STATIC SOLUTIONS OF THE COMBINED DIRAC-ELECTROMAGNETIC-GRAVITATIONAL FIELD EQUATIONS

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ABSTRACT

It is assumed that charged, spin-$\frac{1}{2}$, matter distributions can be described in terms of a Dirac spinor field interacting with the electromagnetic field and a scalar gravitational field. The field equations and the energy-momentum tensor are found from an action principle. The fields are not quantized. The field equations are examined and various limiting forms discussed. This thesis deals particularly with the time-independent spherically-symmetric case. Solutions are found for the exterior region of a charged gravitating sphere. The behaviour of these solutions depend on the value of the charge-mass ratio. When this ratio has the value $(4\pi G)^{\frac{1}{2}}$, where G is the gravitational constant, the entire system can be solved analytically. The ensuing solution, called the Weyl-Majumdar solution, is obtained and discussed. When the charge-mass ratio is smaller than $(4\pi G)^{\frac{1}{2}}$, normalised solutions are found which yield electrostatic and gravitational potentials singular at the origin. The matter density is well-behaved everywhere. Normalised solutions were not found for larger charge-mass ratios. The significance of the solutions, and the accuracy of the numerical technique are discussed. Alternative Lagrangian densities are considered which may yield non-singular solutions.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Our purpose in this thesis is to find static, spherically symmetric solutions of the combined gravitational, electromagnetic and Dirac field equations. In other words we are trying to construct a classical model of an elementary charged particle. We consider only unquantized fields, which means that the solutions cannot be expected to correspond to real physical objects. However, it seems sensible to investigate the classical problem ('Classical' in the sense that we do not consider pair creation) before attempting the much more difficult quantum one. If the classical problem has solutions, they may help us in the other case. There is also the hope, as suggested by Dirac (1951), that the classical solutions, if they exist, may give the correct value for e/m, where e is the charge and m the mass of the electron. (Of course, we would not expect to obtain the values of e and m separately. Dirac believed the value of the electronic charge e to be a purely quantum phenomenon and not derivable in a classical theory.)

The concept of a body in classical field theory is generally treated in one of the following two ways. Either the body, or "particle", is considered as a singularity in the otherwise singularity-free field, or else it is assumed to be a mass of fluid obeying some more or less arbitrary equation of state. Both treatments are obviously unsatisfactory. In the first case we forego the possibility of saying much about the internal structure of the particle. In the second case, we are permitted to have gravitational and other fields which are regular everywhere, which is a great advantage, but, unfortunately, other problems now arise. The equation of state is arbitrary, and the very concept of fluid, which is borrowed from macroscopic physics, probably has no place in the microscopic domain.

There exists, however, a third possibility. We can try to construct solutions of the field equations themselves which are localized in space, are regular everywhere, and which represent concentrations of mass, and possibly, charge. Such solutions were first studied in detail by Wheeler and his co-workers (Wheeler 1955, 1962; Power and Wheeler 1957; Brill and Wheeler 1957). His intention was to draw attention to, and to explore, the extraordinarily rich physics of curved empty space. In this connection he used fields of zero rest mass, since only these had been geometrised. The solutions which he found, and to which he gave the name "geons", were smooth over the whole of 3-space, and represented objects which were extremely large. A classical analysis was valid only when the electric field strength $\mathcal{E}$ was less than the critical field strength, $\mathcal{E}_{CRIT} = m^2 c^3 / e\hbar$, of pair theory. This yields a mass of the order of $c^4/(G^{3/2}\mathcal{E}_{CRIT}) \sim 10^{39}$ g and a radius $\sim 10^{11}$ cm. No such objects have yet been observed. The physics of smaller geons has not been investigated because quantum effects would have to be considered, and as yet no satisfactory quantum theory of gravitation exists. In addition to being excessively large, geons are unstable, although their lifetimes can be very long. The geon, then, as envisaged by Wheeler, constitutes a geometrical model for mass, or, in his own words " mass without mass".

In order to create geons which are smaller in both linear dimensions and mass than the above, it is necessary to re-introduce matter fields. This means that we now have to take into account a new parameter, m_B, the bare mass of the matter field. The analysis, however, remains classical in the sense that the fields are not quantised (although Planck's constant does appear in the field equations).

The neutral Klein-Gordon geon has been studied by Feinblum and McKinley (1968) and by Kaup (1968). These authors examined the time-invariant spherically-symmetric solutions of the coupled Klein-Gordon-Einstein equations. There are certain differences in their approaches so we will consider them separately.

Feinblum and McKinley (1968) sought solutions that would correspond to a spectrum of bound states from a single unobservable "bare" mass, thus indicating a set of observable "physical" masses. Since the problem is too difficult to solve analytically, they used a numerical technique. As boundary conditions they assumed that for large values of the radial co-ordinate the metric should asymptotically approach the Schwarzschild metric, and the wave-function for the Klein-Gordon field should approach the one given by solving for the zeroth-order approximation in the gravitational field. A large value for ρ, the radial co-ordinate, was chosen and a step-by-step integration toward the origin performed. A value of 1.28×10^{-12} g for the bare mass was taken. It was found that for normalized functions the parameters involved in the equations, viz. the eigen-energy E and the normalization constant A for the wave-function, became so small that the equations became ill-behaved and solutions were not obtained. However, an unnormalized solution was found for the ground state. The metric proved to be well-behaved everywhere except at the origin. At this point the curvature tensor diverged. The authors attribute this to the impossibility of solving the eigen-value problem exactly by numerical methods. Kaup (1968), in his discussion of this paper, pointed out that the correct explanation for the occurrence of the divergence might be that they had used an incorrect value for the bare mass.

He based this assertion on a study of their normalisation procedure. However, since this particular solution was unnormalised to begin with, the point seems academic. Finally, we note that the "physical" or observable mass of their solution was equal to 0.07 times the bare mass, or 0.9×10^{-14}g, and its diameter was approximately 4×10^{-30} cm.

In his own work Kaup (1968) obtained solutions which were better behaved. His normalisation was slightly different from that used by the aforementioned authors. For boundary conditions he assumed that the metric approached that of Schwarzschild, and he obtained the asymptotic form of the wave-function by solving the Klein-Gordon equation in the "Coulomb" potential of the gravitational field, i.e. the first-order approximation. Again only the ground state was considered. It was found that there was an upper limit on the value of the bare mass, which was $m_B = 1.75 \times 10^{-5}$ g. This is roughly of the order of $(\hbar c/G)^{\frac{1}{2}}$. The solutions were then examined for stability, and it was shown that for Klein-Gordon geons adiabatic radial perturbations are forbidden. This means that they are therefore resistant to spherically symmetric gravitational collapse.

Although these structures are much smaller than the original geons of Wheeler, they are still too massive to be considered as models for any know particles. Nevertheless they are of considerable interest.

In this spirit we will study geons obtained by solving the Dirac-Maxwell-gravitational field equations. It is necessary to introduce the electromagnetic field because the Dirac equations describe a charged particle. This also means that we will have to consider another parameter, e, the electron charge. It will be shown that we do not encounter the same difficulties with the normalisation as the aforementioned authors since, in our case, the Maxwell equation ensures that all solutions will be automatically normalised. Like Kaup (1968) we find that there is an upper limit of 4.4×10^{-5}g on the value of the bare mass. One further difference between our approach and that of Kaup (1968) and Feinblum and McKinley (1968) is that we do not use the Einstein theory to describe the effects of the gravitational field. We use instead the scalar theory of gravitation (Rastall 1968a, b) which is simpler and more tractable in many respects. All of these points will be discussed in more detail in the body of the text.

One of the reasons why this problem is physically interesting is that there may be solutions for only certain values of e/m. To get a discrete set of values one imposes differentiability conditions on the fields (consider the example of the hydrogen atom). For a particular solution to be physically meaningful we demand that it be correctly normalised and that the energy density be everywhere finite. We also require that the electrostatic and gravitational fields be well-behaved in the region outside the localised matter distribution (particle). Since there is no way to measure these fields inside the particle, there is no physical reason for demanding regularity in this region. Imposing this extra condition may lead, as indicated above, to a set of discrete values for the ratio e/m.

There is one case in which the above equations can be solved analytically. That is when we assume that the component g_{oo} of the metric tensor is a function only of the electrostatic potential Ao. This method of solving the equations of electro-gravitational theory was introduced by Weyl (1917) and further investigated by Majumdar (1947). The method is valid only for the case of static fields. Majumdar showed that, in matter-free space, the functional relationship must be of the form

$$g_{oo} = A + BA_o + 4\pi G c^{-4} A_o^2 \quad , \qquad (1.1)$$

where A and B are constants, G is the gravitational constant, and c the speed of light in vacuo. When the constant B is so chosen that the right-hand side becomes a perfect square, then (1.1) is called the Weyl-Majumdar relation (WMR). The WMR can be used to simplify considerably systems of equations of the type described above (see, for example, Das 1962, 1963; De 1965, 1969; Mukherjee, 1963). Solutions of the Klein-Gordon-Maxwell-Einstein field equations have been found by Das and Coffman (1967) for the case when the WMR is assumed. They showed that, starting from any given static, purely gravitational universe, one can construct universes corresponding to solutions of the above equations, provided only that a single differential equation is satisfied. The WMR was found to imply an equality between the charge and mass parameters of the theory. Starting from the well-known Schwarzschild universe, they obtained solutions corresponding to particles of mass $\sim 3 \times 10^{-5}$g and radius $\sim 2 \times 10^{-33}$cm. The energy E of the matter field was found to be equal to the bare mass m_B, or $E = m_B C^2$, thus giving a binding energy zero to the Klein-Gordon particle. The metric obtained has a co-ordinate singularity at spatial infinity which made its physical interpretation difficult. Other solutions were found which also had singularities at finite values of the radial co-ordinate.

In this thesis we will also, for the sake of completion, consider the Weyl-Majumdar problem for the scalar gravitational field. We will find results similar, in many respects, to those of Das and his co-workers. The method itself, its drawbacks and advantages, will be discussed.

To sum up: in the following we examine the possible states of a Dirac particle at rest in its own electrostatic and gravitational fields. In the first part the field equations are derived and examined; in the second part we investigate the solutions.

CHAPTER II

THE FIELD EQUATIONS

We assume that space-time is a four-dimensional pseudo-Riemannian manifold of signature $+2$, which obeys the Lichnerowicz differentiability conditions, and that there exist co-ordinate systems, called Newtonian charts, in which the metric tensor has components of the form

$$g_{ij} = \delta_{ij} \exp \{-2 c_E^{-2} (\phi - \phi_0)\},$$

$$g_{0\mu} = -\delta_{0\mu} \exp \{2 c_E^{-2} (\phi - \phi_0)\}, \tag{2.1}$$

where Latin indices range from 1 to 3, Greek indices from 0 to 3. The function $\phi(x^1, x^2, x^3, x^0)$ is called the gravitational potential, ϕ_0 is a constant, and c_E is the natural speed of light.

In the above, and in what follows, we use the procedure and notation of Rastall (1968a, b). The geometry of our space is determined by the single real function ϕ, which is arbitrary up to the addition of a constant. The meaning of ϕ_0 is roughly the following: Special Newtonian charts always exist whose tangent vectors are ortho-normal with respect to the metric $g_{\mu\nu}$ at any point where the potential has the value ϕ_0. Charts of this kind are called ϕ_0-charts and are in general determined up to a shift of origin and a constant orthogonal transformation of the spatial co-ordinates (Rastall 1968a). It is clear from (2.1) that if a ϕ_0-chart exists, then a ϕ_0'-chart also exists, for any constant ϕ_0'. The physical predictions of the theory, however, should depend neither on the choice of the constant ϕ_0, nor on the particular

ϕ_0 -chart once this constant is chosen.

ϕ_0 - Quantities

It is possible to define a new metric tensor field η in the following way. Let p by any point in space-time, then, by our first assumption, there exists a ϕ_0 -chart on some neighbourhood of p. The metric tensor $\eta(p)$ is defined at p by requiring

$$\eta(p)\left(X_\mu(p), X_\nu(p)\right) = \eta_{\mu\nu},$$

where

$$\eta_{ij} = \delta_{ij},$$

$$\eta_{0\mu} = -\delta_{0\mu},$$

and $X_\mu(p)$, $X_\nu(p)$ are the tangent vectors of the ϕ_0 -chart at p. Since the ϕ_0 -charts cover space-time, it follows that η is defined globally. It can also be shown that η depends only on the choice of ϕ_0 and not on the choice of ϕ_0-chart (Rastall 1968a).

The metric η can be used to define ϕ_0- lengths and times in analogy with natural lengths and times in special relativity. Consider two neighbouring points, x^i, and $x^i + dx^i$, in three-space, which have the same time co-ordinate $t = x^0/c_E$. The distance between them is given by

$$dl_E = \sqrt{(g_{\mu\nu}\, dx^\mu dx^\nu)},$$

$$= \exp\left\{-c_E^{-2}(\phi - \phi_0)\right\} \sqrt{dx^i dx^i} \qquad (2.3)$$

This is the "natural" length. The meaning of the subscript E will be explained shortly. The ϕ_0 -length, on the other hand, is defined as

$$d\ell = \sqrt{\eta_{\mu\nu}\, dx^{\mu}\, dx^{\nu}} \;,$$

$$= \sqrt{dx^{i}dx^{i}} \;. \qquad (2.4)$$

Similarly for times. The interval between the two times t , $t + dt$, at the one space-time point x^{i} is given by

$$d\tau_{E} = c_{E}^{-1} \sqrt{(-g_{\mu\nu}\, dx^{\mu} dx^{\nu})} \;,$$

$$= \exp\{c_{E}^{-2}(\phi - \phi_{0})\}\, dt \;. \qquad (2.5)$$

This is the "natural" time. The ϕ_0 -time, however, is defined by

$$d\tau = c_{E}^{-1} \sqrt{(-\eta_{\mu\nu}\, dx^{\mu} dx^{\nu})} \;,$$

$$= dt. \qquad (2.6)$$

A word is now in order concerning our notation. Wherever the subscript E appears, it means that the quantity subscripted is measured in natural or "experimental" units. If it does not appear, then the quantity is measured in ϕ_0 -units (to be defined below) and is, therefore, a ϕ_0 -quantity. The one exception to this rule is the gravitational potential ϕ . ϕ is always measured in natural units.

The natural and ϕ_0- units of length and time are related according to the following expressions

$$d\ell_{E} = S^{-1}\, d\ell \;, \qquad (2.7)$$

$$d\tau_{E} = S\; d\tau \;, \qquad (2.8)$$

which are derived from (2.3) - (2.6), and where

$$S = \exp\{c_{E}^{-2}(\phi - \phi_{0})\} \;.$$

Rastall (1968a) has shown that if we add to the above a change also in the unit of mass of the form

$$m_E = S^3 m , \tag{2.9}$$

then the equation of motion for a particle, written in ϕ_0-units, becomes formally identical to the corresponding equation in special relativity. We summarise as follows. Let Q_E be any quantity measured in natural units and let its dimensions be $[Q_E] = [L^\alpha M^\delta T^\beta]$; then its value in ϕ_0 -units is Q , where

$$Q = Q_E \, S^{(\alpha-\beta-3\delta)} . \tag{2.10}$$

Quantities measured in natural units in general do not depend on the choice of ϕ_0 . However, if the quantity Q is a tensor, then its components obviously will depend on the particular co-ordinate system used, regardless of the dimensions of Q . In this work our convention will be that (2.10) holds only for invariant quantities. Thus, if Q is a tensor at the point p , and ($\vec{e}_0$, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$) is a basis of the tangent space at p , then

$$Q = Q^{\mu\nu\ldots\ldots}_{\rho\sigma\ldots\ldots} \, \vec{e}_\mu \otimes \vec{e}_\nu \otimes \ldots\ldots \otimes e^\rho \otimes e^\sigma \ldots\ldots , \tag{2.11}$$

and (2.10) applies to the tensor itself, and not to its components individually. The meaning of ϕ_0 -quantities will become evident in the next section, where we discuss the action principle.

The Action

The Dirac theory of the electron in flat spacetime has been studied in great detail by many authors (see, for example, Rose 1961; Corinaldesi and Strocchi 1963). The field equations may be deduced from a variational principle with an action integral of the form

$$A = \int \mathcal{L}_F (\psi , A_0) \, d^4x , \tag{2.12}$$

where

$$\mathcal{L}_F = \mathcal{L}_D + \mathcal{L}_M + \mathcal{L}_{INT} ,$$

and

$$\mathcal{L}_D = \frac{\hbar c}{2}\left\{ \overline{\Psi}(\tilde{\gamma}^\mu \partial_\mu + \frac{mc}{\hbar})\psi - (\partial_\mu \overline{\Psi}\,\tilde{\gamma}^\mu + \frac{mc}{\hbar}\overline{\Psi})\psi \right\}, \quad (2.13)$$

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \quad (2.14)$$

$$\mathcal{L}_{INT} = \frac{1}{c} J_\mu A^\mu . \quad (2.15)$$

$\mathcal{L}_D$, $\mathcal{L}_M$, and $\mathcal{L}_{INT}$ are, respectively, the Lagrangian densities for the Dirac field, the Maxwell field, and the interaction. The notation is the usual one: ψ is the Dirac wave function, A_μ is the electromagnetic vector potential, the Dirac matrices $\tilde{\gamma}^\mu$ are related to the Minkowski metric tensor $\eta^{\mu\nu}$ by

$$\tilde{\gamma}^\mu \tilde{\gamma}^\nu + \tilde{\gamma}^\nu \tilde{\gamma}^\mu = 2\eta^{\mu\nu} , \quad (2.16)$$

and $\overline{\Psi} = i\,\psi^\dagger \tilde{\gamma}^0$, where $\psi^\dagger$ is the spinor conjugate to ψ .

We can introduce the interaction with the gravitational field in two ways:

(i) We can write out the Lagrangian density $\mathcal{L}_F$ in a generally covariant form, derive the field equations, and then take into account the particular form of the metric (2.1). This method has the advantage of being unambiguous, and shows off the geometrical character of the gravitational interaction.

(ii) We can use the procedure, or "prescription", sketched out by Rastall (1968b). This method is applicable only to those fields whose Lagrangian densities are known in the special relativistic limit, which is the case with (2.12). We take the flat-space Lagrangian density, (the meaning of the subscript F will be explained later), which is a function of the field components q_m and their derivatives $q_{m,\mu}$,

$$\mathcal{L}_F = \mathcal{L}_F(q_m ; q_{m,\mu}) , \tag{2.17}$$

and we make the following re-interpretations. All quantities are now to be considered as ϕ_0 -quantities. We must first however, replace $q_{m,0}$ by $(c^{-1}) \frac{\partial q_m}{\partial t}$. The co-ordinates (x^i, t) are re-interpreted as ϕ_0 - co-ordinates, the components of the fields and the parameters of the theory (such as $e, m, c, \hbar$) are assumed to be measured in ϕ_0-units. Once this re-interpretation has been made, we can derive the field equations in the usual way.

So far we have not mentioned the gravitational field itself. This is included in our theory by the addition of a term $\mathcal{L}_G$ to the Lagrangian density of the other fields $\mathcal{L}_F$. We make the assumption that it is possible to write the total Lagrangian density for the coupled Dirac-electromagnetic-gravitational fields in the form

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_F , \tag{2.18}$$

where $\mathcal{L}_F$ is the Lagrangian density of all the fields other than the gravitational field, and $\mathcal{L}_G$ is the "purely" gravitational part, being a function only of ϕ and its derivatives.

For the time being, we are mainly concerned with $\mathcal{L}_F$. We have outlined above two methods of deriving it. In our case, both methods give the same result.

