ANOMALIES AND THE QUANTUM THEORY OF CHIRAL MATTER
ON A LINE

By

Robert G. Link

B. Sc. (Physics and Mathematics) University of Victoria

M. Sc. (Physics) University of British Columbia

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Department of Physics

The University of British Columbia
Vancouver, Canada

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Abstract

We give a brief review of anomalies in quantum field theory – where they arise, what they imply, and how they have been dealt with in the past. We suggest that theories with a gauge anomaly can be consistently quantized by applying the analogue of Dirac's formalism for classical constrained hamiltonian systems to the quantum theory. This suggestion is implemented for the archetypical chiral Schwinger model. By diagonalizing the hamiltonian, we show that the model is consistent, unitary, and Lorentz invariant. Its vector gauge boson acquires a gauge anomaly generated mass, and the fermionic sector bosonizes.

We review the proposals for describing free chiral bosons and show that there are two which are not the same. The bosonization of a Dirac fermion in background gauge and gravitational fields is used to obtain the coupling of these two theories to backgrounds such that they are both the bosonization of a Weyl fermion in the same backgrounds.

Nonabelian bosonization of gauged Dirac fermions in curved space has been used to obtain a chiral nonabelian Bose theory which corresponds to gauged Weyl fermions in curved space. This theory is the Siegel WZNW model coupled appropriately to background fields.

We perform a canonical quantization of the Siegel WZNW theory using the BRST formalism. We find that by introducing an anomaly cancelling conformal field, it can be quantized for a large class of symmetry groups. For a certain subset of these groups the conformal field vanishes.

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Chapter 1

Introduction

The need to reconcile general relativity with quantum mechanics is fueled by the belief that any complete physical theory is a quantum theory. Classical theory only predicts the average evolution of macroscopic phenomena where the quantum discontinuities can be treated as infinitely small. However, the only way of defining a quantum theory is to start with a classical theory and then quantize it. Basically there are two approaches — canonical quantization and path integral quantization — and at first sight both of these appear to carry the classical symmetry properties and conservation laws over intact to the quantum theory. In the development of ordinary point particle quantum mechanics (*ie* not the quantum mechanics of fields) this preservation of symmetry was generally found to be true [1]. Hence, the discovery by field theorists of the violation of classical conservation laws and symmetries by the quantization procedure was surprising, and was called the anomaly phenomenon. Perhaps a better name, suggested by Roman Jackiw, would be quantum mechanical symmetry breaking.

This most interesting phenomenon has become central in almost any discussion of a quantum field theory which violates parity invariance. Except for the as of yet to be discovered Higgs particle, all matter in accepted elementary particle theory consists of spinor fields which interact via vector boson and gravity fields. The spinor representing a Dirac fermion has four components, corresponding to the fact that it describes a particle or an antiparticle, either of which can have left- or right-handed spin. It can be written

(see appendix A) as a sum of two spinors, called its chiral components, which are independent for the case of massless particles. A given component is called a Weyl spinor, and describes a particle or an antiparticle, where the particle has only one handedness of spin, and the antiparticle has only the opposite handedness of spin. The two chiral components are interchanged under the operation of parity; hence theories with only one chiral component violate parity invariance.

Prior to 1956, when it was discovered that the weak interactions violate parity, it was thought that all of nature's spinor fields were the chirally symmetric Dirac fields. To date only neutrinos of one chirality have been observed, and the parity violating theories of Weyl fermions have become commonplace in the life of a quantum field theorist.

It has become important to study chiral matter, but why on a line? Well, there are three reasons for studying quantum field theory in only two spacetime dimensions.

The first is that the two dimensional theory can be used as a pedagogical toy model for studying the four dimensional theory.¹ Although some aspects of field theory are dimension independent, and some are not, all can have light shed upon them by a study of the theory in a lower spacetime dimension. For example, deep inelastic scattering experiments indicate that hadrons are bound states of point-like quarks which are essentially free at small enough distance scales. Yet free quarks are not seen in isolation — the final scattering states are jets of hadrons. The field theoretic mechanism by which this occurs has only been demonstrated in exactly solvable two dimensional models [2]. In a similar spirit, we will find that the vector boson of chiral electrodynamics has an anomaly generated mass. This hints of a possible alternative to the standard picture of spontaneous symmetry breaking with Higgs mechanism for vector boson mass generation in the standard model.

¹By n dimensional theory, we mean an infinite dimensional field theory in 1 time and $n - 1$ space dimensions.

The second reason is that two dimensional fields are not only pedagogically, but also physically relevant in many circumstances. There are physical systems whose motion is dynamically constrained to lie in a subspace of the full spacetime, and a lower dimensional model adequately describes the reduced dynamics. For example, linear polymers such as polyacetylene have been described by the use of an anomalous two dimensional field theory [3]. Also there are many important applications that arise due to the fact that two dimensional conformally invariant quantum field theories describe the long range behaviour of correlations in planar statistical systems which are undergoing second order phase transitions [4].

The third reason is associated with the idea called string theory. Although string theory has yet to predict something, it may be “the theory of everything”. Instead of basing elementary particle physics on point particles, it takes the fundamental constituents of matter (and indeed of spacetime) to be string-like, that is have one spatial dimension [5]. Other than this, the theory does not relinquish any of the traditional structure of relativistic quantum mechanics, and can therefore be regarded as an extension of the logical framework of elementary particle physics. One of the main reasons for believing it to be relevant, is that it appears to contain quantum gravity in a way which is, for the first time, consistent and finite. It also naturally contains quantum gauge theories of the type that are needed to describe physics at energies well below the Planck mass.

A single classical string propagates through spacetime in a manner which minimizes the area of its world-sheet [6]. The world sheet is a map, with coordinates $X^\mu(\sigma)$, from a two dimensional parameter space, with coordinates σ^α , into physical spacetime. By introducing a metric $h_{\alpha\beta}(\sigma)$ on the parameter space, the action for the propagation of a

bosonic string through spacetime with metric $g_{\mu\nu}$ can be written²

$$S = \int d^2\sigma \sqrt{-h} h^{\alpha\beta}(\sigma) g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu. \quad (1.1)$$

For a d-dimensional Minkowski spacetime metric $g_{\mu\nu} = \eta_{\mu\nu}$, this is precisely the action for d massless scalars in a 2-dimensional curved spacetime, where one of the scalar fields has a “negative kinetic energy”. This action describes a single free string, hence its quantization gives the so called first quantized bosonic string theory.

String interactions are described perturbatively by a series expansion which is a sum of terms analogous to the Feynman diagrams of point particle field theory. Just as the propagators of a Feynman diagram are free field propagators, each string of a string diagram is free. A nonperturbative description of interacting strings involves formulating a string field theory, which upon quantization is said to be second quantized.

The bosonic string is not realistic, and to make it so requires the introduction of internal degrees of freedom propagating along the string. This can be done by introducing world-sheet supersymmetry, which adds fermionic coordinates; and introducing gauge symmetry, which gives the fermions internal quantum numbers. This gives a large class of candidate string theories, but string theorists have found that the need for anomaly cancellation singles out just a few.

A very important property of chirally symmetric two dimensional field theories is that every fermionic theory is equivalent at the quantum level to a bosonic theory. Exhibiting this equivalence, known as bosonization (or fermionization), has proven to be essential in solving a variety of two dimensional problems. It was first used by Tomonaga [7] to solve the one dimensional Coulomb gas problem, and later by Mattis and Lieb [8] to solve a large class of Coulomb-like problems. The nontrivial and nonlinear Schwinger model

²Indices are raised and lowered with the appropriate metric and obey the summation convention eg $X^\mu = g^{\mu\nu} X_\nu$. Both $h_{\alpha\beta}$ and $g_{\mu\nu}$ have Minkowski signature, thus for the flat case of Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$. h is the absolute value of the determinant of $h_{\alpha\beta}$.

[9] upon bosonization becomes a free massive scalar field theory. Similarly the Thirring model (Dirac field with current-current interactions) [10] reduces to a trivially solvable model, with a quadratic action, in its bosonic representation. Ian Affleck has determined exact critical exponents for quantum spin chains by bosonizing the fermionic hamiltonian obtained from the spin chain hamiltonian by a Jordan-Wigner transformation [11].

Bosonization has also been useful in understanding four dimensional phenomena that may be described by an effective two dimensional theory. An example is the Callan-Rubakov effect (anomalous fermion-number breaking in the presence of a magnetic monopole) which is neither perturbative nor quasiclassical and requires an exact solution of the spontaneously broken four dimensional field theory. This is not at present possible, however a natural approximation can be obtained by considering only the dynamics of spherically symmetric fermions. Then the problem becomes effectively two dimensional and can be exactly solved using bosonization [12, 13].

In two spacetime dimensions chiral particles move only in one direction (see appendix A). String theory has motivated people to extend bosonization to chiral field theories because the phenomenologically promising heterotic string has a sector of 32 left-moving fermionic coordinates, a sector of 10 left-moving bosonic coordinates, and a sector of 10 supersymmetric right-moving coordinates.

Let us close this introduction with an overview of this thesis.

In order to put our work into perspective, we give, in chapter two, a general introduction to the well known aspects of anomalies in field theory. Such things as why global anomalies are often welcome, while local anomalies are often not welcome, are explained. We review how the failure of Noether's theorem for a symmetry current is equivalent to the variance of the quantum mechanical effective action under the corresponding symmetry transformation, and how this leads to the nonperturbative phase holonomy

description of the anomaly. We explain which theories may have gauge or gravitational anomalies and why grand unified theories and string theories are formulated in a manner which avoids these anomalies.

In spite of the conventional wisdom that anomalous chiral gauge theories constitute nonsense, we will consider their quantization. To prepare for this work, and later work on chiral bosons, in chapter three we review Dirac's formalism for classical constrained hamiltonian systems. In this chapter we go on to review the hamiltonian formalism for chiral gauge theories, where the anomaly will be seen to manifest itself as an inconsistent constraint algebra. We suggest that this could be dealt with by the analogue of Dirac's procedure applied directly to the quantum theory. In chapter five we will implement this suggestion for the chiral Schwinger model, and show by explicit solution that the model is consistent, unitary, and Lorentz invariant.³ However, there is a loss of gauge invariance, resulting in a quantum theory with one more physical degree of freedom than the classical theory, where the corresponding degree of freedom is gauged away. The gauge boson and the Weyl fermion decouple and give rise to a massive scalar and a massless chiral scalar, respectively.

The fact that the spectrum contains a chiral scalar is no accident, for as we shall see, the equivalence between a Weyl fermion and a chiral boson holds for fairly general theories in two dimensions. To establish this we will, in chapter four, review the bosonization of a free Dirac fermion and see how it is a consequence of the anomalous current commutators, and the Sommerfeld-Sugawara construction of the fermionic energy-momentum tensor.

The Sommerfeld-Sugawara formula says that the hamiltonian of a Weyl fermion is the hamiltonian of a chiral boson. As we will discuss in chapter six, there have been several bosonic non-local lagrangians proposed which yield this hamiltonian. Here we will show that these are obtained by applying the Dirac procedure to a lagrangian which

³This is original work, and has also been published by the author in [14].

is the sum of the usual non-chiral lagrangian for a scalar plus a Lagrange multiplier term which enforces the linear constraint $\partial_+\phi = 0$. The fact that this constraint does not commute with itself introduces the nonlocality. A lagrangian which uses the quadratic chiral constraint $(\partial_+\phi)^2 = 0$ successfully avoids this nonlocality; however, the symmetry generated by this constraint is anomalous, and as a result the quantum theory obtains another chiral degree of freedom.

In chapter seven we will show that both these theories, when appropriately coupled to background gauge and gravitational fields, have the effective action of a Weyl fermion in the same backgrounds.⁴ To do so we need the fact that a scalar and a Dirac fermion in the same backgrounds have the same effective action (a construction which we will review in the first three sections of this chapter).

Theories with several fermion species are often symmetric under nonabelian rotations of the species into one another. We review the bosonization procedure, introduced by Witten, which preserves these symmetries in the first section of chapter eight. The anomalous current commutators here are the famous Kac-Moody algebras, and they, along with the Sommerfeld-Sugawara construction for the nonabelian theory, allow a bosonization to be established which is a generalization of the one established for the single species chirally symmetric theory. The resulting theories are the Wess-Zumino-Novikov-Witten (WZNW) models. In the next section of this chapter we will review how this bosonization is extended to the theory in background gauge and gravitational fields, and in the final section of this chapter we will show how it is used to define a chiral version of the WZNW model which is the bosonization of nonabelian Weyl fermions.

This model is known to have an anomaly in the symmetry which generates the chiral constraint for all but a few of the nonabelian symmetry groups. By introducing one more chiral degree of freedom into the the model we will succeed in quantizing it for a much

⁴The original work here is in section 7.4. This work has also been published by the author in [15].

larger class of nonabelian symmetry groups. This piece of original work is presented in chapter nine and has also been published in [16].

Chapter 2

An Introduction to Anomalies

When the action of a lagrangian field theory is invariant under a continuous group of symmetries, the classical Noether theorem gives a corresponding set of locally conserved currents j^μ .

$$\partial_\mu j^\mu = 0 \quad (2.2)$$

Integrating the time component, j^0 , over all space, then gives a time independent charge, ie a conservation law. The continuous symmetries are usually spacetime transformations or internal symmetry transformations, and any type may either be global (constant parameters) or local (spacetime dependent parameters). The dynamical symmetry is anomalous if the currents of the quantum theory are no longer conserved. This is to be distinguished from spontaneous symmetry breaking, where the current is conserved, but the ground state is not annihilated by the charge [17].

When a global symmetry is anomalous, it simply means that the quantum theory has less symmetry than the classical theory. In fact, many of these symmetries must be broken for phenomenological reasons. For example, the scale invariance of quantum chromodynamics (QCD) with massless quarks is broken; thus allowing the emergence of hadrons as bound states with nonzero masses. The Noether current for the scale transformation,

$$x^\mu \rightarrow tx^\mu, \quad (2.3)$$

is

$$j^\mu = x^\nu T_\nu^\mu, \quad (2.4)$$

where T_ν^μ is the energy-momentum tensor. The conservation of the current implies that T_ν^μ is traceless:

$$\partial_\mu j^\mu = T_\mu^\mu = 0. \quad (2.5)$$

However, in the quantized theory $T_\mu^\mu \neq 0$; hence scale invariance is said to be broken by an anomaly in the trace of the energy-momentum tensor.

Another example of a welcome breaking of a global symmetry is also provided by QCD with n massless quarks. Classically there is a global $U_L(n) \times U_R(n)$ symmetry corresponding to independent rotations of the left-handed and right-handed (*ie* positive and negative chirality) components of the quarks. This is broken to $SU_L(n) \times SU_R(n)$ by the $U(1)$ axial anomaly. It is the nonconservation of the current,

$$j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi, \quad (2.6)$$

obtained from Noether's theorem for the classical chiral symmetry transformation

$$\psi \rightarrow e^{i\gamma_5 \theta} \psi, \quad (2.7)$$

with θ a constant parameter. One obtains the anomalous quantum statement replacing the classical statement 2.2:

$$\partial_\mu j_5^\mu = \frac{1}{8\pi^2} \text{Tr} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} F_{\mu\nu}, \quad (2.8)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the field strength tensor, and $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric Levi-Cevita tensor.

This anomaly was first discovered perturbatively by the violation of the Ward-Takahashi identity (which follows from current conservation) for the fermion triangle diagram of figure (2.1). It has three external photons, with one axial current and two vector current couplings [18]. By the momentum space Feynman rules, this represents an

Figure 2.1: fermion triangle diagram with one axial and two vector currents

amplitude $T_{\lambda\mu\nu}$, by which the Ward-Takahashi expression of vector current conservation is

$$p_1^\mu T_{\lambda\mu\nu} = p_2^\nu T_{\lambda\mu\nu} = 0, \quad (2.9)$$

and axial vector current conservation is

$$(p_1 + p_2)^\lambda T_{\lambda\mu\nu} = 0. \quad (2.10)$$

It is impossible to regulate (ie redefine so as to eliminate infinite results) $T_{\lambda\mu\nu}$ so as to have both the vector and axial vector currents conserved. Regulating so as to leave the vector currents conserved, one has a breakdown of chiral symmetry in the presence of gauge fields. This has been shown to lead to an understanding of the π^0 decay and the mass of the η particle [19].

The analogous diagram with the vector currents replaced by energy momentum tensors (ie two symmetrically coupled external gravitons and one chirally coupled gauge boson) implies the breakdown of chirality conservation in a gravitational field [20]. By an explicit perturbative analysis one obtains the nonconservation of the axial vector

Figure 2.2: fermion triangle diagram with V-A currents

current for massless fermions

$$\nabla_\mu J_5^\mu = \frac{1}{384\pi^2} \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\sigma\tau} R_{\alpha\beta}^{\sigma\tau}, \quad (2.11)$$

where $R_{\mu\nu\sigma\tau}$ is the usual Riemann curvature tensor, and ∇_μ is the coordinate covariant derivative.

In the standard $SU(2) \times U(1)$ model of weak interactions, gauge fields are coupled, not to vector currents, but to linear combinations of vector and axial vector currents. The conservation of these currents is now expected from Noether's theorem applied to a symmetry which has been made local *ie* gauged. However, the fermion triangle diagram of figure (2.2) with three external vector gauge bosons coupled to chiral fermions violates the Ward-Takahashi identity, unless the anomaly cancels when summing over the fermion species running around the loop [21]. Requiring this cancellation has led to the, so far, successful prediction that the number of leptons should equal the number of quarks. This prediction is however not yet fully born out as the top quark has yet to be observed. Similarly, the gauged current of electroweak theory on a curved spacetime is anomalous

(provided one regulates so as to preserve coordinate invariance) unless the sum of the hypercharges of the left-handed fermions vanishes. This is in fact true for each generation of known leptons and quarks (assuming existence of the top quark), and is taken as evidence for grander unification.

The loss of the gauged symmetry was originally understood as unacceptable because the Ward-Takahashi identities, following from gauge invariance, are used to prove renormalizability and unitarity of the four dimensional models. Furthermore, it was argued that gauge theories with an anomaly in the gauge symmetry ("anomalous gauge theories") are inconsistent because the field equation

$$D^\mu F_{\mu\nu} = J_\nu, \quad (2.12)$$

where $D_\mu = \partial_\mu + [A_\mu, \]$ is the gauge covariant derivative, implies

$$D^\nu J_\nu = 0. \quad (2.13)$$

However, because of the anomaly one obtains in fact

$$(D^\nu J_\nu)^a = f^a(A). \quad (2.14)$$

For the case of coupling to a multiplet of Weyl fermions in four dimensions with definite chirality

$$f^a(A) = \frac{\pm i}{24\pi^2} \text{Tr} T^a \epsilon^{\mu\alpha\beta\gamma} \partial_\mu (A_\alpha \partial_\beta A_\gamma + \frac{1}{2} A_\alpha A_\beta A_\gamma), \quad (2.15)$$

where the trace is over the representation matrices, T^a , of the fermions, and the sign depends on the chirality. The above mentioned anomaly cancellation for figure (2.2) corresponds to adjusting the fermion content so that the trace in equation 2.15 vanishes.

To see how gauge anomalies are calculated nonperturbatively [22], consider a theory of Weyl fermions in $2n$ dimensions interacting with an external, nondynamical gauge field. There is no longer an inconsistency with equation 2.12; but more importantly, it

is known that chiral anomalies are independent of the gauge field dynamics. First one demonstrates that the failure of current conservation is equivalent to the loss of the gauge invariance of the quantum mechanical effective action. The lagrangian is ¹

$$L = \int d^{2n}x \bar{\psi} (i\gamma^\mu \partial_\mu - \gamma^\mu A_\mu^a T^a) P_+ \psi, \quad (2.16)$$

where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ are the positive and negative chirality projection operators (see appendix A). The variation of $A_\mu = A_\mu^a T^a$ under an infinitesimal gauge transformation is given by the gauge covariant derivative, $D_\mu = \partial_\mu + A_\mu$,

$$A_\mu \rightarrow A_\mu - D_\mu \epsilon, \quad (2.17)$$

and consequently any functional $\Gamma[A_\mu]$ changes under an infinitesimal gauge transformation as

$$\begin{aligned} \Gamma[A_\mu] &\rightarrow \Gamma[A_\mu - D_\mu \epsilon] \\ &= \Gamma[A_\mu] - \int d^{2n}x \text{Tr} D_\mu \epsilon(x) \frac{\delta \Gamma}{\delta A_\mu(x)} \\ &= \Gamma[A_\mu] + \int d^{2n}x \text{Tr} \epsilon(x) D_\mu \frac{\delta \Gamma}{\delta A_\mu(x)}. \end{aligned} \quad (2.18)$$

Thus the generator of gauge transformations is $D_\mu \delta / \delta A_\mu^a$. Now let $\Gamma[A_\mu]$ be the one-loop fermion effective action:

$$\exp(-\Gamma[A_\mu]) = \int D\bar{\psi} D\psi \exp\left(-\int d^{2n}x \bar{\psi} i\gamma^\mu D_\mu P_+ \psi\right). \quad (2.19)$$

The fermion current induced by an applied gauge field is

$$J_\mu^a = \langle \bar{\psi} \gamma_\mu T^a P_+ \psi \rangle, \quad (2.20)$$

¹For an m -dimensional representation of the gauge group, the T^a matrices are m by m and act upon the m -tuple ψ of Dirac spinors by matrix multiplication ignoring the spinor indices. The Dirac matrices are 2^n by 2^n and multiply each 2^n component Dirac spinor of the m -tuple.

where the expectation value is to be taken in the background field A_μ . So from equation 2.19 it is clear that

$$J_\mu^a = \frac{\delta \Gamma}{\delta A_\mu^a}, \quad (2.21)$$

and the gauge covariant divergence of the current is

$$D_\mu J_\mu^a = D_\mu \frac{\delta \Gamma}{\delta A_\mu^a}. \quad (2.22)$$

Thus the failure of current conservation is equivalent to the loss of gauge invariance. Although $\Gamma[A_\mu]$ is naively gauge invariant, the statement of the gauge anomaly is that the variation of Γ under a gauge transformation is

$$D_\mu \frac{\delta \Gamma}{\delta A_\mu^a} = f^a[A_\mu], \quad (2.23)$$

where in four dimensions $f^a[A_\mu]$ is given by equation 2.15. This formula is not gauge covariant, which is sensible, since it arises in a theory where gauge invariance is lost. Notice that we have no such loss of covariance in equation 2.8 where the non-gauged symmetry has been lost.

Theories with gravity are theories which have local coordinate invariance, called general covariance. Just as in gauge theories, where the gauge field couples to matter via $A_\mu J^\mu$ coupling, in gravitational theories the metric couples to matter via a $T_{\mu\nu} g^{\mu\nu}$ term in the lagrangian [23]. Let $\Gamma[g_{\mu\nu}]$ be the effective action for matter fields propagating in an external gravitational field prescribed by metric $g_{\mu\nu}$. Under the infinitesimal coordinate transformation

$$x^\mu \rightarrow x^\mu + \varepsilon^\mu, \quad (2.24)$$

the variation in the metric tensor is given by the Lie derivative.

$$\delta g_{\mu\nu} = -\nabla_\mu \varepsilon_\nu - \nabla_\nu \varepsilon_\mu. \quad (2.25)$$

By which $\Gamma[g_{\mu\nu}]$ varies as

$$\delta\Gamma[g_{\mu\nu}] = - \int d^{2n}x \sqrt{g} \frac{\delta\Gamma}{\delta g_{\mu\nu}} (\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu). \quad (2.26)$$

But

$$\frac{\delta\Gamma}{\delta g^{\mu\nu}} = \frac{1}{2} \langle T_{\mu\nu} \rangle, \quad (2.27)$$

where the expectation value of the energy-momentum tensor is taken in the background $g_{\mu\nu}$. Integrating equation 2.26 by parts, and using the symmetry of $T_{\mu\nu}$ yields the following variation of Γ :

$$\delta\Gamma = \int d^{2n}x \sqrt{g} \epsilon_\mu \nabla_\nu \langle T^{\mu\nu} \rangle. \quad (2.28)$$

Thus conservation of the energy-momentum tensor is equivalent to general covariance of the effective theory. An anomaly in this symmetry is called a gravitational anomaly.

