FERMION FRACTIONIZATION AND BOUNDARY EFFECTS IN (1 + 1) DIMENSIONS

by

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Abstract

Fermion number fractionization in quantum field theory on a finite interval is studied for a $(1 + 1)$ dimensional fermion-soliton system with explicit charge conjugation symmetry breaking. The effects of boundary conditions on the fractional fermion number and the connection with the corresponding open space problem are investigated. It is argued that the open space fractional charges can be correctly reproduced from the finite interval results only through a careful definition of what is meant by the soliton charge. This definition of the charge distinguishes between the fermionic and boundary induced charges in the system, and isolates the soliton from possibly other charged topological objects in the system. It therefore gives a true measure of the localized fractional fermion number induced on the soliton of interest. It is then rigorously proven that the corresponding charge fluctuations vanish, and hence that the induced fractional charge on the soliton is a quantum observable.
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Introduction

Fermion fractionization first appeared in the physics literature in 1976 in the context of relativistic quantum field theories describing Dirac fermions interacting with certain topological objects called solitons [13]. The study of such a system arose from the important realization that the particle spectrum of a field theory may consist not only of the excitations manifest in perturbation theory, but also of these particles of a non-perturbative topological nature [17]. The original (1 + 1) dimensional model considered a charge conjugation symmetric system, which gave rise to ground states of the system with fermion number $\pm \frac{1}{2}$. The interpretation was that the lowest lying soliton states are doubly degenerate so that fermions "fractionize" over the two degenerate ground states. This novel quantum mechanical and topological phenomenon seems quite remarkable, in that fractional charge can emerge in a theory where all the basic constituents carry integral charges. Further investigation into this phenomenon found that for certain (1 + 1) dimensional fermion-soliton systems without a charge conjugation symmetry, the fermion number of the soliton states could assume arbitrary fractions, even irrational values [7]. The methods used to compute the fermion number in (1 + 1) dimensions were then generalized to (2 + 1) and (3 + 1) dimensional models, and fermion fractionization was also found to occur in fermion-vortex and fermion-monopole systems [15].

In condensed matter physics, it has been realized that certain quasi-one-dimensional
organic polymers, such as polyacetylene, have a broken (discrete) symmetry ground state and consequently solitonic excitations in the form of domain walls [21]. Here orbital electrons couple to the solitons giving rise to fractionally charged states in much the same way as in the field theory examples. Recent experimental and theoretical investigations confirm the soliton picture of organic polymers [1, 8], and thus the conducting polymers of condensed matter physics provide an experimental laboratory for the physics of fermion fractionization.

It has been suggested that the physics of fractionization can be thought of as a vacuum polarization effect. The soliton background fields, which have a non-trivial behaviour at spatial infinity, distort the Dirac sea in such a way that the ground state (that is, the vacuum) in the presence of the external fields has very unusual features, and exhibits anomalous quantum numbers. This can be intuitively seen as follows. Suppose a soliton-antisoliton \((SS)\) pair is created. This configuration has a trivial behaviour at infinity, and hence no fractionization occurs. However, as the pair is separated, the fermionic states are rearranged, the local density of states is modified, and states "pile up" or "thin out" near the region where the fields are rapidly changing (The zero mode in the charge conjugation symmetric model, which yields the two-fold degeneracy of the ground state, can be thought of as localized in the vicinity of the soliton [3]). As the \(SS\) separation becomes infinite and we only "see" one soliton, for example, we find that the charge has been accumulated near its center (this also happens near \(S\)). Of course the total charge of the \(SS\) system is an integer, but the distortion in the Dirac sea is such that fermion-antifermion pairs could have been created by the \(SS\) separation.

Most of the results which suggest the existence of fractionally charged states in fermion theories were obtained by considering models on open (infinite) spaces. The object of this thesis is to examine the fractionization phenomenon in a finite interval in
(1 + 1) dimensions. The results described here appear in the collaboration [22] between this author and G.W. Semenoff. We examine how finite volume effects the fractional fermion number induced on the soliton and how to distinguish this boundary induced fractionization from the physical fermionic induced fractional charge. In particular, we examine how to pass to the open space (infinite volume) limit from the finite interval and obtain consistent results in the two problems. We shall see that for a fermion-soliton system lacking conjugation symmetry, open space anomalous quantum numbers arise as infinite volume limits of the corresponding finite interval results only through a careful definition of the relevant Hermitian operators. This problem was first studied in [3], [12], and [18] for charge conjugation symmetric fermion-soliton systems. The idea is that the fermionic charge which is measured on a localized soliton must be isolated from the other soliton charges which may be present in the system and from the effects of the boundaries of the finite interval (This corresponds to the vacuum polarization view of fermion fractionization described in the preceding paragraph). This is accomplished by localizing the charge operator about the relevant soliton through a definition involving the smearing of the charge operator with some peaked localized test function and identifying that part of the fractional fermion number which is due to boundary conditions on the system in finite volume. This definition then yields consistent results between the open space and finite volume systems. However, the results obtained this way yield only the appropriate expectation values of the fermion number, and the problem which then arises is whether or not the unusually defined charge for the infinite space limit of the finite volume system is in fact an observable of the system; that is, an eigenvalue of the appropriate charge operator. The answer to this question is the main result of this study. We show that for a quite general class of test functions, the fluctuations of the limiting fermion number vanish, which
establishes that the fractional charge induced on the soliton is indeed a sharp quantum number. This shows that the underlying fermion-soliton Fock space admits degenerate soliton states, each an eigenstate of the the appropriate charge operator with fractional fermion number. Thus not only does the fermion number have fractional expectation values, but it also has fractional eigenvalues with vanishing quantum fluctuations.
Chapter I
Open Space Fermion Fractionization
in (1 + 1) Dimensions

In quantum field theory, fermion number \([4, 9]\) is the conserved charge associated with the \(U(1)\) phase invariance of a Lagrangian with Dirac fermions. It is odd under \(C\), or \(CP\). In the condensed matter applications, fermion number coincides with electric charge. The original appearance of fermion number fractionization was in the work of Jackiw and Rebbi \([13]\), who found that certain \((1 + 1)\) dimensional systems of Dirac fermions with charge conjugation symmetry when introduced into the background field of a soliton \([17]\) yielded a normalizable self-conjugate zero energy solution of the corresponding Dirac equation. This zero energy mode implied a degeneracy in the quantum field theory, and the existence of a non-degenerate \(c\)-number zero energy solution gave rise to the interpretation that each soliton ground state was a degenerate doublet, with fermion number \(\pm \frac{1}{2}\). In such field theory models with charge conjugation symmetry, quite general considerations imply that the eigenvalues of the second quantized fermion number operator are always either integers or half-integers. Indeed, the difference in fermion number between any two quantum states in a given sector of the theory is always an integer, \(N - N' = n\). But by charge conjugation symmetry of the system, if a state with fermion number \(N\) exists, then a state with fermion number \(N' = -N\)
also exists, and so \(2N = n\). Thus \(N = \frac{1}{2}n\) and the fermion number of any state is either an integer or a half-integer. For the fermion-soliton system considered in [13], where there exists a zero mode localized about \(x = 0\) in the first quantized theory, the question then is whether in defining the vacuum of the system by “filling the negative energy sea” (according to the usual Dirac hole theory) the first quantized zero energy mode is empty or filled. Evidently, both possibilities are allowed, so the second quantized vacuum is doubly degenerate, with the zero energy solution projecting onto half a positive energy and half a negative energy solution with respect to the normal ground state. Thus a C-invariant definition of charge assigns the values \(\pm \frac{1}{2}\) to the two vacuum states, which can be labelled as a soliton ground state \(|0;+, S\rangle\) and an antisoliton ground state \(|0;-, \bar{S}\rangle\). Subsequent investigations into this phenomenon revealed that for similar systems which do not possess a charge conjugation symmetry, the fermion number may assume even irrational values, the magnitude of the fractionization being parametrized by the conjugation symmetry breaking parameter of the theory [7]. In this case, the loss of charge conjugation symmetry distorts the Dirac sea in such a way that the vacuum exhibits fractional quantum numbers dependent on the symmetry breaking parameter. In Section 1 of this chapter, we describe a special case of the field theory model in [7], and we derive the fractional fermion number for this model in Section 2. We restrict our attention in this chapter to the open space problem, leaving the discussion of the subtle finite interval case to Chapter II. For two concise reviews, see [11] and [15].
1. The Fermion-Soliton System

We consider a Dirac spinor field $\psi$ interacting with a real scalar field $\varphi$ in $(1 + 1)$ dimensions through a Yukawa coupling defined by the Lagrangian

$$L = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - V(\varphi) + \bar{\psi}(i\gamma^\mu \gamma^5 - g\varphi + im\gamma^5)\psi.$$  

Here $V(\varphi)$ is the potential energy density for the scalar field $\varphi$ and $g$ is the Yukawa coupling constant, which we henceforth take to be unity. $m$ is the charge conjugation symmetry breaking parameter which we assume is position independent and positive.