The general relativistic formulation of the Dirac equation has been investigated by many authors (see, for example, the review articles of Brill and Wheeler 1957, and of Bade and Jehle 1953). We will not go into any details here but we will simply state the results. The general relativistic action is given by

$$A_F = \int \mathcal{L}_{FE} \sqrt{-g}\, d^4x \quad , \tag{2.19}$$

where $\mathcal{L}_{FE}$ is the Lagrangian density for the coupled Dirac-Maxwell fields interacting with the gravitational field. In (2.19), and in the next few expressions, we have added the subscript E for later convenience. $\mathcal{L}_{FE}$ can be split up into its component parts,

$$\mathcal{L}_{DE} = -\frac{\hbar c_E}{2} \{ \bar{\Psi}_E \gamma^\mu \nabla_\mu \Psi + \mu_E \bar{\Psi}_E \Psi_E - (\nabla_\mu \bar{\Psi}_E \gamma^\mu - \mu_E \bar{\Psi}_E) \Psi_E \} \quad , \tag{2.20}$$

$$\mathcal{L}_{ME} = -\frac{1}{4} F_{E\mu\nu} F_E^{\mu\nu} \quad , \tag{2.21}$$

and

$$\mathcal{L}_{INTE} = c_E^{-1} A_{E\mu} J_E^\mu \quad . \tag{2.22}$$

$\mathcal{L}_{DE}$ is the (generally covariant) Lagrangian density for the Dirac field. The matrices γ^μ satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \quad , \tag{2.23}$$

which is the generally covariant generalisation of (2.16); the covariant derivatives of the wave function are given by

$$\nabla_\mu \Psi_E = \partial_\mu \Psi_E - \Gamma_\mu \Psi_E \quad , \tag{2.24}$$

where the Γ_μ are the Fock-Ivanenko coefficients (Brill and Wheeler 1957); $F_{E\mu\nu} = (A_{E\nu,\mu} - A_{E\mu,\nu})$ where the $A_{E\mu}$ are the electromagnetic potentials; and the current density $J_{E\mu}$ is given by

$$J_{E\mu} = -i\, e_E c_E \bar{\Psi}_E \gamma_\mu \Psi_E \quad . \tag{2.25}$$

The equations of motion derivable from the action (2.19) are the following,

$$\gamma^{\mu}\left(\nabla_{\mu}+\frac{ie_E}{\hbar c_E}A_{\mu}\right)\psi_E+\mu_E\psi_E=0\,, \tag{2.26}$$

$$\frac{1}{\sqrt{-g}}\partial_{\nu}\left(\sqrt{-g}\,F_E^{\mu\nu}\right)=c_E^{-1}J_E^{\mu}\,, \tag{2.27}$$

and it can be shown from (2.26) and its adjoint that the current J_E^{μ} obeys the equation of continuity,

$$\partial_{\nu}\left(\sqrt{-g}\,\bar{\psi}_E\gamma^{\nu}\psi_E\right)=0\,. \tag{2.28}$$

At this stage the metric form (2.1) is substituted for the (as yet) unspecified $g_{\mu\nu}$. The Fock-Ivanenko co-efficients are calculated in the usual way (see the above references). However, we do not need individual expressions for the Γ_{μ}, since they appear in the equations only in the form $\gamma^{\mu}\Gamma_{\mu}$, and it can easily be shown that

$$\gamma^{\mu}\Gamma_{\mu}=\frac{1}{2}c_E^{-2}\left(\gamma^k\partial_k\phi+3\gamma^0\partial_0\phi\right), \tag{2.29}$$

where $\phi_{,\mu}=\partial\phi/\partial x^{\mu}$, and that

$$g=\det(g_{\mu\nu})=-S^{-4}\,. \tag{2.30}$$

The matrices $\gamma_{\mu}=g_{\mu\nu}\gamma^{\nu}$ are related to the flat space Dirac matrices $\tilde{\gamma}_{\mu}$ by

$$\begin{aligned}\gamma_k&=S^{-1}\tilde{\gamma}_k\,,\\ \gamma_0&=S\,\tilde{\gamma}_0\,.\end{aligned} \tag{2.31}$$

Using (2.29), (2.30), and (2.31) we can rewrite the field equations (2.26) and (2.27) as

$$\begin{aligned}\gamma^{\mu}\left(\partial_{\mu}+\frac{ie_E}{\hbar c_E}A_{E\mu}\right)\psi_E-\frac{1}{2}c_E^{-2}\left(\gamma^k\partial_k\phi+3\gamma^0\partial_0\psi\right)&\\ +\mu_E\psi_E=0\,,&\end{aligned} \tag{2.32}$$

$$\partial_\nu (S^{-2} F_E^{\mu\nu}) = c_E^{-1} S^{-2} J_E^\mu \quad . \tag{2.33}$$

For the sake of later convenience we list the connections between the quantities encountered above and the corresponding ϕ_0 -quantities. These expressions are derived by using the formula (2.10).

$$\begin{aligned}
\hbar &= \hbar_E \ , \\
c &= c_E S^2 \ , \\
\mu &= \mu_E S^{-1} \ , \\
e &= e_E S \ , \\
J &= J_E \ , \\
A &= A_E \ , \\
F &= F_E S^{-1} , \\
\psi &= \psi_E S^{-3/2} , \\
\mathcal{L} &= \mathcal{L}_E S^{-2} \ .
\end{aligned} \tag{2.34}$$

We can apply the "prescription" outlined above to obtain the Lagrangian density from the special relativistic $\mathcal{L}_F$. We obtain

$$\begin{aligned}
\mathcal{L}_D = & -\frac{\hbar c}{2} \{ \bar{\psi} (\gamma^k \partial_k + \mu) \psi - (\partial_k \bar{\psi} \gamma^k - \mu) \psi \\
& + \bar{\psi} (\gamma^0 S^{-2} \partial_0 \psi) - S^{-2} \partial_0 \bar{\psi} \gamma^0 \psi \} ,
\end{aligned} \tag{2.35}$$

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2) \ , \tag{2.36}$$

$$\mathcal{L}_{INT} = c^{-1} J_\mu A^\mu \ , \tag{2.37}$$

where we have written

$$\begin{aligned}
B_k &= \frac{1}{2} S \, \epsilon_{kij} F_{Eij} \ , \\
E_k &= S^{-1} F_{E\,k0} \ .
\end{aligned} \tag{2.38}$$

In the application of the prescription to $\mathcal{L}_M$ we have considered the electric and magnetic field components E_k and B_k to be the basic fields, rather than the potentials A_μ. It is quite easy to see that the $\phi_0 - \mathcal{L}_F$ written above is the same as the generally covariant $\mathcal{L}_{FE}$ (equations (2.20) through (2.22)) except for a factor, i.e.

$$\mathcal{L}_F = \mathcal{L}_{FE} S^{-2} . \tag{2.39}$$

This means that the action integral

$$A_F = \int \mathcal{L}_F \, d^4x \tag{2.40}$$

is the same, and therefore we have the same field equations, (2.32) and (2.33).

The total Lagrangian density is given by (2.18) and the action integral for the coupled Dirac-electromagnetic-gravitational system is

$$A = \int (\mathcal{L}_G + \mathcal{L}_F) \, d^4x . \tag{2.41}$$

It is now necessary for us to specify $\mathcal{L}_G$, the purely gravitational part of the Lagrangian density. Rastall (1968b) has shown that a particularly simple form for $\mathcal{L}_G$ is given by

$$\mathcal{L}_G = \kappa S^{-2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi , \tag{2.42}$$

where $\kappa = -(8\pi G_E)^{-1}$, and G_E is the Newtonian gravitational constant. This choice for $\mathcal{L}_G$ ensures that the Rastall theory will give the same astronomical predictions as the Einstein theory. Other choices for $\mathcal{L}_G$ are possible, of the form

$$\mathcal{L}_G = \kappa S^{-2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \, \exp\{ \alpha c_E^{-2} (\phi - \phi_1) \} ,$$

where ϕ_1 is a constant independent of the choice of Newtonian chart. If $\alpha \neq 0$, then, as Rastall (1968a) has shown, the perihelion advance of test particles will not be the same as in the Einstein theory.

The gravitational equation is derived from the action principle in the usual way. Variation of ϕ yields the Euler equation

$$\frac{\delta \mathcal{L}}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right) = 0 \quad , \tag{2.43}$$

where $\mathcal{L} = \mathcal{L}_F + \mathcal{L}_G$, and, explicitly

$$\frac{\delta \mathcal{L}_G}{\delta \phi} = -2K\left\{ \partial_m \partial_m \phi - S^{-4}\left(\partial_0\partial_0\phi - 2C_E^{-2}(\partial_0\phi)^2\right)\right. , \tag{2.44}$$

$$\frac{\delta \mathcal{L}_M}{\delta \phi} = -C_E^{-2}\left(S^{-2}F_{E0i}F_{E0i} + \tfrac{1}{2}S^2 F_{Eij}F_{Eij}\right)$$

$$= -C_E^{-2}(E^2 + B^2) \tag{2.45}$$

It is easy to show, using (2.35) and its adjoint, that $\mathcal{L}_D + \mathcal{L}_{INT}$ is equal to zero whenever the field equations are satisfied. We make use of this fact to obtain

$$\begin{aligned}\frac{\partial}{\partial \phi}\left(\mathcal{L}_D + \mathcal{L}_{INT}\right) &= 2C_E^{-2}\left(\mathcal{L}_D + \mathcal{L}_{INT}\right)\\
&- \frac{\hbar C_E}{2}\left\{ \bar{\Psi}_E \frac{\partial}{\partial\phi}(\gamma^\mu) D_\mu \Psi_E - D_\mu^* \bar{\Psi}_E \frac{\partial}{\partial \phi}(\gamma^\mu)\Psi_E \right\} S^{-2}\\
&= -\frac{\hbar C_E}{2}\left\{ \bar{\Psi}_E \gamma^k D_k \Psi_E - D_k^* \bar{\Psi}_E \gamma^k \Psi_E \right.\\
&\qquad \left. - \bar{\Psi}_E \gamma^0 D_0 \Psi_E + D_0^* \bar{\Psi}_E \gamma^0 \Psi_E \right\} S^{-2}\end{aligned} \tag{2.46}$$

where $D_\mu = \nabla_\mu + ie_E(\hbar C_E)^{-1} A_\mu$. There is no particular significance in the fact that we have used the natural wave-function Ψ_E . Exactly the same results are obtained using a Lagrangian density $\mathcal{L}_D + \mathcal{L}_{INT}$ written entirely in terms of ϕ_0 -quantities. We can use the vanishing of $\mathcal{L}_D + \mathcal{L}_{INT}$ to simplify (2.46);

$$\begin{aligned}\frac{\delta}{\delta\phi}\left(\mathcal{L}_D + \mathcal{L}_{INT}\right) = \hbar C_E^{-1} S^{-2}\left\{ \bar{\Psi}_E \gamma^0 D_0 \Psi_E - D_0^* \bar{\Psi}_E \gamma^0 \Psi_E \right.\\
\left. + \mu_E \bar{\Psi}_E \Psi_E \right\} .\end{aligned} \tag{2.47}$$

The gravitational field equation becomes

$$\partial_m\partial_m\phi - S^{-4}\left(\partial_0\partial_0\phi - 2C_E^{-2}(\partial_0\phi)^2\right) = -(2\kappa C_E^2)^{-1}(E^2 + B^2)$$

$$+ \hbar(2\kappa C_E)^{-1} S\left(\bar{\psi}\gamma^0 D_0\psi - D_0^*\bar{\psi}\gamma^0\psi + \mu_E\bar{\psi}\psi\right). \qquad (2.48)$$

We shall see later that there a simple connection between the right-hand side of this equation and the energy-momentum tensor of the coupled system, the field equations for which are now given by (2.32), (2.33) and (2.48).

Conserved Quantities and Normalisation

The action integral (2.41) is invariant under gauge transformations of the first kind,

$$\psi \Rightarrow \psi' = \psi \exp(i\alpha(x)) ,$$

$$\psi^\dagger = \psi'^\dagger = \psi^\dagger \exp(-i\alpha(x)) , \qquad (2.49)$$

where α is a real function of the co-ordinates. If we make the above transformation, it is easy to show that

$$\delta A = i\hbar C_E \int_V \alpha\, \partial_\mu (\bar{\psi}_E \gamma^\mu \psi_E S^{-2})\, d^4x = 0 , \qquad (2.50)$$

where we have assumed that α vanishes on the surface S enclosing the space-time volume V. Since α is otherwise an arbitrary function, (2.50) gives us the continuity equation (2.28). This can be written in the form

$$\partial_\mu (c^{-1} J^\mu) = 0 \qquad (2.51)$$

where $J^\mu = -iec\,\bar{\psi}\gamma^\mu\psi$.

The existence of the continuity equation (2.51) means that our theory can be given a probabilistic interpretation. If we integrate (2.51) over the three-space volume V_3 ($x^0 =$ constant), we obtain,

$$C_E \frac{\partial}{\partial t} \int_{V_3} (c^{-1} J^0)\, d^3x + \int_{V_3} \partial_k (c^{-1} J^k)\, d^3x = 0 ,$$

which yields, if we assume that J^k/c vanishes sufficiently fast at spatial infinity,

$$\frac{d}{dt} \int_{V_3} (e\, \overline{\psi} \gamma^0 \psi)\, d^3x = 0 \qquad (2.52)$$

The probability density D is defined as

$$D = \psi^+ \psi = S^{-3} \psi_E^+ \psi_E , \qquad (2.53)$$

and is a positive definite quantity. Equation (2.52) can be written

$$\frac{d}{dt} \left(e_E \int D\, d^3x \right) = 0$$

The expression inside the brackets represents the total natural charge of the system and is independent of the choice of ϕ_0-chart.

In keeping with our interpretation of D as a probability density, we normalise the wave function ψ as follows,

$$\int D\, d^3x = 1 , \qquad (2.54)$$

the integration being over all three-space. It follows that the total natural charge of the system is equal to e_E , and is a constant of the motion.

Energy-Momentum Tensor

We define an energy-momentum tensor in the following way. The action integral (2.41) is invariant under space-time translations of the kind $x^\mu \to x^\mu + \epsilon^\mu$, where the ϵ^μ are constants. In the usual way, by performing an infinitessimal translation and setting $\delta A = 0$ we can show that

$$\partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_m)} \partial_\nu q_m - \delta^\mu{}_\nu \mathcal{L} \right\} = 0 \qquad (2.55)$$

where $\mathcal{L} = \mathcal{L}(q_m, \partial_\mu q_m)$ and the q_m are the field components. We therefore define

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_m)} \partial_\nu q_m - \delta^\mu_\nu \mathcal{L} , \tag{2.56}$$

which is the (mixed) ϕ_0-energy-momentum tensor. By means of (2.35), (2.36), (2.37) and (2.42), we can easily derive the explicit form for $T^\mu{}_\nu$:

$$\begin{aligned} T^\mu{}_\nu = {} & \kappa S^{-2} (2 g^{\mu\rho} \partial_\rho \phi \partial_\nu \phi - \delta^\mu{}_\nu g^{\lambda\rho} \partial_\lambda \phi \partial_\rho \phi) \\ & - \tfrac{1}{4} S^{-2} (4 F^{\mu\tau} A_{\tau,\nu} - \delta^\mu{}_\nu F^{\rho\tau} F_{\rho\tau}) \\ & - \tfrac{1}{2} \hbar c_E S (\bar\Psi \gamma^\mu \partial_\nu \Psi - \partial_\nu \bar\Psi \gamma^\mu \Psi) . \end{aligned} \tag{2.57}$$

The electromagnetic part can be symmetrized if we note that

$$F^{\mu\tau} A_{\tau,\nu} = F^{\mu\tau} F_{\nu\tau} + F^{\mu\tau} A_{\nu,\tau}$$

The second term on the left-hand side has the divergence

$$\begin{aligned} \partial_\mu (S^{-2} F^{\mu\tau} A_{\nu,\tau}) = {} & (S^{-2} F^{\mu\tau} A_\nu)_{,\mu\tau} \\ & - \{ (S^{-2} F^{\mu\tau})_{,\tau} A_\nu \}_{,\mu} \\ = {} & - (c^{-1} J^\mu A_\nu)_{,\mu} , \end{aligned} \tag{2.58}$$

by the Maxwell equation (2.33). We obtain our symmetrized energy-momentum tensor $\hat{T}$ by adding to (2.57) the divergence-free term given by (2.58), or

$$\begin{aligned} \hat{T}^\mu{}_\nu = {} & T^\mu{}_\nu + S^{-2} F^{\mu\tau} A_{\nu,\tau} + c^{-1} J^\mu A_\nu \\ = {} & \kappa S^{-2} (2 g^{\mu\rho} \partial_\rho \phi \partial_\nu \phi - \delta^\mu{}_\nu g^{\lambda\rho} \partial_\lambda \phi \partial_\rho \phi) \\ & - S^{-2} (F^{\mu\tau} F_{\nu\tau} - \tfrac{1}{4} \delta^\mu{}_\nu F^{\lambda\rho} F_{\lambda\rho}) \\ & - \tfrac{1}{2} \hbar c_E S (\bar\Psi \gamma^\mu \nabla_\nu \Psi - \nabla_\nu \bar\Psi \gamma^\mu \Psi) \end{aligned} \tag{2.59}$$

The component $\hat{T}^0{}_0$ of (2.59) is given by

$$\begin{aligned} \hat{T}^0{}_0 = {} & -\kappa \{ (\partial_m \phi)^2 + S^{-4} (\partial_0 \phi)^2 \} + \tfrac{1}{2} (E^2 + B^2) \\ & - \tfrac{1}{2} \hbar c_E S (\bar\Psi \gamma^0 \nabla_0 \Psi - \nabla_0 \bar\Psi \gamma^0 \Psi) , \end{aligned} \tag{2.60}$$

and is a positive define quantity. Using (2.59) we can rewrite the gravitational field equation (2.48) as follows,

$$\partial_{mm}\phi - S^{-4}\left\{ \partial_{00}\phi - 4C_E^{-2}(\partial_0\phi)^2 \right\}$$
$$= -\frac{1}{KC_E^2}\left\{ \hat{T}^0{}_0 - \frac{1}{2}\hat{T}^\mu{}_\mu \right\} , \qquad (2.61)$$

where $\hat{T}^\mu_\mu$ is the trace of the energy-momentum tensor. It is to be noticed that, unlike the case for the neutrino and the electromagnetic fields, the trace of that part of $\hat{T}^\mu{}_\nu$ corresponding to the gravitational field does not vanish. In fact

$$T_G{}^\mu{}_\mu = -2\,\mathcal{L}_G ,$$

and this vanishes only if $\partial_\mu\phi$ is a null vector.

CHAPTER III

TIME-INDEPENDENT SYSTEM

We are attempting to construct a time-invariant model for the electron, and therefore we assume that all physically measurable quantities are time-independent. The static system is characterised by the following:

$$A_j = 0$$

$$\psi_E = \chi_E \exp(iEt/\hbar) , \qquad (3.1)$$

A_0, S, and χ_E are functions only of the space co-ordinates x^i, and E is a real constant which represents the energy of the matter field.