The integrand of the functional integral 2.19 is just the exponential of the lagrangian, and is therefore invariant under the infinitesimal gauge transformation 2.17 plus

$$\begin{aligned} \psi &\rightarrow \psi + \epsilon P_+ \psi \\ \bar{\psi} &\rightarrow \bar{\psi} - P_- \bar{\psi} \epsilon. \end{aligned} \quad (2.29)$$

However, Fujikawa [24] showed that the regulated infinite dimensional fermion measure is not. In this formulation, the anomaly arises from the jacobian factor contributed by the change in the fermion measure under the above change of variables. This factor arises because it is not possible to regulate the measure gauge invariantly.

To obtain the phase holonomy description of the gauge anomaly, one writes the effective action 2.19 as the logarithm of the determinant of the operator $\mathcal{D}$ [25]

$$\mathcal{D}[A] \equiv \gamma^\mu (\partial_\mu + A_\mu P_+) \quad (2.30)$$

$$\exp(\Gamma[A]) \equiv \det i\mathcal{D}[A]. \quad (2.31)$$

The eigenvalues of $\mathcal{D}[A]$ are complex and gauge variant; however, one can regulate their infinite product such that the modulus of the determinant is gauge invariant. Let $g(\theta, x)$ be a one-parameter family of gauge transformations, $g(0, x) = g(2\pi, x) = 1$; and take spacetime to be S^{2n} , so that $x \in S^{2n}$. Now consider how the determinant varies over the one-parameter closed loop of gauge transformed connections

$$A_\mu^\theta = g^{-1}(\theta)(A_\mu + \partial_\mu)g(\theta). \quad (2.32)$$

Since the modulus of the determinant is gauge invariant, only its phase may change as one moves around the loop $g(\theta)$:

$$\det i\mathcal{D}[A_\mu^\theta] = \det i\mathcal{D}[A_\mu]e^{i\omega_1(A, \theta)}. \quad (2.33)$$

The integrated anomaly for this loop of gauge transformations is equal to

$$\int_0^{2\pi} d\theta \frac{\partial \omega_1}{\partial \theta} = 2\pi m, \quad (2.34)$$

where m is the winding number of the phase of the determinant around the loop. A nonzero winding number means the determinant can not be defined gauge invariantly. It was found by Atiyah and Singer [26], using arguments from algebraic topology, that m is equal to the index (ie the number of positive minus the number of negative chirality zero modes) of a Dirac operator on $S^{2n} \times S^2$ with gauge potential $(tA^\theta, 0, 0)$, where the upper hemisphere of the S^2 is the (t, θ) disc with $0 \leq t \leq 1$. By the Atiyah-Singer index theorem, the index of a Dirac operator is equal to the integral of a unique polynomial in the gauge field curvature. This analysis shows that the anomaly can not be removed by any sort of renormalization scheme. Gravitational anomalies are subject to a similar analysis with similar conclusions.

Anomalies (where from now on, anomaly means gauge or gravitational anomaly) arise in parity-violating theories because, in the case of parity-violating gauge or gravitational

interactions, all regularizations spoil gauge invariance or general covariance; and the violation of these symmetries does not necessarily disappear when the regulator is removed. The parity-violating chiral gauge theories are gauge theories in which the left-handed fermions transform differently under the gauge group from the way the right-handed fermions transform. These can only occur in even spacetime dimensions, for only here do we have Weyl spinors. If R and L are the representations of the gauge group furnished by the right and left-handed fermions, respectively; then in a chiral theory R is not equal to L . CPT symmetry in $4k$ dimensions requires R to be the complex conjugate representation of L , so chiral theories can occur only in theories in which R is complex [27, 28]. In $4k + 2$ dimensions, the antiparticle of a left-handed particle is again left-handed, and it is therefore possible to construct theories with fermions of only one chirality. R and L are always real, but need not be equal. The chiral Schwinger model, with $k = 0$, falls into this later category.

The gravitational couplings can violate parity in $4k + 2$ dimensions only; and Alvarez-Gaume and Witten [23] have shown that theories of Weyl fermions of spin $1/2$ or $3/2$, or self-dual antisymmetric tensor fields, coupled to gravity in these dimensions are anomalous. We will be examining the $k = 0$ case, in which a self-dual antisymmetric tensor field is the field strength for a chiral boson.

The anomalies occurring in higher dimensional theories show up perturbatively in diagrams analogous to the Adler-Bell-Jackiw triangle diagram of figure (2.2). In $2n$ dimensions the lowest order anomalous diagrams are chiral fermion loop diagrams with $n + 1$ external gluons or gravitons. A pure gauge anomaly involves only gluons, a pure gravitational anomaly involves only gravitons, and a so called mixed anomaly involves both. In ten dimensions, which is the dimension that superstrings live in, it is the hexagon diagram of figure (2.3) which is essential. If one arranges the fermion content so the anomalies of these one loop diagrams cancel, all other higher order diagrams will

Figure 2.3: hexagon diagram in ten dimensions

be anomaly free.

In string perturbation theory one has Feynman diagrams which are two dimensional surfaces, yet analogous to the Feynman diagrams of ordinary point particle field theory. Here again it is the hexagon diagram which essentially contains the anomaly. By obtaining anomaly cancellation in this diagram, Green, Schwarz and Witten [27] have found the type I superstring theory to be anomaly free if and only if the gauge group is $SO(32)$, the type II superstring theory to be anomaly free, and the heterotic string to be anomaly free if and only if the gauge group is $E_8 \times E_8$, $SO(32)$, or $SO(16) \times SO(16)$. Also $N = 1$ supergravity with $E_8 \times E_8$, $SO(32)$, or $SO(16) \times SO(16)$; and $N = 2$ chiral supergravity, all in ten dimensions, (these are the low energy limits of the above superstring theories) are anomaly free. Furthermore, these four are the only field theories, in ten dimensions, in which the anomalies cancel between fields of different spin.

So far only anomalies for gauge and coordinate transformations which can be continuously obtained from the identity have been discussed. It is also possible to have gauge or

coordinate variance under what are called large transformations. It only makes sense to discuss such anomalies if there are no perturbative anomalies, because a large coordinate or gauge transformation, defined by its topological class, is well defined modulo a local transformation. We will not be dealing with these anomalies, so for an exposition refer to [29, 1]. In [29], Witten shows that the above mentioned superstring and supergravity theories have no variance under large coordinate or gauge transformations.

In light of all that has just been said, it appears that an anomalous theory should either be rejected, or altered so as to remove the anomaly. This reasoning has almost uniquely determined the gauge group for the promising superstring theories, and has thus helped convince people that superstring theory is the correct theory of the elementary matter fields and their interactions. Therefore it is worthwhile to look at this closely. In fact, Faddeev [30] suggested in 1985 that perhaps an anomalous theory can be consistently quantized. As we shall later discuss, authors, including ourselves, have investigated this possibility for some two dimensional models. We will give up gauge invariance in the chiral Schwinger model, and allow a gauge degree of freedom to become a physical degree of freedom. The inconsistency argument encapsulated by equations 2.13 and 2.14 will fail because the analog of $f^a(A)$ will be zero by the Heisenberg equation of motion for the new physical degree of freedom. Furthermore, the model will be found to be unitary, in spite of the usual assertion that unitarity requires gauge invariance. However, because two dimensional field theories are super-renormalizable we will have little to say about renormalizability.

Chapter 3

Constrained Hamiltonians and the Gauge Anomaly

The discussion of anomalies in the previous chapter has been from a lagrangian path integral point of view. We will now present, in detail, the well known equivalent hamiltonian point of view, because it is in this context that we will be addressing the quantization of an anomalous theory. This approach can be regarded as more fundamental, because the path integral is correctly obtained only by first passing through the hamiltonian formalism [31].

3.1 Dirac's Formalism for Constrained Hamiltonian Systems

Many systems, chiral bosons for example, are describable by a lagrangian formalism where constraints are explicitly introduced as equations of motion in a phase space enlarged by the inclusion of Lagrange multipliers. In this larger phase space, a submanifold in which the system lies is specified by a somewhat arbitrary fixed choice of specific values for the multiplier degrees of freedom. In the case of a gauge theory, the given lagrangian is a function on a phase space which is already too large, and a physical submanifold is specified by what is called a gauge choice. These two situations are really the same: A constrained system may be rewritten as a gauge theory by imbedding it in an enlarged phase space; and a gauge theory becomes a constrained theory upon a choice of gauge. A hamiltonian formalism for these theories, with a so called singular lagrangian, has been developed by Dirac [32], and reviewed in [33]. Here is a brief version of the review.

Suppose we have a mechanical system with N degrees of freedom, described by a

lagrangian $\mathcal{L}(q, \dot{q})$. To go to a hamiltonian formalism, we define the canonical momenta

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad 1 = 1, \dots, N. \quad (3.35)$$

If the p_i are not N independent functions of the velocities $\dot{q}_i$, the lagrangian is said to be singular, and by eliminating the velocities from the dependent equations 3.35 we obtain what are called primary constraints:

$$\phi_n(p, q) = 0. \quad (3.36)$$

This does not prevent the definition of a velocity independent preliminary hamiltonian

$$H_o(p, q) \equiv p_i \dot{q}_i - \mathcal{L}(q, \dot{q}). \quad (3.37)$$

We are free to add the primary constraints to H_o and do so:

$$H \equiv H_o + v^n \phi_n(p, q), \quad (3.38)$$

where the v^n are arbitrary functions of p and q .

We say an equality holds weakly and write $\approx$ if it holds after the constraints have been imposed. The time evolution of any function on phase space is given by the Poisson bracket:

$$\dot{g}(p, q) \approx \{g(p, q), H\}. \quad (3.39)$$

(Note that this weak equation does not specify g . We will impose the strong equation after having identified the physical submanifold and an appropriate hamiltonian.) Thus the primary constraints must satisfy the consistency condition

$$\{\phi_n, H\} \approx 0, \quad (3.40)$$

since they are to be zero at all times. To satisfy this equation we solve for as many of the v^n of 3.38 as we can; and we may also have to impose additional constraints,

called secondary constraints. We repeat the procedure of adding the constraints to the hamiltonian and checking for consistency until no more constraints arise, and collectively denote all the constraints, primary and secondary, by $\{\chi_i\}$. Γ will denote the submanifold which they define.

$$\Gamma = \{(p, q) | \chi_i(p, q) = 0 \quad \forall i\} \quad (3.41)$$

The constraints are required to be irreducible, which means that any function of (q, p) that vanishes on Γ is a linear combination of the constraints. In particular, the hamiltonian is written as

$$H(p, q) \equiv H|_{\Gamma}(p, q) + u^m(p, q)\chi_m(p, q). \quad (3.42)$$

Next we divide the constraints into two classes; those that commute weakly with all the constraints, called first class and here denoted as ϕ_i ; and those that do not, called second class and here denoted ψ_a . It is easy to show that the matrix

$$Q_{ab} \equiv \{\psi_a, \psi_b\} \quad (3.43)$$

of second constraints is invertible (then Q_{ab} has a non-zero determinant and the second class constraints are said to be complete), which in turn implies that the u^m corresponding to the second class constraints are determined. However, the u^m corresponding to the first class constraints are completely arbitrary. This arbitrariness is known as gauge freedom, and the gauge transformations are generated by the first class constraints – as can be seen from the following.

Consider the time evolution of a function $g(p, q)$ for a certain initial (p_o, q_o) at t_o . Then by doing a Taylor expansion about t_o to order $t - t_o$, and using 3.39, with hamiltonian 3.42, for the time derivative:

$$g(q(t), p(t)) \approx g(q_o, p_o) + (t - t_o)(\{g, H|_{\Gamma}\} + u^m\{g, \chi_m\}). \quad (3.44)$$

Now if we time evolve from $g(q_o, p_o)$ with a different extension of the hamiltonian, say with $\tilde{u}^m$ instead of u^m , to obtain $\tilde{g}(q(t), p(t))$, the difference between these expressions is the gauge transformation:

$$\delta g = g - \tilde{g} \approx \epsilon^a \{g, \phi_a\}, \quad (3.45)$$

with infinitesimal parameters $\epsilon^a = (t - t_o)(u^a - \tilde{u}^a)$ and generators ϕ_a . A physical observable is a gauge invariant function, and therefore one which commutes weakly with all the first class constraints.

To eliminate the arbitrariness due to the first class constraints, we fix the gauge by introducing gauge fixing constraints, $\gamma_i(p, q) = 0$, equal in number to the first class constraints, so as to make them second class. An important observation to make here is that a first class constraint eliminates two phase space degrees of freedom, while a second class constraint eliminates only one.

To quantize the theory, we might replace the Poisson brackets by commutators and have the variables become operators on states. The constraints are imposed by requiring physical states to obey

$$\chi_i |phys\rangle = 0. \quad (3.46)$$

Then for consistency we must have

$$[\chi_i, \chi_j] |phys\rangle = 0; \quad (3.47)$$

but the second class constraints can not obey this. To remove this inconsistency, Dirac proposed that instead of replacing the Poisson bracket by commutator, we replace the Dirac bracket by commutator. The definition of the Dirac bracket:

$$\{A, B\}_D \equiv \{A, B\} - \{A, \psi_a\} Q_{ab}^{-1} \{\psi_b, B\} \quad (3.48)$$

ensures that the Dirac bracket of any function with a second class constraint is zero. This allows the second class constraints to be set strongly, ie as operator equations in

the quantum theory, to zero. The infinite dimensional field theoretic generalization of this formalism is straightforward. The momentum conjugate to a field $q_i(x)$ is given by a functional derivative of the lagrangian:

$$p^i(x) = \frac{\delta \mathcal{L}}{\delta \dot{q}_i(x)}; \quad (3.49)$$

and the Poisson bracket is defined as

$$\{F, G\} \equiv \int d\vec{x} \frac{\delta F}{\delta q_i(x)} \frac{\delta G}{\delta p^i(x)} - \frac{\delta F}{\delta p^i(x)} \frac{\delta G}{\delta q_i(x)}. \quad (3.50)$$

The constraints become spacetime dependent, so the same goes for the matrix Q_{ab} whose inverse is then defined by

$$\int d\vec{z} Q_{ab}(x, z) Q_{bc}^{-1}(z, y) = \delta_{ac} \delta(\vec{x} - \vec{y}). \quad (3.51)$$

Summations will usually include integrations; for example, the Dirac bracket 3.48 becomes

$$\{A, B\} = \{A, B\} - \int d\vec{x} d\vec{y} \{A, \psi_a(x)\} Q_{ab}^{-1}(x, y) \{\psi_b(y), B\}. \quad (3.52)$$

3.2 The Constrained Hamiltonian Formalism for Chiral Gauge Theories

In this section we will apply the Hamiltonian formalism just described to the infinite dimensional case of a chiral gauge field theory. We remain fairly general, and consider the theory of a multiplet of Weyl fermions, in $2n$ spacetime dimensions, which forms a representation of some nonabelian, compact gauge group with generators T^a obeying the Lie algebra

$$[T^a, T^a] = f^{abc} T^c. \quad (3.53)$$

The lagrangian is given by 2.16 plus the usual Yang-Mills action for the gauge field:

$$L = \int d\vec{x} \left\{ \bar{\psi}_+ (i\gamma^\mu \partial_\mu - \gamma^\mu A_\mu^a T^a) \psi_+ + \frac{1}{2} \text{Tr} F^{\mu\nu} F_{\mu\nu} \right\}, \quad (3.54)$$

where the field strength $F_{\mu\nu} = F_{\mu\nu}^a T^a$ is as in 2.8 and has components

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c. \quad (3.55)$$

This lagrangian is invariant under the gauge transformation

$$\begin{aligned} \psi_+(x) &\rightarrow g(x)\psi_+(x) \\ \bar{\psi}_+(x) &\rightarrow \bar{\psi}_+(x)g^{-1}(x) \\ A_\mu &\rightarrow g(x)(A_\mu(x) + i\partial_\mu)g^{-1}(x), \end{aligned} \quad (3.56)$$

which is the finite version of the infinitesimal transformation 2.17 and 2.29. $g(x)$ represents an element of the gauge group, and, in the case that it is continuously connected to the identity, can be parametrized by functions $\epsilon^a(x)$.

$$g(x) = e^{\epsilon^a(x)T^a}. \quad (3.57)$$

To proceed to a hamiltonian formalism we obtain the canonical momenta

$$\pi_0^a \equiv \frac{\delta L}{\delta \dot{A}_0^a} = 0 \quad \pi_i^a \equiv E_i^a = \frac{\delta L}{\delta \dot{A}_i^a} = F_{0i}^a \quad (3.58)$$

$$\pi_\psi \equiv \frac{\delta L}{\delta \dot{\psi}} = i\psi^\dagger \quad \pi_{\psi^\dagger} \equiv \frac{\delta L}{\delta \dot{\psi}^\dagger} = 0 \quad (3.59)$$

and recognize three primary constraints.

The pair $\pi_\psi - i\psi^\dagger = 0$ and $\pi_{\psi^\dagger} = 0$ are second class; and furthermore, will not generate any secondary constraints. Also, when it becomes time to compute a Dirac bracket they will not have any effect – so we set these two constraints strongly to zero right now.

A preliminary hamiltonian is then

$$H = \int d\vec{x} \left(E_i^a \dot{A}_i^a + i\psi^\dagger \dot{\psi} \right) - L \quad (3.60)$$

which upon the definition of the color magnetic field $B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a$, and an integration by parts, becomes

$$H = H_o + \int \vec{x} A_0^a G^a \quad (3.61)$$

where

$$H_o \equiv \int d\vec{x} \left(\bar{\psi}_+ i \vec{\gamma} \cdot \vec{D} \psi_+ + \frac{1}{2} \vec{E}^a \cdot \vec{E}^a + \frac{1}{2} \vec{B}^a \cdot \vec{B}^a \right) \quad (3.62)$$

and

$$G^a \equiv \vec{\partial} \cdot \vec{E}^a + f^{abc} \vec{A}^b \cdot \vec{E}^c - \psi_+^\dagger T^a \psi_+. \quad (3.63)$$

Demanding the primary constraint $\pi_0^a = 0$ be preserved in time yields the secondary constraint $G^a = 0$, known as Gauss's law. When the gauge group is $U(1)$, it takes the familiar form $\vec{\partial} \cdot \vec{E} = \psi_+^\dagger \psi_+$ for electrodynamics. The Weyl gauge, $A_0 = 0$, eliminates A_0^a and π_0^a from the theory, leaving us with the canonically conjugate variables A_i^a, E_i^a and $\psi_+^\dagger, \psi_+$; the hamiltonian H_o , and the constraints $G^a = 0$. The constraints G^a are (at the classical level) first class, as their Poisson brackets are

$$\{G^a(\vec{x}), G^b(\vec{y})\} = f^{abc} G^c(\vec{x}) \delta(\vec{x} - \vec{y}). \quad (3.64)$$

Fixing $A_0 = 0$ still leaves us free to perform time independent gauge transformations, as can be seen from 3.56, and it is in fact precisely these which are generated by G^a .

In the quantum theory, the field variables become operators, and the statement of gauge invariance is

$$G^a |phys\rangle = 0. \quad (3.65)$$

The finite, time independent transformation 3.57 is implemented by the fixed-time operator

$$U(g) = e^{i \int d\vec{x} \epsilon(x) G^a(x)}. \quad (3.66)$$

Thus 3.65 ensures that physical states are invariant under finite gauge transformations and all would be consistent if the Poisson bracket 3.64 simply went over to a commutator.

However, it is known that for a chiral gauge theory, the anomalous divergence of the current in a lagrangian description manifests itself in anomalous commutators of current components in a Hamiltonian description, so that 3.64 acquires what is called an extension:

$$[G^a(\vec{x}), G^b(\vec{y})] = -if^{abc}G^c(\vec{x})\delta(\vec{x} - \vec{y}) + S^{ab}(\vec{x}, \vec{y}). \quad (3.67)$$

$S_{ab}(\vec{x}, \vec{y})$ is called the Goto-Imamura-Schwinger term [34] and depends on A_i . In general, S_{ab} does not annihilate states, making 3.65 inconsistent and thus spoiling gauge invariance.

In terms of the finite operators 3.66 which represent the gauge group, the commutator 3.67 is the result of the following composition law

$$U(g_1)U(g_2) = e^{2\pi i\omega_2(\vec{A}; g_1, g_2)}U(g_1 \circ g_2), \quad (3.68)$$

where the infinitesimal portion of ω_2 is S_{ab} . In order that the operators $U(g)$ associate, ω_2 must be what is called a two-cocycle (see appendix B). This in turn ensures that the commutator 3.67 obeys the Jacobi identity – and it turns out that S_{ab} does indeed satisfy this. Such a representation of the gauge group is called projective.

In four spacetime dimensions S_{ab} has been found to be [35]

$$S_{ab} = \pm \frac{\epsilon^{ijk}}{24\pi^2} \text{Tr}[T^a, T^b]_+ \partial_i A^j \partial_k \delta(\vec{x} - \vec{y}). \quad (3.69)$$

The calculation was done by considering the quantized fermions in an external background $\vec{A}$. The fermion bilinear term in Gauss's law was regulated by splitting points spatially and subtracting the $\vec{A}$ dependent vacuum expectation value. The sign of 3.69 depends on the chirality; thus for a chirally symmetric theory, S_{ab} for the positive chirality fermions cancels against S_{ab} for the negative chirality fermions. In two dimensions, we will calculate S_{ab} and find it to be $\vec{A}$ independent.

What we have seen here is that the gauge anomaly in a hamiltonian formalism manifests itself as an extension in the algebra of the gauge generators, causing them to

become second class. A possible way of dealing with this is to apply the analogue of Dirac's method for classical systems directly to the quantum theory. This would involve a redefinition of the operators, so as to obtain commutation relations which would allow the second class constraints to be consistently set strongly to zero. This procedure can be very complicated, and we do not know whether it can be accomplished in general. However, as will be seen in chapter 5, we have succeeded in executing it for the $U(1)$ chiral gauge theory in two dimensions.

Chapter 4

Bosonization of a Dirac Fermion

To facilitate the development of chiral theories, we here demonstrate the quantum equivalence of the simplest nonchiral theories. The Sommerfeld-Sugawara formula obtained here will be used in the solution of the chiral Schwinger model. The approach we will take to explain the well known results of this chapter is due primarily to Manton [37]. The method used here to bosonize a free Dirac fermion was developed by him to give another proof of the equivalence of the Schwinger model to a free massive scalar theory. Our analysis will allow both ease of explanation, and a comparison of free field bosonization with gauge field dependent bosonization. In later chapters, various generalizations will be reviewed and used to motivate some chiral Bose theories.

The existence of a nonlocal transformation from local Fermi fields to local Bose fields has been known for some time [36]. It establishes an equivalence between the theory of a free, massless Dirac fermion in two spacetime dimensions, with lagrangian

$$\mathcal{L}_D = \bar{\psi} i \gamma^\mu \partial_\mu \psi, \quad (4.70)$$

and the theory of a free massless scalar in two spacetime dimensions, with lagrangian,

$$\mathcal{L}_S = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi. \quad (4.71)$$

There are several ways to demonstrate this equivalence, and we will expose the most concrete of these, defining the Bose field in terms of the Fermi field. The Sommerfeld-Sugawara formula will arise when we rewrite the hamiltonian for the fermion in terms of the boson. To regulate the theory we will take space to be a circle, Fourier transform

the theory, and work with the discrete momentum modes. The infinities of the theory will be removed by an exponential damping of the high momentum Fourier modes called heat kernel regularization.