We have also introduced the Feynman notation $A \equiv A_\mu \gamma^\mu$ for any space-time vector $A^\mu$, where $\gamma^\mu$ are the Dirac matrices satisfying the $(1 + 1)$ dimensional Minkowski space Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (1.2)$$

and $\gamma^5 = \gamma^0\gamma^1$. The spinor field $\psi$ and its adjoint $\psi^\dagger$ obey the canonical equal-time anticommutation relations

$$\{\psi(x, t), \psi^\dagger(y, t)\} = \delta(x - y)$$

$$\{\psi(x, t), \psi(y, t)\} = \{\psi^\dagger(x, t), \psi^\dagger(y, t)\} = 0. \quad (1.3)$$

If we assume that the $U(1)$ symmetry of the system is not spontaneously broken, then we may use a semiclassical approximation in which $\varphi$ is treated as a classical c-number background field in which the fermions move. Hence we ignore the bosonic quantum fluctuations about the classical background configuration, and for ground state quantum numbers this adiabatic approximation gives very accurate results [10]. Furthermore, since the fractional part of the fermion number is the same for all states in the underlying Fock space (the fermion numbers of two fermionic states can differ
only by an integer), the fractional part of the fermion number is completely determined by this method.

Figure 1. A symmetric potential with doubly degenerate minima.

We now assume that the background field \( \varphi \) is static. The classical Euler-Lagrange equations for \( \varphi \),

\[
\frac{\partial}{\partial \mu} \frac{\delta L(\varphi)}{\delta (\partial_\mu \varphi)} - \frac{\delta L(\varphi)}{\delta \varphi} = 0,
\]

imply that in the static case,

\[
\frac{1}{2} \left( \frac{d}{dx} \varphi(x) \right)^2 = V(\varphi(x)). \tag{1.4}
\]

We further assume that the potential \( V(\varphi) \) is symmetric with doubly degenerate minima (see Figure 1). For example, we could treat \( \varphi \) in the self-interacting \( \varphi^4 \)-theory, in which

\[
V(\varphi) = \frac{\lambda^2}{4\mu^2} (\varphi^2 - \mu^2)^2 = V(-\varphi), \tag{1.5}
\]
where \( \lambda \) gives the mass of \( \varphi \) and \( \mu \) is a constant giving the value of \( \varphi \) in the vacuum sector of the quantum field theory. In this case, we can seek soliton solutions [17] to equation (I.4); that is to say, if, as in (I.5), \( V(\varphi) \) has stable minima at \( \varphi = \pm \mu \), where \( \mu > 0 \), then we seek \( x \)-dependent solutions to (I.4) which minimize the bosonic energy, such that the energy density is finite and localized in space about \( x = 0 \) and \( \varphi \) has the asymptotes

\[
\lim_{x \to \pm \infty} \varphi(x) = \pm \mu. \tag{I.6}
\]

Thus the solitons interpolate between the two degenerate vacua at \( \varphi = \pm \mu \), where the reflection symmetry \( \varphi \leftrightarrow -\varphi \) is spontaneously broken and the fermionic part of the Lagrangian (I.1) becomes the usual translation invariant free fermion Lagrangian with conjugation symmetry explicitly broken. Furthermore, in these vacuum sectors of the quantum field theory, the system of eigenstates is trivial (plane waves with energies \( E = \pm \sqrt{k^2 + m^2 + \mu^2} \)), and can be built on either the \( \varphi = +\mu \) or the \( \varphi = -\mu \) vacuum, which both have zero fermion number. In the case (I.5), the soliton solutions are

\[
\varphi(x) = \pm \mu \tanh(\mu x)
\]

where the plus (minus) sign corresponds to a soliton (antisoliton) at the origin (For a more detailed account of this procedure, see [10] and [15]). For the remainder of this thesis, however, the explicit form of the soliton profile \( \varphi(x) \) is quite arbitrary; all that we need to know about the soliton field is its property (I.6). With such field configurations as described above, the Lagrangian (I.1) models Dirac fermions interacting with a soliton, located at \( x = 0 \), in \((1 + 1)\) dimensions.

In \((1 + 1)\) dimensions, the spinor field \( \psi \) has two components (that is, \( \psi \) is a spinless fermion field), and the Dirac algebra (I.2) can be represented by the \( 2 \times 2 \) matrices

\[
\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = i \sigma^3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^5 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{I.7}
\]
where $\sigma^i$ are the Pauli spin matrices. The Dirac Hamiltonian corresponding to the Lagrangian (I.1) in the first quantized picture is then

$$H = -i\sigma^2 \frac{d}{dx} + \sigma^1 \varphi(x) + \sigma^3 m,$$  

(I.8)

where we assume that the background field $\varphi(x)$ has the soliton profile (I.6). Notice that there is no $2 \times 2$ constant matrix which anticommutes with $H$, and hence the Hamiltonian (I.8) does not admit any norm-preserving symmetry (like charge conjugation) relating its positive and negative energy eigenmodes. This, as we shall see, is the essence of the fractionization of the charge to possibly irrational values.

2. Evaluation of the Open Space Fermion Number

We now proceed to calculate the fermion number of the model (I.1) defined for all real values of the space parameter $x$, that is, for $x$ on the whole of $\mathbb{R}$. The conserved current corresponding to the U(1) symmetry of the Lagrangian (I.1) is

$$j^\mu(x,t) = :\bar{\psi}(x,t)\gamma^\mu \psi(x,t):$$  

(I.9)

$$= \frac{1}{2} [\bar{\psi}(x,t), \gamma^\mu \psi(x,t)]$$

where the double colon symbol denotes normal ordering according to the standard Dirac commutator prescription, defined so that the fermion number is odd under charge conjugation (which, in this model, can be implemented by the Pauli matrix $\sigma^3$) and vanishes in the vacuum sector when $\varphi(x) = \pm \mu$ (This is tantamount to regarding the infinite charge of the negative energy sea as unphysical). The fermion number operator is then

$$Q = \int_{-\infty}^{+\infty} dx \ j^0(x,t)$$  

(I.10)

$$= \frac{1}{2} \int_{-\infty}^{+\infty} dx \ [\psi^\dagger(x), \psi(x)]$$
where $\psi(x) = \psi(x, 0)$.

To evaluate the expression (I.10), we must first second quantize the spinor field $\psi$. For this, we need the eigenmodes $\psi_E$ of the Dirac Hamiltonian (I.8) in the first quantized representation:

$$H\psi_E = E\psi_E.$$  \hspace{1cm} (I.11)

It is convenient to define two elliptic differential operators $D$ and $D^\dagger$ by

$$D = -\frac{d}{dx} + \varphi(x), \quad D^\dagger = \frac{d}{dx} + \varphi(x).$$  \hspace{1cm} (I.12)

The Hamiltonian (I.8) then takes the form

$$H = \begin{pmatrix} m & D \\ D^\dagger & -m \end{pmatrix} \equiv H_0 + m\sigma^3.$$  \hspace{1cm} (I.13)

Since

$$\{H_0, \sigma^3\} = 0,$$

we have

$$H^2 = H_0^2 + m^2 \geq m^2,$$

and as a consequence all the energy eigenvalues $E$ of $H$ satisfy $E^2 \geq m^2$. If we write

$$\psi_E = \begin{pmatrix} u_E \\ v_E \end{pmatrix},$$  \hspace{1cm} (I.14)

then the Dirac equation (I.11) takes the component form

$$D^\dagger u_E = (E + m)v_E$$  \hspace{1cm} (I.15)

$$Dv_E = (E - m)u_E.$$  \hspace{1cm} (I.16)

Iterating the Equations (I.15) and (I.16) we see that $u_E$ and $v_E$ solve the Schrödinger equations

$$\left(-\frac{d^2}{dx^2} + \varphi^2 - \varphi'\right)u_E = (E^2 - m^2)u_E$$  \hspace{1cm} (I.17)
\[
\left(-\frac{d^2}{dx^2} + \varphi^2 + \varphi'\right)v_E = (E^2 - m^2)v_E,
\]  
(I.18)

where \(\varphi' = \frac{d}{dx}\varphi\). From (I.15) and (I.16), we see that every solution of (I.17) or (I.18) yields two solutions of the Dirac equation (I.11) if \(E \neq \pm m\) and one solution if \(E = -m\) (Notice that for the soliton configuration (I.6) the \(E = -m\) solution is normalizable while the \(E = +m\) solution is not). For example, if \(u_E\) is a properly normalized solution of (I.17) with \(E \neq \pm m\), then

\[
\psi_E(x) = \left( \frac{\sqrt{\frac{E+m}{2E}}}{\text{sign}(E)} \left( u_E(x) \right) \right) = \sqrt{\frac{E + m}{2E}} \left( u_E(x) \right)
\]  
(I.19)

is a properly normalized solution of (I.11) with eigenvalue \(E\).

We next introduce the second quantized field operator at \(t = 0\):

\[
\psi(x) = \oint dk \left( b_k \psi_k^{(+)}(x) + c_k^\dagger \psi_k^{(-)}(x) \right)
\]  
(I.20)

where the sum is over both continuum and bound state solutions of the Dirac equation (I.11). Here \(\psi_k^{(+)}(x)\) and \(\psi_k^{(-)}(x)\) are the positive and negative energy solutions, respectively, to (I.11) with \(E = \pm \sqrt{k^2 + m^2 + \mu^2}\), and \(b_k, c_k^\dagger\) and their Hermitian conjugates are annihilation and creation operators for the fermion and antifermion states, respectively, in the second quantized picture. In view of the field algebra (I.3), the operators \(b_k, c_k^\dagger\) and their Hermitian conjugates obey the anticommutation relations

\[
\{b_k, b_q^\dagger\} = \{c_k, c_q^\dagger\} = \delta(k - q)
\]

with all other anticommutators vanishing.

