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ON SOME NON-ARCHIMEDEAN NORMED LINEAR SPACES

ABSTRACT

A class of complete non-Archimedean pseudo-normed linear spaces for which the field of scalars has a trivial valuation is introduced; we call these spaces "V-spaces."

V-spaces differ from the classical normed linear spaces in that the homogeneity of the norm is replaced by the requirement that $\|\alpha x\| = \|x\|$ for all x and all scalars $\alpha \neq 0$; the usual triangle inequality is modified to

$$\|x + y\| \begin{cases} \leq \text{Max } \{\|x\|, \ \|y\|\} & \text{for all } x, y \\ = \text{Max } \{\|x\|, \ \|y\|\} & \text{if } \|x\| \neq \|y\|, \end{cases}$$

and it is assumed that the norm of an element is either zero or is equal to ρ^n for a fixed real $\rho > 1$ and some integer n.

The concept of a "distinguished basis" in a V-space is defined. By use of a modified form of Riesz's Lemma, it is shown that every V-space admits a distinguished basis. Each element of a V-space then has a uniquely determined series expansion in terms of the elements of a given distinguished bases. An analogue of the Paley-Wiener Theorem is proved for distinguished bases. Properties of distinguished bases are exploited throughout this work.

Linear and non-linear operators on V-spaces are also studied. In the usual way, a norm is defined under which the set of bounded operators is a V-space and the set of bounded linear operators is a "V-algebra." A characterization of bounded linear operators is given as well as theorems on spectral decompositions.

Under certain assumptions on the expansions of x, y, A, the existence of solutions to equations of the form xz = y in V-algebras, and of the form Ax = y in arbitrary V-spaces is proved. Approximations of the solutions are obtained.

A representation theorem for continuous linear functionals on a V-space is given. This representation uses an analogue of the classical inner product.

Examples of V-spaces and V-algebras discussed include spaces of functions from a Hausdorff space to a normed linear space, on which the pseudo-norm characterizes the asymptotic behaviour of the functions. Some results of the theory of pure asymptotics are extended to arbitrary V-spaces.

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ON SOME NON-ARCHIMEDEAN NORMED LINEAR SPACES

by

JOSEPH PIERRE ROBERT

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PIERRE ROBERT

Department of Mathematics

The University of British Columbia,
Vancouver 8, Canada

Date April 6, 1965

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INTRODUCTION

The purpose of this work is to initiate the theory of a non-standard type of pseudo-normed linear spaces, herein called V-spaces.

V-spaces depart from the classical normed linear spaces ([7], [36]) in that the usual requirements on the norm function

$$(0.1) \qquad \|\alpha x\| = |\alpha| \, \|x\| \qquad \text{for all } x \text{ and all scalars } \alpha,$$

$$(0.2) \qquad \|x + y\| \le \|x\| + \|y\| \text{ for all } x, y,$$

are replaced by

$$(0.3) \qquad \|\alpha x\| = \|x\| \qquad \text{for all } x \text{ and all scalars } \alpha \neq 0,$$

$$(0.4) \qquad \|x + y\| \begin{cases} \le \operatorname{Max} \{\|x\|, \|y\|\} \text{ for all } x, y, \\ = \operatorname{Max} \{\|x\|, \|y\|\} \text{ if } \|x\| \neq \|y\| , \end{cases}$$

and, also, by the additional condition that the norm of an element is either 0 or is equal to ρ^n for a fixed real ρ, $1 < \rho < \infty$, and some integer n. A V-space is assumed to be complete with respect to its norm and the field of scalars to have characteristic 0 ([10]). Thus, in the usual terminology, a V-space is a complete strongly non-Archimedean pseudo-normed linear space over a field of scalars with characteristic 0 and a trivial valuation.

The author's attention was directed to this abstract structure by the following example. A classical method to obtain information about the asymptotic behaviour of a real valued function is to compare it with the elements of an "asymptotic sequence" of functions (see Erdelyi [9], van der Corput [38], [39]). C. A. Swanson and M. Schulzer [32], [33], have extended this method of comparison to functions defined on some neighbourhood of a non-isolated point of a Hausdorff space and with ranges in an arbitrary Banach space. It is shown, in Chapter 3, that when applied to the elements of a linear space of functions, the results of this method can be expressed by assigning to each function a norm under which the space is a V-space.

Linear spaces satisfying the defining properties of a V-space, except for the retention of (0.1) in place of (0.3), have been systematically investigated by A. F. Monna [24], [25]. Most of the results of Monna are valid under the additional conditions that the space be separable or locally compact. Except in trivial cases, V-spaces are neither locally compact nor separable.

In Chapters 1 and 2 we investigate the basic topological and algebraic properties of V-spaces. A notion of utmost importance in this work is that of "distinguishability". "Distinguished sets" and "distinguished bases" are defined in Section 7, Chapter 1. The concept of distinguishability has been introduced by Monna [24, V], [25, I] under a different name and through another formal definition (see Section 2-5). Monna has shown that in non-Archimedean normed linear spaces over a field with a non-trivial valuation, distinguished bases exist only under restrictive conditions. However, by use of a modified form (Theorem 1-6.1) of the classical Riesz's Lemma ([7], [36]), it is proved in Theorem

2-2.2 that a V-space admits a distinguished basis. It follows (Theorem 1-7.6) that an element belongs to the space if and only if it is a sum of a formal series in terms of the elements of a distinguished basis. Thus, the rôle of a distinguished basis in a V-space is similar to the rôle of a complete orthogonal basis in a Hilbert space.

We also consider V-algebras and give theorems on the existence of inverses and on the spectra of elements of a V-algebra. Most of these theorems are simple modifications of the classical theorems of the theory of normed rings ([7], [26]).

Examples of V-spaces and V-algebras are displayed in Chapter 3. "Asymptotic spaces" are constructed by widening the scope of the method of C. A. Swanson and M. Schulzer [32], [33], referred to above. We also define "moment spaces" in which, for example, one can interpret the methods of Lanczos [21] or Clenshaw [2] for the approximation of the solutions of certain differential equations.

Chapter 4 is devoted to the study of linear and non-linear operators on V-spaces. By setting a proper norm (Definition 1.1) on these operators, the set of bounded operators forms a V-space of which the set of bounded linear operators is a subspace (Theorems 2.1, 3.1).

Elementary theorems (e.g. Theorems 3.3, 3.4) of the theory of bounded linear operators on Banach spaces still apply in V-spaces. However, important differences are exemplified: a continuous linear operator is not necessarily bounded (p. 86); the uniform boundedness theorem does not hold (p. 87).

Theorem 4.1 gives a simple characterization of bounded linear operators. As applications of this important theorem we derive a result of H. F. Davis [4] and indicate how asymptotic expansions of the Laplace transforms of certain functions of two variables can be obtained (see V. A. Ditkin and A. P Prudnikov [6]).

Theorem 6.5 allows the comparison of the spectra of two bounded linear operators when the norm of their difference is less than 1. The result is obtained by showing that an inequality proved by C. A. Swanson [34], [35] for linear transformations with eigenvalues on a Hilbert space can be modified into an equality in V-spaces.

The problem considered by C. A. Swanson and M. Schulzer in [32] and [33] is that of the existence and approximation of "asymptotic solutions" of certain equations in Banach spaces. In Chapter 5, we extend the results of Swanson and Schulzer to arbitrary V-spaces and V-algebras (Theorems 2.2, 3.2, 4.3). Our methods of proof are different than those of [32] and [33]. Our hypotheses are weaker and consequently our proofs are more involved. Possible simplifications of the hypotheses are mentioned.

In Chapter 6 we consider continuous linear functionals. It is known that continuous linear functionals on a V-space are bounded (Monna [24, III]) and that the Hahn-Banach Theorem is valid (Monna [24, III], Cohen [3], Ingleton [17]; we give a new proof of the latter using distinguished bases.

The main result of this chapter is a representation theorem (Theorem 3.5) for linear functionals on certain bounded V-spaces.

The representation theorem is a generalization of a theorem of H. F. Davis [4] which asserts that the space of continuous linear functionals on the space of asymptotically convergent power series in a real variable is isomorphic to the space of polynomials in that variable.

It is shown (Section 6-2) that a new norm, called "*norm", (Definition 2.2) can be defined on the set of finite linear combinations of the elements of a distinguished basis of a V-space X, and that, under this norm, this set is a V-space isomorphic to a subspace of the dual of X. This isomorphism is isometric and is obtained by use of a particular type of inner product (Definition 3.1).

In Chapters 4, 5 and 6, applications of the theorems are shown using some of the examples of asymptotic spaces described in Chapter 3.

CHAPTER 1

VALUED SPACES

1-1 Definitions and notations

In this chapter X denotes a linear space over a field of scalars F. F is a field with characteristic 0, i.e. a field which contains the set of the rational numbers as a subfield. The additive identity (zero-element) of X will be denoted by Θ and that of F by 0.

Definition 1.1. X will be called a pseudo-valued space if there exists a non-negative real valued function defined on all of X, whose value at x will be called the norm of x and denoted by $\|x\|$, and which satisfies:

(1.1) $\|\Theta\| = 0$,

(1.2) $\|\alpha x\| = \|x\|$ for all $x \in X$ and all $\alpha \in F$, $\alpha \neq 0$,

(1.3) $\|x + y\| \leq \text{Max}\,\{\|x\|, \|y\|\}$ for all $x, y \in X$.

Definition 1.2. A pseudo-valued space X will be called a strongly pseudo-valued space if for all $x, y \in X$, $\|x\| \neq \|y\|$ implies

(1.4) $\|x + y\| = \text{Max}\,\{\|x\|, \|y\|\}$.

Definition 1.3. A (strongly) pseudo-valued space X will be called a (strongly) valued space if

(1.5) $\|x\| = 0$ implies $x = \Theta$.

Condition (1.3) is stronger than the usual triangle inequality, since

$$\mathrm{Max}\ \{\|x\|, \|y\|\} \leq \|x\| + \|y\|.$$

A (pseudo-) valued space is a (pseudo-) normed linear space if F is understood to have the trivial valuation

$$|0| = 0, \quad |\alpha| = 1 \quad \text{for all } \alpha \in F, \quad \alpha \neq 0.$$

With this valuation on F, (1.2) can be written as

$$\|\alpha x\| = |\alpha|\ \|x\| \quad \text{for all } x \in X \text{ and all } \alpha \in F.$$

On a (pseudo-) valued space X a (pseudo-) metric is defined by

$$d(x, y) = \|x - y\|.$$

This (pseudo-) metric is non-Archimedean, i.e.

(1.6) $\quad d(x, y) \leq \mathrm{Max}\ \{d(x, z), d(z, y)\}, \quad x, y, z \in X.$

If X is strongly (pseudo-) valued, the (pseudo-) metric is strongly non-Archimedean, i.e. for all $x, y, z \in X$,

(1.7) $\quad d(x, y) = \mathrm{Max}\ \{d(x, z), d(z, y)\}$ when $d(x, z) \neq d(z, y)$.

Every triangle in X is isosceles with the two longest sides being of equal length. Indeed, if $d(x, y) \geq d(y, z) \geq d(z, x)$, it follows from (1.6) that $d(x, y) = d(y, z)$.

The function d is translation invariant, i.e. $d(x + z, y + z) = d(x, y)$.

Example. Let $\lambda \in [0, 1]$ and consider the set of real valued functions

$$\Psi = \{\varphi_r : 0 \leq r < \infty\}, \quad \varphi_r(\lambda) = \lambda^r.$$

Consider the linear space X consisting of the zero function and all formal series of the form

$$x = \alpha_0\varphi_{r_0} + \alpha_1\varphi_{r_1} + \alpha_2\varphi_{r_2} + \cdots, \ \alpha_i \text{ real}, \ \alpha_0 \neq 0,$$

where $r_{i-1} < r_i$ for $i = 1, 2, \cdots$, and where the set $\{r_0, r_1, \cdots\}$ is either finite or is infinite and unbounded.

Let ρ be a fixed real number, $1 < \rho < \infty$, and define on X the function

$$\|0\| = 0, \quad \|x\| = \rho^{-r_0} \quad \text{if } x = \alpha_0\varphi_{r_0} + \cdots .$$

One verifies easily that this function satisfies (1.1), (1.2), $\cdots$, (1.5). Thus, X is a strongly valued space.

The subset Y of X consisting of 0 and of all those points x for which $\{r_0, r_1, \cdots\}$ is a set of rational numbers is a linear subspace of X.

Similarly the subset of X consisting of 0 and of all those points x for which $\{r_0, r_1, \cdots\}$ is a set of integers is a subspace of X.

Other examples are constructed in Chapter 3.

<u>Remark on the terminology</u>. The word "valued" was introduced to avoid the heavy locutions which would have resulted from the use of the generally accepted terminology. A "strongly pseudo-valued" space is a "strongly non-Archimedean pseudo-normed linear space over a field with a trivial valuation " (!). The word "valued" is meant to recall the particular valuation which is imposed upon the field of scalars as well as the similarities of

the defining properties (1.1), (1.3) with those of a valuation in general ([31], [40]).

1-2 Some topological properties

In this section we shall list some of the topological properties of a (pseudo-) valued space X.

For the classical terminology we refer to textbooks on topology or analysis (e.g. [7]; [18]; [19], Vol. I; [36]).

The topology considered is the topology induced on X by the (pseudo-) metric d of 1-1; we recall that this topology is the smallest topology which contains all the balls $S(x, r)$, $x \in X$, $r > 0$. The open ball $S(x, r)$, the closed ball $S'(x, r)$ and the sphere $B(x, r)$, with center x and radius r, are defined by

(1.8a) $\quad S(x, r) = \{y \in X : d(x, y) < r\}, \quad r > 0 ,$

(1.8b) $\quad S'(x, r) = \{y \in X : d(x, y) \leq r\}, \quad r \geq 0 ,$

(1.8c) $\quad B(x, r) = \{y \in X : d(x, y) = r\}, \quad r \geq 0 .$

(i) Since the (pseudo-) metric d is translation invariant (1-1) the neighbourhood system of a point x is the x-translate of the neighbourhood system of Θ, i.e. if V is a neighbourhood of x and W is a neighbourhood of Θ, then $V - x$ is a neighbourhood of Θ and $W + x$ is a neighbourhood of x.

(ii) For any $x, y \in X$ and r,

$$S(x, r) = S(y, r) \text{ or } S(x, r) \cap S(y, r) = \emptyset$$
$$S'(x, r) = S'(y, r) \text{ or } S'(x, r) \cap S'(y, r) = \emptyset$$

where $\emptyset$ denotes the empty set.

To prove the first statement it is sufficient to see that if

$$z \in S(x, r) \cap S(y, r) \text{ and } u \in S(x, r)$$

then $u \in S(y, r)$. Indeed,

$$\begin{aligned} d(y, u) &\leq \operatorname{Max} \{d(y, z), d(z, u)\} \\ &\leq \operatorname{Max} \{d(y, z), d(z, x), d(x, u)\} < r. \end{aligned}$$

The second statement is proved in a similar way.