Before obtaining the hamiltonian, let us mention that the conserved Noether current for global phase transformations of the fermions is the vector current

$$j^\mu = \bar{\psi} \gamma^\mu \psi, \quad (4.72)$$

and that for global chiral rotations 2.11 is the axial current

$$j_5^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi, \quad (4.73)$$

which in the Weyl representation A.332 for the gamma matrices, and the notation

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \quad (4.74)$$

for the spinors, have components

$$\begin{aligned} j^0 &= j_5^1 = \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \\ j^1 &= j_5^0 = \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \end{aligned} \quad (4.75)$$

With the same gamma matrices, the lagrangian 4.70 yields the hamiltonian

$$H = \int_0^{2\pi} i \psi^\dagger \sigma^3 \partial_x \psi dx, \quad (4.76)$$

which upon the Fourier plane wave expansion for the chiral components (We take the radius of the circle representing space to be one.) of ψ

$$\psi_\alpha = \frac{1}{\sqrt{2\pi}} \sum_k a_k^\alpha e^{ikx}, \quad (4.77)$$

where $\alpha = + \text{or} -$, becomes

$$H = \sum_k k \left(a_k^{+\dagger} a_k^+ - a_k^{-\dagger} a_k^- \right). \quad (4.78)$$

We quantize by taking the Poisson brackets over to the following anticommutators:

$$\begin{aligned}\{a_n^\alpha, a_m^\beta\} &= \{a_n^{\alpha\dagger}, a_m^{\beta\dagger}\} = 0 \\ \{a_n^{\alpha\dagger}, a_m^\beta\} &= \delta_{\alpha\beta} \delta_{mn}.\end{aligned}\tag{4.79}$$

Thus $a_k^{+\dagger}$ is the creation operator for a positive chirality particle of energy k (a left-mover), and $a_k^{-\dagger}$ is the creation operator for a negative chirality particle of energy $-k$ (a right-mover). We could now redefine these operators to obtain the usual particle-hole description, in which case the minus sign in H would disappear, as H would then be the sum of the energies of the particles and the antiparticles. This redefinition is however not necessary if we just remember that an antiparticle is an empty negative energy state.

We use the a operators to define the Fock space in the usual manner. The vacuum is defined by

$$\begin{aligned}a_n^+|0\rangle &= 0 \quad \text{for } n > 0 \\ a_n^{+\dagger}|0\rangle &= 0 \quad \text{for } n \leq 0 \\ a_n^-|0\rangle &= 0 \quad \text{for } n < 0 \\ a_n^{-\dagger}|0\rangle &= 0 \quad \text{for } n \geq 0.\end{aligned}\tag{4.80}$$

Then applying these operators appropriately to the vacuum, gives a basis of states in which the energy level of a state is specified as either empty or filled, and in which all but a finite number of negative energy levels are filled, and only a finite number of positive energy levels are filled. Acting on such states, the chiral charge density operators

$$\rho_\alpha(n) \equiv \int_0^{2\pi} \psi_\alpha^\dagger \psi_\alpha e^{inx} dx = \sum_k a_{k+n}^\alpha{}^\dagger a_k^\alpha\tag{4.81}$$

have the commutation relations

$$[\rho_+(q), \rho_+(p)] = -q\delta_{-q,p}$$

$$[\rho_-(q), \rho_-(p)] = q\delta_{-q,p} \quad (4.82)$$

$$[\rho_+(p), \rho_-(q)] = 0.$$

These anomalous (the corresponding Poisson brackets are zero) commutators lie at the heart of almost any theory in two dimensions. They are responsible for the previously mentioned Goto-Imamura-Schwinger term in the gauged theory, and are the key to bosonizing this theory. To prove these relations it is crucial to notice that for $n \neq 0$ $a_{k+n}^\alpha \dagger a_k^\alpha$ annihilates a basis state when the absolute value of k is large enough – in other words the high momenta do not contribute. This observation also shows that the $\rho_\pm(n)$ for $n \neq 0$ are finite. However, the zero momentum operators are infinite and regulated as follows:

$$\begin{aligned} \rho_+^\lambda(0) &= \sum_p a_p^{+\dagger} a_p^+ e^{\lambda p} \\ \rho_-^\lambda(0) &= \sum_p a_p^{-\dagger} a_p^- e^{-\lambda p}. \end{aligned} \quad (4.83)$$

The exponential weights (λ is positive) suppress the contribution of particles with large negative energy, and when $\lambda = 0$ we just have the original formal expressions.

Associated to any basis state, $|F\rangle$, is an unexcited basis state with the same numbers of left and right-movers (see figure 4.4). The right-movers fill the energy levels $\leq M$ and the left-movers fill the energy levels $\geq N$. The chiral charges of this associated unexcited basis state are:

$$\begin{aligned} \rho_+^\lambda(0) &= \sum_{m \leq M} e^{\lambda m} \\ \rho_-^\lambda(0) &= \sum_{n \geq N} e^{-\lambda n}. \end{aligned} \quad (4.84)$$

Evaluating the sums yields

$$\begin{aligned} \rho_+^\lambda(0) &= \frac{1}{\lambda} + (M + \frac{1}{2}) + \frac{\lambda}{2}(M + \frac{1}{2})^2 - \frac{\lambda}{24} + O(\lambda^2) \\ \rho_-^\lambda(0) &= \frac{1}{\lambda} - (N - \frac{1}{2}) + \frac{\lambda}{2}(N - \frac{1}{2})^2 - \frac{\lambda}{24} + O(\lambda^2) \end{aligned} \quad (4.85)$$

Figure 4.4: an excited basis state (a) and its corresponding unexcited state (b) (only energy levels of left-movers are shown)

The absolute regularized charge is defined by subtracting the divergent constant $1/\lambda$, and taking the limit $\lambda \rightarrow 0$:

$$\rho_+^{reg}(0) = M + \frac{1}{2} \quad (4.86)$$

$$\rho_-^{reg}(0) = \frac{1}{2} - N$$

In this $\lambda = 0$ limit, exciting a particle does not change its charge, hence these are the chiral charges of $|F\rangle$. The factor of $1/2$ may appear arbitrary, but is justified on symmetry grounds. One consequence is that an unexcited state with zero electric charge has zero momentum, because the momenta of the left- and right-movers take all integer values precisely once.

Regularizing the hamiltonian 4.78 in the same manner, we have

$$H^\lambda = \sum_{m \leq M} m e^{m\lambda} - \sum_{n \geq N} n e^{-n\lambda} \quad (4.87)$$

for the energy of the unexcited basis state. This is easily evaluated by differentiating equations 4.85 by λ ; and we obtain, after subtracting the divergent constant and taking

λ to zero:

$$H^{reg} = \frac{1}{2} (\rho_+^{reg}(0))^2 + \frac{1}{2} (\rho_-^{reg}(0))^2. \quad (4.88)$$

To get the energy of $|F\rangle$ we must add the excitation energy. This has been shown by Manton [37], by using the anticommutators 4.79, and doing a counting argument, to be

$$\frac{1}{2} \sum_{p \neq 0} (\rho_+(p)\rho_+(-p) + \rho_-(-p)\rho_-(p)). \quad (4.89)$$

Thus we obtain the Sommerfeld-Sugawara formula, with the regularization implied

$$H = \frac{1}{2} \sum_p (\rho_+(p)\rho_+(-p) + \rho_-(-p)\rho_-(p)). \quad (4.90)$$

We define the bosonic operators

$$\begin{aligned} p\phi(p) &= \frac{i}{\sqrt{2}} (\rho_+(p) + \rho_-(p)) \\ \pi(p) &= \frac{1}{\sqrt{2}} (\rho_+(p) - \rho_-(p)), \end{aligned} \quad (4.91)$$

whose commutators and hermiticity properties are that of a scalar field in Fourier space

$$\begin{aligned} [\phi(p), \phi(q)] &= [\pi(p), \pi(q)] = 0 \\ [\pi^\dagger(p), \phi(q)] &= -i\delta_{pq} \\ \pi^\dagger(p) &= \pi(-p) \quad \phi^\dagger(p) = \phi(-p). \end{aligned} \quad (4.92)$$

Notice that for consistency of 4.91 we need $\rho_+(0) + \rho_-(0) = 0$, which is the statement that the total charge is zero. This is a physically sensible condition to impose, because sources and sinks of flux must balance on the circle.

Rewriting the hamiltonian 4.90 in terms of the new operators gives it the form

$$H = \frac{1}{2} \sum_p (\pi^\dagger(p)\pi(p) + p^2\phi^\dagger(p)\phi(p)), \quad (4.93)$$

which is precisely the hamiltonian for a free massless scalar in momentum space. From the definition of the scalar field, we see the bosons to be fermion anti-fermion pairs.

The bosonization of the Schwinger model is very similar. A gauge field dependent, gauge invariant, heat kernel regularization leads to similar derivation of the Sommerfeld-Sugawara formula for the fermion kinetic term. The scalar field is again defined by 4.91 and results in the fermion kinetic term being given by 4.93. However this term now also includes part of the gauge field kinetic energy because the gauge field is fixed down to one global degree of freedom which is included in the definition of the scalar zero modes. The Coulomb interaction energy becomes a mass term, and the bosonized hamiltonian is

$$H = \frac{1}{2} \sum_p \pi^\dagger(p) \pi(p) + \left(p^2 + \frac{e^2}{2\pi} \right) \phi^\dagger(p) \phi(p). \quad (4.94)$$

Thus massless electrodynamics with lagrangian

$$\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu + ie A_\mu) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (4.95)$$

is equivalent to the massive scalar theory

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2, \quad (4.96)$$

with $m = e/\sqrt{2\pi}$.

Notice that in order for this gauge field dependent bosonization to reduce to the free field bosonization, we must not only set $e = 0$, but set $A_\mu = 0$ as well. This later condition is required to make the regularization of the gauged theory reduce to that of the free theory.

Interacting theories can also be analyzed with free field bosonization by translating a perturbation of the free Dirac lagrangian to a perturbation of the free massless scalar lagrangian. To do this, we note that 4.91 transformed back into position space, along with the fact that $\pi(x) = \phi(x)$, gives the identification

$$\begin{aligned} \partial_x \phi(x) &= \frac{1}{\sqrt{\pi}} (\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-) \\ \partial_t \phi(x) &= \frac{1}{\sqrt{\pi}} (\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-) \end{aligned} \quad (4.97)$$

which in a general representation of the Dirac matrices is

$$\begin{aligned} j^\mu &= \bar{\psi} \gamma^\mu \psi = \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi \\ j_5^\mu &= \bar{\psi} \gamma^\mu \gamma_5 \psi = \frac{1}{\sqrt{\pi}} \partial^\mu \phi. \end{aligned} \quad (4.98)$$

The momentum space chiral charge density commutators transformed back into position space yield the current algebra

$$\begin{aligned} [j_0(x), j_0(y)] &= 0 \\ [j_0(x), j_1(y)] &= \frac{i}{\pi} \delta'(x - y) \\ [j_1(x), j_1(y)] &= 0. \end{aligned} \quad (4.99)$$

Current algebras similar to these, for realistic four dimensional theories, form the basis for much of our understanding of elementary particle physics. Although the four dimensional theories do have Sugawara type formulas, these do not allow the theory to be bosonized as in two dimensions. In the fermionic language, the commutators 4.99 arise only upon quantization; whereas in the bosonic language they are simply due to the classical Poisson bracket

$$\{\phi(x), \pi(y)\} = \delta(x - y). \quad (4.100)$$

Thus the bosonized theory has quantum effects already contained in its classical lagrangian.

For illustrative purposes, we close this chapter with the free field bosonization of the Schwinger model. This allows for an easy derivation of the axial anomaly discussed in the third paragraph of the introduction to anomalies chapter.

Using correspondence 4.98, the lagrangian 4.95 becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{e}{\sqrt{\pi}} A_\mu \epsilon^{\mu\nu} \partial_\nu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (4.101)$$

The Euler-Lagrange equations derived from $\mathcal{L}$ are

$$\partial_\mu \partial^\mu \phi + \frac{e}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu A_\mu = 0 \quad (4.102)$$

$$\partial_\mu F^{\nu\mu} - \frac{e}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\mu \phi = 0 \quad (4.103)$$

Equation 4.103 by use of 4.98 is just

$$\partial_\mu F^{\nu\mu} = e j^\mu \quad (4.104)$$

and implies the conservation of the vector current. Upon using $j_5^\mu = \epsilon^{\mu\nu} j_\nu$ equation 4.102 becomes the famous axial anomaly equation:

$$\partial_\mu j_5^\mu = \frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}. \quad (4.105)$$

The question which arises here is: to what extent does this depend on the regularization which was used to bosonize the theory? After all, the lagrangian 4.96 obtained with a gauge field dependent regularization looks quite different from the lagrangian 4.101 obtained with a gauge field independent regularization. The same question arises in the quantum treatment of the fermionic formulation; and the answer is that all regularization procedures must preserve some maximal set of symmetries. In this case the symmetry is gauge invariance, and both regularizations give a lagrangian invariant under the gauge transformations

$$\psi \rightarrow e^{i\theta} \psi \quad (4.106)$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \theta,$$

which in the bosonic language are

$$\phi \rightarrow \phi \quad (4.107)$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \theta.$$

Fortunately the two bosonic lagrangians here in question describe (at least classically) the same physics. In the chiral version of this theory, gauge invariance will be lost, and the resulting ambiguity in how to regularize the theory will introduce another parameter.

Chapter 5

Lorentz Invariant Exact Solution of The Chiral Schwinger Model

5.1 Introduction

The chiral Schwinger model was first introduced by Hagen [38] in 1973 as a new example of an exactly solvable field theory in two spacetime dimensions. The model he proposed is that of a Dirac fermion with only one Weyl component coupled to a gauge field with bare mass μ_o .

$$\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu + i e A_\mu P_+) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\mu_o^2}{2} A_\mu A^\mu \quad (5.108)$$

Besides the mass term, which causes the classical theory to be gauge variant, this differs from the lagrangian 3.54 in that it includes the extra free Weyl component.

By considering the vacuum to vacuum transition amplitude in the presence of external sources (called the generating functional), he was able to obtain all the Green's functions exactly. This allowed him to conclude that the model is relativistic, and also that it has the unusual property that the vector boson mass renormalized to

$$\mu^2 = \frac{\mu_o^2}{1 - \frac{e^2}{4\pi\mu_o^2}} \quad (5.109)$$

Over a decade later, sudden interest in anomalous field theories prompted Jackiw and Rajaraman [39] to examine the gauge anomalous theory without the bare mass term. By studying the field equations and propagator obtained from the effective gauge field action (the action obtained by path integrating out the fermions), they concluded that the theory is not gauge invariant, but is unitary and amenable to particle interpretation. They also found that the vector gauge boson necessarily acquires a mass when consistency

and unitarity are demanded. It has been further argued that the gauge variance of the fermionic part of the path integral measure, upon using the Fadeev-Popov procedure to take into account the integration over gauge orbits, yields a Wess-Zumino term in the effective gauge field action [40].

However, use of the naive path integral representation requires canonical justification. At the hamiltonian level, the theory involves second class gauge constraints which lead to a nontrivial modification of the equations of motion. Several authors [41, 42] have solved the theory (obtained the spectrum exactly) in the canonical hamiltonian formalism using bosonization. The lagrangian 5.108, without the bare mass term bosonizes, using the correspondence 4.98, to

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \sqrt{\hbar} e (\eta^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\mu \phi A_\nu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} a \hbar e^2 A_\mu A^\mu, \quad (5.110)$$

where the mass term is now to be considered as a counterterm reflecting the ambiguity in the regularization procedure used to bosonize the theory. In the vector Schwinger model, demanding the regularization be gauge invariant fixes $a = 0$. Here no choice of a can make the theory gauge invariant, so it is left as a free parameter. The advantage of starting with 5.110 is that the consequences of the anomaly can be dealt with at the classical level by Dirac's formalism for systems with second class constraints. Doing this, the afore mentioned authors found a unitary theory with a relativistic spectrum for $a \geq 1$.

However, bosonization is valid only in the charge zero sector of the theory and it requires the extra $1/2$ degree of freedom of the decoupled Weyl component. It is not obvious whether the latter modification is responsible for the apparent consistency of the model. There have been several recent solutions which avoid bosonization and fail to obtain Lorentz invariance [43, 44]. It is not clear to us that these could not be made Lorentz invariant by adding further Lorentz variant counterterms.

In this chapter we solve the chiral Schwinger model with the minimal degrees of

freedom. At the classical level the lagrangian contains a vector gauge field and a chiral fermion, representing $2\frac{1}{2}$ configuration space degrees of freedom. There are two first class constraints, each of which remove one degree of freedom resulting in a physical $\frac{1}{2}$ degree of freedom – a charged Weyl fermion with a relativistic Coulomb self-interaction. The classical theory can be formulated as a consistent, causal, Lorentz covariant initial value problem.

Upon quantization, the Goto-Schwinger-Imamura term in the gauge constraint algebra changes the two first class constraints into two second class constraints which together remove 1 degree of freedom. Therefore the quantum theory has $1\frac{1}{2}$ physical degrees of freedom. Lorentz invariance is not manifest in this reduction and it is not obvious that it is maintained by the quantization. Furthermore, the presence of the extra physical degrees of freedom would seem to indicate that the quantized theory loses the geometrical interpretation of a gauge theory. It is tempting to conjecture that the quantum theory retains a geometrical interpretation in terms of projective representations of the original gauge symmetry. Although this conjecture is supported by recent investigations using path integrals which suggest that the extra degrees of freedom appear as projective phases in the integration measure [40], there is no formalism where this is manifest at the level of hamiltonian dynamics.

Since it is only at the quantum level that the second class constraints appear, the usual procedure of identifying classical Dirac brackets with commutation relations must here be modified so as to apply directly to the quantum theory. Going back to section 3.1, we see that the Dirac bracket of A with B is really the Poisson bracket of A' with B' , where

$$A' \equiv A - \{A, \psi_a\} Q_{ab}^{-1} \psi_b. \quad (5.111)$$

Thus the analogous way to proceed here would be to make a transformation on the

quantum operators such that they commute with the second class constraints.

We regulate the theory, as in the previous chapter, by taking space to be a circle and damping the high momentum Fourier modes. This will allow us to directly apply the chiral version of the Sommerfeld-Sugawara formula derived there (which incidently, makes no assumption about the total charge). It does however, have a particular implicit regularization, which does not necessarily respect symmetries that the quantum theory might have. To remedy this we add counterterms to the bare lagrangian. We solve the model with the counterterms and find that the hamiltonian is hermitian and positive with a Lorentz invariant spectrum only for a certain range of the parameters. The spectrum contains a single massless chiral scalar and a single massive boson. These resemble the particle content discovered in reference [41] with the exception of the absence of one of the chiral Bose degrees of freedom.

There have been several other attempts to quantize the chiral Schwinger model which have failed to obtain a Lorentz invariant solution [45, 43, 44]. Some of these can be regarded as special cases of the model which we consider here with particular values of the counterterms [45, 44]. Our present work shows how to restore Lorentz invariance in these models by adding counterterms to the action.

A further gauge invariant solution which recognized the origin of the anomaly as an induced quantum curvature and added Lorentz noninvariant nonlocal counterterms obtained a noncovariant spectrum with a hermitian hamiltonian [43]. The solution has a $\frac{1}{2}$ degree of freedom which, with the mass generation inherent in the model, is incompatible with Lorentz invariance. (A chiral particle has a Lorentz invariant spectrum only when it is massless.) It is apparent that the quantum curvature cannot be cancelled by adding local counterterms to the gauge theory action. We conclude that quantization of an anomalous gauge theory necessarily breaks gauge invariance.

5.2 Solution of the Chiral Schwinger Model

The lagrangian 3.54, with gauge group $U(1)$, written in terms of the single component Weyl fermion is

$$L_0 = \int dx \left\{ \psi^\dagger i \sigma^\mu (\partial_\mu + i \sqrt{\pi} e A_\mu) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\}, \quad (5.112)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\sigma^0 = 1$, $\sigma^1 = \epsilon = \pm 1$ with the $+$ ($-$) sign corresponding to a coupling to right- (left-) handed fermions. The fermion bilinear operators which occur in the lagrangian and later in the hamiltonian must be defined with some regularization. We shall choose a particular regularization and parameterize the difference between the one we choose and others by adding local counterterms to the lagrangian

$$L_{ct} = \int dx \left\{ \frac{a}{2} A_\mu A^\mu + \frac{b}{2} (A_\epsilon)^2 \right\} \quad (5.113)$$

where $A_\epsilon = A_0 + \epsilon A_1$. We note that the first term is Lorentz invariant but has the conventionally undesirable feature of breaking gauge invariance. Here, we expect that the quantization of the theory breaks gauge invariance so we cannot exclude dynamical generation of such a term. We shall show that the model is consistent for a large range of this parameter. This counterterm was also needed in the treatment of the bosonized theory, but there the additional Lorentz variant counterterm was not required. The model will be found to have a Lorentz invariant spectrum only for a particular nonzero value of b .

The full lagrangian is:

$$L = \int dx \left\{ \psi^\dagger (i \partial_\epsilon - \sqrt{\pi} e A_\epsilon) \psi + \frac{1}{2} (\partial_0 A_1 - \partial_1 A_0)^2 + \frac{a}{2} (A_0^2 - A_1^2) + \frac{b}{2} (A_\epsilon)^2 \right\} \quad (5.114)$$

The canonical momenta are:

$$\pi_0(x) \equiv \frac{\delta L}{\delta(\partial_0 A_0(x))} = 0 \quad (5.115)$$

$$\pi_1(x) \equiv \frac{\delta L}{\delta(\partial_0 A_1(x))} = \partial_0 A_1(x) - \partial_1 A_0(x) \quad (5.116)$$

$$\pi_\psi(x) \equiv \frac{\delta L}{\delta(\partial_0 \psi(x))} = i\psi^\dagger(x) \quad (5.117)$$

We have one primary constraint, $\pi_0 = 0$. The hamiltonian is

$$\begin{aligned} H_0 &= \int dx \left\{ \pi_1 \partial_0 A_1 + i\psi^\dagger \partial_0 \psi \right\} - L \\ &= \int dx \left\{ \frac{\pi_1^2}{2} + \pi_1 \partial_1 A_0 - \epsilon i\psi^\dagger \partial_1 \psi + \sqrt{\pi} \epsilon \psi^\dagger \psi A_\epsilon + \frac{a}{2} (A_1^2 - A_0^2) - \frac{b}{2} A_\epsilon^2 \right\} \end{aligned} \quad (5.118)$$

We work on the compact one dimensional space S^1 with length $2\pi l$ and do a plane wave expansion of our variables.

$$\psi(x) = \frac{1}{\sqrt{2\pi l}} \sum_{k=-\infty}^{\infty} a_k \exp\left(\frac{i\epsilon_f kx}{l}\right) \quad (5.119)$$

$$A_j(x) = \frac{1}{\sqrt{2\pi l}} \sum_{k=-\infty}^{\infty} A_j(k) \exp\left(\frac{i\epsilon_f kx}{l}\right) \quad (5.120)$$

$$\pi_j(x) = \frac{1}{\sqrt{2\pi l}} \sum_{k=-\infty}^{\infty} \pi_j(k) \exp\left(\frac{i\epsilon_f kx}{l}\right) \quad (5.121)$$

where $\epsilon_f = \pm 1$ (the reason for keeping this sign arbitrary will become clear at the end).