We are now ready to compute the fermion number \(N\) of the ground state \(|0\rangle\) in the second quantized theory (which might be degenerate). In a free fermion theory, \(N = 0\), because the ground state is unique and contains neither fermions nor antifermions.
However, as we shall now see, the presence of the soliton induces a non-trivial vacuum polarization; that is, the Dirac sea of fermions is modified by its interaction with the soliton and the ground state contains particle-like excitations. Using the expression (I.10) for the fermion number operator and the expansion (I.20), we have

\[ N = \langle 0 | Q | 0 \rangle \]

\[ = \frac{1}{2} \int_{-\infty}^{+\infty} dx \langle 0 | [\psi^\dagger(x), \psi(x)] | 0 \rangle \]

\[ = -\frac{1}{2} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \sum_{\pm E} \psi^\dagger_E(x) \psi_E(x) \text{sign}(E) \]

after noting that by definition,

\[ b_k | 0 \rangle = c_k | 0 \rangle = 0 \]

\[ \langle 0 | 0 \rangle = 1. \]

Thus

\[ N = -\frac{1}{2} \int_{-\infty}^{+\infty} dx \psi^\dagger_{-m}(x) \psi_{-m}(x) \]

\[ -\frac{1}{2} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \sum_{\pm E} \psi^\dagger_E(x) \psi_E(x) \text{sign}(E) \]

where \( \psi_{-m}(x) \) is the normalizable \( E = -m \) bound state of (1.8), and the second term above is an integral over the continuum eigenstates of (1.8) with \( E = \pm \sqrt{k^2 + m^2 + \mu^2} \).

That there is no contribution from any other normalizable bound states with \( |E| < m \) follows from the following symmetry argument. We notice that

\[ \left\{ H, \frac{1}{2}[H, \sigma^3] \right\} = \frac{1}{2} [H^2, \sigma^3] = 0 \]

so that the unitary operator

\[ \frac{1}{2}[H, \sigma^3] = \begin{pmatrix} 0 & -D \\ D^\dagger & 0 \end{pmatrix} \]

maps all positive energy eigenmodes of \( H \) not annihilated by \( D \) or \( D^\dagger \) onto negative energy eigenmodes, and vice versa. Thus the contribution from all discrete bound

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states with \(|E| < m\) cancels in (I.21). This symmetry operator, however, does not
preserve the density of continuum states with \(|E| > m\) \([15]\), and as a consequence the
contribution from the continuum modes does not cancel in (I.21). We now substitute
the solutions (I.19) into the above to obtain

\[
N = -\frac{1}{2} - \frac{1}{2} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \sum_{\pm E} \text{sign}(E) \left( \frac{E + m}{2E} u_E^*(x) u_E(x) \\
+ \frac{1}{2E(E + m)} [D^\dagger u_E(x)]^* [D^\dagger u_E(x)] \right)
\]

and so integrating the last term above by parts over \(x\) we find

\[
N = -\frac{1}{2} - \frac{1}{2} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \sum_{\pm E} \text{sign}(E) u_E^*(x) u_E(x) \\
- \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \sum_{\pm E} \frac{\text{sign}(E)}{2E(E + m)} u_E^*(x) D^\dagger u_E(x) \bigg|_{-\infty}^{+\infty}.
\]

Now the functions \(u_E(x)\) are eigenfunctions of (I.17) with eigenvalues \(E^2 - m^2\), and
are therefore even in \(E\). Thus the integrand in the first integral above vanishes when
summed over \(\pm E\), and the fermion number becomes

\[
N = -\frac{1}{2} - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \sum_{\pm E} \frac{\text{sign}(E)}{2E(E + m)} u_E^*(x) D^\dagger u_E(x) \bigg|_{-\infty}^{+\infty}.
\] (I.22)

To evaluate this last integral, we need only consider the scattering solutions to the
Schrödinger equation (I.17):

\[
u_E(x) = \begin{cases} 
e^{ikx} + R(k)e^{-ikx}, & x \to -\infty, \\
T(k)e^{ikx}, & x \to +\infty, \end{cases}
\] (I.23)

where \(E = \pm \sqrt{k^2 + m^2 + \mu^2}\), and \(R(k)\) and \(T(k)\) are reflection and transmission coefficients, respectively. Substituting (I.23) into (I.22), using (I.6) and (I.12), and dropping
rapidly oscillating terms, we find

\[
N = -\frac{1}{2} - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \sum_{\pm E} \frac{\mu \text{sign}(E)}{2E(E + m)} [T(k)]^2 + |R(k)|^2 + 1].
\] (I.24)
Unitarity,

\[ |T(k)|^2 + |R(k)|^2 = 1 \]  

then gives the final result:

\[ N = -\frac{1}{\pi} \arctan \left[ \frac{\mu}{m} \right]. \]  

Thus the fermion number \( N \) may assume arbitrary fractional values, even irrational, depending on the values of the parameters \( \mu \) and \( m \) appearing in the Hamiltonian (1.8). \( N \) is the induced fermionic charge on the soliton located at \( x = 0 \), which exists as a particle excitation in the Fock vacuum of this theory (This is inherent in the condensed matter applications). Notice that the fractionization depends only on the asymptotes of the soliton field \( \varphi(x) \), and arises from the non-triviality of the field; that is, from the fact that \( \varphi(x) \) interpolates between the two degenerate vacua. Notice also that only the continuum eigenmodes of (1.8) contribute to the irrational fractionization in (1.22), which is a result of the lack of conjugation symmetry in the system (I.1). Finally, we note that as \( m \to 0 \), \( N \to \pm \frac{1}{2} \), and the Hamiltonian (I.8) reduces to that originally considered by Jackiw and Rebbi [13]. The above result is therefore in agreement with the original fractional fermion number discovery.

The Hamiltonian (I.8) can be shown [14] to be the continuum limit of the Hamiltonian in condensed matter physics describing a linearly conjugated diatomic polymer, such as polycarbonitrile [19]. In this case the charge conjugation symmetry breaking parameter \( m \) in (I.8) corresponds to the difference of energy levels of the atomic p-orbitals of the two atomic constituents of the polymer. The \( m \to 0 \) limit of (I.8) has been shown [3, 23] to be a continuum version of the Hamiltonian describing the conducting polymer polyacetylene [21]. These results, along with various experimental results [8], provide a physical realization of the fermion fractionization phenomenon.
presented here in a rather abstract mathematical fashion. Furthermore, the existence of fermion fractionization in the two apparently distinct fields of condensed matter physics and relativistic quantum field theory is a strong reaffirmation of the unity of all branches of physics.

Before concluding this chapter, we introduce another method for computing the fermion number (1.21) which will be used in the discussion of the next chapter. The fermion number \( N \) can be related to the spectral asymmetry of the Dirac Hamiltonian (I.8) [16]. Schematically,

\[
N = -\frac{1}{2} \left( \sum_{E>0} 1 - \sum_{E<0} 1 \right)
\]  

(I.27)

where \( E \) are the energy eigenvalues of the Hamiltonian (I.8). However, (I.27) as written above is ill-defined, and must be regularized. We shall use heat kernel regularization, and write

\[
N = -\frac{1}{2} \lim_{\beta \to 0^+} \left( \int_{E>0} dE e^{-\beta E} - \int_{E<0} dE e^{-\beta |E|} \right).
\]  

(I.28)

This formula is particularly useful when the pertinent Dirac Hamiltonian has a discrete set of eigenstates.

The various methods developed to compute the fermion number in \((1 + 1)\) dimensions [15] can also be applied to field theory models in higher dimensions. Thus fermion fractionization arises in a number of phenomenologically important field theories, such as \((2 + 1)\) dimensional quantum electrodynamics where the solitons of the theory are the Abrikosov-Nielson-Oleson vortices, and \((3 + 1)\) dimensional non-Abelian gauge theories where the role of the soliton is played by the Dirac or the 'tHooft-Polyakov monopole [15].
Chapter II

Boundary Dependence of the
Fractional Fermion Number

In this chapter we examine in some detail the effects of finite volume on the fractional charge. We also investigate the quantum observability of the fractional fermion number, and to what extent the soliton behaves as a true charged object. In Section 1, we shall show how certain necessary boundary conditions on the theory in finite volume induce a fractional fermion number. In Section 2, we discuss taking the infinite volume limit, and show that the results of the previous chapter (and other open space theories) are reproducible in such a limit only with a careful limiting definition of the charge operator (1.10). In Section 3 we establish the main result of this thesis, namely that the quantum fluctuations of the fractional charge in this limit vanish, making the soliton’s charge a sharp quantum observable. These problems were originally studied in [12] and [18] for the charge conjugation symmetric ($m = 0$) system, and in [20] for more general field theory models (with $m = m(x)$). The results obtained in this chapter are an extension of these studies to the system (I.1), and give a precise method for computing the charge in the infinite volume limit of a (perhaps more realistic) finite interval system which is equivalent to computing the charge in the open space problem directly [22]. The distinction of the boundary induced fractional fermion number (
which must vanish in the infinite volume system) in Section 1 along with the isolation of the soliton in Section 2 give the precise fractional fermion number induced on a single soliton in the infinite volume limit of the finite interval system, which then coincides with the fermion number computed in Section 2 of Chapter I directly in the corresponding open space problem, as it should. The results of Section 3 then show that the correct fractional charge induced on the soliton in this way is a sharp quantum number, and hence corresponds to a well-defined observable of the fermion-soliton system (I.1). This, of course, is important for the experimental realizations of fermion fractionization. Finally, we conclude in Section 4 with a brief discussion of how well the soliton behaves as an electrically charged object in the sense of its electromagnetic interactions with external gauge fields.

