu} \mathcal{J}^{(3)}_{\mu}(\lambda \lambda'; L_S L_{S'})
This expression contains many terms, both pure transitions and interferences. Note that all terms above contain interferences, even, for example, the first term which may be expanded as

\[ \sum_{L_S} \sum_{L'_S} \sum_{L} \sum_{L'} D^{(1,1)} \left( \frac{L_S}{L'_S} \right) \left( \frac{L}{L'} \right) \left( L \right) \left( L' \right) \]

with pure El transitions written separately from El/El interferences.

**Particular Transitions**

Several particular transitions will now be considered explicitly. A transition from a continuum state with orbital quantum number \( l \) and channel spin \( \sigma \) to a bound state with orbital quantum number \( L \), channel spin \( S \) and total angular momentum quantum number \( J \) will be denoted

\[ (l \sigma) \rightarrow (LSJ) \]

**Pure Transitions:**

(1) Electric dipole, \((l S) \rightarrow (LSJ)\)

\[ \left( \frac{d \sigma}{d \omega} \right)_{el} = \omega \sum_{\mu} \Delta^{(1,1)} \left( \frac{L_S}{L'_S} \right) \left( \frac{L}{L'} \right) \left( L \right) \left( L' \right) \]

from (B.43)
\[
= w \sum_{\mu} \Delta^{(1)}_{\mu} \sum_{m} |^{(1)}q^{m}_{LS; \mu} |^2
\]
using (B.42)

\[
= 3(2L+1) C^2(L) \sum_{\mu} |^{(1)}J_{LS}^{\mu} |^2
\]

\[
x \sum_{\mu} \Delta^{(1)}_{\mu} C^2(L) \sum_{m} C^2(LS; \mu, m)
\]
substituting (B.21)

\[
= w \sum_{\mu} C^2_a \langle L^S_S \rangle \sum_{l} |^{(1)}I^{LS}_{LS} |^2 \frac{3(2J+1)(L+1)(5L+2+3)}{8(2L+1)^2}
\]

\[
x \left\{ |^{(1)}I^{LS}_{LS} |^2 \left[ 1 + \left( \frac{L - 3L - 1}{5L + 2 + 3} \right) \cos^2 \theta_y \right] \right\}, \quad L = L + 1 \quad (B.45)
\]

upon simplifying, using properties of the Clebsch-Gordan coefficients (Rose, 1957) and substituting (B.27).

(2) Electric quadrupole, \((LS) \rightarrow (LSJ)\)

\[
\frac{d\sigma}{d\Omega}_{E2} = w \sum_{\mu} \Delta^{(2)}_{\mu} \Delta^{(2)}_{\mu} \langle L^S_S^{LS} |_{LS} \rangle
\]
from (B.43)

\[
= w \sum_{\mu} C^2 \langle L^S_S^{LS} |_{LS} \rangle \sum_{m} C^2(LZl; \mu, m) \Delta^{(2)}_{\mu}
\]
again using (B.42), (B.22) and simplifying where possible.
(3) Magnetic dipole "spin-flip", $(L \Delta) \rightarrow (LSJ)$

\[
\left( \frac{d\sigma}{d\Omega} \right)_{M_1} = \mathcal{N} \sum_{\mu} \Delta^{(1)} \sum_{\mu} \Delta^{(3,3)} \left( \frac{L \Delta; L S}{L \Delta; L S} \right)
\]

from (B.43)

\[
= 4\pi \mathcal{N} c_3^2 a_{L}^2 \left( \frac{L \Delta; L S}{L \Delta; L S} \right)^2 (2L+1)(2\Delta+1) M_{\Delta S}
\]

\[
\times \sum_{\mu} \Delta^{(1)} \sum_{\mu} C^2(LSJ; 0, m-\mu) C^2(L\Delta S; -\mu, m)
\]

employing (B.42) and (B.23).

(4) Magnetic dipole "orbital", $(LS) \rightarrow (LSJ)$

\[
\left( \frac{d\sigma}{d\Omega} \right)_{M_1'} = \mathcal{N} \sum_{\mu} \Delta^{(1)} \sum_{\mu} \Delta^{(4,4)} \left( \frac{LS; LS}{LS; LS} \right)
\]

from (B.43)

\[
= 2\pi \mathcal{N} c_3^2 c_4^2 a_{L}^2 \left( \frac{L \Delta; L S}{L \Delta; L S} \right)^2 L(L+1)(2\Sigma+1)(1+\cos^2 \theta_y)
\]

using (B.42), (B.23) and (B.27).