(iii) If X is a strongly (pseudo-) valued space, then for any $x \in X$ and $r > 0$, $S(x, r)$, $S'(x, r)$, $B(x, r)$ are all closed and open.

The proofs for $S(x, r)$ and $S'(x, r)$ are similar. For $B(x, r)$ it follows from the equality

$$B(x, r) = S'(x, r) \setminus S(x, r).$$

That $S(x, r)$ is open is guaranteed by the definition of the topology induced on X by d. To show that it is closed, let $y \in X \setminus S(x, r)$. For every $z \in S(y, \frac{1}{2}r)$,

$$d(z, y) < d(x, y),$$

and, by (1.7), $d(z, x) \geq r$. Thus $X \setminus S(x, r)$ contains a neighbourhood of each of its points; it is open, and $S(x, r)$ is closed.

(iv) A component of a topological space is a maximal connected subset of the space ([18], p. 54).

It is a consequence of (iii) that if X is strongly valued, then each component of X consists of a single point; if X is strongly pseudo-valued each component is a translate of $[\theta]$, where

$$[\theta] = S'(\theta, 0) = \{x \in X : \|x\| = 0\}.$$

Spaces whose components consist of single points are called totally disconnected ([28], p. 76).

(v) A space X is said to be 0-dimensional if for any $x \in X$ every neighbourhood of x contains a neighbourhood of x whose boundary is empty ([16], pp. 10, 15).

It follows from (iii) that a strongly (pseudo-) valued space is 0-dimensional, since the balls $S(x, r)$ have empty boundaries.

It has been proved by J. de Groot ([5], Th. II) that a metrizable space admits a non-Archimedean metric if and only if it is strongly 0-dimensional. In de Groot's terminology, a space is strongly 0-dimensional if and only if it is a Hausdorff space and admits a σ-locally finite basis for its topology consisting of subsets which are both closed and open.*

In the case of a strongly valued space X, the family

$$\mathcal{S} = \bigcup\{\mathcal{S}_r : r > 0,\ r \text{ rational}\}$$

where

$$\mathcal{S}_r = \{S(x, r) : x \in X,\ r > 0\}$$

forms a σ-locally finite basis. Indeed $\mathcal{S}$ is a basis, it is the

* The terminology is that of Kelley ([18], pp. 126, 127).
A family of sets is called σ-locally finite if it is the union of a countable number of locally finite subfamilies. A family of sets is called locally finite if every point has a neighbourhood which intersects at most a finite number of sets of the family.

countable union of the $\mathcal{S}_r$'s and, for a fixed r, the family $\mathcal{S}_r$ is locally finite, by (ii) above. (For a given $x \in X$, $S(x, r)$ intersects only one set in $\mathcal{S}_r$: itself.)

Other equivalent definitions of 0-dimensionality and its consequences are studied in [16], Ch. II.

(vi) A field F forms a strongly valued space over itself if the norm function is identical to the trivial valuation:

(1.9) $\|0\| = 0$ and $\|\alpha\| = 1$ for all $\alpha \in F$, $\alpha \neq 0$.

The topology induced on F by this norm is the discrete topology on F ([18], p. 37).

In the sequel, whenever the field F is the field R of the real numbers or the field C of the complex numbers, the symbols (1.9) will be retained to denote the trivial value of the numbers. The symbol $|\alpha|$ will denote the usual absolute value of α, i.e. $|\alpha| = \sqrt{\alpha^2}$ if $\alpha \in R$ and $|\alpha| = |a + bi| = \sqrt{a^2 + b^2}$ if $\alpha \in C$. The topologies induced on R and C by their usual valuations will be called the usual topologies on R and C.

We conclude this section by the following

<u>Theorem 2.1</u>. If X is a strongly (pseudo-) valued space and $r > 0$, then

(i) $S(\theta, r)$ and $S'(\theta, r)$ are subspaces of X;

(ii) The quotient topologies on the quotient spaces $X/S(\theta, r)$ and $X/S'(\theta, r)$ are both discrete.

Proof: (i) is easily verified.

(ii) For the terminology, we refer to [28], pp. 59, 60. The natural mapping from a topological group to one of its quotient groups is a continuous open mapping. The points in the quotient groups $X/S(\theta, r)$ and $X/S'(\theta, r)$ are translates of the balls $S(\theta, r)$ and $S'(\theta, r)$ respectively. The balls $S(\theta, r)$ and $S'(\theta, r)$ were shown to be both closed and open (see (iii) above), thus the points in the quotient groups are both closed and open ([28], p. 59).

1-3 Some properties of the norm function

In this section, X is a strongly (pseudo-) valued space over the field of scalars F.

Theorem 3.1. Let f be a continuous function from D to X, where D is an arbitrary topological space. Let g be the function with domain $F \times D$ defined by:

$$g(\alpha, u) = \| \alpha f(u) \|, \quad \alpha \in F, \quad u \in D.$$

If $v \in D$ is such that $\| f(v) \| \neq 0$, then there exists a neighbourhood $W(v)$ of v such that for all $u \in W(v)$

$$g(\beta, u) = g(1, v) = \| f(v) \|, \quad \beta \in F,$$

provided $\beta \neq 0$.

Proof: If $\beta = 0$ then $g(\beta, u) = 0 \neq \| f(v) \|$.

If $\beta \neq 0$, then by the continuity of f, there exists a neighbourhood $W(v)$ of v such that for all $u \in W(v)$

$$\| f(u) - f(v) \| < \| f(v) \| \neq 0.$$

By (1.4), this implies that $\| f(u) \| = \| f(v) \|$ for all $u \in W(v)$.
By (1.2), since $\beta \neq 0$,

$$g(\beta, u) = \| \beta f(u) \| = \| f(u) \| = \| f(v) \| \text{ for all } u \in W(v).$$

If we let $\alpha = 1$ in the theorem, we obtain:

Corollary 3.2. Let f be a continuous function from D to X, where D is an arbitrary topological space. If $v \in D$ is such that $\| f(v) \| \neq 0$, then $\| f(u) \| = \| f(v) \|$ for all u in some neighbourhood of v.

In Theorem 3.1, let D = X and f be the identity function. We obtain:

Corollary 3.3. Let F = R or F = C. Let $F \times X$ have the cross-product topology induced by the usual topology on F and the topology on X ([18], p. 90).
(i) If $(\alpha, x) \in F \times X$, $\alpha \neq 0$ and $\| x \| \neq 0$, then there exists a neighbourhood V of (α, x) in $F \times X$, such that $\| \beta y \| = \| x \|$ for all $(\beta, y) \in V$.
(ii) The function $g(\alpha, x) = \| \alpha x \|$, defined on $F \times X$, is discontinuous at (α, x) if $\alpha = 0$ and $\| x \| \neq 0$; it is continuous at all other points.

Remark: The conclusion of Corollary 3.3 remains true for an arbitrary field F with characteristic 0 and with the topology induced by some valuation ([40]. Ch. X; [31], Ch. 2), if the restriction of this valuation to the subfield of the rational numbers is identical to the usual valuation of the rationals. This remark also applies to the statements (iii) and (iv) below.

The following properties of a strongly (pseudo-) valued space X can be verified directly or by use of the last corollary:

(i) If $x \in X$ and $\|x\| \neq 0$, then the subspace generated by x, with its relative topology, is a discrete topological space. The distance between any pair of distinct points, αx and βx, is constant and equal to $\|x\|$.

(ii) If F is given the discrete topology, then the function $f: f(\alpha, x) = \alpha x$, defined on $F \times X$, is continuous in α for a fixed x and is continuous in x for a fixed α.

(iii) Let $F = R$ or $F = C$ and let F have its usual topology. The function $f: f(\alpha, x) = \alpha x$, defined on $F \times X$, is continuous in α, for a fixed x, if and only if $\|x\| = 0$; it is continuous in x for a fixed α.

(iv) Let $F = R$ or $F = C$ and let $\{\alpha_n\}$ be a sequence of distinct scalars convergent to α in the usual topology of F. The sequence $\{\alpha_n x\}$ converges to αx in the topology of X, if and only if $\|x\| = 0$.

(v) If $\{x_n\}$ is a sequence in X, convergent to a limit x such that $\|x\| \neq 0$, then every $x_n \in S(x, \|x\|)$ is such that $\|x_n\| = \|x\|$. Further, there are at most a finite number of indices n such that $x_n = \alpha_n x$, $\alpha_n \in F$ and $\alpha_n \neq 1$.

1-4 Convergence of sequences and series

In this section two important theorems concerning the convergence of sequences and series in (pseudo-) valued spaces will be stated.

For the classical terminology, we refer to [7], p. 19; [19], Vol. I, p. 36; or [36], p. 74.

Theorem 4.1. If X is a (pseudo-) valued space:

(i) A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} \| x_n - x_{n+1} \| = 0.$$

(ii) A series $\sum_n x_n$ in X is a Cauchy series if and only if

$$\lim_{n\to\infty} d(x_n, \theta) = \lim_{n\to\infty} \| x_n \| = 0.$$

The proof of this theorem is omitted. It is a mere modification of the proof of a similar theorem for fields with a non-Archimedean valuation. See [31], p. 28 or [40], p. 240.

The proof of the following theorem is also omitted (cf. Lemma 7.5 below). Part (i) is quoted, without proof, in [29], p. 139. Part (ii) follows from Theorem 1.5 (ii) and inequality (1.3).

Theorem 4.2. If X is a (pseudo-) valued space:

(i) A convergent series is unconditionally convergent, i.e. any reordering of its terms converges to the same sum(s).

(ii) If $\sum_n x_n$ is convergent and has sum x, then

$$\| x \| \leq \sup_n \| x_n \|.$$

1-5 Compactness

In this section we give a characterization of the compact subsets of a strongly (pseudo-) valued space X.

For definitions and properties related to compactness we refer to textbooks (e.g. [7]; [18]; [19], Vol. I; [36]).

If the topology of X is discrete, a subset of X is compact if and only if it is finite. Thus X itself is not compact. Since each point forms a neighbourhood of itself, X is locally compact.

If the topology of X is not discrete, then X is neither compact nor locally compact. Indeed every neighbourhood V of θ contains a ball $S(\theta, r)$ for some r. This ball contains a point x such that $\|x\| \neq 0$. Thus V contains the discrete subspace generated by x ((i), page 15).

We shall use the following definition:

Definition 5.1. Let $A \subset X$. The set $\Omega(A)$ defined by

$$\Omega(A) = \{r : \|x\| = r \text{ for some } x \in A\}$$

will be called the norm range of A.

Theorem 5.2. Let X be a strongly (pseudo-) valued space, and A be a subset of X.

(i) A is compact if and only if for each $r > 0$ it is a finite union of disjoint compact subsets $K_1, K_2, \cdots, K_{n(r)}$, such that $x \in K_i$ and $y \in K_j$ implies

$$(1.10) \qquad \begin{aligned} \|x - y\| &< r \qquad \text{for } i = j, \\ \|x - y\| &\geq r \qquad \text{for } i \neq j. \end{aligned}$$

(ii) If A is compact and does not contain θ, except possibly as an isolated point, then its norm range $\Omega(A)$ is finite.

<u>Proof</u>: (i) A set is certainly compact if it is a finite union of compact sets.

For the converse, let A be compact and $r > 0$ be arbitrary. The family

$$\mathcal{S} = \{S(x, r) : x \in A\}$$

is an open cover of A. We can extract from $\mathcal{S}$ a finite subcover $\{S(x_1, r), S(x_2, r), \cdots, S(x_{n(r)}, r)\}$ such that the $S(x_i, r)$ are disjoint. (See (ii), page 9). Then (1.10) is satisfied with K_i replaced by $S(x_i, r)$. Take $K_i = A \cap S(x_i, r)$. K_i is compact since it is the intersection of a compact set and a closed set ((iii), page 10). Then (1.10) holds.

(ii) Since θ is at most an isolated point of A, there exists $r > 0$ such that

$$\|x\| \geq r \quad \text{for all } x \in A, \quad x \neq \theta.$$

Consider, for this particular value of r, the sets K_i, $i = 1, 2, \cdots, n(r)$ of (i). Then, $\theta \notin K_i$, $x \in K_i$ and $y \in K_i$ imply

$$\|x\| \geq r, \quad \|y\| \geq r \quad \text{and } \|x - y\| < r.$$

Since X is strongly (pseudo-) valued, by (1.4), $\|x\| = \|y\|$ and the conclusion follows.

<u>Remarks</u>: (i) The fact that the valuation on the field F

is the trivial valuation is responsible for a high discretization in a (pseudo-) valued space. As a result, we may say, loosely speaking, that compactness is a very restrictive property and that it is rather difficult for a subset of a (pseudo-) valued space to be compact.

No convex set is compact unless it is reduced to a single point (or to a subset of $S(\theta, 0)$). No set with a non-empty interior is compact unless the space is discrete and the set is finite.

One can expect that compactness will not play an important rôle in this theory.

(ii) The results of Theorem 5.2 may be compared with Property 4, in Theorem 2 of Monna, [24], Part I, page 1048.

Monna has shown that if a non-Archimedean normed linear space over a field of scalars with the trivial valuation is locally compact, then the field of scalars is finite. ([24], Part II, p. 1061.)

1-6 Modification of Reisz's Lemma

One can expect that many theorems in the classical theory of normed linear spaces will have analogues in the theory of valued spaces. The theorem of this section is given as an example of a modified statement and its proof.

In the case of a normed linear space X over the real or complex field with the usual valuations, Riesz's Lemma can be stated as follows ([36], p. 96; also [7], p. 578):

"Let Y be a closed, proper subspace of X. Then for each α such that $0 < \alpha < 1$, there exists a point $x_\alpha \in X$ such that $\|x_\alpha\| = 1$ and $\|y - x_\alpha\| > \alpha$ for all $y \in Y$."

If X is a strongly (pseudo-) valued space, the above statement must be modified. The reason for the alteration is the impossibility of normalizing an element in X, i.e. the impossibility of finding, for each x such that $\|x\| \neq 0$, a scalar α such that $\|\alpha x\| = 1$ (unless, of course, $\|x\| = 1$ for all $x \in X$ such that $\|x\| \neq 0$).

<u>Theorem 6.1</u>. (Modified Riesz's Lemma)

Let Y be a closed, proper subspace of a strongly (pseudo-) valued space X. For each α such that $0 < \alpha < 1$, there exists a point $x_\alpha \in X$ such that

$$\|y - x_\alpha\| > \alpha\|x_\alpha\| \quad \text{for all } y \in Y.$$

<u>Proof</u>: (i) If there exists $z \in X$, $\|z\| \neq 0$, such that

$$\|y - z\| \geq \|z\| \quad \text{for all } y \in Y,$$

take $x_\alpha = z$ for all α, $0 < \alpha < 1$.