Using (3.1), the field equations (2.32), (2.33), and (2.48) reduce to the following set

$$m_E c_E^2 \chi_E + i\gamma^0 (E + e_E A_0)\chi_E + \hbar c_E \gamma^j (\partial_j - \tfrac{1}{2} c_E^{-2} \partial_j \phi)\chi_E = 0 , \qquad (3.2)$$

$$\partial_j (S^{-2} \partial_j A_0) = -ie_E S^{-2} \bar{\chi}_E \gamma^0 \chi_E , \qquad (3.3)$$

$$\partial_m \partial_m \phi = -(2\kappa c_E^2)^{-1} S^{-2} (\partial_m A_0)^2$$
$$+ (\kappa c_E^2)^{-1} S^{-2} \{ i(E + e_E A_0)\bar{\chi}_E \gamma^0 \chi_E + \tfrac{1}{2} m_E c_E^2 \bar{\chi}_E \chi_E \} . \qquad (3.4)$$

The remaining Maxwell equations merely state that

$$J^i = 0 . \qquad (3.5)$$

These equations can be simplified still further. We choose the following representation for the Dirac matrices

$$\tilde{\gamma}^0 = i\beta ,$$

$$\tilde{\gamma}^i = -i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} ,$$

where

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{3.6}$$

and the σ_i are the (two-by-two) Pauli spin matrices. We also write

$$\chi_E = S^{\frac{1}{2}}\begin{pmatrix} U \\ V \end{pmatrix} \tag{3.7}$$

where U and V are two-component spinors. Using (2.31), (3.6) and (3.7), the Dirac equation (3.2) can be written as a pair of two-component spinor equations,

$$(E + e_E A_0 - m_E c_E^2 S)U + i\hbar c_E\, S^2 \sigma_j \partial_j V = 0 ,$$

$$(E + e_E A_0 + m_E c_E^2 S)V + i\hbar c_E\, S^2 \sigma_j \partial_j U = 0 . \tag{3.8}$$

In terms of this new notation, the Maxwell and gravitational equations (3.3) and (3.4) take the form

$$\partial_m (S^{-2} \partial_m A_0) = e_E S^{-2} (|U|^2 + |V|^2) , \tag{3.9}$$

$$\partial_m \partial_m \phi = -(2\kappa c_E^2)^{-1} S^{-2} (\partial_m A_0)^2$$

$$- (\kappa c_E^2)^{-1} S^{-2} \left\{ (E + e_E A_0)(|U|^2 + |V|^2) - \tfrac{1}{2} m_E c_E^2 S (|U|^2 - |V|^2) \right\} . \tag{3.10}$$

Equations (3.8), (3.9) and (3.10) represent the time-invariant system, where as yet we have made no assumptions about any particular spatial symmetry. We will use these equations when we discuss the Weyl-Majumdar solutions. However, we are more concerned with the spherically symmetric case.

Spherical Symmetry

In this case we assume that all physically measurable quantities are functions only of the radial co-ordinate r . We write everything in terms of a new co-ordinate-system, the isotropic radial co-ordinates, in which the line element takes the form

$$ds^2 = -S^{-2}(dx^0)^2 + S^2\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right), \tag{3.11}$$

where the relationships between the spherical co-ordinates r , θ , φ and the Cartesians x^1 , x^2 , x^3 are the same as in flat space. We make use of the following identities

$$r^2 \underline{\nabla} = \underline{r}\,(\underline{r}\cdot\underline{\nabla}) - \underline{r}\times(\underline{r}\times\underline{\nabla}), \tag{3.12}$$

where $\underline{\nabla}$ is the gradient vector, and

$$(\underline{\sigma}\cdot\underline{r})(\underline{\sigma}\cdot\underline{\ell}) = \underline{r}\cdot\underline{\ell} + i\,\underline{\sigma}\cdot(\underline{r}\times\underline{\ell}), \tag{3.13}$$

where $\underline{\ell} = -i\,\underline{r}\times\underline{\nabla}$ is the usual angular momentum operator. Using (3.12) and (3.13) we obtain

$$\underline{\sigma}\cdot\underline{\nabla} = \sigma_r \frac{\partial}{\partial r} - \frac{1}{r}\sigma_r\,\underline{\sigma}\cdot\underline{\ell}. \tag{3.14}$$

The Dirac equations can be written (3.8) in Hamiltonian form

$$E\begin{pmatrix} u \\ v \end{pmatrix} = H\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.15}$$

where H , the Hamiltonian, is given by

$$\begin{aligned} H = {} & -e_E A_0 + \beta m_E c_E^2 S \\ & -i\hbar c_E S^2 \alpha_r\left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r}\beta \mathcal{K}\right), \end{aligned} \tag{3.16}$$

and $\mathcal{K}$ is the angular momentum operator for the Dirac field:

$$\mathcal{K} = \beta(\underset{\sim}{\sigma}\cdot\underset{\sim}{\ell} + 1) , \tag{3.17}$$

and

$$\alpha_r = \begin{pmatrix} 0 & \sigma_r \\ \sigma_r & 0 \end{pmatrix} . \tag{3.18}$$

In this co-ordinate system it can be seen that, except for the factors in S, H has the same form as the special relativistic Hamiltonian. It is easy to show that the three operators J^2, J_3, and $\mathcal{K}$ commute with H and with each other and therefore determine three constants of the motion.

We choose a representation in which the operators H, $\mathcal{K}$, J^2 and J_3 are diagonalised. If we follow closely the flat-space treatment of the same problem (the modifications are obvious), then it can be shown (Corinaldesi and Strocchi 1963) that the eigen-functions of these operators are given by

$$S^{-\frac{1}{2}}\chi_E^{(1)} = \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} = \begin{pmatrix} f(r)\, X_j^{(m_j)+} \\ g(r)\, X_j^{(m_j)-} \end{pmatrix} ,$$

$$S^{-\frac{1}{2}}\chi_E^{(2)} = \begin{pmatrix} u^{(2)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} f(r)\, X_j^{(m_j)-} \\ g(r)\, X_j^{(m_j)+} \end{pmatrix} . \tag{3.19}$$

The eigenvalues of the operators H, $\mathcal{K}$, J^2 and J_3 with respect to the above eigenfunctions are, respectively, E, k, $j(j+1)$, and m_j. In the first case we have

$$\mathcal{K}(S^{-\frac{1}{2}}\chi_E^{(1)}) = (j+\tfrac{1}{2})\, S^{-\frac{1}{2}}\chi_E^{(1)} , \tag{3.20}$$

i.e. $k = j + \frac{1}{2}$, and

$$X_j^{(m_j)+} = \begin{pmatrix} \left(\dfrac{\ell + m_j + \frac{1}{2}}{2\ell+1}\right)^{\frac{1}{2}} Y_\ell^{(m_j - \frac{1}{2})} \\ \left(\dfrac{\ell - m_j + \frac{1}{2}}{2\ell+1}\right)^{\frac{1}{2}} Y_\ell^{(m_j + \frac{1}{2})} \end{pmatrix} ,$$

$$\chi_j^{(m_j)-} = \begin{bmatrix} \left(\dfrac{\ell - m_j + \frac{3}{2}}{2\ell + 3}\right)^{\frac{1}{2}} Y_\ell^{(m_j - \frac{1}{2})} \\ \left(\dfrac{\ell + m_j + \frac{3}{2}}{2\ell + 3}\right)^{\frac{1}{2}} Y_\ell^{(m_j + \frac{1}{2})} \end{bmatrix} . \tag{3.21}$$

We have put $\ell = j - \frac{1}{2}$. In the second case $k = -(j + \frac{1}{2})$, $\ell = j + \frac{1}{2}$, and the expressions for the $\chi_j^{(m_j)\pm}$ are the same except that we must replace ℓ by $(\ell - 1)$.

To obtain the equations for the radial functions $f(r)$ and $g(r)$ we expand the Hamiltonian equation (3.15),

$$E \begin{bmatrix} f\ \chi_j^{(m_j)\pm} \\ g\ \chi_j^{(m_j)\mp} \end{bmatrix}$$

$$= \begin{bmatrix} \left[(-e_E A_0 + m_E c_E^2 S) f - i\hbar c_E S^2 \left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{k}{r}\right) g\right] \chi_j^{(m_j)\pm} \\ \left[(-e_E A_0 - m_E c_E^2 S) g - i\hbar c_E S^2 \left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{k}{r}\right) f\right] \chi_j^{(m_j)\mp} \end{bmatrix} , \tag{3.22}$$

where we have replaced $\mathcal{K}$ by its eigenvalue k and we have used the result (Corinalderi and Strocchi 1963) that

$$\sigma_r \chi_j^{(m_j)\pm} = \chi_j^{(m_j)\mp} . \tag{3.23}$$

From (3.22) it is easy to see that the radial equations are as follows

$$\hbar c_E \left(\frac{d}{dr} + \frac{k}{r}\right) G = - S^{-2} \left(E + e_E A_0 - m_E c_E^2 S\right) F , \tag{3.24}$$

$$\hbar c_E \left(\frac{d}{dr} - \frac{k}{r}\right) F = S^{-2} \left(E + e_E A_0 + m_E c_E^2 S\right) G , \tag{3.25}$$

where we have written $f = r^{-1}F$ and $g = -ir^{-1}G$.

It follows from our assumption of spherical symmetry that the expressions

$$S^{-1}\chi_E^{+}\chi_E = r^{-2}\left(F^2|\chi_j^{(m_j)\pm}|^2 + G^2|\chi_j^{(m_j)\mp}|^2\right),$$

$$S^{-1}\chi_E^{+}\beta\chi_E = r^{-2}\left(F^2|\chi_j^{(m_j)\pm}|^2 - G^2|\chi_j^{(m_j)\mp}|^2\right),$$

(3.26)

which appear in the gravitational and electromagnetic field equations, must be functions only of the radial co-ordinate. This means that $|\chi_j^{(m_j)}|^2$ and $|\chi_j^{(m_j)-}|^2$ are constants, i.e. are independent of the angular co-ordinates θ, φ. We can use this fact to determine the allowed values of the eigenvalue k. For the first group of solutions ($S^{-\frac{1}{2}}\chi_E^{(1)}$), where $k = -(j+\frac{1}{2}) = -\ell$, it can be shown, by considering the properties of the spherical harmonics $Y_\ell^{(m)}$, that only the first case $j = \frac{1}{2}$ fulfills the above condition. We have $\ell = 1$, $m_j = \pm\frac{1}{2}$ and

$$|\chi_{\frac{1}{2}}^{(\frac{1}{2})\pm}|^2 = |\chi_{\frac{1}{2}}^{(-\frac{1}{2})\pm}|^2 = (4\pi)^{-1} \qquad (3.27)$$

For larger values of j these expressions will in general be functions of θ and φ. Similarly, for the solutions ($S^{-\frac{1}{2}}\chi_E^{(2)}$), where $k = j + \frac{1}{2} = \ell + 1$, only the case $j = \frac{1}{2}$, $\ell = 0$ is allowed. Equation (3.27) is again valid in this case. To sum up, we have two solutions, of differing parity, which obey the criteria of time-invariance and spherical symmetry;

$$S^{-\frac{1}{2}}\chi_E^{(1)} = \begin{pmatrix} f(r)\,\chi_{\frac{1}{2}}^{(\frac{1}{2})+} \\ g(r)\,\chi_{\frac{1}{2}}^{(\frac{1}{2})-} \end{pmatrix},$$

(3.28)

where $\quad k = -\ell = -1$, and

$$S^{-\frac{1}{2}}\chi_E^{(2)} = \begin{pmatrix} f(r)\, X_{\frac{1}{2}}^{(\frac{1}{2})-} \\ g(r)\, X_{\frac{1}{2}}^{(\frac{1}{2})+} \end{pmatrix} \tag{3.29}$$

where $\quad k = \ell + 1 = 1$

The electromagnetic and gravitational field equations (3.9) and (3.10) can be re-written in terms of the radial co-ordinates. Using the above results, we have,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\, S^{-2}\frac{dA_0}{dr}\right)$$

$$= \frac{e_E}{4\pi} S^{-2}\, r^{-2}\left(F^2 + G^2\right) , \tag{3.30}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = -(2\kappa c_E^2)^{-1} S^{-2}\left(\frac{dA_0}{dr}\right)^2$$

$$- (4\pi\kappa c_E^2)^{-1} S^{-2} r^{-2}\left\{ (E + e_E A_0 - \tfrac{1}{2} m_E c_E^2 S)F^2 \right.$$

$$\left. + \ (E + e_E A_0 + \tfrac{1}{2} m_E c_E^2 S)G^2 \right\} . \tag{3.31}$$

The system of time-invariant spherically symmetric equations is now given by (3.24), (3.25), (3.30), and (3.31). We have four equations for four unknown functions of r . In the following chapters we will attempt to solve these equations using a variety of methods, including numerical integration.

For the sake of future convenience we make a transition to dimensionless notation. All lengths will be written in terms of the (bare) Compton wavelength of the Dirac particle, $\hbar\,(m_E c_E)^{-1}$, and all energies in terms of the (bare) rest energy $m_E c_E^2$. Furthermore, we define and will use later the following dimensionless constants

$$\tau = G_E m_E^2 / (\hbar c_E) ,$$

$$\alpha = e_E^2 / (4\pi \hbar c_E) ,$$

$$\varepsilon = E / (m_E c_E^2) . \tag{3.32}$$

τ is, essentially, the square of the bare mass written in relativistic units, α is the fine-structure constant, and ε is the dimensionless energy eigenvalue.

To achieve a dimensionless notation, the following substitutions are made,

$$A_0 \Rightarrow a_0 = e_E A_0 / (m_E c_E^2) ,$$

$$F \Rightarrow \tilde{F} = [e_E^2 / (4\pi m_E c_E^2)]^{+1/2} F ,$$

$$G \Rightarrow \tilde{G} = [e_E^2 / (4\pi m_E c_E^2)]^{+1/2} G ,$$

$$r \Rightarrow \rho = m_E c_E \hbar^{-1} r . \tag{3.33}$$

The field equations become

$$\left(\frac{d}{d\rho} + \frac{k}{\rho}\right)\tilde{G} = -S^{-2}(\varepsilon + a_0 - S)\tilde{F} , \tag{3.34}$$

$$\left(\frac{d}{d\rho} - \frac{k}{\rho}\right)\tilde{F} = S^{-2}(\varepsilon + a_0 + S)\tilde{G} , \tag{3.35}$$

$$\frac{d}{d\rho}\left(\rho^2 S^{-2} \frac{da_0}{d\rho}\right) = S^{-2}(\tilde{F}^2 + \tilde{G}^2) , \tag{3.36}$$

$$\begin{aligned} \frac{d}{d\rho}\left(\rho^2 \frac{d\phi}{d\rho}\right) &= \tau\alpha^{-1} S^{-2} \rho^2 \left(\frac{da_0}{d\rho}\right)^2 \\ &+ 2\tau\alpha^{-1} S^{-2}(\varepsilon + a_0 - \tfrac{1}{2}S)\tilde{F}^2 \\ &+ 2\tau\alpha^{-1} S^{-2}(\varepsilon + a_0 + \tfrac{1}{2}S)\tilde{G}^2 . \end{aligned} \tag{3.37}$$

The normalisation condition (2.54) can be expanded to yield

$$(4\pi)^{-1} \int S^{-2}(F^2 + G^2) \sin^2\theta \, d\theta \, d\varphi \, dr = 1$$

which, in the previous notation, becomes

$$\int_0^\infty S^{-2}(\tilde{F}^2 + \tilde{G}^2)\, d\rho \;=\; \alpha \; . \tag{3.38}$$

We can use the Maxwell equation (3.36) to integrate this expression and to obtain the normalisation condition in the form of conditions on the boundary values of the gravitational and electromagnetic potentials:

$$\left(\rho^2 S^{-2} \frac{da_0}{d\rho} \right)\Big|_0^\infty \;=\; \alpha \; . \tag{3.39}$$

CHAPTER IV

EXTERIOR FIELD OF A CHARGED SPHERE

In this section we obtain solutions for the gravitational and electrostatic field equations in the region outside a static spherically symmetric distribution of charged matter. Essentially, we are solving the problem of a charged point particle. The corresponding Riessner-Nordstrom solution in general relativity is well known. It will be shown that for points very distant from the centre of the sphere, the two theories yield metrics which agree up to first order in r^{-1}, where r is the radial co-ordinate. Only in one case, when the gravitational and electromagnetic fields are related in a specific way, do the two theories predict the same space-time structure.

The solution obtained is examined for various values of the ratio $e_E^2(4\pi G_E M_E^2)^{-1}$, where e_E is the charge and M_E the gravitational mass of the sphere as seen by a distant observer. It is found that, when this ratio is greater than one, the metric is well-behaved only outside a certain radius, the "Schwarzschild radius" of the electric charge. On the other hand, when the ratio is less than one, this singularity does not occur and the metric is well-behaved everywhere except at the origin. In this latter case the electrostatic potential is everywhere finite.

The region of space in which the asymptotic solutions are valid is called the "exterior region". The solutions of the Dirac equation in which we are most interested are localised in the "strong" sense, i.e. the matter density contains a factor of the form $\exp(-a^2 r)$, where a^2 is a constant which depends on the binding energy of the field.

Therefore the exterior region is that part of space for which $r \gg a^2$. In this region the matter density is so small that it has a negligible effect on the electrostatic and gravitational fields. The situation can be compared to that of a very thin atmosphere surrounding a very dense body. Because the gravitational and electrostatic fields are long-range, the effect of the dense concentration of matter, which may be at a distance, completely overshadows that of the small amount of matter in the neighbourhood. However, the inverse problem is quite different. The "athmosphere" is very much influenced by the electro-gravitational fields. This problem will be examined in the next chapter.

In the exterior region, the equations for the system are

$$\frac{d}{dr}\left(r^2 S^{-1}\frac{dS}{dr}\right) = \frac{4\pi G_E}{C_E^4} r^2 \left(\frac{dA_0}{dr}\right)^2 , \tag{4.1}$$

$$\frac{d}{dr}\left(r^2 S^{-2}\frac{dA_0}{dr}\right) = 0 . \tag{4.2}$$

(4.2) is integrated immediately to yield

$$\frac{dA_0}{dr} = \mu S^2 r^{-2} , \tag{4.3}$$

where μ is a constant of integration. Inserting (4.3) into (4.1), we obtain

$$\frac{d}{dr}\left(r^2 S^{-1}\frac{dS}{dr}\right) = \frac{4\pi G_E}{C_E^4} \mu^2 r^{-2} S^2 . \tag{4.4}$$

Writing $z = S^{-1}$, $u = (4\pi G_E \mu^2 C_E^{-4} \qquad) r^{-1}$, (4.4) becomes

$$z\frac{d^2 z}{du^2} - \left(\frac{dz}{du}\right)^2 + 1 = 0 . \tag{4.5}$$

The general solution of (4.5) has been given by Kamke (1943) and is

$$z = a^{-1}\sin(au + b) , \tag{4.6}$$

where a, b are constants to be determined from the boundary conditions. In terms of u, and using (4.6), (4.3) becomes

$$\frac{dA_0}{du} = -\left(\frac{c_E^4}{4\pi G_E}\right)^{\frac{1}{2}} \frac{a^2}{\sin^2(au+b)} , \tag{4.7}$$

which is integrated to yield

$$A_0 = c_1 + a\left(\frac{c_E^4}{4\pi G_E}\right)^{\frac{1}{2}} \cot(au+b) , \tag{4.8}$$

where c_1 is a constant also to be determined from our boundary conditions, which we obtain by assuming that for very large values of r, the gravitational and electrostatic potentials have the form

$$A_0 = -\frac{e_E}{4\pi}\frac{1}{r} + O(r^{-2}) ,$$

$$\phi = -\frac{G_E M_E}{r} + O(r^{-2}) . \tag{4.9}$$

We recall that, for large r (small ϕ), we have

$$z = S^{-1} = 1 - c_E^{-2}\phi . \tag{4.10}$$

Expanding (4.6), we obtain

$$\begin{aligned} z &= \frac{\cos b}{a}\sin\left(a\sqrt{\frac{4\pi G_E \mu^2}{c_E^4}}\,\frac{1}{r}\right) \\ &+ \frac{\sin b}{a}\cos\left(a\sqrt{\frac{4\pi G_E \mu^2}{c_E^4}}\,\frac{1}{r}\right) , \\ &= \frac{\sin b}{a} + \cos b\sqrt{\frac{4\pi G_E \mu^2}{c_E^4}}\,\frac{1}{r} \\ &- a\sin b\left(\frac{4\pi G_E \mu^2}{c_E^4}\right)\frac{1}{r^2} + O(r^{-3}) , \end{aligned} \tag{4.11}$$

where we have retained the r^{-2} dependence for future comparison with the Riessner-Nordstrom metric. In the same way we can show

$$A_0 = c_1 + a\cot b\left(\frac{c_E^4}{4\pi G_E}\right)^{\frac{1}{2}} - \frac{\mu}{r} + O\left(\frac{1}{r^2}\right) . \tag{4.12}$$

It is clear from (4.9), (4.10), (4.11) and (4.12) that

$$a = \sin b ,$$

$$c_1 = -\cos b\left(\frac{c_E^4}{4\pi G_E}\right)^{\frac{1}{2}} ,$$

$$\mu = e_E (4\pi)^{-1}$$

$$\cos b = \frac{G_E M_E}{c_E^2} \left(\frac{c_E^4}{4\pi G_E \mu^2} \right)^{\frac{1}{2}} = \left(\frac{4\pi G_E M_E^2}{e_E^2} \right)^{\frac{1}{2}} \tag{4.13}$$

where we must take, in all cases, the positive square root. Inserting these values for the various constants into the solution (4.6) and (4.8) gives

$$z = \cos\left\{ \sin b \left(\frac{4\pi G_E \mu^2}{c_E^4} \right)^{\frac{1}{2}} \frac{1}{r} \right\}$$

$$+ \frac{1}{\sin b} \left(\frac{4\pi G_E M_E^2}{e_E^2} \right)^{\frac{1}{2}} \sin\left\{ \sin b \left(\frac{4\pi G_E \mu^2}{c_E^4} \right)^{\frac{1}{2}} \frac{1}{r} \right\}, \tag{4.14}$$

$$A_0 = -\frac{M_E c_E^2}{e_E}$$

$$+ \sin b \left(\frac{c_E^4}{4\pi G_E} \right)^{\frac{1}{2}} \cot\left\{ \sin b \left(\frac{4\pi G_E \mu^2}{c_E^4} \right) \frac{1}{r} \right\}. \tag{4.15}$$

<u>CASE ONE</u>: $0 < \cos b < 1$

In this case a, b are real, and $4\pi G_E M_E^2 < e_E^2$. The metric is well-behaved and regular outside a certain radius r_0, given by

$$r_0 = \frac{\sin b}{\pi - b} \left(\frac{G_E e_E^2}{4\pi c_E^4} \right)^{\frac{1}{2}} \tag{4.16}$$

where we have assumed, without loss of generality, that $0 < b < \pi/2$. The metric function $z = (-g^{00})^{\frac{1}{2}}$ has zeros at r_0 and at a countable infinity of points in $r < r_0$. For $r \gg r_0$, the expansions (4.11) and (4.12) are very accurate. Using (4.13), these become

$$z = 1 + \frac{G_E M_E}{c_E^2} \frac{1}{r} - \left(1 - \cos^2 b\right)^{\frac{1}{2}} \frac{G_E e_E^2}{4\pi c_E^4} \frac{1}{r^2}$$

$$+ O(r^{-3}), \tag{4.17}$$

$$A_0 = -\frac{e_E}{4\pi} \frac{1}{r} + O(r^{-2}). \tag{4.18}$$

In general relativity the corresponding Riessner-Nordstrom solution is given by

$$A_0 = -\frac{e_E}{4\pi}\frac{1}{r} ,$$

$$-g_{00} = 1 - \frac{2G_E M_E}{c_E^2}\frac{1}{r} + \frac{G_E e_E^2}{4\pi c_E^4}\frac{1}{r^2} . \tag{4.19}$$

For large r , we can see that the solutions are equivalent up to first order in r^{-1} . For smaller r , and for a nonzero cos b, the theories predict different gravitational fields.