Interferences:

(1) $E1/E1$, $(L S) \rightarrow (LSJ) / (L' S) \rightarrow (LSJ)$, $(L \neq L')$

\[
\left( \frac{d\sigma}{d\Omega} \right)_{E1/E1} = \mathcal{N} \sum_{\mu} \Delta^{(1)} \left( \sum_{\mu} \left( \Delta^{(1,1)} \left( \frac{L \Delta; L S}{L \Delta; L S} \right) + \sum_{\mu} \left( \frac{L' S; L S}{L' S; L S} \right) \right) \right)
\]

from (B.43)
since
\[ \delta^{(\mu)}(l^s; L^s) = \delta^{(\mu)*}(l^s; L^s) \]

= \( w 2 \text{Re} \left\{ \sum \Delta^{(\mu)} \delta^{(\mu)}(l^s; L^s) \right\} \)

using (B.42)

\[ = w c_1^2 a_\ell^2 L^S L_S (l^s; L^s) \cos (\varphi^S - \varphi^S_{2}) \]

\[ \times \frac{3l(l+1)(2j+1)}{(2l+1)^2} \left( 3 \cos^2 \theta - 1 \right) \]  (B.49)
simplifying, once (B.21) is employed, and using (B.27).

(2) \( E_2/E_2, (l^s) \rightarrow (l^s) \rightarrow (l^s) \rightarrow (L^s) \)

\[ \left( \frac{\partial S}{\partial \lambda} \right)_{E_2/E_2} = w \sum \Delta^{(2)} \left( \delta^{(2,2)}(l^s; L^s) + \delta^{(2,1)}(l^s; L^s) \right) \]

from (B.43)

= \( w 2 \text{Re} \left\{ \sum \Delta^{(2)} \sum (2) q^{m_m}(l^s; L^s) q^{m_m*}(l^s; L^s) \right\} \)

as above

= \( w 10 (2j+1) c_1^2 a_\ell^2 \sum L^S L_S (l^s; L^s) \)

\[ \times C(l^s l^s; 00) C(l^s l^s; 00) \cos (\varphi^S - \varphi^S_{2} + (l-l')\frac{\pi}{2}) \]

\[ \times \sum \Delta^{(2)} C(l^s l^s; -\mu, \mu) C(l^s l^s; -\mu, \mu) \]  (B.50)
simplifying as above using (B.22).
(3) $E1/E2, (L'S) \rightarrow (LS) / (L'S) \rightarrow (LS)$

$$
\left( \frac{d\sigma}{dt} \right)_{E1/E2} = 2 \mathfrak{w} \ Re \left\{ \Delta^{(1,2)} \ \mathcal{J}^{(1,2)} (L'S; L'S) \right\}
$$

- from (B.43)

$$
= \mathfrak{w} \ 2 \ Re \left\{ \sum_{m} \Delta^{(1,2)} \ \sum_{m} \mathcal{J}^{(1,2)} (L'S; L'S) \right\}
$$

using (B.42)

$$
= \mathfrak{w} C_1 C_2 \ a L^2 \ \left( ^{(1)} I_{L'} L_S \right) \left( ^{(1)} I_{L'} L_S \right) \cos (\phi^S - \phi^S')
$$

$$
\times (-)^{\frac{j + l' + 1}{2}} 2 \sqrt{\mathfrak{w} (2J + 1) C (L1L; 00) C (L2L'; 00)}
$$

$$
\times \sum_{m} \Delta^{(1,2)} C (L1L; -\mu, \mu) C (L2L'; -\mu, \mu)
$$

(8.51)

using (B.21) and (B.22) and simplifying.

(4) $E1/M1, (L'S) \rightarrow (LS) / (L'S) \rightarrow (L'S'J)$

$$
\left( \frac{d\sigma}{dt} \right)_{E1/M1} = \mathfrak{w} \ 2 \ Re \left\{ \sum_{m} \Delta^{(1,3)} \ \mathcal{J}^{(1,3)} (L'S; L'S') \right\}
$$