1. Fractional Fermion Number in a Finite One Dimensional Box

We now consider the system (1.1) defined on a finite interval \([L_1, L_2]\) with \(L_1 < 0 < L_2\). The problem which now arises is that in finite volume, the Hamiltonian (1.8) is not necessarily a Hermitian operator. Indeed, a straightforward computation shows that for any two Dirac spinors \(\psi^{(1)} = \begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \end{pmatrix}\) and \(\psi^{(2)} = \begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \end{pmatrix}\) in \(L^2([L_1, L_2], dx)\),

\[
(H\psi^{(2)}, \psi^{(1)}) = (H\psi^{(2)}, \psi^{(1)}) + [\psi_2^{(2)}(x)\psi_1^{(1)}(x) - \psi_1^{(2)}(x)\psi_2^{(1)}(x)]|_{L^1_{L_1}}^{|L^2_{L_2}}
\]

where \((\cdot, \cdot)\) denotes the usual inner product on \(L^2([L_1, L_2], dx)\). Thus we see that the Hamiltonian (I.8) defined on \([L_1, L_2]\) will be Hermitian if and only if

\[
[\psi_2^{(2)}(x)\psi_1^{(1)}(x) - \psi_1^{(2)}(x)\psi_2^{(1)}(x)]|_{L^1_{L_1}}^{|L^2_{L_2}} = 0
\]

for any two spinors \(\psi^{(1)}\) and \(\psi^{(2)}\) in the domain of \(H\). If we set \(\psi^{(1)} = \psi^{(2)}\), we see further that for any Dirac spinor \(\psi = \begin{pmatrix} u \\ v \end{pmatrix}\),

\[
(H\psi) - (H\psi, \psi) = 2i \text{Im}[(u, Dv) - (D^\dagger u, v)]
\]
from the expression (1.13) for the Hamiltonian. Thus, interestingly enough, the self-
adjointness of the Hamiltonian (1.13) does not require that the operators $D$ and $D^\dagger$ be
the adjoints of one another. That $D$ and $D^\dagger$ are not necessarily adjoints of each other
in finite volume is easily seen from their definition (1.12), which gives

$$(u, Dv) - (D^\dagger u, v) = u^*(x)v(x)\big|_{L_1}^{L_2}$$

for any two $L^2$-functions $u(x)$ and $v(x)$ defined on $[L_1, L_2]$. The Hermiticity conditions
then give for the self-adjointness of the Hamiltonian the domain of $H$ as

$$\text{dom}(H) = \text{dom}(H^\dagger) \subset \{\psi = \begin{pmatrix} u \\ v \end{pmatrix} \in L^2([L_1, L_2], dx) : \text{Im}[u^*(x)v(x)]_{L_1}^{L_2} = 0\},$$

and for $D$ to be the adjoint of $D^\dagger$

$$\text{dom}(D) = \{u, v \in L^2([L_1, L_2], dx) : u^*(x)v(x)\big|_{L_1}^{L_2} = 0\} \subset \text{dom}(H).$$

The reason why the domains of $H$ and $D$ do not coincide in the above sense when
these Hermiticity conditions are imposed in finite volume is that there is more of a
freedom in imposing boundary conditions on two component spinors than on just sin-
gle $L^2$-functions alone. On a spinor $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$ we can impose independent boundary
conditions on the functions $u$ and $v$ which yield $\psi \in \text{dom}(H)$, while the boundary
conditions specified in $\text{dom}(D)$ must be the same for both $u$ and $v$ since the conditions
specified for $D$ are fixed for each $u \in \text{dom}(D)$. That is to say, in $\text{dom}(H)$ we essen-
tially have the freedom to specify two sets of independent boundary conditions, one
for each spinor component, while in $\text{dom}(D)$ we may only specify one such set. As a
consequence, the boundary conditions required for $H$ to be Hermitian are weaker than
those required for $D$ to be the adjoint of $D^\dagger$. For example, the boundary conditions

$$u(L_2) = -u(L_1) = c$$
\[ v(L_2) = v(L_1) = c \]  

(II.1)

where \( c \in \mathbb{R}, c \neq 0 \), give \( u^*(x)v(x)|^L_1 = 2c^2 \) and hence \( \begin{pmatrix} u \\ v \end{pmatrix} \in \text{dom}(H) \), but \( (D^\dagger u, v) \neq (u, Dv) \); that is, (II.1) gives \( H = H^\dagger \) in (I.13) but does not give \( D \) as the adjoint of \( D^\dagger \). The boundary conditions (II.1) however lead to inconsistencies in the spectral analysis of the Hamiltonian. For example, in the free fermion theory \((m = g = 0 \text{ in (I.1)})\), the eigenmodes of the Hamiltonian are \( \psi_E = \begin{pmatrix} u_E \\ v_E \end{pmatrix} \) where

\[ u_E(x) = e^{-iEx} \]
\[ v_E(x) = -ie^{-iEx}, \]

(II.2)

and therefore the boundary conditions (II.1) yield inconsistent values for the energy eigenvalues \( E \). Boundary conditions of the form (II.1), and others whereby \( u \) and \( v \) have different boundary dependences, are therefore undesirable in general for our considerations here, since the eigenmodes of \( H \) span the function space \( \text{dom}(H) \) when \( H \) is Hermitian. One may also try to impose boundary conditions of the form

\[ u(L_1) = v(L_1) = c_1 \]
\[ u(L_2) = v(L_2) = c_2 \]

with \( c_i \in \mathbb{C} \) for \( i = 1, 2, |c_1| \neq |c_2| \), but then the eigenmodes, such as (II.2), in general force \( |c_1| = |c_2| \). Moreover, if \( D \) and \( D^\dagger \) are not Hermitian conjugates of each other, then \( D^\dagger D \) and \( DD^\dagger \) are not positive Hermitian operators and hence the Schrödinger equations (I.17) and (I.18) need not yield complete orthonormal sets of eigenfunctions. Consequently, the eigenmodes \( \psi_E = \begin{pmatrix} u_E \\ v_E \end{pmatrix} \) may not be normalizable in the usual way, which would then contradict the Hermiticity of the Hamiltonian. Thus the boundary conditions which give a Hermitian Hamiltonian but do not give \( D \) as the adjoint of \( D^\dagger \) in general lead to disasters in the spectral analysis of the theory and the function space.
dom(\(H\)) is in general then not a well-defined Hilbert space. The above considerations along with the fact that we will eventually take the limit to the open space theory where the Hermiticity conditions hold true trivially imply that we must take

\[
\text{dom}(H) = \text{dom}(D)
\]
in the above sense, so that the domains of \(H\) and \(D\) coincide, \(H\) is Hermitian, and \(D\) and \(D^\dagger\) are the adjoints of each other. Therefore a necessary boundary condition on any Dirac spinor \(\psi = \begin{pmatrix} u \\ v \end{pmatrix}\) (and in particular the eigenmodes of \(H\)) defined on \([L_1, L_2]\) is

\[
u^*(x)v(x)\bigg|_{L_1}^{L_2} = 0.
\]

(II.3)

The boundary conditions (II.3), as it turns out, produce a fractional fermion number even in the case where \(m = g = 0\) in (I.1) (the corresponding trivial free fermion theory). The magnitude of the fractionization is parametrized by the boundary condition parameters. That this is quite natural can be seen intuitively from the following rudimentary example. Consider the model of a one dimensional finite chain of atoms whereby we have an odd number \(2M + 1\) of lattice sites each with one orbiting electron. If we omit the spin of the orbital electrons, then the resulting system has only half as many degrees of freedom as originally, and we effectively then have only half as many electrons as in the original physical system. This system then corresponds to a lattice regulated version of \((1 + 1)\) dimensional free Dirac field theory. In this case, the charge at each site must be defined with half a unit of electronic charge subtracted off [3] (not one) so that the chain as a whole is electrically charge neutral, as required in the original real physical system where there are also ionic charges that keep the physical system as a whole more or less neutral (in the field theory context, this corresponds to the normal ordering (I.9)). If we now consider the particular vacuum state \(|0\rangle\) of
the chain defined so that all negative energy wavefunctions are occupied (say, at lattice sites 1, 2,..., \( M \)) while the zero and positive energy wavefunctions are unoccupied (at sites \( M + 1, M + 2,..., 2M + 1 \)) in the first quantized theory (this definition ensures neutrality of the entire chain), then the total fermionic charge \( Q_0 \) for this state is

\[
Q_0 = \text{(number of occupied states)} - \frac{1}{2}\text{(number of lattice sites)}
\]

\[
= M - \frac{1}{2}(2M + 1)
\]

\[
= -\frac{1}{2}.
\]

Thus the chain has a net half-integral fractional charge. But note, however, that if we chose an even number of sites or identified the end points of the odd chain (periodic boundary conditions), then the total charge \( Q_0 \) would be integral. Thus in this case, the fractional charge arises as an end effect, dependent upon the boundary conditions imposed on the system. This example provides a nice physical and intuitive understanding of the boundary dependence of the fermion number. However, as we shall now show, this type of charge fractionization even in the absence of solitonic excitations is not limited to lattice models; it occurs even in the continuum systems we have been discussing.