(ii) If (i) fails, let $x_o \in X \setminus Y$ and choose $y_o \in Y$ such that $\|y_o - x_o\| < \|x_o\|$ (so that $\|y_o\| = \|x_o\|$).

Define

$$\delta(y) = \frac{\|y - x_o\|}{\|y\|}, \qquad y \in Y, \quad \|y\| \neq 0.$$

$$\delta = \inf_{y \in Y} \delta(y)$$

Then, $\delta \leq \delta(y_o) < 1$. Moreover $\delta(y) < 1$ implies $\|y - x_o\| < \|y\|$ and hence $\|y\| = \|x_o\|$; therefore, since Y is closed,

$\delta(y) = \dfrac{\|y - x_o\|}{\|x_o\|}$ is bounded away from zero for $y \in Y$. Thus

$$0 < \delta < 1.$$

Let α be given, $0 < \alpha < 1$, and let

$$\delta' = \min\{\alpha^{-1}\delta,\ \tfrac{1}{2}(1 + \delta)\}.$$

$\delta < \delta' < 1$. There exists $y_1 \in Y$ such that

$$\delta(y_1) = \frac{\|y_1 - x_o\|}{\|x_o\|} < \delta'.$$

Let $x_\alpha = x_o - y_1$. Then

$$\|x_\alpha\| < \delta'\|x_o\| = \delta'\|y_1\|. \tag{1.11}$$

Now let $y \in Y$. If $\|y - x_\alpha\| \geq \|x_\alpha\|$ the proof is finished, so we may assume

$$\|y - x_\alpha\| < \|x_\alpha\|.$$

Then $\|y\| = \|x_\alpha\| < \|y_1\|$ (by (1.11)) so that $\|y + y_1\| = \|y_1\|$. Hence since $y + y_1 \in Y$, we have by the definition of δ:

$$\|y - x_\alpha\| = \|y + y_1 - x_o\| \geq \delta\|y + y_1\| = \delta\|y_1\|,$$

and by (1.11)

$$\|y - x_\alpha\| > \frac{\delta}{\alpha\delta'}\,\alpha\|x_\alpha\| \geq \alpha\|x_\alpha\|.$$

This completes the proof.

<u>Example</u>. Consider the strongly valued space X of the Example of page 7 and its subspace Y (page 8).

To verify that Y is closed, let $x \in X \setminus Y$; then

$$x = \beta_o \varphi_{s_o} + \beta_1 \varphi_{s_1} + \cdots , \qquad \beta_o \neq 0,$$

where for some integer p, $\beta_p \neq 0$, s_p is irrational and s_i is rational for all i, $0 \leq i < p$. Clearly $\| x \| \geq \rho^{-s_p}$.

If $y \in Y$, $y \neq 0$, we have

$y = \alpha_o \varphi_{r_o} + \alpha_1 \varphi_{r_1} + \cdots$, $\alpha_o \neq 0$, r_i rational for all i. Therefore, $\| x - y \| \geq \rho^{-s_p}$ since

$$x - y = \cdots + \beta_p \varphi_{s_p} + \cdots$$

Thus, no sequence in Y can converge to a point of $X \setminus Y$; Y is a closed proper subspace of X.

Given α, $0 < \alpha < 1$, let s be irrational and $\alpha < s < 1$. The same argument as above shows that

$$\| y - \varphi_s \| \geq \| \varphi_s \| > \alpha \| \varphi_s \| \quad \text{for all } y \in Y.$$

1-7 Distinguished bases

In the previous sections we were concerned with properties of valued spaces which were mostly of topological nature. In this section we introduce algebraic concepts which depend on linearity.

We recall a few classical definitions ([7], pp. 36, 46, 50; [36], pp. 44, 45). Let A be a subset of a topological linear space X, over a field F. The subspace (A) generated by A is the set of all the finite linear combinations of elements of A. The topological closure of (A) will be denoted by [A] and be called the closed subspace generated by A. The set A is said to be linearly independent if for any finite subset $\{x_1, x_2, \cdots, x_n\}$ of A,

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \theta, \quad \alpha_i \in F,$$

implies $\alpha_i = 0$ for all i. A is called a Hamel basis of X if A is a linearly independent set and $(A) = X$.

It is known ([7], [36]) that a linear space has a Hamel basis; that a subset is a Hamel basis if and only if it is a maximal linearly independent set; that all the Hamel bases of a space X have the same cardinality.

We add the following definition:

Definition 7.1. Let A be a subset of a topological linear space X.

(i) A is said to be a completely independent set if

$$x \notin [A \setminus \{x\}] \quad \text{for each } x \in A.$$

(ii) A is called a complete basis if it is a completely independent set and $[A] = X$.

Clearly a completely independent set is also linearly independent. The converse is not true as is shown by the following example. Let $C[0, 1]$ be the space of all the real valued continuous functions f on $[0, 1]$, with the uniform norm: $\|f\| = \sup_{0 \leq \lambda \leq 1} |f(\lambda)|$; let X be the subspace of $C[0, 1]$ generated by $\Phi_0 \cup E_0$, where

$$\Phi_0 = \{\varphi_n : n = 0, 1, 2, \cdots\}, \quad \varphi_n(\lambda) = \lambda^n,$$
$$E_0 = \{e_{-r} : r > 0\}, \quad e_{-r}(\lambda) = e^{-r\lambda}.$$

It is known (Weierstrass' Theorem, [36], [37]) that the set Φ_0 is a completely independent set which is a complete basis but not a Hamel basis of X. Thus $\Phi_0 \cup E_0$ is a linearly independent but not completely independent set.

<u>Theorem 7.2</u>. A complete basis A of a topological linear space X is also a Hamel basis of X if and only if (A) contains an open set.

<u>Proof</u>: If A is a Hamel basis $(A) \supset X$. Conversely, suppose that (A) contains an open set. Then (A) contains an interior point and (A) is a subgroup of X. It is known ([18], p. 106) that any subgroup of a topological group which contains an interior point is closed (and open). Thus, A is a linearly independent set and $(A) = [A] = X$.

Returning to the theory of valued spaces, we introduce the notion of distinguishability in the following way.

<u>Definition 7.3</u>. Let A be a non-empty subset of a (pseudo-) valued space X.

(i) A is said to be a <u>distinguished set</u> if no element of A has norm equal to 0, and, if for any finite subset of distinct points $x_1, x_2, \cdots, x_n$ of A,

$$\| \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \| = \operatorname{Max}_i \| x_i \|,$$

$\alpha_i \in F$, whenever $\alpha_i \neq 0$ for $i = 1, 2, \cdots, n$.

(ii) A is called a <u>distinguished basis of X</u> if A is a distinguished set and a complete basis of X.

In section 4 of Chapter 3, we show that there exists a norm on the space X of the example of page 23, under which X is a strongly pseudo-valued space. The Hamel basis $\Phi_o \cup E_o$ will be shown to be neither a complete nor a distinguished basis.

The essential feature of a distinguished set A in a strongly (pseudo-) valued space is the following: if $x, y \in A$, $x \neq y$ and $\| x \| = \| y \| = r$, then $\| \alpha x + \beta y \| = r$, except when $\alpha = \beta = 0$.

The author has not been able to show the existence of distinguished bases in arbitrary (pseudo-) valued spaces. Nevertheless, under an important additional assumption on the norm range of the space, we shall prove, in Chapter 2, that a strongly (pseudo-) valued space has a distinguished basis. This assumption will be satisfied in all the examples in Chapter 3 and applications in Chapters 4, 5 and 6.

In the case of an arbitrary (pseudo-) valued space, we have:

<u>Theorem 7.4</u>. A (pseudo-) valued space admits a Hamel basis which is a distinguished set.

The proof is identical to the proof of the existence of a Hamel basis in a linear space ([36], p. 45). A distinguished Hamel basis is a maximal distinguished set.

In the remainder of this section we restrict our attention to strongly (pseudo-) valued spaces. In strongly (pseudo-)

valued spaces, Definition 7.3(ii) is slightly redundant. Indeed, we shall prove in Theorem 7.6(i) below that in those spaces a distinguished set is completely independent. Thus a distinguished basis A in a strongly (pseudo-) valued space X is a distinguished subset such that $[A] = X$. To prove Theorem 7.6 we shall need the following lemma, which is an improvement over Theorem 4.2(ii).

Lemma 7.5. Let A be a distinguished subset of a strongly (pseudo-) valued space X. Let $\{x_n\}$ be an at most countable subset of A. If $\alpha_n \in F$, $\alpha_n \neq 0$ for each n, and $x = \sum_n \alpha_n x_n$, then

$$\|x\| = \sup_n \|x_n\|.$$

Proof: By Theorem 4.1(ii), given $r < \|x_1\|$, there exists N such that for all $n > N$, $\|x_n\| < r$. By Theorem 4.2(ii)

$$\Big\| \sum_{n \geq N+1} \alpha_n x_n \Big\| \leq r,$$

and, since A is distinguished,

$$\Big\| \sum_{n=1}^{N} \alpha_n x_n \Big\| = \max_{1 \leq n \leq N} \|x_n\| \geq \|x_1\| > r.$$

Thus,

$$\|x\| = \Big\| \sum_{n=1}^{N} \alpha_n x_n + \sum_{n > N} \alpha_n x_n \Big\| = \max_{1 \leq n \leq N} \|x_n\|$$

$$= \sup_n \|x_n\|.$$

Note that this supremum is attained.

Theorem 7.6. Let X be a strongly (pseudo-) valued space.

(i) A distinguished subset A of X is completely independent.

(ii) If A is a distinguished basis of X, then every $x \in X$ can be represented uniquely (except for order) by a series $\sum_{n=1}^{\infty} \alpha_n x_n$, with $x_n \in A$, $\alpha_n \in F$, $n = 1, 2, \cdots$.

Proof: (i) Let x_o be an arbitrary point of A. Suppose that $x_o \in [B]$, where $B = A \setminus \{x_o\}$. Then there exists a sequence $\{y_n\}$ in (B) which converges to x_o. It follows that there exists an element y of (B)

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m, \quad \alpha_i \neq 0, \quad x_i \in B,$$
$$i = 1, 2, \cdots, m,$$

such that $\| y - x_o \| < \| x_o \|$, i.e.

$$\left\| \sum_{i=0}^{m} \alpha_i x_i \right\| < \| x_o \| \leq \max_{0 \leq i \leq m} \| x_i \|, \quad \alpha_o = -1.$$

This contradicts the distinguishability of A.

(ii) Since $x \in [A]$, there exists a sequence $\{y_n : n = 1, 2, \cdots\}$ in (A) which converges to x. Let $z_1 = y_1$ and $z_n = y_n - y_{n-1} \in (A)$ for $n = 2, 3, \cdots$. Then the series $\sum_{n=1}^{\infty} z_n$ converges to x.

Let

$$z_n = \alpha_{n1} x_{n1} + \alpha_{n2} x_{n2} + \cdots + \alpha_{n,p(n)} x_{n,p(n)}, \quad \alpha_{nj} \neq 0, \quad x_{nj} \in A.$$

The set $\{x_{nj} : n = 1, 2, \cdots, j = 1, 2, \cdots, p(n)\}$ is a countable set. Let its elements be ordered into a sequence $\{x_m : m = 1, 2, \cdots\}$ such that $\| x_m \| \geq \| x_{m+1} \|$ for all m.

For each integer $m \geq 1$, there exists an integer $N(m)$ such that

$$\| z_n \| < \| x_m \| \text{ for all } n \geq N(m).$$

Therefore, for each m, the number of integers n such that $x_m = x_{nj}$ for some j, $1 \leq j \leq p(n)$, is finite. The series $\sum_{n=1}^{\infty} z_n$ can thus be reordered by grouping the terms in x_m, for each integer m.

The uniqueness (except for order) follows from Lemma 7.5.

Consequences of the above theorem are:

(i) If A is a distinguished subset of X, then

$$\Omega(A) = \Omega((A)) \setminus \{0\} = \Omega([A]) \setminus \{0\}.$$

(ii) If A is a distinguished basis of X, then $\inf \Omega(A) = 0$ when X is not discrete, and in any case $\Omega(A) = \Omega(X) \setminus \{0\}$, i.e. for every $r \in \Omega(X)$, $r \neq 0$, there exists $x \in A$ such that $\| x \| = r$.

If A is a distinguished basis of a strongly (pseudo-) valued space X, the unique series

$$(1.12) \qquad \sum_{n=1}^{\infty} \alpha_n x_n, \quad x_n \in A, \quad \alpha_n \in F, \quad \alpha_n \neq 0,$$

which converges to a given point $x \in X$, will be called the expansion of x in terms of A. According to Theorem 4.2(i), the terms of such an expansion can be reordered to give a non-increasing series, i.e. a series such that $\| x_n \| \geq \| x_{n+1} \|$ for all $n \geq 1$.

<u>Notation</u>: If $x \in X$ and $y \in A$, we shall denote by $(x, y)_A$ the coefficient of y in the expansion of x in terms of A. With this notation (1.12) becomes

$$(1.13) \qquad x = \sum_{n=1}^{\infty} (x, x_n)_A \, x_n.$$

Assuming that a strongly (pseudo-) valued space X admits a distinguished basis, we can state

Theorem 7.7. All distinguished bases of X have the same cardinality.

This theorem justifies:

Definition 7.8. If a strongly (pseudo-) valued space admits a distinguished basis, the cardinality of this basis will be called the (algebraic) dimension of the space.

The proof of Theorem 7.7 is omitted; it is similar to the proof given by Dunford and Schwartz ([7], p. 253) for the invariance of the cardinality of complete orthonormal bases of a Hilbert space.*

Theorems 7.6 and 7.7 indicate that, to some extent, the rôles of distinguished sets and distinguished bases in a strongly (pseudo-) valued space are similar to the rôles of orthogonal sets and orthonormal bases in a Hilbert space ([7], pp. 252-253).

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* If A and B are two distinguished bases of X, the only modification to [7], p. 253, is the replacement of the words "...the inner product of a and b is non-zero...", or of the symbol "... $(a, b) \neq 0$ ..." by "...$(a, b)_B \neq 0$ and $(b, a)_A \neq 0$ ..." .

CHAPTER 2

V-SPACES

2-1 Definitions

A systematic study of non-Archimedean normed linear spaces has been made by A. F. Monna ([24], [25]). Other references are [3], [12], [17].

Monna obtains interesting results when the norm range of the non-Archimedean normed linear space is assumed to have at most one accumulation point: 0. We shall retain this assumption. In most of his work, Monna requires that the valuation of the field of scalars be non-trivial; this, of course, is impossible in the case of a valued space.