- from (B.43)

$$
= 4 \sqrt{3} \pi (2L + 1)(2L' + 1)(2S + 1) \ \mathcal{C} (L1L; 00) \ \mathcal{M}_{SS'}
$$

$$
\times Re \left\{ \left( ^{(1)} J_{L'S} \right) \left( ^{(1)} J_{L'S'} \right) \sum_{m} \Delta^{(1,3)} C (L1L; -\mu, \mu) \ (-)^{\mu}
$$

$$
\times \sum_{m} \mathcal{C} (LSJ; -\mu, m) \mathcal{C} (L'S'J; 0, m - \mu) \mathcal{C} (1SS'; -\mu, m)
$$

using (B.42), (B.21) and (B.23)
\[
\begin{align*}
&\sum_{\mu} \Delta^{(1,3)}_{\mu} C(L11; \mu, \mu) C(L1L'; \mu, \mu) \\
&= P_1(x) \left\{ C(L11; -1, 1) C(L1L'; -1, 1) \\
&\quad - C(L11; 1, -1) C(L1L'; 1, -1) \right\} \\
&= P_1(x) C(L11; -1, 1) C(L1L'; -1, 1) \\
&\quad \left( 1 - (-)^{L+1-L'} (-)^{L+1-L'} \right) \\
&= P_1(x) C(L11; -1, 1) C(L1L'; -1, 1) \\
&\quad \left( 1 + (-)^{L-L'} \right) \\
&= 0 \quad \text{unless } L = L', \text{ using the fact that} \\
&L - L \text{ is odd (because of } C(L11; 00)). \text{ In addition, the triangle relation} \\
&\Delta(L1L') \\
&\text{must be satisfied by those arguments of the Racah coefficient}
\end{align*}
\]
(Rose, 1957). However, if $L$ or $L'$ is 0, then the other must be 1 to satisfy the triangle relation; but then $L \neq L'$ and there can be no interference. The result of all this is that if either final state (for the $E_1$ or $M_1$ transition) is an $S$-state, then there can be no $E_1/M_1$ interference.

Finally

$$
\left( \frac{d\sigma}{d\Omega} \right)_{E_1/M_1} = \sum L L' \sum a_L a_{L'} \left( \frac{d}{d\omega} \right)_{L L'} ^{L L'} \times \frac{\sqrt{3} \pi (2S+1)(2S'+1) (2J+1) (L+L'+1)}{\sqrt{L(L+1)(2L+1)}} \times \frac{(L(L+1) - L(L+1) + 2) M_{SS'} P_1 (\cos \theta_y)}{(L'(L'+1) - L'(L'+1) + 2) M_{SS'} P_1 (\cos \theta_y)}
$$

subject to $L = L' \neq 0$ and $L = L \pm 1$.

(5) $E_2/M_1$, $(LS) \rightarrow (LSJ)$ / $(L'S) \rightarrow (L'S')$

This case is treated in a manner completely analogous to the preceding one

$$
\left( \frac{d\sigma}{d\Omega} \right)_{E_2/M_1} = \sum L L' \sum a_L a_{L'} \left( \frac{d}{d\omega} \right)_{L L'} ^{L L'} \times \frac{\sqrt{3} \pi (2S+1)(2S'+1) (2J+1) (L+L'+1)}{\sqrt{L(L+1)(2L+1)}} \times \frac{(L(L+1) - L(L+1) + 2) M_{SS'} P_1 (\cos \theta_y)}{(L'(L'+1) - L'(L'+1) + 2) M_{SS'} P_1 (\cos \theta_y)}
$$

from (E.43)
\[
\begin{align*}
\sigma_{E1} &= \frac{4\pi}{2L+1} c_1^2 a_s^2 \left( \langle^{1s} I_{Ls}^J \rangle \right)^2 \\
\sigma_{E2} &= 8\pi (2J+1) c_2^2 (L2L;00) c_s^2 a_s^2 \left( \langle^{1s} I_{Ls}^J \rangle \right)^2
\end{align*}
\]
(3) Magnetic dipole "spin-flip" $(L \Delta) \rightarrow (LSJ)$

\[ \sigma_{M1} = \frac{32}{3} \pi^2 (2L+1)(2J+1) \omega_c^2 \alpha_L^2 (\langle 3 \rangle I_{LS}^L)^2 M_{\Delta S}^2 \] (8.56)

(4) Magnetic dipole "orbital" $(LS) \rightarrow (LSJ)$

\[ \sigma_{M1'} = \frac{32}{3} \pi^2 L(L+1)(2J+1) \omega_c^2 c_4^2 \alpha_L^2 (\langle 3 \rangle I_{LS}^L)^2 \] (8.57)
APPENDIX C