In our model (I.8) we now consider \( m = g = 0 \) in (I.1), so that the pertinent Dirac Hamiltonian is

\[
H_f = -i\sigma^2 \frac{d}{dx}.
\]  

(II.4)

As a simple example, we consider the eigenmodes (I.14) of \( H_f \) with the boundary conditions satisfying (II.3)

\[
\psi_E(L_2) = e^{i\theta} \psi_E(L_1)
\]  

(II.5)

where \( \theta \in [0, 2\pi) \). Solving the Dirac equation (I.11) with \( H_f \) and imposing the bound-

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The provided text is a continuation of the discussion on the fractional charge in a quantum system, specifically focusing on the calculation and interpretation of the charge for states on a chain with given boundary conditions. The text elaborates on the fractional charge arising from boundary conditions and contrasts this with the integral charge that can arise in cases of even numbers of sites or periodic boundary conditions. It concludes with a mention of the applicability of this phenomenon to continuum systems, not limited to lattice models. The mathematical expressions are derived to show the calculation of the charge, with particular attention to the role of boundary conditions in determining the nature of the charge.
ary conditions (II.5) yields the energy eigenvalues

\[ E_n = \frac{2\pi n - \theta}{L_2 - L_1}, \quad n \in \mathbb{Z} \]  

(II.6)

with corresponding eigenfunctions

\[ \psi_{E_n}(x) = \frac{1}{\sqrt{2(L_2 - L_1)}} \begin{pmatrix} e^{-iE_n x} \\ -ie^{-iE_n x} \end{pmatrix}. \]  

(II.7)

We now substitute the energy eigenvalues (II.6) into the expression (I.28) for the fermion number. After evaluating the simple infinite series and taking the appropriate limit, we obtain the fermion number \( N_0 \) for the Hamiltonian (II.4) as

\[
N_0 = -\frac{1}{2} \lim_{\beta \to 0^+} \left( \sum_{n=\lceil \theta/2\pi \rceil + 1}^{+\infty} e^{-\beta E_n} - \sum_{n=-\infty}^{-\lceil \theta/2\pi \rceil} e^{-\beta |E_n|} \right)
\]

\[ = -\left( \frac{\theta}{2\pi} - \left\lfloor \frac{\theta}{2\pi} \right\rfloor \right) \]  

(II.8)

where \([a]\) denotes the integer part of \( a \in \mathbb{R} \) with \( a - [a] \in [0, 1) \) (Notice that if a Dirac Hamiltonian has a symmetry relating positive and negative energy states, then only zero energy modes can contribute to (I.27). For the sake of illustration, however, we have not included this contribution in (II.8)). Thus the fermion number in the trivial vacuum sector with zero coupling and conjugation symmetry exhibits fractionization to arbitrary values dependent on the value of \( \theta \in [0, 2\pi) \). The boundary conditions (II.5) thus also distort the Dirac sea and give the vacuum state a non-trivial polarization by producing a definite asymmetry in the energy eigenvalue spectrum of (II.4), as given in (II.6). We note, however, that the charge density

\[ \rho_{E_n}(x) = \psi_{E_n}^\dagger(x) \psi_{E_n}(x) = \frac{1}{L_2 - L_1} \]  

(II.9)

is uniformly distributed in \([L_1, L_2]\). Clearly these same considerations hold for the Hamiltonian (II.4) and arbitrary boundary conditions of the form (II.3), so that the
boundary induced fractional fermion number (such as (II.8)) appears as a "background" charge uniformly distributed over the interval \([L_1, L_2]\) in the soliton sector.

It has been shown [20] more generally (for a Dirac Hamiltonian of the form (I.8) with an \(x\)-dependent \(m\)) that the allowed self-adjoint extensions of a Dirac Hamiltonian defined on a finite interval \([L_1, L_2]\) are labelled by an arbitrary \(2 \times 2\) unitary matrix \(U\). The boundary induced fractional charge is then given by \(\alpha/\pi\), where \(\det U = -e^{2i\alpha}\). As pointed out before, this effect can be regarded as quite natural, since the topological effects (that is, the asymptotic limits of the soliton field) which induce the fractional charge in the open space problem correspond to the far away situation. For the finite space problem, the analogous role is played by boundary conditions.

2. Definition of the Soliton Charge

We now give a precise definition of the fermion number induced on a single soliton in the finite volume system. The reason why this definition is a subtle point is best illustrated by a few examples.

First consider the fermion number (II.8). If we take \(\theta = 0\) in (II.5), then we are imposing periodic boundary conditions on the system, and the fermion number (II.8) in the vacuum sector becomes \(N_0 = 0\) (Actually, \(N_0 = -1\), since we must include the contribution from the zero mode in this case). It can then be shown [20] that for the system (I.8) in the soliton sector, the total fermion number is always an integer, even in the infinite volume limit. The reason why the fractional fermion number does not appear here is that the imposition of periodic boundary conditions automatically introduces an antisoliton into the system at \(x = L_2\); that is, the field \(\varphi(x)\) interpolates between \(-\mu\) and \(+\mu\) (soliton at \(x = 0\)) and then from \(+\mu\) to \(-\mu\) again (antisoliton.
at $x = L_2$). This must be so with periodic boundary conditions in order that the soliton field $\varphi(x)$ be a single-valued function of $x$. Thus the total charge of the soliton-antisoliton system is an integer and we would like to formulate a way to “measure” the fractional charge that appears on the soliton alone, which we know exists from the infinite volume problem. As we shall see, this can be accomplished by first isolating the soliton from the antisoliton (and other possible solitons in the system) and then allowing the soliton-antisoliton separation to become infinite (that is, let $L_2 \to \infty$).

Next, consider the boundary conditions (II.3). If, instead of using the box quantization procedure of Section 1, we treat the system in the continuum limit by assuming the energy eigenvalues $E$ are independent of $L_2$ and $L_1$ (this is done in order to work properly with the continuum spectrum of $H$ and to get the contribution of the continuum states to the fermionic charge), then letting $|L_1|, L_2 \to \infty$ in (II.3) and using (I.19), we see that a condition guaranteeing Hermiticity of the Hamiltonian (I.13) is

$$u_E^*(x)D^\dagger u_E(x)|^{+\infty}_{-\infty} = 0. \quad (II.10)$$

However, if we substitute (II.10) into the expression (I.22) for the fermion number, we see that the integral in (I.22) vanishes, and the fermion number is

$$N = -\frac{1}{2} \quad (II.11)$$

and can assume only the half-integer value as obtained originally by Jackiw and Rebbi [13]. This result is therefore a contradiction to the open space results obtained in [7], [14], [15], and [19], where arbitrary charge fractions were obtained as in (I.26) for the Hamiltonian (I.8). What has happened here is that the contribution to the fermion number from the continuum modes has cancelled and the charge fraction $-\frac{1}{2}$ has appeared because of the existence of the normalizable $E = -m$ mode, just as in the
charge conjugation symmetric system which possessed a normalizable zero energy mode. The fractionization (II.11) is therefore the same as for the corresponding open space fermion-soliton system with conjugation symmetry. However, as shown in Chapter I, the loss of conjugation symmetry distorts the Dirac fermion sea quite differently than the distortion from just the existence of a normalizable bound state. Thus an additional charge fractionization has been induced here by the boundaries (like that in (II.8)), and this cancels out the continuum contribution to the fermionic charge induced on the soliton; that is to say, the fractionization (II.11) is boundary induced and "masks" the fermionic charge fractionization (I.26). The resolution of this apparent contradiction lies entirely in the definition of the charge on the soliton, which we will now motivate with the following example from atomic physics [12].

Consider a two-centered molecule with one electron passing between the two atoms (Figure 2). Here the expectation value of the charge near one atom is $\frac{1}{2}$ when the two centers of the molecule become separated by a large distance. However, any measurement finds the electron associated with one atom or the other, regardless of their separation. To see precisely how the charge $\frac{1}{2}$ fraction occurs, consider the ground
state electronic wavefunction, which is a symmetric superposition of two functions, each localized about one of the two atoms:

$$\psi_+ = \frac{1}{\sqrt{2}} (\psi_1 + \psi_2).$$

We define a charge operator which is localized at the first atom,

$$Q_f = \int dx f(x) \rho(x)$$ (II.12)

where $\rho(x)$ is the electron density, and $f(x)$ is a function peaked at the position of the first atom. The charge on the first atom is then obtained by first letting the two atoms become infinitely separated, and then letting $f \to 1$, in that order. This is important since it is this order which properly isolates the first atom, and the reverse order, as we shall see, does not yield the same result. Then the expected charge, localized at the first atom, is one-half of the total charge:

$$< \psi_+ | Q_f | \psi_+ > = \int dx f(x) \psi_+^*(x) \psi_+ (x) \approx \frac{1}{2} \int dx f(x) \psi_1^*(x) \psi_1 (x) \frac{1}{2}$$

by orthonormality of the electronic wavefunctions, where in the second integral above we have let the two atoms become infinitely separated. Thus the charge is half-integral, but only in an appropriately defined limit. If we had set $f \to 1$ first and then let the atomic separation become infinite, then by orthonormality of the electronic wavefunctions we would have obtained $< \psi_+ | Q_f | \psi_+ > \to 1$. Note that here, however, the variance, or quantum fluctuations, of the charge does not vanish when the atoms are separated by an arbitrarily large amount, because there is another state which is almost degenerate with the ground state:

$$\psi_- = \frac{1}{\sqrt{2}} (\psi_1 - \psi_2),$$
and hence
\[
\langle \psi_+ | Q_f^2 | \psi_+ \rangle \geq \langle \psi_+ | Q_f | \psi_+ \rangle \langle \psi_+ | Q_f | \psi_+ \rangle + \langle \psi_+ | Q_f | \psi_- \rangle \langle \psi_- | Q_f | \psi_+ \rangle
\]
\[
\to \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]
Thus, we see that the variance
\[
(\Delta Q_f)^2 = \langle \psi_+ | Q_f^2 | \psi_+ \rangle - \left( \langle \psi_+ | Q_f | \psi_+ \rangle \right)^2 \to \frac{1}{2} - \frac{1}{4} = \frac{1}{4}
\]
does not vanish even in the limit of infinite separation, because there is a state degenerate with the ground state. Therefore the fractional charge here is not an observable of the system, but merely a quantum average of the observable charges on each atom (Notice that the eigenvalues of \( Q_f \) in the above limit are 1 and 0 in the ground states).