Definition 1.1. A V-space X is a strongly pseudo-valued or a strongly valued space which is complete in its norm topology and for which there exists a set of integers $\omega(X)$ and a real number $\rho > 1$, such that

$$(2.1) \qquad \Omega(X) = \{0\} \cup \{\rho^{-n} : n \in \omega(X)\}.$$

Definition 1.2. A discrete V-space is a V-space such that the set $\omega(X)$ of Definition 1.1 satisfies

$$(2.2) \qquad \sup \omega(X) = M \quad \text{for some } M < \infty .$$

The topology of a discrete V-space is discrete. A V-space such that

$$(2.3) \qquad \sup \omega(X) = \infty$$

has a proper sequence convergent to Θ and hence its topology is not discrete.

Examples. In the space X defined on pages 7, 8, the set of formal series for which the set $\{r_0, r_1, \cdots\}$ is a set of integers is a V-space satisfying (2.3).

Many other examples are constructed in Chapter 3.

Conventions. (i) In all of this work, the symbol "ρ" will retain the meaning attached to it in Definition 1.1.

(ii) In the sequel, whenever two or more V-spaces will be considered simultaneously, it will be assumed that the value of ρ is the same for all of these spaces.

Remark: A normed linear space cannot be complete if its field of scalars is not complete with respect to its valuation. For this reason there is no complete normed linear space over the field R_o of the rational numbers with the usual valuation on R_o. In the case of a V-space, any field F (of characteristic 0 -- in particular R_o itself) is acceptable since the valuation of F is trivial. Under the trivial valuation, any field is complete.

The definitions, theorems and remarks of the following sections of this chapter, except Th. 2.4(ii), do not depend on the completeness of the V-space. They remain valid for any space which satisfies Definition 1.1 except for the completeness requirement. Part (ii) of Theorem 2.4 requires completeness.

2-2 Existence of distinguished bases

In this section we prove that a V-space admits a distinguished basis (Theorem 2.2).

The proof of this statement is analogous, in part, to the proof of the existence of an orthonormal basis in a Hilbert space ([7], p. 252; see also [36], p. 117). It is made possible by the following improvement over Riesz's Lemma (Theorem 1 - 6.1).

Lemma 2.1. Let Y be a proper, closed subspace of a V-space X. There exists $z \in X$ such that

$$\|y - z\| \geq \|z\|, \quad \text{for all } y \in Y.$$

Proof: Let α satisfy $\rho^{-1} < \alpha < 1$. Then

$$\|x\| > \alpha\|z\| \text{ implies } \|x\| \geq \|z\|$$

for any pair $x, z \in X$.

By Theorem 1-6.1, there exists $z \in X\setminus Y$, such that

$$\|y - z\| > \alpha\|z\|, \quad \text{for all } y \in Y.$$

Thus $\|y - z\| \geq \|z\|$ for all $y \in Y$.

Theorem 2.2. A V-space admits a distinguished basis.

Proof: Let D be the family of all distinguished subsets of a V-space X. D is not empty since a single point with non-zero norm forms a distinguished subset of X. Let D be ordered by set inclusion. It is easy to see that a linearly ordered subfamily of D satisfies the conditions of Zorn's Lemma ([36], pp. 39-40; [18], p. 33). Therefore D contains at least one maximal element H.

We shall show that $[H] = X$ (see pp. 23, 25). Suppose the contrary. Then by Lemma 2.1 there exists $z \in X \setminus [H]$ such that

$$\| y - z \| \geq \| z \| \quad \text{for all } y \in [H].$$

a) If for each $y \in (H)$, $\| y \| \neq \| z \|$, then

$$\| y + z \| = \operatorname{Max} \{ \| y \|, \| z \| \} \quad \text{for all } y \in (H).$$

b) If a) fails, then for each $y \in (H)$ such that $\| y \| = \| z \|$, we have

$$-\frac{\alpha}{\beta} y \in [H] \quad \text{for all } \alpha, \beta \in F, \quad \alpha \neq 0, \ \beta \neq 0,$$

and by the above inequality:

$$\| \alpha y + \beta z \| = \| -\frac{\alpha}{\beta} y - z \| = \| z \| = \operatorname{Max} \{ \| y \|, \| z \| \}.$$

From a) and b) it follows that $H \cup \{z\}$ is a distinguished subset of X, contradicting the maximality of H. Hence $[H] = X$ and H is a distinguished basis of X.

The same argument applies as usual to yield the following:

Corollary 2.3. A V-space admits a distinguished basis which contains any given distinguished set.

In a Banach space B a complete basis is a sequence $\{b_n\}$ such that for every $b \in B$ there exists a unique sequence of scalars $\{\alpha_n\}$ such that $b = \sum_n \alpha_n b_n$. The classical Paley-Wiener theorem ([1], [27], [30]) asserts that every sequence in B, which is "sufficiently close" to a complete basis, is itself a basis.

Arsove [1] has extended the Paley-Wiener theorem to arbitrary complete metric linear spaces over the real or complex field with the usual valuation. Theorem 2.4(ii) is the V-space analogue of Arsove's Theorem 1 ([1], p. 366). We do not require that H be countable.

<u>Theorem 2.4.</u> Let H be a distinguished subset of a V-space X and f be a mapping of H into X such that, for each $h \in H$,

$$(2.4) \qquad \| h - \alpha_h f(h) \| < \| h \|$$

for some scalar $\alpha_h \neq 0$. Then

(i) f(H) is a distinguished set of X;

(ii) f(H) is a distinguished basis if H is a distinguished basis.

<u>Proof:</u> (i) $\| f(h) \| = \| h \|$ for all $h \in H$. Let $\{h_i : i = 1, 2, \cdots, n\}$ be a subset of H such that $\| f(h_i) \| = r$ for $i = 1, 2, \cdots, n$ and some $r > 0$. Let $\{\beta_i : i = 1, 2, \cdots, n\}$ be any set of non-zero scalars. Then, from (2.4), $\| h_i \| = \| f(h_i) \| = r$ and

$$\left\| \left(\frac{\beta_1}{\alpha_1} h_1 + \frac{\beta_2}{\alpha_2} h_2 + \cdots + \frac{\beta_n}{\alpha_n} h_n \right) - (\beta_1 f(h_1) + \cdots + \beta_n f(h_n)) \right\|$$

$$= \left\| \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} (h_i - \alpha_i f(h_i)) \right\| < r .$$

Since H is distinguished, $\left\| \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} h_i \right\| = r$. It follows that

$$\left\| \sum_{i=1}^{n} \beta_i f(h_i) \right\| = r.$$

This proves that f(H) is a distinguished set (see p. 25).

(ii) The proof is a rewording of Arsove [1], page 367, in which "y_n" and "λ" must be replaced by "$\alpha_{h_n} f(h_n)$" and "ρ^{-1}" respectively. Note that the proof requires the completeness of the space (see page 31).

For examples and applications, see Section 4 of Chapter 3 and Theorem 4-6.5.

2-3 Distinguished families of subsets

The notion of distinguishability was introduced for subsets of a (pseudo-) valued space. It will now be extended to families of subsets.

In all of this section X is a V-space.

A set will be called trivial if it is a subset of $[\theta]$, i.e. if all its elements have norms equal to zero.

Definition 3.1. A family $\{A_\alpha\}$ of subsets of X is a distinguished family of subsets of X if

(i) $A_{\alpha_1} \cap A_{\alpha_2}$ is trivial for $\alpha_1 \neq \alpha_2$,

(ii) every non-empty subset B of $\bigcup_\alpha A_\alpha$ such that

a) $\|x\| \neq 0$ for each $x \in B$,

b) no two elements of B belong to the same A_α,

is a distinguished subset of X.

Clearly, a trivial set and any other subset of X form a distinguished family. Also, if $\{A_\alpha\}$ is a distinguished family,

$\{B_\alpha\}$ is a distinguished family of subsets if $B_\alpha \subset A_\alpha$ for all α.

The following theorem gives a characterization of distinguished families of non-trivial subspaces of X.

Theorem 3.2. A family $\{X_\alpha\}$ of non-trivial (closed or open) subspaces of X is a distinguished family of subsets of X if and only if:

(i) $X_{\alpha_1} \cap X_{\alpha_2}$ is trivial for $\alpha_1 \neq \alpha_2$,

(ii) any union of distinguished subsets of some or all of the X_α's is a distinguished subset of X.

Proof: The sufficiency is obvious. To prove the necessity, let $B = \bigcup_\alpha B_\alpha$, where B is not empty and B_α is either empty or a distinguished subset of X_α.

Consider a finite linear combination of elements of B:

$$x = \sum_{i=1}^{n} \sum_{j=1}^{p_i} \alpha_{ij} x_{ij},$$

where no α_{ij} is equal to 0 and where for each $i \in \{1, 2, \cdots, n\}$, $x_{ij} \in B_{\alpha_i}$ for $j = 1, 2, \cdots, p_i$.

Define $x_i = \sum_{j=1}^{p_i} \alpha_{ij} x_{ij}$. Then $x_i \in B_{\alpha_i}$.

Since the B_{α_i}'s are distinguished sets and $\{x_1, x_2, \cdots, x_n\}$ is by (ii) of Definition 3.1 a distinguished set:

$$\| x \| = \left\| \sum_{i=1}^{n} x_i \right\| = \max_{1 \leq i \leq n} \| x_i \| = \max_{1 \leq i \leq n} \left\{ \max_{1 \leq j \leq p_i} \| x_{ij} \| \right\}.$$

This shows that B is a distinguished set.

If the family of subspaces considered in Theorem 3.2 is a finite family of closed subspaces, the necessary and sufficient condition of this theorem can be considerably weakened.

Theorem 3.3. A finite family $\mathcal{A} = \{X_1, X_2, \cdots, X_n\}$ of non-trivial, closed subspaces of a V-space X forms a distinguished family of subspaces of X if and only if there exists a family $\mathcal{B} = \{H_1, H_2, \cdots, H_n\}$ such that:

(i) H_i is a distinguished basis of X_i; $i = 1, 2, \cdots, n$;

(ii) $H_i \cap H_j$ is empty for $i \neq j$;

(iii) $H_o = \bigcup_{i=1}^{n} H_i$ is a distinguished basis for the closed subspace $X_o = [X_1 \cup X_2 \cup \cdots \cup X_n]$.

Proof: Necessity. For each i, X_i is a V-space and admits a distinguished basis, H_i. By (i) of Definition 2.5, the assumption that $\mathcal{A}$ is a distinguished family implies that $X_i \cap X_j$ is trivial for $i \neq j$. Since distinguished bases do not contain any trivial element, (ii) is satisfied. By (ii) of Theorem 3.2, H_o is a distinguished set. Clearly $[H_o] = X_o$.

Sufficiency. It is easy to see that $X_i \cap X_j$ is trivial for $i \neq j$. We must show that (ii) of Definition 3.1 is satisfied.

Let $\{x_1, x_2, \cdots, x_m\}$, $m \leq n$, be such that $\| x_i \| \neq 0$ and assume that the X_i's are reindexed in such a way that $x_i \in X_i$, for $i = 1, 2, \cdots, m$.

For each fixed i, $1 \leq i \leq m$, there exists a non-increasing expansion of x_i in terms of H_i:

$$x_i = \sum_{j\geq 1} \alpha_{ij} y_{ij}, \quad \alpha_{ij} \neq 0.$$

According to Lemma 1-7.5

$$\| x_i \| = \sup_j \| y_{ij} \|.$$

Suppose that

$$\| y_{ik} \| = \sup_j \| y_{ij} \| \text{ for } k \leq p_i,$$

$$\| y_{ik} \| < \sup_j \| y_{ij} \| \text{ for } k > p_i.$$

p_i is necessarily finite.

Consider now any set of scalars $\{\beta_1, \beta_2, \cdots, \beta_m\}$. We can assume without loss of generality that $\beta_i \neq 0$ for each $i \leq m$. Let $x = \sum_{i=1}^{m} \beta_i x_i$. If we can show that

$$\| x \| = \operatorname{Max}_{1\leq i\leq m} \| x_i \|, \tag{2.5}$$

we will have proved that $\mathcal{A}$ is a distinguished family of subsets of X.

Since H_o is a distinguished set by assumption, we have:

$$\| x \| = \left\| \sum_{i=1}^{n} \sum_{j=1}^{p_i} \beta_i \alpha_{ij} y_{ij} + \sum_{i=1}^{n} \sum_{j=p_i+1}^{\infty} \beta_i \alpha_{ij} y_{ij} \right\|, \tag{2.6}$$

$$\left\| \sum_{i=1}^{n} \sum_{j=1}^{p_i} \beta_i \alpha_{ij} y_{ij} \right\| = \operatorname{Max}_{\substack{1\leq i\leq n \\ 1\leq j\leq p_i}} \| y_{ij} \| = \operatorname{Max}_{1\leq i\leq n} \| x_i \|, \tag{2.7}$$

$$\left\| \sum_{i=1}^{n} \sum_{j=p_i+1}^{\infty} \beta_i \alpha_{ij} y_{ij} \right\| < \operatorname{Max}_{\substack{1\leq i\leq n \\ 1\leq j\leq p_i}} \| y_{ij} \| = \operatorname{Max}_{1\leq i\leq n} \| x_i \|. \tag{2.8}$$

The equality (2.7) is guaranteed by the fact that no cancellation of terms can arise in the finite sum $\sum_{i=1}^{n} \sum_{j=1}^{p_i}$ since the

H_i's are assumed to be disjoint.

From (2.6), (2.7) and (2.8), it follows that (2.5) is true, and the proof of sufficiency is completed.

If a finite family $\{X_1, X_2, \cdots, X_n\}$ of non-trivial, closed subspaces of a V-space is a distinguished family, the closed subspace $[X_1 \cup X_2 \cup \cdots \cup X_n]$ will be called the <u>direct sum</u> of $X_1, X_2, \cdots, X_n$. This direct sum will be denoted by $X_1 \oplus X_2 \oplus \cdots \oplus X_n$. Conversely, whenever in the sequel the symbol $\oplus$ will be used, it will be understood that the subspaces involved form a distinguished family of closed subspaces. (Compare with [7], pp. 37, 256).

The following corollary to Theorem 3.3 is easily proved:

<u>Corollary 3.4.</u> The decomposition of any point of $X_1 \oplus X_2 \oplus \cdots \oplus X_n$ as a sum of elements of the X_i's is unique, except for order and addition of trivial elements.

In this section we have exhibited some analogy between Hilbert spaces and V-spaces. The analogy is a consequence of the similarity of the rôles played by the concepts of orthogonality and distinguishability in the two types of spaces. The definitions and theorems of this section should be compared with the definitions and theorems on complete orthonormal bases, orthogonal complements, etc., in the theory of Hilbert spaces; [7], [14], [19], [36]. In the next section an important difference between the two structures will become apparent.

2-4 Distinguished complements

In a Hilbert space, the orthogonal complement of a set A is defined to be the set of all the elements of the space which are orthogonal to all the elements of A. In a V-space, we introduce a corresponding notion: The distinguished adjunct A^d of a subset A of a V-space X is defined by

$$A^d = \{x \in X : \{x\} \text{ and } A \text{ form a distinguished pair of subsets of } X\}.$$

In Theorem 4.1, simple properties of distinguished adjuncts are stated. (ii) expresses the fact that the distinguished adjunct of a set A is the largest set which forms with A a distinguished pair of subsets of X. Parts (i), (iii), (iv) and (v) should be compared with Theorems 1, 2, 3, 4 of [14], p. 24. The proofs of the statements follow directly from our definitions and are omitted.