DIRECT CAPTURE INVOLVING
PARTICLES OF SPINS 0 AND 1/2
WITH SPIN-ORBIT POTENTIALS

In this case the initial and final states have fixed channel spin \( \Delta = S = 1/2 \), but no longer are degenerate for fixed \( l \) and different \( j \) (\( j = l \pm 1/2 \)). Transitions from a continuum state (A.10) with orbital angular momentum quantum number \( l \), total angular momentum quantum number \( j \) and magnetic quantum number \( m \) to a bound state (A.11) with corresponding quantum numbers \( L \), \( J \) and \( M \) are considered. These transitions proceed with the emission of a gamma-ray of multipolarity \( \mathcal{L} \), parity \( \Pi \) and magnetic quantum number \( \mu \), i.e. the gamma-ray has character \( k \) (E1, E2 or M1) and \( \mu \). Such a transition is shown schematically in Fig. C.1.
The interaction Hamiltonian is given by (A.32), where

\[ q \hat{S}_z + q \hat{S}_y = 2 \mu, \hat{S} \]

with \( \hat{S} = 1/2 \hat{S} \) being the spin operator for spin 1/2 and with \( \mu_3 \) being the magnetic moment (in nuclear magnetons) for the spin-1/2 particle. The direct capture cross sections can then be obtained using (A.34).

The development of this appendix proceeds along the same lines as did Appendix B and consequently will be somewhat abbreviated.

Matrix Elements

The matrix elements to be evaluated were defined in (B.1-4). As in Appendix B these matrix elements are considered individually.

(1) Electric dipole

\[ Q^{(1)}_{\lambda m m} = \langle \Phi_m | i c_i \mathcal{O}_{E1} Y_{\lambda i}^{m*} | \Psi_m \rangle \]
\[ = \frac{\mu_{lJ}}{r} \sum_{\beta} C(L + J; M - \beta, \beta) Y_{LM - \beta} \chi_{l\beta}^{\beta} \]

\[ = \iota c_{l} \Theta_{El} Y_{i}^{m*} \sum_{L} \sum_{j} \sqrt{4\pi (2L + 1)} i^{L+1} e^{i\phi_{l}j} \]

\[ \times \frac{R_{lj}}{kr} Y_{lm - \beta} \chi_{l\beta}^{\beta} C(l + j; 0, m) C(l + j; m - \alpha, \alpha) \]

\[ = \sum_{L} \sum_{j} \sum_{\beta} \sqrt{4\pi (2L + 1)} C(L + j; M - \beta, \beta) \]

\[ \times C(l + j; 0, m) C(l + j; m - \alpha, \alpha) \chi_{l\beta}^{\beta} \chi_{l\beta}^{\beta} \]

\[ \times \int_{0}^{r} dr \frac{u_{lJ}^{*}}{r} \Theta_{El} R_{lj} \int_{0}^{2\pi} Y_{lm - \beta}^{*} Y_{i}^{m*} Y_{lm - \alpha} \]

and letting

\[ I_{lj}^{(l)} L'J' = \int_{0}^{r} dr \frac{u_{lJ}^{*}}{r} \Theta_{El} R_{lj} \]

\[ J_{lj}^{(l)} L'J' = c_{l} i^{L+1} e^{i\phi_{l}j} I_{lj}^{(l)} L'J' \]

using Gaunt's formula and relationships for Racah coefficients (Rose, 1957)

\[ Q_{\mu m}^{(l)} = \sum_{l} \sum_{j} (-)^{l-M-\frac{1}{2}} \sqrt{(2l+1)(2j+1)(2l+1)(2j+1)} \]

\[ \times C(l - l; 0, 0) C(l + j; 0, m) C(j + j; m - m, m) \]

\[ \times \delta_{m, m - m} W(lj; L'J'; \frac{1}{2}) J_{lj}^{(l)} L'J' \]

where the M has been suppressed.
(2) Electric quadrupole

Similarly, letting

\[ I_{ij;j, LJ}^{(2)} = \int_0^\infty r^2 dr \frac{u_{ij}^0}{r} O_{E2} \frac{R_{ij}}{kr} \]  

and \[ J_{ij;j, LJ}^{(2)} = c_2 i^l e^{i\varphi_{ij}} I_{ij;j, LJ}^{(2)} \]  

we have

\[ Q_{\mu\nu}^{(2)} = \langle \Phi_m | c_2 O_{E2} Y_{\mu\nu} | \Phi_m \rangle \]

\[ = \sum_{\ell,j} (-)^{L-M+j} \delta_{\mu,\ell-M} \frac{\sqrt{(2\ell+1)(2j+1)(2\ell+1)(2j+1)}}{\ell} \]

\[ \times C(\ell j; \ell j | \ell j; \ell j) C(\ell j; \ell j | \ell j; \ell j) W(\ell j; \ell j | \ell j; \ell j) \]  

(3) Magnetic dipole, "spin-flip"