CASE TWO: $1 < \cos b$

In this case the constants a , b are pure imaginary, and $e_E^2 < 4\pi G_E M_E^2$. The solutions for z and A_0 are then

$$z = \frac{1}{a'} \sinh (a'u + b') , \tag{4.20}$$

$$A_0 = -\frac{M_E c_E^2}{e_E} + a'\left(\frac{c_E^4}{4\pi G_E}\right)^{\frac{1}{2}} \coth (a'u + b') , \tag{4.21}$$

where $a' = -ia = (\cos^2 b - 1)^{\frac{1}{2}}$, $b' = -ib$, and $\cos b$ is given by (4.13). The metric function z has no singularities except at the origin, and is regular everywhere. For large r the expansions (4.17) and (4.18) again are applicable provided the factor $(1 - \cos^2 b)^{\frac{1}{2}}$ is replaced by $(\cos^2 b - 1)^{\frac{1}{2}}$.

An important special case of this group of solutions is when we put $e_E = 0$. (4.20) reduced to

$$z = \exp\left(G_E M_E c_E^{-2} r^{-1}\right) , \tag{4.22}$$

and is the exterior metric function for a point particle with no charge. (4.21) becomes simply $A_0 = 0$.

CASE THREE: $\cos b = 1$

In this case $e_E^2 = 4\pi G_E M_E^2$. This situation occurs in most of the Weyl-Majumdar models. A simple calculation shows that (4.14), (4.15) reduce to

$$Z = 1 + \frac{G_E M_E}{c_E^2} \frac{1}{r} , \tag{4.23}$$

$$A_0 = -\frac{e_E}{4\pi} \left(r + \frac{G_E M_E}{c_E^2} \right)^{-1} , \tag{4.24}$$

where $G_E M_E c_E^{-2} = \left(G_E e_E^2 c_E^{-4} / 4\pi \right)^{\frac{1}{2}}$ is the Schwarzschild radius of the sphere. The line element is given by

$$\begin{aligned} ds^2 = & - \left(1 + \frac{G_E M_E}{c_E^2 r} \right)^{-2} (dx^0)^2 \\ & + \left(1 + \frac{G_E M_E}{c_E^2 r} \right)^2 \left(dr^2 + r^2 d\Omega^2 \right) , \end{aligned} \tag{4.25}$$

where $d\Omega^2 = \sin^2\theta \, d\varphi^2 + d\theta^2$. It is interesting to note that general relativity predicts the same results as the above. The solution (4.23) has been studied by Bonnor (1960, 1964), and by Papapetrou (1947).

CASE FOUR: $\cos b = 0$

In this case $M_E = 0$. Bonnor (1960) has pointed out a very peculiar feature of the Riessner-Nordstrom solution in general relativity. If we put $M_E = 0$ and at the same time retain $e_E \neq 0$, we obtain a solution for a charged point particle which has no gravitational mass. The gravitational potential is of order r^{-2} and it appears as if the electrostatic energy does not contribute to the gravitational mass.

A similar situation occurs in the present theory, for, if we take $M_E = 0$, the solutions (4.14) and (4.15) become

$$Z = \cos (r_1 / r) , \tag{4.26}$$

$$A_0 = \left(\frac{c_E^4}{4\pi G_E} \right)^{\frac{1}{2}} \tan (r_1 / r) , \tag{4.27}$$

where

$$r_1 = \left(G_E e_E^2 / 4\pi c_E^4 \right)^{\frac{1}{2}} . \tag{4.28}$$

For $r \gg r_1$, the expansion in z yields no r^{-1} dependence and therefore the system exerts no long-range gravitational attraction.

CHAPTER V

THE WEYL-MAJUMDAR METHOD FOR OBTAINING STATIC SOLUTIONS OF THE FIELD EQUATIONS

Before proceeding with our investigation of the system of equations (3.34)-(3.37) in the general case, it is instructive to examine the one case in which the system can be solved analytically.

Consider a distribution of charged matter in equilibrium. Provided no other forces are present we can say that the equilibrium is maintained by a balance of electrostatic and gravitational forces. Such a balance implies, for time-invariant systems, a relation between the electrostatic and gravitational potentials. Weyl (1917) postulated that this relation could be in the form of a functional relationship between the component g_{00} of the metric tensor and the electrostatic potential A_0. Assuming this, he showed that

$$g_{00} = A + BA_0 + \frac{4\pi G_E}{c_E^4} A_0^2 , \tag{5.1}$$

where A and B are constants. He obtained this result by studying axially symmetric electrovac universes in general relativity. Majumdar (1947) extended this work to the case where there is no spatial symmetry, and showed that (5.1) remains valid. If the constant B is chosen so that (5.1) reduces to a perfect square, then the whole system of the combined Einstein-Maxwell field equations reduces to a single Laplace equation. These ideas were carried over to the case of non-empty space by Das (1962) who showed that, in certain cases, the imposition of the Weyl-Majumdar relation,

$$g_{00} = \left\{ A^{\frac{1}{2}} \pm \left(\frac{4\pi G_E}{c_E^4} \right)^{\frac{1}{2}} A_0 \right\}^2 , \tag{5.2}$$

implies, and is implied by, the equality of the dimensionless charge and mass parameters of the system in question.

In this chapter the Weyl relation (5.1) is examined in the framework of the scalar theory of gravitation. All of the results and all the examples shown here have counterparts in general relativity. First of all, the relation analogous to (5.1) valid for the scalar theory is obtained. An attempt is then made to understand the physical meaning of the constant B . This is first done for the spherically symmetric case, for which the solution in the exterior region is given in Chapter IV. It is shown that, irrespective of the charge-mass ratio, every (exterior) solution obeys the relation (5.1), and that $B = 8\pi G_E M_E c_E^{-2} e_E^{-1}$, where M_E is the total gravitational mass, and e_E the total charge of the sphere. This result is then extended to the general case where it is shown that, even without sphericaly symmetry, B represents the mass-charge ratio of the system.

It is interesting to note that, with this value for B , (5.1) reduces to (5.2) only if

$$e_E = \pm (4\pi G_E)^{\frac{1}{2}} M_E . \tag{5.3}$$

This is probably why, in previous work on this problem, the assumption (5.2) always led to the result (5.3).

Difficulties arise when we attempt to extrapolate (5.1) into the interior region. It turns out that, both for the Klein-Gordon and Dirac fields, imposing (5.1) in this region leads to an over-determined system of equations, unless (5.3) is valid.

The Weyl-Majumdar relation for our theory is obtained by considering the free-space electrostatic-gravitational equations. These are, from (3.9), (3.10),

$$\partial_m \left(S^{-2} \partial_m A_o \right) = 0 \ , \tag{5.4}$$

$$\partial_{mm} \phi = \frac{4\pi G_E}{c_E^2} S^{-2} \left(\partial_m A_o \right)^2 . \tag{5.5}$$

We assume that ϕ is a function only of A_0, which means that

$$\partial_{mm} \phi = \phi'' \left(\partial_m A_o \right)^2 + \phi' \partial_{mm} A_o \ , \tag{5.6}$$

where the prime denotes differentiation with respect to A_0. Inserting (5.6) into (5.4), (5.5) yields

$$\partial_{mm} A_o = 2 \phi' c_E^{-2} \left(\partial_m A_o \right)^2 , \tag{5.7}$$

$$\partial_{mm} A_o = \phi'^{-1} \left(\frac{4\pi G_E}{c_E^2} S^{-2} - \phi'' \right) \left(\partial_m A_o \right)^2 , \tag{5.8}$$

and hence

$$\phi'' + 2 c_E^{-2} (\phi')^2 = \frac{4\pi G_E}{c_E^2} S^{-2} \ . \tag{5.9}$$

The solution of (5.9) is given by

$$S^2 = A + B A_o + \frac{4\pi G_E}{c_E^4} A_o^2 \ , \tag{5.10}$$

which corresponds to the relation (5.1) in general relativity. The constant A has no physical significance, and, without loss of generality, may be taken to be equal to 1. The constant B is at present undetermined. If, however, it is so chosen that (5.10) becomes a perfect square, then the field equations (5.7), (5.8) reduce to a single Laplace equation.

$$\partial_{mm} (S^{-1}) = 0 \ .$$

In Chapter IV we obtained solutions for A_o and S in the exterior region of a spherically symmetric charged matter distribution. If e_E is the charge, and M_E the gravitational mass of the sphere, then these solutions are given by (4.14), (4.15) in the case where $a = \sin b \neq 0$, by (4.23), (4.24) in the case where $a = 0$, and by (4.26), (4.27) in the case where $\cos b = 0$. It is easy to show that all these sets of solutions obey the relation (5.10). In the first case

$$S^2 = 1 + \frac{8\pi G_E M_E}{c_E^2 e_E} A_0 + \frac{4\pi G_E}{c_E^4} A_0^2 , \tag{5.11}$$

where $e_E^2 \neq 4\pi G_E M_E^2$, in the second case

$$S^2 = \left(1 \pm \sqrt{\frac{4\pi G_E}{c_E^4}} A_0\right)^2 , \tag{5.12}$$

where $e_E^2 = 4\pi G_E M_E^2$, and in the last case

$$S^2 = 1 + \frac{4\pi G_E}{c_E^4} A_0^2 \tag{5.13}$$

where $M_E = 0$. We see that, in all cases

$$B = \frac{8\pi G_E M_E}{c_E^2 e_E} . \tag{5.14}$$

If we assume that (5.11) is valid even in the presence of charged matter, then this implies a simple relationship between the charge and mass densities. In such a case the field equations can be written

$$\partial_m (S^{-2} \partial_m A_0) = \sigma , \tag{5.15}$$

$$\partial_{mm} \phi - \frac{4\pi G_E}{c_E^4} S^{-2} (\partial_m A_0)^2 = \frac{4\pi G_E}{c_E^4} \rho , \tag{5.16}$$

where σ, ρ are respectively the charge and gravitational mass densities. For the moment we have left these quite general. Using (5.10), we easily obtain

$$\sigma = S^{-2} \phi'^{-1} \left(\frac{4\pi G_E}{c_E^4} \rho\right) , \tag{5.17}$$

which yields,

$$\rho = \left(\frac{c_E^4 B}{8\pi G_E} + A_0\right) \sigma . \tag{5.18}$$

Previously we have obtained the expression (5.14) for B in the case of spherical symmetry. Using (5.18) we can extend this result to the more general case. From (5.15), (5.16), by considering the fields far from the source, it can be shown that the charge and gravitational mass parameters are given by

$$\begin{aligned} e_E &= \int \sigma \, d^3x \,, \\ M_E c_E^2 &= \int \{ S^{-2}(\partial_m A_0)^2 + \rho \} \, d^3x \,. \end{aligned} \tag{5.19}$$

Using (5.15), (5.18), we obtain

$$\begin{aligned} M_E c_E^2 = & \frac{c_E^4 B}{8\pi G_E} \int \sigma \, d^3x \\ & + \int \partial_m \left(S^{-2} A_0 \partial_m A_0 \right) d^3x \,. \end{aligned} \tag{5.20}$$

Provided that $S^{-2} A_0 \partial_m A_0$ falls off sufficiently rapidly at spatial infinity (and this is always true for a localised distribution), it follows from (5.19) that (5.20) yields $B = 8\pi G_E M_E c_E^{-2} e_E^{-1}$ which is again (5.14).

As a first example we consider a scalar (Klein-Gordon) matter field. For such a system the time-independent field equations are

$$\begin{aligned} \partial_{mm}\chi &= \frac{S^{-2}}{\hbar^2 c_E^2} \left\{ m_E^2 c_E^4 - S^{-2}(E + e_E A_0)^2 \right\} \chi \,, \\ \partial_m \left(S^{-2} \partial_m A_0 \right) &= \frac{e_E S^{-4}}{m_E c_E^2} \left(E + e_E A_0 \right) \chi^2 \,, \\ \partial_{mm}\phi &= \frac{4\pi G_E}{c_E^4} \Big[S^{-2} (\partial_m A_0)^2 \\ & \quad + \frac{S^{-2}}{m_E c_E^2} \left\{ 2 S^{-2} (E + e_E A_0)^2 - m_E^2 c_E^4 \right\} \chi^2 \,. \end{aligned} \tag{5.21}$$

The Klein-Gordon wave-function is given by

$$\psi = \chi \exp\left(i E t / \hbar c_E \right) ,$$

where χ is real and E represents the energy of the matter field. For such a system

$$\rho = \frac{S^{-2}}{m_E c_E^2}\left\{2S^{-2}(E+e_E A_o)^2 - m_E c_E^4\right\}\chi^2,$$

$$\sigma = \frac{e_E}{m_E c_E^2} S^{-4}(E+e_E A_o)\chi^2 . \tag{5.22}$$

Inserting (5.22) into (5.18), and cancelling out the common factor χ^2, we obtain

$$S^2 = \frac{2}{m_E^2 c_E^4}\left(E + e_E A_o\right)^2 - \frac{1}{m_E^2 c_E^4}\left(M_E c_E^2 + e_E A_o\right)\left(E + e_E A_o\right) \tag{5.23}$$

Comparing (5.23) with (5.11), and equating the coefficients of the various powers of A_o, we find that the only Weyl-type solution for the Klein-Gordon field is the Weyl-Majumdar solution with

$$E = M_E c_E^2 = m_E c_E^2 = \left(\frac{e_E^2 c_E^4}{4\pi G_E}\right)^{\frac{1}{2}} \tag{5.24}$$

The binding energy, defined by $E_B = (m_E c_E^2 - E)$ is equal to zero, and the system of equations (5.21) can be reduced to two equations,

$$\partial_{mm}\chi = 0 ,$$

$$\partial_{mm}(S^{-1}) = -e_E S^{-3}\chi^2 . \tag{5.25}$$

Similar results hold for the Dirac field, but the derivation is more laborious. The time-independent field equations (for the Dirac-electrostatic-gravitational fields) are given by (3.8), (3.9) and (3.10). In our present notation

$$\sigma = e_E S^{-2}\left(|u|^2 + |v|^2\right)$$

$$\rho = 2S^{-2}(E + e_E A_o)\left(|u|^2 + |v|^2\right) - m_E c_E^2 S^{-1}\left(|u|^2 - |v|^2\right) . \tag{5.26}$$

The relation (5.18) becomes

$$S^{-2}(2E - M_E c_E^2 + e_E A_0)(|u|^2 + |v|^2) = m_E c_E^2 S^{-1}(|u|^2 - |v|^2), \tag{5.27}$$

and this expression holds for all x . For points very distant from the centre of the matter distribution $A_0 \approx 0$, $S \approx 1$, and the Dirac field is essentially free. It can be shown that, as we approach spatial infinity

$$|u|^2 - |v|^2 = \frac{E}{m_E c_E^2}(|u|^2 + |v|^2), \tag{5.28}$$

and hence, in this region (5.27) yields

$$2E - M_E c_E^2 = E, \tag{5.29}$$

which means that $E = M_E c_E^2$. (For a demonstration of (5.28) see, for example, Corinalderi and Strocchi 1963 p. 156, and our own results in Chapter 6). Inserting (5.29) in (5.27) we obtain

$$\begin{aligned} S^{-2}(E + e_E A_0)(|u|^2 + |v|^2) \\ = m_E c_E^2 S^{-1}(|u|^2 - |v|^2). \end{aligned} \tag{5.30}$$

Comparing (5.30) with the corresponding expression for the Klein-Gordon field, we find that the wave-function does not cancel out. This is due to the interaction between the gravitational field and the spin of the Dirac particle. Since S is a function of A_0 only, (5.30) gives us an expression relating A_0, $|u|^2$, and $|v|^2$. There is no reason why the Dirac equations (3.8) should be consistent with this relation and therefore, in general, we have an over-determined system. To show this we will examine in detail the above equations for the case of spherical symmetry. If we write

$$\alpha(r) = S^{-2} \hbar^{-1} c_E^{-1} (E + e_E A_0),$$

$$\beta(r) = S^{-1} \hbar^{-1} c_E^{-1} (m_E c_E^2),$$

$$\sigma_{\pm} = \left(F^2 \pm G^2 \right) , \tag{5.31}$$

the field equations (3.24), (3.25), (3.30) and (3.31) become

$$(\alpha - \beta) F = -\left(\frac{d}{dr} + \frac{k}{r} \right) G , \tag{5.32}$$

$$(\alpha + \beta) G = \left(\frac{d}{dr} - \frac{k}{r} \right) F , \tag{5.33}$$

$$\frac{d}{dr}\left(r^2 S^{-2} \frac{dA_0}{dr} \right) = \frac{e_E}{4\pi} S^{-2} \sigma_+ , \tag{5.34}$$

$$\frac{d}{dr}\left(r^2 \frac{d\phi}{dr} \right) = \frac{4\pi G_E}{C_E{}^4} S^{-2} r^2 \left(\frac{dA_0}{dr} \right)^2 + \frac{k C_E G_E}{C_E{}^4} \left(2\alpha \sigma_+ - \beta \sigma_- \right) . \tag{5.35}$$