We also define the momentum space chiral charge density as before

$$\rho^\dagger(n) = \int_0^{2\pi l} \psi^\dagger \psi \exp\left(\frac{i\epsilon_f nx}{l}\right) dx = \sum_{k=-\infty}^{\infty} a_{k+n}^\dagger a_k \quad (5.122)$$

This allows us to write

$$\begin{aligned} H_0 &= \sum_n \left\{ \frac{1}{2} \pi_1^\dagger(n) \pi_1(n) + \frac{i\epsilon_f n}{l} \pi_1^\dagger(n) A_0(n) + \frac{\epsilon \epsilon_f n}{l} a_n^\dagger a_n - \frac{b}{2} A_\epsilon^\dagger(n) A_\epsilon(n) \right. \\ &\quad \left. + \frac{a}{2} (A_1^\dagger(n) A_1(n) - A_0^\dagger(n) A_0(n)) \right\} \end{aligned} \quad (5.123)$$

The field variables are quantum operators obeying the canonical commutations relations:

$$[A_j(x), \pi_k(y)] = \delta_{jk} \delta(x-y) \longleftrightarrow [A_j(n), \pi_k^\dagger(m)] = i\delta_{jk} \delta_{nm} \quad (5.124)$$

$$[\psi(x), \psi^\dagger(y)]_+ = i\delta(x-y) \longleftrightarrow [a_n^\dagger, a_m]_+ = \delta_{nm} \quad (5.125)$$

Defining the Fock vacuum as the state annihilated by

$$a_{\varepsilon_0 n} | 0 \rangle = 0 \quad n \geq 0 \quad (5.126)$$

$$a_{\varepsilon_0 n}^\dagger | 0 \rangle = 0 \quad n < 0 \quad (5.127)$$

with $\varepsilon_0 = \pm 1$, we use the anomalous commutator for the charge density 4.71

$$[\rho(p), \rho^\dagger(q)] = \varepsilon_0 p \delta_{p,q}, \quad (5.128)$$

and the chiral version of the Sommerfeld-Sugawara formula

$$\sum_n n a_n^\dagger a_n = \frac{1}{2} \varepsilon_0 \sum_n \rho^\dagger(n) \rho(n). \quad (5.129)$$

These lead to the bosonization of the Weyl fermion upon the the definition:

$$\sigma(n) = \frac{\rho(n)}{\sqrt{|n|}}, \quad (5.130)$$

which is analogous to the definition of the Bose field in the chirally symmetric theory.

The bosonic commutator is

$$[\sigma(p), \sigma^\dagger(q)] = \varepsilon_0 \delta_{p,q} \quad (5.131)$$

Then combining the momentum slices $-n$ and n results in the hamiltonian

$$H_0 = h_0(0) + \sum_{n>0} h_0(n) \quad (5.132)$$

where

$$h_0(0) = \frac{1}{2} \pi_1(0)^2 + \frac{\varepsilon \varepsilon_0 \varepsilon_f}{2l} \rho(0)^2 + \frac{e}{\sqrt{2l}} \rho(0) A_\varepsilon(0) + \frac{a}{2} (A_1(0)^2 - A_0(0)^2) - \frac{b}{2} A_\varepsilon(0)^2 \quad (5.133)$$

and

$$\begin{aligned} h_0(n) = & \pi_1^\dagger(n) \pi_1(n) + \frac{\varepsilon \varepsilon_0 \varepsilon_f n}{l} \sigma^\dagger(n) \sigma(n) + \frac{e \sqrt{n}}{\sqrt{2l}} (\sigma^\dagger(n) A_\varepsilon(n) + A_\varepsilon^\dagger(n) \sigma(n)) \\ & + \frac{i \varepsilon_f n}{l} (\pi_1^\dagger(n) A_0(n) - A_0^\dagger(n) \pi_1(n)) - b A_\varepsilon^\dagger(n) A_\varepsilon(n) \\ & + a (A_1^\dagger(n) A_1(n) - A_0^\dagger(n) A_0(n)) \end{aligned} \quad (5.134)$$

We do not worry at this point about the possible appearance of constant terms arising from commuting operators; this is taken care of by field independent normal ordering.

We now proceed to implement Dirac's procedure directly at the quantum level. We first look at $h_0(n)$ for $n \neq 0$ and set $l = 1$. Consistency of our constraint $\pi_0 = 0$ under time evolution requires $[\pi_0, H_0] = 0$. We thus define (dropping the irrelevant n label)

$$G \equiv -i[h_0, \pi_0] = -i\epsilon_f n \pi_1 + \frac{e\sqrt{n}}{\sqrt{2}}\sigma - aA_0 - bA_e = 0 \quad (5.135)$$

so that

$$[G, \pi_0^\dagger] = a + b \quad (5.136)$$

and

$$[G, G^\dagger] = \frac{1}{2}\epsilon_0 e^2 n + 2nb\epsilon\epsilon_f. \quad (5.137)$$

$G = 0$ is Gauss's law for this system, and the non-vanishing commutator with itself is the Goto-Schwinger-Imamura term plus a term due to the Lorentz variant counterterm. When $a + b \neq 0$, $\pi_0 = 0$ and $G = 0$ form a complete set of second class constraints. The $a + b = 0$ case will be dealt with later.

We can find a canonical transformation that leaves π_0 unchanged, while the transformed A_0 is a linear combination of the two constraints [46]. We easily show that the hamiltonian can then be written as

$$h_0(A_0, \pi_1, A_1, \sigma) = h_{\text{physical}}(\pi_1, A_1, \sigma) + h_{\text{constraints}}, \quad (5.138)$$

where $h_{\text{constraints}}$ contains only terms proportional to the untransformed constraints. Furthermore, the transformed operators π_1, A_1, σ all commute with both of the second class constraints. Since the constraints have decoupled, we may consistently set them to zero without modifying the commutators (the analogue in Dirac's procedure would be that the Poisson brackets and the Dirac brackets are identical). This justifies the algebraically simpler procedure of obtaining the same physical hamiltonian —up to a canonical

transformation- by simply using the constraint $G = 0$ to eliminate A_0 directly

$$A_0 = (a + b)^{-1}(-i\varepsilon_f n \pi_1 + \frac{e\sqrt{n}}{\sqrt{2}}\sigma - b\varepsilon A_1) \quad (5.139)$$

so that we have

$$\begin{aligned} h = & \pi_1^\dagger \pi_1 \left(1 + \frac{n^2}{a+b}\right) + \sigma^\dagger \sigma \left(\varepsilon \varepsilon_0 \varepsilon_f + \frac{\varepsilon^2}{2(a+b)}\right) n + i \left(A_1^\dagger \pi_1 b - \pi_1^\dagger A_1\right) \frac{nb\varepsilon\varepsilon_f}{a+b} \\ & + A_1^\dagger A_1 \frac{a^2}{a+b} + \left(\sigma^\dagger A_1 + A_1^\dagger \sigma\right) \frac{eae\sqrt{n}}{\sqrt{2}(a+b)} + i \left(\pi_1^\dagger \sigma - \sigma^\dagger \pi_1\right) \frac{en^{3/2}\varepsilon_f}{\sqrt{2}(a+b)} \end{aligned} \quad (5.140)$$

Performing the following canonical transformation to diagonalize the hamiltonian:

$$\begin{aligned} A_1 &= A - \beta \Sigma - \frac{i\beta^2 \varepsilon_0}{2} \pi \\ \sigma &= \Sigma + i\varepsilon_0 \beta \pi \end{aligned} \quad (5.141)$$

$$\pi_1 = \pi$$

with

$$\beta = \frac{e\sqrt{n\varepsilon}}{\sqrt{2}a} \quad (5.142)$$

we find after some tedious algebra

$$h = \pi^\dagger \pi \left(1 + \frac{n^2(4a - \varepsilon\varepsilon_0\varepsilon_f e^2)^2}{16a^2(a+b)}\right) + i(A^\dagger \pi - \pi^\dagger A) \frac{\varepsilon_0 n(e^2 + 4b\varepsilon\varepsilon_0\varepsilon_f)}{4(a+b)} \quad (5.143)$$

$$+ \Sigma^\dagger \Sigma n\varepsilon\varepsilon_0\varepsilon_f + A^\dagger A \frac{a^2}{a+b} \quad (5.144)$$

Defining the creation operators

$$f_{1,2} = \frac{\gamma A}{\sqrt{2}} \pm \frac{i\pi}{\gamma\sqrt{2}} \quad \text{where} \quad \gamma = \sqrt{\frac{16a^4(a+b)}{16a^2(a+b) + (4a - \varepsilon\varepsilon_0\varepsilon_f e^2)^2}} \quad (5.145)$$

such that

$$[f_1, f_1^\dagger] = -[f_2, f_2^\dagger] = 1 \quad [f_1, f_2^\dagger] = 0 \quad (5.146)$$

we get

$$h = E_1 f_1^\dagger f_1 + E_2 f_2^\dagger f_2 + n\varepsilon\varepsilon_0\varepsilon_f \Sigma^\dagger \Sigma \quad (5.147)$$

where

$$E_{2,1} = \pm \frac{\epsilon_0 n (e^2 + 4\epsilon\epsilon_0\epsilon_f b)}{4(a+b)} + \sqrt{\frac{(4a - \epsilon\epsilon_0\epsilon_f e^2)^2 n^2}{16(a+b)^2} + \frac{a^2}{(a+b)}}. \quad (5.148)$$

$f_1^\dagger$, $f_2^\dagger$, and $\Sigma^\dagger$ are the creation operators for elementary excitations of momentum n , $-n$, and $\epsilon_0 n$ respectively.

So we see that in general, the spectrum is not Lorentz invariant, for it does not have a relativistic dispersion relation. This lack of Lorentz invariance is deeply connected with the fact that this theory has an anomaly which is manifest in the non-vanishing commutator $[G, G^\dagger]$. It is intriguing that with the choice of b such that $4b + \epsilon\epsilon_0\epsilon_f e^2 = 0$ the gauge generators commute, $[G, G^\dagger] = 0$, and we recover a Lorentz invariant spectrum

$$h = E f_1^\dagger f_1 + E f_2^\dagger f_2 + n \Sigma^\dagger \Sigma, \quad (5.149)$$

with

$$E = \sqrt{\frac{n^2}{l^2} + m^2} \quad \text{where} \quad m^2 = \frac{a^2}{a - \frac{e^2}{4}}. \quad (5.150)$$

Note that we had to choose $\epsilon\epsilon_0\epsilon_f = +1$ in order to obtain a positive definite hamiltonian (which is necessary for a unitary theory). We also see that a is restricted to values greater than $\frac{e^2}{4}$, for a positive mass squared), which corresponds to the restriction $a \geq 1$ for the 5.110 theory.

The analysis of the $n = 0$ sector is trivial. Proceeding as above, one easily obtains

$$h(0) = m \left(f^\dagger(0) f(0) + \frac{1}{2} \right) + \frac{1}{2l} \rho(0)^2 \quad (5.151)$$

which is consistent with the $n \neq 0$ sector. The extension of our result from S^1 to the real line is quite trivial as we simply replace $\frac{n}{l}$ by p the momentum.

The above analysis is singular for the particular combination of counterterms where $a+b = 0$, which we must analyze separately. This critical value corresponds to the critical value $a = 1$ of references [39, 42] where the mass diverges. We consider only the critical

value $4b = -e^2$ as, once again, only for this value do we get a relativistic hamiltonian. Now

$$[\pi_0, G^\dagger] = [G, G^\dagger] = 0. \quad (5.152)$$

However a third constraint arises from commuting Gauss's law with the hamiltonian. The third constraint turns out to be $\pi_1 = 0$, which along with $G = 0$ forms a complete set of second class constraints. The other constraint, $\pi_0 = 0$, is a first class constraint which is associated to a gauge freedom for our system. We can fix the corresponding gauge by setting $A_0 = 0$, and use the other two constraints to eliminate both A_1 and π_1 from the system, so that we get quite simply

$$H = \frac{1}{2l} \rho(0)^2 + \sum_{n>0} \frac{n}{l} \sigma^\dagger(n) \sigma(n). \quad (5.153)$$

Looking back at the case $a + b \neq 0$ and taking the limit as a goes to $-b$, we see that the mass goes to infinity and that the corresponding boson effectively decouples. This is consistent with the result obtained above if we had put $a + b = 0$ at the outset.

5.3 Discussion

In summary we have obtained a Lorentz invariant unitary solution of the chiral Schwinger model with minimal degrees of freedom by adding gauge and Lorentz variant counterterms to the bare lagrangian. This can be interpreted as a particular choice of regularization which is implemented at the hamiltonian level and is implicit in the gauge field independent Sommerfeld-Sugawara formula 5.129. It requires that the renormalization of the fermionic charge and the fermionic hamiltonian operators are independent of the gauge fields. The difference between this and other regularizations is then parameterized by the local counterterms.

The spectrum we obtain agrees with that found previously by Jackiw and Rajaraman, and Girotti et al. [39, 42] except we find a massless chiral scalar instead of a massless

scalar. The extra $1/2$ degree of freedom found previously is due to the bosonization of the decoupled Weyl fermion. Incidentally this results in a different constraint structure as we have 2 second class or 2 second class plus 1 first class where Girotti et al. find 2 second class or 4 second class constraints. However as a first class constraint eliminates as many degrees of freedom as two second class constraints it is not surprising that the resulting theories show equivalence. A point that is not clear to us though is the connection between the extra $1/2$ degree of freedom and the absence of a Lorentz variant counterterm.

Our methods can be regarded as providing a canonical justification for the previous results found using path integral methods. It is not surprising that the naive path integral representation can be applied since the theorem of Fradkin and Vilkoviski [47] indicates that for a system with second class constraints the naive path integral still correctly describes the dynamics. However, that theorem holds for constrained systems where the character of the constraints appears at the classical level and is not modified by quantum effects. Here, as in the case of all gauge theories with perturbative anomalies, the algebra of the gauge constraints is modified at the quantum level, their second class nature is not exhibited at the classical level. We conjecture that the path integral is still valid in this case and that the quantum effects are exhibited by the failure of the fermion measure to be gauge invariant [24]. This means that we can parameterize the different acceptable regularizations of the measure by using counterterms with arbitrary parameters. We have found that only certain regularizations are allowed by the physical requirements of unitarity and Lorentz invariance.

The bosonized lagrangian 5.110 can also be obtained by localizing the gauge field effective action [39]. As mentioned in the introduction, authors [40] have found that Fadeev-Popov procedure results in a Wess-Zumino term in the effective action as well.

Upon localization this yields an action

$$\mathcal{L} = \mathcal{L}_o + \mathcal{L}_{WZ}, \quad (5.154)$$

where $\mathcal{L}_o$ is given by 5.110, and

$$\mathcal{L}_{WZ} = \frac{1}{2}(a-1)\partial_\mu\theta\partial^\mu\theta - e\theta((a-1)\partial_\mu A^\mu + \epsilon^{\nu\mu}\partial_\mu A_\nu). \quad (5.155)$$

θ is the Wess-Zumino field [48]. This term restores the gauge invariance of the theory, as $\mathcal{L}$ is now explicitly invariant under the transformation

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \partial_\mu\epsilon \\ \theta &\rightarrow \theta - e\epsilon \\ \phi &\rightarrow \phi - e\epsilon \end{aligned} \quad (5.156)$$

However, now that the theory is gauge invariant, we can go to the gauge $\theta = 0$ and recover the previous theory. In [48] it is shown that with a gauge covariant quantization (instead of fixing the gauge, constraints are imposed on the Hilbert space), the physical gauge invariant correlation functions in the theory with the Wess-Zumino term are the same as the physical correlation functions of the theory without the Wess-Zumino terms.

Finally let us point out that the physics here is a faint image of the standard model. We have a massive vector gauge boson and chiral fermion anti-fermion pairs. However, here the gauge boson acquires a mass through anomaly generated symmetry breaking, whereas the standard model uses the Higgs mechanism with spontaneous symmetry breaking. This latter method is somewhat ad hoc and introduces a large number of parameters into the model. If the top quark did not exist (it has not yet been found up to 70 GeV) the standard model would be anomalous, and perhaps the W and Z particles would acquire their mass via anomaly generated gauge symmetry breaking.

Chapter 6

Chiral Bosons

Chiral bosons (the same thing as self-dual scalars in two dimensions) are very important objects in string theory. Probably their most useful property is that they represent chiral Fermi theories. Here we will review some of the proposals for describing a single boson in Minkowski space, and end with a discussion of the curved space generalization.

6.1 Chiral Bosons as Charge Density Solitons

In the previous chapter we had found that the hamiltonian for a Weyl fermion coupled to a vector gauge field could be transformed to the hamiltonian for a massive boson (the gauge field) plus a massless chiral boson (fermion anti-fermion pairs). Stepping slightly backwards, we have found that a free right-moving Weyl fermion with lagrangian density

$$\mathcal{L} = i\sqrt{2}\psi^\dagger\partial_+\psi \equiv i\psi^\dagger(\partial_t + \partial_x)\psi, \quad (6.157)$$

yields, via the Sommerfeld-Sugawara formula, a hamiltonian describing a right-moving scalar

$$H = \frac{1}{2} \sum_n \rho^\dagger(n)\rho(n), \quad (6.158)$$

with commutator

$$[\rho(n), \rho^\dagger(m)] = n\delta_{n,m}. \quad (6.159)$$

Transforming back to position space, we have a boson field ρ which has the equal time commutator

$$[\rho(x), \rho(y)] = i\delta'(x - y), \quad (6.160)$$

and dynamics governed by the local hamiltonian

$$H = \frac{1}{2} \int dx \rho^2(x). \quad (6.161)$$

We ask what bosonic lagrangian could give this theory, and are answered by Floreanini and Jackiw [49] who say that it is the following nonlocal one:

$$L = \frac{-1}{4} \int dx dy \rho(x) \epsilon(x-y) \dot{\rho}(y) - \frac{1}{2} \int dx \rho^2(x), \quad (6.162)$$

where $\epsilon(x)$ is the heavyside function

$$\epsilon(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} \quad (6.163)$$

This lagrangian does describe a self-dual scalar, for the Euler-Lagrange equations are

$$\rho(x) + \frac{1}{2} \int dy \epsilon(x-y) \dot{\rho}(y) = 0, \quad (6.164)$$

which imply $\partial_+ \rho = 0$. In addition, we have the soliton type boundary conditon $\rho(+\infty) = -\rho(-\infty)$.

To check that the lagrangian does give the desired hamiltonian formalism, we go through the Dirac procedure [50]. The momentum canonically conjugate to $\rho(x)$ is

$$\pi(x) = \frac{-1}{4} \int dy \rho(y) \epsilon(y-x), \quad (6.165)$$

which implies that the system posses the infinite continuous set of primary constraints

$$\chi(x) \equiv \pi(x) + \frac{1}{4} \int dy \rho(y) \epsilon(y-x) \approx 0. \quad (6.166)$$

The canonical hamiltonian then derived from lagrangian 6.162 is indeed 6.161. The Dirac algorithm does not yield secondary constraints, and since

$$Q(x, y) \equiv [\chi(x), \chi(y)] = \frac{1}{2} \epsilon(x-y), \quad (6.167)$$

we conclude that all the constraints are second class. Furthermore, since $Q^{-1}(x, y) = \delta'(x - y)$, the Dirac bracket 3.52 yields the equal time commutator 6.160 for ρ .

A local action is obtained by a redefinition of the dynamical variable [49]

$$\phi(x) = \frac{1}{2} \int dy \epsilon(x - y) \rho(y); \quad (6.168)$$

because substitution of this variable into lagrangian 6.162 gives a local lagrangian with density

$$\mathcal{L} = -\frac{1}{2} \partial_x \phi \dot{\phi} - \frac{1}{2} (\partial_x \phi)^2. \quad (6.169)$$

The nonlocality hidden in 6.168 reappears in the equal-time commutator. In fact, use of 6.160 or canonical quantization of 6.169 yields

$$[\phi(x), \phi(y)] = \frac{-i}{2} \epsilon(x - y). \quad (6.170)$$

While the hamiltonian formulation of this theory is, by virtue of the fixed time definition 6.168, the same as the previous, and leads to the self-dual equation for ϕ , the Euler-Lagrange equations that follow from $\mathcal{L}$ are $\partial_x \partial_+ \phi = 0$, which have solution $\partial_+ \phi = f(t)$. Thus to obtain the self-dual equation we must impose the boundary condition $f(t) = 0$.

To summarise thus far, the local, manifestly Poincare-invariant theory of a single Weyl fermion can be represented by two nonlocal versions of the theory of a chiral boson, both with the same hamiltonian formalism. This theory is Poincare invariant [49], however not manifestly so. It is therefore difficult to couple to background gravity or to guarantee Lorentz invariance when coupling to external gauge fields. The latter would be necessary to examine gravitational and gauge anomalies of the chiral fields or to construct covariant chiral string theories. The version 6.162 is a nonlocal lagrangian in terms of a local field, while the version 6.169 is a local lagrangian in terms of a nonlocal field.

6.2 The Siegel Action

We would like a completely local lagrangian field theory of chiral bosons. This cannot be done with the minimal left-moving or right-moving degrees of freedom. In general both must be included in the action and the unwanted modes are eliminated by a chiral constraint. The following lagrangian appears to work:

$$\mathcal{L} = \partial_+ \phi \partial_- \phi + \lambda \partial_+ \phi. \quad (6.171)$$

The first term is just the usual lagrangian for a free massless scalar, while the second term has a Lagrange multiplier enforcing the linear constraint $\partial_+ \phi = 0$ as the field equation derived from a stationary variation with respect to λ . For this to be a theory of a single chiral scalar, λ must simply be a gauge degree of freedom. However, because the linear chiral constraint is second class it is not a gauge generator, and therefore λ cannot be gauged away from the equations of motion.

This can be partially cured by adding the counterterm $\lambda^2/4$ to the lagrangian 6.171. We then obtain the hamiltonian for a scalar

$$\mathcal{H} = \frac{1}{2} \pi_\phi^2 + \frac{1}{2} \phi'^2, \quad (6.172)$$

along with the chiral constraint ,

$$\chi(x) = \pi_\phi(x) + \phi'(x) = 0, \quad (6.173)$$

whose Poisson bracket algebra,

$$\{\chi(x), \chi(y)\} = -2\delta'(x - y), \quad (6.174)$$

permits the computation of the equal time commutator via the Dirac bracket

$$[\phi(x), \phi(y)] = \frac{-i}{4} \epsilon(x - y). \quad (6.175)$$

Then setting the second class constraint strongly to zero results in the hamiltonian being again given by 6.161, and we just have the theory of the previous section. It appears that this is known in the literature; however nowhere is it explicitly stated. The advantage of lagrangian 6.171 over those of the previous section is that it is more susceptible to a curved space generalization.

It was first pointed out by Siegel [51] that if we square the second class chiral constraint $\partial_+\phi = 0$, we obtain the first class constraint $(\partial_+\phi)^2 = 0$. The lagrangian he proposed for describing chiral bosons is

$$\mathcal{L}_s = \partial_+\phi\partial_-\phi - \lambda^{++}(\partial_+\phi)^2. \quad (6.176)$$

It is invariant under the gauge transformations

$$\delta\phi(x) = \epsilon^+(x)\partial_+\phi(x) \quad (6.177)$$

$$\delta\lambda^{++} = \epsilon^+(x)\partial_+\lambda^{++}(x) - \partial_+\epsilon^+(x)\lambda^{++}(x) + \partial_-\epsilon^+(x),$$

which allow λ^{++} to be gauged to zero.

In the corresponding hamiltonian formalism the constraint $\pi_\lambda = 0$ is first class, allowing the gauge $\lambda^{++} = 0$ to be imposed. The secondary constraint

$$T(x) \equiv \frac{1}{2}(\pi + \phi')^2 = 0 \quad (6.178)$$

is first class at the classical level since its Poisson brackets generate the one-dimensional diffeomorphism algebra

$$\{T(x), T(y)\} = (T(x) + T(y))\delta'(x - y). \quad (6.179)$$

Thus at the classical level the chiral constraint can be consistently imposed without altering the canonical brackets, and we have the desired local theory of a self-dual scalar.