Letting

\[ I_{ij;j, LJ}^{(3)} = \int_0^\infty r^2 dr \frac{u_{ij}^0}{r} O_{M1} \frac{R_{ij}}{kr} \]  

and \[ J_{ij;j, LJ}^{(3)} = c_3 i^l e^{i\varphi_{ij}} I_{ij;j, LJ}^{(3)} \]  

and evaluating the matrix element of the spin operator between spin functions as in Appendix B (B.16-18), we obtain

\[ Q_{\mu\nu}^{(3)} = \langle \Phi_m | (-)^M c_3 O_{M1} \gamma_{\mu} \gamma_{\nu} | \Phi_m \rangle \]

\[ = - \sum_{j} 2 \frac{\sqrt{6\pi (2\ell+1)(2j+1)}}{\ell} \delta_{\mu,\ell-M} \gamma_{\nu} \gamma_{\mu} \]

\[ \times C(\ell j; \ell j | \ell j; \ell j) W(\ell j; \ell j | \ell j; \ell j) J_{ij;j, LJ}^{(3)} \]  

(c.9)
(4) Magnetic dipole, "orbital"

Again, evaluating the matrix element of the orbital angular momentum operator between spherical harmonics, we obtain

\[
Q_{\mu m}^{(4)} = \langle \Phi_m | (-)^m c_3 \Theta_{M1} c_4 L_{-\mu} | \Psi_m \rangle
\]

\[
= \sum_{l} c_4 (-)^{J-j} \sqrt{4\pi l(l+1)(2J+1)} (2l+1)
\]

\[
\times C(l \frac{1}{2} J_j; m_0) C(1 J_j; \mu, m-m) W(l l J_j, \frac{1}{2}; 1 \frac{1}{2}) J^{(3)}_{L_j}, L J
\]

Adopting the notation

\[
q_{l j; L J}^{m m} = (-)^{L-M} \sqrt{(2l+1)(2J+1)(2l+1)(2J+1)} C(l+l; 00)
\]

\[
\times C(l \frac{1}{2} j; 0 m) C(1 j; \mu, m-m, m) W(l J l J; \frac{1}{2} 1) J^{(1)}_{L_j}, L J \delta\mu, m-M
\]

\[
q_{l j; L J}^{m m} = (-)^{L-M} \frac{1}{2} \sqrt{(2l+1)(2J+1)(2l+1)(2J+1)} C(l+2 l; 00)
\]

\[
\times C(l \frac{1}{2} j; 0 m) C(1 j; \mu, m-m, m) W(l J l J; \frac{1}{2} \frac{1}{2}) J^{(2)}_{L_j}, L J \delta\mu, m-M
\]

\[
q_{l j; L J}^{m m} = -2 \sqrt{6\pi (2l+1)(2J+1)} m_0 C(l \frac{1}{2} l j; m_0)
\]

\[
\times C(1 J_j; m, m-m) W(l \frac{1}{2} j J; 1 L) J^{(3)}_{L_j}, L J \delta\mu, m-M \delta L L
\]

\[
q_{l j; L J}^{m m} = (-)^{J-j} \sqrt{4\pi l(l+1)(2J+1)} (2l+1) c_4 C(l \frac{1}{2} j j; m_0)
\]

\[
\times C(1 J_j; m, m-m) W(l l J j; \frac{1}{2}) J^{(3)}_{L_j}, L J \delta\mu, m-m \delta L L
\]

\[
(C.11-14)
\]
then the above matrix elements can be written collectively as

\[ Q^{(k)}_{lm} = \sum_{l} \sum_{j} (k) q_{l,j;L,j}^{lm} \text{, } k = 1, 4 \]  
\[ \text{(C.15)} \]

Cross Sections

Letting

\[ J^{(k,k')}_{m} (l_j; l'_j) = \sum_{m} (k) q_{l,j;L,j}^{mm} (k') q_{l'_j;L,j}^{mm} \ast \]  
\[ \text{(C.16)} \]

and using (C.15) and (B.14) with expressions (A.32-34), the direct capture cross section can be written