In contrast, as we shall see in the next section, the fluctuations for the soliton system (I.8) do vanish in the above limit, which will establish that fermion fractionization in the fermion-soliton system is a true quantum effect. The difference in the soliton problem is that the localized number operator cannot reach the states degenerate with the ground state in the second quantized theory.

![Figure 3. Profiles of the soliton configuration (solid line) and of the sampling function (dashed line). The soliton is located at the origin, and the sampling function is peaked there.](image)

The localized charge operator (II.12) and the appropriate limiting procedure dis-
cussed above were first introduced in [12] and [18]. In particular, the above discussion shows that one must distinguish between the global fermion number and the fermion number which is localized in the neighbourhood of an isolated soliton. In a finite interval \([L_1, L_2]\) the appropriate charge operator giving the charge on the soliton is

\[
Q_f = \int_{L_1}^{L_2} dx f(x) j^0(x,t),
\]

where \(f(x) = 1\) in the region where the soliton field \(\varphi(x)\) deviates from its asymptotic values and vanishes at and near the boundaries of the interval (Figure 3). In the next section we shall show that for a certain general class of test functions \(f(x)\) the expectation value of \(Q_f\) is a sharp quantum number in the limit \(|L_1|, L_2 \to \infty\), then \(f \to 1\). This is the limit where the soliton can be considered isolated from the other solitons and antisolitons that may implicitly be present in the system due to boundary conditions, and thus gives a correct “measure” of the charge on the soliton located at \(x = 0\). For the special case of charge conjugation symmetry, in terms of the eigenmodes \(\psi_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix}\) of (1.8) with \(m = 0\), from (1.10) and (1.20) we see that \(Q_f\) for \(m = 0\) has the explicit form

\[
Q_f = \oint dk dk' \int_{L_1}^{L_2} dx f(x) \left[ (u_k^*(x)u_{k'}(x) + v_k^*(x)v_{k'}(x))(a_k^\dagger a_{k'} - b_k^\dagger b_{k'}) \\
+ (u_k^*(x)u_{k'}^*(x) - v_k^*(x)v_{k'}^*(x))a_{k'}^\dagger a_k + (u_k u_{k'} - v_k v_{k'}) b_k b_{k'} \right].
\]

Notice that as \(f \to 1\) and then \(|L_1|, L_2 \to \infty\), \(Q_f\) becomes diagonal with integer eigenvalues,

\[
Q_{f=1} = \oint dk (a_k^\dagger a_k - b_k^\dagger b_k),
\]

so that we clearly must consider this limit in the reverse order to obtain the true fractional charge induced on the single soliton. Notice also that since \(Q_f\) is not diagonal, it doesn’t commute with the Hamiltonian and hence is not a constant of the motion. Thus \(Q_f\) fluctuates about its mean in finite volume for \(f \neq 1\).
Equation (II.13) is the definition of the soliton charge in finite volume which we shall now adhere to. In the appropriate limiting procedure described above, it isolates the fermionic charge induced on the soliton at \( x = 0 \) from the effects of the boundaries in the finite volume system, in addition to isolating this soliton from other solitons in the system. It therefore resolves the apparent contradictions which were discussed above.

In the first example of this section, where we imposed periodic boundary conditions, it has been shown [12, 18] that for \( m = 0 \) the charge (II.13) in the appropriate limit measured on the soliton is indeed \( \pm \frac{1}{2} \), as it should be. The second example, which evidently contradicted the result (I.26), is incorrect only in its interpretation. The value (II.11) obtained for the fermion number is in fact the *global* fermion number since, by implicit construction, it was obtained in the limit \( f \rightarrow 1 \) first, then \( |L_1|, L_2 \rightarrow \infty \) (that is, the soliton was allowed to interact with the boundaries). That the two limits \( \lim_{f \rightarrow 1} \lim_{|L_1|, L_2 \rightarrow \infty} \) and \( \lim_{|L_1|, L_2 \rightarrow \infty} \lim_{f \rightarrow 1} \) do not commute has already been noted in [12] and [18]. The local fermion number for the system (I.8) with the various boundary conditions discussed in Section 1 will then be to a good approximation (not exact only because of finite volume) the number quoted in (I.26). The \( |L_1|, L_2 \rightarrow \infty \) limit then correctly reproduces the result (I.26). We must also remember to subtract off the corresponding "background" vacuum charge (such as (II.8)), which is present even in the absence of the soliton, in order to obtain the true charge on the soliton (Notice that the corresponding vacuum charge density, such as (II.9), vanishes in infinite volume and so does not contribute to the open space fermion number). In the next section we will discuss what classes of functions \( f(x) \) can be used in (II.13) to isolate the soliton and yet maintain the fractional fermion number as a sharp quantum observable.
3. Quantum Fluctuations of the Fractional Fermion Number

The definition (II.13) of the fermion number operator will yield the correct open space fractional expectation values $<0|Q|0>$ of the charge in the appropriate limiting procedure discussed in Section 2. The question which now arises is whether the fractional fermion number obtained in this way is in fact a true quantum observable (that is, an eigenvalue of the charge operator $Q$), or just a quantum average over various integer charged soliton states (as in the two-centered molecule example above). This can be answered by considering the quantum fluctuations of the charge, or the variance, defined by

$$(\Delta Q)^2 \equiv <0|(Q - <0|Q|0>)^2|0>$$

$$(\Delta Q)^2 = <0|Q^2|0> - (<0|Q|0>)^2.$$  \hspace{1cm} (II.14)

If (II.14) vanishes, then since all higher order moments of $Q$ can be expressed in terms of the second order moments of $Q$ this establishes that the fractional charge is indeed an eigenvalue of the charge operator $Q$, and hence a sharp quantum number. It will then also follow that fermion fractionization is a true quantum effect, in contrast to the two-molecule example of Section 2 where the fluctuations did not disappear. In this section we establish the main result of this thesis: With the definition (II.13) for the partial charge operator and the limiting procedure discussed in Section 2, the quantum fluctuations of the soliton charge vanish.

It should be noted that the choice of sampling functions $f(x)$ in (II.13) is not completely arbitrary. $f(x)$ must be sufficiently smooth in its fall to zero to make the fluctuations vanish as $f \to 1$. For example, suppose we tried the obvious choice for $f(x)$, a step function localized in $[L_1, L_2]$ and symmetric about $x = 0$. Then $f'(x) = \frac{d}{dx}f(x)$ would be a sum of two Dirac delta functions at some $x = \pm x_0 \in (L_1, L_2)$. Now it will be shown below that the fluctuations of the partial charge operator $Q_f$ depend in
fact on $f'(x)$ and not $f(x)$ itself. The delta functions therefore make the fluctuations constant, independent of $L_1$ and $L_2$ (as in the two-centered molecule example), and hence the quantum fluctuations do not vanish in the required limit. That is to say, the fluctuations at the sharp cutoff points of the sampling function are much larger than anywhere else in the finite volume system and therefore dominate the total charge variance. Thus the region where $f(x)$ changes from 1 to 0 must be diffuse and not a sharp cutoff.

To be definite, we let the interval $[L_1, L_2]$ of consideration contain the region where the soliton interpolates between the two vacua $\varphi(x) = \pm \mu$ (see Figure 3). Let us consider a one-parameter family of real-valued differentiable test functions $\{f_\Lambda(x)\} \subset C^1_0(\mathbb{R})$ defined on $\mathbb{R}$ with compact support in $[L_1, L_2]$ (This is possible below for all $\Lambda$ because we shall let $|L_1|, L_2 \to \infty$ first). We assume that $f_\Lambda(0) = 1$ for all $\Lambda$ and

$$\lim_{\Lambda \to \infty} f_\Lambda = 1$$

on $\mathbb{R}$. We further assume that $f'_\Lambda \equiv \frac{d}{dx} f_\Lambda \in L^2(\mathbb{R}, dx)$ with

$$\lim_{\Lambda \to \infty} \|f'_\Lambda\|_2 = 0 \quad (11.15)$$

where $\| \cdot \|_2$ denotes the usual norm on $L^2(\mathbb{R}, dx)$. (11.15) is the required "smoothness" condition on $f(x)$ discussed above (Notice that the condition (11.15) is violated if $f_\Lambda(x)$ are step functions). For example, we could take $\{f_\Lambda(x)\}$ to be the class of Gaussian test functions $f_\Lambda(x) = e^{-x^2/\Lambda^2}$.

