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of

WAH-CHUN CHAN

B.Sc., National Taiwan University, 1958

M.Sc., The University of New Brunswick, 1961

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# THE DERIVATION OF OPTIMAL CONTROL LAWS AND THE SYNTHESIS OF REAL-TIME OPTIMAL CONTROLLERS FOR A CLASS OF DYNAMIC SYSTEMS


## ABSTRACT

A method for the solution of a class of optimal control problems based on a modified steepest descent method is discussed. This method is suitable for the solution of problems in variational calculus of the Mayer type, and can be used to realize comparatively simple on-line optimal controllers by means of analogue computer techniques.

The essence of the modified steepest descent method is to search for the optimum value of a performance function by replacing a search in function space by a search in parameter space. In general, an iterative type of search for the optimum value of the performance function is required. However, in certain classes of problems the optimal control variable can be expressed as a function of the system state variables and no iteration is necessary.

Several optimal control problems for the rocket flight problem are studied and optimal control laws are derived as functions of the system state variables. Experimental results show that the method is very satisfactory. A PACE 231-R analogue computer is used to solve the sounding rocket problem. A more complex problem, the two-dimensional zero-lift rocket flight problem, is solved using the modified method of steepest descent and an electromechanical flight simulator. The experimental results obtained with the flight simulator show that the modified steepest

descent method is practical and show promise of being useful in the design of real-time optimal controllers.

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by

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We accept this thesis as conforming to the required standard

Members of the Department
of Electrical Engineering

The University of British Columbia

JANUARY, 1965

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# 1. INTRODUCTION

## 1.1 Historical Note on the Theory of Optimal Processes

The classical theory of the calculus of variations was developed by Euler and Lagrange at the end of the eighteenth century. Euler obtained the necessary condition for a relative weak minimum in the form of an equation, now known as the Euler equation. Lagrange introduced the Lagrange multiplier to facilitate the formulation of minimum problems subject to constraints. The Lagrange equation in mechanics has the same form as the Euler equation. The Euler equation is, therefore, also referred to as the Euler-Lagrange equation. In this thesis the name Euler-Lagrange equation instead of Euler equation is used.

The method of dynamic programming was developed by Bellman in the last decade and is essentially a numerical technique suited for digital computation.

Recently Pontryagin developed a mathematically rigorous theory of optimal control which is called the maximum principle.

A further computational technique available to solve minimum problems is the gradient method or the method of steepest descent. The gradient method has been applied by Kelley for solving optimal flight path problems[1]. A similar scheme has been developed by Bryson and his colleagues[2]. Bohn[3,4] has presented a modified approach for solving optimal control problems which appears suitable for computing the instantaneous control policy in real time . This thesis is concerned with the development of this method which, for reasons that will be given later in the thesis, is called the modified steepest descent method. Chapter 1 gives a brief review of the various techniques mentioned above.

## 1.2 The Principle of Optimality

The principle of optimality[5] states that "an optimal policy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision". This principle plays the fundamental role in the theory of dynamic programming.

## 1.3 The Method of Dynamic Programming[6]

The theory of dynamic programming is based on the principle of optimality. It gives a systematic approach for determining a numerical solution to minimum problems. In theory, dynamic programming is a very general approach, however, in practice, it has restricted applicability because of the problem of dimensionality.

In this section the basic technique of dynamic programming is discussed.

Consider the problem of minimizing the functional J

$$J(x) = \int_0^T F(t,x,\dot{x})\, dt \tag{1.1}$$

where the vector notation

$$x = (x_1,\ldots,x_n), \quad \dot{x} = \frac{dx}{dt}$$

and

$$x(0) = c = (c_1,\ldots,c_n)$$

is used. The dynamic programming approach to minimizing J is to consider the function

$$f(t,x) = \underset{\{\dot{x}\}}{\text{Min}} \int_t^T F(\tau, x, \frac{dx}{d\tau})\, d\tau \tag{1.2}$$

It is evident that

$$f(T,x(T)) = 0$$

and that

$$f(0,c) = \text{Min } J(x)$$

The principle of optimality applied to (1.2) yields

$$f(t,x) = \underset{\{\dot{x}\}}{\text{Min}} \left[ \int_t^{t+\Delta t} F(\tau, x, \frac{dx}{d\tau})\, d\tau + \int_{t+\Delta t}^{T} F(\tau, x, \frac{dx}{d\tau})\, d\tau \right] \tag{1.3}$$

$$\text{Thus } f(t,x) = \underset{\{\dot{x}\}}{\text{Min}} \left[ F(t,x,\dot{x})\Delta t + f(t+\Delta t, x+\dot{x}\Delta t) + O(\Delta t) \right] \tag{1.4}$$

where $O(\Delta t)$ indicates terms of the order of $(\Delta t)^2$. Expanding (1.4) in a power series about $(t,x)$ and letting $\Delta t \to 0$, yields

$$0 = \underset{\{\dot{x}\}}{\text{Min}} \left[ F(t,x,\dot{x}) + \frac{\partial f}{\partial t} + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \dot{x}_j \right] \tag{1.5}$$

The solution of (1.5) must satisfy the following two nonlinear partial differential equations

$$F + \frac{\partial f}{\partial t} + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \dot{x}_j = 0 \tag{1.6}$$

and

$$\frac{\partial F}{\partial \dot{x}_j} + \frac{\partial f}{\partial x_j} = 0, \quad j = 1,2,\ldots,n. \tag{1.7}$$

Thus the original problem of minimizing the functional J of (1.1) is transformed into the problem of solving the nonlinear partial differential equations, (1.6) and (1.7) for f. In general these nonlinear partial differential equations can not be solved directly.

### 1.3.1 The Principle of Optimality as a Numerical Technique

Most problems in optimal control are far too complex for an analytical solution. A numerical solution may be obtained by the use of digital computers. In order to employ digital computers for the numerical solution of (1.6) and (1.7), it is necessary to convert the nonlinear partial differential equations into a finite-difference equation. A more convenient method of solution is to solve for the functional f of (1.2) by minimizing a discrete approximation of the form

$$J_k(x) = \sum_{i=k}^{N-1} F\left(i\Delta t, x^{(i)}, \frac{x^{(i+1)} - x^{(i)}}{\Delta t}\right) \Delta t \tag{1.8}$$

where

$x^{(i)} = x(i\Delta t)$ and where the derivative $\dot{x}$ is approximated by

$$\dot{x}^{(i)} = (x^{(i+1)} - x^{(i)})/\Delta t$$

Let $u^{(i)} = \dot{x}^{(i)}$, and introduce the sequence of functions

$$\begin{aligned} f_k(k\Delta t, c) &= \min_{\{u\}} J_k(x) \\ &= \min_{\{u\}} \sum_{i=k}^{N-1} F(i\Delta t, x^{(i)}, u^{(i)})\Delta t \end{aligned} \tag{1.9}$$

for $-\infty < c < \infty$ , $k = 0,1,\ldots,N-1$. Then

$$f_N(T,c) = 0 \tag{1.10}$$

and

$$\begin{aligned} f_k(k\Delta t,c) &= \underset{\{u\}}{\mathrm{Min}}\left[F(k\Delta t,c,u)\Delta t + \sum_{i=k+1}^{N-1} F(i\Delta t,x^{(i)},u^{(i)})\Delta t\right] \\ &= \underset{\{u\}}{\mathrm{Min}}\left[F(k\Delta t,c,u)\Delta t + f_{k+1}((k+1)\Delta t,c+u\Delta t)\right] \end{aligned} \tag{1.11}$$

Equation (1.11) is the basis of the dynamic programming method for the solution of minimum problems[5].

## 1.3.2 The Problem of Dimensionality

The numerical solution of (1.11) requires the tabulation and storage of sequences of functions of n variables. This introduces some complications. To illustrate this, consider the case of a two-dimensional problem where

$$c = (c_1,c_2)$$
$$u = (u_1,u_2)$$

Assume that $c_1$ and $c_2$ are both allowed to have one hundred values. Since the number of different values for $c_1$ and $c_2$ is $10^4$, the tabulation of the values of $f(c_1,c_2,T)$ for a particular value of T requires a memory capable of storing $10^4$ numbers. Moreover, since the recurrence relation requires that $f(c,T)$ is stored while the values for $T+\Delta t$ are calculated, and since the values of $u_1$ and $u_2$ must also be stored, the memory must be capable of storing at least $4 \times 10^4$ numbers.

Generally speaking, with current digital computers having memories of 32,000 words, only two-dimensional minimum problems can be handled unless some method for reducing dimensionality is found. The problem becomes difficult to cope with for higher dimensions. As pointed out by Bellman, a three-dimensional trajectory problem involving three position variables and three velocity variables, treated by the dynamic programming approach results in functions of six state variables. In this case, even if each variable is allowed to take only 10 different values, this leads to $10^9$ values requiring an extremely large computer memory.

### 1.3.3 The Euler-Lagrange Equations

All the necessary conditions in the classical theory of calculus of variations can be derived from the principle of optimality. Consider the variational problem discussed in Section 1.3. The principle of optimality yields the nonlinear partial differential equations (1.6) and (1.7). Differentiating (1.7) with respect to t, gives

$$\frac{d}{dt}\left(F_{\dot{x}_j}\right) + \frac{\partial^2 f}{\partial x_j \partial t} + \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}\,\dot{x}_i = 0 \qquad (1.12)$$

and partial differentiation of (1.6) with respect to $x_j$ gives

$$F_{x_j} + \sum_{i=1}^{n} F_{\dot{x}_i}\frac{\partial \dot{x}_i}{\partial x_j} + \frac{\partial^2 f}{\partial t \partial x_j}\,\dot{x}_j + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_j}\frac{\partial \dot{x}_i}{\partial x_j} + \frac{\partial^2 f}{\partial x_i \partial x_j}\,\dot{x}_i\right) = 0$$

Thus

$$F_{x_j} + \frac{\partial^2 f}{\partial t \partial x_j} + \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\,\dot{x}_i = 0 \qquad (1.13)$$

Substituting (1.13) in (1.12) yields

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_j}\right) - \frac{\partial F}{\partial x_j} = 0, \qquad j = 1,2,\ldots,n. \tag{1.14}$$

which are the Euler-Lagrange equations.

It is also possible to derive (1.14) from the nonlinear partial differential equations for f using the method of characteristics.

### 1.3.4 The Legendre-Clebsch Condition

The necessary condition for a minimum of (1.5) is that the second derivative of the square brackets with respect to $\dot{x}_i$ must be positive. This leads to the Legendre-Clebsch condition

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} \, \delta \dot{x}_i \, \delta \dot{x}_j > 0 \tag{1.15}$$

or

$$\frac{\partial^2 F}{\partial \dot{x}_1^{\,2}} > 0,$$

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial \dot{x}_1 \partial \dot{x}_1} & \dfrac{\partial^2 F}{\partial \dot{x}_1 \partial \dot{x}_2} \\[2ex] \dfrac{\partial^2 F}{\partial \dot{x}_2 \partial \dot{x}_1} & \dfrac{\partial^2 F}{\partial \dot{x}_2 \partial \dot{x}_2} \end{vmatrix} > 0, \quad \ldots,$$

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial \dot{x}_1 \partial \dot{x}_1} & \cdots & \dfrac{\partial^2 F}{\partial \dot{x}_1 \partial \dot{x}_n} \\ \cdot & \cdots & \cdot \\ \dfrac{\partial^2 F}{\partial \dot{x}_n \partial \dot{x}_1} & \cdots & \dfrac{\partial^2 F}{\partial \dot{x}_n \partial \dot{x}_n} \end{vmatrix} > 0.$$

### 1.3.5 The Weierstrass Condition

The Legendre-Clebsch Condition does not rule out the possibility of a relative minimum. If $F(t,x,\dot{x})$ is an absolute minimum, it follows from (1.6) that the following inequality must satisfy

$$F(t,x,\dot{x})+\frac{\partial f}{\partial t}+\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\dot{x}_j \leq F(t,x,\dot{X})+\frac{\partial f}{\partial t}+\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\dot{X}_j$$

or

$$F(t,x,\dot{X}) - F(t,x,\dot{x}) + \sum_{j=1}^{n}(\dot{X}_j - \dot{x}_j)\frac{\partial f}{\partial x_j} \geq 0 \qquad (1.16)$$

for all functions $\dot{X}$.

From (1.7),

$$\frac{\partial f}{\partial x_j} = -\frac{\partial F}{\partial \dot{x}_j}$$

and (1.16) yields the Weierstrass condition for an absolute minimum.

$$F(t,x,\dot{X}) - F(t,x,\dot{x}) - \sum_{j=1}^{n}(\dot{X}_j - \dot{x}_j)\frac{\partial F}{\partial \dot{x}_j} \geq 0 \qquad (1.17)$$

### 1.3.6 The Transversality Condition

So far the discussion of the minimization of a functional is restricted to the case of fixed end points.

Suppose now that the end points are variable. The necessary condition for a minimum of the functional is that the differential of the function $f(t,x)$ must vanish. Therefore

$$df = \frac{\partial f}{\partial t}dt + \sum_{j=1}^{n}\frac{\partial f}{\partial x_j}dx_j = 0$$

Thus

$$\frac{\partial f}{\partial t}\, dt = - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, dx_j \qquad (1.18)$$

Multiplying (1.6) by dt gives

$$F dt + \frac{\partial f}{\partial t}\, dt + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, \dot{x}_j dt = 0$$

Substituting (1.7) and (1.18) in the above equation yields

$$(F - \sum_{j=1}^{n} \dot{x}_j F_{\dot{x}_j})\, dt + \sum_{j=1}^{n} F_{\dot{x}_j}\, dx_j = 0$$

This holds at both end points. Thus

$$\left[ \sum_{j=1}^{n} F_{\dot{x}_j}\, dx_j + (F - \sum_{j=1}^{n} \dot{x}_j\, F_{\dot{x}_j})dt \right]_0^T = 0 \qquad (1.19)$$

Equation (1.19) is called the transversality condition.

## 1.3.7 The Weierstrass-Erdmann Corner Conditions

Many variational problems of engineering interest have solutions which may have a finite number of corner points, where one or more of the derivatives $\dot{x}_j$ have a discontinuity. Suppose that $\dot{x}_k$ is discontinuous, then, since $\frac{\partial f}{\partial x_k}$ is continuous, it follows from (1.7) that $\frac{\partial F}{\partial \dot{x}_k}$ must be continuous at a corner. Similarly, $\frac{\partial f}{\partial t}$ is continuous and substituting (1.7) in (1.6) yields

$$F - \sum_{j=1}^{n} \dot{x}_j\, F_{\dot{x}_j} = - \frac{\partial f}{\partial t}$$

which is also continuous at a corner. Therefore

$$\left( F_{\dot{x}_k} \right)_- = \left( F_{\dot{x}_k} \right)_+ \qquad (1.20)$$

and

$$(F - \sum_{j=1}^{n} \dot{x}_j F_{\dot{x}_j})_- = (F - \sum_{j=1}^{n} \dot{x}_j F_{\dot{x}_j})_+ \tag{1.21}$$

where the negative and positive signs denote trajectory positions immediately before and after a corner point, respectively. Equations (1.21) and (1.20) are called the Weierstrass-Erdmann corner conditions.

### 1.3.8 The Inequality Constraint

In many problems there may be inequality constraints on the independent variable u of (1.11) (the so-called control variable). If, for example,

$$|u| \leq U$$

where U is the upper bound for the magnitude of u, then the choice of $u_k$ at each iteration stage in the dynamic programming approach is restricted and the computational aspect of the problem is thereby simplified.

### 1.3.9 The Lagrange Mutlipliers[6]

The Lagrange multiplier method is the most suitable means for handling a minimum problem subject to constraints. Two different kinds of Lagrange multipliers which depend on the type of constraints are discussed in this section.

Consider the problem of minimizing the functional

$$J(x) = \int_0^T H(t, x, \dot{x})dt, \; x(0) = c \tag{1.22}$$

subject to the constraint

$$\int_0^T G(t,x,\dot{x})dt = y \qquad (1.23)$$

where y is a given value. To solve the minimum problem the lower limit is considered variable so that the minimum f of J(x) becomes a function of three variables, t,x, and y. In other words, y is considered as an additional variable. The solution of the minimum problem is given by

$$f(t,x,y) = \underset{\{\dot{x}\}}{\text{Min}} \int_t^T H(\tau,x,\frac{dx}{d\tau})d\tau \qquad (1.24)$$

where y is determined by the equation of constraint

$$\int_t^T G(\tau,x,\frac{dx}{d\tau})d\tau = y \qquad (1.25)$$

Equation (1.24) can be treated in the same manner as was done previously for (1.2) yielding

$$f(t,x,y) = \underset{\{\dot{x}\}}{\text{Min}} \left[ H(t,x,\dot{x})\Delta t + f(t+\Delta t,\ x+\dot{x}\Delta t,\ y-G(t,x,\dot{x})\Delta t) + O(\Delta t) \right] \qquad (1.26)$$

Proceeding as before, the following functional equation for f(t,x,y) is obtained

$$0 = \underset{\{\dot{x}\}}{\text{Min}} \left[ H(t,x,\dot{x}) + \frac{\partial f}{\partial t} + \sum_{j=1}^{n} \dot{x}_j \frac{\partial f}{\partial x_j} - G(t,x,\dot{x}) \frac{\partial f}{\partial y} \right] \qquad (1.27)$$

The solution of (1.27) must satisfy the equations

$$0 = H_{\dot{x}_j} + \frac{\partial f}{\partial x_j} - G_{\dot{x}_j} \frac{\partial f}{\partial y} \qquad (1.28)$$

and

$$0 = H + \frac{\partial f}{\partial t} + \sum_{j=1}^{n} \dot{x}_j \frac{\partial f}{\partial x_j} - G \frac{\partial f}{\partial y} \tag{1.29}$$

Differentiation of (1.28) with respect to t, and partial differentiation of (1.29) with respect to $x_j$ yields

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}_j} (H - \frac{\partial f}{\partial y} G) - \frac{\partial}{\partial x_j} (H - \frac{\partial f}{\partial y} G) = 0 \tag{1.30}$$

Partial differentiation of (1.29) with respect to y yields the following results:

$$0 = \frac{\partial^2 f}{\partial t \partial y} + \sum_{j=1}^{n} \dot{x}_j \frac{\partial^2 f}{\partial x_j \partial y} - G \frac{\partial^2 f}{\partial y^2} \tag{1.31}$$

or

$$0 = \frac{d}{dt} (\frac{\partial f}{\partial y}) \tag{1.32}$$

Thus

$$\frac{\partial f}{\partial y} = \text{constant} \tag{1.33}$$

It can be seen from (1.30) that if a new variable

$$\lambda = - \frac{\partial f}{\partial y} \tag{1.34}$$

is introduced, (1.30) results in the Euler-Lagrange equations

$$\frac{d}{dt} (F_{\dot{x}_j}) - F_{x_j} = 0, \quad j = 1, 2, \ldots, n. \tag{1.35}$$

where

$$F = H + \lambda G \tag{1.36}$$

This shows that $-\frac{\partial f}{\partial y}$ plays the role of the Lagrange multiplier. In the case of the constraint being an integral form of (1.23), the Lagrange multiplier is a constant.

In general, the Lagrange multiplier is not a constant. Consider the constraint to be of the form

$$h(t,x,\dot{x},u) = 0 \tag{1.37}$$

or

$$\dot{x} = g(t,x,u),\ x(0) = c \tag{1.38}$$

where the control variable $u = u(u_1,\ldots,u_m)$ is to be chosen so as to minimize the functional $J(x)$. In this case, the Lagrange multiplier is no longer a constant. For example, consider the problem of minimizing the time required to transfer the system described by (1.38) from the initial state $(c_1,\ldots,c_n)$ to the final state $(b_1,\ldots,b_n)$. The functional $T = T(u)$ to be minimized is subject to the oonstraints

$$x_j(T) = b_j,\ j = 1,2,\ldots,n. \tag{1.39}$$

This is a minimum-time problem. By introducing the function

$f(t,x)$ = time required to transfer the system described by (1.38) from x to b

and applying the principle of optimality the equation

$$f(t,x) = \underset{\{u\}}{\text{Min}} \left[\Delta t + f(t+\Delta t, x+g\Delta t) + O(\Delta t)\right] \tag{1.40}$$

is obtained. Expanding the second term in a power series and letting the limit as $\Delta t \to 0$ yields the relation

$$0 = \underset{\{u\}}{\operatorname{Min}} \left[ 1 + f_t + \sum_{j=1}^{n} f_{x_j} g_j \right] \tag{1.41}$$

The solution of (1.41) must satisfy the equations

$$0 = 1+f_t+\sum_{j=1}^{n} f_{x_j} g_j \tag{1.42}$$

and

$$0 = \sum_{j=1}^{n} f_{x_j} \frac{\partial g_j}{\partial u_i}, \; i = 1,2,\ldots,m. \tag{1.43}$$

Partial differentiation of (1.42) with respect to $x_j$ yields

$$\frac{\partial^2 f}{\partial t \partial x_j} + \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k \partial x_j} g_k + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \frac{\partial g_k}{\partial x_j} = 0 \tag{1.44}$$

Since

$$\frac{d}{dt} f_{x_j} = \frac{\partial}{\partial t} f_{x_j} + \sum_{k=1}^{n} \frac{\partial}{\partial x_k} (f_{x_j}) \frac{dx_k}{dt}$$

$$= \frac{\partial^2 f}{\partial x_j \partial t} + \sum_{k=1}^{n} \frac{\partial}{\partial x_k} (f_{x_j}) g_k \tag{1.45}$$

it follows by substituting (1.44) into (1.45) that

$$\frac{d}{dt} f_{x_j} + \sum_{k=1}^{n} \frac{\partial g_k}{\partial x_j} f_{x_k} = 0, \; j = 1,2,\ldots,n. \tag{1.46}$$

Introducing the Lagrange multipliers

$$\lambda_j = f_{x_j} \tag{1.47}$$

into (1.46) yields

$$\frac{d\lambda_j}{dt} + \sum_{k=1}^{n} \frac{\partial g_k}{\partial x_j} \lambda_k = 0, \ j = 1,2,\ldots,n. \tag{1.48}$$

The solution of the 2n+m equations (1.38), (1.43) and (1.48) gives the 2n+m unknown functions which are $\lambda_j$, $x_j$ and $u_i$. The trajectory defined by these variables satisfies the necessary conditions for a minimum-time trajectory.

### 1.3.10 The Dynamic Programming Approach to the Case of Two Fixed End Points

The numerical technique discussed in Section 1.2.1 allows a problem with two fixed end points to be replaced by an initial-value problem.

Consider the problem of minimizing the functional

$$J(x) = \int_0^T F(t,x,\dot{x})dt \tag{1.49}$$

subject to the two end conditions

$$x(0) = a, \ x(T) = b \tag{1.50}$$

Proceeding as in Section 1.3.1 where $u = \dot{x}$ yields the relation

$$f(c+ u\Delta t \ ,t+\Delta t) = \min_{\{u\}} \left[ F(c,u)\Delta t + f(c,t) \right] \tag{1.51}$$

The condition that the final values of $x(t)$ be the assigned values b must be satisfied. This means in effect that at the last stage of the process, for any values of $x_j$, the choice of the control variables $u_j$ must be such as to result in $x_j(T) = b_j$.