\[ \frac{1}{W} \frac{d\Sigma}{d\Omega} = \sum_{l} \sum_{j} \sum_{l'} \sum_{j'} \sum_{m} \left\{ \Delta_{m}^{('')} D_{m}^{('')}(l_j; l'_j) + \Delta_{m}^{(2,2)} D_{m}^{(2,2)}(l_j; l'_j) + \Delta_{m}^{(1,1)} D_{m}^{(1,1)}(l_j; l'_j) + 2 \text{Re} \{ \Delta_{m}^{(1,2)} D_{m}^{(1,2)}(l_j; l'_j) \} + \Delta_{m}^{(1,3)} D_{m}^{(1,3)}(l_j; l'_j) + \Delta_{m}^{(1,4)} D_{m}^{(1,4)}(l_j; l'_j) + \Delta_{m}^{(2,1)} D_{m}^{(2,1)}(l_j; l'_j) + \Delta_{m}^{(2,2)} D_{m}^{(2,2)}(l_j; l'_j) + \Delta_{m}^{(2,3)} D_{m}^{(2,3)}(l_j; l'_j) + \Delta_{m}^{(2,4)} D_{m}^{(2,4)}(l_j; l'_j) + \Delta_{m}^{(3,1)} D_{m}^{(3,1)}(l_j; l'_j) + \Delta_{m}^{(3,2)} D_{m}^{(3,2)}(l_j; l'_j) + \Delta_{m}^{(3,3)} D_{m}^{(3,3)}(l_j; l'_j) + \Delta_{m}^{(3,4)} D_{m}^{(3,4)}(l_j; l'_j) \} \]  
\[ \text{(C.17)} \]
Particular Transitions

As in Appendix B several particular transitions will now be considered explicitly. The notation

\[(\ell J) \rightarrow (L J)\]

is used to denote a transition between a continuum state with quantum numbers \(\ell\) and \(J\) and a bound state with corresponding quantum numbers \(L\) and \(J\).

Pure Transitions:

1. Electric dipole, \((\ell J) \rightarrow (L J)\)

\[
\left(\frac{d\sigma}{d\Omega}\right)_{E1} = \pi \sum_{\mu} \Delta_{\mu}^{(\ell J)} \, S^{(\mu')}_{\mu}(\ell J; l J) \\
\text{from (C.17)}
\]

\[
= \pi \sum_{\mu} \Delta_{\mu}^{(\ell J)} \sum_{\mu} \left| \left< \mu' | \frac{1}{\sqrt{2L+1}} \right| \ell J \right|^2 \\
\text{using (C.16)}
\]

\[
= \pi \sum_{\mu} \Delta_{\mu}^{(\ell J)} \sum_{\mu} \left| (-)^{L-M-\frac{1}{2}} \sqrt{(2L+1)(2J+1)(2\ell+1)(2\mu+1)} \right|
\times C(L1l; 00) C(l\ell j; 0m) C(Jj1; \mu-m,m) W(ljLJ;\frac{1}{2}1) \\
\times \overline{J_{\ell j}}(LJ;LJ)^2 \\
\text{using (C.11)}
\]

\[
= \frac{3}{2} (\ell + L+1)(2\ell +1)^2 \pi C_1^2 \left( I_{\ell j}^{(\ell J)} \right)^2 W(ljLJ;\frac{1}{2}1) \\
\times \sum_{\mu} \Delta_{\mu}^{(\ell J)} C^2(j1J; \frac{1}{2},-\mu), \quad \ell = 0 \pm 1 \quad (C.18)
\]
using relations for the Clebsch-Gordan coefficients
(Rose, 1957)

(2) Electric quadrupole, \( (l_j) \rightarrow (L J) \)

Similarly, using (C.17), (C.16) and (C.12) as for the El case,

\[
\left( \frac{\partial \sigma}{\partial \epsilon} \right)_{E2} = W \sum_{\mu} \Delta^{(2)}_\mu \Delta^{(2,2)}_\mu (l_j; l_j)
\]

\[
= 5(2L+1)(2j+1)^2 \left( \begin{array}{c}
I_{(j)}^{(2)}; \mu
\end{array} \right) \sum_{\mu} \Delta^{(2)}_\mu \left( \begin{array}{c}
\frac{1}{2}, \frac{1}{2}
\end{array} \right) W^2 (l_j; L J; \frac{1}{2})
\]

\[
\times C^2 (L2 J; 00) \sum_{\mu} \Delta^{(2)}_\mu C^2 (j2 J; \frac{1}{2}, -\mu) \quad (C.19)
\]

(3) Magnetic dipole "spin-flip", \( (L j) \rightarrow (L J) \)

\[
\left( \frac{\partial \sigma}{\partial \epsilon} \right)_{M1} = W \sum_{\mu} \Delta^{(1)}_\mu \Delta^{(3,3)}_\mu (L j; L j)
\]

\[
= 24 \pi (2j+1)^2 \left( \begin{array}{c}
J_{(j)}^{(3)}; \mu
\end{array} \right) \sum_{\mu} \Delta^{(1)}_\mu C^2 (j1 J; \frac{1}{2}, -\mu)
\]

\[
\times \sum_{\mu} \Delta^{(3,3)}_\mu C^2 (L1 J; \frac{1}{2}, L L) \quad (C.20)
\]

where (C.17), (C.16) and (C.13) have been used.