We define the partial charge operator for the soliton in $[L_1, L_2]$ as

$$Q_{\Lambda, L_1, L_2} = \int_{L_1}^{L_2} dx f_\Lambda(x) j^0(x, t) \quad (11.16)$$

and the open space fermion number operator (1.10) is then

$$Q = \int_{-\infty}^{+\infty} dx j^0(x, t) = \lim_{\Lambda \to \infty} \lim_{|L_1|, L_2 \to \infty} \int_{L_1}^{L_2} dx f_\Lambda(x) j^0(x, t). \quad (11.17)$$
The partial charge fluctuations are

$$\left(\Delta Q_{A,L_1,L_2}\right)^2 \equiv <0|(Q_{A,L_1,L_2} - <0|Q_{A,L_1,L_2}|0>)^2|0>, \quad (II.18)$$

so that the fractional fermion number fluctuations are given by

$$\left(\Delta Q\right)^2 = \lim_{A \to \infty} \lim_{|L_1|,|L_2| \to \infty} \left(\Delta Q_{A,L_1,L_2}\right)^2. \quad (II.19)$$

We shall now evaluate (II.18). From (II.16) we have

$$\left(\Delta Q_{A,L_1,L_2}\right)^2 = \int_{L_1}^{L_2} dx_1 dx_2 f_A(x_1)f_A(x_2) <j^0(x_1,t_1)j^0(x_2,t_2)>_0 \bigg|_{\Delta t \to 0^+} \quad (II.20)$$

where $\Delta t \equiv t_1 - t_2$, and we have introduced the ground state current correlation function

$$<j^\mu(x_1,t_1)j^\nu(x_2,t_2)>_0 = <0|j^\mu(x_1,t_1)j^\nu(x_2,t_2)|0> - <0|j^\mu(x_1,t_1)|0><0|j^\nu(x_2,t_2)|0>. \quad (II.21)$$

Current conservation,

$$\partial_\mu j^\mu(x,t) = 0,$$

gives

$$-\partial^2_{\Delta t} <j^0(x_1,t_1)j^0(x_2,t_2)>_0 = \partial_{x_1} \partial_{x_2} <j^1(x_1,t_1)j^1(x_2,t_2)>_0$$

and so imposing the boundary conditions that $<j^0(x_1,t_1)j^0(x_2,t_2)>_0 \to 0$ at $\Delta t \to +\infty$ (for example, in a uniform vacuum the density-density correlation vanishes as $|\Delta t|^{-2}$ for $\Delta t \to \infty$), we can write $(\Delta Q_{A,L_1,L_2})^2$ in terms of derivatives of $f_A(x)$ as

$$\left(\Delta Q_{A,L_1,L_2}\right)^2 = \int_{L_1}^{L_2} dx_1 dx_2 f'_A(x_1)f'_A(x_2) \int_0^{+\infty} (d\Delta t)^2 <j^1(x_1,t_1)j^1(x_2,t_2)>_0. \quad (II.22)$$

In (II.22) we have integrated by parts over $x$ and set the boundary term to zero in view of the region of support of $f_A(x)$. We now need the correlation function appearing in (II.22). Using (I.9), we can write the current-current correlation as

$$<j^1(x_1,t_1)j^1(x_2,t_2)>_0 = \text{tr}\left[\gamma^1 S(x_1 - x_2; t_1 - t_2)\gamma^1 S(x_2 - x_1; t_2 - t_1)\right]$$

$$= -\text{tr}\left[\sigma^3 S(x_1 - x_2; t_1 - t_2)\sigma^3 S(x_2 - x_1; t_2 - t_1)\right]$$
from the representation (I.7), where tr denotes the trace over spinor indices and $S(x - y; t_x - t_y)$ is the fermion propagator (the Green’s function for the Dirac operator $(i\partial + i\sigma^2m - \varphi(x))$) given by

$$S(x - y; t_x - t_y) = \langle 0|T[\bar{\psi}(x, t_x)\psi(y, t_y)]|0 \rangle.$$  \hspace{1cm} (II.23)

Here $T$ denotes the usual time-ordering operator and the Green’s function (II.23) satisfies

$$(i\partial_x + i\sigma^2m - \varphi(x))S(x - y; t_x - t_y) = \delta(x - y)\delta(t_x - t_y).$$  \hspace{1cm} (II.24)

Thus

$$(\Delta Q_A)^2 = -\int_{-\infty}^{+\infty} dx_1 dx_2 f'_A(x_1)f'_A(x_2)$$

$$\times \int_0^{+\infty} (d\Delta t)^2 \text{tr}[\sigma^3 S(x_1 - x_2; \Delta t)\sigma^3 S(x_2 - x_1; -\Delta t)]$$

where we have now taken the limit to the open space problem and set

$$(\Delta Q_A)^2 \equiv \lim_{|L_1|,|L_2|\to\infty} (\Delta Q_{A,L_1,L_2})^2.$$  \hspace{1cm} (II.25)

To analyse the functional (II.25) it is convenient to first analyse the fluctuations $(\Delta Q_A^{(0)})^2$ in the vacuum sector where $\varphi(x) = \mu$ for all $x \in \mathbb{R}$. This is intuitively useful because $f'_A \simeq 0$ in the region where the field $\varphi(x)$ deviates from its asymptotes (see Figure 3) and so the inclusion of a non-trivial background shouldn’t drastically alter the vacuum form of the fluctuation functional (II.25). Fourier transforming in both sides of (II.24) with $\varphi(x) = \mu$, the free fermion propagator takes the momentum space form

$$S_0(\omega, p) = (\partial + i\sigma^2m - \mu)^{-1}$$

$$= (\partial + i\sigma^2m + \mu)[(\partial + i\sigma^2m - \mu)(\partial + i\sigma^2m + \mu)]^{-1}$$

$$= \frac{\partial + i\sigma^2m + \mu}{\omega^2 - p^2 - m^2 - \mu^2 + i\epsilon}$$  \hspace{1cm} (II.26)
where we have used (1.2) and (1.7) in the second line above. In (II.26) we have introduced the usual \( i\epsilon \) prescription, where \( \epsilon \rightarrow 0^+ \), for the poles of the matrix \( S_0 \) in the complex plane, and by definition

\[
S(x - y; t_x - t_y) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{dp}{2\pi} e^{-i\omega(t_x-t_y)+ip(x-y)} S(\omega, p). \tag{II.27}
\]

Substituting (II.27) and (II.26) into (II.25), using (1.2) and (1.7) to evaluate the traces, and performing the time integrations, we find the fluctuation functional in the vacuum sector to be

\[
(\Delta Q^0_A)^2 = -2 \int_{-\infty}^{+\infty} dx_1 dx_2 \left[ f'_A(x_1)f'_A(x_2) \right] 
\]

\[
\times \left[ \int_{-\infty}^{+\infty} \frac{dp_1 dp_2 dp\omega_1 dp\omega_2}{2\pi 2\pi 2\pi 2\pi} e^{-i(\omega_1+\omega_2)\Delta t + i(p_1-p_2)(x_1-x_2)} \right]_{\Delta t \rightarrow 0^+}
\]

\[
\times \frac{\omega_1\omega_2 - p_1p_2 + m^2 + \mu^2}{(\omega_1 + \omega_2 - \epsilon)^2(\omega_1^2 - E_1^2 + \epsilon)(\omega_2^2 - E_2^2 + \epsilon)},
\]

where \( E_i = \sqrt{p_i^2 + m^2 + \mu^2} \) for \( i = 1, 2 \). Evaluation of the frequency integrals in (II.28) gives

\[
(\Delta Q^0_A)^2 = \int_{-\infty}^{+\infty} dx_1 dx_2 f'_A(x_1)f'_A(x_2) \int_{-\infty}^{+\infty} \frac{dp_1 dp_2}{2\pi} e^{i(p_1-p_2)(x_1-x_2)}
\]

\[
\times \frac{E_1E_2 - p_1p_2 + m^2 + \mu^2}{2E_1E_2(E_1 + E_2)^2}.
\tag{II.29}
\]

The inequality

\[
E_1E_2 \geq |m^2 + \mu^2 - p_1p_2|
\]

then yields, after integrating over the momenta, the following bounds on the fluctuation functional:

\[
(\Delta Q^0_A)^2 \leq \int_{-\infty}^{+\infty} dx_1 dx_2 f'_A(x_1)f'_A(x_2) \int_{-\infty}^{+\infty} \frac{dp_1 dp_2}{2\pi} \frac{e^{i(p_1-p_2)(x_1-x_2)}}{(p_1 + p_2)^2 + 4(m^2 + \mu^2)}
\]

\[
\leq \frac{1}{4} \int_{-\infty}^{+\infty} dx_1 dx_2 f'_A(x_1)f'_A(x_2) \int_{-\infty}^{+\infty} \frac{dp_2}{2\pi} \frac{e^{-2ip_2(x_1-x_2)}}{\sqrt{m^2 + \mu^2}},
\]

\[35\]
and so integrating over the final momentum variable we see that

\[
(\Delta Q_A^{(0)})^2 \leq \frac{1}{8\sqrt{m^2 + \mu^2}} \| f'_A \|^2.
\]

Substituting (II.30) into (II.19) and using (II.15), we conclude that the charge fluctuations \((\Delta Q^{(0)})^2\) in the vacuum sector vanish.