However, at the quantum level this algebra contains an anomaly, and is essentially the Virasoro algebra because $T(x)$ is the positive light-cone component of the energy momentum tensor for a scalar.

$$[T(x), T(y)] = i(T(x) + T(y))\delta'(x - y) + \frac{i}{48\pi}(\partial_x + \partial_x^3)\delta(x - y). \quad (6.180)$$

There are several ways in which we could deal with this anomaly. One way would be, as we did with the chiral Schwinger model, is apply the analogue of Dirac's procedure directly to the quantum theory. However, this would be fairly complicated and also reintroduce the same type of nonlocality that we wished to avoid in the first place. The other way would be, as has also been done with the chiral Schwinger model, is to add a Wess-Zumino term to the Siegel lagrangian which cancels the anomaly. Imbimbo and Schwimmer [52] have observed that a scalar field can serve as its own Wess-Zumino term and have found that the term

$$\mathcal{L}_{ano} = \sqrt{\frac{25}{6\pi}} \lambda^{++} \partial_+^2 \phi \quad (6.181)$$

explicitly breaks the classical Siegel invariance, but restores the invariance at the quantum level. However, as we will later discuss, this does not generalize to the nonabelian theory.

The term which Labastida and Pernici [53] added is

$$\mathcal{L}_{ct} = \partial_+ \rho \partial_- \rho - \lambda^{++} \partial_+ \rho \partial_+ \rho + 2\partial_+ \lambda^{++} \partial_+ \rho, \quad (6.182)$$

where the Wess-Zumino field ρ is now called the conformal field, and transforms as

$$\delta \rho = \epsilon^+ \partial_+ \rho + \partial_+ \epsilon^+. \quad (6.183)$$

The conformal field breaks the invariance classically, however the authors [53] showed that the quantum theory described by $\mathcal{L}_s + \mathcal{L}_{ct}$ is anomaly-free. Furthermore, by a BRST analysis they found that it describes two right-moving scalars. This result will be

reestablished here in a later chapter when we generalize their analysis to the nonabelian theory of chiral scalars.

It is natural to inquire about the relationship between this theory, and the quantum theory of Floreanini and Jackiw's model which describes a single right-moving chiral scalar and is the bosonized version of a Weyl fermion. We will address this issue in chapter 8 and find that both theories appropriately coupled to background gauge and gravitational fields are the bosonization of a Weyl fermion in the same backgrounds. The conformal field can be viewed as a decoupled auxiliary field.

Another way to cancel the the anomaly of the Siegel symmetry 6.177 has been discovered by Hull [54]. It consists of adding fermionic "no-mover" fields described by lagrangian

$$\mathcal{L}_{NM} = \frac{-i}{2}(\psi\partial_-\psi + \lambda^{++}\psi\partial_+\psi). \quad (6.184)$$

It is invariant under the "Siegel transformation". Its equations of motion for the Weyl-Majorana spinor field, ψ , imply that it is a constant (which we take to be zero). However, at the quantum level this field becomes nontrivial, and in fact fifty of these spinorial no-mover fields will cancel the anomaly due to one chiral boson described by the Siegel action.

6.3 Coupling Chiral Bosons to Background Gravity

In a curved space, chirality can only be defined locally with respect to the orthonormal frames originally introduced by Cartan. Suppose our manifold has a metric $g_{\mu\nu}(x)$, in terms of which the distance ds between two infinitesimally nearby points x^μ and $x^\mu + dx^\mu$ is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \quad (6.185)$$

The metric may always be decomposed into vielbeins $e_\mu^a(x)$ (zweibeins in two dimensions) satisfying

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad (6.186)$$

so that

$$ds^2 = \eta_{ab} e^a e^b, \quad (6.187)$$

where $e^a = e_\mu^a dx^\mu$. η_{ab} is the flat metric which we take to have signature $(+1, -1)$. Latin indices refer to the orthonormal tangent frame, while greek indices refer to the coordinate frame. The vielbein e_μ^a , and its inverse E_a^μ , defined by

$$E_a^\mu(x) = \eta_{ab} g^{\mu\nu} e_\nu^b \quad (6.188)$$

and satisfying

$$E_a^\mu e_\mu^b = \delta_a^b, \quad E_a^\mu e_\nu^a = \delta_\nu^\mu, \quad (6.189)$$

are used to change tensor quantities referred to coordinate frames to tangent frames, and vice versa.

A geometrical interpretation can be given to Siegel's action as follows: Begin with the action for bosons coupled to external gravity

$$S = \int d^2x \sqrt{g} \frac{1}{2} g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x), \quad (6.190)$$

where g is the absolute value of the determinant of $g_{\mu\nu}$. Then decompose the metric into zweibeins as in 6.186, set one of the light-cone components flat and orthonormal,

$$e_\mu^-(x) \equiv \frac{1}{\sqrt{2}} (e_\mu^0 - e_\mu^1) = \delta_\mu^-, \quad (6.191)$$

and treat the other components e_μ^+ as dynamical degrees of freedom. Then with the definition

$$\lambda^{++}(x) = \frac{e_-^+(x)}{e_+^+(x)} \quad (6.192)$$

the action 6.190 becomes Siegel's action 6.176.

The gauge transformations 6.177 are then seen as coordinate transformations which preserve condition 6.191. Indeed, under the coordinate transformation

$$\delta x^\mu = \epsilon^\mu(x) \quad (6.193)$$

the zweibein transforms like a vector field,

$$\delta e_\mu^a(x) = \epsilon^\nu(x) \partial_\nu e_\mu^a(x) + \partial_\mu \epsilon^\nu(x) e_\nu^a(x), \quad (6.194)$$

and the scalar like

$$\delta \phi(x) = \epsilon^\nu(x) \partial_\nu \phi(x). \quad (6.195)$$

Condition 6.191 is preserved when $\epsilon^-(x) = 0$, in which case transformations 6.194 and 6.195 reduce to transformations 6.177. Thus Siegel's gauge symmetry is seen as freedom to make coordinate transformations on the part of the light cone in which the field is constrained to zero.

If e^a forms an orthonormal frame basis, then so does $L_b^a(x)e^b$, where L_b^a is a Lorentz matrix:

$$\eta_{cd} L_a^c L_b^d = \eta_{ab}. \quad (6.196)$$

Thus local Lorentz transformations are defined to be the above orthonormal tangent frame rotations, which in light cone coordinates are

$$\begin{aligned} e_\mu^+(x) &\rightarrow \Lambda(x) e_\mu^+(x) \\ e_\mu^-(x) &\rightarrow \Lambda^{-1}(x) e_\mu^-(x), \end{aligned} \quad (6.197)$$

where $\Lambda = L_+^\dagger = (L_-^-)^{-1}$. Siegel's action is trivially invariant under Lorentz transformations since it depends only on the manifestly invariant ratio λ^{++} . However to cancel the anomaly in Siegel's symmetry, the authors of [55] found it necessary to add the Lorentz

variant counterterm 6.182 where

$$\rho(x) = \ln e_+^+(x). \quad (6.198)$$

As observed by the authors of [55], this geometrical picture of the chiral constraint indicates a natural way to couple chiral bosons to a background two-metric. Just use the action 6.190, rewritten in terms of the zweibein, with $e_\mu^-(x)$ left as an external classical field. $\phi(x)$ and $e_\mu^+(x)$ are the dynamical fields, with $e_\mu^+(x)$ no longer thought of as gravity, but rather as a Lagrange multiplier which enforces the tangent space chiral constraint

$$E_+^\mu E_+^\nu \partial_\mu \phi \partial_\nu \phi = 0. \quad (6.199)$$

Another motivation for authors [55] to propose such a model is the fact (as we will see in the next chapter) that right-handed fermions in background gravity also only couple to $e_\mu^-(x)$.

Chapter 7

Bosonization in Background Gauge and Gravitational Fields

Here we will demonstrate, via path integral techniques, that the correspondence between a Weyl fermion and a chiral boson holds in suitably coupled arbitrary background gauge and gravitational fields. To do so, we must first review the correspondence between a Dirac fermion and a nonchiral boson. We do so in the first two sections following reference [55]. The next section reviews original work done by Sanielevici, Semenoff, and Wu in reference [55], and the final section presents original work by the present author.

7.1 The Effective Action for a Scalar in Background Fields

The action of a scalar field coupled to a background metric $g_{\mu\nu}(x)$ and vector and axial vector gauge fields $V_\mu(x)$ and $A_\mu(x)$ respectively, is

$$S_B = \int d^2x \sqrt{g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{\sqrt{\pi}} g^{\mu\nu} A_\mu \partial_\nu \phi + \frac{1}{\sqrt{\pi}} \tilde{\epsilon}^{\mu\nu} V_\mu \partial_\nu \phi \right\}, \quad (7.200)$$

where g is the absolute value of the determinant of $g_{\mu\nu}$ and

$$\tilde{\epsilon}^{\mu\nu} = \frac{1}{\sqrt{g}} \epsilon^{\mu\nu}, \quad \tilde{\epsilon}_{\mu\nu} = \sqrt{g} \epsilon_{\mu\nu} \quad (7.201)$$

is the Levi-Civita tensor, and $\epsilon^{01} = -\epsilon^{10} = -\epsilon_{01} = \epsilon_{10} = 1$ with all other components zero. The axial and vector currents are given by

$$\begin{aligned} j_A^\mu &= \frac{1}{\sqrt{g}} \frac{\delta S_B}{\delta A_\mu} = \frac{1}{\sqrt{\pi}} g^{\mu\nu} \partial_\nu \phi \\ j_V^\mu &= \frac{1}{\sqrt{g}} \frac{\delta S_B}{\delta V_\mu} = \frac{1}{\sqrt{\pi}} \tilde{\epsilon}^{\mu\nu} \partial_\nu \phi, \end{aligned} \quad (7.202)$$

and the energy momentum currents are

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{g}} \frac{\delta S_B}{\delta g^{\mu\nu}} \\ &= \partial_\mu \phi \partial_\nu \phi + \frac{1}{\sqrt{\pi}} A_\mu \partial_\nu \phi + \frac{1}{\sqrt{\pi}} A_\nu \partial_\mu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{\sqrt{\pi}} g^{\alpha\beta} A_\alpha \partial_\beta \phi \right). \end{aligned} \quad (7.203)$$

In the quantum theory, we are interested in the effective action obtained by path integrating the scalar,

$$S_B^{eff}[g, A, V] = \frac{1}{i} \ln \int d\phi(x) \exp \{i S_B[\phi, g, A, V]\}; \quad (7.204)$$

because it allows us to compute current correlation functions. For example, the vacuum expectation of the energy momentum tensor is:

$$\langle T_{\mu\nu}(x) \rangle = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}(x)} S_B^{eff}[g, A, V]. \quad (7.205)$$

By completing the square in the quadratic form 7.200 (a simple procedure documented in many texts on field theory) we obtain

$$\begin{aligned} S_B^{eff} &= \frac{1}{\pi} \int d^2x \sqrt{g} (\nabla_\mu A^\mu - \tilde{\epsilon}^{\mu\nu} \nabla_\mu V_\nu) \frac{1}{\nabla^2} (\nabla_\alpha A^\alpha - \tilde{\epsilon}^{\alpha\beta} \nabla_\alpha V_\beta) \\ &+ \frac{1}{i} \ln \int d\phi(x) \exp i S_B[\phi, g, 0, 0], \end{aligned} \quad (7.206)$$

where the coordinate covariant derivative on vectors is

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} V^\mu). \quad (7.207)$$

It was observed by Polyakov [56] that the result of the final integral is determined by the trace anomaly [57]

$$g_{\mu\nu} \langle T^{\mu\nu} \rangle [g, A = V = 0] = \frac{-1}{12\pi} R, \quad (7.208)$$

where R is the scalar curvature. This enabled him to obtain the exact effective action

$$\begin{aligned} S_B^{eff} &= \frac{-1}{96\pi} \int d^2x \sqrt{g} R \frac{1}{\nabla^2} R \\ &+ \frac{1}{\pi} \int d^2x \sqrt{g} (\nabla_\mu A^\mu - \tilde{\epsilon}^{\mu\nu} \nabla_\mu V_\nu) \frac{1}{\nabla^2} (\nabla_\alpha A^\alpha - \tilde{\epsilon}^{\alpha\beta} \nabla_\alpha V_\beta). \end{aligned} \quad (7.209)$$

In the next section it is shown that this is also the effective action for a Dirac fermion in the same background fields, regulated so as to be coordinate covariant and vector gauge invariant. In fact, the coupling to gauge fields has here been chosen so as to obtain the same axial anomaly as fermions regulated in the vector conserving scheme.

7.2 The Effective Action for a Dirac Fermion in Background Fields

The action for a Dirac fermion coupled to background gravitational and vector and axial vector gauge fields is constructed so as to be invariant under general coordinate, local Lorentz, gauge, and chiral gauge transformations. It is

$$S_D = \int d^2x |e| \bar{\psi} \gamma^a E_a^\mu \left(i \partial_\mu + \omega_{\mu ab} \sigma^{ab} + \gamma_5 A_\mu + V_\mu \right) \psi, \quad (7.210)$$

where $|e| = \det e_\mu^a = \sqrt{g}$ is the usual factor introduced to make the measure coordinate invariant. The spin connection $\omega_{\mu ab}$ can be written in two dimensions, as $\omega_{\mu ab} = \omega_\mu \epsilon_{ab}$ with

$$\omega_\mu = \frac{1}{2} e_\mu^a \tilde{\epsilon}^{\alpha\beta} \partial_\alpha e_\beta^a, \quad (7.211)$$

in terms of which the scalar curvature is

$$R = 2 \tilde{\epsilon}^{\mu\nu} \partial_\mu \omega_\nu. \quad (7.212)$$

The spin matrix is

$$\sigma^{ab} = \frac{1}{4\pi} [\gamma^a, \gamma^b] = \frac{-i}{2} \gamma_5 \epsilon^{ab}, \quad (7.213)$$

where γ^a are the flat space gamma matrices

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \quad (7.214)$$

and $\gamma_5 = i\gamma^0\gamma^1$ is the chirality matrix.

The axial vector and vector currents are given by

$$\begin{aligned} j_A^\mu &= \frac{1}{|e|} \frac{\delta S_D}{\delta A_\mu} = \bar{\psi} \gamma^\mu \gamma_5 \psi \\ j_V^\mu &= \frac{1}{|e|} \frac{\delta S_D}{\delta V_\mu} = \bar{\psi} \gamma^\mu \psi, \end{aligned} \quad (7.215)$$

and the energy-momentum currents are

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{g}} \frac{\delta S_D}{\delta g^{\mu\nu}} = \frac{1}{|e|} E_{a\mu} \frac{\delta S_D}{\delta E_a^\nu} \\ &= \frac{i}{4} \left(\bar{\psi} \gamma_\mu D_\nu \psi + \bar{\psi} \gamma_\nu D_\mu \psi - (D_\mu \bar{\psi}) \gamma_\nu \psi - (D_\nu \bar{\psi}) \gamma_\mu \psi \right), \end{aligned} \quad (7.216)$$

where $\gamma_\mu \equiv e_\mu^a \gamma_a$ are the curved space gamma matrices, and $D_\mu = \partial_\mu + A_\mu \gamma_5 + V_\mu$ is the gauge covariant derivative.

To find the effective action we first use the identity

$$\gamma^a E_a^\mu A_\mu \gamma_5 = -\gamma^a E_a^\mu \tilde{\epsilon}_{\mu\nu} A^\nu, \quad (7.217)$$

so that the action can be rewritten as

$$S_D = \int d^2x |e| \bar{\psi} \gamma^a E_a^\mu (i\partial_\mu + \omega_{\mu ab} \sigma^{ab} + V_\mu - \tilde{\epsilon}_{\mu\nu} A^\nu) \psi, \quad (7.218)$$

and the fact that a two-dimensional vector field can be written as the sum of the gradient of a scalar and the dual gradient of another scalar,

$$V_\mu - \tilde{\epsilon}_{\mu\nu} A^\nu = \partial_\mu \xi + \tilde{\epsilon}_{\mu\nu} \partial^\nu \chi, \quad (7.219)$$

where

$$\xi = \frac{1}{\nabla^2} \nabla^\mu (V_\mu - \tilde{\epsilon}_{\mu\nu} A^\nu) \quad (7.220)$$

and

$$\chi = \frac{1}{\nabla^2} \tilde{\epsilon}^{\mu\nu} \nabla_\mu (V_\nu - \tilde{\epsilon}_{\nu\lambda} A^\lambda). \quad (7.221)$$

Then performing the change of variables

$$\psi \rightarrow e^{i(\xi + \gamma_5 \chi)} \psi \quad (7.222)$$

removes the gauge fields from S_D , but reintroduces them in the path integral measure:

$$d\psi(x)d\bar{\psi}(x) \rightarrow d\psi(x)d\bar{\psi}(x)e^{2i\text{Tr}\gamma_5\chi}, \quad (7.223)$$

where the trace in the exponent is over the function space on which ψ and $\bar{\psi}$ are defined.

Using a vector gauge invariant heat-kernel regularization, Fujikawa [24] obtained

$$\begin{aligned} S_D^{eff} &= \frac{1}{i} \ln \int d\psi(x)d\bar{\psi}(x) \exp iS_D [e_\mu^a, A, V] \\ &= \frac{1}{\pi} \int d^2x \sqrt{g} (\nabla_\mu A^\mu - \tilde{\epsilon}^{\mu\nu} \nabla_\mu V_\nu) \frac{1}{\nabla^2} (\nabla_\alpha A^\alpha - \tilde{\epsilon}^{\alpha\beta} \nabla_\alpha V_\beta) \\ &\quad + \frac{1}{i} \ln \int d\psi(x)d\bar{\psi}(x) \exp iS_D [\psi, g, A = V = 0]. \end{aligned} \quad (7.224)$$

As in the case of bosons, the result of the final integration can be deduced from the trace anomaly of the energy momentum tensor; and the resulting effective action is identical to 7.209 for the scalar field. This implies that the correlation functions of the vector and axial vector currents 7.202 and 7.215 are identical, thus generalizing the identification 4.98 to curved space. The correspondence between the bosonic and fermionic energy-momentum tensors 7.203 and 7.216 generalizes the Sommerfeld-Sugawara formula 4.90 for free fields to those in background gauge and gravitational fields. In the next section we will use the expression 7.209 for the effective action to obtain the effective action for chiral fermions. This will also turn out to be the effective action for chiral bosons.

7.3 The Chiral Effective Action

As we have alluded to in the previous chapter, the action for a Dirac fermion splits into a sum of left- and right-handed actions. In fact, using the definitions

$$A_\mu^R = V_\mu + A_\mu, \quad A_\mu^L = V_\mu - A_\mu \quad (7.225)$$

$$\psi_L = \frac{1}{2}(1 + i\gamma_5)\psi, \quad \psi_R = \frac{1}{2}(1 - i\gamma_5)\psi, \quad (7.226)$$

along with the identity

$$|e|E_a^\mu = -\epsilon^{\mu\nu}\epsilon_{ab}e_\nu^b, \quad (7.227)$$

and the fact that the Minkowski metric in light-cone coordinates is

$$\eta_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (7.228)$$

the action 7.210 becomes

$$\begin{aligned} S_D &= S_L + S_R \\ &= \sqrt{2} \int d^2x \left\{ e_\mu^- \epsilon^{\mu\nu} \psi_R^\dagger (i\vec{\partial}_\nu + A_\nu^R) \psi_R - e_\mu^+ \epsilon^{\mu\nu} \psi_L^\dagger (i\vec{\partial}_\nu + A_\nu^L) \psi_L \right\}, \end{aligned} \quad (7.229)$$

where $\vec{\partial} = \frac{1}{2}(\vec{\partial} - \vec{\partial})$ and acts only on the spinors (in the above expression).

This implies that the Dirac fermion determinant must factorize into Weyl determinants which depend only on the background fields which couple to each of the Weyl operators:

$$\begin{aligned} \exp iS_D^{eff} [e_\mu^a, V_\mu, A_\mu] &= \int d\psi_R(x) d\psi_R^\dagger(x) \exp iS_R \times \int d\psi_L(x) d\psi_L^\dagger(x) \exp iS_L \\ &= \det \left(\sqrt{2}i(i\nabla_{\frac{1}{2}}^+ + A^{+L}) \right) \times \det \left(\sqrt{2}i(i\nabla_{\frac{1}{2}}^- + A^{-R}) \right) \times \exp iS_{CT}, \end{aligned} \quad (7.230)$$

where $A^{\pm L,R} = \pm e_\mu^\pm \epsilon^{\mu\nu} A_\nu^{L,R}$, $\nabla_{\frac{1}{2}}^\pm = \pm(e_\mu^\pm \epsilon^{\mu\nu} \partial_\nu - \frac{1}{2}\epsilon^{\mu\nu}(\partial_\mu e_\nu^\pm))$, and S_{CT} are some counterterms depending on the backgrounds. This factorization implies that the effective action can be written as a sum of two nonlocal terms which are functionals exclusively of those background fields which couple to each Weyl field, and the local counterterms:

$$S_D^{eff} [e_\mu^a, V_\mu, A_\mu] = \Gamma^+[e_\mu^+, A^{+L}] + \Gamma^-[e_\mu^-, A^{-R}] - S_{CT}, \quad (7.231)$$

where Γ^+ and Γ^- are the effective actions for left- and right-handed Weyl fermions respectively:

$$\Gamma^+[e_\mu^+, A^{+L}] = \frac{1}{i} \ln \det \left\{ \sqrt{2}i \left(i\nabla_{\frac{1}{2}}^+ + A^{+L} \right) \right\} \quad (7.232)$$

$$\Gamma^-[e_\mu^-, A^{-R}] = \frac{1}{i} \ln \det \left\{ \sqrt{2}i \left(i\nabla_{\frac{1}{2}}^- + A^{-R} \right) \right\}. \quad (7.233)$$

And sure enough, the authors of [55] have found that use of the variables

$$\begin{aligned} \lambda_{++} &= \frac{e_+^-}{e_-^-}, & \lambda_{--} &= \frac{e_-^+}{e_+^+} \\ \rho &= \ln e_+^+, & \sigma &= \ln e_-^- \end{aligned} \quad (7.234)$$

in the effective action 7.209 confirms this and yields the chiral effective actions

$$\begin{aligned} \Gamma^+[e_\mu^+, A^{+L}] &= -\frac{1}{48\pi} \int d^2x \left(\partial_+^2 \lambda_{--} \right) \frac{1}{\partial_+ \partial_- - \partial_+ \lambda_{--} \partial_+} \left(\partial_+^2 \lambda_{--} \right) + \\ &+ \frac{1}{2\pi} \int d^2x \left(\frac{A^{+L}}{e_+^+} \right) \bar{\partial}_+ \frac{1}{\partial_+ \partial_- - \partial_+ \lambda_{--} \partial_+} \bar{\partial}_+ \left(\frac{A^{+L}}{e_+^+} \right) \end{aligned} \quad (7.235)$$

$$\begin{aligned} \Gamma^-[e_\mu^-, A^{-R}] &= -\frac{1}{48\pi} \int d^2x \left(\partial_-^2 \lambda_{++} \right) \frac{1}{\partial_+ \partial_- - \partial_- \lambda_{++} \partial_-} \left(\partial_-^2 \lambda_{++} \right) + \\ &+ \frac{1}{2\pi} \int d^2x \left(\frac{A^{-R}}{e_-^-} \right) \bar{\partial}_- \frac{1}{\partial_+ \partial_- - \partial_- \lambda_{++} \partial_-} \bar{\partial}_- \left(\frac{A^{-R}}{e_-^-} \right). \end{aligned} \quad (7.236)$$

and the local counterterms

$$\begin{aligned} S_{CT} &= \frac{-1}{96\pi} \int d^2x \left\{ -2\partial_+ \lambda_{--} \frac{1}{1 - \lambda_{++} \lambda_{--}} \partial_- \lambda_{++} + \partial_- \lambda_{++} \frac{\lambda_{--}}{1 - \lambda_{--} \lambda_{++}} \partial_+ \lambda_{--} \right. \\ &+ \partial_+ \lambda_{--} \frac{\lambda_{++}}{1 - \lambda_{--} \lambda_{++}} \partial_+ \lambda_{--} + \frac{1}{2}(\rho + \sigma) |e| \nabla^2 (\rho + \sigma) \\ &+ \frac{2}{1 - \lambda_{++} \lambda_{--}} ((\partial_+ \lambda_{--})(\partial_+ - \lambda_{++} \partial_-) + (\partial_- \lambda_{++})(\partial_- - \lambda_{--} \partial_+))(\rho + \sigma) \\ &\left. + \frac{1}{2\pi(1 - \lambda_{++} \lambda_{--})} \left(\left(\frac{A^{-R}}{e_-^-} \right)^2 \lambda_{--} + \left(\frac{A^{+L}}{e_+^+} \right)^2 \lambda_{++} + 2 \left(\frac{A^{-R}}{e_-^-} \right) \left(\frac{A^{+L}}{e_+^+} \right) \right) \right\}. \end{aligned} \quad (7.237)$$

Local counterterms in an effective action may be thought of as parametrizing the choice of regularization of the measure in the path integral. Once the essential nonlocal part of the effective action containing the propagators is determined, any local polynomial in the external fields and their derivatives may be added. Usually one adds the minimum number of terms required to preserve a maximal subset of the classical symmetries. The

counterterms S_{CT} above are those which we must add to $\Gamma^+[e_\mu^+, A^{+L}] + \Gamma^-[e_\mu^-, A^{-R}]$ to obtain the effective action 7.209 for a Dirac fermion which is coordinate, local Lorentz, and vector gauge invariant. It is not possible to add further counterterms which make this effective action invariant under chiral gauge transformations, unless we sacrifice vector gauge invariance. Similarly, we cannot add counterterms which restore the Weyl invariance (ie remove the trace anomaly) unless we give up general coordinate invariance.