Consequently, the terminal constraints fix the function

$f(c,T)$ given by the relation

$$f(c,(N-1)\Delta t) = F(c,u) \tag{1.52}$$

where

$$u = \frac{b-c}{\Delta t} \tag{1.53}$$

thus

$$f(c,(N-1)\Delta t) = F(c, \frac{b-c}{\Delta t}) \tag{1.54}$$

Here, b is taken to be fixed and c is considered to be variable. This is shown in Fig. 1.1.

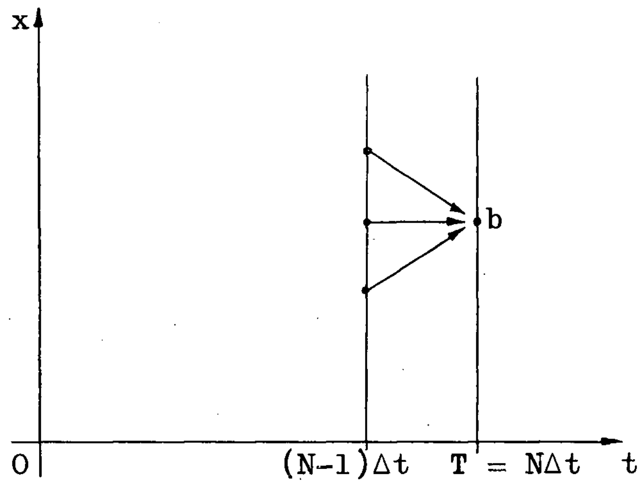


Fig. 1.1. The final stage and the terminal condition

In dynamic programming the terminal constraint simplifies the computation. Since $f(c,T)$ is determined by the terminal conditions, the remaining functions of the sequence $f(c+u\Delta t,\ t+\Delta t)$ are determined by means of (1.51) with no further reference to the terminal conditions.

## 1.4 The Gradient Method[7]

The gradient method or the method of steepest descent is an elementary concept suitable for the solution of minimum problems. In recent years the computational convenience of the gradient method has led to a variety of applications.

In order to present the basic idea of the gradient method, consider the problem of minimizing a continuous function

$$f = f(x_1, \ldots, x_n)$$

If an arc length is defined by

$$ds^2 = \sum_{j=1}^{n} dx_j^{\,2} \tag{1.55}$$

the derivative of f along the arc is

$$\frac{df}{ds} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \cdot \frac{dx_j}{ds} \tag{1.56}$$

Introducing the constraint

$$1 - \sum_{j=1}^{n} \left(\frac{dx_j}{ds}\right)^2 = 0 \tag{1.57}$$

by means of a Lagrange multiplier $\lambda$ yields

$$\frac{df}{ds} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \cdot \frac{dx_j}{ds} + \lambda \left[1 - \sum_{j=1}^{n} \left(\frac{dx_j}{ds}\right)^2\right]$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} y_j + \lambda \left[1 - \sum_{j=1}^{n} y_j^{\,2}\right] \tag{1.58}$$

where $\qquad y_i = \dfrac{dx_i}{ds}$

Partial differentiation of $\frac{df}{ds}$ with respect to $y_j$ yields

$$\frac{\partial}{\partial y_j}\left(\frac{df}{ds}\right) = \frac{\partial f}{\partial x_j} - 2\lambda\ y_j \tag{1.59}$$

For $\frac{df}{ds}$ to be a maximum, the above equation must vanish:

$$\frac{\partial f}{\partial x_j} - 2\lambda y_j = 0 \tag{1.60}$$

Hence
$$y_j = \frac{1}{2\lambda} \cdot \frac{\partial f}{\partial x_j} \tag{1.61}$$

Substituting $y_j$ into (1.57) yields

$$1 - \sum_{j=1}^{n} \left(\frac{1}{2\lambda} \cdot \frac{\partial f}{\partial x_j}\right)^2 = 0$$

Hence

$$\lambda = \pm \frac{1}{2}\left[\sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\right)^2\right]^{\frac{1}{2}} \tag{1.62}$$

Substituting $\lambda$ into (1.61) yields

$$y_j = \frac{dx_j}{ds} = \pm \frac{\partial f}{\partial x_j}\left[\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2\right]^{-\frac{1}{2}}, \quad j = 1,2,\ldots,n. \tag{1.63}$$

and the maximum derivative of f with respect to s is

$$\frac{df}{ds} = \pm\left[\sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\right)^2\right]^{\frac{1}{2}} \tag{1.64}$$

For the steepest descent direction, the negative sign is taken, while the positive sign is taken for the steepest ascent direction. Now consider $x_j$ as components of a vector x, the directions $\frac{dx_j}{ds}$ as components of the unit vector $\frac{dx}{ds}$, and the partial derivatives $\frac{\partial f}{\partial x_j}$ as components of a gradient vector, then

$$\frac{df}{ds} = \text{grad } f \cdot \frac{dx}{ds} \tag{1.65}$$

Introducing the function

$$V = \frac{ds}{d\tau}$$

where $\tau$ is a parameter into (1.55) yields

$$V = \left[ \sum_{j=1}^{n} \left(\frac{dx_j}{d\tau}\right)^2 \right]^{\frac{1}{2}} \tag{1.66}$$

Since

$$\frac{dx_j}{d\tau} = \frac{dx_j}{ds} \cdot \frac{ds}{d\tau}$$

it follows from (1.63) and (1.66) that

$$\frac{dx_j}{d\tau} = \pm \frac{\partial f}{\partial x_j} \left[ \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 \right]^{-\frac{1}{2}} V \tag{1.67}$$

If

$$V = k \left[ \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 \right]^{\frac{1}{2}}$$

where k is a positive constant, it follows that

$$\frac{dx_j}{d\tau} = \pm k \frac{\partial f}{\partial x_j} \tag{1.68}$$

For the steepest descent, the negative sign is taken. This relation is the basic condition of the steepest descent direction for f.

### 1.4.1 Numerical Computation by the Steepest Descent Method

The numerical computation of the minimum of the function $f(x_1,\ldots,x_n)$ requires that the equation of steepest descent be approximated as a finte-difference equation, that is, (1.68) is written as

$$\Delta x_j \cong -k\Delta\tau \frac{\partial f}{\partial x_j}, \quad j = 1,2,\ldots,n.$$

The proportionality constant k can be absorbed by the step size $\Delta\tau$, hence $x_j$ may be written as

$$\Delta x_j \cong -h \frac{\partial f}{\partial x_j}, \quad j = 1,2,\ldots,n.$$

or

$$x_j^{(i+1)} \cong x_j^{(i)} - h^{(i)}\left(\frac{\partial f}{\partial x_j}\right)^{(i)}, \quad j = 1,2,\ldots,n. \tag{1.69}$$

where $h = k\Delta\tau$ and $h^{(i)} = k^{(i)}\Delta\tau$. The process is repeated until a minimum of $f(x_1,\ldots,x_n)$ is obtained at $P_m(x^{(m)})$. Equation (1.69) is a general formula for iteration. The step size h may be adjusted to reduce the number of steps required.

### 1.4.2 The Steepest Descent Method for Finding the Minimum of a Functional

Consider the problem of minimizing the functional

$$J(x) = \int_0^T F(t,x,\dot{x})dt, \quad x(0) = c \tag{1.70}$$

where x belongs to a class of admissible functions.

Let $x(t) = y(t) + h\, u(t),\ u(0) = u(T) = 0$ (1.71) where h is a parameter, $y(t)$ is a first approximation and where u is to be found so that $J(x) < J(y)$.

Equation (1.70) can be written as

$$J(h) = \int_0^T F(t, y+hu, \dot{y}+h\dot{u})dt \tag{1.72}$$

The derivative of $J(h)$ with respect to h is

$$\frac{dJ}{dh} = \int_0^T \sum_{j=1}^{n} (F_{x_j} u_j + F_{\dot{x}_j} \dot{u}_j)dt \tag{1.73}$$

Integrating the second term of (1.73) by parts yields

$$\frac{dJ}{dh} = \int_0^T \sum_{j=1}^{n} (F_{x_j} - \frac{d}{dt}(F_{\dot{x}_j}))\, u_j dt \tag{1.74}$$

For the path of steepest descent (1.74) must be negative which is the case if $u_j$ is chosen so that

$$u_j(t) = \frac{d}{dt}(F_{\dot{x}_j}) - F_{x_j} \tag{1.75}$$

At the minimum of J, $u_j(t) = 0$.

## 1.5 The Calculus of Variations and the Theory of Optimal Control

The general problem of the calculus of variations can be formulated as a problem of Bolza, Lagrange or Mayer. These three formulations are theoretically equivalent and the problem of Lagrange and Mayer can be considered as particular cases of the problem of Bolza[8].

The problem of Bolza can be formulated as follows: Consider the set of functions

$$x_j(t), \quad j = 1,2,\dots,n.$$

satisfying the set of constraints

$$\varphi_i(t,x,\dot{x}) = 0, \; i = 1,2,\dots, \quad m < n \tag{1.76}$$

which involves (n-m) degrees of freedom.

Assuming that the functions $x_j(t)$ and t are consistent with the boundary conditions at t=0 and at t=T, that is,

$$\psi_r\left[0,x(0)\right] = 0, \; r = 1,2,\dots,q. \tag{1.77}$$

$$\psi_p\left[T,x(T)\right] = 0, \; p = q+1,\dots,s \leqslant 2n+2 \tag{1.78}$$

then the problem is to find the special set of functions $x_j(t)$ which results in a minimum for the functional

$$J = \Big[G(t,x)\Big]_0^T + \int_0^T H(t,x,\dot{x})dt \tag{1.79}$$

If the function G of (1.79) is identically zero, that is if,

$$G(t,x) \equiv 0$$

then the functional of (1.79) reduces to

$$J = \int_0^T H(t,x,\dot{x})dt \tag{1.80}$$

This is the problem of Lagrange.

On the other hand, if the integrand of (1.79) is identically zero, that is if,

$$H(t,x,\dot{x}) \equiv 0$$

then the functional of (1.79) becomes

$$J = \Big[ G(t,x) \Big]_0^T$$

This is the problem of Mayer.

It is of primary interest to interpret the general problem of Bolza from the point of view of optimal control. The essential difference between the calculus of variations and the theory of optimal control is that the derivatives in the integrand of the functional J in the calculus of variations are replaced by the control variables $u_k(t)$.

Thus, instead of considering the minimization of the functional

$$J = \Big[ G(t,x) \Big]_0^T + \int_0^T H(t,x,\dot{x})dt$$

subject to the constraints

$$\varphi_i(t,x,\dot{x}) = 0, \; i = 1,2,\ldots, \quad m < n. \tag{1.81}$$

the minimization of the Functional

$$J = \Big[ G(t,x) \Big]_0^T + \int_0^T H(t,x,u)dt$$

subject to the constraints of the form

$$\dot{x}_j = f_j(t,x,u), \; j = 1,2,\ldots,n. \tag{1.82}$$

is considered. Where u is the set $(u_1,\ldots,u_m)$.

In general the optimal control problem can be stated as follows: Given an initial state $(0,x(0))$, find the corresponding admissible control variables $u_k$ defined in the interval $[0,T]$ for which the functional J assumes its minimum.

If the set of control variables $u_k$ can be determined as functions of the state variables $x_j$ so that the functional J is minimum, then the set of control variables $u_k$ can be obtained by feedback from the state variables at the output. In this case the control variables are of the form

$$u_k = L_k(x), \ k=1,2,\ldots,m. \tag{1.83}$$

and the functions $L_k(x)$ are referred to as the control laws. The problem can therefore be stated as an optimal feedback control problem: Find the control laws such that when (1.83) is substituted in (1.82), the functional J assumes its minimum with regard to the set of all admissible control laws.

## 1.6 The Adjoint System and the Euler-Lagrange Equation

The equations of constraints (1.82) are, in general, first order nonlinear differential equations. If these nonlinear differential equations are linearized, one obtains a system of linear differential equations of the form

$$\delta\dot{x}_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \delta x_j + \sum_{k=1}^{m} \frac{\partial f_i}{\partial u_k} \delta u_k \tag{1.84}$$

where the partial derivatives are evaluated on the optimal trajectory.

The adjoint system of (1.84) is defined by

$$\dot{\lambda}_i = -\sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i}, \ i = 1,2,\ldots,n. \tag{1.85}$$

Consider now the problem of Mayer of Section 1.5, where the Euler-Lagrange equations are given by

$$\frac{d}{dt}\left(F_{\dot{x}_j}\right) - F_{x_j} = 0, \quad j = 1,2,\ldots,n. \tag{1.86}$$

and where

$$F = \sum_{i=1}^{n} \lambda_i \left[\dot{x}_i - f_i(t,x,u)\right]$$

substituting this function F in the Euler-Lagrange equations yields

$$\dot{\lambda}_i = -\sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i}, \quad i=1,2,\ldots,n. \tag{1.87}$$

The equations of (1.87) are exactly the same as equations of (1.85), thus the Euler-Lagrange equations in the calculus of variations are the same as the adjoint system for the linearized equations of constraints. It should also be noted that the equations of (1.48) are the Euler-Lagrange equations, where the Lagrange multipliers have the special meaning in dynamic programming given by (1.47).

## 1.7 The Maximum Principle(9)

Pontryagin and his co-authors have stated in the book "The Mathematical Theory of Optimal Processes" that the method of dynamic programming lacks a rigorous logical basis in those cases where it is successfully made use of as a heuristic tool. The maximum principle gives a rigorous mathematical theory for optimal processes. Therefore, it is of theoretical interest to discuss briefly the minimum problem as it is formulated by the maximum principle.

Consider the functional

$$J = \int_0^T F(t,x,\dot{x})dt \tag{1.88}$$

where $x = (x_1, \dots, x_n)$

and the problem is to find the minimum of J for all the admissible control variables $u_k$ which transfer the point from $x_j(0)$ to $x_j(T)$.

Let
$$\dot{x}_0 = F(t,x,u) \tag{1.89}$$
$$\dot{x}_j = u_j, \qquad j = 1,2,\dots,n. \tag{1.90}$$

and form the H-function

$$H(p,x,u) = p_0 F + \sum_{j=1}^{n} p_j u_j \tag{1.91}$$

where the variables p are defined by the relations

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}, \qquad i = 0,1,\dots,n. \tag{1.92}$$

Hence
$$\frac{dp_i}{dt} = -\frac{\partial F}{\partial x_i} p_0, \quad i = 0,1,\dots,n. \tag{1.93}$$

then the relation of (1.93) gives

$$\frac{dp_0}{dt} = 0 \tag{1.94}$$

$$\frac{dp_j}{dt} = -p_0 \frac{\partial F}{\partial x_j}, \qquad j = 1,2,\dots,n. \tag{1.95}$$

The maximum principle states that in order for u and x to define an optimal trajectory it is necessary that there exists a continuous vector function $p = (p_0,\dots,p_n)$ corresponding to u and x, such that

1. for every t, $0 \leqslant t \leqslant T$, the function H attains its maximum at the point u,

$$M(p,x) = \sup_{\{u\}} H(p,x,u) \tag{1.96}$$

2. at the terminal time T, the relations

$$p_0(T) \leqslant 0, \; M\left[p(T),\; x(T)\right] = 0 \tag{1.97}$$

are satisfied.

The equation of (1.96) implies that

$$\frac{\partial H}{\partial u_j} = 0, \quad j = 1,2,\dots,n. \tag{1.98}$$

Partial differentiation of (1.96) with respect to $u_j$ yields

$$\frac{\partial H}{\partial u_j} = p_0 \; \frac{\partial F}{\partial u_j} + p_j, \; j = 1,2,\dots,n. \tag{1.99}$$

By the equation of (1.98), the above equation becomes

$$p_0 \frac{\partial F}{\partial u_j} + p_j = 0, \quad j = 1,2,\dots,n. \tag{1.100}$$

It follows from (1.100) that $p_0 \neq 0$, otherwise all the $p_i = 0$, $i = 0,1,\dots,n$. It is seen from (1.94) and (1.97) that $p_0$ is a negative constant. It is convenient to choose

$$p_0 = -1$$

so that (1.100) becomes

$$p_j = \frac{\partial F}{\partial u_j}, \quad j = 1,2,\dots,n. \tag{1.101}$$

On the other hand, if $p_0 = -1$ is substituted in (1.95) and then integrated, it gives

$$p_j = p_j(0) + \int_0^t \frac{\partial F}{\partial x_j}\, ds, \; j = 1,2,\ldots,n. \qquad (1.102)$$

replacing $u_j$ by $\dot{x}_j$ in (1.101) and substituting into (1.102), yields

$$\frac{\partial F}{\partial \dot{x}_j} = p_j(0) + \int_0^t \frac{\partial F}{\partial x_j}\, ds \qquad (1.103)$$

Differentiating this equation with respect to t yields the Euler-Lagrange equations

$$\frac{d}{dt}\,(F_{\dot{x}_j}) - F_{x_j} = 0, \quad j = 1,2,\ldots,n. \qquad (1.104)$$

## 1.8 The First Integral

The solution of the Euler-Lagrange equations satisfies the relation,

$$\frac{d}{dt}\,(F - \sum_{j=1}^{n} \dot{x}_j F_{\dot{x}_j}) = \frac{\partial F}{\partial t} \qquad (1.105)$$

If F does not depend on the independent variable t explicitly,

$$\frac{\partial F}{\partial t} = 0$$

and the following first integral is obtained.

$$F - \sum_{j=1}^{n} \dot{x}_j\, F_{\dot{x}_j} = C \qquad (1.106)$$

where C is the constant of integration. This relation is called the first integral of the Euler-Lagrange equations.

## 1.9 The Modified Steepest Descent Method

The essence of the modified steepest descent method for solving minimum problems is to consider a general process which is described by a system of ordinary differential equations of the form

$$\dot{x} = f(x,u), \; x_i(0) = c_i, \; i = 1,2,\ldots,n. \qquad (1.107)$$

where $x = (x_1,\ldots,x_n)$

$u = (u_1,\ldots,u_m)$

and $f = (f_1,\ldots,f_n)$

The system under consideration is assumed to move from a point $x(0)$ to another terminal point $x(T)$. Some of the terminal conditions of $x(T)$ may be unspecified. The problem is to minimize the performance function $P(T,x(T))$ by choosing a special set of control variables $u_k$. This is a problem of Mayer. The basic idea of the modified steepest descent method is to consider the function P as a function of a set of unknown parameters which are functions of the unknown initial conditions of the state variables and the Lagrange multipliers. Thus

$$P = P(a) \qquad (1.108)$$

where $a = (a_1,\ldots,a_n)$

$$= \left[\lambda_1(0),\ldots,\lambda_r(0), \; x_{r+1}(0),\ldots,x_n(0)\right]$$

and where $\lambda_i(0)$ are the unknown initial conditions for the

Lagrange multipliers.

The problem under consideration can be formulated as follows: The function

$$F = \sum_{j=1}^{n} \lambda_j \, (\dot{x}_j - f_j) \tag{1.109}$$

is formed where $\lambda_j$ are the Lagrange multipliers.

At a minimum, the Euler-Lagrange equations

$$\frac{d}{dt} (F_{\dot{x}_j}) = F_{x_j} \quad , \quad j = 1,\ldots,n. \tag{1.110}$$

and

$$0 = F_{u_k}, \quad k = 1,\ldots,m. \tag{1.111}$$

must be satisfied.

Substituting (1.109) into (1.110) and (1.111), yields the following equations

$$\frac{d\lambda_j}{dt} = - \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial x_j} \quad , \quad j = 1,\ldots,n. \tag{1.112}$$

$$0 = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial u_k} \quad , \quad k = 1,\ldots,m. \tag{1.113}$$

By solving the system of (2n+m) differential equations of (1.107), (1.112) and (1.113), the (2n+m) unknown variables $x_j$, $\lambda_j$, and $u_k$ can be determined. The general scheme for the solution is represented in Fig. 1.2. The initial values are sampled and introduced into a high speed repetitive trajectory computer. The performance function P is determined and the

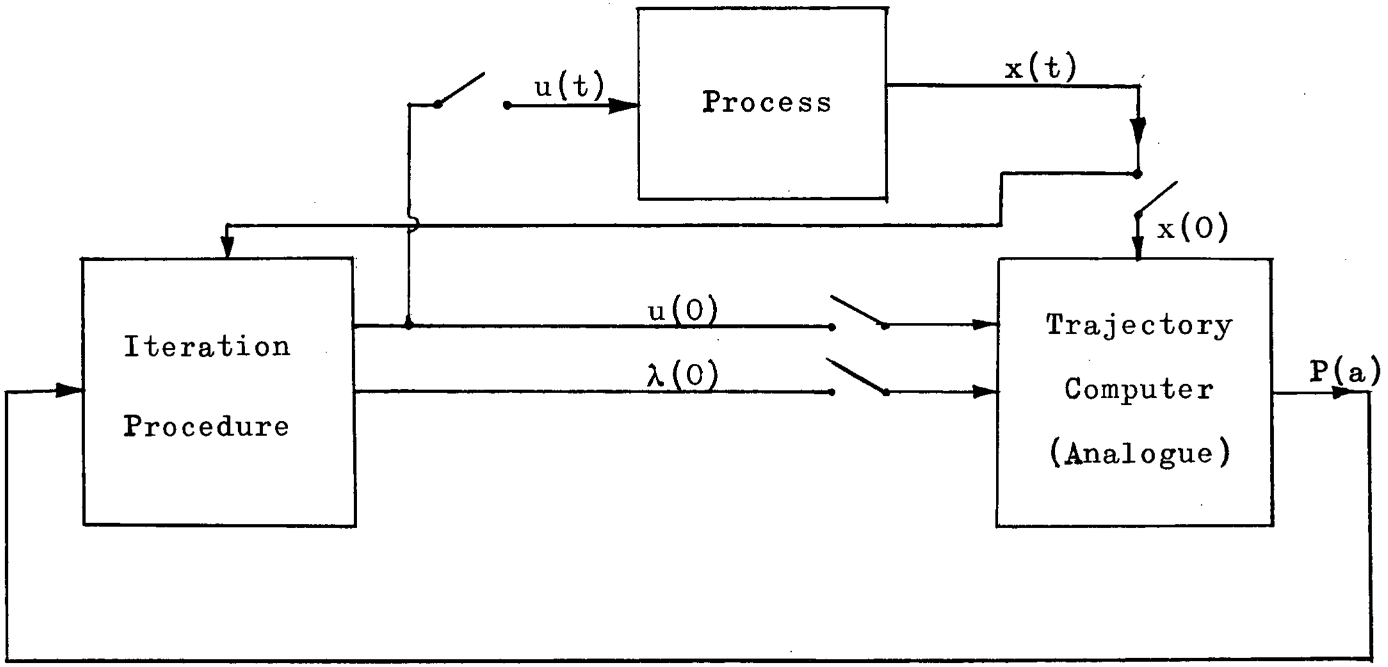


Fig. 1.2 <u>A general optimal process</u>

unknown initial values are adjusted by an iterative procedure to minimize P. The sampled value of u is introduced into the process. If there are no disturbances the state $x(t)$ of the process will correspond in real time to the computed trajectory. In the above system the initial values for the trajectory are the real-time values of the process variables.

In most problems not all the initial conditions are given and therefore a search procedure for the minimum of the function P must be employed. The important idea of the modified steepest descent method is to solve the preceeding (2n+m) equations subject to the condition that the derivatives of the performance function P with respect to the parameters $a_j$ are always negative, that is,

$$\frac{\partial P}{\partial a_j} < 0, \quad j = 1,\ldots,n. \tag{1.114}$$

The values of $a_j$ are unknown and can be determined by iteration. For each iteration the condition of (1.114) must be satisfied. The modified steepest descent method does not rule out the possibility of a local minimum unless the entire range of parameter values are used which may not be practical (see Fig. 1.3 where $a_k$ results in a true minimum and $a_k'$ results in a local minimum).