The Weyl condition (5.30) can be written

$$\alpha \sigma_+ = \beta \sigma_- . \tag{5.36}$$

Multiplying (5.32) by $(\alpha + \beta) G$, (5.33) by $(\alpha - \beta) F$, and subtracting we get

$$\frac{\alpha}{2} \frac{d\sigma_+}{dr} - \frac{\beta}{2} \frac{d\sigma_-}{dr} - \frac{k}{r} \left(\alpha \sigma_- - \beta \sigma_+ \right) = 0 . \tag{5.37}$$

If instead we add, we get

$$\frac{\alpha}{2} \frac{d\sigma_-}{dr} - \frac{\beta}{2} \frac{d\sigma_+}{dr} - \frac{k}{r} \left(\alpha \sigma_+ - \beta \sigma_- \right) = 2 \left(\alpha^2 - \beta^2 \right) FG , \tag{5.38}$$

where $4F^2G^2 = \sigma_+{}^2 - \sigma_-{}^2$. A combination of (5.36), (5.37) yields

$$\frac{\beta}{2} \sigma_+ \left\{ \frac{d}{dr}\left(\frac{\alpha}{\beta} \right) - \frac{2k}{r} \left(1 - \frac{\alpha^2}{\beta^2} \right) \right\} = 0 . \tag{5.39}$$

The solutions $\beta = 0$, $\sigma_+ = 0$ are trivial. We recall that $\sigma_+{}^2 \geq \sigma_-{}^2$ and therefore $\beta^2 \geq \alpha^2$. Assuming $\beta \neq 0$, $\sigma_+ \neq 0$, from (5.39) we obtain

$$\frac{\alpha}{\beta} = k \left(\frac{\mu^2 r^4 - 1}{\mu^2 r^4 + 1} \right) , \tag{5.40}$$

where μ is a constant of integration. Using (5.11), (5.31), (5.40), and recalling that $k^2 = 1$ for spherically symmetric solutions, we obtain the following expression which is valid for all r,

$$\left\{E + e_E A_0(r)\right\}^2 S(r)^{-2} = m_E c_E^{\,2}\left(\frac{\mu^2 r^4 - 1}{\mu^2 r^4 + 1}\right)^2 . \tag{5.41}$$

There are two ways of looking at this equation. If $\mu \neq 0$ then we must assume it is an ordinary equation and solve accordingly for A_0. (Recall that S is a function only of A_0.) On the other hand, if $\mu = 0$, then the expression on the right-hand-side is a constant, which means that if we are to obtain a non-trivial solution for A_0 we must assume that (5.41) is an identity in A_0 and equate the coefficients of the various powers of A_0. In the first case, $\mu \neq 0$, we find

$$A_0 = -\frac{E}{e_E}\left(1 \mp \sqrt{\frac{\rho^2}{\varepsilon^2}\,\frac{(1-\delta\varepsilon^2)}{(1-\delta\rho^2)}}\right), \tag{5.42}$$

where $\rho = \alpha\beta^{-1}$, $\varepsilon = E\,(m_E c_E^2)^{-1}$ and $\delta = 4\pi\, G_E m_E^2 e_E^{-2}$. S must always be positive and therefore the solutions for S, A_0 are as follows

$$S = \left(\frac{1-\delta\varepsilon^2}{1-\delta\rho^2}\right)^{\frac{1}{2}},$$

$$A_0 = -\frac{E}{e_E}\left(1 - \frac{\rho}{\varepsilon}S\right). \tag{5.43}$$

These expressions are regular for all r. However, when we use (5.43) to solve for σ_+, σ_- individually, we obtain expressions which are not compatible with the second of the Dirac equations (5.38). The proof of this is long and tedious so we will not reproduce it here. It is enough to state that $\mu \neq 0$ does not lead to a consistent set of solutions.

However, if $\mu = 0$, then we can look upon (5.37) as an identity in A_0. Expanding S^2 by means of (5.11) and equating coefficients of the various powers of A_0, we obtain

$$E = m_E c_E^2 ,$$

$$e_E^2 = 4\pi G_E E^2 c_E^{-4}. \tag{5.44}$$

A_0 itself remains undetermined and we now have the correct number of equations and unknowns. The results (5.44) are the same as were obtained for the Klein-Gordon field, and correspond to a solution of Weyl-Majumdar type.

Returning briefly to the case of no spatial symmetry we find that the conditions (5.44) imply, from (5.11), (5.26)

$$S^2 = \left(1 + \sqrt{\frac{4\pi G_E}{c_E^4}}\, A_0 \right)^2 ,$$

$$E\left(|u|^2 + |v|^2\right) = E\left(|u|^2 - |v|^2\right) . \tag{5.45}$$

Obviously $|v|^2 = 0$, and the equations for the system become

$$i\, \sigma_j \partial_j u = 0 , \tag{5.46}$$

$$\partial_{mm}(S^{-1}) = -4\pi G_E E c_E^{-4} S^{-2} |u|^2 . \tag{5.47}$$

There is a close relationship between (5.46), (5.47) and the corresponding equations for the Klein-Gordon field (5.25). We have a different power of S on the right hand side of (5.47) but this is due to the interaction of the spin with the gravitational field.

The equation (5.46) can be solved. We can write

$$\sigma_j \partial_j = \sigma_r \left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{K}{r} \right) \tag{5.48}$$

where $K = \underline{\sigma} \cdot \underline{l} + 1$ is the angular momentum operator. We have only to consider the following equation

$$\left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{K}{r} \right) u = 0 , \tag{5.49}$$

for the two-spinor u. The solutions can be expressed in terms of the eigenfunctions of K (Corinalderi and Strocchi 1963)

$$u_j^{(m_j)+} = f^+(r) \begin{pmatrix} C_1^+ \; Y_{j-\frac{1}{2}}^{(m_j-\frac{1}{2})} \\ C_2^+ \; Y_{j-\frac{1}{2}}^{(m_j+\frac{1}{2})} \end{pmatrix} ,$$

$$u_j^{(m_j)-} = f^-(r) \begin{pmatrix} C_1^- \; Y_{j+\frac{1}{2}}^{(m_j-\frac{1}{2})} \\ C_2^- \; Y_{j+\frac{1}{2}}^{(m_j+\frac{1}{2})} \end{pmatrix} , \tag{5.50}$$

where the $C^{\pm}$ are normalisation constants. The eigenvalues for k are

$$k^+ = j + \tfrac{1}{2} = \ell + 1 ,$$

$$k^- = -(j+\tfrac{1}{2}) = -\ell , \tag{5.51}$$

and the functions f obey the radial equations

$$\left(\frac{d}{dr} + \frac{1}{r} - \frac{(\ell+1)}{r} \right) f^+ = 0 ,$$

$$\left(\frac{d}{dr} + \frac{1}{r} + \frac{\ell}{r} \right) f^- = 0 . \tag{5.52}$$

Inserting the value for $|u|^2$ into the second equation of (5.46), we obtain a countable infinity of possible equations for S^{-1}, each one corresponding to a different angular momentum state. This is in contrast to the single equation described by Das (1962), who assumed that u was constant.

For the case of spherical symmetry we must have $k = \pm 1$, and the only solution of (5.51) which tends to zero for large r is given by

$$f = f_0 \, r^{-2} , \tag{5.53}$$

where f_0 is a constant and $k = -\ell = -1$. From (5.53) we have $|u|^2 = (4\pi)^{-1} r^{-4} f_0^{\,2}$, and the equation for S^{-1} becomes

$$\frac{d}{dr}\left(r^2 \frac{dS^{-1}}{dr}\right) = -\lambda r^{-2} S^{-2} , \tag{5.54}$$

where $\lambda = G_E E\, C_E^{-4} f_0^{\,2}$. Writing $z = S^{-1}$, $y = r^{-1}$, (5.54) can be simplified

$$\frac{d^2 z}{dy^2} = -\lambda z^2 . \tag{5.55}$$

A particular solution of (5.55) is given by

$$z = -\frac{6}{\lambda} y^{-2} . \tag{5.56}$$

This solution diverges for large r and must be discarded. The general solution (of (5.55)) can be written down in terms of elliptic integrals. Transforming (5.55) yet again, we find

$$\frac{dz}{dy} = \pm \left(\tilde{B} - \lambda \tfrac{2}{3} z^3\right)^{\frac{1}{2}} \tag{5.57}$$

$\tilde{B}$ is a constant of integration and is determined by the normalisation. We require

$$r^2 \frac{dz}{dr}\bigg|_{r=0} = 0 , \tag{5.58}$$

and hence, if we write $z_0 = z(r=0)$,

$$\tilde{B} = \frac{2}{3} \lambda z_0^{\,3} . \tag{5.59}$$

(5.57) can be transformed to integral form. It becomes

$$\tilde{B}^{\frac{1}{2}} \int_0^y dy = \int_{t_0}^{t} \frac{dt}{(1-t^3)^{\frac{1}{2}}} , \tag{5.60}$$

where $t = z z_0^{-1}$ and $t_0 = z_0^{-1}$. Since z is always positive, so is t, and therefore $z \leq z_0$ for all y. We use the general formula

$$\int_t^1 \frac{dt}{(1-t^3)^{\frac{1}{2}}} = \frac{1}{\sqrt[4]{3}} \, cn^{-1}\left(\frac{\sqrt{3}-1+t}{\sqrt{3}+1-t}, k'\right) \tag{5.61}$$

where $k' = \sin 75°$, to obtain a formal solution for our problem. We have, then

$$t = \frac{(\sqrt{3}+1)\, cn\{\sqrt[4]{3}\,\tilde{B}^{\frac{1}{2}}(y_0-y), k'\} - (\sqrt{3}-1)}{1 + cn\{\sqrt[4]{3}\,\tilde{B}^{\frac{1}{2}}(y_0-y), k'\}}, \tag{5.62}$$

where

$$\sqrt[4]{3}\,\tilde{B}^{\frac{1}{2}} y_0 = cn^{-1}\left(\frac{\sqrt{3}-1+z_0^{-1}}{\sqrt{3}+1-z_0^{-1}}, k'\right) \tag{5.63}$$

The solution (5.62) has the correct asymptotic behaviour for large r : z tends to 1 for $r \to \infty$. However, for small r, the function becomes periodic and there are singularities in the metric. The system has, from (5.54), (5.59),

$$\int_0^\infty z^2 f^2 r^2 dr = 1 . \tag{5.64}$$

To sum up: we have shown that solutions of the Dirac-Maxwell-gravitational field equations can be obtained by using the Weyl-Majumdar method and that these solutions imply that the metric tensor is not regular over the whole range.

CHAPTER VI

ASYMPTOTIC SOLUTION FOR THE DIRAC WAVE-FUNCTION

In deriving the asymptotic forms for A_0 and S (Chapter IV) we assumed that in the exterior region the matter density is negligible. In the present chapter we will demonstrate the validity of that assumption by finding the asymptotic form of the Dirac wave-functions. These solutions will be used in the numerical integration in Chapter VII to determine the boundary conditions on the wave-function.

The Dirac equations in dimensionless notation are given by (3.34) and (3.35). We consider terms in the interaction only up to first order in r^{-1}. Using (3.32), (3.33) and (4.9) we obtain

$$a_0 = -\frac{\alpha}{\rho} + O(\rho^{-2}) \ ,$$

$$z = S^{-1} = 1 + \frac{G_E M_E m_E}{\hbar c_E}\frac{1}{\rho} + O(\rho^{-2}) \ . \qquad (6.1)$$

Inserting these expressions for S and a_0 into (3.34), (3.35), we find

$$\left(\frac{d}{d\rho} + \frac{k}{\rho}\right)\tilde{G} = -\left\{\varepsilon - 1 - \frac{\alpha_1}{\rho}\right\}\tilde{F} \ , \qquad (6.2)$$

$$\left(\frac{d}{d\rho} - \frac{k}{\rho}\right)\tilde{F} = \left\{\varepsilon + 1 - \frac{\alpha_2}{\rho}\right\}\tilde{G} \ , \qquad (6.3)$$

where

$$\alpha_1 = \alpha - \frac{G_E M_E m_E}{\hbar c_E}\left(2\varepsilon - 1\right) \ , \qquad (6.4)$$

$$\alpha_2 = \alpha - \frac{G_E M_E m_E}{\hbar c_E}\left(2\varepsilon + 1\right) \ . \qquad (6.5)$$

It is interesting to note that, even when we consider only the r^{-1} dependence of the gravitational field, we cannot simply replace the flat-space expression ($E + e_\varepsilon A_0$) by ($E + e_\varepsilon A_0 + E\zeta_\varepsilon^{-2}\phi$), where $E\zeta_\varepsilon^{-2}\phi$ is the gravitational potential energy of the particle in the field of the source: the interaction is more complicated than a simple Coulomb potential. In order to illustrate this point, to which we will return later, we have left the mass M_ε of the source in its original form and only at the end of our calculations will we make the specification that M_ε corresponds to the self-field of the Dirac particle.

If we write

$$\tilde{F} = F' \exp(-\sigma) ,$$

$$\tilde{G} = G' \exp(-\sigma) , \tag{6.6}$$

where $\sigma = (1-\varepsilon^2)^{\frac{1}{2}} \rho$, the equations (6.2) and (6.3) become,

$$\left(\frac{d}{d\sigma} + \frac{k}{\sigma} - 1 \right) G_\varepsilon' = - \left(\frac{\varepsilon - 1}{\sqrt{1-\varepsilon^2}} - \frac{\alpha_1}{\sigma} \right) F' , \tag{6.7}$$

$$\left(\frac{d}{d\sigma} - \frac{k}{\sigma} - 1 \right) F' = \left(\frac{\varepsilon + 1}{\sqrt{1-\varepsilon^2}} - \frac{\alpha_2}{\sigma} \right) G' . \tag{6.8}$$

Our next step is to construct power series solutions for F' and G' . We require a localised solution (bound state) which means that F', G' must approach zero for large r . We write

$$F' = \sigma^\beta \sum_{n=0}^{\infty} a_n' \sigma^n ,$$

$$G' = \sigma^\beta \sum_{n=0}^{\infty} b_n' \sigma^n , \tag{6.9}$$

where $a_0' \neq 0$, $b_0' \neq 0$. Substituting (6.9) into (6.7) and (6.8) we obtain, by equating coefficients of $\sigma^{\nu+\beta-1}$

$$\left(\nu + \beta + k \right) b_\nu' - b_{\nu-1}' - \frac{(1-\varepsilon)}{\sqrt{1-\varepsilon^2}} a_{\nu-1}' - \alpha_1 a_\nu' = 0 ,$$

$$\left(\nu + \beta - k \right) a_\nu' - a_{\nu-1}' - \frac{(1+\varepsilon)}{\sqrt{1-\varepsilon^2}} b_{\nu-1}' + \alpha_2 b_\nu' = 0 . \tag{6.10}$$

For $\nu = 0$ we have

$$(\beta + k)\, b_0' - \alpha_1 a_0' = 0 ,$$
$$(\beta - k)\, a_0' + \alpha_2 b_0' = 0 , \tag{6.11}$$

which implies

$$\beta = \pm \left(k^2 - \alpha_1 \alpha_2 \right)^{\frac{1}{2}} . \tag{6.12}$$

The equations (6.10) lead to the following recurrence relation for the a_ν',

$$\left\{ \nu + \beta - k + \alpha_2 \frac{\alpha_1 \sqrt{1-\varepsilon^2} + (1-\varepsilon)(\nu+\beta-k)}{\sqrt{1-\varepsilon^2}\,(\nu+\beta+k) - \alpha_2 (1-\varepsilon)} \right\} a_\nu'$$
$$= \left\{ 1 + \frac{1+\varepsilon}{\sqrt{1-\varepsilon^2}} \frac{\alpha_1 \sqrt{1-\varepsilon^2} + (1-\varepsilon)(\nu+\beta-k-1)}{\sqrt{1-\varepsilon^2}\,(\nu+\beta+k+1) - \alpha_2 (1-\varepsilon)} \right\} a_{\nu-1}' . \tag{6.13}$$

A similar recurrence relation for the b_ν' is obtained by replacing

$$a_\nu' = \frac{\sqrt{1-\varepsilon^2}\,(\nu+\beta+k) - \alpha_2 (1-\varepsilon)}{\alpha_1 \sqrt{1-\varepsilon^2} + (1-\varepsilon)(\nu+\beta-k)}\, b_\nu' . \tag{6.14}$$

For very large ν, we have from (6.13)

$$a_\nu' = \frac{2}{\nu}\, a'_{\nu-1} , \tag{6.15}$$

and this would imply that the functions F', G' increase like exp (2σ) for large σ. This contradicts our hypothesis of a localised matter distribution. Therefore the series must terminate at some finite value of n, say n', and this means that

$$a'_{n'+1} = b'_{n'+1} = 0 . \tag{6.16}$$

Putting $\nu = n'+1$, n' in (6.10) and (6.14) respectively, and solving for β we obtain

$$\beta = -n' - (1-\varepsilon^2)^{-\frac{1}{2}} \left\{ \varepsilon\alpha - \frac{G_E M_E m_E}{\hbar c_E} \left(2\varepsilon^2 - 1 \right) \right\} . \tag{6.17}$$

On the other hand, we have, from (6.12)

$$\beta = \pm\left\{ k^2 - \left[\alpha - \frac{G_E M_E m_E}{\hbar c_E}(2\varepsilon - 1)\right]\left[\alpha - \frac{G_E M_E m_E}{\hbar c_E}(2\varepsilon + 1)\right]\right\}^{\frac{1}{2}} . \tag{6.18}$$

From (6.17) and (6.18) we can obtain an expression for ε. However, before continuing with the present problem we shall consider, as a corollary and an example of the above work, the "hydrogen-atom" problem in gravitational theory. We treat the central body as a fixed point-particle of mass M_E and we consider only terms up to order r^{-1} in the interaction. (6.17) and (6.18) apply with $\alpha = 0$. We must take the positive sign for β in (6.18). Since ε is contained in the expression for β, we have a much more complicated system than the one which occurs in the electrostatic case. The eigenvalues are given by the solutions of the following equation, which is third order in ε^2.

$$\begin{aligned} &16 n'^2\mu^2\varepsilon^6 - \left\{32 n'^2\mu^2 + (k^2 - n'^2)^2 + 2\mu^2(k^2 - n'^2) + \mu^4\right\}\varepsilon^4 \\ &+ \left\{20 n'^2\mu^2 - 2(k^2 - n'^2)^2 - 2(k^2 - n'^2)\mu^2\right\}\varepsilon^2 \\ &+ \left\{(k^2 - n'^2)^2 - 4n'^2\mu^2\right\} = 0 , \end{aligned} \tag{6.19}$$

where $\mu = G_E M_E m_E \hbar c_E^{-1}$. The eigenvalues are labelled by the values of the angular quantum number k and the radial quantum number n', which refers to the number of nodes of the wave-function. In contrast with the usual (electrostatic) hydrogen atom problem, there is no natural way to define a "principal quantum number". For $n' = 0$, we have

$$\varepsilon_{k,0} = \left(\frac{k^2}{k^2 + \mu^2}\right)^{\frac{1}{2}} . \tag{6.20}$$

Brill and Wheeler (1957) have also considered this problem. They examined the behaviour of a Dirac electron in a Schwarzschild gravitational field. Up to order r^{-1}, it can be shown that their radial equations are equivalent to ours. However, in solving their equations, they used a different approximation, and neglected one term of order r^{-1}, which we have included, and therefore their results differ from ours. Explicityly, their radial equations were of the form (Equation (39) of their paper)

$$\frac{1}{\hbar c_E}\left[\, e^{-\frac{1}{2}\nu}\left(E + e_E A_0\right) + m_E c_E^2 \right] \tilde{F} - e^{-\frac{1}{2}\lambda} \frac{d\tilde{G}}{dr} - \frac{k}{r}\tilde{G} = 0\,, \tag{6.21}$$

where $e^{\nu} = e^{-\lambda} = 1 - 2G_E M_E c_E^{-2} r^{-1}$. Expanding, and keeping all terms in r^{-1}, we obtain

$$\frac{1}{\hbar c_E}\left(E + e_E A_0 - \phi + m_E c_E^2\right) \tilde{F} - \frac{d\tilde{G}}{dr} - \frac{k}{r}\tilde{G} - \phi E^{-1}\frac{d\tilde{G}}{dr} = 0\,, \tag{6.23}$$

where $\phi = -G_E M_E E c_E^{-2} r^{-1}$ is the gravitational potential energy of a particle of energy E. From (6.6) we have

$$\phi E^{-1}\frac{d\tilde{G}}{dr} \cong \frac{1}{\hbar c_E}\frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}\,\phi\,\tilde{G}\,, \tag{6.24}$$

again to first order in r^{-1}. Brill and Wheeler dropped this term since, in their case, the coefficient of $r^{-1}\tilde{G}$ was small compared to k. In our case, on the other hand, we have included all terms of order r^{-1}, in order to obtain a more general result.