The chiral effective actions 7.235, 7.236 are trivially invariant under the local Lorentz transformations 6.195, and also under the Weyl transformations

$$e_\mu^a(x) \rightarrow \kappa(x) e_\mu^a(x). \quad (7.238)$$

However, they are not invariant under general coordinate transformations $\delta x^\mu = \epsilon^\mu(x)$, whereby coordinate indices transform according to the Lie derivative. The general coordinate invariance of $\Gamma^-[e_\mu^-, A^{-R}]$ can be restored by adding local counterterms which break the Lorentz and Weyl invariance. In fact the chiral effective action computed by Leutwyler [58]

$$\begin{aligned} \tilde{\Gamma}^-[e_\mu^a, A_\mu, V_\mu] &= \frac{-1}{96\pi} \int d^2x \frac{|e|}{2} R \frac{1}{\nabla^2} \nabla_\mu (g^{\mu\nu} + \tilde{\epsilon}^{\mu\nu}) \omega_\nu \\ &+ \frac{1}{\pi} \int d^2x |e| \nabla_\mu (g^{\mu\nu} + \tilde{\epsilon}^{\mu\nu}) A_\nu^R \frac{1}{\nabla^2} \nabla_\alpha (g^{\alpha\beta} + \tilde{\epsilon}^{\alpha\beta}) A_\beta^R \end{aligned} \quad (7.239)$$

differs from $\Gamma^-[e_\mu^-, A^{-R}]$ by local counterterms [55]. This effective action is invariant under general coordinate transformations, and has the covariant Lorentz anomaly

$$\tilde{\Gamma}^-[\Lambda e_\mu^+, \Lambda^{-1} e_\mu^-, \Lambda A^{+L}, \Lambda^{-1} A^{-R}] = \tilde{\Gamma}^-[e_\mu^+, e_\mu^-, A^{+L}, A^{-R}] - \frac{1}{96\pi} \int d^2x |e| R(x) \ln \Lambda(x). \quad (7.240)$$

This is an example in two dimensions of the general method in $2n$ dimensions of obtaining the Lorentz anomaly from the Einstein (general coordinate) anomaly which has been given by Bardeen and Zumino [59].

Although there is no obvious local and Lorentz invariant decomposition of the classical bosonic action 7.200 into left-moving and right-moving parts, since 7.231 is also the effective action for bosons, the boson partition function also factorizes. We will use this fact in the next section to define chiral Bose theories with the chiral effective action $\Gamma^-[e_\mu^-, A^{-R}]$.

7.4 Two Methods of Chiral Bosonization

In the last chapter we saw that there are two ways of obtaining a theory of chiral bosons, in two spacetime dimensions, from the non-chiral theory defined by Lagrange density $\partial_+\phi\partial_-\phi$. One way is to set one light-cone component of the gauge current to zero, that is impose the linear chiral constraint $J_+ = \partial_+\phi = 0$. The other way is to set one light-cone component of the energy-momentum tensor current to zero, that is impose the quadratic chiral constraint $T_{++} = (\partial_+\phi)^2 = 0$. Although the linear constraint is second class, the quantum theory obtained does indeed describe a single right-mover. The quadratic constraint although first class classically, is second class at the quantum level. To cancel this anomaly we must give the quantum theory another 1/2 degree of freedom, hence it describes two right-movers. Even though these two theories have different spectra, we will prove that suitably regularized they have equal correlation functions of J_- and T_{--} . Furthermore this holds in both external gauge and external gravitational fields for suitably defined modifications of the gauge and energy momentum currents. These correlation functions are those of the gauge and energy momentum currents of Weyl fermions in the same backgrounds.

Consider the theory defined by S_B , given by lagrangian 7.200 plus S_{CT} , given by equation 7.237, with ϕ and e_μ^+ considered to be dynamical and A_μ, V_μ, e_μ^- left as external. That the action be stationary with respect to variations of e_μ^+ gives the constraint $T_{++} = 0$

ie

$$E_+^\mu E_+^\nu (\partial_\mu \phi \partial_\nu \phi + \frac{2}{\sqrt{\pi}} A_\mu \partial_\nu \phi + T_{CT\mu\nu}) = 0 \quad (7.241)$$

where T_{CT} is the part of the energy-momentum tensor due to S_{CT} and is of order $\hbar$. With the external fields set zero $S_{CT} = 0$, $T_{++} = (\partial_+ \phi)^2$, and the theory is just the one considered in section 6.2 :

$$S_B = \int (\partial_+ \phi \partial_- \phi + \lambda_{--} (\partial_+ \phi)^2 + \mathcal{L}_{ct}) d^2 x. \quad (7.242)$$

Where $\mathcal{L}_{ct}$ is given by 6.182 and depends only on e_μ^+ , which will here be integrated out shortly. Recall $\mathcal{L}_{ct}$ is precisely the counterterm we need to cancel the anomaly in the Siegel symmetry.

Consider again the theory defined by S_B of lagrangian 7.200 plus S_{CT} , except this time take ϕ and A^{+L} to be dynamical and leave e_μ^a and A^{-R} external. That the action be stationary with respect to variations of A^{+L} gives the constraint $J_+ = 0$ ie

$$E_+^\mu (\partial_\mu \phi + J_{CT\mu}) = 0, \quad (7.243)$$

where J_{CT} is the part of the gauge current due to S_{CT} and is of order $\hbar$. With the external fields set zero $S_{CT} = 0$, $J_+ = \partial_+ \phi$ and the theory is just the one considered in section 6.1

$$S_B = \int \partial_+ \phi \partial_- \phi + \frac{1}{\pi} A^{+L} \partial_+ \phi. \quad (7.244)$$

This is missing the $(A^{+L})^2/4\pi^2$ counterterm which we could now add, but will be integrated out shortly.

We will now obtain the generating functional $W_1[e_\mu^-, A^{-R}]$ for the latter of these two theories by path integrating $S_B + S_{CT}$ over ϕ and A^{+L} . First note that from equation 7.231 that

$$\begin{aligned} & \int d\phi(x) \exp \left\{ iS_B[e_\mu^a, A_\mu, V_\mu] + iS_{CT}[e_\mu^a, A_\mu, V_\mu] \right\} \\ &= \exp \left\{ i\Gamma^+[e_\mu^+, A^{+L}] + i\Gamma^-[e_\mu^-, A^{-R}] \right\}. \end{aligned} \quad (7.245)$$

Therefore integrating this over A^{+L} yields (up to an overall constant and a change of measure induced local counterterm due to rescaling A^{+L} by e_+^+)

$$W_1[e_\mu^-, A^{-R}] = \exp \left\{ i\Gamma^-[e_\mu^-, A^{-R}] \right\} \exp \left\{ i\Gamma^+[e_\mu^+, A^{+L} = 0] \right\} \det^{\frac{1}{2}}(\partial_+\partial_- - \partial_+\lambda_{--}\partial_+). \quad (7.246)$$

However

$$\begin{aligned} \exp \left\{ i\Gamma^+[e_\mu^+, A^{+L} = 0] \right\} &= \int d\phi(x) \exp \left\{ iS_B[e_\mu^+, e_\mu^- = \delta_\mu^-, A_\mu = V_\mu = 0] \right\} \\ &= \int d\phi(x) \exp \left\{ -i \int d^2x \phi (\partial_+\partial_- - \partial_+\lambda_{--}\partial_+) \phi \right\} \\ &= \det^{-1/2}(\partial_+\partial_- - \partial_+\lambda_{--}\partial_+). \end{aligned} \quad (7.247)$$

So that

$$W_1[e_\mu^-, A^{-R}] = \exp \left\{ i\Gamma^-[e_\mu^-, A^{-R}] \right\}, \quad (7.248)$$

which is the generating functional for Weyl fermions. Correlation functions for the chiral gauge and energy-momentum currents are given by

$$\begin{aligned} &< J_-(x_1) \dots J_-(x_m) T_-^{\mu_1}(y_1) \dots T_-^{\mu_n}(y_n) >= \\ &\left\{ \frac{\delta}{\delta A^{-R}(x_1)} \dots \frac{\delta}{\delta A^{-R}(x_m)} \frac{\delta}{\delta e_{\mu_1}^-(y_1)} \dots \frac{\delta}{\delta e_{\mu_n}^-(y_n)} \right\} \Gamma^-[e_\mu^-, A^{-R}]. \end{aligned} \quad (7.249)$$

The chiral Bose theory with constraint equation 7.241 is obtained by path integrating $S_B + S_{CT}$ over ϕ and e_μ^+ . Integrating equation 7.245 over e_μ^+ yields (up to an overall constant and a local counterterm due to rescaling e_μ^+ by A^{+L}) once again the generating functional $W_1[e_\mu^-, A^{-R}]$ given by equation 7.248. Again correlation functions are given by equation 7.249.

In conclusion we have two ways of defining chiral bosons which have identical correlation functions (namely those of Weyl fermions) for the chiral gauge and energy-momentum currents. These two theories of chiral bosons have been gauged and made coordinate covariant in such a manner as to obtain the same gauge and gravitational

anomalies as Weyl fermions regulated in the decoupled left-right scheme. In zero backgrounds this amounts to using $\partial_+ \phi = 0$ or $(\partial_+ \phi)^2 = 0$ as a constraint on the left-right symmetric scalar theory.

Chapter 8

Nonabelian Bosonization

Both theories of elementary particles and theories of condensed matter often have several species of fermions. These theories usually are invariant under “rotations” of the fermion species into one another, hence any bosonization procedure should preserve this symmetry. Nonabelian bosonization of fermion theories leads to various Bose theories, depending on which symmetries are to be preserved. We will begin by describing the procedure for free fermions, due to Witten, and progress to theories in background fields (this generalization is due to a number of authors). As in the previous chapter, the chirally symmetric theories in backgrounds lead naturally to two chirally asymmetric bosonic theories in the same backgrounds. One of these theories was proposed in [55], the other proposal is due to the present author and is unpublished.

8.1 Bosonization of Free Fermions

The lagrangian for a single free massless Dirac fermion split into its Weyl components is

$$\mathcal{L} = i\sqrt{2}\psi_-^\dagger \partial_+ \psi_- + i\sqrt{2}\psi_+^\dagger \partial_- \psi_+. \quad (8.250)$$

This form makes obvious the statements of vector and axial vector current conservation, which in light cone coordinates are

$$\partial_+ J_- = 0 = \partial_- J_+, \quad (8.251)$$

where

$$J_- = \psi_-^\dagger \psi_- \quad , \quad J_+ = \psi_+^\dagger \psi_+ \quad (8.252)$$

are the light cone components of the vector current $J^\mu = \bar{\psi}\gamma^\mu\psi$.

The bosonization of this theory amounted to the identifications

$$J_- = \frac{1}{\sqrt{\pi}}\partial_-\phi \quad J_+ = \frac{-1}{\sqrt{\pi}}\partial_+\phi, \quad (8.253)$$

where ϕ is a scalar field with dynamics determined by the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi, \quad (8.254)$$

which has the correct equations of motion for the currents, namely $\partial_+\partial_-\phi = 0$.

What made these identifications possible were the anomalous Fermi current commutators

$$[J_\pm(x), J_\pm(y)] = \frac{\pm i}{2\pi}\delta'(x-y), \quad (8.255)$$

and the Sommerfeld-Sugawara formula for the Fermi energy momentum tensor, which in light cone coordinates is

$$\begin{aligned} T_{++} &= \pi J_+ J_+ \\ T_{--} &= \pi J_- J_- \end{aligned} \quad (8.256)$$

Both the commutators and the expressions for the energy-momentum follow canonically from the free massless scalar theory.

Consider generalizing this to a free massless fermion theory with N species of fermions.

$$\mathcal{L} = i\sqrt{2}\psi_-^{k\dagger}\partial_+\psi_-^k - i\sqrt{2}\psi_+^{k\dagger}\partial_-\psi_+^k. \quad (8.257)$$

We could just introduce a separate boson, ϕ^k , for each fermion. However, only the diagonal fermion bilinears such as $\psi_-^{k\dagger}\psi_-^k$ would have a simple bosonized form. The off-diagonal fermion bilinears would be complicated and non-local, making it difficult to extend the bosonization to interacting theories. Furthermore, this theory would not have the chiral $U(N) \times U(N)$ symmetry of the lagrangian 8.257

$$\psi_-^k \rightarrow U^{kj}\psi_-^j, \quad \psi_+^k \rightarrow V^{kj}\psi_+^j, \quad (8.258)$$

with $U, V \in U(N)$.

An alternate bosonization procedure which keeps manifest the full chiral $U(N) \times U(N)$ symmetry has been introduced by Witten [60] in 1984. Since then his method has been generalized to theories of free massless Fermi fields which form a representation of any nonabelian group $\mathcal{G}$ which satisfies the criterion of Goddard, Nahm, and Olive [61]. For example, if the Weyl fermions had a Majorana condition imposed on them (ie $\psi_{\pm}^{\dagger} = \psi_{\pm}$), the symmetry would be $SO(N) \times SO(N)$, and $\mathcal{G}$ would be $SO(N)$. For another example, suppose the Weyl fermions had two indices, say flavour $SU(N)$ and color $SU(M)$. Then to preserve the full chiral $(SU(N) \otimes SU(M)) \times (SU(N) \otimes SU(M))$ symmetry we would take $\mathcal{G} = SU(N) \otimes SU(M)$; however, if we wished (as is often the case in applications) we could preserve only a subgroup of this symmetry by taking, for example, $\mathcal{G} = SU(N) \oplus SU(M)$. Let us first take $\mathcal{G}$ to be either $U(1)$ or simple, and later explain what to do with semisimple groups, and groups with $U(1)$ factors. As in the single species $U(1)$ theory, the rules are derived by focussing on the currents.

The chiral $\mathcal{G}_+$ and chiral $\mathcal{G}_-$ transformations (ie the transformations for the positive and negative chirality fermions) are generated by the $\mathcal{G}$ algebra valued currents

$$\begin{aligned} J_+^a &= \psi_+^{i\dagger} (T^a)_{ij} \psi_+^j \\ J_-^a &= \psi_-^{i\dagger} (T^a)_{ij} \psi_-^j, \end{aligned} \quad (8.259)$$

where the matrices T^a are a basis of generators of $\mathcal{G}$, normalized such that

$$\text{Tr} T^a T^b = \frac{1}{2} \delta^{ab}. \quad (8.260)$$

For the case $\mathcal{G} = U(1)$, $T^a = \frac{1}{\sqrt{2}}$, the theory is just the previous single fermion theory. Given lagrangian 8.257 it is clear that the field equations for the currents are

$$\partial_- J_+^a = 0 = \partial_+ J_-^a. \quad (8.261)$$

The nonabelian generalization of the anomalous current commutators have been obtained in various places [62, 63] and are

$$\begin{aligned} [J_{\pm}^a(x), J_{\pm}^b(y)] &= if^{abc} J_{\pm}^c(x) \delta(x-y) \pm \frac{ik}{4\pi} \delta^{ab} \delta'(x-y), \\ [J_+^a(x), J_-^b(y)] &= 0, \end{aligned} \quad (8.262)$$

where f^{abc} are the structure constants for the algebra.

$$[T^a, T^b] = if^{abc} T^c \quad (8.263)$$

This current algebra is two commuting copies of the level k Kac-Moody algebra associated to $\mathcal{G}$ [64]. With the choice of normalization 8.260, k is equal to one.

The nonabelian generalization of the Sommerfeld-Sugawara formula is also available [65]. It is

$$T_{++} = \frac{2\pi}{1+q} J_+^a J_+^a, \quad (8.264)$$

with a similar expression for T_{--} . q is the quadratic Casimir constant for the adjoint representation of $\mathcal{G}$:

$$q\delta^{ab} = f^{acd} f^{bcd}. \quad (8.265)$$

The multiplicative factor $(1+q)^{-1}$ was missed in the original work of Sugawara, but had later been seen to arise upon careful consideration of normal ordering in the regularized expression for the energy-momentum tensor [66].

To represent the $\mathcal{G}$ algebra valued currents 8.259 by a bosonic field, Witten proposed that we write

$$\begin{aligned} J_+ &= \frac{i}{4\pi} G^{-1} \partial_+ G = J_+^a T^a \\ J_- &= \frac{-i}{4\pi} (\partial_- G) G^{-1} = J_-^a T^a, \end{aligned} \quad (8.266)$$

where the field G takes values in the group $\mathcal{G}$. Then the chiral $\mathcal{G} \times \mathcal{G}$ transformations will act on G by matrix multiplication

$$G \rightarrow UGV^{-1}, \quad (8.267)$$

Figure 8.5: A mapping from S^2 into the group manifold $\mathcal{G}$.

with $U, V \in \mathcal{G}$. The factor ordering in 8.266 is important, for it insures the compatibility of the equations $\partial_- J_+ = 0 = \partial_+ J_-$.

We might be tempted to choose the principal σ -model associated with $\mathcal{G}$ to be the action governing G :

$$L_o = \frac{1}{4\lambda^2} \int d^2x \text{Tr} \partial_\mu G^{-1} \partial^\mu G. \quad (8.268)$$

However, to obtain the desired field equations and commutators for the currents, we must add the Wess-Zumino functional W .

The definition of W is topological. By choosing the appropriate boundary conditions we may think of G as a mapping from a large two-sphere, S^2 , into the manifold $\mathcal{G}$. This S^2 can be considered to be the boundary of a ball in R^3 , and provided that $\Pi_2(\mathcal{G}) = 0$, the mapping G can be extended to a mapping of B into $\mathcal{G}$. If y_1, y_2, y_3 are coordinates for B , the Wess-Zumino functional is

$$W = \frac{1}{12\pi} \int d^3y \epsilon^{ABC} \text{Tr} (G^{-1} \partial_A G G^{-1} \partial_B G G^{-1} \partial_C G). \quad (8.269)$$

Locally the integrand can be written as a total divergence, so by use of Stoke's theorem

we can write W as an integral over ordinary spacetime. The resulting integrand is local, but complicated and nonpolynomial.

W is only well defined modulo $W \rightarrow W + 2\pi$ because of the existence of topologically inequivalent ways to extend G into a mapping on B . However, the lagrangian

$$L = L_o + nW \quad (8.270)$$

gives a well defined path integral for n an integer.

This lagrangian does give the desired field equations 8.261 when $\lambda^2 = 2\pi/n$. Furthermore, for this value of λ^2 , the canonical commutators for the currents are the level n Kac-Moody algebra 8.262. Thus we should take $n = 1$ to reproduce the algebra of the free fermions. Finally, the energy-momentum tensor canonically derived from L , after a multiplicative renormalization by $(1+q)^{-1}$, is given by 8.264.

Since the free fermion theory and the Wess-Zumino-Novikov-Witten (WZNW) theory have the same currents, obeying the same algebra, and have the same energy-momentum tensor quadratic in the currents, they are equivalent. The equivalence goes beyond the currents, because in both theories there are no operators which commute with all the currents [67, 68]. Hence the Hilbert space of the theory, in both cases, can be built as an irreducible representation of the current algebra. Furthermore, for $k = \pm 1$ and a particular finite choice for the total energy-momentum of the vacuum (so as to obtain a highest weight representation) the Kac-Moody representation is essentially unique.

In the case that $\mathcal{G}$ is the direct sum of simple groups satisfying the criterion of Goddard, Nahm, and Olive, we just introduce a separate WZNW field for each group in the sum. Currents from the different factors commute, and the energy-momentum tensor is just a sum of terms like 8.264. In the case that the otherwise simple group has a $U(1)$ factor, everything goes as in the case of a simple group, except that the energy-momentum tensor, which turns out to be a sum of abelian and nonabelian parts, requires the $(1+q)^{-1}$

multiplicative renormalization only in its nonabelian part. This will become clear in the next chapter.

8.2 Nonabelian Bosonization in Background Gauge Fields

Nonabelian bosonization can be extended to theories in background gauge and gravitational fields by a generalization of the procedure for a single Dirac fermion. Consider first the theory of N Dirac fermions, in Minkowski space, coupled to background $U(N)$ axial vector and vector gauge fields:

$$S_D = \int d^2x \left\{ \psi_R^\dagger (i\vec{\partial}_+ + A_+^R) \psi_R - \psi_L^\dagger (i\vec{\partial}_- + A_-^L) \psi_L \right\}, \quad (8.271)$$

where $\psi_{R,L}$ now denote N -tuples of Weyl fermions. The notation is the same as in previous chapters – for example $A_\mu^R = (A_\mu^R)^a T^a = V_\mu + A_\mu$.

The effective action for this lagrangian can be determined exactly [69, 70], and in terms of the Wess-Zumino-Novikov-Witten action

$$S[G] = \frac{1}{8\pi} \int d^2x \operatorname{Tr} \partial_\mu G^\dagger \partial^\mu G + \frac{1}{12\pi} \int d^3x \epsilon^{ABC} \operatorname{Tr} (G^\dagger \partial_A G G^\dagger \partial_B G G^\dagger \partial_C G) \quad (8.272)$$

is

$$S^{eff}[A^{R,L}] = -S[BA^\dagger] + \int d^2x \left(\alpha \operatorname{Tr} A_+^R A_-^L + \beta \operatorname{Tr} A_+^R \operatorname{Tr} A_-^L \right), \quad (8.273)$$

where the $U(N)$ matrices A and B are solutions to

$$\begin{aligned} iA_-^L &= A^\dagger \partial_- A \\ iA_+^R &= B^\dagger \partial_+ B. \end{aligned} \quad (8.274)$$

The parameters α, β depend on the regularization used in performing the functional integral. Under the vector gauge transformations

$$\begin{aligned} A_-^L &\rightarrow g^{-1}(A_-^L + \partial_-)g \\ A_+^R &\rightarrow g^{-1}(A_+^R + \partial_+)g, \end{aligned} \quad (8.275)$$

the matrices A, B transform as

$$A \rightarrow Ag, \quad B \rightarrow Bg, \quad (8.276)$$

so that with the choice $\alpha = \beta = 0$ the effective action is vector gauge invariant. However, then it is not invariant under the chiral gauge transformations

$$\begin{aligned} A_-^L &\rightarrow g(A_-^L + \partial_-)g^{-1} \\ A_+^R &\rightarrow g^{-1}(A_+^R + \partial_+)g \end{aligned} \quad (8.277)$$

under which the matrices A, B transform as

$$A \rightarrow Ag^{-1}, \quad B \rightarrow Bg. \quad (8.278)$$

By use of the identity

$$S[BA^\dagger] = S[B] + S[A^{-1}] - \frac{1}{4\pi} \int d^2x \operatorname{Tr} A_+^R A_-^L, \quad (8.279)$$

and the choice $\alpha = 1/4\pi$, $\beta = 0$ we obtain

$$S^{eff}[A^{L,R}] = -S[B] - S[A^\dagger], \quad (8.280)$$

which corresponds to a regularization which decouples the right and left sectors. A bosonic theory which has the same effective action is [71]

$$S_B = S[G] + \frac{1}{4\pi} \operatorname{Tr} \int d^2x (G^\dagger \partial_+ G A_-^L - G \partial_- G^\dagger A_+^R). \quad (8.281)$$

By taking functional derivatives of the effective action with respect to $(A_-^L)^a$ and $(A_+^R)^a$ we obtain correlation functions for either the currents 8.259 of the fermionic theory, or the currents 8.266 of the bosonic theory which are identical. For example

$$\langle J_-^a(x) \rangle = \frac{\delta}{\delta(A_+^R)^a(x)} S^{eff}[A^{L,R}] \quad (8.282)$$

where

$$e^{iS^{eff}[A^{L,R}]} = \int d\psi_R d\psi_L e^{iS_D} = \int dG e^{iS_B}. \quad (8.283)$$

This generalizes Witten's identifications for the currents in the free theory, to the theory coupled to background gauge fields. It also suggests that to obtain a chiral theory of nonabelian bosons, say right-movers, we should elevate the status of A_-^L to a dynamical field, and leave A_+^R external. However, as in the previous chapter, we have not dealt with the gauge fixing of this theory, and as we know from a canonical analysis of the abelian theory, this can present problems. Thus the purpose of the next generalization, namely including gravity, is two-fold. It will extend nonabelian bosonization to theories which are gauged and coordinate invariant, and will allow us to define nonabelian chiral theories in these backgrounds that use the quadratic chiral constraint.