As for the numerical computation, it is assumed that the computation starts from a point $A_0 = (a_{j0})$ which may be arbitrary. The parameter $a_{10}$ is adjusted so that P decreases to a minimum. The remaining parameters can then be adjusted in sequence in the same manner. Proceeding in this way, a new

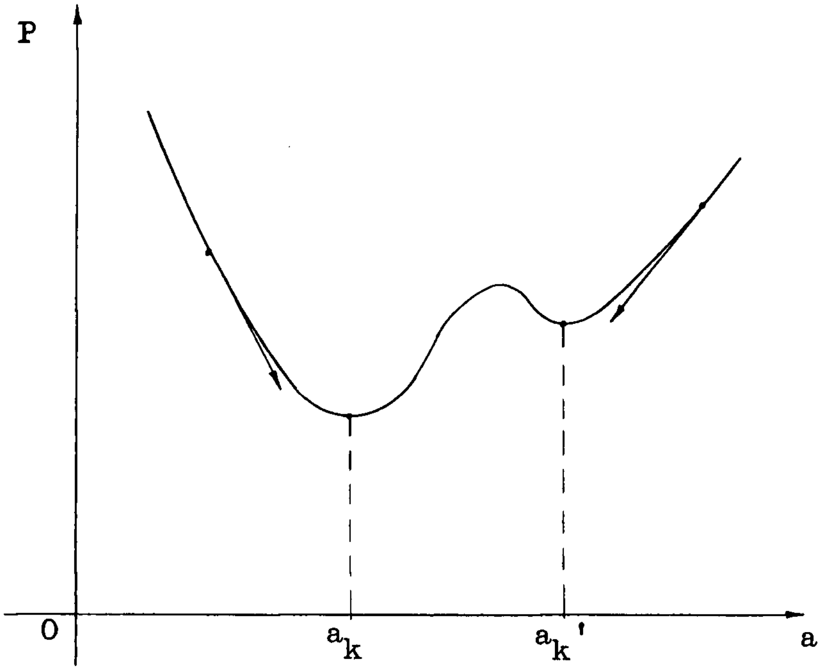


Fig. 1.3 True minimum and local minimum

point $A_1 = (a_{j1})$ is obtained. The general step may be summarized in the following way. From a point $A_r = (a_{jr})$ to the next point $A_{r+1} = (a_{j(r+1)})$ is found by a step-by-step procedure.

1. Adjust $a_{1r}$ by a small amount to have a smaller P until P starts to increase.
2. Repeat 1 for $a_{2r},\dots,a_{nr}$, each time adjusting one parameter only.
3. Now a new point $A_{r+1} = (a_{j(r+1)})$ is obtained and the steps 1 and 2 are repeated until a minimum of P is obtained.

It is important to note that for the adjustment of each $a_{j(r)}$ the conditions

$$P(a_{1(r+1)}, a_{2r}, \ldots, a_{nr}) < P(a_{1r}, \ldots, a_{nr})$$

$$P(a_{1(r+1)}, a_{2(r+1)}, a_{3r}, \ldots, a_{nr}) < P(a_{1(r+1)}, a_{2r}, \ldots a_{nr})$$

..................................................

$$P(a_{1(r+1)}, \ldots, a_{n(r+1)}) < P(a_{1(r+1)}, \ldots, a_{(n-1)(r+1)}, a_{nr})$$

apply.

## 1.10 Remarks

It is of interest to compare the modified steepest descent method studied in this thesis with other computational techniques. The standard variational technique of the calculus of variations transforms the original variational problem into a problem in the solution of ordinary differential equations involving two-point boundary conditions. To solve a two-point boundary value problem is usually difficult from the computational point of view.

Dynamic programming, in theory, eliminates the two-point boundary value problem. However, it introduces a new difficulty, the problem of dimensionality, which means that an extremely large digital computer memory is required.

The gradient method or the steepest descent method was developed by Cauchy and has been independently applied to variational problems dealing with flight paths by Kelley and Bryson. This technique has been very successful. However, it requires extensive digital computing facilities and does not appear suitable for developing comparatively simple real-time optimal controllers.

The modified steepest descent method is particularly suitable for the solution of certain classes of minimum problems by means of digital or analogue computers. The analogue computer is very convenient for solving trajectory problems. Another advantage of employing the analogue computer is that it is then possible to construct comparatively simple real-time optimal controllers. Since the analogue computer solves problems in a continuous manner, it is suitable for high-speed computation and feedback methods can be used for obtaining iterative solutions.

# 2. OPTIMAL CONTROL PROCESSES FOR ROCKET FLIGHT PROBLEMS

## 2.1 Introduction

Analytical studies may facilitate the computation of the solution for optimal control problems. The iterative approach used in the modified steepest descent method may also be greatly simplified if an analytical expression for the optimal control law in terms of state variables can be found.

The calculus of variations is the only suitable method for obtaining analytic information about the properties of the optimal control law and the optimal trajectory and is therefore, of fundamental importance. This chapter is devoted to the application of the calculus of variations to the problem of rocket flight and to analytical studies for deriving optimal control laws.

It is also of theoretical interest to have a complete analytical solution of a problem. This allows a study of the properties of the Lagrange multipliers which play an important role in the determination of optimal control laws. On the other hand, the analytical solution can serve as a means for checking the accuracy of the analogue computations used in the modified steepest descent method discussed in Chapter 3.

## 2.2 Formulation of Rocket Flight Problems by Means of the Calculus of Variations

The determination of optimal trajectories for missiles, aircrafts and satellites is an important application of optimization theory. Goddard recognized the calculus of variations as an important tool in the analysis of rocket performance in 1919. A general theory of rocket flight problems was recently developed by Breakwell, Fried, Lawden, Miele, Leitman and others. A brief review of the rocket flight problem will now be given.

### 2.2.1 Basic Assumptions and Equations of Motion

For the general formulation of the rocket flight problem, the following assumptions are made (see Fig. 2.1):

(1) The rocket is considered as a particle or a point mass.

(2) The power plant of the rocket engine is considered as an ideal engine, so that the equivalent exit velocity $V_e$ for the fuel is a constant. The thrust is taken as $V_e\beta$, where $\beta$ is a control parameter.

(3) The Earth is assumed to be flat, and the acceleration due to gravity is taken to be constant.

(4) The rocket moves in a vertical two-dimensional plane.

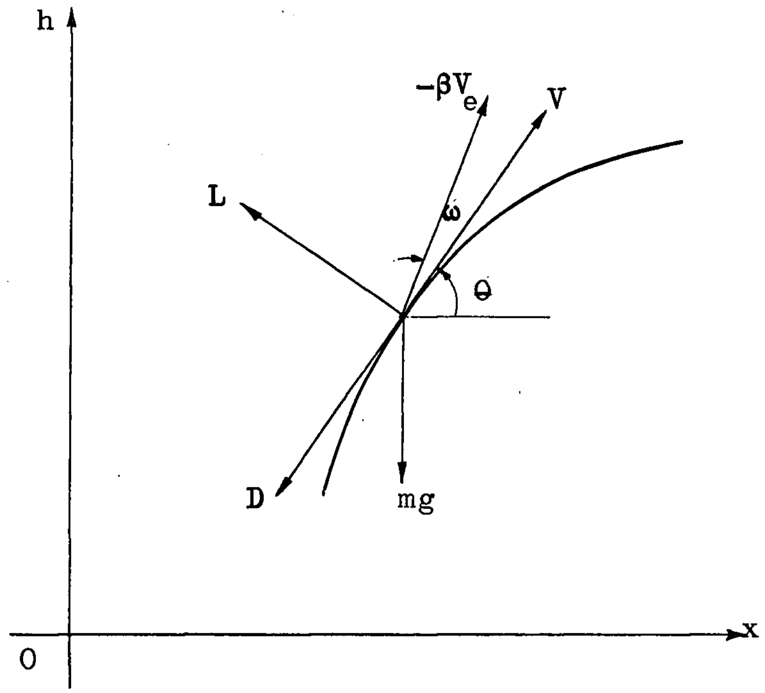


Fig. 2.1 The forces acting on a rocket

By these hypotheses the equations of motion for a rocket can be written[10] as

$$\varphi_1 = \dot{x} - V \cos \Theta = 0 \tag{2.1}$$

$$\varphi_2 = \dot{h} - V \sin \Theta = 0 \tag{2.2}$$

$$\varphi_3 = \dot{V} + g \sin \Theta + \frac{D - V_e \beta \cos \omega}{m} = 0 \tag{2.3}$$

$$\varphi_4 = \dot{\Theta} + \frac{g}{V} \cos \Theta - \frac{L + V_e \beta \sin \omega}{mV} = 0 \tag{2.4}$$

$$\varphi_5 = \dot{m} + \beta = 0 \tag{2.5}$$

where x is the range, h is the altitude, V is the velocity, g is the acceleration due to gravity, L is the lift, D is the drag, m is the mass, $\Theta$ is the path inclination, and $\omega$ is the angle between the thrust and the velocity. The drag is assumed to have the general form

$$D = D(h,V,L) \tag{2.6}$$

and the engine characteristics of the rocket are represented as a function of a parameter $\alpha$, that is, the control parameter is

$$\beta = \beta(\alpha) \tag{2.7}$$

### 2.2.2 Formulation of the Rocket Flight Problem

The set of five equations of motion, (2.1) to (2.5), involves one independent variable, the time t, and eight dependent variables, they are: x, h, V, $\Theta$, m, $\omega$, L and $\beta$. Thus, the problem under consideration has three degrees of freedom, and three conditions for optimal performance can be imposed. In this connection, the optimal control problem of Mayer type, can be stated as follows:

Among all sets of functions x(t), h(t), V(t), $\Theta$(t), m(t),

$\omega(t)$, $L(t)$ and $\beta(t)$, satisfying the equations of motion, (2.1) to (2.5), and certain prescribed end conditions, to determine the special set which minimizes the performance function $\left[P\right]_{t_o}^{t_f}$, where

$$P = P(x,h,V,\Theta,m,t)$$

The end conditions are constraints imposed on the initial and the final values of x, h, V, Θ, m and t. In general, not all the end conditions are known.

In the case that two additional constraining equations of the form

$$\varphi_6 = \Phi(x,h,V,\Theta,m,L,\beta,\omega,t) = 0 \tag{2.8}$$

$$\varphi_7 = \Psi(x,h,V,\Theta,m,L,\beta,\omega,t) = 0 \tag{2.9}$$

are present, the problem has only one remaining degree of freedom, and one condition for optimal performance can be imposed.

By introducing a set of Lagrange multipliers $\lambda_i(t)$, $i = 1,2,\ldots,7$, the so-called augmented function can be formed

$$F = \sum_{i=1}^{7} \lambda_i \varphi_i \tag{2.10}$$

and the Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_j}\right) = \frac{\partial F}{\partial x_j}, \quad j = 1,\ldots,8. \tag{2.11}$$

where $x_1 = x$, $x_2 = h$, $x_3 = V$, $x_4 = \Theta$, $x_5 = m$, $x_6 = L$, $x_7 = \alpha$, and $x_8 = \omega$.

As discussed in the last chapter, if the augmented function F of (2.10) does not depend on the time t explicitly, the first integral

$$F - \sum_{i=1}^{7} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i} = C \tag{2.12}$$

exists.

The Euler-Lagrange equations and the first integral for the rocket flight problem are given in the Appendix.

Several possibilities exist for modifying the trajectory of a rocket. The elevator position, the thrust magnitude, and the thrust direction can be controlled. Thus, for a given set of end conditions, an infinite number of trajectories exist which are mathematically and physically possible. Among all the possible trajectories it is of interest to find those trajectories which meet a requirement for optimal performance.

Particular forms of the performance function P are:

(1) $P = \Big[-m\Big]_{t_o}^{t_f}$ , problems of minimizing the fuel consumption.

(2) $P = \Big[t\Big]_{t_o}^{t_f}$ , problems of minimizing the flight time.

(3) $P = \Big[-x\Big]_{t_o}^{t_f}$ , problems of maximizing the range.

## 2.3 Analytical Study of Optimal Control for the Sounding Rocket Problem[11,12]

The equations of motion for the rocket flight, (2.1) to (2.5), are nonlinear differential equations, and the associated Euler-Lagrange equations, (A.1) to (A.8), are linear differential equations whose coefficients are functions of the state variables. If the equations of motion can be solved so that the state variables are functions of time, the Euler-Lagrange equations may be considered as linear differential equations with time varying coefficients.

Since there is no systematic analytical method for solving nonlinear differential equations, the determination of an analytical solution for the rocket flight problem is extremely difficult and, in general, is not possible. However, analytical solutions may be obtained in special simple cases.

A problem of interest is the case of rocket flight in a resisting medium. This problem can be solved analytically in the case of vertical flight with a drag function of the form

$$D = kV^2 \exp(-ah) \tag{2.13}$$

where k and a are constants.

The sounding rocket problem has been studied by many scientists, such as, Hamel (1927), Oberth (1929), Malina and Smith (1938), Tsien and Evans (1951), and Leitmann (1957), etc. Much work, both numerical and analytical, has been done on this problem. However, with the exception of trivial cases, no complete analytical solution has yet been obtained. The partial analytical results published in the literature will therefore be extended as far as possible in an attempt to obtain a complete analytical solution.

It is assumed that the following end conditions are specified:

$$h(t_o) = h_o = 0 \quad , \quad h(t_f) = h_f = \text{final altitude (given)}$$

$$V(t_o) = V_o = 0 \quad , \quad V(t_f) = V_f = 0$$

$$m(t_o) = m_o = \text{unknown}, \quad m(t_f) = m_f = \text{payload (given)}$$

where $m_o$ is the initial mass which includes the mass of the fuel.

The problem is to minimize the fuel consumption required to reach a specified altitude by controlling the thrust. The performance function P is $(m_o - m_f)$. Since $m_f$ is fixed, the problem is equivalent to minimizing the initial mass $m_o$.

The Euler-Lagrange equation (A.17) shows that two different classes of subarcsexist for the optimal trajectory:

(1) $\frac{d\beta}{d\alpha} = 0$, subarcswith constant thrust.

(2) $\lambda_5 - \lambda_3 \frac{V_e}{m} = 0$, subarcswith variable thrust.

For the sounding rocket problem it can be shown that impulsive boosting is always required. In this case the equation of motion (A.12) may be approximated for the boosting period by the equation

$$\dot{V} - \frac{V_e \beta}{m} \cong 0, \quad t_o \leqslant t \leqslant t_1. \tag{2.14}$$

where $t_o$ is the initial time and $t_1$ is the end of the boosting interval.

Solving (2.14) together with (A.13) yields

$$m \cong m_o \exp\left(- \frac{V}{V_e}\right), \quad t_o \leqslant t \leqslant t_1. \tag{2.15}$$

where $m_o$ is the initial mass and $m_1$ is the mass at the end of the boosting interval.

The boosting interval is often very short and the impulsive thrust is extemely large. The total time for the boosting period may then be taken as $t_1 - t_o = \Delta t$, and the velocity V is suddenly increased from zero to $V_1$ while the mass decreases from $m_o$ to $m_1$. The entire optimal trajectory has only three subarcs: The boosting subarc, the variable thrust subarc, and the coasting subarc (zero thrust).

Integrating (A.11) from $t_o$ to $t_1$ yields

$$h_1 = \int_{t_o}^{t_1} V \, dt = \int_{t_o}^{t_o + \Delta t} V \, dt$$

since $\Delta t$ is very small and V is finite, the above integral is negligible and

$$h_1 = \Delta h \cong 0 \tag{2.16}$$

Let the mass flow of the impulsive boosting be $\beta_m$. Integration of (A.13) gives

$$m_1 - m_o = \int_{t_o}^{t_1} \beta_m dt$$

$$= \beta_m \Delta t \tag{2.17}$$

Since $\beta_m$ is extremely large, the product $\beta_m \Delta t$ is a finite quantity.

Solving the Euler-Lagrange equations (A.14) to (A.16) yields

$$\lambda_{21} = \lambda_{20} + \int_{t_o}^{t_1} \frac{\lambda_3}{m} \frac{\partial D}{\partial h} = dt \cong \lambda_{20} \tag{2.18}$$

$$\lambda_{31} = \lambda_{30} + \int_{t_o}^{t_1} \left[ -\lambda_2 + \frac{\lambda_3}{m} \frac{\partial D}{\partial V} \right] dt \cong \lambda_{30} \tag{2.19}$$

and

$$\lambda_{51} = \lambda_{50} + \int_{t_o}^{t_1} \frac{\lambda_3}{m^2} (V_e \beta_m - D) dt$$

$$= \lambda_{50} + \int_{m_o}^{m_1} \frac{\lambda_3 V_e}{m^2} (-dm) - \int_{t_o}^{t_1} \frac{\lambda_3 D}{m^2} dt$$

$$\cong \lambda_{50} + \lambda_{30} V_e \left(\frac{1}{m_1} - \frac{1}{m_o}\right) \tag{2.20}$$

where the second subscript denotes the value of $\lambda_i$ at the time $t = t_k$, that is, $\lambda_i(t_k) = \lambda_{ik}$. The above approximations are obtained by neglecting all integrals with respect to t since the time interval $t_1 - t_o$ is negligible. The drag function and its derivatives, $\frac{\partial D}{\partial V}$ and $\frac{\partial D}{\partial h}$ are finite during this interval. This can be seen from the drag function (2.13).

Information about the end conditions of the Lagrange multipliers may be obtained from the transversality condition and the first integral.

The transversality condition is

$$\left[ dm + \lambda_2 dh + \lambda_3 dV + \lambda_5 dm + C\ dt \right]_{t_o}^{t_f} = 0 \qquad (2.21)$$

where C is the first integral.

Since $m_o$, $t_o$, and $t_f$ are free, the transversality condition yields

$$\lambda_{50} = -1 \qquad (2.22)$$

and

$$C = 0 \qquad (2.23)$$

The transversality condition does not give any information about the final values of the Lagrange multipliers for this problem. However, the first integral (A.18) gives

$$\lambda_{3f} = 0 \qquad (2.24)$$

For the variable thrust subarc, $\frac{d\beta}{d\alpha} \neq 0$, and it follows from (A.17) that the condition

$$\lambda_5 - \lambda_3 \frac{V_e}{m} = 0, \ t_1 \leqslant t \leqslant t_2 \ , \qquad (2.25)$$

must be satisfied, where $t_2$ is the time at which the thrust is cut off.

The first integral (A.18) now reduces to

$$\lambda_2 V - \lambda_3 \ (g + \frac{D}{m}) = 0, \ t_1 \leqslant t \leqslant t_2 \ . \qquad (2.26)$$

It is obvious that (2.26) also holds for the coasting subarc where $\beta = 0$.

Differentiating (2.25) with respect to t yields

$$m \ \dot{\lambda}_5 + \lambda_5 \dot{m} - \dot{\lambda}_3 \ V_e = 0 \qquad (2.27)$$

Substituting (2.5), (A.15) and (A.16) into (2.27) gives

$$\lambda_2 - \frac{\lambda_3}{m} \ (\frac{D}{V_e} + \frac{\partial D}{\partial V}) = 0, \ t_1 \leqslant t \leqslant t_2 \ . \qquad (2.28)$$

Substituting (2.13) into (2.28) yields

$$\frac{\lambda_2}{\lambda_3} = \frac{D}{m} \ (\frac{2}{V} + \frac{1}{V_e}), \ t_1 \leqslant t \leqslant t_2 \ . \qquad (2.29)$$

Eliminating $\lambda_2$ and $\lambda_3$ between (2.26) and (2.29) gives

$$mg - D(1 + \frac{V}{V_e}) = 0, \quad t_1 \leqslant t \leqslant t_2 \ . \qquad (2.30)$$

Equation (2.30) shows that the velocity V can not be zero during the variable thrust period. Therefore impulsive boosting is required. Moreover, equation (2.30) can be used to determine the switching time $t_1$ for the actual flight, and it will be used as a control law in the next chapter for the analogue computation of the sounding rocket problem.