For the case of a Dirac electron in its own gravitational field, we make the assumption that $M_E = E c_E^{-2}$, or, in other words, that E is the total energy of the Dirac field. On this point Kaup (1968) and Feinblum and McKinley (1968) differed. Kaup defined M_E as follows

$$M_E = \frac{1}{c_E^2}\,\frac{m}{N}\,m_E\,, \tag{6.25}$$

where

$$\mathfrak{m} = (m_E c_E^2)^{-1} \int T^0{}_0 \, d^3x \, ,$$

$$N = \int \sqrt{-g} \, J^0 \, d^3x \quad . \qquad (6.26)$$

$\mathfrak{m}$ is the integral of the zero-zero component of the energy-momentum tensor of the Klein-Gordon field, and N is a conserved quantity. (For the case of a charged field J^0 is the charge density.) He found that, in general, with the definition (6.25), $M_E c_E^2 \neq E$.

Feinblum and McKinley, however, used a different normalisation, which generated a different N , and therefore Kaup's results do not hold in their case. They simply assumed, but did not prove, that E = $M_E c_E^2$.

In the present theory, if we use (6.25), then it is easy to show, from (2.54), (2.57) that

$$\mathfrak{m} = (m_E c_E^2)^{-1} \int (-\tfrac{1}{2} \ell c_E) \, (\bar{\psi} \gamma^0 \partial_0 \psi - \partial_0 \bar{\psi} \gamma^0 \psi) \, S \, d^3x \, ,$$

$$= \frac{E}{m_E c_E^2} \int \psi^+ \psi \, d^3x \quad ,$$

$$N = \int \psi^+ \psi \, d^3x \quad , \qquad (6.27)$$

and therefore $M_E c_E^2 = E$. Unfortunately, due to the complications of the field equations, there is no simple way to prove that the M_E as defined by (6.25) is in fact the total gravitational mass of the system. We simply assume that it is.

For a spherically symmetric solution $k = \pm 1$ and hence $n' = 0$ . . From (6.17) and (6.18) we obtain

$$1 - \alpha^2 - \varepsilon^2 + 2\alpha\tau\varepsilon^2 - \tau^2\varepsilon^4 = 0 \, , \qquad (6.28)$$

where τ is given by (3.32). The solution of (6.28) is

$$\varepsilon^2 = (2\tau^2)^{-1} \left\{ (1 - 4\alpha\tau + 4\tau^2)^{\frac{1}{2}} - 1 + 2\alpha\tau \right\}. \tag{6.29}$$

Hence, if we know τ, which is essentially the square of the bare mass of the Dirac particle, we can calculate ε. It is remarkable that only for one value of m_ε, given by letting $\alpha = \tau$, do we obtain $\varepsilon = 1$, and therefore only in that case does the binding energy of the field vanish. (This is in fact the Weyl-Majumdar solution which has been examined in a previous section, and in which case it is possible to obtain an analytic solution and to prove that $E = M_\varepsilon c_\varepsilon^2$.)

From (6.9), (6.10), the asymptotic solutions for $\tilde{F}$, $\tilde{G}$ are

$$\tilde{F} = a_0'(1-\varepsilon^2)^{\beta/2} \rho^\beta \exp\left(-\sqrt{1-\varepsilon^2}\,\rho\right), \tag{6.30}$$

$$\tilde{G} = -\frac{(1-\varepsilon)}{(1-\varepsilon^2)^{1/2}} \tilde{F} \tag{6.31}$$

where ε and β are given by (6.29) and (6.17). In the following chapter we will use these expressions as boundary conditions for the numerical integration.

CHAPTER VII

NUMERICAL RESULTS AND DISCUSSION

The non-linear system of equations (3.34)-(3.37) has been solved by numerical methods. Briefly, our procedure was as follows. We first chose a value ρ_0 of the radial co-ordinate so large that the asymptotic solutions obtained previously were valid. A step-by-step numerical integration toward the origin was then begun. In the integration process, the four equations (3.34)-(3.37) were replaced by the six first-order equations,

$$\frac{d\tilde{G}}{d\rho} = -\frac{k}{\rho}\tilde{G} - S^{-2}(\varepsilon + a_0 - S)\tilde{F}, \tag{7.1}$$

$$\frac{d\tilde{F}}{d\rho} = \frac{k}{\rho}\tilde{F} + S^{-2}(\varepsilon + a_0 + S)\tilde{G}, \tag{7.2}$$

$$\frac{da_0}{d\rho} = S^2\rho^{-2}y_1, \tag{7.3}$$

$$\frac{d\Phi}{d\rho} = \rho^{-2}y_2, \tag{7.4}$$

$$\frac{dy_1}{d\rho} = S^{-2}(\tilde{F}^2 + \tilde{G}^2), \tag{7.5}$$

$$\frac{dy_2}{d\rho} = \frac{\tau}{\alpha}\Big\{S^2\rho^{-2}y_1^2 + 2S^{-2}(\varepsilon + a_0 - \tfrac{1}{2}S)\tilde{F}^2 + 2S^{-2}(\varepsilon + a_0 + \tfrac{1}{2}S)\tilde{G}^2\Big\}, \tag{7.6}$$

where y_1, y_2 are defined by (7.3) and (7.4). These were integrated by a numerical method, the details of which we willconsider later in the chapter. The integrations were repeated for various values of the parameters a_0' and m_E. In all cases we assumed that e_E was the electronic charge, or, in other words, that α had the numerical value that is usually accepted for the fine-structure constant.

As boundary data we use the following

$$\begin{aligned}
\Phi &= -\frac{\tau\varepsilon}{\rho} \,, \\
\frac{d\Phi}{d\rho} &= \frac{\tau\varepsilon}{\rho^2} \,, \\
a_0 &= -\frac{\alpha}{\rho} \,, \\
\frac{da_0}{d\rho} &= \frac{\alpha}{\rho^2} \,, \\
\tilde{F} &= A1\,\rho^{\beta} \exp\left(-\sqrt{1-\varepsilon^2}\,\rho\right) , \\
\tilde{G} &= -\frac{(1-\varepsilon)}{\sqrt{1-\varepsilon^2}}\,\tilde{F} \,,
\end{aligned} \tag{7.7}$$

where $A1$, defined by $A1 = a_0'(1-\varepsilon^2)^{\beta/2}$, can now be used instead of a_0' as a parameter of the theory.

Previously we have shown that for spherically symmetric solutions $k = \pm 1$. Following a procedure similar to that used by Bethe and Salpeter (1957, p. 153), we now determine which is the correct sign to choose for various values of the interaction parameters. For the case of an electrostatic potential alone the rule is quite simple. If the Coulomb potential is attractive (repulsive), we must take $k = +1$ (-1). When we include the gravitational interaction, certain modifications are necessary. From (6.9), (6.10), we can write down three expressions for the ratio ,

$$\frac{a_0'}{b_0'} = \frac{\beta + k}{\alpha - \tau\varepsilon(2\varepsilon - 1)} \,, \tag{7.8}$$

$$\frac{a_0'}{b_0'} = \frac{\tau\varepsilon(2\varepsilon + 1) - \alpha}{\beta - k} \,, \tag{7.9}$$

$$\frac{a_0'}{b_0'} = -\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon} \,. \tag{7.10}$$

(7.10) is obtained from (6.9) by setting $\nu = 1 + n'$, where $n' = 0$. For bound states this expression is always negative. Imposing this condition on (7.8), (7.9) leads to a contradiction if we take the wrong sign for k. For $\tau < \alpha$ there are two cases;

(i) $\tau \lesssim \alpha(3-5\delta)^{-1}$, where $\delta = 1 - \varepsilon \approx 10^{-5}$. In this case $-1 \leq \beta < 0$, $\tau\varepsilon(2\varepsilon+1) < \alpha$, and so we must take $k = -1$.

(ii) $\alpha(3-5\delta)^{-1} < \tau < \alpha$. Here the denominator of (7.8) and the numerator of (7.9) are positive, while $\beta < -1$. This means that we can take either sign for k, or, $k = \pm 1$.

For $\tau > \alpha$ we have three cases;

(iii) $\alpha < \tau < 1.44$. In this region $0 < \beta < 1$, the denominator of (7.8) is negative, the numberator of (7.9) positive, and therefore $k = +1$.

(iv) $1.44 \lesssim \tau \lesssim 4.00$. Here $-1 < \beta < 0$, the other quantities have the same sign as in (iii), and we must take $k = +1$.

(v) $4.00 < \tau < \infty$. In this last case, where $\beta < -1$, it is easy to show there there is no choice of sign for k which makes (7.8) and (7.9) compatible. Hence, there exists an upper limit of 4.4×10^{-5} grams for the bare mass.

In Fig. 1 we have plotted $\bar{M}_\varepsilon$ against $\bar{m}_\varepsilon$, where the bar indicates we have used dimensionless notation, or

$$\bar{M}_\varepsilon = \left(\frac{G_\varepsilon}{\hbar c_\varepsilon}\right)^{\frac{1}{2}} M_\varepsilon ,$$

$$\bar{m}_\varepsilon = \sqrt{\tau} = \left(\frac{G_\varepsilon}{\hbar c_\varepsilon}\right)^{\frac{1}{2}} m_\varepsilon . \tag{7.11}$$

The graph is also divided into four regions which exhibit the various required values of k as outlined above. In region I, $k = -1$, in II $k = \pm 1$, in III $k = +1$ and in IV there is no allowed value for k.

We see that no matter how large the bare mass may be, the observed mass is always less than 2.2×10^{-5} grams. Of course, as we have seen above, there are other reasons why we may not take an arbitrarily large value for the bare mass. In his study of the Klein-Gordon geon, Kaup (1968) found that no solutions existed which had a bare mass larger than 1.76×10^{-5} grams. (This corresponds to a maximum value of 1.70×10^{-5} grams for the observed mass.)

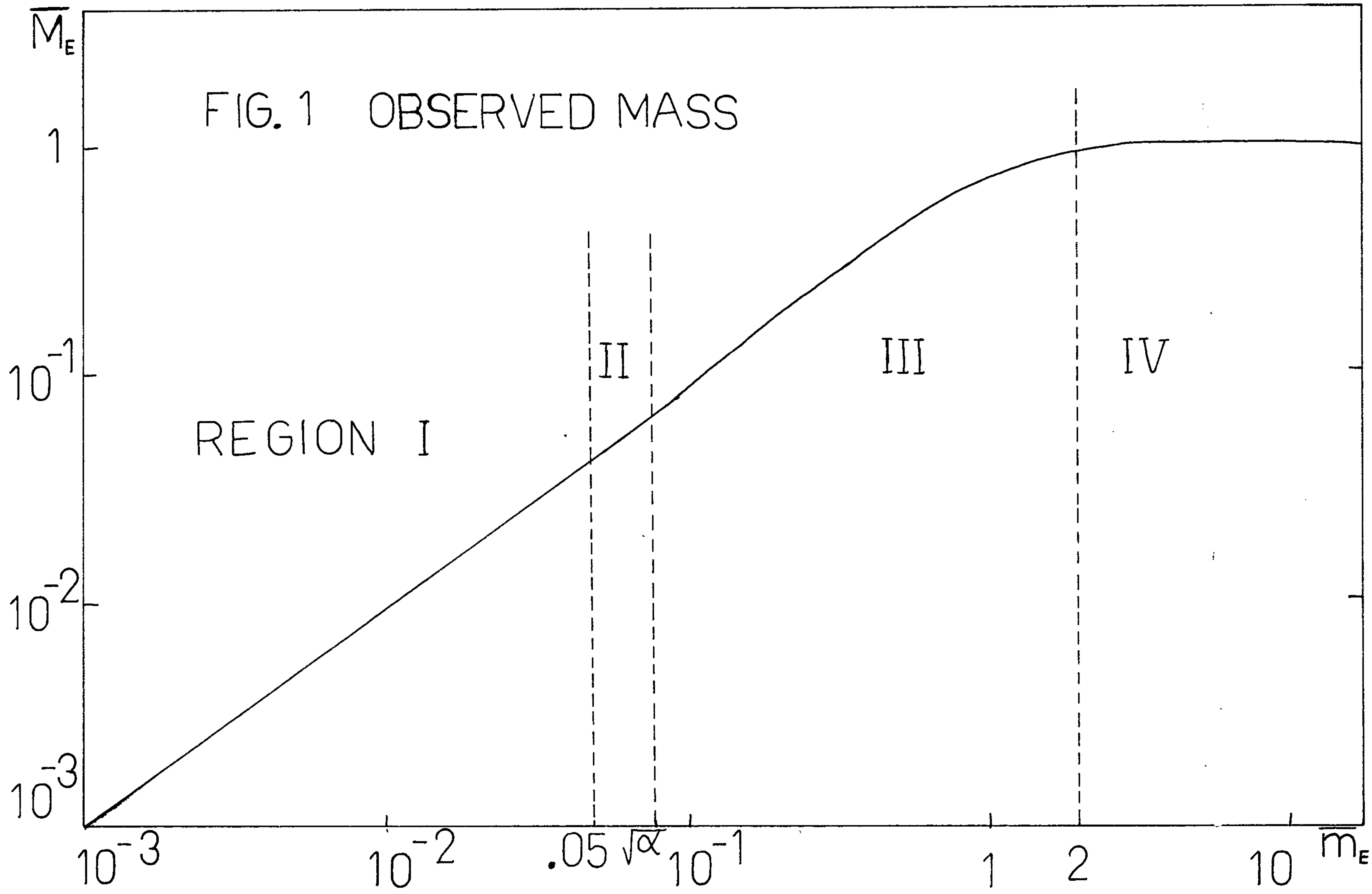

FIG. 1 OBSERVED MASS

In the present work, solutions were found for various values of $\bar{m}_E$ and $A1$. As an example Fig. 2 shows the shape of the mass density (see expression (2.53)) for the case $\tau = \bar{m}_E^2 = 0.05$ ($\bar{m}_E = 0.2236$) and $A1 = 0.002$. The normalisation is defined by

$$n = \int_0^\infty D\rho^2 d\rho \quad , \tag{7.12}$$

where we have included the factor 4π in our dimensionless D . For a correctly normalised solution, from (3.38), we need

$$n = \alpha \, .$$

In order to obtain normalised solutions we fixed the value of $\tau = \bar{m}_E^2$ and integrated the system of equations for the whole range of possible values of $A1$. We repeated this process for several values of τ . The results of this study are summarised in Figs. 2 - 5.

We found that for all $\bar{m}_E$, $A1$, the mass density D has the general shape exhibited by Fig. 2. Normalised solutions were found in the range $\alpha < \tau < 4.00$. For $A1$ small we found $n > \alpha$, and for $A1$ large $n < \alpha$. For the case $\tau = 0.005$ the best solutions (from the point of view of the normalisation) were given by $A1 = 1.750461 \times 10^{-3}$, in which case

$$n - \alpha = 0.104 \times 10^{-6} \quad , \tag{7.13}$$

and $A1 = 1.750465 \times 10^{-3}$, when

$$n - \alpha = -0.763 \times 10^{-7} \quad , \tag{7.14}$$

or $|n - \alpha| \doteq 10^{-5}\alpha$, which is as good an agreement as can be expected in numberical work. For other values of τ comparable accuracy was obtained.

Contrary to what we had hoped, however, we found that in every case the electrostatic and gravitational potentials were singular near the