(4) Magnetic dipole "orbital", \( (L j) \rightarrow (L J) \)

\[
\left( \frac{\partial \sigma}{\partial \epsilon} \right)_{M1'} = W \sum_{\mu} \Delta^{(1)}_\mu \Delta^{(4,4)}_\mu (L j; L j)
\]
where (C.17), (C.16) and (C.14) have been used.

Total Cross Sections:

The total cross sections are again obtained by integrating the above over solid angles.

(1) Electric dipole, \((\ell j) \rightarrow (L J)\)

\[
\sigma_{E1} = 4\pi (2j+1)(2\ell+1)(\ell+1) W_c^2 W^2 (L J, \ell; \frac{1}{2}, 0) (I_{\ell j, \ell J}^{(1)})^2
\]

\[
\ell = L \pm 1
\]

(2) Electric quadrupole, \((\ell j) \rightarrow (L J)\)

\[
\sigma_{E2} = 8\pi (2L+1)(2j+1)(2\ell+1) W_c^2 W^2 (L J, \ell; \frac{1}{2}, 2) \times C^2 (L, 2, 0, 0) (I_{\ell j, \ell J}^{(2)})^2
\]

(3) Magnetic dipole "spin-flip", \((L j) \rightarrow (L J)\)

\[
\sigma_{M1} = 64\pi^2 (2j+1)(2\ell+1) W_c^2 M_s^2 W^2 (\ell \frac{1}{2}, j J, 1 L) (I_{\ell j, \ell J}^{(3)})^2
\]

(4) Magnetic dipole "orbital", \((L j) \rightarrow (L J)\)

\[
\sigma_{M1'} = \frac{32}{3} \pi^2 L (L+1)(2L+1)(2j+1)(2\ell+1) W_c^2 C_4^2
\]

\[
x W^2 (L L j J, \ell; \frac{1}{2}) (I_{\ell j, \ell J}^{(3)})^2
\]
Reduction in case of no spin-orbit force:

In this case \( I_{l,j;L}^{(k)} \) is independent of \( j, J \) and may be written \( I_{l,j;L}^{(k)} \). Summing the above cross sections over \( j \), using

\[
\sum_e (2e+1)(2f+1) W(abcd;ef) W(abcd;eg) = 8S_0
\]

produces

\[
\sigma_{E1} = \frac{4\pi (2J+1)(l+L+1)}{(2l+1)} \ \ & \ \text{c}^2 (I_{l,j;L}^{(1)})^2 \quad (c.27) \\
\sigma_{E2} = 8\pi (2J+1) c^2 (L2l;00) \ \ & \ \text{c}^2 (I_{l,j;L}^{(2)})^2 \quad (c.28) \\
\sigma_{M1} = 32\pi^2 (2J+1) \ \ & \ \text{c}^2 \left[ \begin{array}{c} \text{c}^2 \left( I_{l,j;L}^{(3)} \right)^2 \\
\mu_s^2
\end{array} \right] \quad (c.29) \\
\sigma_{M1}' = \frac{32}{3} \pi^2 (l+1)(2J+1) \ \ & \ \text{c}^2 \left( I_{l,j;L}^{(3)} \right)^2 \quad (c.30)
\]

which agree with the corresponding formulae in Appendix B with

\( I_1 = 1/2, I_2 = 0, g_1 = 2\mu_s \) and \( \mu = S = 1/2 \).

Interferences: The following interferences are considered, making use of (C.17), (C.16) and (C.11-14).

(1) \( E1/E1, \ (l,j) \rightarrow (L,j) \ / \ (l',j') \rightarrow (L,j) \)

\[
\left( \frac{d\sigma}{d\Omega} \right)_{E1/E1} = \mathcal{W} \sum_{l,j} \Delta_{l,j}^{(ii)} \Delta_{l,j}^{(i')i'} (l,j; l',j')
\]
Using properties of the Clebsch-Gordan coefficients, the cross section may be written

\[
\frac{d\sigma}{d\Omega} \mid_{\epsilon_1/\epsilon_1} = \sqrt{2} \sum_{M_M} \Delta(M,M)(L+1, j, L-1, j') \cos(\varphi_{L+1,j} - \varphi_{L-1,j'})
\]