Theorem 4.1. If A and B are subsets of a V-space, then each of the following statements is valid:

(i) $A \cap A^d$ is trivial.

(ii) If (A, B) is a distinguished family of subsets of X, then $A \subset B^d$ and $B \subset A^d$.

(iii) $A \subset A^{dd}$.

(iv) If $A \subset B$, then $B^d \subset A^d$.

(v) $A^d = A^{ddd}$.

In a Hilbert space the orthogonal complement of any set is a closed subspace ([14], p. 24, Th. 2.6). However, the same is not true of the distinguished adjunct of a set in a V-space. The following example illustrates this fact. Let $A = S(\theta, r)$ and $\|z\| > r$; then $z \in A^d$ and for all $y \in A$, $y + z \in A^d$; clearly $y = (y + z) - z$ does not belong to A^d. Thus, A^d is not, in general, a subspace of the V-space, even when A itself is a subspace.

Two consequences of the discrepancy just mentioned are, first, that the notion of distinguished adjunct will not be useful in the sequel; and, secondly, that the non-uniqueness of the distinguished complement of a closed subspace (as described in the following definition) will have a rôle in the theory.

<u>Definition 4.2.</u> Two closed subspaces Y, Z, of a V-space X are said to be <u>distinguished complements</u> of one another if

(i) Y and Z form a distinguished pair of subsets of X,

(ii) $Y \oplus Z = X$.

It is clear that the only distinguished complement of $[\theta]$ is X, and conversely. If Y is a non-trivial, closed, proper subspace of X, Y admits a distinguished complement, but, in general, it is not unique. This is expressed in Theorem 4.4, in which we use the following terminology.

<u>Definition 4.3.</u> Let Y and Z be closed subspaces of a V-space, with distinguished bases H(Y) and H(Z) respectively. H(Z) is called an <u>extension of H(Y) to Z</u> if $H(Y) \subset H(Z)$. (This implies $Y \subset Z$).

Theorem 4.4. Let Y be a non-trivial, closed, proper subspace of a V-space X. Let $H(Y)$ be any distinguished basis of Y and H be any extension of $H(Y)$ to X. The subspace $[H \setminus H(Y)]$ is a non-trivial, closed, proper subspace of X which is a distinguished complement of Y.

The theorem is easily proved, using Corollary 2.3 and Theorem 3.3. It does not state that two different pairs $(H_1(Y), H_1)$ and $(H_2(Y), H_2)$ necessarily will generate distinct distinguished complements of Y.

As a simple example, let X have a distinguished basis formed by three elements x_1, x_2, x_3 with $\| x_1 \| = \| x_2 \| = \| x_3 \|$. Let $Y = [x_1]$ and $H(Y) = \{x_1\}$. Three possible extensions of $H(Y)$ to X are:

$H_1 = \{x_1, x_2, x_3\}$, $H_2 = \{x_1, x_1 + x_2, x_3\}$, $H_3 = \{x_1, x_2 + x_3, x_3\}$.

The distinguished complements of Y generated by the pairs $(H(Y), H_1)$ and $(H(Y), H_3)$ are both equal to $[x_2, x_3]$ but that generated by $(H(Y), H_2)$ differs from $[x_2, x_3]$.

2-5 Notes

(i) The concept of distinguishability has been introduced by Monna under a different name and through another formal definition.

In his early papers, [24], Monna uses the term "pseudo-orthogonal"; in his later work [25], he uses the word "orthogonal". In a strongly non-Archimedean normed linear space, a point x is said to be orthogonal to a point y if the distance from x to the linear subspace (y) is equal to the norm of x. ([24], V, p. 197; [25], I, p. 480). It is easily verified that y is then orthogonal to x.

The equivalence of this definition of orthogonality and of our definition of distinguishability is indicated in the following theorem (cf. Monna, [24]):

Theorem 5.1. Let A be a subset of a V-space X. For $x \in A$, let A_x denote the linear subspace $(A \setminus \{x\})$. Then, A is a distinguished subset of X if and only if, for all $x \in A$:

$$x \notin [\theta] \quad \text{and distance } (x, A_x) = \|x\|.$$

The proof is omitted.

From the notion of orthogonality, Monna constructs a theory of orthogonal sets and orthogonal complements quite analogous to our theory of distinguished sets and complements.

In [25], it is assumed that the valuation of the field of scalars is not trivial. Use is not made very extensively of "complete orthogonal" (distinguished) bases, which exist only under special assumptions, such as local compactness and separability.

An important tool used by Monna is the concept of a "projection". We have postponed the introduction of projections in our theory until linear operators are studied.

(ii) Ingleton ([17], p. 42; see also [25], I, p. 475) defines a spherically complete totally non-Archimedean metric space (field) as a totally non-Archimedean metric space (field) in which every family of closed balls linearly ordered by inclusion has non-void intersection. Spherical completeness implies completeness ([25], I, p. 476). In general, completeness does not imply spherical completeness, but, if the norm (valuation) satisfies (2.1) and (2.3) of Definition 1.1, then completeness implies spherical completeness ([25], II, p. 486). Therefore, a V-space is spherically complete.

Monna ([25], III, p. 466) has shown that the existence of a complete orthogonal (distinguished) basis in a non-Archimedean normed linear space is related to the completeness of the space and the spherical completeness of the field of scalars, when the valuation of the field is non-trivial.

It is possible that a reformulation of the arguments of Monna to the case of a field of scalars provided with a trivial valuation could lead to a proof of existence of a distinguished basis for a V-space. Our proof (Theorem 2.2) is more direct and shows that completeness conditions are unnecessary (see page 31).

2-6 V-Algebras

In this section we shall consider V-spaces X on which a multiplication is defined, i.e. such that to each pair $(x, y) \in X \times X$ there corresponds a unique "product" $xy \in X$.

Remark: In this Section we shall define some elements of a

V-space by use of sequences and series. Since a V-space can be a pseudo-normed space, the limit of a sequence or the sum of a series are not necessarily unique. For this reason we make the following notational convention:

Convention. In the sequel, the relation "$x = y$" means that $\|x - y\| = 0$; strict identity between x and y is indicated by the symbol "$x \equiv y$".

The definitions and theorems of this Section are simple modifications of the definitions and theorems of the classical theory of normed rings ([22], [26]).

Definition 6.1. A V-space X with a multiplication is called a V-algebra if for all $x, y \in X$ and all scalars α:

(2.9) $\alpha(xy) \equiv (\alpha x)y \equiv x(\alpha y)$,

(2.10) $x(yz) \equiv (xy)z$,

(2.11) $x(y + z) \equiv xy + xz, \ (y + z)x \equiv yx + zx$,

(2.12) $\|xy\| \leq \|x\| \cdot \|y\|$.

We also assume the existence of an identity, i.e. of an element e such that

(2.13) $xe \equiv ex \equiv x$ for all $x \in X$,

(2.14) $\|e\| = 1$.

X is said to be a commutative V-algebra if

(2.15) $\quad xy \equiv yx \quad$ for all $x, y \in X$.

Condition (2.12) implies the continuity of the multiplication, in both variables. Thus, if $x_n \to x$ and $y_n \to y$, $n = 1, 2, \cdots$, then $x_n y_n \to xy$.

If a V-space X has all the properties of a V-algebra except for the existence of an identity, the identity e can be formally adjoined. The classical process of adjunction of an identity to an algebra is described in [22], p. 59 and in [26], p. 157. Condition (2.14) is not essential but is made to simplify the proofs of certain statements. From (2.12), $\|e\| \geq 1$ in all cases. Therefore, dividing all norms by $\|e\|$ will not change the topology and will give to the identity a new norm satisfying (2.14).

As usual, we denote by x^n the product $xx\cdots x$ of n elements equal to x. $x^0 \equiv e$ for all $x \in X$.

Definition 6.2. An element e' will be called a pseudo-identity if $e' = e$.

Definition 6.3. Let x be an element of a V-algebra X.

(i) x is said to be pseudo-regular if there exists an element x^{-1} such that

$$xx^{-1} = x^{-1}x = e.$$

x^{-1} is called a pseudo-inverse of x.

(ii) x is said to be regular if there exists an element x^{-1} such that

$$xx^{-1} \equiv x^{-1}x \equiv e.$$

x^{-1} is called the <u>inverse</u> of x. (It can be proved that such an element is unique).

(iii) If x is not (pseudo-) regular it is said to be <u>singular</u>.

No element of $[\theta]$ is (pseudo-) regular for otherwise $1 = \|xx^{-1}\| \leq \|x\|\|x^{-1}\| = 0$.

<u>Theorem 6.4</u>. Let x^{-1} be a pseudo-inverse of an element x of a V-algebra X. Then y is a pseudo-inverse of x if and only if $y = x^{-1}$. Consequently, any two pseudo-inverses of x have equal norms.

<u>Proof</u>: The sufficiency is obvious. The necessity follows from (2.10) - (2.12) and the fact that for some $\theta', \theta'' \in [\theta]$:

$$(x^{-1} - y)x \equiv \theta';$$

$$\|(x^{-1} - y)xx^{-1}\| = \|\theta' x^{-1}\| = 0;$$

$$\|(x^{-1} - y)xx^{-1}\| = \|(x^{-1} - y)(e + \theta'')\| = \|x^{-1} - y\|.$$

<u>Lemma 6.5</u>. Let X be a V-algebra, $x \in X$ and $\|x\| < 1$. Then

(i) the sums of the series $\sum_{n=0}^{\infty} x^n$ are pseudo-inverses of $(e - x)$;

(ii) for every pseudo-inverse x' of $(e - x)$:

$$x + (e - x') = x(e - x'),$$

$$x + (e - x') = (e - x')x.$$

The proof is obtained by direct verification. (See [22], pp. 64-66).

Theorem 6.6. Let Y denote the set of pseudo-regular points of X. If $y \in Y$, $x \in X$ and $\| x - y \| < \| y^{-1} \|^{-1}$, then

(i) $x \in Y$;

(ii) $\| x^{-1} \| = \| y^{-1} \|$;

(iii) $\| x^{-1} - y^{-1} \| \leq \| y^{-1} \|^{2} \cdot \| x - y \| < \| y^{-1} \|$;

(iv) Y is an open subset of X and the mapping $y \to y^{-1}$, defined on Y, is continuous.

Proof: (i) $x = (x - y) + y = y[y^{-1}(x - y) + e]$

Since $\| y^{-1}(x - y) \| < 1$, it follows from Lemma 6.5 that

$$z = y^{-1}(x - y) + e = y^{-1}x$$

has a pseudo-inverse z^{-1}. Thus, x has a pseudo-inverse $x^{-1} = z^{-1}y^{-1}$.

(ii) From Lemma 6.5,

$$x^{-1} = \left(\sum_{n=0}^{\infty} \{-y^{-1}(x - y)\}^{n} \right) y^{-1} = y^{-1} + \cdots$$

Since

$$\| \{y^{-1}(x - y)\}^{n+1} \| < \| \{y^{-1}(x - y)\}^{n} \|, \quad n = 0, 1, 2, \cdots$$

$$\| x^{-1} \| = \| y^{-1} \|.$$

(iii) The conclusion follows from the equality

$$x^{-1} - y^{-1} = \left(\sum_{n=1}^{\infty} \{-y^{-1}(x - y)\}^{n} \right) y^{-1}$$

$$= -y^{-1}(x - y)y^{-1} + \cdots$$

(iv) is a consequence of (iii).

Definition 6.7. The spectrum $\sigma(x)$ of an element x of a V-algebra X is the set of scalars λ for which $(x - \lambda e)$ is singular.

Theorem 6.8. Let $x \in X$, $0 < \| x \| \leq 1$. If for some scalar μ, $\| x - \mu e \| < 1$, then

(i) $\sigma(x)$ is empty or $\sigma(x) = \{\mu\}$;

(ii) for $\lambda \neq \mu$, the pseudo-inverses of $(x - \lambda e)$ are the sums of the series

$$(2.16) \qquad - \sum_{n=0}^{\infty} \frac{(x - \mu e)^n}{(\lambda - \mu)^{n+1}} ;$$

and satisfy

$$(2.17) \qquad \| (x - \lambda e)^{-1} \| = \| x - \lambda e \| = 1.$$

Proof: We note first that this theorem is an extension of Lemma 6.5. Indeed, if $\| x \| < 1$, we take $\mu = 0$.

(i) is a consequence of (ii).

(ii) Since $\| (x - \mu e)^n \| \leq \| x - \mu e \|^n$, the series (2.16) converges. Using the continuity of the multiplication, we verify directly that if y is a sum of (2.16),

$$y(x - \mu e) - (\lambda - \mu)y = y(x - \lambda e) = e,$$

$$(x - \mu e)y - (\lambda - \mu)y = (x - \lambda e)y = e.$$

(2.17) follows from (2.16) and the fact that a V-algebra is a strongly valued space.

A direct proof of the following theorem is similar to that of Theorem 6.8.

Theorem 6.9. Let $x \in X$, $\| x \| > 1$. If for some scalar μ, $(x - \mu e)$ is pseudo-regular and $\| (x - \mu e)^{-1} \| < 1$, then

(i) $\sigma(x)$ is empty;

(ii) for $\lambda \neq \mu$, the pseudo-inverses of $(x - \lambda e)$ are the sums of the series

$$\sum_{n=0}^{\infty} (\lambda - \mu)^n [(x - \mu e)^{-1}]^{n+1}$$

and satisfy

$$\| (x - \lambda e)^{-1} \| = \| (x - \mu e)^{-1} \|.$$

Remarks. (i) As in the classical theory of normed rings, the singularity or regularity of an element of a V-algebra depends on its belonging to some maximal ideal of the algebra. (For the terminology, see [26], p. 159 or [7], p. 38.)

One can prove that a) any ideal of a V-algebra is contained in a maximal ideal ([22], p. 58; [26], p. 159); b) a maximal ideal is closed ([22, p. 68]; c) an element of a V-algebra is pseudo-regular if and only if it does not belong to any maximal ideal of the algebra ([22, p. 64; [26], p. 159).

It is also easy to prove that if Y is an ideal then $\| e - y \| \geq 1$ for all $y \in Y$. (Compare with [22], Th. 22D, p. 68).

(ii) The theory of Banach algebras ([22], [26]) is partially based on the fact that any Banach field is completely isomorphic to the field of the complex numbers (with its usual topology). From this result one attempts to characterize the maximal ideals of the algebra.

There does not seem to be any interesting analogue of this theory in the case of V-algebras.