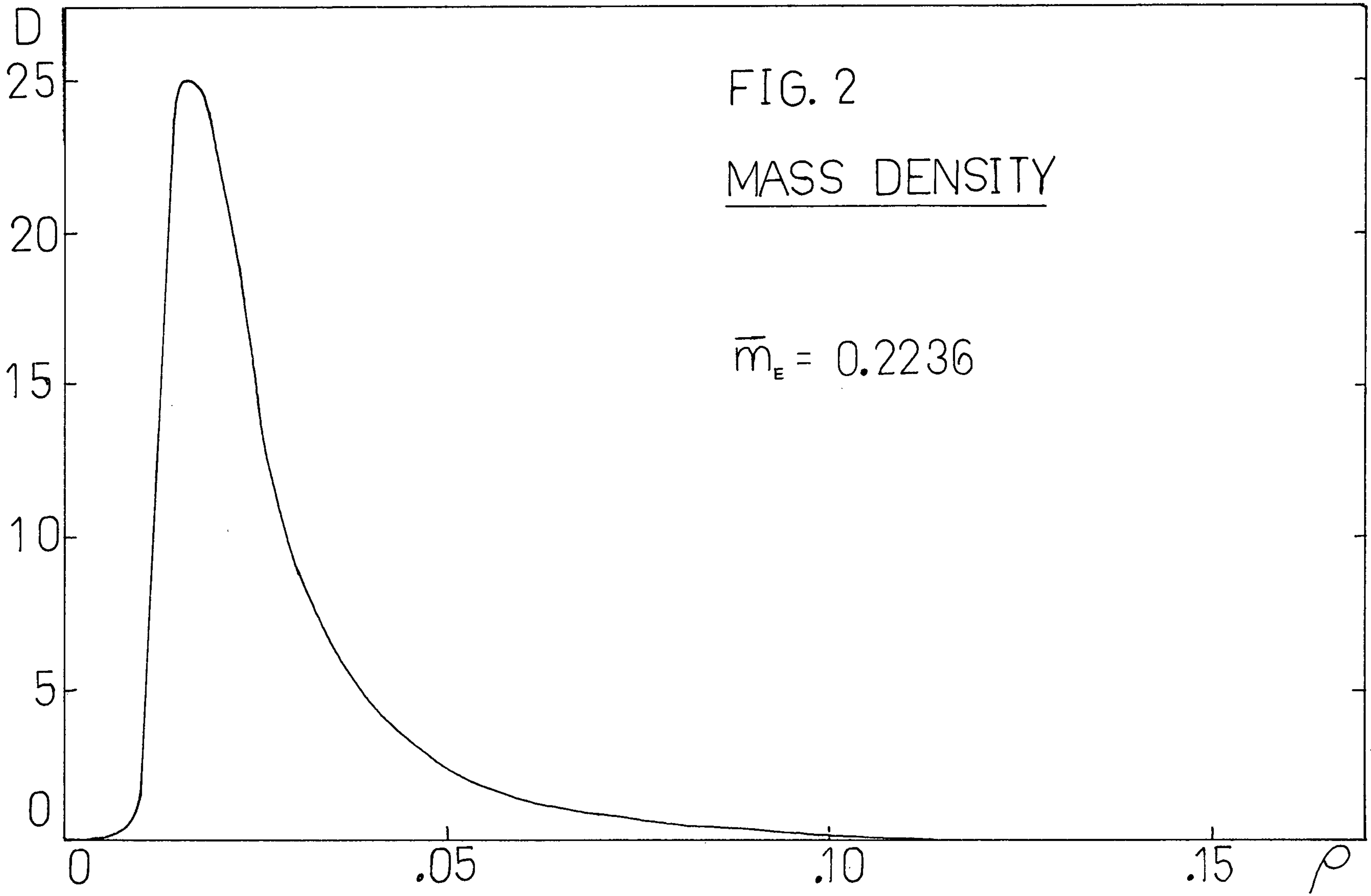

FIG. 2

MASS DENSITY

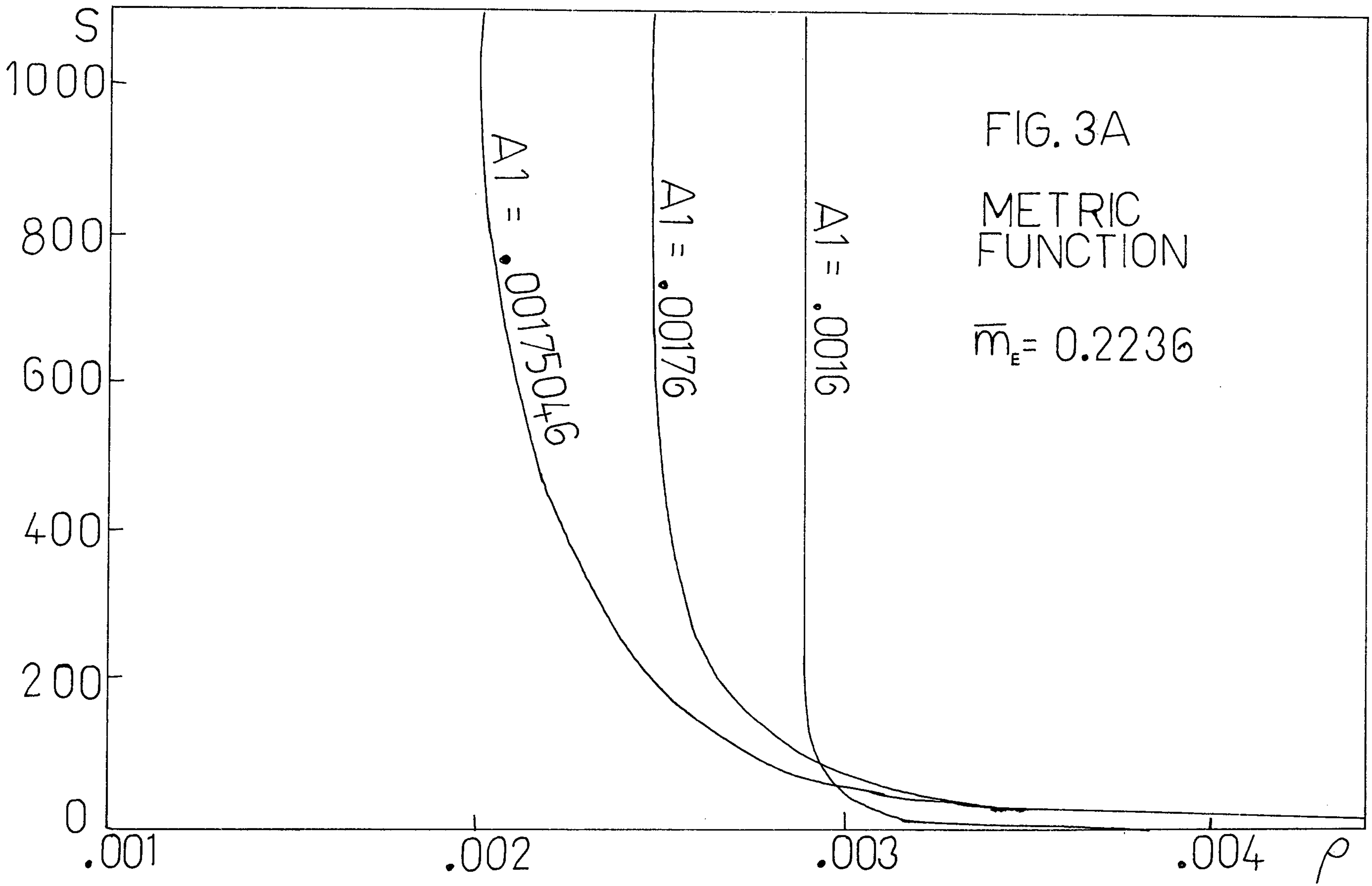
FIG. 3A
METRIC FUNCTION
$\overline{m}_E = 0.2236$
A1 = .0017504б
A1 = .00176
A1 = .0016
S
1000
800
600
400
200
0
.001
.002
.003
.004
ρ

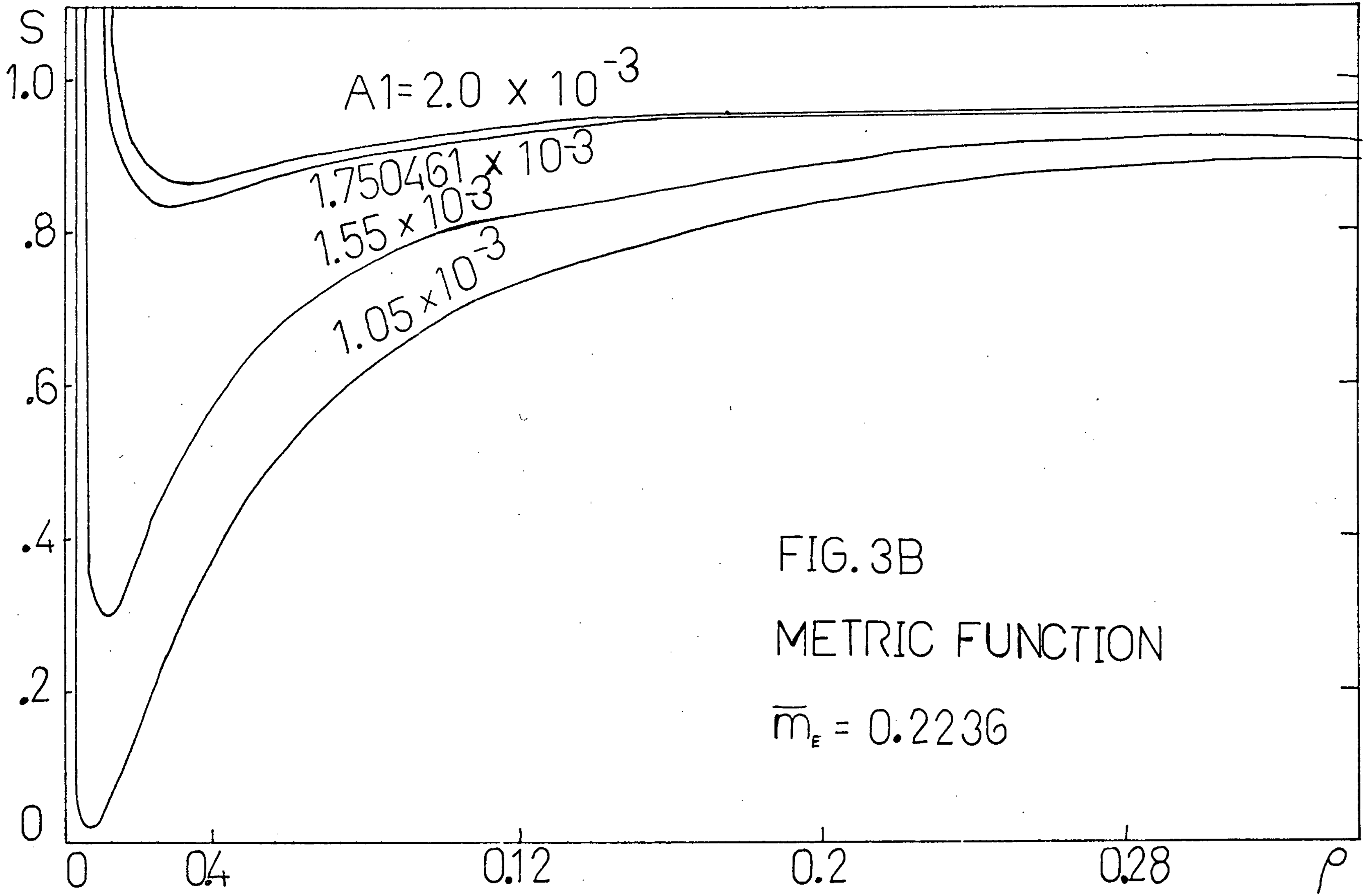

FIG. 3B

METRIC FUNCTION

$\overline{m}_E = 0.2236$

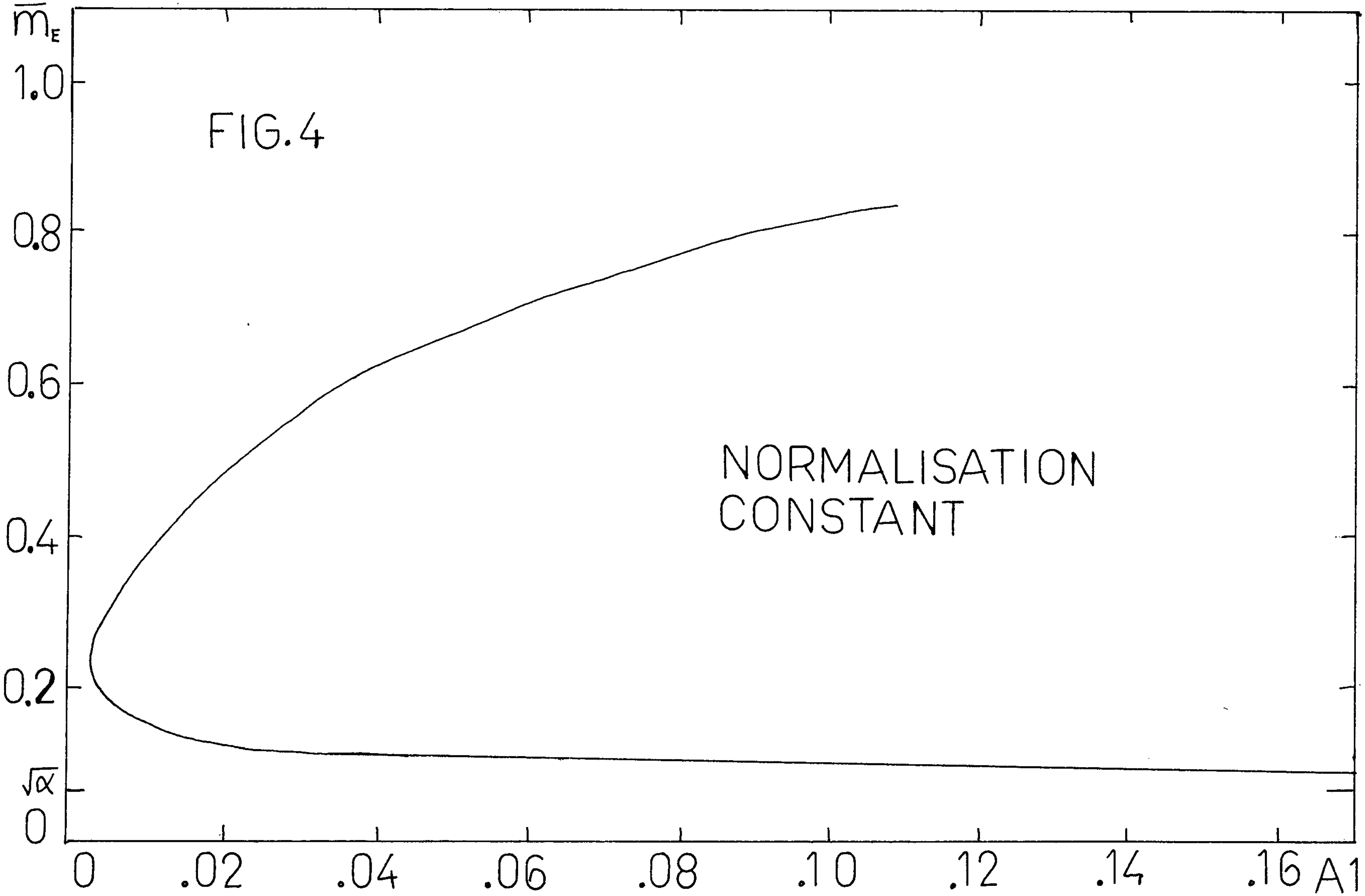
$\overline{m}_E$
1.0
0.8
0.6
0.4
0.2
$\sqrt{\alpha}$
0
0
.02
.04
.06
.08
.10
.12
.14
.16
A1
NORMALISATION CONSTANT

FIG. 4

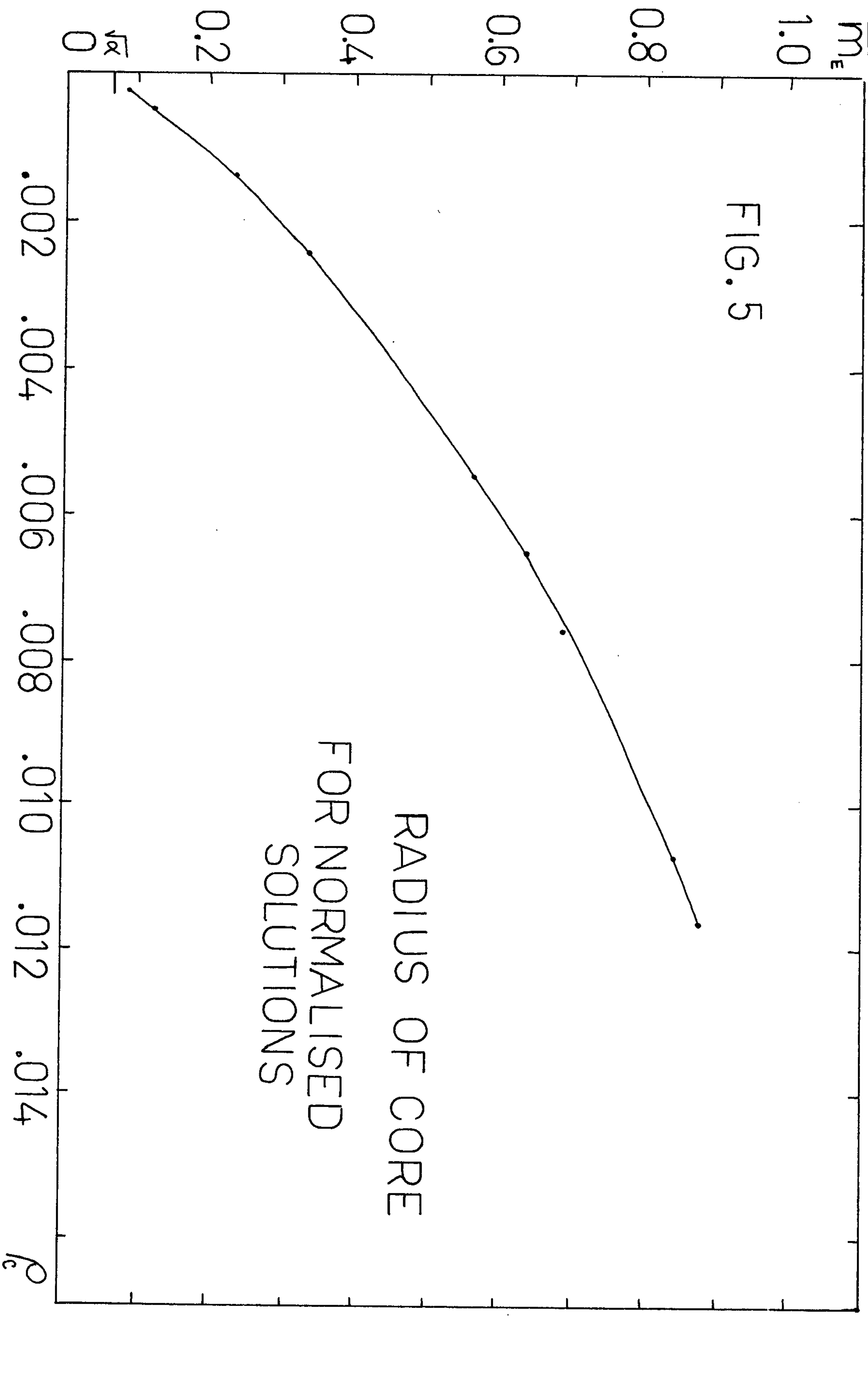
FIG. 5
RADIUS OF CORE
FOR NORMALISED
SOLUTIONS
$\overline{m}_E$
1.0
0.8
0.6
0.4
0.2
0
$\sqrt{\alpha}$
.002
.004
.006
.008
.010
.012
.014
ρ_c

origin. Considered as classical models of charged particles, these solutions corresponded to objects comprising a thin shell of matter surrounding a highly singular region of space. In the following detailed description of our solutions, we shall make use of the terminology of such a "shell" model.

Discussion of the Solutions

The system of equations (7.1)-(7.6) with the initial conditions (7.7) was integrated numerically using, as our basic method, a Runge-Kutta subroutine of order 4 (Fowler 1964). A second method, based on an extrapolation procuedure using rational functions due to Bulirsch and Stoer (1966), was used on certain selected examples to check our results and to evaluate the errors.

The Runge-Kutta method used is that due to Gill (1951). As step-size we used $h = \rho/50$ where ρ is the radial co-ordinate. Carr (1958) has shown that for h small enough (but not too small, to avoid excessive round-off error) this program is very accurate. The second method, translated into Fortran IV by M. Leslie (1966) from the Algol procedure of Bulirsch and Stoer (1966), involved the use of an automatic step-size correction procedure. This meant that after each integration step h was changed to the optimal step size for the next integration step. The program also contained a subroutine for controlling the accuracy of the computed values of the functions being integrated. If y was one such function then the computation of y at each integration step was repeated until two successive computed values of y differed at most by an amount εS where $\varepsilon = 10^{-6}$ and S was of the order y. Burlirsch and Stoer (1966) examined in detail the errors involved using this method and showed that it gave results superior to most other commonly used methods, including the Runge-Kutta. We did not use it all the time because of the high cost in computer time and storage.

Assuming that the difference in the results obtained using the two methods was of the order of magnitude of the errors involved, we compared the integrated solutions for several cases. As expected we found that in the early stages of the integration the difference was negligible. As $\rho \to 0$, however, the error increases somewhat until for the last few integration steps we had lost three or four digits. This means that, estimated in this rough fashion, our solutions were accurate to at least four places of decimals. Hence no qualitative errors in the shapes of the solution-curves was indicated. For example, for the case $\overline{m}_E$ = 0.6325, $A7$ = 0.001, both methods gave the same value for Φ at ρ = 86.938180, whereas at ρ = 0.11258820 the Runge-Kutta subroutine gave Φ = 3.3022970 while the Bulirsch-Stoer procedure gave Φ = 3.3025360. Normalised solutions were found for values of the bare mass range from 0.1 to 1.0 (in units of 2.2 x 10^{-5} grams). In all cases these solutions had the same basic properties.

Figure 3A shows the form of the metric function $S = (-g_{00})^{\frac{1}{2}}$ for several values of $A7$ (with $\overline{m}_E$ = 0.2236), while Figure 3B shows the behaviour of three representative cases for small ρ. We have discussed elsewhere the form of S for large ρ. It is seen that all the solutions have the same basic shape, and differ only in the position $\rho = \rho_{MIN}$ where S has a minimum and in the radius ρ_c of the inner core. (The inner core is defined as that region of space close to the origin where the matter density is negligibly small.) Of course only the normalised solutions are of interest. We include these graphs merely to illustrate the result obtained which was that, for constant $\overline{m}_E$, the solutions behaved in a smooth way with variations in $A7$ and exhibited no qualitative differences.

Fig. 4 graphs $\bar{m}_E$ against the value of $A1$ which yields a correctly normalised solution. It is seen that, in the region $\bar{m}_E > 0.23$, $A1$ increases for increasing $\bar{m}_E$. It is to be recalled that elsewhere we have shown the existence of a maximum value for the bare mass. For $0.23 > \bar{m}_E > \sqrt{\alpha}$, $A1$ increases with decreasing $\bar{m}_E$ and approaches infinity asymptotically as $\bar{m}_E \to \sqrt{\alpha}$. For example when $\bar{m}_E = .0946$ we found that $A1 = -1.2 \times 10^9$ gave a correctly normalised solution. At the point $\bar{m}_E = \sqrt{\alpha}$ (which corresponds to the Weyl-Majumdar case) our asymptotic solution is no longer valid. We have discussed this case in Chapter 5.