We now consider the fluctuations \((\Delta Q_A^{(s)})^2\) in the soliton sector. In this case we use the eigenfunction expansion of the fermion propagator, which from (II.23) and (I.20) is given as

\[
S(x - y; t_x - t_y) = \int dE \left( \theta(E) \theta(t_x - t_y) e^{-iE(t_x - t_y)} \psi_E(x) \overline{\psi}_E(y) - \theta(-E) \theta(t_y - t_x) e^{iE(t_x - t_y)} \psi_E(y) \overline{\psi}_E(x) \right)
\]

where \(\psi_E(x)\) are the eigenmodes of (I.8). Substitution of (II.31) into (II.25) then yields

\[
(\Delta Q_A^{(s)})^2 = -\int_{-\infty}^{+\infty} dx_1 dx_2 f'_A(x_1) f'_A(x_2)
\times \int dE_1 dE_2 \frac{\theta(E_1) \theta(-E_2)}{(E_1 + E_2)^2} \text{tr} [\sigma^3 \psi_{E_1}(x_1) \overline{\psi}_{E_1}(x_2) \sigma^3 \psi_{E_2}(x_1) \overline{\psi}_{E_2}(x_2)].
\]

Now by definition, in the finite volume case we have \(f'_A(x) \simeq 0\) for all \(A\) except near the boundaries at \(x \simeq L_1, L_2\). Thus the only significant contribution of (II.31) to the fluctuation functional \((\Delta Q_A^{(s), L_1, L_2})^2\) is at \(x \simeq L_1, L_2\), and so upon taking the infinite volume limit we may approximate (II.31) by using the asymptotic forms of the eigenfunctions (I.19) in the soliton sector. Hence we compute (II.31) using (I.19) and (I.12), and substituting in the scattering states (I.23) and the asymptotic limits (I.6) of the soliton field at the four successive open space boundary points \(x \to \pm \infty\) and \(y \to \pm \infty\). We then substitute the sum of these four boundary point contributions of (II.31) into (II.32) above and evaluate the traces. The computation is straightforward but very tedious, and so we only quote the final results of this calculation. After many
simplifications, we arrive at the bound

\[
(\Delta Q_A^{(s)})^2 \leq \int_{-\infty}^{+\infty} dx_1 dx_2 f'_A(x_1)f'_A(x_2) \int_{-\infty}^{+\infty} \frac{dp_1 dp_2}{2\pi} e^{i(p_1-p_2)(x_1-x_2)}
\times \left( \frac{E_1 E_2 - p_1 p_2 + m^2 + \mu^2}{2E_1 E_2 (E_1 + E_2)^2} \right) \left( |T(p_1)|^2 |T(p_2)|^2 + |R(p_1)|^2 |R(p_2)|^2 + |R(p_1)|^2 + |R(p_2)|^2 + 2 \text{Re}\{T(p_1)^* T(p_2) + R(p_1)^* R(p_2) \}
+ R(p_1)^* R(p_2)^* + R(p_1) R(p_2) + T(p_1) T(p_2)^* R(p_1)^* R(p_2) \} + 1 \right).
\]

(II.33)

Notice that except for the reflection and transmission coefficients, the right hand side of inequality (II.33) is just the vacuum form (II.29) of the fluctuation functional. Thus, as expected on intuitive grounds, the form of the vacuum fluctuations in (II.29) is not altered tremendously by the inclusion of a soliton. From (I.25), we see that all reflection and transmission coefficient terms in (II.33) can be bounded by unity, and so from (II.29) we obtain

\[
(\Delta Q_A^{(s)})^2 \leq 15 (\Delta Q_A^{(0)})^2.
\]

(II.34)

From (II.19), we see that the soliton sector charge fluctuations \((\Delta Q^{(s)})^2\) also vanish, establishing the claimed results. Notice that the above calculations also show how much of the fluctuations are due to the soliton alone, aside from the usual vacuum fluctuations which are always present in finite volume.
4. Electromagnetic Properties of the Soliton Charge

In Sections 1 through 3 of this chapter we have concluded that there are essentially three contributions to the charge induced in a system composed of a soliton interacting with a Dirac fermion in finite volume in \((1 + 1)\) dimensions. First of all, there is the charge induced from the boundaries of the interval. In order for the Hamiltonian to be a Hermitian operator, it is necessary to impose the boundary conditions \((\text{II.3})\) on the Dirac spinors of the theory. These boundary conditions modify the eigenvalue spectrum of the Hamiltonian and since the fermion number is essentially determined by the spectral asymmetry of the Dirac Hamiltonian, a (possibly fractional) nonzero fermion number may result from the boundaries (this happens even in the free fermion theory). Next, there is the contribution to the charge from other topological excitations in the system. Whether solitons other than the relevant soliton at \(x = 0\) are specified \textit{ab initio} or are present implicitly because of the boundary conditions \((\text{II.3})\) imposed on the system, the total fermion number as determined by the charge operator \((\text{I.10})\) directly is the sum of the charges induced on each such object in the system. Finally, there is the charge of interest, namely the fermionic charge induced on the single soliton at \(x = 0\) from its interaction with the fermions. This charge is due solely to a rearrangement of the Dirac sea caused by the symmetry and spectral properties of the Hamiltonian. In the previous sections, we have shown how to isolate this physical charge from the rest of the charges present in the system. We also showed that the open space fermion number obtained in this way is in fact an observable of the system and corresponds to the fermion number calculated directly in the corresponding open space problem, as it should because in infinite volume the effects of boundaries and other solitons don't contribute to the charge of the localized soliton. Moreover, we determined what class of sampling functions can be used to isolate the charge on the soliton in this way,
namely the subset of $C_0^1(\mathbb{R})$ of differentiable test functions with compact support in
the finite interval whose first derivatives have vanishing $L^2$-norm in the appropriate
limit described in Section 2 of this chapter. In this final section, we briefly address the
seemingly final problem to consider here, namely whether or not the charged soliton
obtained in this way is a true charged object.

In [2], it has been argued that for the open space model (I.1) (and more general
models in $(1 + 1)$ dimensions) the fractional charge residing on the soliton is a bonafide
charge in the sense of its electromagnetic interactions. This was established by gauging
the fractional charge and studying its electromagnetic properties, and by verifying
that fractionally charged solitons behave like charged particles in the presence of a
background electric field. The gauging of the fermionic charge is achieved by adding
the minimal coupling $e j^\mu A_\mu$ with an external gauge field $A_\mu(x, t)$ to the Lagrangian
(I.1), and the introduction of a constant electric field is obtained by adding a term
$\frac{1}{2} \frac{\theta e}{2\pi} \epsilon^{\mu\nu} F^{\mu\nu}$ to (I.1) [5, 6]:

$$L = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - V(\varphi) + \overline{\psi}(i \partial - g \varphi + i m \gamma^5) \psi$$

$$+ e \overline{\psi} \gamma^\mu \psi A_\mu + \frac{1}{2} \frac{\theta e}{2\pi} \epsilon^{\mu\nu} F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

(II.35)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and $\theta \in [-\pi, \pi]$ is an angle totally independent of the other parameters in the theory. In [2], it is shown
that for $\theta = 0$ the ground state degeneracy is lifted and the charged soliton is confined
in $(1 + 1)$ dimensions. The tension of the string confining two solitons is examined in a
semi-classical analysis of (II.35), and it is deduced that fractionally charged solitons are
confined by a tension weaker than the tension confining ordinary fermions by a factor
equal to the product of their fractional charges. Since a string confining two objects has
tension proportional to the product of their charges, this establishes that the solitons
behave like charged particles in the presence of an external gauge field. For $\theta \neq 0$, it is
shown that there exists a value of \( \theta \), determined by the value of the fractional charge, at which the fractionally charged soliton becomes liberated. The existence of such a liberation angle suggests that the induced fractional charge behaves as a normal charge [5].

Now consider the system (II.35) in a finite interval \([L_1, L_2]\) as before. The discussion in Sections 1 and 2 of this chapter clearly show that the fermion number \( N \) which is measured on the soliton via (I.10) in finite volume is not a true charge in the above sense. This is because the nonzero contributions to the charge from the boundaries of the interval are uniformly distributed over the entire region of interest and do not correspond to a sharp, physical fermionic charge localized on the soliton. These charges do not couple to external electromagnetic fields and therefore do not induce the dynamics produced by nonzero charges in electromagnetic interactions. That is, the boundary induced fractional charges appear as background charges in the presence of external fields, and just correspond to a constant polarization which is always present in finite volume. Moreover, the interaction of the soliton with the boundaries of the interval also induces fractional charges which do not share the usual properties of fermionic charge, again since these charges aren’t sharply localized in the interval. The fermionic induced charge in the finite interval, however, was shown in the previous sections to coincide (in the appropriate limit) with the sharp, localized fractional fermion number induced directly on the soliton in the open space model. This physical charge is obtained by localizing the distributed charge density \( j^0(x, t) \) on the soliton and it measures the sharp fermionic charge residing on the soliton. Also, the uniform background polarization vanishes in the open space limit described in Section 2 of this chapter. The soliton charge obtained in this way should therefore behave like the fractional charges discussed in [2]. More precisely, if the gauge field \( A_\mu(x, t) \) in (II.35) is slowly varying
in space like the sampling functions $f_A(x)$, and the overlap of the supports of $A_\mu(x,t)$ and $f_A(x)$ is non-empty about the origin, then $A_\mu(x,t)$ couples in the appropriate way to the partial charge $Q_A$, which is insensitive to the explicit form of $f_A(x)$ as long as the region where $f_A(x) \approx 1$ contains the "width" of the soliton (that is, the region where the soliton field $\varphi(x)$ deviates from its asymptotes). Thus the localized fermion number induced on the soliton can be coupled to slowly varying external gauge fields which interact with the soliton at $x = 0$, and the arguments of [2] then imply that this localized charge is a true charge in the sense of its electromagnetic behaviour. This establishes that the soliton, in addition to having a true observable fractional fermionic induced charge, also behaves as a charged object in the usual sense of fermionic charge.

In conclusion, the localized fermion number residing on the soliton has been shown in this chapter to be a well-defined quantum observable and a true electromagnetic charge. Such a charge should therefore be experimentally measurable, especially through investigation of electromagnetic effects. In a conducting polymer system, for example, one could measure the charge localized on a finite length domain wall soliton and study its behavior in an external electric field. This observation therefore yields possible experimental realizations of the theoretical ideas presented in this thesis.
References