8.3 Chiral Nonabelian Bosonization in Background Fields

As in the abelian theory, the various actions in curved space are obtained by covariantizing the flat space expressions [72]. The nonabelian coordinate covariant Dirac action is

$$S_D = \sqrt{2} \int d^2x \left\{ e_-^\mu \epsilon^{\mu\nu} \psi_R^\dagger (i\vec{\partial}_\nu + (A_\nu^R)^a T^a) \psi_R - e_+^\mu \epsilon^{\mu\nu} \psi_L^\dagger (i\vec{\partial}_\nu + (A_\nu^L)^a T^a) \psi_L \right\}, \quad (8.284)$$

and the WZNW action 8.281 coupled to background gravity is

$$\begin{aligned} S_B &= \frac{1}{8\pi} \int d^2x \sqrt{g} g^{\mu\nu} \text{Tr} \partial_\mu G^\dagger \partial_\nu G + \frac{1}{12\pi} \int d^3x \epsilon^{ABC} \text{Tr} G^\dagger \partial_A G G^\dagger \partial_B G G^\dagger \partial_C G \\ &+ \frac{1}{4\pi} \int d^2x \sqrt{g} \left\{ (g^{\mu\nu} + \tilde{\epsilon}^{\mu\nu}) \text{Tr} G^\dagger \partial_\mu G A_\nu^L - (g^{\mu\nu} - \tilde{\epsilon}^{\mu\nu}) \text{Tr} G \partial_\mu G^\dagger A_\nu^R \right\}, \end{aligned} \quad (8.285)$$

where both $A^{R,L}$ and G take values in the fundamental representation of $U(N)$.

In [55] it was shown that the gauge fields $A^{R,L}$ can be represented by $U(N)$ matrices A, B similarly as in 8.274

$$\begin{aligned} (g^{\mu\nu} + \tilde{\epsilon}^{\mu\nu}) A_\nu^L(x) &= (g^{\mu\nu} + \tilde{\epsilon}^{\mu\nu}) A^\dagger(x) \partial_\nu A(x) \\ (g^{\mu\nu} - \tilde{\epsilon}^{\mu\nu}) A_\nu^R(x) &= (g^{\mu\nu} - \tilde{\epsilon}^{\mu\nu}) B^\dagger(x) \partial_\nu B(x) \end{aligned} \quad (8.286)$$

where $A(x)$ depends only on A^{+L}/e_+^+ and λ_{--} , and $B(x)$ depends only on A^{-R}/e_-^- and λ_{++} . These authors also showed that the quantum field theories governed by 8.284 and 8.285 can be regulated so that they have the same effective action

$$S^{eff}[g, A^{L,R}] = \frac{-N}{96\pi} \int d^2x \sqrt{g} R \frac{1}{\nabla^2} R - S[B] - S[A^\dagger], \quad (8.287)$$

where $S[G]$ is given by 8.285 with the gauge fields set to zero. This implies that all correlation functions of the gauge and energy momentum currents are equal, thus generalizing Witten's nonabelian bosonization to gauged fields in curved spacetime.

It turns that the nonlocal part of $S[A^\dagger]$ depends only on A^{+L}/e_+^+ and λ_{--} , and the nonlocal part of $S[B]$ depends only on A^{-R}/e_-^- and λ_{++} . This allows the definition of a chiral nonabelian Bose theory with a quadratic constraint, similarly as in the previous chapter.

Consider the theory defined by the bosonic action 8.285 with G and e_μ^+ considered to be dynamical, and $A_\mu^{L,R}$ and e_μ^- left as external. Demanding that the action be stationary with respect to variations of e_μ^+ constrains the positive light cone component of the energy-momentum to zero:

$$E_+^\mu E_+^\nu \text{Tr} \left(\partial_\mu G^\dagger \partial_\nu G + 2G^\dagger \partial_\mu G A_\nu^L - 2G \partial_\mu G^\dagger A_\nu^R + T_{CT\mu\nu} \right) = 0, \quad (8.288)$$

where $T_{CT\mu\nu}$ is due to the local counterterms, S_{CT} , that must be added to 8.285 so that it yields the effective action

$$\tilde{S}^{eff}[g, A^{L,R}] = N\Gamma^+[e_\mu^+, 0] + N\Gamma^-[e_\mu^-, 0] - \tilde{S}[A^\dagger] - \tilde{S}[B], \quad (8.289)$$

and $\tilde{S}[A]$ is the nonlocal part of $S[A]$. Then path integrating over G and e_μ^+

$$\int dG(x) de_\mu^+(x) e^{iS_B[G, e_\mu^a, A^{L,R}] + S_{CT}[e_\mu^a, A^{L,R}]} \quad (8.290)$$

yields (up to an overall constant and a local counterterm due to rescaling e_μ^+ by A^{+L}) the effective action for nonabelian right-handed Weyl fermions:

$$N\Gamma^-[e_\mu^-, 0] - \tilde{S}[B]. \quad (8.291)$$

Again this implies that the correlation functions of the gauge currents, J_-^a , and energy-momentum currents, T_-^μ , for the fermionic and bosonic theories are equal, since they are obtained by functional differentiation of the effective action by $(A^{-R})^a$ and e_μ^- respectively. Thus $S_B + S_{CT}$, with e_μ^+ now used as a lagrange multiplier, is a Bose theory which is quantum mechanically equivalent to the $U(N)$ gauge theory of Weyl fermions. Although we have only discussed $U(N)$, the authors of [72] have shown that the chirally symmetric version of this theory in curved space bosonizes similarly for any gauge theory whose symmetry group is one obeying the criterion of Goddard, Nahm, and Olive. The chiral bosonization described here works for those theories as well in an almost identical manner. For a simple group N in 8.287 just gets replaced by $C(N) = d_N/(1 + C_N)$, where d_N is the dimension of the adjoint representation, and C_N is the dual Coxeter number (we will define $C(N)$ better in the next chapter).

In the $U(1)$ theory we had two methods of bosonization in background gauge and gravitational fields. This suggests that by treating A^{+L} as dynamical and leaving the remaining backgrounds external, we should again obtain the chiral effective action 8.291. In order to obtain this, path integration of $S[A^\dagger]$ over A^{+L} must yield a nonlocal part which cancels against $\Gamma^+[e_\mu^+, 0]$. We have not been able to show this. One way out of this would be to truncate the background gravitational field by setting $e_\mu^+ = 0$; in which case we would obtain the effective action 8.291 trivially. The chiral Bose theory S_B would then be only coupled to e_μ^- and therefore not be manifestly coordinate covariant. The corresponding Weyl theory is coupled to only e_μ^- as well, however in a manifestly coordinate covariant manner.

Again, we have not dealt with the gauge fixing of the Siegel symmetry. To address this issue and put this work on a more solid foundation, we will in the next chapter perform a canonical analysis of the zero background chiral Bose theory proposed here. This theory is the nonabelian version of the Siegel action 6.176. We will find that for

$N \leq 24$ the counterterm 6.182 will remove the anomaly in the Siegel symmetry. Since this counterterm depends only on e_μ^+ (recall 6.198), which is path integrated out anyways, the only modification to the argument here would be another term in T_{CT} .

Chapter 9

BRST Quantization of Nonabelian Chiral Bosons

9.1 Introduction

We have seen that the Wess-Zumino-Novikov-Witten models arise upon nonabelian bosonization of fermion fields which form a representation of some nonabelian group satisfying the criterion of Goddard, Nahm, and Olive. This has proved useful in analyzing systems such as quantum spin chains and the quantum Hall effect. In addition to the fact that these theories represent fermion theories, they are important field theories in their own right. They describe two-dimensional Goldstone bosons which have many of the essential properties of the Goldstone bosons in four-dimensional spontaneously broken particle physics theories [73]. Also they are theories of string moving in the space $E_d \times \mathcal{G}$, where E_d is a d -dimensional euclidean space.

The phenomenologically interesting heterotic superstring with its built-in left-right asymmetry has made the chiral version of the WZNW theories important. The $U(1) \times \dots \times U(1)$ (10 times) $\oplus E_8 \times E_8$ describes the left sector of the heterotic string [74].

In the previous chapter we had found that the Siegel form of the WZNW action has an effective action corresponding to a chiral quantum theory. To cancel the anomaly in the Siegel symmetry we will add a conformal field to the theory similarly as in the Polyakov string theory [75]. There the Liouville field was added to the bosonic string action to obtain a quantization away from critical dimension 26. Here a canonical analysis will confirm that the Siegel WZNW action does indeed describe a theory of right-moving

bosons, but only for certain symmetry groups. Also, the boson describing the conformal field disappears from the theory only for a subset of these groups.

Our analysis will consist of a generalization of the BRST analysis that Labastida and Pernici [53] have performed on the $U(1)$ Siegel action. There has been an objection to this [76] because the first class chiral constraint $(\partial_+\phi(x))^2 = 0$ is not irreducible, and thus its square root should be taken so as to make it so. However, this constraint should be considered a quantum mechanical one, and hence a priori only a nilpotency statement can be made about the operator $\partial_+\phi(x)$.

As we have mentioned in chapter 6, instead of introducing the conformal field, we could cancel the anomaly in Siegel's symmetry by the addition of a $\lambda_{--}\partial_+^2\phi$ counterterm. In the nonabelian theory, Imbimbo and Schwimmer have added

$$\lambda_{--}\text{Tr}\partial_+(G^{-1}\partial_+G) \quad \text{or} \quad \lambda_{--}\text{Tr}G^{-1}\partial_+^2G \quad (9.292)$$

to cancel the anomaly. However, unless $\mathcal{G}$ has a $U(1)$ factor, the first term is identically zero, and the second term integrates by parts to a term which can be absorbed into the original lagrangian, plus a term which is zero. For the $U(N)$ theory these authors have found results which are similar to ours [52].

Another possibility is to add the Hull lagrangian 6.184. We will comment on this later in the chapter.

As discussed in chapter 6, we do have a lagrangian formulation of a single chiral boson that uses the linear constraint $\partial_+\phi = 0$. Wu and McClain have given a BRST formulation of this theory that deals quite nicely with the fact that the constraint is second class [77]. Their trick was to split the original set of constraints into a set of commuting first class constraints, and a set of gauge fixing conditions. The nonlocality encountered in other methods here only appears as the fact that it is the Fourier components of the constraints that are split into two groups.

Abdalla and Abdalla [78] have given a constrained hamiltonian formalism for the nonabelian chiral theory that uses the current $J_+ = G^{-1}\partial_+G$ as the constraint. However, they obtain canonical momenta from the unconstrained lagrangian, and then declare $J_+ = 0$ as the constraint. How to obtain their hamiltonian formalism from a lagrangian which already contains the constraint remains unknown. Nevertheless, their formulation can easily be cast into the BRST language by the method given by Wu and McClain. As expected, since both theories correspond to the same fermionic theory (for the appropriate symmetry groups), the spectrum agrees with that obtained here, except they do not require the conformal field, and do not have critical dimensions.

The BRST formalism for quantizing gauge theories is very convenient because the effects of choosing a gauge are obtained without sacrificing any of the advantages of having an exact symmetry. We lose the gauge symmetry upon a choice of gauge; but in the BRST method the symmetry is regained in a larger phase space that includes ghost degrees of freedom. The BRST formulation of the bosonic string has proved itself indispensable for formulating a covariant bosonic string field theory. It is quite likely that the BRST formulation of the chiral WZNW theories will be useful in formulating a covariant string field theory for the heterotic string as well.

9.2 Analysis of the Siegel Action

The Siegel form of the WZNW action in Minkowski space is

$$\begin{aligned} I_o = & \frac{1}{4\pi} \int d^2x \left(\text{Tr} \partial_+ G^{-1} \partial_- G - \lambda_{--} \text{Tr} \partial_+ G^{-1} \partial_+ G \right) \\ & + \frac{1}{12\pi} \int_B d^3x \epsilon^{ABC} \text{Tr} \left(G^{-1} \partial_A G G^{-1} \partial_B G G^{-1} \partial_C G \right), \end{aligned} \quad (9.293)$$

where $G(x)$ is an $N \times N$ matrix in the fundamental representation of the group $\mathcal{G}$. In the abelian $U(1) \times \dots \times U(1)$ (N times) case we take $G = e^{i\phi}$, where $\phi = \text{diag}(\phi_1 \phi_2 \dots \phi_N)$.

Then the last term disappears and this reduces to the usual Siegel action for N bosons. Since $\text{Tr} \partial_+ G^{-1} \partial_+ G$ is a positive definite form, λ_{--} is a lagrange multiplier forcing G to be a function of x^- only.

The action 9.293 is invariant under the gauge transformations

$$\begin{aligned}\delta G &= \epsilon^+ \partial_+ G \\ \delta \lambda_{--} &= \partial_- \epsilon^+ + \epsilon^+ \partial_+ \lambda_{--} - \lambda_{--} \partial_+ \epsilon^+.\end{aligned}\tag{9.294}$$

Recall from chapter 6 that these may be viewed as coordinate transformations on the positive light cone ie $\delta x^+ = \epsilon^+$, $\delta x^- = 0$. This gauge symmetry is the symmetry that we will elevate to BRST symmetry.

In the $U(1)$ case the effective action obtained by path integrating out ϕ is $\Gamma^+[\lambda_{--}]$ (see equations 7.249), which is gauge variant under the transformation 9.294. To obtain a gauge invariant quantum theory the authors of [53] added the counterterm 6.182 which breaks the classical invariance. We add the same counterterm with an overall numerical factor α to be determined

$$I_{ct} = \alpha \int d^2x (\partial_+ \rho \partial_- \rho - \lambda_{--} \partial_+ \rho \partial_+ \rho + 2 \partial_+ \lambda_{--} \partial_+ \rho),\tag{9.295}$$

where the conformal field ρ transforms under the gauge transformations 9.294 as

$$\delta \rho = \epsilon^+ \partial_+ \rho + \partial_+ \epsilon^+.\tag{9.296}$$

This transformation is obtained by recalling that $\rho = \ln e^+$.

The BRST transformation is obtained [79] by replacing ϵ^+ in transformations 9.294 and 9.296 by a Grassmanian ghost field c^+ . A Grassmanian antighost field b_{++} and an auxilliary field B_{++} are also introduced, with the transformation of these new fields determined by demanding the BRST transformation be nilpotent.

$$\bar{\delta} G = c^+ \partial_+ G$$

$$\begin{aligned}
\bar{\delta}\rho &= c^+\partial_+\rho + \partial_+c^+ \\
\bar{\delta}\lambda_{--} &= \partial_-c^+ + c^+\partial_+\lambda_{--} - \lambda_{--}\partial_+c^+ \\
\bar{\delta}c^+ &= c^+\partial_+c^+ \\
\bar{\delta}b_{++} &= B_{++} \\
\bar{\delta}B_{++} &= 0.
\end{aligned} \tag{9.297}$$

$\lambda_{--} = 0$ is an appropriate gauge choice, and to implement it in a covariant manner at the canonical level we add the ghost plus gauge-fixing BRST term as in [53]

$$I_g = -i\bar{\delta}(b_{++}\lambda_{--}). \tag{9.298}$$

This formalism is due to Kugo and Ojima [80] and was used in [81] to quantize the bosonic string. Our theory defining action is the sum of 9.293, 9.295, 9.296

$$I = I_o + I_{ct} + I_g. \tag{9.299}$$

To eliminate λ_{--} from the theory we shift the auxilliary field B_{++}

$$B_{++} = B_{++}' + \frac{i}{4\pi} \text{Tr}(\partial_+ G^{-1} \partial_+ G) + i\alpha \partial_+ \rho \partial_+ \rho + 2i\alpha \partial_+^2 \rho - 2b_{++} \partial_+ c^+ + c^+ \partial_+ b_{++} \tag{9.300}$$

and obtain

$$\begin{aligned}
I &= \int d^2x \left(\frac{1}{4\pi} \text{Tr} \partial_+ G^{-1} \partial_- G + \alpha \partial_+ \rho \partial_- \rho + i b_{++} \partial_- c^+ \right) \\
&+ \frac{1}{12\pi} \int d^3x \epsilon^{ABC} \text{Tr} G^{-1} \partial_A G G^{-1} \partial_B G G^{-1} \partial_C G.
\end{aligned} \tag{9.301}$$

Where the term $B_{++}' \lambda_{--}$ has been omitted since B_{++}' and λ_{--} may now be consistently set (strongly in the quantum theory) to zero. This shifts away the classical gauge variance of 9.299 (ie 9.301 is invariant under 9.297) and the BRST transformation is now (as we will see) nilpotent only in the quantum theory. The Euler-Lagrange equations derived from 9.301 are

$$\partial_+ \partial_- \rho = \partial_- c^+ = \partial_- b_{++} = 0$$

$$\partial_+ J_- = \partial_- J_+ = 0, \quad (9.302)$$

where the algebra valued currents $J_\pm$ are defined as before

$$\begin{aligned} J_+ &= \frac{i}{4\pi} G^{-1} \partial_+ G = J_+^a T^a \\ J_- &= \frac{-i}{4\pi} \partial_- G G^{-1} = J_-^a T^a. \end{aligned} \quad (9.303)$$

Again the matrices T^a $a = 0, 1, 2, \dots, N^2 - 1$ are the generators of $\mathcal{G}$ normalized such that $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$. Since $J_\pm$ obey appropriate field equations we will rewrite the theory in terms of these currents. The BRST transformation 9.297 becomes

$$\begin{aligned} \bar{\delta} J_-^a &= 0 \\ \bar{\delta} J_+^a &= c^+ \partial_+ J_+^a + \partial_+ c^+ J_+^a \\ \bar{\delta} \rho &= c^+ \partial_+ \rho + \partial_+ c^+ \\ \bar{\delta} c^+ &= c^+ \partial_+ c^+ \\ \bar{\delta} b_{++} &= -2\pi i J_+^a J_+^a + i\alpha \partial_+ \rho \partial_+ \rho + 2i\alpha \partial_+^2 \rho \\ &\quad - 2b_{++} \partial_+ c^+ + c^+ \partial_+ b_{++}. \end{aligned} \quad (9.304)$$

Since the BRST symmetry is now exact, we can obtain the BRST charge Q as the integrated zero component of the Noether current for the BRST transformation. The Poisson brackets obtained from 9.301 are

$$\begin{aligned} \left\{ \rho(\sigma), \frac{\partial \rho}{\partial \tau}(\sigma') \right\} &= \frac{1}{\alpha} \delta(\sigma - \sigma') \\ \left\{ c^+(\sigma), b_{++}(\sigma') \right\}_+ &= -i\sqrt{2} \delta(\sigma - \sigma') \\ \left\{ J_\pm^a(\sigma), J_\pm^a(\sigma') \right\} &= f^{abc} J_\pm^c(\sigma) \delta(\sigma - \sigma') \pm \frac{\delta^{ab}}{4\pi} \delta'(\sigma - \sigma'). \end{aligned} \quad (9.305)$$

By demanding that Q generate the BRST transformation, we obtain

$$Q = \int c^+ \left(2\pi J_+^a J_+^a + i\partial_+ c^+ b_{++} + \alpha \partial_+ \rho \partial_+ \rho + 2\alpha \partial_+^2 \rho \right) d\sigma. \quad (9.306)$$

The structure here is generic to the BRST formalism. In general, the BRST charge is the sum of the ghosts times the symmetry generators, and here the symmetry generator is the positive light-cone component of the energy momentum tensor. It generates coordinate transformations, in the enlarged phase space, on the positive light-cone.

To proceed to the quantum theory we replace the Poisson brackets by commutators. The operator version of Q has singularities which must be dealt with carefully; in particular the nonabelian part requires a multiplicative renormalization [82] in order that Q generate the operator transformation 9.304. For the purpose of regularization we take space to be a circle of length 2π and Fourier expand in terms of oscillators for the right- and left-movers:

$$\begin{aligned}
 c^+(\tau, \sigma) &= \frac{1}{\sqrt{2\pi}} \sum_n c_n e^{-in(\sigma+\tau)} \\
 b_{++}(\tau, \sigma) &= \frac{-1}{\sqrt{\pi}} \sum_n b_n e^{-in(\sigma+\tau)} \\
 J_+^a(\tau, \sigma) &= \frac{1}{2\sqrt{2\pi}} \sum_n J_n^a e^{-in(\sigma+\tau)} \\
 J_-^a(\tau, \sigma) &= \frac{1}{2\sqrt{2\pi}} \sum_n \bar{J}_n^a e^{in(\sigma-\tau)} \\
 \sqrt{|\alpha|}\rho(\tau, \sigma) &= \rho_o + \frac{1}{2\sqrt{\pi}} \alpha_o(\tau + \sigma) + \frac{1}{2\sqrt{\pi}} \tilde{\alpha}_o(\tau - \sigma) \\
 &\quad + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n e^{-in(\sigma+\tau)} + \tilde{\alpha}_n e^{in(\sigma-\tau)} \right)
 \end{aligned} \tag{9.307}$$

with commutators

$$\begin{aligned}
 [\rho_o, \alpha_o] &= [\rho_o, \tilde{\alpha}_o] = \frac{i \operatorname{sgn}(\alpha)}{2\sqrt{\pi}} \\
 [\alpha_n, \alpha_m] &= [\tilde{\alpha}_n, \tilde{\alpha}_m] = n\delta(n+m) \operatorname{sgn}(\alpha) \\
 [J_n^a, J_m^b] &= i\sqrt{2} f^{abc} J_{m+n}^c + n\delta^{ab} \delta(n+m) \\
 [\bar{J}_n^a, \bar{J}_m^b] &= i\sqrt{2} f^{abc} \bar{J}_{m+n}^c + n\delta^{ab} \delta(n+m) \\
 [b_n, c_n]_+ &= \delta(n+m).
 \end{aligned} \tag{9.308}$$

For definiteness, take $\mathcal{G}$ to be $U(N)$ and choose the generators such that T^0 is proportional to the identity and T^a $a = 1, 2, \dots, N^2 - 1$ generate $SU(N)$. Accordingly $f^{abc} = 0$ if any of abc are zero and the remaining structure constants are those of $SU(N)$. With our normalization conventions $f^{abc}f^{abd} = N\delta^{cd}$ and

$$Q = \sum_n : c_{-n} \left(L_n^\alpha + L_n^A + L_n^J + \frac{1}{2} L_n^g + \sqrt{|\alpha|} l_n - \beta \delta(n) \right) : \quad (9.309)$$

where β is a constant taking into account normal ordering ambiguities. The Virasoro operators are the Fourier components of T_{++} :

$$\begin{aligned} L_n^\alpha &= \frac{\text{sgn}(\alpha)}{2} \sum_m \alpha_m \alpha_{n-m} & L_n^A &= \frac{1}{2} \sum_m J_m^0 J_{n-m}^0 \\ L_n^J &= \frac{1}{2(N+1)} \sum_m \sum_{a=1}^{N^2-1} J_m^a J_{n-m}^a & L_n^g &= \sum_m (n-m) b_{m+n} c_{-m} \\ l_n &= -\text{sgn}(\alpha) i \sqrt{2n} \alpha_n. \end{aligned} \quad (9.310)$$

Notice the $1/(N+1)$ renormalization factor in L_n^J , it is required for $[L_n^J, J_m^a] = -m J_{n+m}^a$ $a \geq 1$.