Differentiating (2.30) with respect to t yields

$$\dot{m} = \frac{D}{g} \left[\frac{2\dot{V}}{V} + \frac{3\dot{V}}{V_e} - a \ V(1 + \frac{V}{V_e})\right] \qquad (2.31)$$

Substituting (2.30) and (2.31) into (A.12) gives

$$\frac{\dot{V}}{V_e} = \frac{V}{V_e}\, a\, V_e \left[ \frac{\dfrac{V^2}{V_e^{\,2}} + \left(1 - \dfrac{g}{aV_e^{\,2}}\right)\dfrac{V}{V_e} - \dfrac{2g}{aV_e^{\,2}}}{\dfrac{V^2}{V_e^{\,2}} + 4\dfrac{V}{V_e} + 2} \right] \tag{2.32}$$

Let $v = \dfrac{V}{V_e}$ and $b = \dfrac{g}{aV_e^{\,2}}$, then (2.32) can be written as

$$\dot{v} = \frac{gv}{bV_e}\,\frac{v^2 + (1-b)v - 2b}{v^2 + 4v + 2} \tag{2.33}$$

or

$$dt = \frac{bV_e}{g}\,\frac{v^2 + 4v + 2}{v\left[v^2 + (1-b)v - 2b\right]}\,dv \tag{2.34}$$

Integrating this equation from $t_1$ to t gives

$$t = t_1 + \frac{V_e}{g}\left[\ln\frac{v_1}{v} + \frac{(1+b)}{2}\ln\frac{v^2 + (1-b)v - 2b}{v_1^{\,2} + (1-b)v_1 - 2b} + \frac{K}{2}\ln\left\{\frac{2v_1 + (1-b) + K}{2v + (1-b) + K}\;\frac{2v + (1-b) - K}{2v_1 + (1-b) - K}\right\}\right] \tag{2.35}$$

where $K = \sqrt{(1-b)^2 + 8b}$

Since

$$\dot{h} = V = vV_e$$

it follows that $dh = V_e v\,dt$

Substituting (2.34) into this equation yields

$$dh = \frac{1}{a}\,\frac{v^2 + 4v + 2}{v^2 + (1-b)v - 2b}\,dv$$

Integrating this equation from $h_1$ to h gives

$$h = h_1 + \frac{1}{a}\left[v - v_1 + \frac{3+b}{2}\ln\frac{v^2+(1-b)v-2b}{v_1{}^2+(1-b)v_1-2b}\right.$$

$$\left.+\frac{K}{2}\ln\left\{\frac{2v_1+(1-b)+K}{2v+(1-b)+K}\;\frac{2v+(1-b)-K}{2v_1+(1-b)-K}\right\}\right] \tag{2.36}$$

The mass m can be determined as a function of v and t by rewriting (A.12) in the form

$$\frac{\dot{m}}{m} = -\frac{1}{V_e}\left(\dot{V} + g + \frac{D}{m}\right)$$

and then substituting (2.30) for $\frac{D}{m}$ into the last equation. Thus

$$\frac{\dot{m}}{m} = -\dot{v} - \frac{g}{V_e} - \frac{g}{V_e(1+v)}$$

or

$$\frac{dm}{m} = -\left(dv + \frac{g}{V_e}\,dt\right) - \frac{g\,dt}{V_e(1+v)}$$

Now substituting (2.34) for dt in the above equation gives

$$\frac{dm}{m} = -\left(dv + \frac{g}{V_e}\,dt\right) - \frac{b}{v(1+v)}\;\frac{v^2+4v+2}{v^2+(1-b)v-2b}\;dv$$

which can be integrated to the form

$$\ln m\Big|_{m_1}^{m} = -\left(v + \frac{g}{V_e}\,t\right)\Big|_{t_1}^{t} + \ln\frac{v^2+v}{v^2+(1-b)v-2b}\Bigg|_{v_1}^{v}$$

or

$$m=m_1 \frac{v^2+v}{v_1^2+v_1} \frac{v_1^2+(1-b)v_1-2b}{v^2+(1-b)v-2b} \exp\left[-(v-v_1)-\frac{g}{V_e}(t-t_1)\right] \quad (2.37)$$

To solve the Euler-Lagrange equation (A.16), the following equations

$$\frac{V_e\beta-D}{m} = \dot{V} + g$$

$$\frac{\lambda_3}{m} = \frac{\lambda_5}{V_e}$$

which are obtained from (A.12) and (2.25) are required. Substituting these two equations into (A.16) gives

$$\dot{\lambda}_5 = \lambda_5(\dot{v} + \frac{g}{V_e})$$

where $\dot{v} = \dot{V}/V_e$. Integrating this equation from $t_1$ to t yields

$$\lambda_5 = \lambda_{51} \exp\ (v + \frac{g}{V_e}\ t) \quad (2.38)$$

Substituting this into (2.25) gives

$$\lambda_3 = \frac{m\lambda_{51}}{V_e} \exp\ (v + \frac{g}{V_e}\ t) \quad (2.39)$$

The Lagrange multiplier $\lambda_2$ can be determined by the first integral (2.26):

$$\lambda_2 = \frac{m\lambda_{51}}{v\ V_e^2} (g + \frac{D}{m}) \exp\ (v + \frac{g}{V_e}\ t) \quad (2.40)$$

For the coasting subarc, the thrust is cut off, so that $\beta = 0$. Thus $\dot{m} = 0$ and the mass m is constant. Let $m = m_2$ at

$t = t_2$, then $m_2 = m_f$, and the equations of motion and the Euler-Lagrange equations become

$$\dot{h} - V = 0 \tag{2.41}$$

$$\dot{V} + g + \frac{D}{m_f} = 0 \tag{2.42}$$

$$\dot{m} = 0 \tag{2.43}$$

and

$$\dot{\lambda}_2 = \frac{\lambda_3}{m_f}\frac{\partial D}{\partial h} \tag{2.44}$$

$$\dot{\lambda}_3 = -\lambda_2 + \frac{\lambda_3}{m_f}\frac{\partial D}{\partial V} \tag{2.45}$$

$$\dot{\lambda}_5 = -\lambda_3 \frac{D}{{m_f}^2} \tag{2.46}$$

where $D = k\, V^2 \exp(-ah)$

Since
$$\dot{V} = \frac{dV}{dt} = \frac{dV}{dh}\ \frac{dh}{dt}$$

$$= V\frac{dV}{dh} = v{V_e}^2\frac{dv}{dh}$$

substituting this equation into (2.42) gives

$$v\frac{dv}{dh} + \frac{g}{{V_e}^2} + \frac{kv^2}{m_f}\exp(-ah) = 0$$

or
$$\frac{d}{dh}(v^2) + \frac{2k}{m_f}v^2\exp(-ah)+2\,ab = 0 \tag{2.47}$$

where
$$b = \frac{g}{a{V_e}^2}$$

Equation (2.47) is a linear differential equation with respect to $v^2$. It has an integrating factor of the form $\exp(-\frac{2k}{m_f}e^{-ah})$, and can be written as

$$\frac{d}{dh}\left[v^2\exp(-\frac{2k}{m_f}e^{-ah})\right] = 2ab\ \exp\ (-\frac{2k}{m_f}e^{-ah}) \tag{2.48}$$

In order to integrate the right hand side of this equation, let

$$y = e^{-ah},\quad dy = -aydh$$

and
$$\int \exp\ (-\frac{2k}{m_f}e^{-ah})dh = -\int \exp\ (-\frac{2k}{m_f}y)\ \frac{dy}{ay}$$

The integration can be performed by expanding the exponential function in a Taylor series. Thus

$$\int \exp\ (c\ y)\ \frac{dy}{y} = \ln\ (c\ y) + \sum_{n=1}^{\infty} \frac{(c\ y)^n}{n.n\ !}$$

and integrating (2.48) yields

$$v^2 = 2\ b\ \exp\ (\frac{2k}{am_f}e^{-ah})\left[-ah + \sum_{n=1}^{\infty} \frac{(-\frac{2k}{am_f})^n\ e^{-anh}}{n\ .\ n\ !} + C_1\right]$$

$$\triangleq f(h) \tag{2.49}$$

where $C_1$ is the constant of integration and is given by

$$C_1 = ah_f - \sum_{n=1}^{\infty} \frac{(-\frac{2k}{am_f})^n\ e^{-anh_f}}{n\ .\ n\ !}$$

Thus $C_1$ is a known constant since $h_f$ is given. From (2.41)

$$dt = \frac{dh}{V} = \frac{dh}{V_e v}$$

and (2.49) gives
$$v = \sqrt{f(h)}$$

thus
$$dt = \frac{1}{V_e}\ \frac{dh}{\sqrt{f(h)}}$$

Integrating this equation gives

$$t = t_2 + \frac{1}{V_e}\int_{h_2}^{h} \frac{dh}{\sqrt{f(y)}} \tag{2.50}$$

Substituting the first integral (2.26) into (2.45) for $\lambda_2$ yields

$$\dot{\lambda}_3 = \frac{\lambda_3}{V} \left(-g + \frac{D}{m_f}\right)$$

Since $$\dot{\lambda}_3 = \frac{d\lambda_3}{dt} = \frac{d\lambda_3}{dh} \cdot \frac{dh}{dt} = V \frac{d\lambda_3}{dh}$$

thus $$V \frac{d\lambda_3}{dh} = \frac{\lambda_3}{V} \left(-g + \frac{kV^2}{m_f} e^{-ah}\right)$$

or $$\frac{d\lambda_3}{\lambda_3} = \frac{k}{m_f} e^{-ah} dh - \frac{g}{V_e^{\,2}} \frac{dh}{v^2}$$

But (2.49) gives $$v^2 = f(h)$$

thus

$$\frac{d\lambda_3}{\lambda_3} = \frac{k}{m_f} e^{-ah} dh - \frac{g}{V_e^{\,2}} \frac{dh}{f(h)}$$

Integrating this equation yields

$$\lambda_3 = \exp\left[-\frac{k}{am_f} e^{-ah} - \frac{g}{V_e^{\,2}} \int_{h_2}^{h} \frac{dy}{f(y)}\right] + C_2$$

$$\triangleq F(h) \tag{2.51}$$

where $C_2$ is the constant of integration and is given by

$$C_2 = -\exp\left[-\frac{k}{am_f} e^{-ah_f} - \frac{g}{V_e^{\,2}} \int_{h_2}^{h_f} \frac{dv}{f(y)}\right] \tag{2.52}$$

The Lagrange multiplier $\lambda_2$ can be obtained from the first integral

$$\lambda_2 = \frac{F(h)}{vV_e} \left(g + \frac{kv^2V_e^{\,2}}{m_f} e^{-ah}\right) \tag{2.53}$$

Substituting (2.51) into (2.46) gives

$$\dot{\lambda}_5 = - F(h) \frac{kV^2}{m_f^2} e^{-ah}$$

Since $$\dot{\lambda}_5 = \frac{d\lambda_5}{dt} = \frac{d\lambda_5}{dh} \cdot \frac{dh}{dt} = V \frac{d\lambda_5}{dh}$$

and $$V = V_e v = V_e \sqrt{f(h)}$$

thus $$\frac{d\lambda_5}{dh} = \frac{-kV_e}{m_f^2} \sqrt{f(h)}\, F(h)\, e^{-ah}$$

Integrating this equation gives

$$\lambda_5 = \frac{-kV_e}{m_f^2} \int_{h_2}^{h} F(y) \sqrt{f(y)}\, e^{-ay}\, dy + C_3 \qquad (2.54)$$

where $C_3$ is the constant of integration. For the further discussion it will be convenient to give a summary for the solution of the sounding rocket problem.

(1) For the boosting subarc $(0 \leq t \leq t_1)$, where $t_o = 0$.

$$h_1 = 0 \qquad (2.16)$$

V suddenly increases from zero to $V_1$

$m = m_o \exp(-\frac{V}{V_e})$, where $m_o$ is unknown. (2.15)

$$t_1 \cong \Delta t \cong 0$$

$$\lambda_2 \cong \lambda_{20} \qquad (2.18)$$

$$\lambda_3 \cong \lambda_{30} \qquad (2.19)$$

$$\lambda_5 \cong \lambda_{50} + \lambda_{30} V_e \left(\frac{1}{m} - \frac{1}{m_o}\right) \qquad (2.20)$$

where $$\lambda_{50} = -1 \qquad (2.22)$$

The first integral is

$$\lambda_2 V - \lambda_3 (g + \frac{D}{m}) - (\lambda_5 - \lambda_3 \frac{V_e}{m}) = 0$$

(2) For the variable thrust subarc, $(t_1 \leqslant t \leqslant t_2)$,

$$h = \frac{1}{a}\left[v - v_1 + \frac{3+b}{2} \ln \frac{v^2 + (1-b)v - 2b}{v_1^2 + (1-b)v_1 - 2b}\right.$$

$$\left. + \frac{K}{2} \ln \left\{ \frac{2v_1 + (1-b) + K}{2v + (1-b) + K} \cdot \frac{2v + (1-b) - K}{2v_1 + (1-b) - K} \right\}\right] \tag{2.36}$$

$$m = m_1 \cdot \frac{v^2 + v}{v_1^2 + v_1} \cdot \frac{v_1^2 + (1-b)v_1 - 2b}{v^2 + (1-b)v - 2b} \exp\left[-(v - v_1)\right.$$

$$\left. - \frac{g}{V_e} t \right] \tag{2.37}$$

$$t = \frac{V_e}{g}\left[\ln \frac{v_1}{v} + \frac{(1+b)}{2} \ln \frac{v^2 + (1-b)v - 2b}{v_1^2 + (1-b)v_1 - 2b}\right.$$

$$\left. + \frac{K}{2} \ln \left\{ \frac{2v_1 + (1-b) + K}{2v + (1-b) + K} \; \frac{2v + (1-b) - K}{v_1 + (1-b) - K} \right\}\right] \tag{2.35}$$

where $b = \dfrac{g}{aV_e^2}$, $K = \sqrt{(1-b)^2 + 8b}$ and $v = \dfrac{V}{V_e}$

$$\lambda_5 - \lambda_3 \frac{V_e}{m} = 0 \tag{2.25}$$

$$\lambda_2 = \frac{m\lambda_{51}}{v\,V_e^2} (g + \frac{D}{m}) \exp\,(v + \frac{g}{V_e} t) \tag{2.40}$$

$$\lambda_3 = \frac{m\,\lambda_{51}}{V_e} \exp\,(v + \frac{g}{V_e} t) \tag{2.39}$$

$$\lambda_5 = \lambda_{51} \exp\,(v + \frac{g}{V_e} t) \tag{2.38}$$

The first integral is $\lambda_2 V - \lambda_3 (g + \frac{D}{m}) = 0$ (2.26)

$$mg - D(1 + \frac{V}{V_e}) = 0 \tag{2.30}$$

(3) For the coasting subarc, $(t_2 \leqslant t \leqslant t_f)$,

$$v^2 = 2b \exp\left(\frac{2k}{am_f} e^{-ah}\right) \left[ -ah + \sum_{n=1}^{\infty} \frac{\left(-\frac{2k}{am_f}\right)^n e^{-anh}}{n \cdot n!} + C_1 \right]$$

$$\triangleq f(h) \qquad (2.49)$$

where
$$C_1 = ah_f - \sum_{n=1}^{\infty} \frac{\left(-\frac{2k}{am_f}\right)^n e^{-anh_f}}{n \cdot n!}$$

$$t = t_2 + \frac{1}{V_e} \int_{h_2}^{h} \frac{dh}{\sqrt{f(y)}} \qquad (2.50)$$

$$m = m_f = \text{constant}$$

$$\lambda_2 = \frac{F(h)}{v\, V_e} \left(g + \frac{kv^2\, V_e^2}{m_f} e^{-ah}\right) \qquad (2.53)$$

$$\lambda_3 = \exp\left[ -\frac{k}{am_f} e^{-ah} - \frac{g}{V_e^2} \int_{h_2}^{h} \frac{dy}{f(y)} \right] + C_2$$

$$\triangleq F(h) \qquad (2.51)$$

$$\lambda_5 = \frac{-kV_e}{m_f^2} \int_{h_2}^{h} F(y) \sqrt{f(y)}\ e^{-ay}\, dy + C_3 \qquad (2.54)$$

where
$$C_2 = -\exp\left[ -\frac{k}{am_f} e^{-ah_f} - \frac{g}{V_e^2} \int_{h_2}^{h_f} \frac{dy}{f(y)} \right]$$

$$(2.52)$$

The first integral is

$$\lambda_2 V - \lambda_3 \left(g + \frac{D}{m_f}\right) = 0 \qquad (2.26)$$

It is evident that the form of the analytical solution is very complicated. On the coasting subarc, the analytical solution cannot be expressed in a closed form. However, by the use of digital computers an accurate numerical solution may be obtained. For example, Leitmann[12] has obtained the optimal thrust program as a function of time, using a digital computer and the analytical results to obtain the optimal trajectory. In Leitmann's method the trajectory was solved in reverse time, starting at the final point.

Although the analytical solution has a complicated form it still yields interesting information about the optimal trajectory of the sounding rocket problem. This will be discussed in the following section.

(1) The Optimal Controller

The entire optimal trajectory has three subarcs (the impulsive boosting subarc, the variable thrust subarc and the coasting subarc) and associated with these subarcs are three different types of thrust programs. These are impulsive thrust, variable thrust and zero thrust. This means that the optimal controller has three modes of operation. The first and the last modes are ones of maximum and zero thrust respectively. The variable thrust mode is controlled by the optimal controller which must also determine the instants at which modes are switched. A possible optimal controller can be obtained by means of (2.30). The method whereby (2.30) is used to obtain the optimal control law is to consider (2.30)

$$\varepsilon \triangleq mg - D\left(1 + \frac{V}{V_e}\right) \qquad (2.55)$$

as an error signal. The signal $\varepsilon$ is fed into a high gain amplifier and the amplifier output is used to control the fuel flow. A detailed discussion and some other possible optimal control laws will be studied in the next chapter.

(2) The Initial Values of the Lagrange Multipliers

The Lagrange multipliers play an important role in the present study of optimal controllers. In the general case, the control law depends on the Lagrange multipliers. Usually the initial conditions of Lagrange multipliers are not all known and the controller must then compute the unknown initial conditions.

The sounding rocket problem has two unknown initial Lagrange multipliers, $\lambda_{20}$ and $\lambda_{30}$.

It follows from the analytical study that both $\lambda_{20}$ and $\lambda_{30}$ are negative. This statement can be proved by the following argument:

At the end of boosting, that is at the time $t_1$, the analytical solution gives

$$\lambda_{21} \cong \lambda_{20} \tag{2.18}$$

$$\lambda_{31} \cong \lambda_{30} \tag{2.19}$$

$$\lambda_{51} \cong -1 + \lambda_{30} V_e \left(\frac{1}{m_1} - \frac{1}{m_o}\right) \tag{2.20}$$

and

$$\lambda_{51} - \lambda_{31} \frac{V_e}{m_1} = 0 \tag{2.25}$$

The last equation can be approximated:

$$\lambda_{51} \cong \lambda_{30} \frac{V_e}{m_1}$$

Substituting the above equation into (2.20) and solving for $\lambda_{30}$ yields

$$\lambda_{30} \cong -\frac{m_o}{V_e} \tag{2.56}$$

Equation (2.56) shows that $\lambda_{30}$ must be negative, since $m_o$ and $V_e$ are positive quantities. It follows from (2.19) that $\lambda_{31}$ must be negative. Furthermore, the first integral (2.26) shows that

$$\lambda_{21} V_1 - \lambda_{31}(g + \frac{D_1}{m_1}) = 0$$

where $V_1$, g, $D_1$ and $m_1$ are positive, and $\lambda_{31}$ is negative. Thus $\lambda_{21}$ must be negative and from (2.18) $\lambda_{20}$ must be negative. In conclusion, all the Lagrange multipliers in the sounding rocket problem must have negative initial values.

(3) A Qualitative Study of the Motion of the Sounding Rocket Problem

A qualitative study often gives a better understanding of a problem. The general behaviour of the state variables and the Lagrange multipliers may be obtained from the analytical solution. The altitude h is always increasing along the entire trajectory.

For the boosting subarc, the analytical solution shows that V is increasing and that both m and $\lambda_5$ are decreasing, but $\lambda_2$ and $\lambda_3$ are almost constant.

For the variable thrust subarc, the optimum thrust gives an optimum velocity. Equation (2.36) shows that V must increase since h is increasing all the time. The mass m is determined by the equation (see (2.30)).

$$m = \frac{D}{g}\left(1 + \frac{V}{V_e}\right)$$

$$= \frac{k}{g}\,\frac{V^2\left(1 + \frac{V}{V_e}\right)}{\exp(ah)}$$

Since m is decreasing, it follows from the above equation that the denominator, $ge^{ah}$, increases faster than the numerator $kV^2\left(1 + \frac{V}{V_e}\right)$. The Lagrange multipliers $\lambda_2$ and $\lambda_3$ increase because they have positive time derivatives and $\lambda_5$ decreases because it has a negative derivative with respect to t (see (A.14), (A.15) and (A.16)).

For the coasting subarc, the drag is small at high altitude, and the thrust is zero, thus the velocity is approximately equal to $V_2 - g(t-t_2)$ (see (A.12)). The altitude h increases until V becomes zero. The Lagrange multipliers $\lambda_2$ and $\lambda_5$ remain almost constant for the coasting subarc, since their time derivatives are negligible (see (A.14) and (A.16)) and $\lambda_3$ increases to its final value $\lambda_{3f}$ with a slope approximately equal to $-\lambda_2$ (see(A.15)). The analytical solution for the coasting subarc contains an integral. The integrand is $1/f(h)$ and is infinite at $h = h_f$ since $f(h_f) = v_f^{\,2} = 0$. The integrals $\int_{h_2}^{h_f} \frac{dy}{\sqrt{f(y)}}$ and $\int_{h_2}^{h_f} \frac{dy}{f(y)}$ are, however, finite. The singular nature of the integrand makes a direct digital computation using the analytical results difficult. If the approximation $V \cong V_2 - g(t - t_2)$ for the coasting subarc is made, the function $f(h) = v^2 \cong V_e^{-2}\left[V_2 - g(t - t_2)\right]^2$ can be used to compute the above two integrals.

The following curves in Fig. 2.2 and Fig. 2.3 illustrate the general behaviour of the state variables and Lagrange multipliers.

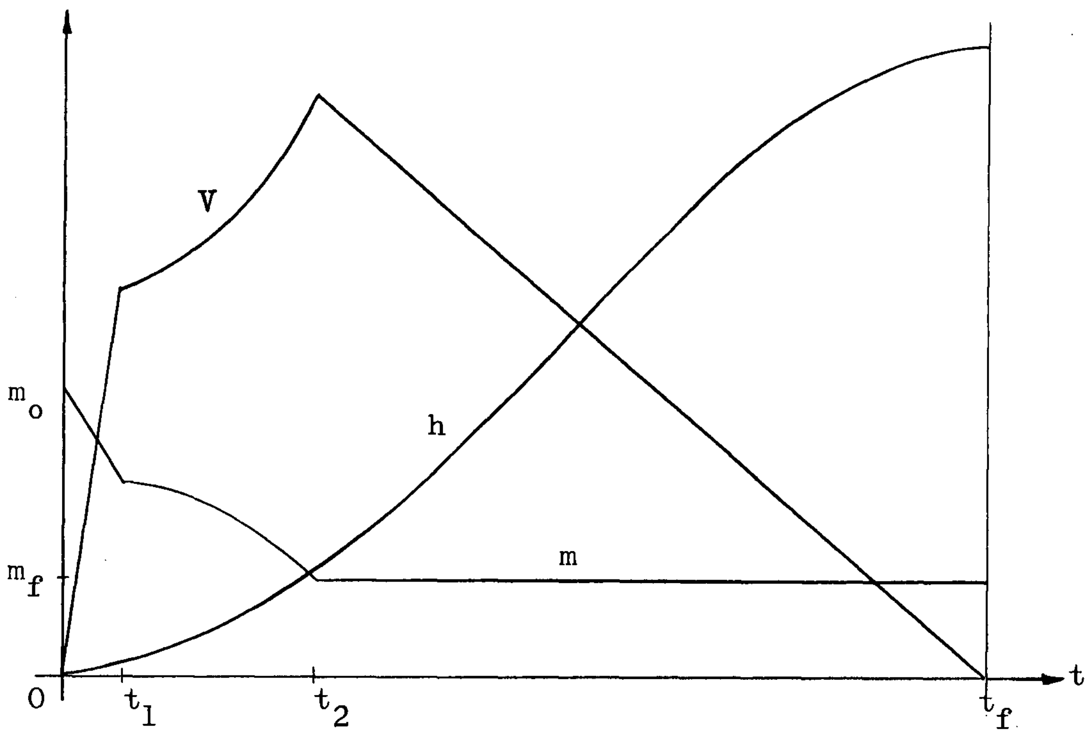


Fig. 2.2 The state variables

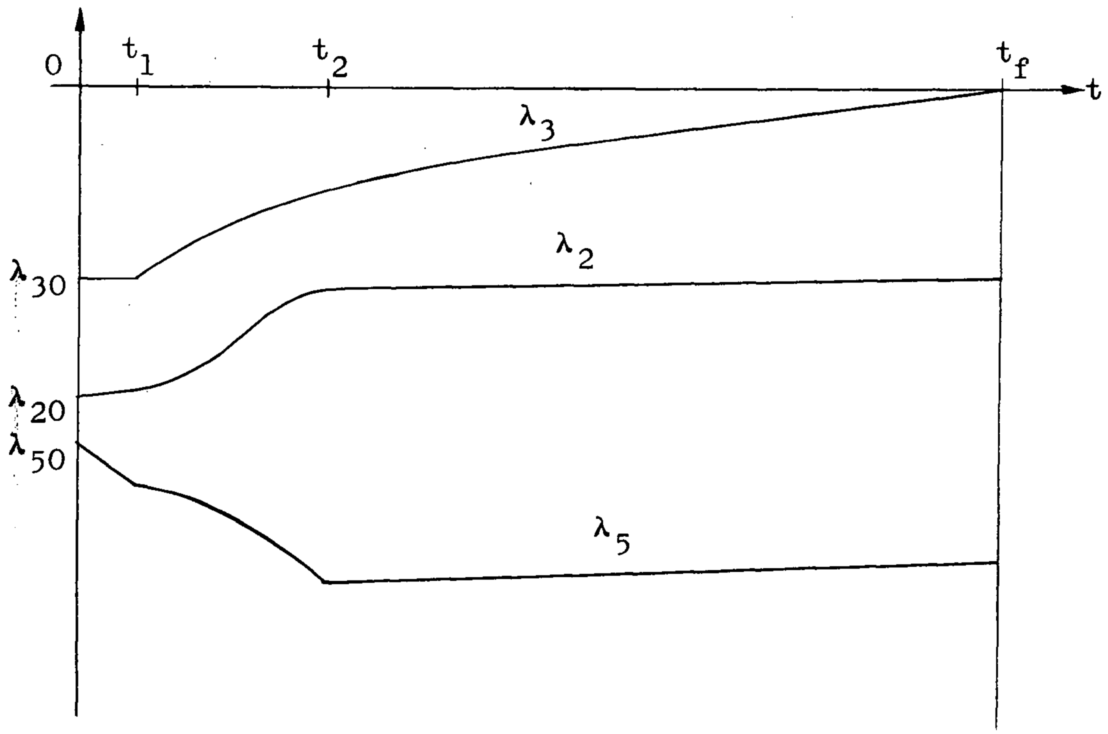


Fig. 2.3 The Lagrange multipliers

# 3. OPTIMAL FEEDBACK CONTROL SYSTEMS

## 3.1 Introduction

The general problem in optimal control is the determination of the inputs to a system subject to certain constraints so that the state of the system follows a trajectory resulting in the optimization of a given performance criterion. In other words, the problem is to determine the control variable as a function of time so that the system satisfies the specified criterion. This is essentially an open loop control system and, from the control engineering point of view, may not be satisfactory. The control variable resulting in optimum performance can be determined analytically only for very simple systems, for example, the constant coefficient linear system. Furthermore, the open loop control has the disadvantage that disturbances existing in a physical system results in non-optimum performance. Therefore, a closed loop feedback control system is desirable.