CHAPTER 3

EXAMPLES OF V-SPACES

ASYMPTOTIC AND MOMENT SPACES

3-1 Introduction

Certain spaces of functions, mapping a Hausdorff space into a normed linear space, can be normed in such a way that they become V-spaces. In this chapter we shall describe two methods to generate such V-spaces.

In the first type of V-spaces, the norm considered will characterize the asymptotic behaviour of the functions; the resulting spaces will be called "asymptotic spaces". In the second type, we shall associate with each function a sequence of scalars, called "moments", and the norm of a function will depend on the first non-zero moment; the resulting spaces will be called "moment spaces".

3-2 The O and o relations

a) Let $\mathbf{\Lambda}$ be a Hausdorff space and let P and S be arbitrary sets. We consider functions of the three variables $\lambda \in \mathbf{\Lambda}$, $p \in P$ and $s \in S$. The variable λ will be called the asymptotic variable; p and s will be called the primary and secondary parameters respectively.

b) Let λ_o be a fixed non-isolated point of $\mathbf{\Lambda}$.

The abbreviation "cd-nbhd of λ_o" will stand for "closed neighbourhood of λ_o in $\mathbf{\Lambda}$, deleted of the point λ_o itself". A cd-nbhd of λ_o has non-void interior. A finite intersection of cd-nbhds of λ_o is a cd-nbhd of λ_o.

c) If f and g are two functions defined on $V \times P \times S$, where V is some cd-nbhd of λ_o, and with range in a (pseudo-) normed space (which may be different for the two functions), then the relation

$$f = O(g), \quad \lambda \to \lambda_o,$$

will mean that there exist, for each $s \in S$, a positive constant $\alpha(s)$ and a cd-nbhd $V(s)$ of λ_o such that

$$(3.1) \qquad \|f(\lambda, p, s)\| \leq \alpha(s)\|g(\lambda, p, s)\| \quad \text{for all } \lambda \in V(s) \text{ and all } p \in P.$$

In this inequality the norms are those of the appropriate range spaces.

Similarly, the relation

$$f = o(g), \quad \lambda \to \lambda_o,$$

will mean that for any $\varepsilon > 0$, there exists, for each $s \in S$, a cd-nbhd $V(s, \varepsilon)$ of λ_o such that

$$\|f(\lambda, p, s)\| \leq \varepsilon\|g(\lambda, p, s)\| \quad \text{for all } \lambda \in V(s, \varepsilon) \text{ and all } p \in P.$$

In using the O and o symbols, the specification $\lambda \to \lambda_o$ will usually be omitted.

These O and o relations have the following properties:

$$(3.3)\qquad O(O(f)) = O(f),$$

$$(3.4)\qquad O(o(f)) = o(O(f)) = o(o(f)) = o(f),$$

$$(3.5)\qquad O(f) + O(f) = O(f) + o(f) = O(f),$$

$$(3.6)\qquad o(f) + o(f) = o(f),$$

$$(3.7)\qquad O(f)\cdot O(g) = O(fg),$$

$$(3.8)\qquad O(f)\cdot o(g) = o(f)\cdot o(g) = o(fg).$$

Properties (3.7) and (3.8) apply when the range spaces are (pseudo-) normed rings. The proofs are immediate and the formulae can be extended to combinations of any finite number of order symbols. For those and other relations, see [9], Chapter 1.

d) <u>Definition 2.1.</u> A sequence $\{f_n\}$ of functions is called an <u>asymptotic sequence</u> (as $\lambda \to \lambda_o$) if

(i) f_n is defined on $V_n \times P \times S$, where V_n is some cd-nbhd of λ_o;

(ii) all f_n have the same range space; and

(iii) $f_{n+1} = o(f_n)$ for each n.

3-3 <u>Asymptotic spaces: Definition</u>

a) Let $\boldsymbol{\Lambda}$, P, S be as in Section 2 and $\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}\setminus\{\lambda_o\}$.

Let B and B_o be two (pseudo-) normed linear spaces. The (pseudo-) norms on both spaces will be denoted by $\|\cdot\|$.

Let N be a set of integers such that sup N $= \infty$. The ordering on N is the natural ordering of the integers. For $n \in N$, $\sigma^o(n) = n$, $\sigma^1(n) = \sigma(n)$ denotes the successor of n in N and $\sigma^j(n)$, $j = 2, 3, \cdots$ denotes the j^{th} successor of n in N; the element $m \in N$ such that $\sigma^j(m) = n$ is denoted by $\sigma^{-j}(n)$.

b) <u>Definition 3.1.</u> A family of functions $\Phi = \{\varphi_n : n \in N\}$ is called an <u>asymptotic scale</u> (as $\lambda \to \lambda_o$) if for each $n \in N$:

(i) φ_n is defined on $\Lambda' \times P \times S$ and have range in B_o; and

(ii) the sequence $\{\varphi_{\sigma^j(n)} : j = 0, 1, 2, \cdots\}$ is an asymptotic sequence.

In analogy with the terminology of J. G. van der Corput [38], [39] we use the following

<u>Definition 3.2.</u> A function f defined on $V \times P \times S$, where V is some cd-nbhd of λ_o, and with range in a (pseudo-) normed space is said to be <u>asymptotically finite with respect to an asymptotic scale Φ</u> $= \{\varphi_n : n \in N\}$ if there exists $n \in N$ such that $f = O(\varphi_n)$.

c) Let X be the linear space of all functions defined on $\Lambda' \times P \times S$, with range in B, which are asymptotically finite with respect to a given asymptotic scale Φ.

For each $x \in X$, define

$$\omega(x) = \sup\{n \in N : x = O(\varphi_n)\}, \tag{3.9}$$

and, for some fixed real ρ, $1 < \rho < \infty$,

$$\| x \| = \rho^{-\omega(x)}. \tag{3.10}$$

The function defined on X by (3.9) and (3.10) will be called the <u>Φ-asymptotic norm on X.</u>

Since the asymptotic behaviour of a function, as $\lambda \to \lambda_o$, is entirely determined by its values on any set of the form $V \times P \times S$, where V is a cd-nbhd of λ_o, the Φ-asymptotic norm of the difference of two functions in X which are identical on $V \times P \times S$ is equal to zero. Conversely, if a function is defined on $V \times P \times S$ and is asymptotically finite with respect to Φ, then it can be arbitrarily extended to all of $\Lambda' \times P \times S$ and the Φ-asymptotic norm of the difference of any two of its extensions is equal to zero.

d) Theorem 3.3. The space X, under the Φ-asymptotic norm, is a V-space.

Proof: Using (3.3) - (3.9) one easily verifies that, under the norm (3.10), X has the properties of a strongly pseudo-valued space (Definitions 1-1.1 and 1-1.2). The Φ-asymptotic norm satisfies (2.1) of Definition 2-1.1 and also (2.3).

To prove the completeness of X, consider an arbitrary Cauchy sequence $\{s_i : i = 0, 1, 2, \cdots\}$. Let

$$y_o = s_o, \quad y_i = s_i - s_{i-1} \quad \text{for} \quad i = 1, 2, \cdots$$

From Theorems 1-4.1 and 1-4.2, it is sufficient to prove the convergence of a particular rearrangement of the series

$$\sum_{i=0}^{\infty} y_i . \tag{3.11}$$

Without loss of generality, we can assume that none of the y_i's has zero-norm. Let

$$M = \{n \in N : \|y_i\| = \rho^{-n} \text{ for some } i\},$$
$$q = \inf M.$$

Since $\{s_i\}$ is a Cauchy sequence, it follows that $q > -\infty$. Also, for each $n \in M$, the number of y_i's with norms equal to ρ^{-n} is finite. Let x_n be their sum. It follows that

$$(3.12) \qquad \|x_n\| \leq \rho^{-n} \quad \text{for all } n \in N$$

and that the series

$$(3.13) \qquad \sum_{\substack{n=q \\ n \in M}}^{\infty} x_n$$

can be considered as a rearrangement of (3.11).

We now fix an arbitrary value $s \in S$ for the secondary parameter.

It follows from (3.12) that for each $n \in M$, there exist a constant $\alpha[n, s] > 0$ and a cd-nbhd $V[n, s]$ of λ_o such that

$$\begin{aligned} \|x_{\sigma(n)}(\lambda, p, s)\| &\leq \alpha[\sigma(n), s]\|\varphi_{\sigma(n)}(\lambda, p, s)\| \\ &\leq \tfrac{1}{2}\alpha[n, s]\|\varphi_n(\lambda, p, s)\| \end{aligned}$$

for all $\lambda \in V[\sigma(n), s]$ and for all $p \in P$.

We can assume without loss of generality that these $V[n, s]$'s are nested and are selected in such a way that their intersection is void.*

Then, for all $j \geq 0$,

$$\|x_{\sigma^j(n)}(\lambda, p, s)\| \leq 2^{-j}\alpha[n, s]\|\varphi_n(\lambda, p, s)\|$$

for all $\lambda \in V[\sigma^j(n), s]$ and for all $p \in P$.

We shall now define an element x of X by specifying its values on $\Lambda' \times P \times \{s\}$.

For $\lambda \in \Lambda' \setminus V[q, s]$ we define

$$x(\lambda, p, s) = 0 \text{ for all } p \in P.$$

For $\lambda \in V[q, s]$, there exists an integer $N(\lambda, s)$:

$$N(\lambda, s) = \operatorname{Max}\{n \in N : \lambda \in V[n, s]\}$$

If $\lambda \in V[n, s]$ then $N(\lambda, s) \geq n$.

For $\lambda \in V[q, s]$ we define

$$x(\lambda, p, s) = \sum_{\substack{q \leq n \leq N(\lambda, s) \\ n \in M}} x_n(\lambda, p, s) \text{ for all } p \in P.$$

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*In addition to these requirements, the choice of the cd-nbhds $V[n, s]$ is guided by the condition that for $\lambda \in V[\sigma^j(q), s]$, $j = 1, 2, \cdots$

$$\|\varphi_{\sigma^j(q)}(\lambda,p,s)\| \leq \frac{1}{2}\,\frac{\alpha[\sigma^{j-1}(q),s]}{\alpha[\sigma^j(q),s]}\|\varphi_{\sigma^{j-1}(q)}(\lambda,p,s)\| \quad \text{for all } p \in P;$$

which implies

$$\|\varphi_{\sigma^j(q)}(\lambda,p,s)\| \leq 2^{-j}\,\frac{\alpha[q, s]}{\alpha[\sigma^j(q),s]}\|\varphi_q(\lambda, p, s)\| \quad \text{for all } p \in P.$$

We shall now show that x is a limit of the sequence (3.13) and thus of (3.11).

Let $\varepsilon > 0$ be given; there exists J such that for all $j \geq J$, $\rho^{-j} < \varepsilon$. We assert that, for all $j \geq J$, $j \in M$,

$$\left\| x - \sum_{\substack{q \leq n \leq j \\ n \in M}} x_n \right\| \leq \rho^{-j} < \varepsilon,$$

or, equivalently, that

$$(x - \sum_{\substack{q \leq n \leq j \\ n \in M}} x_n) = O(\varphi_j).$$

Indeed, for each $s \in S$, for $\lambda \in V[j, s]$, $(j \in M)$, $\lambda \in V[\sigma(j), s]$, $\lambda \in V[\sigma^2(j), s]$, $\ldots$, $\lambda \in V[N(\lambda, s), s]$ and

$$\| x(\lambda, p, s) - \sum_{\substack{q \leq n \leq j \\ n \in M}} x_n(\lambda, p, s) \|$$

$$= \| x_{\sigma(j)}(\lambda, p, s) + x_{\sigma^2(j)}(\lambda; p, s) + \cdots + x_{N(\lambda, s)}(\lambda, p, s) \|$$

$$\leq 2\alpha[j, s] \| \varphi_j(\lambda, p, s) \|, \quad \text{for all } p \in P.$$

This completes the proof.

The above proof is modelled after the second part of the proof of [33], Theorem 1 (also [32], Th. 4.2). In [33], the range space B is a Banach space; we have shown that the completeness of X does not require the completeness of B.

If B is a (pseudo-) normed ring, the product xy of two functions $x, y \in X$ is defined by

$$xy(\lambda, p, s) = x(\lambda, p, s) \cdot y(\lambda, p, s) \quad \text{for all } \lambda, p, s.$$

If B_o is a (pseudo)-normed ring, a similar definition of the product of two elements of Φ can be given.

Theorem 3.4. Let B and B_o be (pseudo-) normed rings. If for all m, n $\in$ N, $\varphi_m\varphi_n = O(\varphi_{m+n})$, then X satisfies the properties (2.9), (2.10) and (2.11) of a V-algebra.

Proof: If x, y $\in$ X and $\|x\| = \rho^{-m}$, $\|y\| = \rho^{-n}$, then by (3.7) and (3.3):

$$xy = O(\varphi_m)\cdot O(\varphi_n) = O(\varphi_m\varphi_n) = O(O(\varphi_{m+n})) = O(\varphi_{m+n})$$

Hence, $\|xy\| \leq \rho^{-(m+n)} = \|x\|\cdot\|y\|$ and (2.11) is satisfied.

One verifies easily that (2.9) and (2.10) hold.

e) Let x $\in$ X have, in the Φ-asymptotic norm on X, the following expansion:

(3.15) $\quad x = a_o x_o + a_1 x_1 + a_2 x_2 + \cdots, \quad x_i \in X, \alpha_i \in F.$

The expansion (3.15) is said to be an expansion of the "Poincaré type" ([11], pp. 218-219) if the sequence $\{x_n : n = 0, 1, 2, \cdots\}$ is an asymptotic sequence (see Definition 2.1).

When the range spaces B and B_o are identical, $\Phi \subset X$. A convergent expansion in terms of the elements of Φ is said to be an expansion "essentially of the Poincaré type".

In [10] and [11] the desirability, in a theory of asymptotics, of accepting expansions which are not of the Poincaré type is highly stressed. In an asymptotic space, expansions which are not of the Poincaré type can occur if there exist countable distinguished sets with elements having arbitrarily small norms and which cannot be ordered into an asymptotic sequence. Specific examples will be given in the next sections.

3-4 Asymptotic spaces: Example I

a) Our first examples of asymptotic spaces are simple and do not involve any primary or secondary parameters: $P = S = \emptyset$.

The Hausdorff space Λ is the real interval $[0, \bar{\lambda}]$, $0 < \bar{\lambda} < \infty$, and $\lambda_o = 0$. A cd-nbhd of 0 is an interval of the form $(0, \lambda']$, $0 < \lambda' \leq \bar{\lambda}$. The range spaces B and B_o are both the space of the real numbers. N is the set of all integers and the asymptotic scale to be used is $\Phi = \{\varphi_n : n \in N\}$, where $\varphi_n(\lambda) = \lambda^n$.

Let $X = \mathcal{R}$ be the space of all real valued functions x on Λ' which are asymptotically finite with respect to Φ and with norm defined by (3.9) and (3.10).