Fig. 5 depicts the radius of the inner core in each case in which a normalised solution was found. For $\bar{m}_E > \sqrt{\alpha}$ we found that the radius of the core increased with increasing bare mass. For $\bar{m}_E < \sqrt{\alpha}$ we were unable to find any normalised solutions. It may be significant that, in this region, the electrostatic self-repulsive force, as estimated from the asymptotic forms, is greater than the gravitational self-attraction. The case $\bar{m}_E = 0$ gave solutions which could not be normalised, and which had an oscillatory behaviour at spatial infinity. This is caused by the fact that the binding energy $E_B = m_E c_E^2 - E = -E$ is in this case negative. The only exception is the trivial case $m_E c_E^2 = E = 0$.

In our search for solutions with S regular at the origin, we constructed a power series solution for the whole system for small ρ and attempted, by a least square method, to fit this solution to our numerically integrated one. Of course, since our equations are non-linear, we could only derive the coefficients for the first few powers of ρ, and therefore we had no proof that the series were convergent. As a working hypothesis, we assumed that they were. The results of our investigations were that we could not make such a fit and that it was highly unlikely that a solution for S regular at the origin existed. This conclusion is reinforced by the smoothness of the curve in Fig. 5 which suggests that the radius of the singular core never shrinks to zero.

Of course it is impossible, in principle, to prove a negative statement like the above by using a numerical technique. The best we can do is to show that, within the capacities of our method, no solution regular at the origin can be obtained, and no solution is indicated. There is always the possibility that the problem is an eigenvalue one, giving a regular solution only for very precise values of the parameters. We investigated this possibility very thoroughly, going to seven places of decimals, until we reached the limit of accuracy of the computer. That is, until the computer output became insensitive to changes in the input. We found no evidence of a regular solution.

An attempt was also made to integrate outwards from $\rho = 0$ assuming a regular solution. Unfortunately, due to the lack of any definite intial conditions, there were too many unknown parameters and the attempt was abandoned.

In the figures we have concentrated mainly on showing the behaviour of the metric function S. The electrostatic potential a_0 was found to have properties very similar to S. The matter density D always has the form indicated in Fig. 2.

CHAPTER VIII

ALTERNATIVE LAGRANGIAN DENSITIES

In Chapter II we discussed how to introduce the gravitational interaction by a simple generalisation of the flat-space theory. We assumed that the total Lagrangian density for the system could be written in the form $\mathcal{L} = \mathcal{L}_F + \mathcal{L}_G$, where $\mathcal{L}_G$ depended only on the gravitational potential ϕ and its first partial derivatives. The "field" part $\mathcal{L}_F$ was known in the special relativistic limit and its generalisation was unambiguous. However, the one uncertainty in our theory lay in the choice of $\mathcal{L}_G$. In the choice we made, we were guided by a desire for simplicity and by a desire to have a theory whose predictions for the perihelion advance of test particles were the same as those of the Einstein theory. A possibility exists, however, that the study of the oblateness of the Sun may indicate an error in the Einsteinian prediction for the perihelion advance of Mercury, and in that case the choice of an alternative Lagrangian density for our theory would be justified. In the context of the present work, moreover, the modified equations may lead to solutions which are regular everywhere. Consider, then, instead of (2.42) the gravitational Lagrangian density

$$\begin{aligned}\mathcal{L}_G &= K S^{-2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \exp\left(\lambda c_\epsilon^{-2}(\phi-\phi_0)\right), \\ &= K S^{\lambda-2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} , \qquad (8.1)\end{aligned}$$

where λ is a constant. If $\lambda = 0$ then we are back to the case which we have studied already, and which predicts the same perihelion advance for test-particles as the Einstein theory. If $\lambda = -1$, then, as Rastall (1968b) has shown, the energy densities of the gravitational and matter fields behave in the same way as sources of the gravitational field. The perihelion advance of test-particles is 8% less than that predicted by general relativity and is compatible with Dicke's measurements of the solar oblateness (Dicke 1967). Finally, with $\lambda = -2$, the energy densities of the gravitational and electromagnetic fields behave in the same way as sources of the gravitational field and the prediction for the perihelion advance of test particles is 16% less than occurs in general relativity.

The modified gravitational equation is obtained in the usual way by varying $\mathcal{L}$ with respect to ϕ. The variation of $\mathcal{L}_F$ is the same as before, but (2.44) becomes, from (8.1)

$$\frac{\delta \mathcal{L}_G}{\delta \phi} = -2\kappa S^{\lambda} \Big\{ \phi_{,mm} + \frac{\lambda}{2c_E^2} \phi_{,m}^2 - S^{-4}\Big(\phi_{,00} - \frac{(4-\lambda)}{2c_E^2}\phi_{,0}^2\Big)\Big\} . \tag{8.2}$$

From (8.2) the gravitational field equation (2.48) becomes

$$\begin{aligned} S^{\lambda}\Big\{\phi_{,mm} + \frac{\lambda}{2c_E^2}\phi_{,m}^2 - S^{-4}\Big(\phi_{,00} - \frac{(4-\lambda)}{2c_E^2}\phi_{,0}^2\Big)\Big\} \\ = -\frac{1}{2\kappa c_E^2}(E^2+B^2) \\ + \frac{\hbar}{2\kappa c_E^2} S\big(\bar\Psi \gamma^0 D_0 \Psi - D_0^* \bar\Psi \gamma^0 \Psi + \mu_E \bar\Psi\Psi\big) . \end{aligned} \tag{8.3}$$

The Maxwell and Dirac equations of course remain unchanged.

The energy-momentum tensor is altered in a similar way. Using (8.1), we find for the symmetrized energy-momentum tensor

$$\begin{aligned} \hat{T}^{\mu}_{\ \nu} = \kappa S^{\lambda-2}\big(2 g^{\mu\lambda}\phi_{,\lambda}\phi_{,\nu} - \delta^{\mu}_{\nu} g^{\lambda\rho}\phi_{,\rho}\phi_{,\lambda}\big) \\ - S^{-2}\big(F^{\mu\tau}F_{\nu\tau} - \tfrac{1}{4}\delta^{\mu}_{\nu} F^{\lambda\rho}F_{\lambda\rho}\big) \\ - \tfrac{1}{2}\hbar c_E S\big(\bar\Psi\gamma^{\mu}\nabla_{\nu}\Psi - \nabla_{\nu}\bar\Psi\gamma^{\mu}\Psi\big) , \end{aligned} \tag{8.4}$$

which takes the place of (2.59), and

$$\hat{T}^0_0 = -\kappa S^\lambda \left(\phi^2_{,m} + S^{-4}\phi^2_{,0} \right) + \tfrac{1}{2}(E^2+B^2)$$

$$- \tfrac{1}{2}\hbar c_E S \left(\bar{\psi}\gamma^0\nabla_0\psi - \nabla\bar{\psi}\gamma^0\psi \right) , \tag{8.5}$$

$$S^\lambda\left\{\phi_{,mm} + \frac{\lambda}{2c_E^2}\phi^2_{,m} - S^{-4}\left(\phi_{,00} - \frac{(8-\lambda)}{2c_E^2}\phi^2_{,0}\right)\right\}$$

$$= -\frac{1}{\kappa c_E^2}\left(\hat{T}^0_0 - \tfrac{1}{2}\hat{T}^\mu_\mu \right) . \tag{8.6}$$

(8.5) and (8.6) replace equations (2.60) and (2.61). The gravitational equation for the time-independent system is now given by

$$S^\lambda\left\{\phi_{,mm} + \frac{\lambda}{2c_E^2}\phi^2_{,m}\right\} = -\frac{1}{2\kappa c_E^2} S^{-2}\left(A_{0,m}\right)^2$$

$$+ \frac{S^{-2}}{\kappa c_E^2}\left\{ i(E + e_E A_0)\bar{\chi}_E\gamma^0\chi_E + \tfrac{1}{2} m_E c_E^2 \bar{\chi}_E\chi_E \right\} , \tag{8.7}$$

instead of (3.4), while (3.10) is replaced by

$$S^\lambda\left(\phi_{,mm} + \frac{\lambda}{2c_E^2}\phi^2_{,m}\right) = -\frac{1}{2\kappa c_E^2} S^{-2}\left(A_{0,m}\right)^2$$

$$- \frac{S^{-2}}{\kappa c_E^2}\left\{ (E + e_E A_0)(|U|^2 + |V|^2) - \tfrac{1}{2} m_E c_E^2 (|U|^2 - |V|^2) \right\} . \tag{8.8}$$

For the case of spherical symmetry it can easily be shown that (3.31) becomes

$$\frac{2c_E^2}{\lambda} S^{\lambda/2} \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dS^{\lambda/2}}{dr}\right) = -\frac{1}{2\kappa c_E^2} S^{-2}\left(\frac{dA_0}{dr}\right)^2$$

$$- \frac{1}{4\pi\kappa c_E^2} S^{-2}\frac{1}{r^2}\left\{ (E + e_E A_0 - \tfrac{1}{2} m_E c_E^2 S) F^2 \right.$$

$$\left. + (E + e_E A_0 + \tfrac{1}{2} m_E c_E^2 S) G^2 \right\} , \tag{8.9}$$

or, in dimensionless notation, in place of (3.37)

$$\frac{2}{\lambda} S^{\lambda/2}\frac{d}{d\rho}\left(\rho^2 \frac{dS^{\lambda/2}}{d\rho}\right) = \frac{\tau}{\alpha} S^{-2}\rho^2\left(\frac{da_0}{d\rho}\right)^2$$

$$+ \frac{2\tau}{\alpha} S^{-2}\left(\varepsilon + a_0 - \tfrac{1}{2}S\right)\tilde{F}^2$$

$$+ \frac{2\tau}{\alpha} S^{-2}\left(\varepsilon + a_0 + \tfrac{1}{2}S\right)\tilde{G}^2 . \tag{8.10}$$

Asymptotic Solutions

In the region of negligible matter density the equations for the system (4.1) and (4.2) become

$$\frac{d}{dr}\left(r^2 \frac{dS^{\lambda/2}}{dr}\right) = \frac{4\pi G_E}{c_E^4} \frac{\lambda}{2} S^{-\lambda/2 - 2} r^2 \left(\frac{dA_0}{dr}\right)^2 , \tag{8.11}$$

$$\frac{d}{dr}\left(r^2 S^{-2} \frac{dA_0}{dr}\right) = 0 . \tag{8.12}$$

From (8.12) we get

$$\frac{dA_0}{dr} = \mu S^2 r^{-2} , \tag{8.13}$$

where μ is a constant of integration. Inserting (8.13) into (8.11) we obtain

$$\frac{d}{dr}\left(r^2 \frac{dS^{\lambda/2}}{dr}\right) = \frac{4\pi G_E}{c_E^4} \frac{\lambda}{2} \frac{\mu^2}{r^2} S^2 . \tag{8.14}$$

writing $y = S^{\lambda/2}$, $u = (2\pi G_E \mu^2 |\lambda| c_E^{-4})^{\frac{1}{2}} r^{-1}$, (8.14) becomes

$$y'' - \frac{\lambda}{|\lambda|} y^{(4/\lambda - 1)} = 0 , \tag{8.15}$$

where the primes denote differentiation by u . This equation is valid for all λ except $\lambda = 0$, which case has been studied previously. Multiplying across by y' and integrating we find

$$\frac{1}{2} y'^2 - \frac{\lambda^2}{4|\lambda|} y^{4/\lambda} = c_1 , \tag{8.16}$$

where c_1 is a constant. Integrating once again, we obtain

$$\int^y \frac{dy}{(c_1 + \frac{|\lambda|}{2} y^{4/\lambda})^{1/2}} = c_2 + u \tag{8.17}$$

where c_2 is another constant. μ , c_1 and c_2 are determined by considering the forms of the electrostatic and gravitational potentials very far from the centre. We require, for large r

$$\Phi = - \frac{G_E M_E}{c_E^2} \frac{1}{r} + O(r^{-2}) , \tag{8.18}$$

$$A_0 = -\frac{e_E}{4\pi}\frac{1}{r} + O(r^{-2}) , \tag{8.19}$$

where, as before, e_E is the charge and M_E the mass of the central body.

As an example we consider the case $\lambda = -2$. From (8.17) we obtain

$$y^2 = \frac{1}{c_1}\left\{ (c_1 c_2 + c_1 u)^2 - 1 \right\} \tag{8.20}$$

in the case where $c_1 \neq 0$, and

$$y^2 = 2u + c_3 \tag{8.21}$$

where c_3 is a constant, in the special case $c_1 = 0$. To determine c_1 , c_2 , c_3 we recall that $y^2 = S^{-2}$ and use (8.18). We find that

$$\begin{aligned} c_1 &= \frac{G_E M_E}{4\pi\mu^2} - 1 , \\ c_1 c_2 &= \sqrt{\frac{G_E M_E}{4\pi\mu^2}} , \\ c_3 &= 1 . \end{aligned} \tag{8.22}$$

For the solution (8.21) we require further more

$$\frac{G_E M_E^2}{4\pi\mu^2} = 1 , \tag{8.23}$$

and, as we will see shortly, this means that the special case (8.21) corresponds to the Weyl-Majumdar solution. From (8.13) we have

$$\frac{dA_0}{du} = - S^2 \sqrt{\frac{e_E^4}{4\pi G_E}} , \tag{8.24}$$

where

$$S^2 = (1 + 2c_1 c_2 u + c_1 u^2)^{-1} , \tag{8.25}$$

in the first case, and

$$S^2 = (1 + 2u)^{-1} , \tag{8.26}$$

in the second case. Solving (8.24) we obtain

$$A_0 = C_4 + \sqrt{\frac{C_E^4}{4\pi G_E\,(c_1^2 c_2^2 - c_1)}}\,\tanh^{-1}\left(\frac{c_1 c_2 + c_1 u}{\sqrt{c_1^2 c_2^2 - c_1}}\right), \tag{8.27}$$

for $c_1 \neq 0$, and

$$A_0 = C_5 - \frac{1}{2}\sqrt{\frac{C_E^4}{4\pi G_E}}\,\log(1 + 2u), \tag{8.28}$$

in the Weyl-Majumdar case. For large r (small u) we expand these series and use (8.19) to derive the constants. We find

$$C_4 = -\sqrt{\frac{C_E^4}{4\pi G_E\,(c_1^2 c_2^2 - c_1)}}\,\tanh^{-1}\left(\frac{c_1 c_2}{\sqrt{c_1^2 c_2^2 - c_1}}\right),$$

$$C_5 = \frac{1}{2}\sqrt{\frac{C_E^4}{4\pi G_E}},$$

$$\mu = \frac{e_E}{4\pi}. \tag{8.29}$$

We can substitute the value of μ thus obtained into (8.22) and (8.23) to obtain the final values for the constants c_1 and c_2. To sum up, the solutions obtained above are valid in the region where the mass density is negligible. We can use them in the same way as the solutions of Chapter IV to determine the boundary conditions for the electrostatic and gravitational fields.

Weyl-Majumdar Relation

The Weyl-Majumdar relation valid for the modified Lagrangian density is obtained, as in Chapter V, by considering the free-space electrostatic-gravitational equations. These are, from (3.9), (8.8)

$$\partial_m\left(S^{-2}\partial_m A_0\right) = 0, \tag{8.30}$$

$$\phi_{,mm} + \frac{\lambda}{2C_E^2}\,\phi_{,m}^2 = \frac{4\pi G_E}{C_E^2}\,S^{-2-\lambda}\left(\partial_m A_0\right)^2. \tag{8.31}$$

Using (5.6) we easily obtain

$$\partial_{mm} A_0 = 2\phi' c_E^{-2} (\partial_m A_0)^2 , \tag{8.32}$$

$$\partial_{mm} A_0 = \frac{1}{\phi'} \left\{ \frac{4\pi G_E}{c_E^4} S^{-2-\lambda} - \phi'' - \frac{\lambda}{2c_E^2} (\phi')^2 \right\} (\partial_m A_0)^2 , \tag{8.33}$$

and hence

$$\phi'' + \frac{1}{c_E^2}\left(2 + \frac{\lambda}{2}\right)(\phi')^2 = \frac{4\pi G_E S^{-2-\lambda}}{c_E^2}$$

which yields, finally

$$\left\{ S^{(2+\lambda/2)} \right\}'' = \left(\frac{4+\lambda}{2}\right) \frac{4\pi G_E}{c_E^4} S^{-\lambda/2} . \tag{8.34}$$

We can solve this equation in the following way. Letting $y = S^{(2+\lambda/2)}$, (8.34) becomes

$$y'' = (4 + \lambda) \frac{2\pi G_E}{c_E^4} y^{-\frac{\lambda}{4+\lambda}} . \tag{8.35}$$

Multiplying across by y' , and integrating, we obtain

$$\frac{1}{2}(y')^2 - \frac{2\pi G_E}{c_E^4}(4 + \lambda)\left(\frac{4+\lambda}{4}\right) y^{4/4+\lambda} = \frac{1}{2} A , \tag{8.36}$$

where A is a constant. The solution can be written as

$$\int^y \frac{dy}{\sqrt{A + b y^{4/4+\lambda}}} = B + A_0 , \tag{8.37}$$

where B is a constant also, and

$$b = \frac{4\pi G_E}{c_E^4} \frac{(4+\lambda)^2}{4} . \tag{8.38}$$

For example, if $\lambda = 0$, (8.37) can be immediately integrated to yield (5.10). On the other hand, if $\lambda = -2$, then (8.37), (8.38) become

$$\int^y \frac{dy}{(A + by)^{1/2}} = B + A_0 , \tag{8.39}$$

$$b = \frac{4\pi G_E}{c_E^4} . \tag{8.40}$$

Integrating (8.39), and solving for y we obtain

$$y = \frac{A}{b} \sinh\left(B\sqrt{b} + A_0\sqrt{b}\right),$$

$$= F\cosh(\sqrt{b}A_0) + G\sinh(\sqrt{b}A_0) \qquad (8.41)$$

where $F = Ab^{-1}\sinh(\sqrt{b}B)$, $G = Ab^{-1}\cosh(\sqrt{b}B)$.

Without loss of generality we can set $F = 1$ and obtain, finally

$$y = \cosh(\sqrt{b}A_0) + G\sinh(\sqrt{b}A_0) \qquad (8.42)$$

By methods similar to those used in Chapter V we can show that

$$G = \frac{4\pi G_E M_E^2}{e_E^2} \qquad (8.43)$$

When $G = 1$, the Weyl-Majumdar solution is involved and the system of equations (8.30), (8.31) reduces to the single Laplace equation

$$\partial_{mm}(S^{-2}) = 0. \qquad (8.44)$$

Conclusion

In this work we have investigated and found static, spherically symmetric solutions of the combined Dirac-electromagnetic-gravitational field equations.. We have shown the existence of normalised solutions which describe spherical, shell-like models for particles. Unfortunately, these solutions involve electrostatic and gravitational fields which are not regular at the origin. Our investigations have led us to the conclusion that such solutions do not exist, at least within the framework of the present theory. It is possible that the choice of a different Lagrangian density for the gravitational field might lead to equations which do produce regular solutions. Another possibility is that we might abandon the assumption, $\mathbf{E} = M_E C_E^2$, of Chapter VI. (In Chapter VIII, some alternative Lagrangian densities are investigated.)

The 'solutions' found correspond to objects with mass of the order of 4.8×10^{-6} grams, inner radius 7.5×10^{-36} cms, and outer radius 3.7×10^{-34} cms. They are thus much heavier and more compact than any of the known elementary particles. Of course, since we are using an unquantized theory, it would be unreasonable to expect models which correspond to actual physical objects.

Another possible explanation for the properties of our solutions is that our (implicit) assumptions regarding the topology of space in the inner region are incorrect. In this context it is instructive to consider the ideas of Wheeler (1968) on the geometrodynamical description of electric charge. In his view electric charges are nothing but sets of lines of force trapped in "wormholes". Our inner singular region should perhaps be replaced by the mouth of such a wormhole.

Wheeler (1968) has also shown that, if one deals in distances of the order of the Planck length, 1.6×10^{-33} cms, then, strictly speaking, one should use quantum geometrodynamics, if one knew how. In the present state of our knowledge, however, we can only explore, as deeply as possible, the classical theory in the hope that some day the results may be of use in the study of the more complete theory.

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