This chapter is devoted to the study of feedback optimal control systems. Specific problems are studied and the optimal control for each case is derived as a function of the system state variables.

## 3.2 The Concept of Optimal Feedback Control and the Synthesis of Optimal Controllers

Optimal controllers synthesized by use of the calculus of variations result in a multivariable type of control systems. In general, a multivariable optimal control system consists of two subsystems. These are the plant and the so-called adjoint system. The plant is usually described by a set of differential equations and the adjoint system corresponds to the Euler-Lagrange equations. The interrelationship between these two subsystems is shown in Fig. 3.1.

The system illustrated in Fig. 3.1 may be considered as an n by m optimal feedback control system, where n refers to the number of the state variables x(t), and m refers to the number of the control variables u(t). The following matrix notations are used in Fig. 3.1.

$$x(t) = \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$$ , n x 1 matrix of state variables.

$$\lambda(t) = \begin{bmatrix} \lambda_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n(t) \end{bmatrix}$$ , n x 1 matrix of the Lagrange multipliers.

$$u(t) = \begin{bmatrix} u_1(t) \\ \cdot \\ \cdot \\ \cdot \\ u_m(t) \end{bmatrix}$$ , m x 1 matrix of control variables.

$$P(a_1,\ldots,a_n) = \begin{bmatrix} P_1 \\ \cdot \\ \cdot \\ \cdot \\ P_m \end{bmatrix}$$ , m x 1 matrix of the terminal values of x(t) and t.

The performance function P is to be optimized. The number of elements of the u(t) matrix is always the same as that

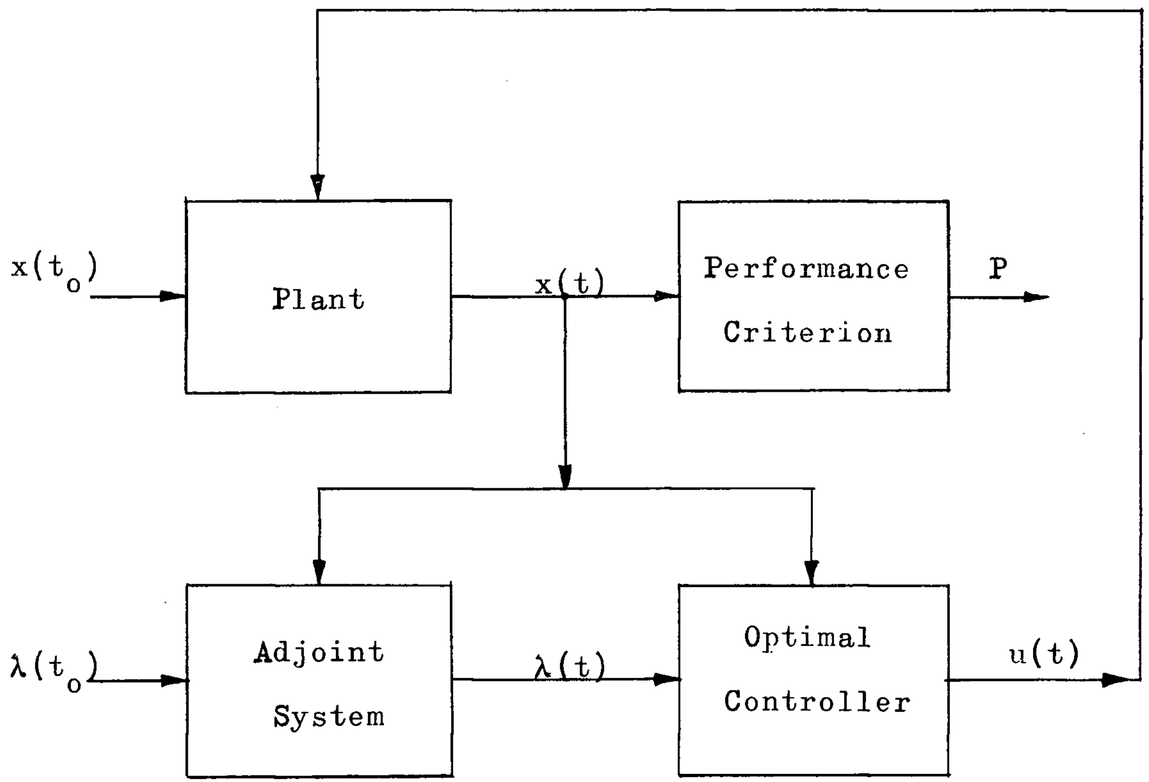


Fig. 3.1 A general multivariable optimal feedback control system

of the P matrix.

### 3.2.1 A Multivariable Optimal Feedback Control System

In some cases the optimal control law may not contain the Lagrange multiplier $\lambda(t)$ explicitly. The control variable $u(t)$ may then be determined as a function of the state variable $x(t)$. In this case the general multivariable feedback control system described in Fig. 3.1 reduces to the form shown in Fig. 3.2. The following sections discuss optimal controllers of this type for a variety of flight conditions.

### 3.2.2 Synthesis of Optimal Control Laws for Rocket Flight

In the study of optimal control systems the synthesis of the optimal controller is a major problem. In the case of optimal feedback control systems the determination of the optimal control law is of primary importance.

The simplified problems of rocket flight have been formulated in the Appendix, and they will be studied in this section. These simplified problems have one degree of freedom. Thus there exists only one optimal control variable in these problems.

(1) The Vertical Flight (Sounding Rocket) Problem

It follows from Chapter 2 that optimal condition for the variable thrust subarc is

$$\lambda_5 - \lambda_3 \frac{V_e}{m} = 0 \tag{2.25}$$

Actually, this condition holds true for all the four problems

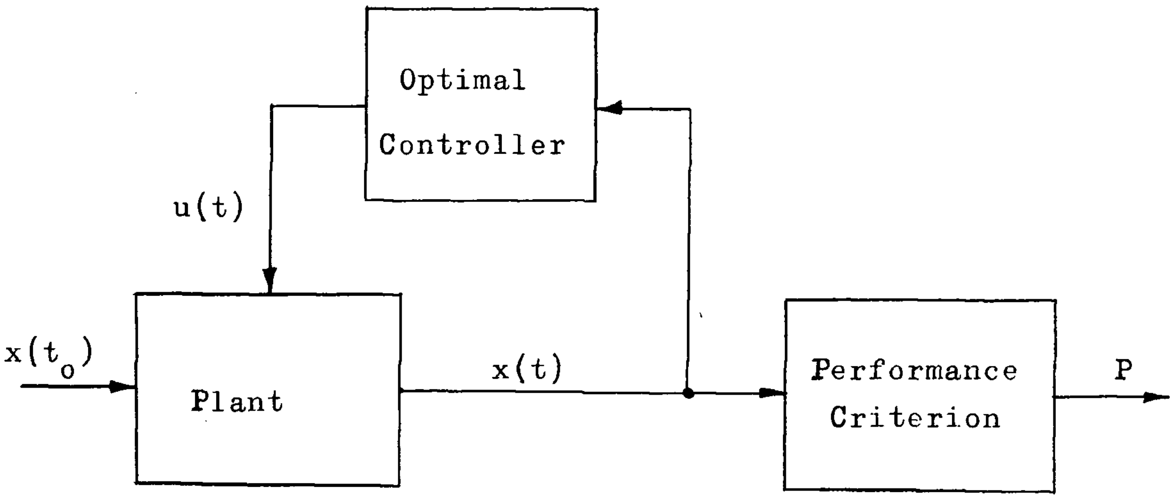


Fig. 3.2 A multivariable optimal feedback control system

discussed in this chapter. Differentiating (2.25) with respect to t yields

$$\dot{m}\,\lambda_5 + m\,\dot{\lambda}_5 - V_e\,\dot{\lambda}_3 = 0 \qquad (3.1)$$

It follows from Chapter 2, Section 2.3 that (3.1) leads to equation (2.30), that is

$$f_s \stackrel{\Delta}{=} mg - D\left(1 + \frac{V}{V_e}\right) = 0 \qquad (3.2)$$

where $f_s$ is called the switching function. The boosting stage terminates when $f_s$ goes through zero. Differentiating (3.2) with respect to t gives

$$\dot{m}g - D\,\frac{\dot{V}}{V_e} - \left(1 + \frac{V}{V_e}\right)\left(\frac{2D}{V}\,\dot{V} - aD\dot{h}\right) = 0 \qquad (3.3)$$

The equations of motion, (A.11), (A.12) and (A.13) can be used to eliminate $\dot{m}$, $\dot{V}$ and $\dot{h}$ in the above equation resulting in

$$u = \beta$$

$$= \frac{D\left[\left(g + \frac{D}{m}\right)\left(2 + \frac{3V}{V_e}\right) + aV^2\left(1 + \frac{V}{V_e}\right)\right]}{g\,V + \frac{D}{m}\,(2V_e + 3V)}$$

(3.4)

which gives the optimal control variable as a function of the state variables for the variable thrust subarc.

(2) The Horizontal Flight Problem

The equations for optimal horizontal flight are derived in a manner similar to the problem of vertical flight. After substituting (A.23), (A.25) and (A.26) into (3.1) the following equation results

$$\lambda_4 \frac{L}{mV} + \lambda_1 V_e - \lambda_3 \left(\frac{D}{m} + \frac{V_e}{m}\frac{\partial D}{\partial V}\right) = 0 \qquad (3.5)$$

The first integral for the variable thrust subarc is

$$\lambda_1 V - \lambda_3 \frac{D}{m} = 0 \qquad (A.29)$$

Solving this equation for $\lambda_1$, and (A.27) for $\lambda_4$ and then substituting into (3.5), yields the condition

$$f_s \triangleq V\left(D + V_e \frac{\partial D}{\partial V} - mg \frac{\partial D}{\partial L}\right) - V_e D = 0 \qquad (3.6)$$

which must be satisfied by the optimal variable thrust subarc. Here $L = mg$ and $D = D(V,L)$

Expressing (3.6) in the form

$$D(V - V_e) + V V_e \frac{\partial D}{\partial V} - mgV \frac{\partial D}{\partial L} = 0$$

and then differentiating with respect to t yields

$$\dot{V} D + (V - V_e)\left(\frac{\partial D}{\partial V}\dot{V} + \frac{\partial D}{\partial L}\dot{L}\right) + \dot{V} V_e \frac{\partial D}{\partial V}$$

$$+ V V_e \left(\frac{\partial^2 D}{\partial V^2}\dot{V} + \frac{\partial^2 D}{\partial V \partial L}\dot{L}\right) - \dot{m}gV \frac{\partial D}{\partial L} - mg\dot{V}\frac{\partial D}{\partial L}$$

$$- mgV\left(\frac{\partial^2 D}{\partial L \partial V}\dot{V} + \frac{\partial^2 D}{\partial L^2}\dot{L}\right) = 0$$

Substituting $\dot{L} = \dot{m}g$ into the above equation gives

$$\dot{V}\left[D + V\frac{\partial D}{\partial L} - mg\frac{\partial D}{\partial L} + V V_e \frac{\partial^2 D}{\partial V^2} - mg V \frac{\partial^2 D}{\partial L \partial V}\right] - \dot{m}g\left[- V_e \frac{\partial D}{\partial L} + V V_e \frac{\partial^2 D}{\partial V \partial L} - mg V \frac{\partial^2 D}{\partial L^2}\right] = 0$$

Let $A(m,V,L) \triangleq D + V\frac{\partial D}{\partial L} - mg\frac{\partial D}{\partial L} + V V_e \frac{\partial^2 D}{\partial V^2} - mg V \frac{\partial^2 D}{\partial L \partial V}$

$$B(m,V,L) \triangleq -V_e \frac{\partial D}{\partial L} + V V_e \frac{\partial^2 D}{\partial V \partial L} - mg V \frac{\partial^2 D}{\partial L^2}$$

and substituting (A.22) and (A.23) into the previous equation yields the optimal control variable

$$u = \beta$$

$$= \frac{AD}{AV_e - mgB} \tag{3.7}$$

(3) The Arbitrary Inclined Rectilinear Flight Problem.

This is a more general case and includes the vertical and horizontal flight problems. The derivation of the optimal control variable is the same. Substituting (A.34), (A.37) and (A.38) into (3.1) and using the optimal condition (2.25) for the variable thrust subarc, the following equation is obtained.

$$\lambda_4 \frac{L}{mV} + \lambda_1 V_e \cos \theta + \lambda_2 V_e \sin \theta - \lambda_3 \left(\frac{D}{m} + \frac{V_e}{m}\frac{\partial D}{\partial V}\right) = 0 \tag{3.8}$$

The first integral for this problem along the variable thrust subarc is given by (A.41)

$$\lambda_1 \cos \theta + \lambda_2 \sin \theta - \frac{\lambda_3}{V}\left(\frac{D}{m} + g \sin \theta\right) = 0$$

The Euler-Lagrange equation (A.39) gives

$$\lambda_4 = \lambda_3 V \frac{\partial D}{\partial L}$$

It follows from the above two equations and (3.8) that the optimal variable thrust subarc must satisfy the condition

$$f_s \triangleq D(V - V_e) + V V_e \frac{\partial D}{\partial V} - mg\,(V_e \sin\theta + V \cos\theta \frac{\partial D}{\partial L}) = 0 \tag{3.9}$$

where $L = mg \cos\theta$, $D = D(h,V,L)$ and $\theta$ is a constant. It can be seen that (3.2) and (3.6) are special cases of (3.9). Differentiating (3.9) with respect to t yields

$$D\dot{V} + (V - V_e)\,(\frac{\partial D}{\partial V}\dot{V} + \frac{\partial D}{\partial h}\dot{h} + \frac{\partial D}{\partial L}\dot{L}) + \dot{V} V_e \frac{\partial D}{\partial V}$$

$$+ V V_e(\frac{\partial^2 D}{\partial V^2}\dot{V} + \frac{\partial^2 D}{\partial V \partial h}\dot{h} + \frac{\partial^2 D}{\partial V \partial L}\dot{L}) - \dot{m}g\,(V_e \sin\theta + V \cos\theta \frac{\partial D}{\partial L})$$

$$- mg\,\dot{V} \cos\theta \frac{\partial D}{\partial L} - mg\,V \cos\theta\,(\frac{\partial^2 D}{\partial L \partial V}\dot{V} + \frac{\partial^2 D}{\partial L \partial h}\dot{h} + \frac{\partial^2 D}{\partial L^2}\dot{L}) = 0$$

By means of (A.32), (A.33), (A.34) and the equation

$$\dot{L} = \dot{m}\,g \cos\theta$$

The previous expression can be solved for $\beta$ yielding the optimal control variable

$$u = \beta$$

$$= \frac{mC - A(mg \sin\theta + D)}{mB - V_e A} \tag{3.10}$$

where $$A \triangleq D + VV_e \frac{\partial^2 D}{\partial V^2} - mg \cos\theta \frac{\partial D}{\partial L} - mg\,V \cos\theta \frac{\partial^2 D}{\partial V \partial L}$$

$$B \triangleq g \cos\theta \left[(V - V_e)\frac{\partial D}{\partial L} + VV_e \frac{\partial^2 D}{\partial V \partial L} - mg \cos\theta \frac{\partial^2 D}{\partial L^2} - V_e \tan\theta - V \frac{\partial D}{\partial L}\right]$$

$$C \triangleq (V - V_e)V \sin \Theta \frac{\partial D}{\partial h} + V^2 V_e \sin \Theta \frac{\partial^2 D}{\partial V \partial h} - mg\, V^2 \sin \Theta \cos \Theta \frac{\partial^2 D}{\partial L \partial h}$$

(4) The Zero-lift Flight Problem.

Substituting (A.48) and (A.53) into (3.1) yields the equation

$$\dot{\lambda}_3 = -\lambda_3 \frac{D}{m\, V_e} \tag{3.11}$$

The optimal condition for the variable thrust subarc is given by (A.54)

$$\lambda_5 - \lambda_3 \frac{V_e}{m} = 0$$

Substituting this into (A.53) gives

$$\dot{\lambda}_5 = \frac{\lambda_5}{mV_e} \quad (V_e \beta - D) \tag{3.12}$$

It follows from (A.51) that

$$\dot{\lambda}_3 = -\lambda_1 \cos \Theta - \lambda_2 \sin \Theta + \frac{\lambda_3}{m} \frac{2D}{V} - \lambda_4 \frac{g}{V^2} \cos \Theta$$

Substituting (3.11) into the above equation yields

$$-\lambda_3 \frac{DV}{mV_e} + \lambda_1 V \cos \Theta + \lambda_2 V \sin \Theta - \lambda_3 \frac{2D}{m} + \lambda_4 \frac{g}{V} \cos \Theta = 0 \tag{3.13}$$

The first integral for the variable thrust subarc is

$$\lambda_1 V \cos \Theta + \lambda_2 V \sin \Theta - \lambda_3 \left(\frac{D}{m} + g \sin \Theta\right) - \lambda_4 \frac{g}{V} \cos \Theta = 0 \tag{3.14}$$

The Euler-Lagrange equation (A.50),

$$\dot{\lambda}_2 = -\lambda_3 \frac{aD}{m} \tag{3.15}$$

with the aid of (3.11), can be written as

$$\dot{\lambda}_2 = a\, V_e\, \dot{\lambda}_3 \tag{3.16}$$

Integrating (3.16) gives

$$\lambda_2 = a\, V_e\, \lambda_3 + C_2 \tag{3.17}$$

where $C_2$ is the initial condition of $\lambda_{20} - a\, V_e\, \lambda_{30}$.

Subtracting (3.13) from (3.14) and solving for $\lambda_4$ yields

$$\lambda_4 = \frac{\lambda_3\, V}{2\, g \cos \theta} \left(\frac{DV}{mV_e} + \frac{D}{m} - g \sin \theta\right) \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.13) and solving for $\lambda_3$ results in

$$\lambda_3 = \frac{2V(C_1 \cos \theta + C_2 \sin \theta)}{\frac{DV}{mV_e} + \frac{3D}{m} + g \sin \theta - 2aVV_e \sin \theta} \tag{3.19}$$

where $\lambda_1 = C_1$ is a constant, a result which follows from the Euler-Lagrange equation (A.49).

Now letting

$$\lambda_3 = \frac{A(\theta, V)}{B(V,h,m,\theta)} \tag{3.20}$$

where
$$A(\theta,V) \triangleq 2\, V(C_1 \cos \theta + C_2 \sin \theta) \tag{3.21}$$

$$B(V,h,m,\theta) \triangleq \frac{DV}{mV_e} + \frac{3D}{m} + g \sin \theta - 2aVV_e \sin \theta \tag{3.22}$$

and differentiating (3.20) gives

$$\dot{\lambda}_3 = \frac{B\dot{A} - A\dot{B}}{B^2} \tag{3.23}$$

It follows from (3.11) and (3.20) that

$$\dot{\lambda}_3 = -\lambda_3 \frac{D}{mV_e}$$

$$= -\frac{AD}{m\, V_e B} \tag{3.24}$$

Eliminating $\dot{\lambda}_3$ by the aid of (3.23) gives

$$- \frac{A\ B\ D}{mV_e} = B\ \dot{A} - A\ \dot{B} \tag{3.25}$$

where

$$\dot{A} = \frac{\partial A}{\partial V}\dot{V} + \frac{\partial A}{\partial \Theta}\dot{\Theta}$$

$$= \frac{\partial A}{\partial V}\left(-g \sin \Theta - \frac{D}{m}\right) + \frac{\partial A}{\partial V}\frac{V_e \beta}{m} - \frac{\partial A}{\partial \Theta}\frac{g}{V}\cos \Theta$$

$$\dot{B} = \frac{\partial B}{\partial V}\dot{V} + \frac{\partial B}{\partial h}\dot{h} + \frac{\partial B}{\partial m}\dot{m} + \frac{\partial B}{\partial \Theta}\dot{\Theta}$$

$$= \frac{\partial B}{\partial V}\left(-g \sin \Theta - \frac{D}{m}\right) + \frac{\partial B}{\partial V}\frac{V_e \beta}{m} + \frac{\partial B}{\partial h} V \sin \Theta$$

$$- \frac{\partial B}{\partial m}\beta - \frac{\partial B}{\partial \Theta}\frac{g}{V}\cos \Theta$$

Substituting $\dot{A}$ and $\dot{B}$ into (3.25) and solving for $\beta$ results in the optimal control variable for the variable thrust subarc

$$\begin{aligned} u &= \beta \\ &= \frac{1}{F}\Bigg[\frac{A\ B\ D}{m\ V_e} + A\ g \sin \Theta \frac{\partial B}{\partial V} + \frac{AD}{m} + AV \sin \Theta \frac{\partial B}{\partial h} \\ &\qquad - \frac{A\ g \cos \Theta}{V}\ \frac{\partial B}{\partial \Theta} \\ &\qquad - B\ g \sin \Theta \frac{\partial A}{\partial V} - \frac{BD}{m}\frac{\partial A}{\partial V} - \frac{Bg}{V}\cos \Theta \frac{\partial A}{\partial \Theta}\Bigg] \end{aligned} \tag{3.26}$$

where

$$F \triangleq \frac{A\ Ve}{m}\frac{\partial B}{\partial V} - A\frac{\partial B}{\partial m} - \frac{B\ Ve}{m}\frac{\partial A}{\partial V}$$

and

$$\frac{\partial A}{\partial V} = 2\ C_1 \cos \Theta + 2\ C_2 \sin \Theta$$

$$\frac{\partial A}{\partial \Theta} = -2\ C_1\ V \sin \Theta + 2\ C_2\ V \cos \Theta$$

$$\frac{\partial B}{\partial V} = \frac{3D}{mV_e} + \frac{6D}{mV} - 2a\ V_e \sin \Theta$$

$$\frac{\partial B}{\partial h} = - \frac{aD}{m}\left(3 + \frac{V}{V_e}\right)$$

$$\frac{\partial B}{\partial \Theta} = g \cos \Theta - 2\ a\ V\ V_e \cos \Theta$$

$$\frac{\partial B}{\partial m} = - \frac{D}{m^2} \left(3 + \frac{V}{V_e}\right)$$

By the aid of equations (3.17), (3.18), (3.20) and (3.14), the switching function $f_s$ can be obtained

$$f_s \triangleq C_1 V \cos \Theta + C_2 V \sin \Theta + a V_e V \sin \Theta \frac{A}{B} - \frac{A}{2B} g \sin \Theta$$

$$- \frac{ADV}{2mBV_e} - \frac{3AD}{2mB} = 0 \qquad (3.27)$$

The optimal control law for the four different problems of rocket flight has been derived. For this class of optimal control problems the fuel consumption has been minimized. However, the technique can also be applied to problems of maximum range and minimum flight time, etc. The following block diagram represents the control scheme for all four problems. There are in each problem three modes of control corresponding to the boosting subarc, the variable thrust subarc and the coasting subarc (see Fig. 3.3).