\[
= \sqrt{2} \sum_{M_M} \Delta(M,M)(L+1, j, L-1, j') \cos(\varphi_{L+1,j} - \varphi_{L-1,j'})
\]

\[
\times W(L+1, j, L, \frac{1}{2}) W(L-1, j, L, \frac{1}{2}) \sum_{M_M} \Delta(M,M)(L+1, j, L-1, j') \cos(\varphi_{L+1,j} - \varphi_{L-1,j'})
\]

\[
\times C(L-1, j', 0, m) C(J, j', m, m) C(J, j', m, m)
\]
(2) $E_1/E_2, \quad (Lj) \rightarrow (L' j') \rightarrow (L J)$

$$
\left( \frac{d\sigma}{d\Omega} \right)_{E_1/E_2} = \mathcal{W}^2 \text{Re} \left\{ \sum_{\mu} \Delta^{(1,2)}_{\mu} \Delta^{(1,2)}_{\mu} (Lj, L'j') \right\} \\
= (-) \frac{l-l' \pm 1}{2} \mathcal{W} c_1 c_2 \left[ I^{(1)}_{L_j, L_j} I^{(2)}_{L_j', L_j'} \right] \frac{L \cos (\varphi_{Lj} - \varphi_{L'j'})}{L(L+1) \sqrt{(2L+1)(2L'+1)(2j+1)(2j'+1)}} \\
\times C(L, L' ; 00) C(L, L' ; 00) W(Lj, Lj, \frac{1}{2}) W(L'j', Lj, \frac{1}{2}) \\
\times \sum_{\mu} \Delta^{(1,2)}_{\mu} \sum_{\mu} C(L\frac{1}{2} j; 0 m) C(L\frac{1}{2} j'; 0 m) \\
\times C(JJj; \mu-m, m) C(Jj'; 2; \mu-m, m) \\
\tag{C.33}
$$

(3) $E_1/M_1 "$spin-flip") $, \quad (Lj) \rightarrow (L J) \rightarrow (L J)$

$$
\left( \frac{d\sigma}{d\Omega} \right)_{E_1/M_1} = \mathcal{W}^2 \text{Re} \left\{ \sum_{\mu} \Delta^{(1,3)}_{\mu} \Delta^{(1,3)}_{\mu} (Lj, Lj') \right\} \\
= (-)^{j-j'} (-) \frac{l-l' \pm 1}{2} \mathcal{W} c_1 c_2 \left[ I^{(1)}_{L_j, L_j} I^{(3)}_{L_j, L_j'} \right] \frac{L \cos (\varphi_{Lj} - \varphi_{Lj'})}{L(L+1) \sqrt{(2L+1)(2L+1)(2j+1)}} \frac{2\pi}{(2L+1)} \\
\times C(L, L ; 00) W(Lj, Lj, \frac{1}{2}) W(Lj', Lj', \frac{1}{2}) \\
\times \sum_{\mu} \Delta^{(1,3)}_{\mu} \sum_{\mu} C(L\frac{1}{2} j; 0 m) C(L\frac{1}{2} j'; 0 m) \\
\times C(JJj; \mu-m, m) C(Jj'; 2; \mu-m, m) \\
\tag{C.34}
$$
(4) El/Ml ("orbital"), \((l_j) \rightarrow (L J) / (l_j') \rightarrow (L J)\)

\[
\left(\frac{d\sigma}{d\Omega}\right)_{\text{El/Ml}} = 2 \Re \sum_{\mu} \Delta_{\mu}^{(1,3)} \delta_{\mu}^{(1,4)} (l_j, l_j')
\]

\[
= (-)^{\frac{3L+l+1}{2}} \frac{4}{3} \pi c_l c_3 c_4 I_{l_1}^{(1)} I_{j_1}^{(3)} I_{l_1'}^{(3)} I_{j_1'}^{(3)} \cos(\varphi_{l_1} - \varphi_{l_1'})
\]

\[
\times \sqrt{\frac{3}{2}} \pi L(L+1)(2L+1)(2l+1)(2l'q+1)(2L+1)(2L+1)(2L+1)(2L+1)
\]

\[
\times C(l_1l_1; 00) W(l_j L J; \frac{1}{2} 1) W(l_j' L J; \frac{1}{2} 1)
\]

\[
\times \sum_{\mu} \Delta_{\mu}^{(1,3)} \sum_{\mu} C(l_1 l_1 j_1; 0 m) C(l_1 l_1 j_1'; 0 m)
\]

\[
\times C(J j_1; \mu-m, m) C(J j_1'; \mu-m, m)
\]

(C.35)

Following these procedures, other interferences can likewise be considered in detail.