Clearly: $\Phi \subset X$; $\varphi_o(\lambda) \equiv 1$ and $\|\varphi_o\| = 1$; $\varphi_m\varphi_n = O(\varphi_{m+n})$ for all $m, n \in N$. From this and Theorems 3.3 and 3.4, it follows that X is a V-algebra. Direct proofs of this result were given by A. Erdélyi [9], J. Popken [29], J. G. van der Corput [38], [39].

b) To illustrate the results of Chapters 1 and 2, we consider the following subsets of $\mathcal{R}$:

$$\overline{\Phi} = \{\varphi_\alpha : -\infty < \alpha < \infty\}, \qquad \varphi_\alpha(\lambda) = \lambda^\alpha;$$

$$\Phi_k = \{\varphi_n : n \in N,\ n \geq k\};$$

(3.16) $$E = \{e_\alpha : -\infty < \alpha < \infty\}, \qquad e_\alpha(\lambda) = \exp \alpha\lambda;$$

$$J = \{J_n : n = 0, 1, 2, \cdots\}, \text{ where}$$

$J_n : \lambda \to J_n(\lambda)$ is the Bessel function of the first kind of order n;

$$G = \{z_n : n = 0, 1, 2, \cdots\}, \quad \text{where}$$

(3.17) $$z_n(\lambda) = \begin{cases} \lambda^n \sin (n+1)\frac{\pi}{\lambda} & \text{if } n \text{ is even,} \\ \lambda^n \cos (n+1)\frac{\pi}{\lambda} & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to verify that $\overline{\Phi}$, Φ and Φ_k are distinguished subsets of $\mathcal{R}$.

For n = 0, 1, 2, ..., ([8], Vol. II)

(3.18) $$\|\varphi_n - 2^n n! J_n\| = \|\varphi_{n+2}\| < \|\varphi_n\|.$$

Thus, by the Paley-Wiener Theorem (Theorem 2-2.4) J is also a distinguished subset of $\mathcal{R}$.

The set G is also a distinguished subset of $\mathcal{R}$ since its elements have distinct norms: $\|z_n\| = \rho^{-n}$. Since $z_n = o(z_m)$ is not true for any n, m > 0, the sequence $\{z_n\}$ is not an asymptotic sequence. An expansion in terms of the elements of G:

(3.19) $$\alpha_o \sin\frac{\pi}{\lambda} + \alpha_1\lambda \cos\frac{2\pi}{\lambda} + \alpha_2\lambda^2 \sin\frac{3\pi}{\lambda} + \cdots, \quad \alpha_n \text{ real,}$$

always converges to some element of $\mathcal{R}$. Yet, it is not an expansion of the Poincaré type (see Section 3, e). Expansions such as (3.19) are mentioned in [11].

For all $\beta \neq 0$,

$$\varphi_\alpha + \beta z_n = O(\varphi_m) \text{ if and only if } m \geq \min\{\alpha, n\}.$$

Thus $(\overline{\Phi}, G)$ is a distinguished pair of subsets of $\mathcal{R}$. A consequence is that there exists a distinguished basis of $\mathcal{R}$ which contains $\overline{\Phi}$ and G. Another consequence is that a function which is the sum of a series (3.19) cannot admit an expansion in terms of elements of $\overline{\Phi}$, i,e. in terms of powers of λ.

The set E is contained in the closed subspace generated by $\overline{\Phi}_o$. Indeed, it is well known that for any real number α, the series $\sum_{n=o}^{\infty} \frac{\alpha^n}{n!} \varphi_n$ converges asymptotically, as $\lambda \to 0$, to the function e_α.

E is not a distinguished subset of $\mathcal{R}$, since for $\alpha \neq \beta$:

$$\| e_\alpha \| = 1, \quad \| e_\beta \| = 1, \quad \| e_\alpha - e_\beta \| = \rho^{-1} < 1.$$

c) Let $\mathcal{G}$ denote the closed subspace generated by G, i.e. the set of functions which admit expansions of the form (3.19). (See Theorem 1-7.6)

Let $\mathcal{P}$ denote the closed subspace generated by $\overline{\Phi}$, i.e. the set of functions which admit expansions of the form

(3.20) $\qquad \alpha_n\varphi_n + \alpha_{n+1}\varphi_{n+1} + \cdots , \quad n \in N, \quad \alpha_n$ real.

$\mathcal{P}$ is a subalgebra of $\mathcal{R}$. A non-trivial element of $\mathcal{P}$ is pseudo-regular. From the above discussion, $\mathcal{G} \cap \mathcal{P}$ is trivial.

Let $\mathcal{P}_k$ be the closed subspace generated by $\overline{\Phi}_k$, i.e. the set of sums of expansions (3.20) with $n \geq k$.

E is contained in $\mathcal{P}_k$ for all $k \leq 0$.

$\mathcal{P}_o$ is a V-algebra. From (3.18) and Theorem 2-2.4, J is a distinguished basis of $\mathcal{P}_o$.

E and Φ_o do not form a distinguished pair of subsets of $\mathcal{P}_o$ but their union is a linearly independent set. Therefore, there exists a Hamel basis of $\mathcal{P}_o$ which contains $E \cup \Phi_o$. In [4], continuous linear functionals on the subspace $(E \cup \Phi_o)$ are studied; see Chapter 6.

d) To construct the space $\mathcal{R}$, we selected $\lambda_o = 0$. Clearly, we would have obtained a similar space by choosing any other finite value for λ_o and the asymptotic scale $\Phi = \{\varphi_n : n \in N\}$ where $\varphi_n(\lambda) = (\lambda - \lambda_o)^n$.

One may also consider $\Lambda = [\bar{\lambda}, \infty]$, $0 \leq \bar{\lambda} < \infty$, $\lambda_o = \infty$ and the asymptotic scale $\Phi = \{\varphi_n : n \in N\}$, where $\varphi_n(\lambda) = \lambda^{-n}$.

e) Spaces such as $\mathcal{R}$ can be constructed in which Λ is some subset of the complex plane and for more sophisticated asymptotic scales. Examples are given in [11], with asymptotic scales such as

$$\Psi = \{\psi_n : n = 0, 1, 2, \cdots\},$$
$$\psi_n(\lambda, z, r, s) = \Gamma\left(z + \frac{n}{r}\right) \lambda^{-z-ns};$$

λ is the complex asymptotic variable, $\lambda \to \lambda_o = \infty$; z is a complex number considered as a primary parameter, i.e. order relations must hold uniformly in z; the positive real numbers r and s are secondary parameters. Proper conditions must be placed on the domains of λ and z. (See [11]; also [9], [10].)

3-5 Asymptotic spaces: Example II

In this section we give two examples involving formal power series in two real or complex variables. The spaces to be constructed will be used in Chapter 4 to obtain asymptotic expansions of some functions defined as two-dimensional Laplace transforms [6].

b) In this example the Hausdorff space Λ is the set of points $\lambda = (u, v)$ of R^2 for which $0 \leq u, v \leq \infty$; $\lambda_o = (0, 0)$. The range spaces B and B_o are the spaces of complex numbers and of real numbers respectively. N is the set of all non-negative integers and the asymptotic scale to be used is

$$\Phi = \{\varphi_n : n \in N\}, \quad \varphi_n(u, v) = (u + v)^n.$$

Let X = Q be the space of all complex valued functions on Λ' which are asymptotically finite with respect to Φ.

As for the space $\mathcal{R}$ of Section 4, one verifies that Q is a V-algebra, under the Φ-asymptotic norm defined by (3.9) and (3.10).

Let $x_{\ell j} \in X$ be the function defined by

$$x_{\ell j}(u, v) = u^{\ell} v^j, \quad \ell, j \in N.$$

For all non-zero complex numbers α, β, and all integers ℓ, j such that $0 \leq \ell \leq n$, $\ell < j \leq n$:

$$(3.21) \qquad x_{n-\ell,\ell} = O(\varphi_m) \text{ if and only if } m \leq n,$$

$$(3.22) \qquad \alpha x_{n-\ell,\ell} + \beta x_{n-j,j} = O(\varphi_m) \text{ if and only if } m \leq n.$$

Indeed, to verify (3.22), suppose firstly that $m \leq n$. Then:

$$|\alpha u^{n-\ell}v^{\ell} + \beta u^{n-j}v^{j}| \leq \mathrm{Max}\{|\alpha|, |\beta|\}\cdot(u^{n-\ell}v^{\ell} + u^{n-j}v^{j})$$
$$\leq \mathrm{Max}\{|\alpha|, |\beta|\}(u + v)^{n}.$$

Thus:

$$\alpha x_{n-\ell,\ell} + \beta x_{n-j,j} = O(\varphi_n) = O(\varphi_m)$$

and (3.22) is satisfied if $m \leq n$. Conversely, suppose that the relation is true for some $m = n + k$, $k > 0$. Since every cd-nbhd of $(0, 0)$ must contain points (u, v) for which $v = u$, there exists a constant $A > 0$ such that for all u small enough

$$|\alpha + \beta|u^{n} \leq A\cdot 2^{n+k}u^{n+k}.$$

This is impossible. This completes the verification of (3.22).

Consider the set

$$H_k = \{x_{\ell j} : \ell + j \geq k\}, \quad k \in N.$$

It follows from (3.21) and (3.22) that H_k is a distinguished subset of Q. Let $\mathcal{K}_k$ denote the closed subspace of Q generated by H_k, i.e. the set of all functions which admit expansions of the form

$$(3.23) \qquad \sum_{\ell}\sum_{j} \alpha_{\ell j}x_{\ell j}, \quad \ell + j \geq k, \quad \alpha_{\ell j} \text{ complex}.$$

Unlike the space $\mathcal{P}_k$ of Example I (Section 4.c), the elements of a distinguished basis of $\mathcal{K}_k$ do not have distinct norms: from (3.21)

$$\| x_{n-j,j} \| = \rho^{-n} \quad \text{for } 0 \leq j \leq n.$$

A particular subspace of $\mathcal{K}_k$ is the subspace of functions which admit expansions (3.23) such that for some sequence $\{\alpha_n\}$ of complex numbers ($i^2 = -1$):

$$\alpha_{n-j,j} = \binom{n}{j} i^j \alpha_n \quad \text{for } 0 \leq j \leq n \text{ and all } n.$$

Setting $z = u + iv$, the expansions of such functions are of the form $\sum_{n \geq k} \alpha_n z^n$.

c) We now let the Hausdorff space Λ be a set of points $\lambda = (z, w)$ where z and w each belong to a subset of the complex Riemann sphere which contains the point at infinity. Let $\lambda_o = (\infty, \infty)$. N is the set of all non-negative integers and we denote by Ψ the asymptotic scale

$$\Psi = \{\psi_n : n \in N\}, \quad \psi_n(z, w) = \left(\frac{1}{|z|} + \frac{1}{|w|}\right)^n .$$

Thus, B_o is the space of the real numbers.

Let B be the space of the complex numbers and consider the space Q' of complex valued functions on Λ' which are asymptotically finite with respect to Ψ.

It can be shown (as in b)) that the set

$$H_k' = \{y_{\ell j} : \ell + j \geq k\}, \; k \in N, \; y_{\ell j}(z,w) = z^{-\ell} w^{-j},$$

is a distinguished subset of Q'. $\mathcal{K}_k'$ will denote the closed subspace generated by H_k'.

3-6 Asymptotic spaces: Example III

a) $\mathcal{B}(D)$ will denote the set of all bounded transformations from a closed subset D of a Banach space S into S itself; i.e. the set of transformations from D to S for which $\|A\|_D < \infty$, where

$$(3.24) \qquad \|A\|_D = \sup_{\substack{s_1, s_2 \in D \\ s_1 \neq s_2}} \frac{\|As_1 - As_2\|}{\|s_1 - s_2\|} .$$

The function (3.24) is a pseudo-norm on $\mathfrak{B}(D)$. If $\|A - B\| = 0$, then $As = Bs + s_o$ for all $s \in D$ and some fixed $s_o \in S$.

Under the assumption that D is a linear subspace of S, $\mathfrak{J}(D)$ will denote the set of all bounded linear transformations from D to S. On $\mathfrak{J}(D)$, (3.24) is equivalent to

$$\|A\|_D = \sup_{\substack{s \in D \\ s \neq 0}} \frac{\|As\|}{\|s\|}$$

and is a norm under which $\mathfrak{J}(D)$ is a Banach space ([7], p. 61; [26], p. 75).

The convergence induced by (3.24) on $\mathfrak{B}(D)$ and $\mathfrak{J}(D)$ will be called the uniform convergence on D ([7], p. 475; [26], p. 444). Thus, a sequence $\{A_n\}$ in $\mathfrak{B}(D)$ converges uniformly to A on D if and only if $A \in \mathfrak{B}(D)$ and $\lim_{n\to\infty} \|A-A_n\|_D = 0$, i.e.:

$$\lim_{n\to\infty} \frac{\|(A-A_n)s_1 - (A-A_n)s_2\|}{\|s_1 - s_2\|} = 0 \quad \text{for all } s_1, s_2 \in D,\ s_1 \neq s_2.$$

b) We now construct an asymptotic space of functions with range in $\mathfrak{B}(D)$, i.e. $B = \mathfrak{B}(D)$.

The Hausdorff space Λ is the real interval $[0, 1]$ and $\lambda_0 = 0$. The space B_0 is the space of the real numbers. N is the set of all non-negative integers and we use the asymptotic scale

$$\Phi = \{\varphi_n : n \in N\}, \quad \varphi_n(\lambda) = \lambda^n.$$

Let X be the space of all mappings x defined on $\Lambda' = (0, 1]$, with range in $\mathcal{B}(D)$ and which are asymptotically finite with respect to Φ, i.e. such that for some $n \in N$,

$$\|x\|_D = O(\varphi_n). \tag{3.25}$$

Suppose that $y \in X$, y is independent of λ and $\|y\|_D \neq 0$. Then, one verifies easily that $\|y\| = 1$. Furthermore, if $x \in X$ satisfies $\|y - x\| < 1$, then, as a function of λ, x converges uniformly to y on D, as $\lambda \to 0$, since for λ small enough and some constant $\alpha > 0$

$$\|y - x\|_D \leq \alpha\lambda.$$

If $x \in X$ has a Φ-asymptotic norm strictly less than 1 and for λ in some cd-nbhd of 0, x maps D into itself, then $x(\lambda)$ is a contraction mapping on D for all λ in some cd-nbhd of 0, i.e. for some λ', $0 < \lambda' \leq 1$, $x(\lambda)$ maps D into itself and $\|x(\lambda)\|_D < 1$ when $\lambda \leq \lambda'$ ([19], Vol. I, p. 43).

Spaces of this type have been considered by C. A. Swanson and M. Schulzer [32], [33]. In these references, transformations satisfying (3.25) are said to be "of Class Lip (φ_n)." Specific examples are given in [32], pp. 28-38.

c) The space X of b) consists of mappings from (0, 1] to the set $\mathcal{B}(D)$ of bounded transformations from D to S. In the space X' to be constructed now, unbounded transformations will also be considered. The range space B is the Banach space S.