The switching time $t_1$ is determined when the switching function $f_s$ goes through zero (see (3.2), (3.6), (3.9) and (3.27)). The controller then operates to keep $f_s = 0$ until the cut-off time is reached. In the problem of zero-lift flight, the initial values of the Lagrange multipliers $\lambda_1$, $\lambda_2$ and $\lambda_3$ enter into the optimal control law. The method for evaluating the initial values is discussed in Chapter 4.

## 3.3 Analogue Computer Technique for the Synthesis of Optimal Controllers

The conditions for optimal control derived in the last

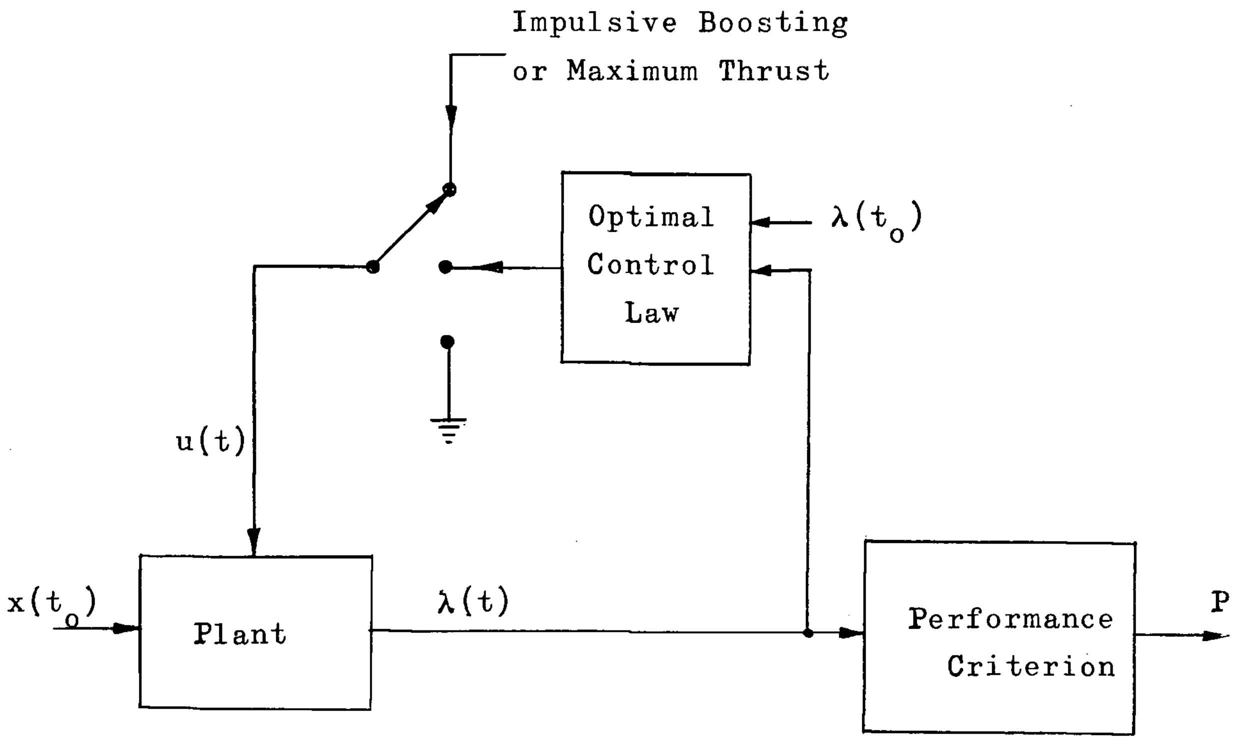


Fig. 3.3 The modes of control for optimum rocket flight

section can be used to synthesize optimal controllers. Digital computers are suitable for numerical computation. However, analogue computers appear better suited for the synthesis of comparatively simple real-time controllers. The lengthy iterative computations of the digital computer are replaced by relatively high-speed feedback loops where an error signal is applied to a high-gain amplifier and the amplifier output can be used as the optimal control variable. The block diagram of Fig. 3.4 shows this technique.

## 3.4 Analogue Computer Study of the Sounding Rocket Problem

The analogue computer technique discussed in Section 3.3 will now be applied to the sounding rocket problem. A PACE 231-R analogue computer was used and a schematic diagram of the computer program is illustrated in Fig. 3.5. The problem is computed backward in time.

In Fig. 3.5 the error signal is given by the switching function

$$f_s \triangleq \varepsilon(t) \triangleq mg - D(1 + \frac{V}{V_e}) \tag{3.28}$$

and the control variable by

$$u(t) = -K\,\varepsilon(t) \tag{3.29}$$

The reason for computing the problem backward in time is that the final velocity, altitude, and mass are known. Thus for backward time computation no iteration is required for determining the optimal trajectory.

The numerical values chosen are the following:

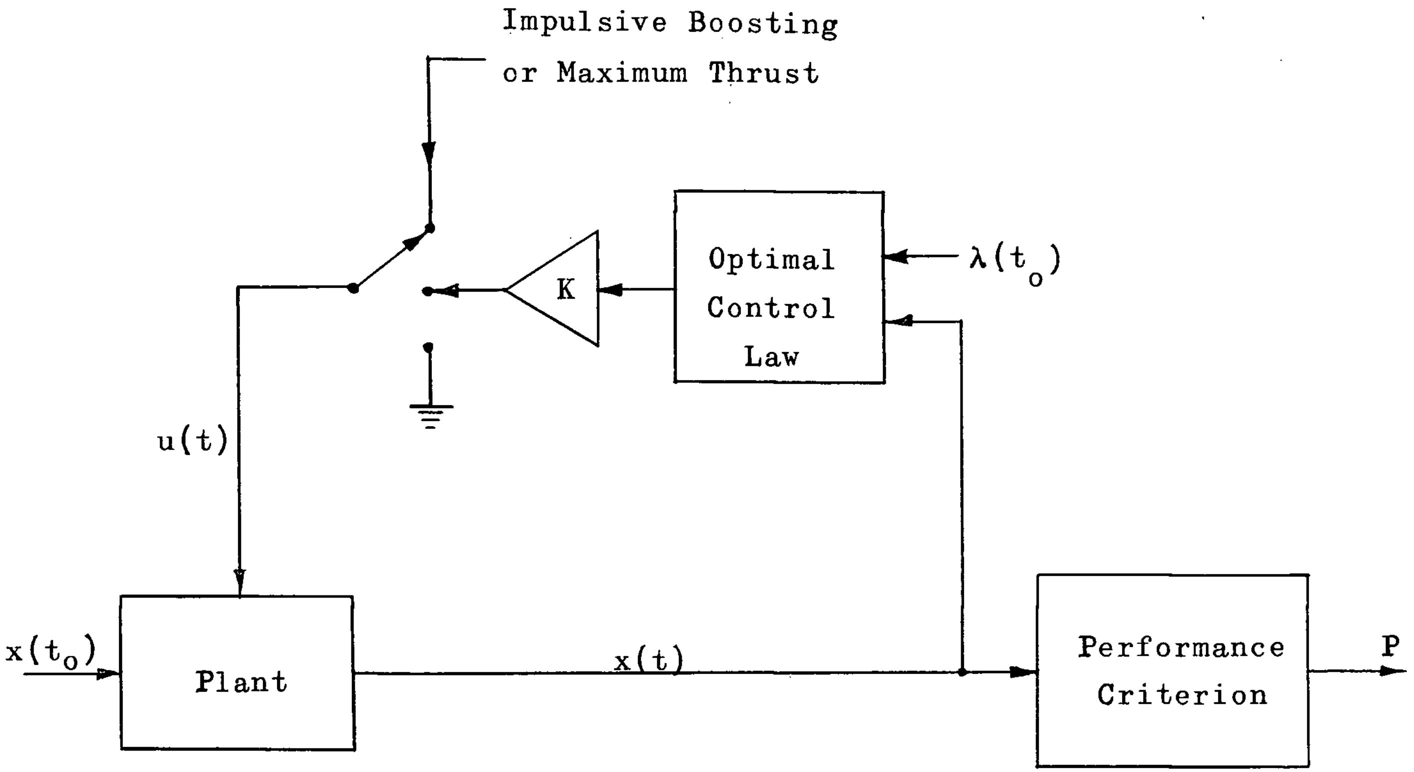


Fig. 3.4 Synthesis of optimal controllers by means of analogue computers

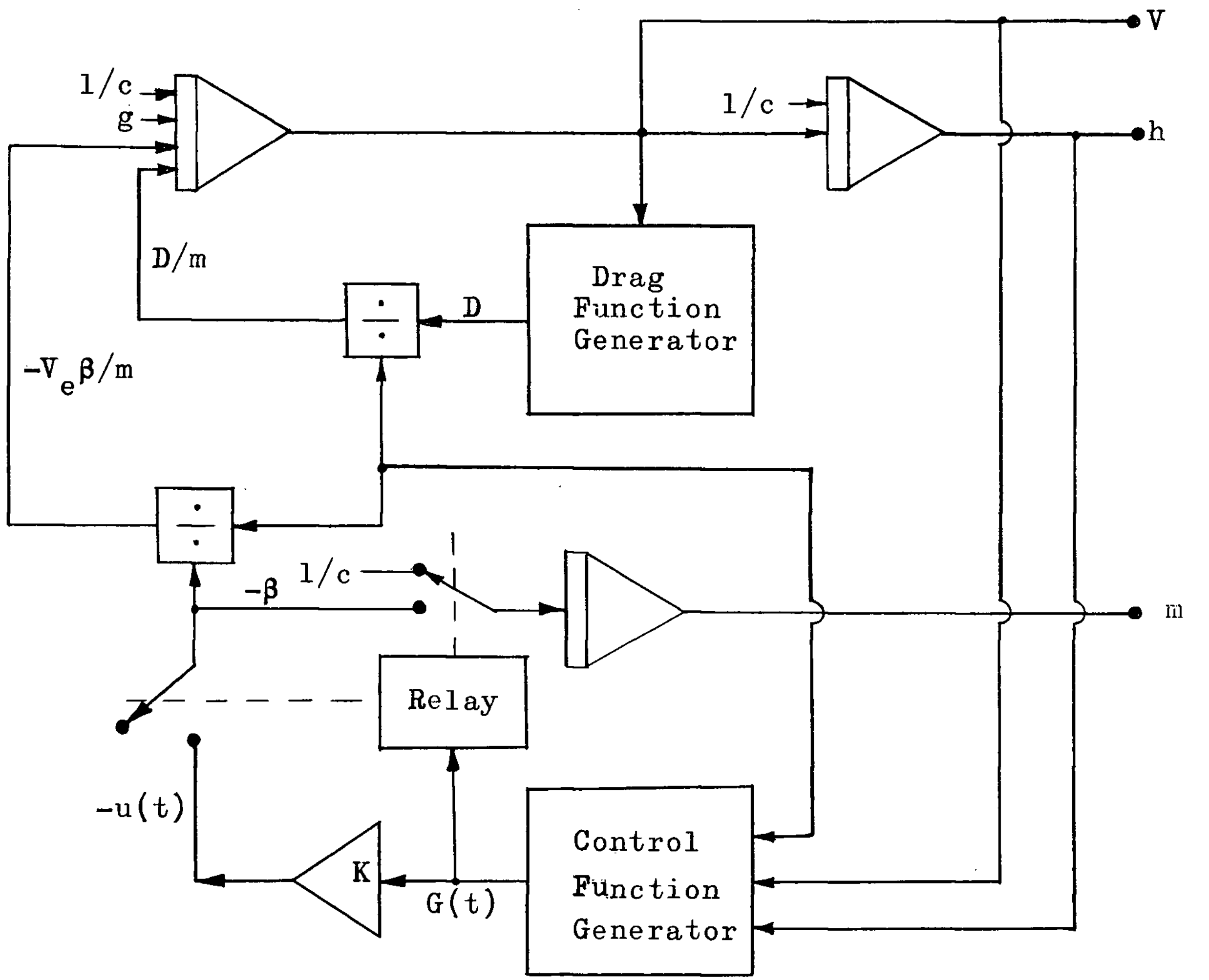


Fig. 3.5 Analogue computer program for the sounding rocket problem

$$h_f = 4{,}889{,}500 \text{ ft.}$$
$$m_f = 10 \text{ slug}$$
$$V_f = 0 \text{ ft/sec}$$
$$D = k\, V^2 e^{-ah}$$
$$V_e = 5500 \text{ ft/sec}$$
$$k = 10^{-4} \text{ slug - ft.}$$
$$a = 1/22000 \text{ ft}^{-1}$$
$$K = 100$$

The resulting state variables are shown in Fig. 3.6 where $\tau = t_f - t$ is the backward time variable.

The function $\varepsilon(\tau)$ is used to determine the instant $\tau_2$, when $\varepsilon(\tau_2) = 0$. At $\tau = \tau_2$ the following values are obtained:

$$h_2 = 62{,}600 \text{ ft.}$$
$$V_2 = 5{,}313 \text{ ft/sec}$$
$$m_2 = 10 \text{ slug}$$
$$\tau_2 = 161.3 \text{ sec.}$$
$$u_2 = 0.72 \text{ slug/sec.}$$

and the feedback computation of thrust based on $\varepsilon(\tau) = 0$ is introduced by means of a relay. At $\tau = \tau_1$, the following values are obtained:

$$h_1 = 0$$
$$V_1 = 2275 \text{ ft/sec}$$
$$m_1 = 20.85 \text{ slug}$$
$$\tau_1 = 179.5 \text{ sec.}$$
$$u_1 = 0.5 \text{ slug/sec}$$

At $\tau = \tau_0$, the initial mass including fuel is

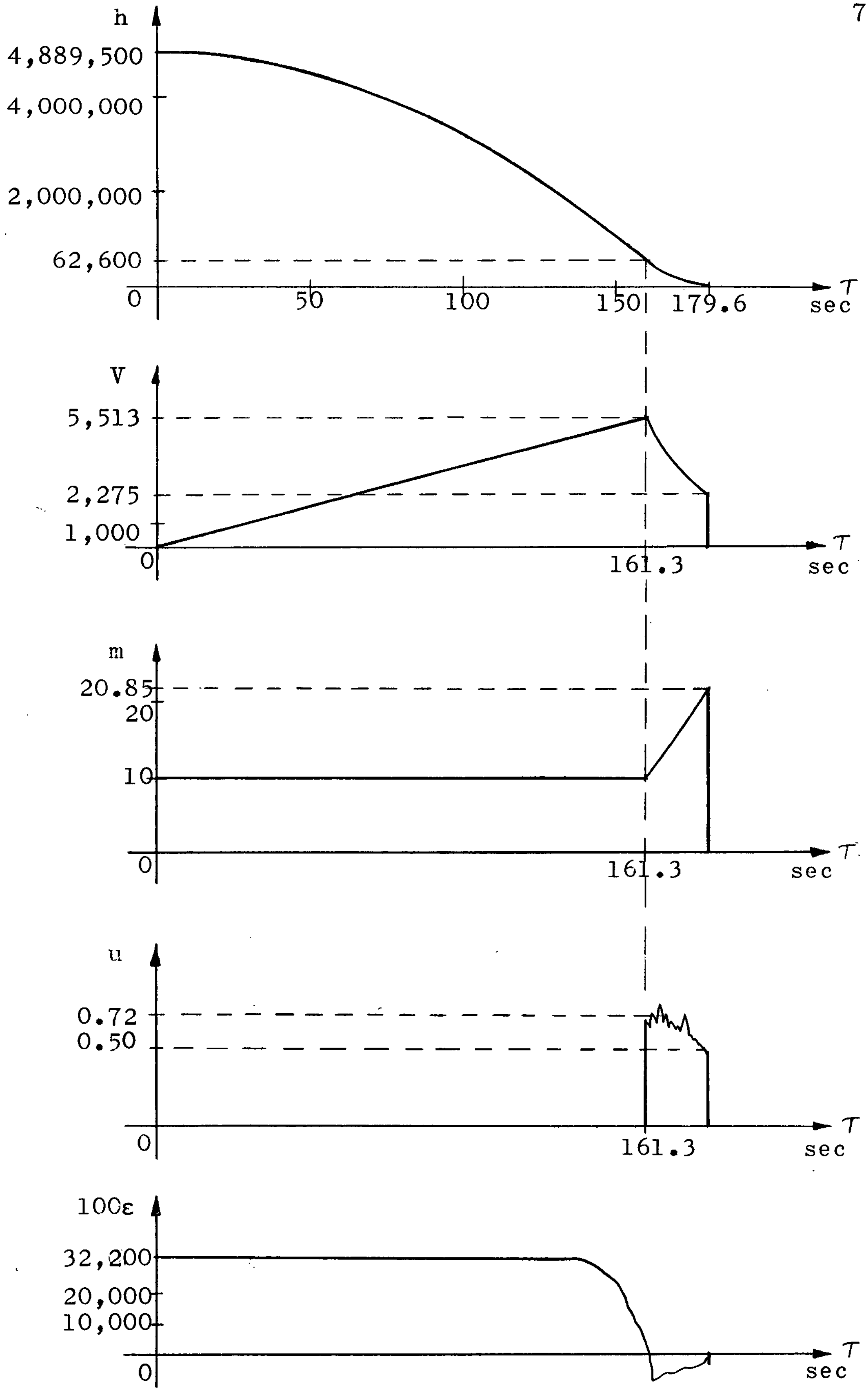


Fig. 3.6 Experimental results for the sounding rocket problem

$$m_o \cong m_1 \exp \left(\frac{V_1}{V_e}\right)$$

$$= 31.5 \text{ slug}$$

At the instant $\tau = \tau_2$, a relay switches the control variable u into the input of the mass integrator. For the coasting subarc the input to the mass integrator is zero and the mass is constant. At the final altitude $h_f$ the velocity is zero and the error signal $\varepsilon(\tau)$ is $m_f g$. Since both D and V increase with $\tau$ it can be seen from (3.28) that the error signal decreases to zero. At $\tau = \tau_2$ the relay operates and the rocket enters the variable thrust subarc. When h = 0, a second relay is used to clamp all integrator inputs at zero, freezing the operation.

Leitmann[12] has used the analytical results (see Section 2.3) and an IBM 701 digital computer for the solution of the sounding rocket problem with the same given data as was used in this section. His results are

$$h_2 = 62{,}576 \text{ ft.}$$
$$V_2 = 5{,}308 \text{ ft/sec}$$
$$m_2 = 10 \text{ slug}$$
$$u_2 = 0.74 \text{ slug/sec}$$
$$\tau_1 - \tau_2 = 18.7 \text{ sec.}$$
$$m_1 = 21 \text{ slug}$$
$$m_o = 31.4 \text{ slug}$$
$$u_1 = 0.51 \text{ slug/sec.}$$

In general this approach of using the analytical result to compute the solution is not possible, since the analytical result is not obtainable. However, the approach of Fig. 3.4 has general applicability. Comparison of the results shows that the experimental results for the sounding rocket problem are very satisfactory.

## 3.5 Some Other Possible Optimal Controllers

In the preceding section the switching function given by (3.28) has been used for the synthesis of the optimal control variable u by an analogue computer. The switching instant $\tau_2$ separating the coasting subarc from the variable thrust subarc is determined by $f_s(\tau_2) = 0$. On the variable thrust subarc a feedback loop around a high-gain amplifier is used to satisfy the condition for optimal control which requires that $\varepsilon(\tau) = 0$. It should be noted that the switching function $f_s(\tau)$ is a function of state variables. In the general case of Fig. 3.1 such a switching function may not be obtainable. In this case some other means must be used in order to determine the control variable u for the optimal trajectory. These can be obtained from the switching function

$$\varepsilon_2 \triangleq m\,\lambda_5 - V_e\,\lambda_3 \tag{3.30}$$

and the first integral (provided it exists, see (A.18)).

$$\varepsilon_3 \triangleq C - \lambda_2 V - \lambda_3\left(g + \frac{D}{m}\right) - \beta\left(\lambda_5 - \lambda_3 \frac{V_e}{m}\right) \tag{3.31}$$

Therefore there are three possible functions which can be used for the synthesis of control variable u for the optimal trajectory by means of a high-gain amplifier. These are

$$\varepsilon_1 = mg - D\left(1 + \frac{V}{V_e}\right) \tag{3.32}$$

$$\varepsilon_2 = m\,\lambda_5 - V_e\,\lambda_3 \tag{3.33}$$

$$\varepsilon_3 = C - \lambda_2 V - \lambda_3\left(g + \frac{D}{m}\right) - \beta\left(\lambda_5 - \lambda_3 \frac{V_e}{m}\right) \tag{3.34}$$

A switching function of the type given by (3.32) is preferable since it results in an extremely simple controller. Otherwise the Lagrange multipliers must be computed. In such a case $\varepsilon_2$ and $\varepsilon_3$ can be used in the same manner as $\varepsilon_1$ was used. It should be noted, however, that $\varepsilon_3 = 0$ for the complete trajectory and is not, therefore, a switching function even though it can be used to synthesize the control variable u.

In order to use (3.33), the Lagrange multipliers $\lambda_3$ and $\lambda_5$ must be solved simultaneously with the equations of motion. It is of interest to note that $\lambda_3$ and $\lambda_5$ can be obtained by solving the two differential equations (see (3.11) and (3.12))

$$\dot{\lambda}_3 = -\frac{\lambda_3}{m}\frac{D}{V_e} \tag{3.35}$$

$$\dot{\lambda}_5 = \frac{\lambda_5}{m\,V_e}\,(V_e\beta - D) \tag{3.36}$$

If the first integral is to be used for synthesizing the control variable u for the optimal trajectory, the complete set of Euler-Lagrange equations must be solved. This is much more complicated than the case of solving equations (3.35) and (3.36).

# 4. THE MODIFIED STEEPEST DESCENT METHOD

## 4.1 Introduction

Computational methods for the solution of optimization problems have had two primary directions in the past: The direct approach and the indirect approach. In the direct approach, equations of motion are solved by selecting an initial control variable and then performing an iteration on the control variable so that each new iteration improves the performance function to be optimized. The indirect approach involves the development of an iterative technique for solving the equations of motion and the Euler-Lagrange equations. The direct approach is usually associated with the gradient method or the method of steepest descent.

In this chapter a modified steepest descent method is described for the solution of optimization problems which can be programmed on analogue computers.

## 4.2 Basic Concept of the Modified Steepest Descent Method

The Mayer formulation of variational problems has been discussed in Chapter 2. In the case of the four rocket flight problems studied in Chapter 3, the optimal control variable can be determined as a function of state variables and feedback control methods can be employed. In general, the control variable u for the optimal trajectory may involve Lagrange multipliers and the computation of u becomes much more complicated.