1. Normal ordering is required only for $n = 0$ and gives the definitions

$$\begin{aligned} L_0^\alpha &= \text{sgn}(\alpha) \left(\frac{1}{2} \alpha_0^2 + \sum_{m \geq 1} \alpha_{-m} \alpha_m \right) \\ L_0^g &= \sum_{m \geq 1} m (c_{-m} b_m + b_{-m} c_m) \\ L_0^J &= \frac{1}{2(N+1)} J_0^2 + \frac{1}{N+1} \sum_{m \geq 1} \sum_{a=1}^{N^2-1} J_{-m}^a J_m^a, \end{aligned} \quad (9.311)$$

where J_0^2 is the Casimir operator for the fundamental representation of $SU(N)$. The commutators of the Virasoro operators are known [61] for a large class of groups and in our case

$$\begin{aligned} [L_n^\alpha, L_m^\alpha] &= (n-m) L_{n+m}^\alpha + \frac{1}{12} (n^3 - n) \delta(n+m) \\ [L_n^A, L_m^A] &= (n-m) L_{n+m}^A + \frac{1}{12} (n^3 - n) \delta(n+m) \end{aligned}$$

$$\begin{aligned}
[L_n^g, L_m^g] &= (n-m)L_{n+m}^g + \frac{1}{6}(n-13n^3)\delta(n+m) \\
[l_n, l_m] &= 2\operatorname{sgn}(\alpha)n^3\delta(n+m) \quad [l_n, L_m^\alpha] = -i\operatorname{sgn}(\alpha)\sqrt{2}n^2\alpha_{n+m} \\
[L_n^J, L_m^J] &= (n-m)L_{n+m}^J + \frac{(N-1)}{12}(n^3-n)\delta(n+m).
\end{aligned} \tag{9.312}$$

It is remarkable that the $N^2 - 1$ $SU(N)$ currents contribute to the anomaly the same as $N - 1$ abelian currents. Defining

$$L_n = L_n^\alpha + L_n^A + L_n^J + L_n^g + \sqrt{|\alpha|}l_n - \beta\delta(n) \tag{9.313}$$

we find that

$$Q^2 = \frac{1}{2} \sum_{m,n} c_{-n}c_{-m} ([L_n, L_m] - (n-m)L_{m+n}) \tag{9.314}$$

and from 9.312 we have

$$[L_n, L_m] = (n-m)L_{n+m} + \left(2\beta n + \frac{N-25}{12}n^3 + \frac{1-N}{12}n + 2\alpha n^3\right)\delta(n+m). \tag{9.315}$$

We demand the central term in the Virasoro algebra vanish so as to obtain a nilpotent Q . This is another generic feature of BRST: a nilpotent BRST operator is equivalent to an anomaly free constraint algebra. The anomaly is the same in the abelian $U(1) \times \dots \times U(1)$ and nonabelian $U(N)$ theories, so we may consider N to be the number of bosons or we can consider N to be the number of fermions (which the $U(N)$ theory corresponds to). For $N = 26$ and $\beta = 1$ the theory is anomaly free already and we do not add the conformal field, while for $N = 25$ we add another ordinary boson (or fermion) and for $N \leq 24$ we choose

$$\alpha = \frac{25-N}{24} \quad \beta = \frac{N-1}{24}. \tag{9.316}$$

Note that in the $U(1)$ case we have $\alpha = 1$ $\beta = 0$ which is in agreement with [53]. In Hwang's [83] quantization of the bosonic string with a Liouville term (Polyakov's string theory) $Q^2 = 0$ is obtained for a coefficient $26 - N$ and intercept $N - 2$. This is because he

assumes the Liouville field contributes to the anomaly only with the αn^3 term, whereas we have the $(n^3 - n)/12$ contribution as well from the original $\alpha \partial_+ \rho \partial_- \rho$ term.

For a general compact Lie group, $\mathcal{G}$, the analysis involving the Virasoro algebra is very similar. If $\mathcal{G}$ is semi-simple, ie $\mathcal{G} = \mathcal{G}^1 \oplus \mathcal{G}^2 \oplus \dots$ where $\mathcal{G}^i$ is simple, then each factor in the sum contributes an L_n^J term to L_n . If $\mathcal{G}$ is not semisimple and there are terms in the direct sum like $U(1)$ or $U(1) \times \mathcal{G}^i$, then each $U(1)$ factor contributes an L_n^A term to L_n . The contribution to the central term in the Virasoro algebra from all the L_n^J s and L_n^A s is $C(N)/12$, where

$$C(N) = \sum_i \frac{d_N^i}{1 + \frac{q_\psi^i}{k_i \psi_i^2}} + d. \quad (9.317)$$

d is the total number of $U(1)$ factors, d_N^i is the dimension of the adjoint representation of $\mathcal{G}^i$, q_ψ^i is the quadratic Casimir for the adjoint representation of $\mathcal{G}^i$, ψ_i^2 is the length squared of the longest root of the fundamental representation of $\mathcal{G}^i$ (we have taken this to be one), and k_i is the level of the Kac-Moody algebra associated to $\mathcal{G}^i$. For $\mathcal{G} = U(1)$ $C(N) = 1$, for $\mathcal{G} = SU(N)$ $C(N) = N - 1$, for $\mathcal{G} = SO(N)$ $C(N) = N/2$, etc. [61].

N in equations 9.316 just gets replaced by $C(N)$. So in general for $C(N) = 25$ there is no conformal field yet the anomaly is not canceled, and quantization fails. For $C(N) = 26$ the Seigel symmetry is anomaly-free in agreement with reference [84] where a quantization was carried out with no conformal field. The symmetry groups that we are limited to in this case correspond to the critical dimensions of superstring theory.

Having found the conditions for a consistent constraint algebra, it remains to check that physical states, ie those that are annihilated by the constraints, are indeed right-moving excitations only. The physical state condition in the BRST formalism can be written very economically, it is

$$Q|\Phi\rangle = 0, \quad (9.318)$$

and we will see shortly that the states obeying this can be written as

$$|\Phi\rangle = |\Phi\rangle_{phys} + Q|J\rangle, \quad (9.319)$$

where $|\Phi\rangle_{phys}$ contains only right moving modes. A state written in the form $Q|J\rangle$ is pure gauge in the BRST formalism, because the states of the original theory (without the ghosts) are identified with BRST cohomology classes of a particular ghost number.

We demand the vacuum obey 9.318, hence any state built exclusively from right-movers will also, since Q is built with left oscillators only. Furthermore, any such right sector state can not possibly be written as $Q|J\rangle$, and is therefore physical. Since the left and right sectors commute, to show 9.319 we need only show the left sector is cohomologically trivial. The vacuum is defined in the usual manner: $\alpha_n|0\rangle = J_n^a|0\rangle = c_n|0\rangle = b_n|0\rangle = 0$ for $n \geq 1$, it must form a representation of $[J_o^a, J_o^b] = if^{abc}J_o^c$, we choose it to be annihilated by b_o , and we choose its α_o eigenvalue such that $L_o|0\rangle = 0$. This last choice is necessary to obtain $Q|0\rangle = 0$. A basis for the left sector is then obtained by applying $\alpha_n, J_n^a, c_n, b_n$ for $n \leq -1$.

Following Labastida and Pernici we expand Q in the zero modes of the ghosts

$$Q = c_o H + Q_B + b_o M, \quad (9.320)$$

and we can verify that

$$[Q, H] = [Q_B, H] = [Q_B, M] = [H, M] = 0. \quad (9.321)$$

Now $H = L_o$ and is positive definite on the left sector provided $\alpha \geq 0$. So if $|\Phi\rangle$ is a state in this sector obeying the physical state condition then

$$|\Phi\rangle = Q|J\rangle \quad \text{where} \quad |J\rangle = b_o H^{-1}|\Phi\rangle. \quad (9.322)$$

In other words the left sector is pure gauge for $C(N) \leq 24$ or for $C(N) = 26$.

For $C(N) > 26$ H is no longer positive definite and one must consider states which are annihilated by H . However, in this case $\tilde{\alpha}_n$ creates states of negative norm so the right sector certainly has ghosts. Furthermore, as pointed out in [53], the abelian theory with one negative metric field and with $N > 1$ has a cohomologically nontrivial left sector. As the abelian theory is simply a nonabelian theory with all the structure constants set zero, it is impossible that the nonabelian theory with the negative metric conformal field has a cohomologically trivial left sector, and is therefore non-chiral. We conclude that canceling the anomaly by adding a conformal field provides a satisfactory method for quantization of the Siegel WZNW model only for $C(N) \leq 24$. For $C(N) = 26$ the quantization can be carried out without the conformal field.

Groups obeying the criterion of Goddard, Nahm, and Olive have integer or half-integer $C(N)$ and hence Siegel WZNW lagrangians based on these groups, with $C(N) \leq 26$, are also anomaly free upon the addition of $2(26 - C(N))$ fermionic no-movers described by lagrangian 6.184. The interpretation of the new quantum degrees of freedom has yet to be made.

Chapter 10

Closing Discussion

The bosonization of fermion theories coupled to background gauge and gravitational fields is elevated to the fully interacting theory by adding the gauge and gravitational actions to both the fermionic and the bosonic actions, and declaring all variables to be dynamical. The gauge field action is just the usual Yang-Mills action; while the usual Einstein-Hilbert action for gravity is just a number (the Euler characteristic), because $\sqrt{g}R$ is a total divergence in two dimensions.

Jackiw [85] has suggested that the appropriate lagrangian for two dimensional gravity is $N(R - \Lambda)$, where N is a lagrange multiplier field enforcing the constant curvature field equation $R = \Lambda$, where Λ is a constant. This model coupled to chiral fermions is the gravitational analogue of the chiral Schwinger model. Despite the fact that it has a gravitational anomaly, as in the chiral Schwinger model, a consistent quantization can be obtained by allowing some of the gauge degrees of freedom to become physical at the quantum level. However, unlike the chiral Schwinger model, here the unitarity is lost [86].

The main purpose of studying these anomalous theories was to question whether a fundamental theory must be gauge anomaly free. Although the successful quantization of the chiral Schwinger model indicates not, Ball [87] has argued that this anomalous model is just the low-energy effective theory derived from a more fundamental anomaly-free theory. However, this theory can also be viewed as a regularization of the chiral Schwinger model that restores gauge invariance by adding degrees of freedom. His theory

has additional chiral fermions which cancel the anomaly, and when their mass is taken to infinity they do not decouple completely, but leave behind a degree of freedom. In the bosonized picture, this degree of freedom is the anomaly canceling Wess-Zumino field. The bosonized chiral Schwinger model is then seen as the gauge-fixed version of the gauge invariant theory that includes a Wess-Zumino term.

Any method of quantization of an anomalous gauge theory yields a theory with more degrees of freedom than the classical theory that it was derived from. In recognition of this fact, we have chosen to quantize Siegel chiral scalars (those that use the constraint $T_{++} = 0$) by adding a Wess-Zumino field to the theory to cancel the anomaly. Because of its geometrical interpretation, it is called a conformal field.

We have found that for chiral scalars with nonabelian symmetry groups, adding the conformal field cancels the Siegel anomaly, and results in a chiral theory, only for symmetry groups that have $C(N) \leq 24$. In the case that $C(N) = 25$, we can achieve the anomaly-free case $C(N) = 26$ by adding one more $U(1)$ field. Thinking of this field as the special conformal field that we need when $C(N) = 25$, we can say that our quantization of nonabelian chiral bosons is successful for groups with $C(N) \leq 26$. A similar thing happens in the quantization of the bosonic string away from critical dimensions, where a Liouville field is added to the theory. This similarity is due to the fact that the $U(N)$ Siegel lagrangian is the lagrangian for a euclidean space bosonic string theory with a truncated world sheet metric.

The BRST quantization we have performed on the Siegel WZNW theory in zero backgrounds can be extended to a quantization of the theory coupled to dynamical gauge and gravitational fields. We elevate the gauge or diffeomorphism symmetry to BRST symmetry by introducing another set of ghosts for each symmetry. The resulting gauge-fixed lagrangian will be the sum of our lagrangian, plus additional ghost terms. The ghosts introduced to gauge the zweibein flat will give a contribution to the central term

in the Virasoro algebra which increases the critical dimension.

The path integral analysis we did to establish the equivalence between chiral bosons and Weyl fermions did not deal with the gauge-fixing of the symmetry generated by the chiral constraint. Our BRST analysis fills this gap because the canonical BRST lagrangian is the lagrangian that we would obtain by a Fadeev-Popov procedure to fix the gauge in the path integral. The path integral measure and lagrangian now include the ghosts, but since they are decoupled, the arguments of the path integral analysis remain essentially unchanged. However, when the BRST quantization fails to give a chiral theory, the Fadeev-Popov procedure fails as well, because the integration over the Lagrange multiplier degrees of freedom fails to decouple. (Establishing this assertion requires an explicit regularization of the path integral measure.) Thus we conclude that the Seigel form of the WZNW action represents nonabelian Weyl fermions when and only when the nonabelian symmetry group obeys the criterion of Goddard, Nahm, and Olive, and obeys $C(N) \leq 26$.

To complete the bosonization program for Weyl fermions, and obtain bosonized actions for all symmetry groups which obey the criterion of Goddard, Nahm, and Olive, it appears that we should use the WZNW theory with the constraint $J_+ = 0$, because this leads to a theory without critical dimensions. This has been solved in the abelian case; however, there are still several open problems to be dealt with in the nonabelian theory.

Appendix A

Chirality of Matter Fields

Recall that the spinor representations of the Lorentz group, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, are realized by two component complex spinors, $\psi_+(x)$ and $\psi_-(x)$ respectively, called Weyl spinors [31]. To build a four-component Dirac spinor, we may simply take a direct sum

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad (\text{A.323})$$

and call ψ_+ and ψ_- the chiral components. To define chiral four-component spinors define the chiral operator

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (\text{A.324})$$

which has eigenvalues ± 1 ; and define a positive chirality spinor to be an eigenvector of γ_5 with eigenvalue $+1$, and a negative chirality spinor to be an eigenvector of γ_5 with eigenvalue -1 .

The two spinor representations are interchanged by the operation of parity, so the parity operator is

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (\text{A.325})$$

If we define

$$\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad (\text{A.326})$$

where σ^i are the Pauli matrices, then we have the anticommutator

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (\text{A.327})$$

and the relation

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \quad (\text{A.328})$$

What we have here is the Weyl representation of the Dirac matrices. For an arbitrary matrix representation in four spacetime dimensions we let γ^μ be any four by four, hermitian matrices satisfying A.327. The chirality of a four-component spinor is again defined with respect to γ_5 , where γ_5 is now defined by A.328. The positive and negative chiral components of ψ are obtained by the projection operators $P_\pm$:

$$\psi_\pm \equiv P_\pm \psi \equiv \frac{1}{2}(1 \pm i\gamma_5)\psi. \quad (\text{A.329})$$

In the case of massless fermions, the Dirac equation is equivalent to a decoupled pair of two-component Weyl equations. Furthermore, the positive energy solutions have the property that the chirality equals the helicity; while the negative energy solutions have the property that the chirality is the negative of the helicity.

This notion of chirality generalizes to any even, say $2n$, spacetime dimension. The Dirac matrices are 2^n by 2^n , and γ_5 is defined as

$$\gamma_5 \equiv i\gamma^0\gamma^1 \dots \gamma^{2n-1}, \quad (\text{A.330})$$

and satisfies $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma^\mu\} = 0$.

The massless free Dirac equation in two dimensions

$$\gamma^\mu \partial_\mu \psi = 0, \quad (\text{A.331})$$

in the Weyl representation

$$\gamma^0 = \sigma^1, \quad \gamma^1 = i\sigma^2, \quad \gamma_5 = i\gamma^0\gamma^1 = -i\sigma^3 \quad (\text{A.332})$$

has solution

$$\psi = \begin{pmatrix} \psi_+(x^+) \\ \psi_-(x^-) \end{pmatrix}, \quad (\text{A.333})$$

where $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$. Thus the Weyl spinors, ψ_+ and ψ_- , represent left- and right-movers respectively.

Free massless scalars obey the Klein-Gordon equation

$$\partial^\mu \partial_\mu \phi = 0 \quad (\text{A.334})$$

which in two spacetime dimensions has the solution

$$\phi = \phi_+(x^+) + \phi_-(x^-). \quad (\text{A.335})$$

So analogously to the fermionic case, we call ϕ_+ and ϕ_- the positive and negative chirality components of ϕ .

In Minkowski spacetimes of dimension $4n + 2$, an antisymmetric tensor gauge field of rank $2n$ can have a self-duality condition imposed on its rank $2n + 1$ field strength. In two dimensions ($n = 0$) the “antisymmetric tensor” is just a scalar ϕ , with field strength, $F_\mu = \partial_\mu \phi$, invariant under the gauge transformation $\delta\phi = \text{constant}$. The self-duality condition on the field strength

$$F_\mu = \epsilon_\mu^\nu F_\nu \quad (\text{A.336})$$

means ϕ has positive chirality: $\phi = \phi(x^+)$. Thus the quantization of a chiral boson is the $n = 0$ case of the controversial quantization of a self-dual antisymmetric tensor field.

Appendix B

Projective Representations in Quantum Mechanics

A projective representation is a generalization of a group representation. Here the idea will be introduced along with some simple examples from quantum mechanics. This discussion is taken from lecture III of *Anomalies and Topology* by R. Jackiw [1].

Consider a group with elements g that act upon some variable q according to $q \rightarrow q^g$. Then functions of this variable, $F(q)$, and operators on these functions, $U(g)$, give a representation of the group if the operators act on the functions as

$$U(g)F(q) = F(q^g), \quad (\text{B.337})$$

and compose as

$$U(g_1)U(g_2) = U(g_{12}), \quad (\text{B.338})$$

where $g_{12} = g_1 g_2$ is the composition of g_1 and g_2 .

Ordinary quantum mechanics, in the position space representation, provides a representation of the abelian translation group which acts on the coordinates as

$$\vec{r} \rightarrow \vec{r} + \vec{a}. \quad (\text{B.339})$$

The operators that effect translations, namely

$$U(\vec{a}) = e^{i\vec{a} \cdot \vec{p}}, \quad (\text{B.340})$$

where the momentum operator is $p^j = -i\partial/\partial r^j$, act on the wavefunctions, $\psi(\vec{r})$, according to B.337

$$U(\vec{a})\psi(\vec{r}) = \psi(\vec{r} + \vec{a}). \quad (\text{B.341})$$

Also, the operators compose according to B.338

$$U(\vec{a})U(\vec{b}) = U(\vec{a} + \vec{b}). \quad (\text{B.342})$$

However, this simplest situation can be complicated by the introduction of phases, and these phases are called cocycles. The first generalization is to allow a phase in B.337

$$U(g)F(q) = e^{-2\pi i \omega_1(q;g)} F(q^g) \quad (\text{B.343})$$

However, consistency with B.338 imposes a constraint on ω_1 :

$$U(g_1)U(g_2)F(q) = U(g_1) \left(e^{-2\pi i \omega_1(q;g_2)} F(q^{g_2}) \right) \quad (\text{B.344})$$

$$= e^{-2\pi i \omega_1(q;g_1)} e^{-2\pi i \omega_1(q^{g_1};g_2)} F(q^{g_{12}}) \quad (\text{B.345})$$

$$U(g_{12})F(q) = e^{-2\pi i \omega_1(q;g_{12})} F(q^{g_{12}}). \quad (\text{B.346})$$

The right-hand sides will be equal, provided the phase is what is called a 1-cocycle.

$$\omega_1(q;g_1) - \omega_1(q;g_{12}) + \omega_1(q^{g_1};g_2) = 0 \quad \text{mod integer}. \quad (\text{B.347})$$

1-cocycles appear in ordinary quantum mechanics. For example, a Galileo transformation,

$$\vec{r} \rightarrow \vec{r} + \vec{v}t \quad (\text{B.348})$$

$$\vec{p} \rightarrow \vec{p} + m\vec{v}, \quad (\text{B.349})$$

is implemented by the unitary operator

$$U(\vec{v}) = e^{i\vec{v} \cdot (\vec{p}t - m\vec{r})}, \quad (\text{B.350})$$

which has an action on wavefunctions which introduces a phase obeying the 1-cocycle condition.

$$U(\vec{v})\psi(\vec{r}) = e^{-2\pi i \omega_1(\vec{r};\vec{v})} \psi(\vec{r} + \vec{v}t). \quad (\text{B.351})$$

In the next generalization, a phase is introduced into the composition law.

$$U(g_1)U(g_2) = e^{-2\pi i \omega_2(q; g_1, g_2)} U(g_{12}). \quad (\text{B.352})$$

A consistency condition on the phase follows from the assumed associativity of the composition law. If

$$(U(g_1)U(g_2))U(g_3) = U(g_1)(U(g_2)U(g_3)), \quad (\text{B.353})$$

then one easily shows that ω_2 satisfies

$$\omega_2(q^{g_1}; g_2, g_3) - \omega_2(q; g_{12}, g_3) + \omega_2(q; g_1, g_{23}) - \omega_2(q; g_1, g_2) = 0 \pmod{\text{integer}}. \quad (\text{B.354})$$

Such an object is called a 2-cocycle; and representations which obey B.352 are called projective.

Again, ordinary quantum mechanics uses 2-cocycles. For example, translations on phase space,

$$\vec{r} \rightarrow \vec{r} + \vec{a} \quad (\text{B.355})$$

$$\vec{p} \rightarrow \vec{p} + \vec{b}, \quad (\text{B.356})$$

are effected by the operator

$$U(\vec{a}, \vec{b}) = e^{(\vec{a} \cdot \vec{p} - \vec{b} \cdot \vec{r})}, \quad (\text{B.357})$$

which composes according to B.352 with a 2-cocycle

$$2\pi\omega_2 = \frac{1}{2}(\vec{a}_1 \cdot \vec{b}_2 - \vec{a}_2 \cdot \vec{b}_1). \quad (\text{B.358})$$

Higher cocycles can be defined, by abandoning associativity, but are not allowed in the present Hilbert space formulation of quantum mechanics. A trivial cocycle is one which can be removed by a unitary transformation on the states and operators.

In the gauge theory defined by lagrangian 2.16, the analogous variables to q of B.337, are the gauge potentials A_μ , which are acted upon by the gauge group according to

2.32. By performing a second gauge transformation, and using their composition law, one can easily show that $\omega_1(A, \theta)$ of equation 2.25 is a one cocycle. Furthermore, if it has a nonzero winding number it is nontrivial. Thus the gauge anomaly for Weyl fermions may be viewed as the appearance of a 1-cocycle in the effective action. The condition that it is in fact a 1-cocycle, taken in its infinitesimal form, is called the Wess- Zumino consistency condition for the anomaly [88]. In the hamiltonian formulation of field theory, the anomaly manifests itself as a 2-cocycle in the algebra of the gauge generators.

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