The basis of the modified steepest descent is to search for the optimum value of the performance function by replacing a search in function space by a search in parameter space. This greatly reduces the dimensionality of the problem. The performance function is considered as a function of unknown terminal conditions. The final state of the system is determined by the solution of the equations of motion and the initial values of the state variables. The control variable for the optimal trajectory is determined by the state variables and Lagrange multipliers. The performance function may, therefore, be considered as a function of the unknown terminal conditions for the state variables and Lagrange multipliers. In theory, if the terminal conditions for the state variables and Lagrange multipliers are all known, the optimization problem can be solved by the method discussed in Section 3.2.

In many practical problems the terminal conditions are usually not all known. This complicates the synthesis of the control variable u for the optimal trajectory. In such cases some of the terminal conditions may be approximately determined by some means, and then the performance function is optimized with respect to the remaining terminal conditions, using the gradient method. This is the essential feature of the modified method of steepest descent.

Consider the problem of minimizing the performance function

$$\begin{aligned} P &= P(a_1, \ldots, a_n) \\ &= \Big[ P(t,x) \Big]_{t_0}^{t_f} \end{aligned} \tag{4.1}$$

subject to the equations of motion

$$\dot{x}_j = f_j(t,x,u), \; j = 1,\ldots,n. \tag{4.2}$$

where $x = (x_1,\ldots,x_n)$, $u = (u_1,\ldots,u_m)$, and the functions P and $f_j$ are given functions of their arguments.

Following the theory of calculus of variations, the augmented function

$$F = \sum_{j=1}^{n} \lambda_j(\dot{x}_j - f_j) \tag{4.3}$$

is formed which satisfies the Euler-Lagrange equations

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_j}\right) - \frac{\partial F}{\partial x_j} = 0 \;, \quad j = 1,\ldots,n. \\ &\frac{\partial F}{\partial u_k} = 0, \qquad\qquad k = 1,\ldots,m. \end{aligned} \tag{4.4}$$

and the transversality condition

$$\left[ dP + \left(F - \sum_{j=1}^{n} \frac{\partial F}{\partial \dot{x}_j} \dot{x}_j\right) dt + \sum_{j=1}^{n} \frac{\partial F}{\partial \dot{x}_j} dx_j \right]_{t_0}^{t_f} = 0 \tag{4.5}$$

Substituting the function F into equations (4.4) and (4.5) gives

$$\dot{\lambda}_j = - \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial x_j} \tag{4.6}$$

and

$$\left[ dP - \sum_{j=1}^{n} \lambda_j f_j \, dt + \sum_{j=1}^{n} \lambda_j \, dx_j \right]_{t_0}^{t_f} = 0 \tag{4.7}$$

If the function F does not depend on t explicitly, the first integral exists:

$$F - \sum_{j=1}^{n} \frac{\partial F}{\partial \dot{x}_j} \; \dot{x}_j = C \tag{4.8}$$

that is

$$\sum_{j=1}^{n} \lambda_j f_j = C \tag{4.9}$$

It follows from the transversality condition that if either $t_f$ or $t_0$ is free the first integral is equal to zero.

The computational technique for the solution of the optimization problem is to solve equations (4.2) and (4.6) subject to the conditions (4.7) and (4.9) so that the performance function P is a minimum. Note that the transversality condition yields information about the terminal values of the $\lambda$. If the first integral is known, it may give some information about the terminal values of x and $\lambda$. However, usually not all terminal values of x are given and not all terminal values of $\lambda$ can be determined by the transversality condition and the first integral.

For a minimum problem having n state variables $x_j$ the performance function P will, in general, have n unknown parameters $a_j$. If the first integral is known (provided it exists), only (n-1) unknown parameters are independent. In order to reduce the dimensionality a first approximation of these (n-1) unknown parameters may be obtained by computing a subclass of admissible trajectories which satisfy the equations of motion and the known terminal conditions of the state variables. The subclass of admissible trajectories is taken to satisfy some, but not necessarily all, the terminal conditions for $\lambda$. The initial values of x and $\lambda$ for the optimal trajectory can now be determined by the method of steepest descent.

In general, a computer program using the modified steepest descent method could proceed as follows. In order to simplify the discussion it is assumed that more initial values of the state variables than final values are known.

(1) A suitable control $u_0$ is selected as a first approximation and the equations of motion are solved forward in time. If $x_k(t_f)$ is known and $x_k(t_i)$ is unknown, an approximation to $x_k(t_i)$ can be obtained by adjusting $x_k(t_i)$ until the final value of $x_k$ takes on the prescribed value $x_k(t_f)$. If both terminal values $x_k(t_i)$ and $x_k(t_f)$ of a state variable $x_k(t)$ are unknown, a first approximation to $x_k(t_i)$ can be determined by minimizing the performance function P by the steepest descent method. The trajectories determined in this manner form a subclass of admissible trajectories.

(2) With the previously determined admissible trajectory the equations of motion and Euler-Lagrange equations are simultaneously solved backward in time. The unknown terminal values $\lambda_j(t_f)$ are adjusted at $t = t_f$ by iteration until the prescribed initial values of the corresponding $\lambda_j$ are obtained. A first approximation of initial values for x and $\lambda$ has now been determined.

(3) The equations of motion and Euler-Lagrange equations are simultaneously solved by the feedback control method (see Fig. 1.2) forward in time. The controller is introduced by the feedback control

technique and the value of the performance function is noted. This subclass of trajectories have a variable thrust subarc and the thrust for this subarc is determined by the optimal control law.

(4) The unknown initial values of x and $\lambda$ are adjusted according to the modified method of steepest descent until the performance function is minimized.

## 4.3 Possibility of Practical Applications

In practice, there is often a need for a low cost and comparatively simple on-line method for the solution of optimal control problems. At the present time many of the computational techniques existing in various industries often require the use of a large capacity general purpose digital computer. For economical reasons, this may not be acceptable in many possible applications. However, the modified steepest descent method can be used to realize comparatively simple on-line controllers.[4] The instantaneous control policy in real time may be obtained from an analogue computer which operates on a fast time scale. The trajectory in state space is solved by an analogue computer and a digital computer stores the data for the steepest descent adjustment of the unknown parameters. This modified steepest descent method takes account of random disturbances since a new control policy is computed for each trajectory. (see Fig. 4.1).

## 4.4 Further Investigations

The general idea of the modified steepest descent method based on the indirect approach of the calculus of variations

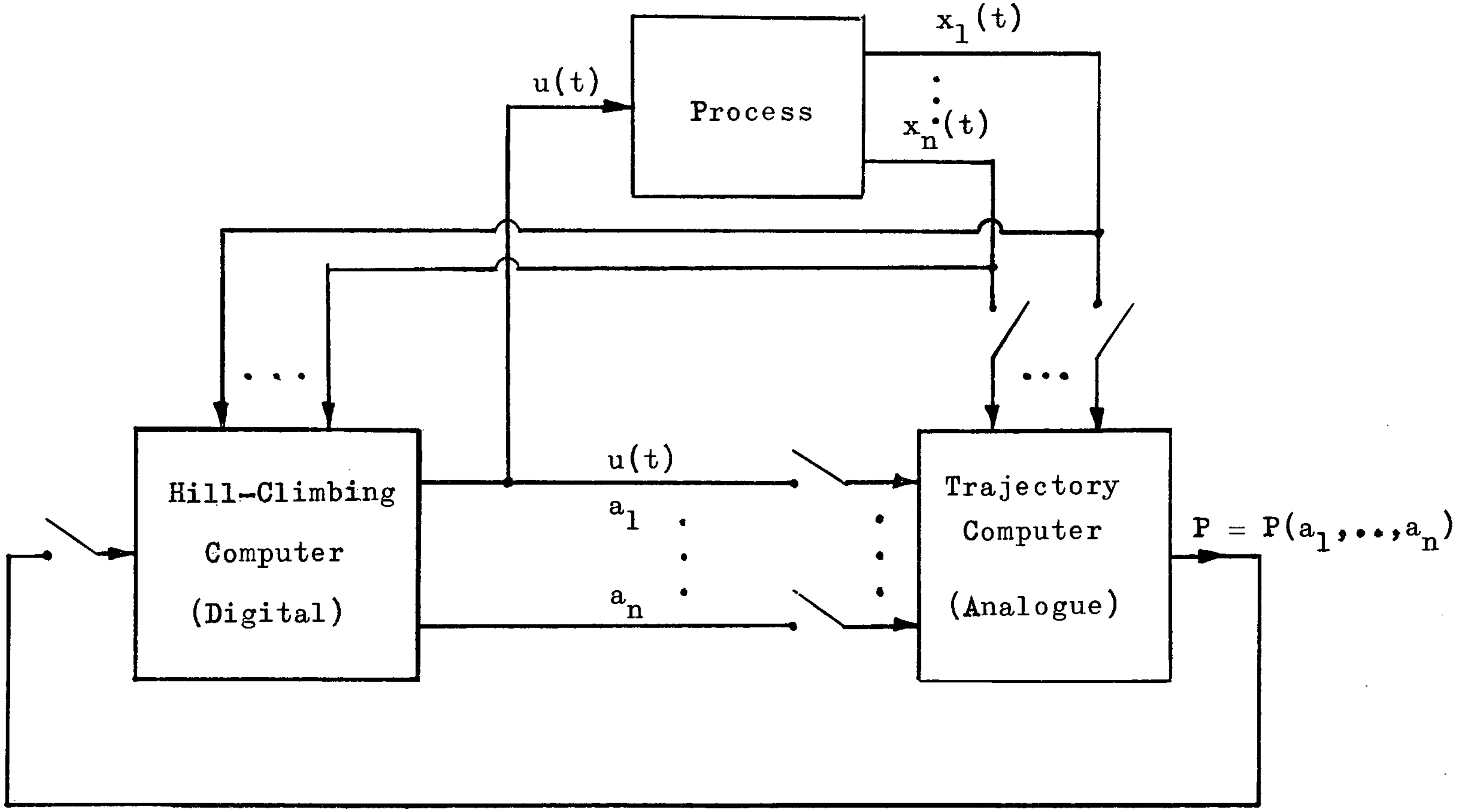


Fig. 4.1 An optimal controller for a general process

seems a very effective computational method. The high speed analogue computer is particularly suitable for the determination of trajectories and feedback methods can be used to synthesize the control variable. While computational experience with this method is limited at the present time, its potential as a computational scheme for practical applications deserves further studies.

It is suggested that further investigations in this method should be pursued to facilitate practical applications to the following problems:

1. The application of digital hill-climbing or gradient methods for automatically optimizing the performance function.
2. Hybrid computational methods for automatically adjusting the unknown parameters.
3. The extension of the method to problems of many degrees of freedom.

All these problems must be left open for future investigations.

# 5. FLIGHT SIMULATOR AND ANALOGUE SIMULATION

## 5.1 Introduction

Analogue computers may be divided broadly into direct analogues and indirect, or functional, analogues. The principle of operation of the direct analogue computer is based on a one-to-one correspondence between the behaviour of the analogue system and that of the physical system under study. In the indirect or functional analogue computer, the equations which describe a physical system are formulated by components, such as summers, integrators, multipliers, etc.

The flight simulator is a functional analogue computer of the electromechanical type and is ideally suited for the solution of trajectory problems. In order to study the rocket flight problem, a CF-100 flight simulator has been suitably modified.

## 5.2 Basic Components of the Flight Simulator

There are five basic components of the flight simulator. These are the summer, servo-amplifier, resolver, phase sensitive detector and relay. By means of these components mathematical operations can be performed. The summing amplifier, or the summer, carries out the arithmetic operations of sign inversion, multiplication by a constant and summation. The integration is carried out by an electromechanical integrator. This integrator consists of a servo-amplifier, a servo-motor and a tachometer. A gear box is used to couple the servo-motor to a linear potentiometer which converts the shaft angle into a voltage. Furthermore, the integrator is also used to generate functions and to carry out multiplication and division. The resolver performs trigonometric operations involving the transformation of coordinates. The phase sensitive detector is a device used to detect the phase change of an input signal with respect to a reference signal. A relay is energized when the input signal changes its phase.

## 5.3 Simulation of the Optimal Control Law

This section is devoted to the simulation of the optimal control law for the zero-lift rocket flight problem discussed in Chapter 3. For the programming of this problem a large number of multipliers and function generators are required. This cannot be handled by most ordinary analogue computers since only a small number of multipliers and function generators are normally available. The electromechancial computing units of a flight simulator are ideally suited for this type of problem. In the study of the theory of optimal rocket flight, it has been shown that the optimal trajectory consists of three subarcs. Associated with each subarc is a mode of control for the control parameter $\beta$. If impulsive boosting is assumed, one of the subarcs may be computed analytically. If the thrust program consists of maximum thrust, variable thrust and zero thrust, the maximum thrust mode must be included in the simulator. In general there are, therefore, three modes of thrust control.

It can be seen from the Appendix that the control parameter

$\beta$, appears in both equations (A.46) and (A.48). The three modes of thrust control must, therefore, be applied to these two equations.

The sequence of the modes is important. It follows from the theory of rocket flight that the sequence of these modes are:

Mode 1: $\beta = \beta_{max}$, constant thrust. Fuel consumption is at a constant rate and the mass is a linear function of time. $m = m_0 - \beta_{max} t$.

Mode 2: $\lambda_5 - \lambda_3 \frac{V_e}{m} = 0$, variable thrust. The mass is constrained to satisfy the variable thrust condition for optimal flight.

Mode 3: $\beta = 0$, zero thrust. The mass is constant.

The zeros of the function

$$f(m,\lambda) \triangleq \lambda_5 - \lambda_3 \frac{V_e}{m} \tag{5.1}$$

can be used to define the three subarcs (see Fig. 5.1).

The switching from Mode 1 to Mode 2 is performed in the simulator by a phase sensitive detector and a relay. In Mode 1, the relay is in the position for maximum thrust. When $f(m,\lambda)$ becomes zero, the relay switches to Mode 2. During Mode 2 the control parameter $\beta$ is implicitly constrained so that $f(m,\lambda) = 0$. For Mode 3, the signal representing the control parameter $\beta$ is shorted to ground.

## 5.4 Analysis of a Test Problem

In order for the simulator to perform satisfactorily,

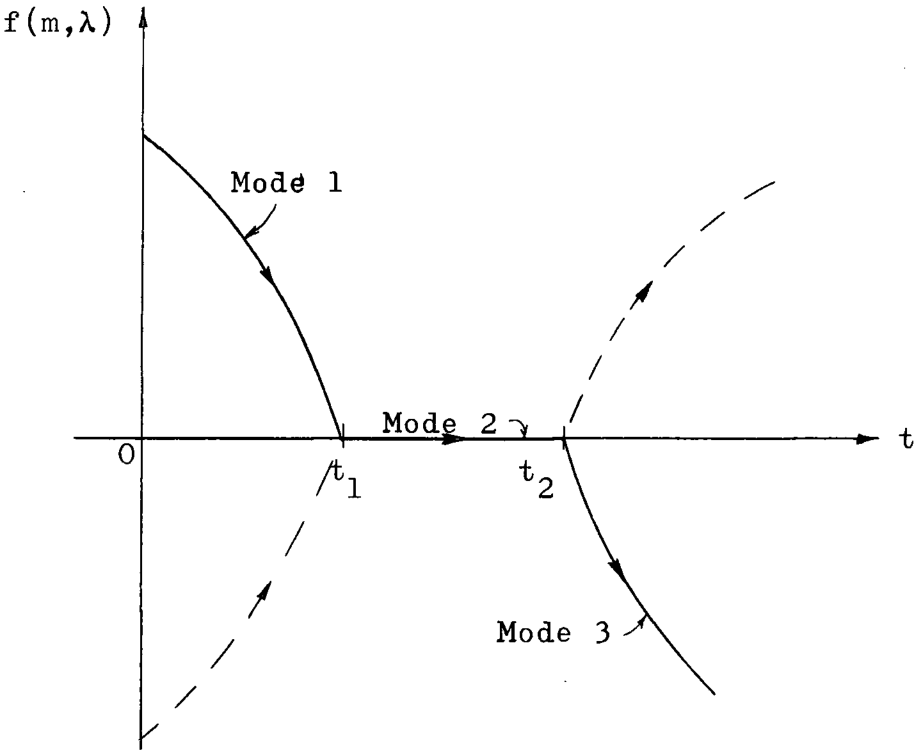


Fig. 5.1 Three modes of thrust control

various units must be calibrated. The calibration can be best performed by solving a simple problem of free motion described by the following differential equations:

$$\begin{aligned}\dot{x} &= V\cos\theta \\ \dot{h} &= V\sin\theta \\ \dot{V} &= -g\sin\theta \\ \dot{\theta} &= -\frac{g}{V}\cos\theta\end{aligned} \qquad (5.2)$$

The initial conditions at t = 0 are

$$\begin{aligned}x(0) &= 0 \\ h(0) &= 0 \\ V(0) &= V_o \\ \theta(0) &= \theta_o\end{aligned}$$

where $0 < \Theta_o < \frac{\pi}{2}$.

The solution of this set of differential equations is

$$x = V_o \cos \Theta_o \, t$$

$$h = V_o \sin \Theta_o t - \frac{1}{2} g t^2$$

$$V^2 = V_o^2 + g^2 t^2 - 2 g V_o \sin \Theta_o \, t \qquad (5.3)$$

$$\tan\left(\frac{\Theta}{2} + \frac{\pi}{4}\right) = \tan\left(\frac{\Theta_o}{2} + \frac{\pi}{4}\right) \Big/ \sqrt{t^2 - 2 \frac{V_o}{g} \sin \Theta_o t + \frac{V_o^2}{g^2}}$$

Eliminating the sin $\Theta$ from the second and the third equations of (5.2) gives

$$\dot{h} = -V \dot{V}/g$$

Integrating the above equation yields

$$V^2 = V_o^2 - 2 g h \qquad (5.4)$$

Since V cannot be zero, it follows from the second equation of (5.2) that sin $\Theta$ must be zero at $h_{max}$. Furthermore, because of (5.4), V is a minimum when h is a maximum.

From the solution for the velocity of (5.3), it is seen that

$$V_{min} = V_o \cos \Theta_o \qquad (5.5)$$

which is extremely useful for calibration purposes.

Another important fact is that the velocity in the x-direction, that is, $\dot{x}$ is always constant. This gives a good check for the operation of the simulator.

Differentiating the solution for the velocity and equating it to be zero gives

$$t = \frac{V_o}{g} \sin \theta \tag{5.6}$$

and at this instant the velocity reaches its minimum.

The above equations were used to scale the voltages on the simulator so that for the mass used the trajectory covered a convenient range of an xy-recorder.

## 5.5 Experimental Test of the Modified Steepest Descent Method

The basic idea for the method of modified steepest descent has been discussed in Chapter 4. It would evidently be profitable to study a particular problem which can lead to a better understanding of the nature of the method.

Consider the zero-lift rocket flight problem. The performance function to be minimized is the fuel consumption. If the initial mass $m_o$ is assumed to be given, the problem is equivalent to maximizing the final mass $m_f$. The initial and final conditions are

$$\begin{aligned} x(t_o) &= 0, & x(t_f) &= x_f \\ h(t_o) &= 0, & h(t_f) &= h_f \\ V(t_o) &= 0, & & \\ m(t_o) &= m_o, & & \end{aligned} \tag{5.7}$$

where $m_o$, $x_f$ and $h_f$ are given values. The following values of the state variables are unknown at the terminal points: $\theta_o$, $\theta_f$, $V_f$, $m_f$. Here $m_f$ is to be maximized.

The transversality condition for this problem is

$$\left[ - C\ dt + \lambda_1\ dx + \lambda_2\ dh + \lambda_3\ dV + \lambda_4\ d\Theta + (\lambda_5 - 1) dm \right]_{t_o}^{t_f} = 0$$

(5.8)

The quantities $t_o$, $t_f$, $V_f$, $\Theta_o$, $\Theta_f$, $m_f$ are free, so that $C = 0$, $\lambda_{40} = 0$, $\lambda_{3f} = 0$, $\lambda_{4f} = 0$ and $\lambda_{5f} = 1$, and $\lambda_{10}$, $\lambda_{20}$, $\lambda_{30}$, $\lambda_{50}$, $\lambda_{1f}$ and $\lambda_{2f}$ are unknown.

The first integral (see (A.55)) is

$$\lambda_1 V \cos\Theta + \lambda_2 V \sin\Theta - \lambda_3 (\frac{D}{m} + g \sin\Theta) - \lambda_4 \frac{g}{V} \cos\Theta - \beta(\lambda_5 - \lambda_3 \frac{V_e}{m}) = 0$$

and for $t = t_f$, $\beta = 0$, $\lambda_{3f} = 0$ and $\lambda_{4f} = 0$. Hence

$$\tan\Theta_f = -\frac{\lambda_{1f}}{\lambda_{2f}} \tag{5.10}$$

Equation (5.10) gives a relation between $\Theta_f$, $\lambda_{1f}$ and $\lambda_{2f}$. From the Euler-Lagrange equation (A.49) it is seen that $\lambda_1$ is a constant for the entire optimal trajectory.

For this particular problem $\Theta_o$ can not be $90^o$, as can be seen from the equation of motion (A.47) for $\Theta$. If $\Theta_o = 90^o$, and the lift is zero, $\dot{\Theta}$ is zero if $V_o$ is not zero, thus the final point $(x_f, h_f)$ cannot be reached. If $\Theta_o < 90^o$, then $V_o$ cannot be zero, otherwise $\dot{\Theta}$ will be infinite at the initial point. Thus an initial velocity is essential which can be obtained by impulsive boosting. In this case, the computation starts with the variable thrust subarc, since the boosting subarc is very short and may be neglected.

Consider now the case of impulsive boosting where there is no constraint on the magnitude of the thrust. Let $t_1$ be the time at the end of boosting, then

$$
\begin{aligned}
& t_1 - t_o = \Delta t \cong 0 \\
& x_1 \cong x_o = 0 \\
& h_1 \cong h_o = 0 \\
& V_1 \neq 0 \\
& m_1 \cong m_o \exp\left(-\frac{V_1}{V_e}\right) \\
& \lambda_{11} \cong \lambda_{10} \\
& \lambda_{21} \cong \lambda_{20} \\
& \lambda_{31} \cong \lambda_{30} \\
& \lambda_{41} \cong \lambda_{40} = 0 \\
& \lambda_{51} \cong \lambda_{50} + \lambda_{30} V_e\left(\frac{1}{m_1} - \frac{1}{m_o}\right)
\end{aligned}
\tag{5.11}
$$

At $t = t_1$, the variable thrust subarc starts, and

$$\lambda_{51} = \lambda_{31} \frac{V_e}{m_1} \tag{5.12}$$

If the computation starts at $t = t_1$, the initial values for the state variables: $V_1$, $m_1$ and $\Theta_1$ are unknown. However, $V_1$ and $m_1$ are related by the relation

$$m_1 \cong m_o \exp\left(-\frac{V_1}{V_e}\right) \tag{5.13}$$

If the magnitude of the thrust is constrained by the condition

$$0 \leqslant V_e\beta \leqslant V_e\beta_{max} \tag{5.14}$$

where $\beta_{max}$ is the maximum control parameter, the approximation of

(5.11) still can be applied, but the optimal trajectory will start with maximum thrust subarc. Since the initial velocity $V_o$ is zero, some auxiliary device is required to avoid that $\dot{\Theta}$ be infinite at the start. This can be done by holding the rocket on a launcher with maximum thrust for a negligibly short time, and the rocket then starts with a maximum thrust subarc with an initial angle $\Theta_o$ less than $90^o$. This is equivalent to the problem of starting with an initial velocity $V_i \neq 0$ and an initial mass given by

$$m_i \cong m_o \exp\left(-\frac{V_i}{V_e}\right) \tag{5.15}$$

Thus the optimal trajectory starts with the following initial conditions:

$$\begin{aligned}
&t_i - t_o = \Delta t \cong 0 \\
&x_i \cong x_o = 0 \\
&h_i \cong h_o = 0 \\
&V_i \neq 0 \\
&m_i \cong m_o \exp\left(-\frac{V_i}{V_e}\right) \\
&\lambda_{1i} \cong \lambda_{10} \\
&\lambda_{2i} \cong \lambda_{20} \\
&\lambda_{3i} \cong \lambda_{30} \\
&\lambda_{4i} \cong \lambda_{40} \\
&\lambda_{5i} \cong \lambda_{50} + \lambda_{30} V_e \left(\frac{1}{m_i} - \frac{1}{m_o}\right)
\end{aligned} \tag{5.16}$$

In this case the switching function may not reach zero at $t = t_i$. The optimal trajectory must then start with a maximum thrust subarc. When the switching function (5.1) is zero, the trajectory enters the variable thrust subarc. The computation starts at

$t = t_i$ with the initial values of the state variables $V_i$, $m_i$ and $\Theta_i$ unknown. However, $V_i$ and $m_i$ are related by the equation

$$m_1 \cong m_o \exp\left(-\frac{V_i}{V_e}\right) \tag{5.17}$$

For simplicity, the drag function D used in the simulation is assumed to have the form

$$\begin{aligned} D &= D(V,h) \\ &= k\, V^2\, e^{-ah} \\ &\cong k\, \frac{V^2}{1 + ah} \end{aligned} \tag{5.18}$$

To determine a first approximation for the initial values of $V_i$, $m_i$ and $\Theta_i$, the trajectory is considered to consist of a suitable constant thrust subarc or a maximum thrust subarc and a zero thrust subarc. A value $V_i$ is selected and $m_i$ computed by (5.17). A suitable initial value $\Theta_i$ is chosen and the length of the constant thrust subarc varied so that the final point $(x_f, h_f)$ is reached. Fig. 5.2 illustrates the results obtained for various $\Theta_i$. The value of $m_f$ for each of these trajectories is noted and the results are plotted as shown in Fig. 5.3.

In this manner $\Theta_i$, $V_i$ and $m_i$ are approximately determined. A particular set of data is shown in Fig. 5.4. All quantities on the simulator are in terms of degrees of shaft rotation.

Since $\Theta_f$ is now known at the final point, it follows that $\lambda_{1f}$ and $\lambda_{2f}$ are related by

$$\lambda_{2f} = -\cot \Theta_f\, \lambda_{1f} \tag{5.19}$$

Note that at the final point, $\lambda_{1f}$ and $\lambda_{2f}$ are the only unknowns.

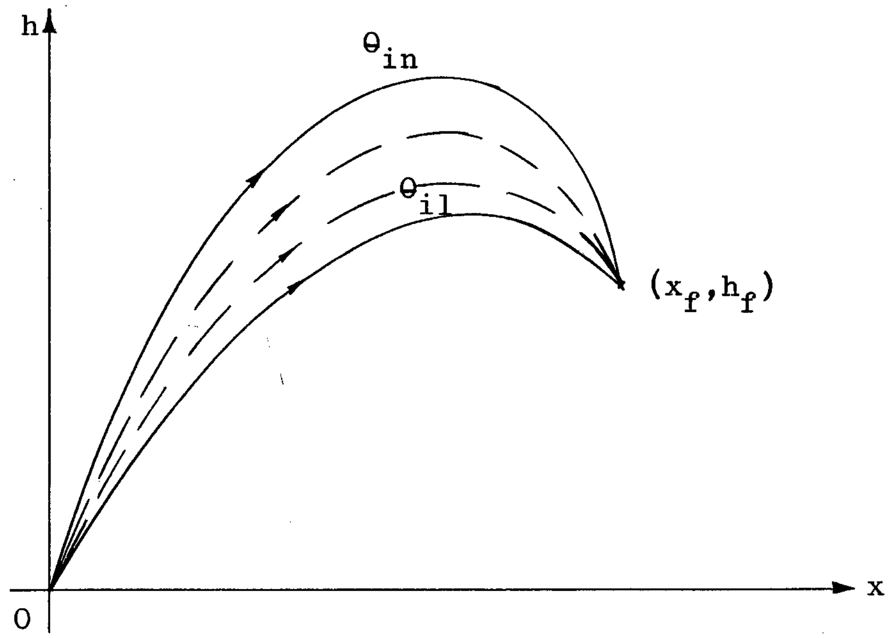


Fig. 5.2 A subclass of admissible trajectories

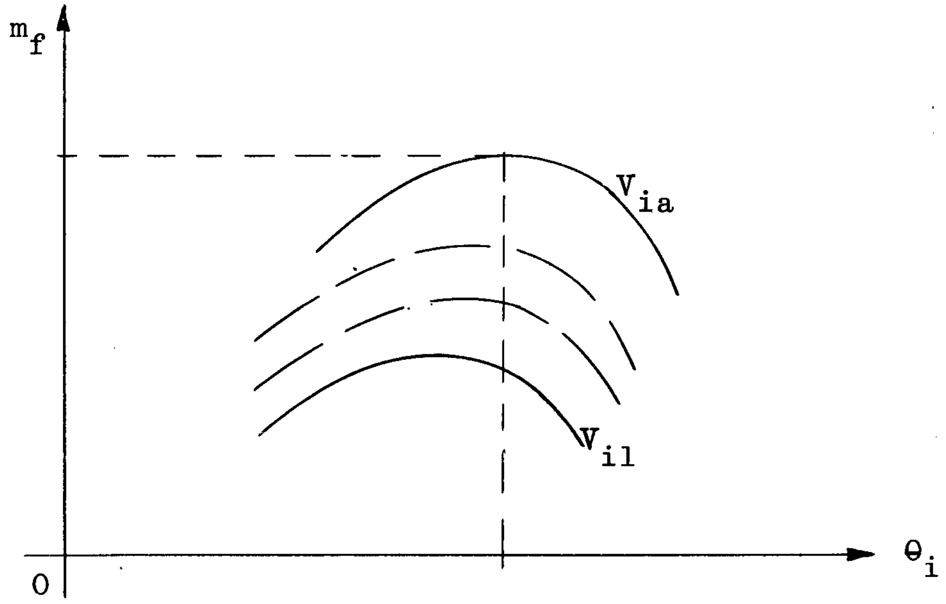


Fig. 5.3 Determination of approximate initial values for the state variables

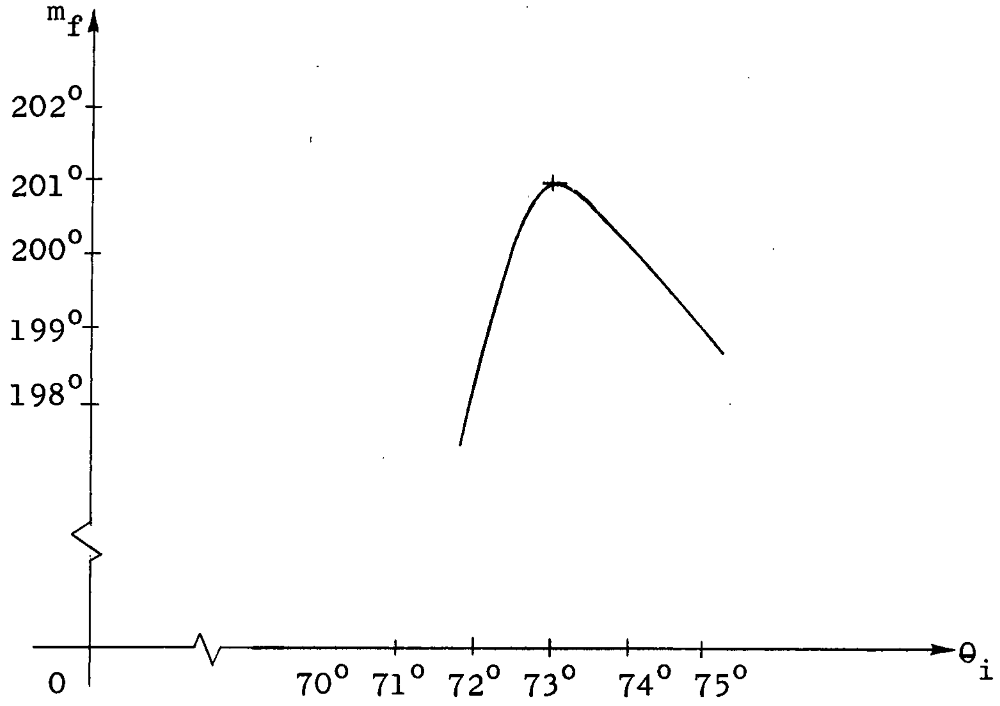


Fig. 5.4 A particular set of approximate initial values of the state variables

If $\lambda_{1f}$ is known, $\lambda_{2f}$ can be computed by (5.19). Therefore, by selecting a $\lambda_{1f}$, the equations of motion and the Euler-Lagrange equations can be solved backwards in time. The Lagrange multiplier $\lambda_{1f}$ is varied until the condition $\lambda_{4i} = 0$ is satisfied. All initial values are now specified and it is then possible to compute improved trajectories by introducing the optimal control for the trajectory and solving it forward in time. The final mass $m_f$ is now considered as a function of the parameters: $\theta_i$, $\lambda_{1i}$, $\lambda_{2i}$, $\lambda_{3i}$, and optimum values of these parameters can be determined by the modified steepest descent method. The adjustment of the parameter values terminates when $m_f$ reaches a maximum. This approach proved fairly successful on the flight simulator. The numerical result is in terms of degrees of shaft rotation. Since the flight simulator does not have a high accuracy, no precise numerical results have been obtained. However, a set of trajectories similar to Fig. 5.2 consisting of a maximum thrust subarc, a variable thrust subarc and a zero thrust subarc can be obtained. Fig. 5.5 illustrates the performance function $m_f$ considered as a function of the parameter $a_k$.

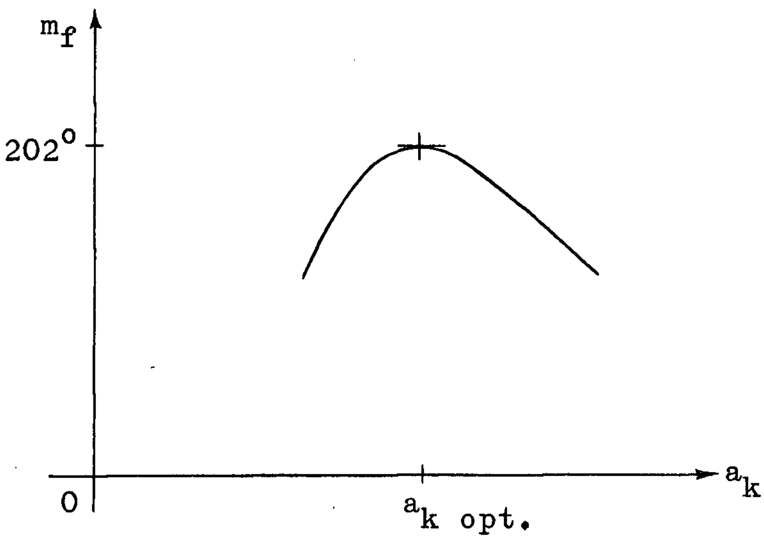


Fig. 5.5 <u>Optimum performance function</u>

At the point $a_k = a_{k\ opt.}$, the initial values of the parameters are

$$a_1 = \theta_i = 73^o$$
$$a_2 = V_i = 50^o$$
$$a_3 = m_i = 330^o$$
$$a_4 = \lambda_{1i} = 168^o$$
$$a_5 = \lambda_{2i} = 219^o$$
$$a_6 = \lambda_{3i} = 253^o$$
$$a_7 = \lambda_{4i} = 0^o \quad \text{(This is known)}$$

For this problem the Lagrange multiplier $\lambda_5$ is obtained from the first integral. Therefore, $\lambda_{5i}$ is fixed by the first integral.

# 6. CONCLUSION

General optimal control problems formulated by the method of the calculus of variations with particular emphasis on the problem of Mayer have been studied. Special cases of optimal control can be realized by means of feedback control. The Lagrange multipliers can be eliminated and the control variable for the optimal trajectory is then a function of the state variables only. In this case the optimal control system can be treated as an optimal feedback control system. Analogue computer methods are convenient for the solution of such problems.

The modified steepest descent method is suitable for the solution of certain classes of optimal control problems.

(1) For very complex problems the dimensionality of the problem can be reduced by using conventional iterative and gradient methods to determine subclasses of admissible trajectories satisfying some, but not necessarily all, of the terminal conditions. The modified steepest descent method can then be used to optimize the performance function which is considered to be a function of the remaining terminal conditions.

(2) Simulator and analogue computer results show that the method is practical and can be used to synthesize real-time optimal controllers.

(3) For complex problems hybrid-computers are essential and are of considerable future interest. This thesis has dealt mainly with the analogue portion of the optimal controller. The optimization of the performance function has

been performed by a manual search. In an actual system the optimization would be performed by a digital computer (see Fig. 4.1). The analogue computer is suitable for high speed trajectory computations while the digital computer is suitable for the logical operations involved in the optimization of the performance function. The results of the research undertaken show that analogue computers can be used to synthesize the control variable for optimal control once the correct initial values are known. It is well known that digital computers can readily optimize a performance function P of several variables by some type of gradient method. The optimization of P is used to determine the correct initial values. It can therefore be concluded that it is possible to synthesize optimal controllers for a variety of systems by hybrid computational means.

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# APPENDIX

## 1. The Euler-Lagrange Equations for Rocket Flight Problems

Substituting the augmented function F of (2.10) into (2.11) yields the set of Euler-Lagrange equations

$$\dot{\lambda}_1 = \lambda_6 \frac{\partial \Phi}{\partial x} + \lambda_7 \frac{\partial \Psi}{\partial x} \tag{A.1}$$

$$\dot{\lambda}_2 = \frac{\lambda_3}{m} \frac{\partial D}{\partial h} + \lambda_6 \frac{\partial \Phi}{\partial h} + \lambda_7 \frac{\partial \Psi}{\partial h} \tag{A.2}$$

$$\begin{aligned}\dot{\lambda}_3 = &-\lambda_1 \cos \Theta - \lambda_2 \sin \Theta + \frac{\lambda_3}{m} \frac{\partial D}{\partial V} \\ &+ \lambda_4 \left(\frac{L + V_e\beta \sin \omega}{m\, V^2} - \frac{g}{V^2} \cos \Theta\right) \\ &+ \lambda_6 \frac{\partial \Phi}{\partial V} + \lambda_7 \frac{\partial \Psi}{\partial V}\end{aligned} \tag{A.3}$$

$$\begin{aligned}\dot{\lambda}_4 = &\lambda_1 V \sin \Theta - \lambda_2 V \cos \Theta + \lambda_3 g \cos \Theta \\ &- \lambda_4 \frac{g}{V} \sin \Theta + \lambda_6 \frac{\partial \Phi}{\partial \Theta} + \lambda_7 \frac{\partial \Psi}{\partial \Theta}\end{aligned} \tag{A.4}$$

$$\begin{aligned}\dot{\lambda}_5 = &\frac{\lambda_3}{m^2} (V_e\beta \cos \omega - D) + \frac{\lambda_4}{m^2 V}(L + V_e\beta \sin \omega) \\ &+ \lambda_6 \frac{\partial \Phi}{\partial m} + \lambda_7 \frac{\partial \Psi}{\partial m}\end{aligned} \tag{A.5}$$

$$0 = \frac{\lambda_3}{m} \frac{\partial D}{\partial L} - \frac{\lambda_4}{mV} + \lambda_6 \frac{\partial \Phi}{\partial L} + \lambda_7 \frac{\partial \Psi}{\partial L} \tag{A.6}$$

$$\begin{aligned}0 = \frac{d\beta}{d\alpha}\Big(&-\lambda_3 \frac{V_e}{m} \cos \omega - \lambda_4 \frac{V_e}{mV} \sin \omega + \lambda_5 + \lambda_6 \frac{\partial \Phi}{\partial \beta} \\ &+ \lambda_7 \frac{\partial \Psi}{\partial \beta}\Big)\end{aligned} \tag{A.7}$$

$$0 = \lambda_3 \frac{V_e\beta}{m} \sin \omega - \lambda_4 \frac{V_e\beta}{mV} \cos \omega + \lambda_6 \frac{\partial \Phi}{\partial \omega} + \lambda_7 \frac{\partial \Psi}{\partial \omega} \tag{A.8}$$

2. The Vertical Flight (The Sounding Rocket) Problem

Assume that the thrust direction is vertical and that the two additional constraints are

$$\Phi = \theta - \frac{\pi}{2} = 0 \tag{A.9}$$

$$\Psi = \omega = 0 \tag{A.10}$$

The equations of motion become

$$\varphi_2 = \dot{h} - V = 0 \tag{A.11}$$

$$\varphi_3 = \dot{V} + g + \frac{D - V_e\beta}{m} = 0 \tag{A.12}$$

$$\varphi_5 = \dot{m} + \beta = 0 \tag{A.13}$$

The Euler-Lagrange equations are

$$\dot{\lambda}_2 = \frac{\lambda_3}{m}\frac{\partial D}{\partial h} \tag{A.14}$$

$$\dot{\lambda}_3 = -\lambda_2 + \frac{\lambda_3}{m}\frac{\partial D}{\partial V} \tag{A.15}$$

$$\dot{\lambda}_5 = \frac{\lambda_3}{m^2}(V_e\beta - D) \tag{A.16}$$

$$0 = \frac{d\beta}{d\alpha}\left(\lambda_5 - \lambda_3\frac{V_e}{m}\right) \tag{A.17}$$

The first integral is

$$\lambda_2 V - \lambda_3\left(g + \frac{D}{m}\right) - \beta\left(\lambda_5 - \lambda_3\frac{V_e}{m}\right) = C \tag{A.18}$$

3. The Horizontal Flight Problem

If the flight path is assumed to be horizontal and if the thrust direction is parallel to V, the additional constraints are

$$\Phi = \theta = 0 \tag{A.19}$$

$$\Psi = \omega = 0 \tag{A.20}$$

The equations of motion are

$$\varphi_1 = \dot{x} - V = 0 \tag{A.21}$$

$$\varphi_3 = \dot{V} + \frac{D - V_e\beta}{m} = 0 \tag{A.22}$$

$$\varphi_5 = \dot{m} + \beta = 0 \tag{A.23}$$

The Euler-Lagrange equations are

$$\dot{\lambda}_1 = 0 \tag{A.24}$$

$$\dot{\lambda}_3 = -\lambda_1 + \frac{\lambda_3}{m}\frac{\partial D}{\partial V} \tag{A.25}$$

$$\dot{\lambda}_5 = \frac{\lambda_3}{m^2}(V_e\beta - D) + \lambda_4 \frac{L}{m^2 V} \tag{A.26}$$

$$0 = \lambda_3 \frac{\partial D}{\partial L} - \frac{\lambda_4}{V} \tag{A.27}$$

$$0 = \frac{d\beta}{d\alpha}\left(\lambda_5 - \lambda_3 \frac{V_e}{m}\right) \tag{A.28}$$

The first integral is

$$\lambda_1 V - \lambda_3 \frac{D}{m} - \beta\left(\lambda_5 - \lambda_3 \frac{V_e}{m}\right) = C \tag{A.29}$$

4. The Arbitrarily Inclined Rectilinear Flight Problem

If the flight path is rectilinear at an arbitrary angle $\Theta$ with respect to a horizontal plane and if the thrust direction is parallel to the flight path, the additional constraints are

$$\Phi = \Theta - \text{constant} = 0$$

$$\Psi = \omega = 0 \tag{A.30}$$

The equations of motion are

$$\varphi_1 = \dot{x} - V\cos\Theta = 0 \tag{A.31}$$

$$\varphi_2 = \dot{h} - V\sin\Theta = 0 \tag{A.32}$$

$$\varphi_3 = \dot{V} + g\sin\Theta + \frac{D - V_e\beta}{m} = 0 \tag{A.33}$$

$$\varphi_5 = \dot{m} + \beta = 0 \tag{A.34}$$

The Euler-Lagrange equations are

$$\dot{\lambda}_1 = 0 \tag{A.35}$$

$$\dot{\lambda}_2 = \frac{\lambda_3}{m}\frac{\partial D}{\partial h} \tag{A.36}$$

$$\dot{\lambda}_3 = -\lambda_1 \cos \Theta - \lambda_2 \sin \Theta + \frac{\lambda_3}{m}\frac{\partial D}{\partial V} \tag{A.37}$$

$$\dot{\lambda}_5 = \frac{\lambda_3}{m^2}(V_e\beta - D) + \frac{\lambda_4}{m^2}\frac{L}{V} \tag{A.38}$$

$$0 = \lambda_3 \frac{\partial D}{\partial L} - \frac{\lambda_4}{V} \tag{A.39}$$

$$0 = \frac{d\beta}{d\alpha}(\lambda_5 - \lambda_3 \frac{V_e}{m}) \tag{A.40}$$

The first integral is

$$\lambda_1 V \cos \Theta + \lambda_2 V \sin \Theta - \lambda_3(g \sin \Theta + \frac{D}{m}) - \beta(\lambda_5 - \lambda_3 \frac{V_e}{m}) = C \tag{A.41}$$

5. The Zero-lift Flight Problem

If the thrust direction is tangent to the flight path and if the lift is assumed to be zero, the additional constraints are

$$\Phi = L = 0 \tag{A.42}$$

$$\Psi = \omega = 0 \tag{A.43}$$

The equations of motion are

$$\varphi_1 = \dot{x} - V \cos \Theta = 0 \tag{A.44}$$

$$\varphi_2 = \dot{h} - V \sin \Theta = 0 \tag{A.45}$$

$$\varphi_3 = \dot{V} + g \sin \Theta + \frac{D - V_e \beta}{m} = 0 \tag{A.46}$$

$$\varphi_4 = \dot{\Theta} + \frac{g}{V} \cos \Theta = 0 \tag{A.47}$$

$$\varphi_5 = \dot{m} + \beta = 0 \tag{A.48}$$

The Euler-Lagrange equations are

$$\dot{\lambda}_1 = 0 \tag{A.49}$$

$$\dot{\lambda}_2 = \frac{\lambda_3}{m} \frac{\partial D}{\partial h} \tag{A.50}$$

$$\dot{\lambda}_3 = -\lambda_1 \cos \Theta - \lambda_2 \sin \Theta + \frac{\lambda_3}{m} \frac{\partial D}{\partial V} - \lambda_4 \frac{g}{V^2} \cos \Theta \tag{A.51}$$

$$\dot{\lambda}_4 = \lambda_1 V \sin \Theta - \lambda_2 V \cos \Theta + \lambda_3 g \cos \Theta - \lambda_4 \frac{g}{V} \sin \Theta \tag{A.52}$$

$$\dot{\lambda}_5 = \frac{\lambda_3}{m^2} (V_e \beta - D) \tag{A.53}$$

$$0 = \frac{d\beta}{d\alpha} (\lambda_5 - \lambda_3 \frac{V_e}{m}) \tag{A.54}$$

The first integral is

$$\lambda_1 V \cos \Theta + \lambda_2 V \sin \Theta - \lambda_3 (g \sin \Theta + \frac{D}{m}) - \lambda_4 \frac{g}{V} \cos \Theta - \beta(\lambda_5 - \lambda_3 \frac{V_e}{m}) = C \tag{A.55}$$