Let $\boldsymbol{\Lambda}$, λ_o, N and Φ be as in b). Let X' be the space of all mappings from (0, 1] $\times$ D to S which are asymptotically finite with respect to Φ when $s \in D$ is considered as a secondary parameter.

The Φ-asymptotic norm of $x \in X'$ is less than or equal to ρ^{-n} if for each fixed $s \in D$, there exist a constant $\alpha[s] > 0$ and a cd-nbhd V[s] of 0 such that

$$\|x(\lambda, s)\| \leq \alpha[s]\lambda^n \text{ for all } \lambda \in V[s];$$

equivalently, $\|x\| < \rho^{-n}$ if for all $s \in D$,

$$\|x(\lambda, s)\| \leq \beta[s]\cdot\lambda^n\cdot\|s\|, \text{ for all } \lambda \in V[s],$$

where $\beta[s] = \alpha[s]\cdot\|s\|^{-1}$ if $s \neq 0$.

Suppose that $y \in X'$, y is independent of λ and $\|y\|_D \neq 0$. Then for each $s \in D$, $\|y(\lambda, s)\| = \|y(s)\|$ for all $\lambda \in \boldsymbol{\Lambda}'$ and, hence, $\|y\| = 1$. Furthermore, if $x \in X'$ satisfies $\|x - y\| < 1$, then, as a function of λ, x converges strongly ([7], p. 475) to y on D, when $\lambda \to 0$; indeed, for each $s \in D$, there exists $\alpha[s] > 0$ such that for all λ small enough

$$\|x(\lambda, s) - y(\lambda, s)\| \leq \alpha[s]\cdot\lambda.$$

3-7 Moment spaces

a) For simplicity we restrict our definition of moment spaces to spaces of real valued functions defined on a finite interval

$[a, b]$, $-\infty < a < b < \infty$. All integrals considered are Riemann-Stieltjes integrals ([41], p. 1).

Let α be a real valued function of bounded variation on $[a, b]$ ([41], p. 6). Let $\Phi = \{\varphi_n(t) : n = 0, 1, 2, \cdots\}$ be a sequence of non-zero, real functions on $[a, b]$ such that all integrals

$$(3.26) \qquad \mu_n(1) = \int_a^b \varphi_n(t)d\alpha(t), \quad n = 0, 1, 2, \cdots,$$

exist and are finite.

Let X' be the linear space of all real functions x, defined on $[a, b]$ and such that all integrals

$$\mu_n(x) = \int_a^b x(t)\varphi_n(t)d\alpha(t), \quad n = 0, 1, 2, \cdots,$$

exist and are finite. $\mu_n(x)$ is called the <u>n-th moment of x relative to</u> Φ.

For $x \in X'$, define

$$\omega(x) = \inf\{n : \mu_n(x) \neq 0\}$$

and, for some fixed ρ, $1 < \rho < \infty$,

$$(3.27) \qquad \|x\| = \rho^{-\omega(x)} .$$

It is immediate that X', with the norm (3.27), is a V-space, except possibly for completeness. In X' the distance of two functions x, y is less than or equal to ρ^{-n} if and only if $\mu_i(x) = \mu_i(y)$ for $i = 0, 1, 2, \cdots, n-1$.

X' admits a distinguished basis (Theorem 2-2.2). Two elements of a distinguished basis of X' cannot have the same norm. Indeed, if $\|x\| = \|y\| = \rho^{-n}$ for some n, then

$$\mu_i(x) = \mu_i(y) = 0 \text{ for } i < n;\ \mu_n(x) \neq 0;\ \mu_n(y) \neq 0;$$

therefore

$$\mu_i(\mu_n(y)x - \mu_n(x)y) = 0 \quad \text{for } i \leq n.$$

This implies that $\|\mu_n(y)x - \mu_n(x)y\| < \rho^{-n}$. Thus x and y are not distinguished.

Let N be the set of integers defined by

$$N = \{n : \text{for some } x_n \in X',\ \mu_i(x_n) = \delta_{in} \text{ for } i \leq n\}.$$

For each $n \in N$, let x_n be a function such that $\mu_i(x_n) = \delta_{in}$ for $i \leq n$. The set $H = \{x_n : n \in N\}$ forms a distinguished basis of X'.

The completion X of X', i.e. the set of formal expansions in terms of H, is a V-space (Theorem 1-7.6).

V-spaces constructed in the manner described above will be called "moment spaces".

b) For the remainder of the Section we suppose that the function α is strictly increasing on $[a, b]$ and that Φ is a linearly independent set of continuous functions contained in X.

These assumptions imply that all the integrals (3.26) and the integrals

$$\int_a^b \varphi_m(t)\cdot\varphi_n(t)\cdot d\alpha(t), \quad m, n = 0, 1, 2, \cdots,$$

exist and are finite ([41], p. 7).

A sequence $\{f_n\}$ of functions defined on $[a, b]$ is said to be orthonormal with respect to α if

$$\langle f_m, f_n \rangle = \delta_{mn}, \qquad m, n = 0, 1, 2, \cdots,$$

where

$$\langle f, g \rangle = \int_a^b f(t) \cdot g(t) d\alpha(t).$$

Lemma 7.1. If f is a non-negative continuous function on $[a, b]$ and $\int_a^b f(t) d\alpha(t) = 0$, then $f(t) \equiv 0$.

The proof is identical to that of Proposition (5.2) in [37], p. 41.

Theorem 7.2. There exists a unique sequence of functions $\{p_n\}$ of the form

$$p_n(t) = a_{nn}\varphi_n(t) + a_{n,n-1}\varphi_{n-1}(t) + \cdots + a_{no}\varphi_o(t), \; a_{nn} > 0,$$

which is orthonormal with respect to α.

The proof is an easy modification of [37], pp. 41-42.

Theorem 7.3. (i) $\| p_n \| = \rho^{-n}$, $n = 0, 1, 2, \cdots$

(ii) $\{p_n\}$ is a distinguished basis of X.

(iii) If $f \in X$, then, in the norm of X,

$$(3.28) \qquad f(t) = \sum_{n=0}^{\infty} \langle f, p_n \rangle p_n(t).$$

Proof: (i) For all m, there exist coefficients b_{mi} such that

$$\varphi_m(t) = \sum_{i=0}^{m} b_{mi} p_i(t), \quad b_{mm} = \frac{1}{a_{mm}} > 0.$$

Thus, $\| p_n \| = \rho^{-n}$ follows from

$$(3.29) \qquad \int_a^b p_n(t)\varphi_m(r)d\alpha(t) = \begin{cases} 0 & \text{if } m < n, \\ \frac{1}{a_{mm}} & \text{if } m = n. \end{cases}$$

(ii) follows from (i) and a previous remark, page 72.

(iii) For $m < n$,

$$\mu_m\left(f - \sum_{i=0}^{n-1} \langle f, p_i\rangle p_i\right) =$$

$$= \int_a^b f(t)\varphi_m(t)d\alpha(t) - \sum_{i=0}^{n-1} \langle f,p_i\rangle \int_a^b p_i(t)\varphi_m(t)d\alpha(t)$$

$$= \sum_{i=0}^{m} b_{mi} \langle f,p_i\rangle - \sum_{i=0}^{m} \langle f,p_i\rangle b_{mi} = 0.$$

Thus $\| f - \sum_{i=0}^{n-1} \langle f,p_i\rangle p_i \| \le \rho^{-n}$ which proves the convergence.

The series (3.28) is usually called the <u>Fourier series</u> of f with respect to $\{p_n\}$ ([37], p. 45). In the moment space X, the distance of two functions f and g is less than or equal to ρ^{-n} if and only if the first n Fourier coefficients of f: $\langle f,p_i\rangle$, $i = 0, 1, 2, \cdots, n-1$, are equal to the corresponding Fourier coefficients $\langle g, p_i\rangle$ of g.

c) Whenever α is strictly increasing on $[a, b]$ and for all n, $\varphi_n(t) = [\varphi(t)]^n$, where φ is a non-constant continuous function on $[a, b]$, the results of b) are valid. Furthermore, we have the following Theorems 7.4 and 7.5.

Theorem 7.4. The orthonormal sequence $\{p_n\}$ satisfies a recurrence formula of the form

$$p_{n+1}(t) = [c_n\varphi(t) + d_n]p_n(t) + e_np_{n-1}(t),\ n \geq 1,$$

where c_n, d_n, e_n are real constants. (Set $p_{-1}(t) \equiv 0$).

The proof is a modification of the proof of Proposition (5.4), [37], p. 43.

Theorem 7.5. $\|p_m \cdot p_n\| \leq \rho^{-|m-n|}$ and for some coefficients c_{mni},

$$p_m(t)\cdot p_n(t) = \sum_{i=|m-n|}^{m+n} c_{mni}p_i(t).$$

Proof: Suppose $m \geq n$. Then

$$\mu_i(p_m \cdot p_n) = \int_a^b p_m(t)p_n(t)\varphi_i(t)d\alpha(t)$$

$$= \sum_{r=i}^{n+i} a_{m,r-i} \int_a^b p_m(t)\varphi^r(t)d\alpha(t).$$

From (3.29), $\mu_i(p_m \cdot p_n) = 0$ for $n + i < m$. The conclusion follows from this and the fact that p_mp_n is a polynomial in φ of degree $m + n$.

For additional properties of the orthonormal sequence, Fourier series and coefficients, see [37], Chapter 5.

d) Examples. Let $\varphi_n(t) = t^n$ and $[a, b] = [-1, 1]$. The following are three examples of moment spaces (See [37], p. 50. Also [8], [21].)

$d\alpha(t)$	$p_n(t)$	Reference
$(1-t^2)^{-\frac{1}{2}}dt$	$\frac{1}{\sqrt{\pi}}\, T_n(t)$	$T_n(t)$: Chebyshev polynomials
$(1+t^2)^{\frac{1}{2}}dt$	$\sqrt{\frac{2}{\pi}}\, U_n(t)$	$U_n(t)$: Chebyshev polynomials of the second kind
dt	$\left(n+\frac{1}{2}\right)^{\frac{1}{2}} P_n(t)$	$P_n(t)$: Legendre polynomials

To illustrate how a problem can be interpreted within the scope of a moment space, we consider the differential equation

$$(3.30) \qquad Lx = 2(t+3)\frac{dx}{dt} + x = 0, \quad x(-1) = 1.$$

Two methods have been proposed for the approximation of the solution of a differential equation which take advantage of the special properties (given in b) and c) above) of the Chebyshev polynomials. One is due to Lanczos [20], [21], the other to Clenshaw [2]. See L. Fox [13]. In both methods, the equation (3.30) is replaced by the equation

$$(3.31) \qquad My(t) = Ly(t) - \tau T_n(t) = 0, \quad y(-1) = 1.$$

It can be shown that for a certain value τ_0 of the parameter τ, (3.31) has a solution of the form

$$y(t) = \tau_0 \sum_{i=0}^{n} \beta_i T_i(t), \text{ with } \tau_0 = \left(\sum_{i=0}^{n} (-1)^i \beta_i\right)^{-1}.$$

By Theorem 7.5, if $z \in X$, then Lz and Mz belong to X and

$$\| Lz - Mz \| \leq \rho^{-n}.$$

A study of this method of substitution of a perturbed equation for the original one, if conducted within the frame of the theory of moment spaces, may lead to interesting results and interpretations.

CHAPTER 4

BOUNDED OPERATORS ON V-SPACES

4-1 Definitions and notations

In this Chapter, unless otherwise specified, X and Y will denote two V-spaces over the same field of scalars F; Z will denote a closed subset of X. An operator from Z to Y is a single valued mapping defined on all of Z with range in Y.

The conventions of pages 31 and 45 apply in this Chapter.

Definition 1.1. An operator A from Z to Y is said to be *linear on Z* if

$$A(\alpha u + \beta v) = \alpha Au + \beta Av$$

for all $\alpha, \beta \in F$ and all $u, v \in Z$ such that $\alpha u + \beta v \in Z$.

Definition 1.2. Let A be an operator from Z to Y.

(i) The *norm of A on Z*, denoted by $\|A\|_Z$ is defined by

$$\|A\|_Z = \inf\{M \geq 0 : \|Au-Av\| \leq M\|u-v\| \text{ for all } u,v \in Z\} \tag{4.1}$$

(ii) If Z = X, the *norm of A on X* is denoted by $\|A\|$, i.e. $\|A\|_X = \|A\|$.

(iii) A is said to be *bounded on Z* if $\|A\|_Z < \infty$.

It follows that $\|A\|_Z = 0$ if and only if for some fixed $y \in Y$ and all $u \in Z$, $\|Au - y\| = 0$.

If Z is a linear subspace of X and A is linear on Z, (4.1) is equivalent to

$$(4.2) \qquad \| A \|_Z = \inf\{M \geq 0 : \| Au \| \leq M \| u \| \text{ for all } u \in Z\}.$$

In X, the balls $S(\theta, r)$, $S'(\theta, r)$ are closed subspaces of X and the quotient spaces $X/S(\theta, r)$, $X/S'(\theta, r)$ are discrete V-spaces (See Theorem 1-2.1). Consequently, the norm on X of an operator, even a linear operator, cannot be determined by consideration of its values on these balls only (unless, of course, $X = S(\theta, r)$ or $X = S'(\theta, r)$). This is in striking contrast to the case of a linear operator on a Banach space ([7], [36]) where

$$\|A\| = \inf\{M \geq 0 : \|Ax\| \leq M\|x\| \text{ for all } x \in S'(\theta, r)\}$$

In a V-space, if $Z \supset Z'$, then $\| A \|_Z \geq \| A \|_{Z'}$.

$\mathfrak{O}(Z, Y)$ will denote the set of all bounded operators from Z to Y. $\mathfrak{J}(Z, Y)$ will denote the set of all bounded linear operators from Z to Y.

If $Z = X = Y$, we shall use the notations $\mathfrak{O}(X)$ and $\mathfrak{J}(X)$ in place of $\mathfrak{O}(X, X)$ and $\mathfrak{J}(X, X)$.

The product AB of two elements A, B of $\mathfrak{O}(X)$ is defined by $(AB)x \equiv A(Bx)$ for all $x \in X$. It is simple to verify that

$$(4.3) \qquad \| AB \| \leq \| A \| \cdot \| B \| .$$

In general, the product of non-linear operators does not satisfy conditions (2.9) and (2.10) of Definition 2-6.1, and hence $\mathfrak{O}(X)$ is not an algebra. These conditions are satisfied for linear operators and hence $\mathfrak{J}(X)$ is a non-commutative subalgebra of the space $\mathfrak{O}(X)$. (See [7], [36].)

O will denote the zero-operator in $\mathcal{O}(Z, Y)$: $Ou \equiv \theta \in Y$ for all $u \in Z$. I will denote the identity operator in $\mathcal{O}(X)$, i.e. $Ix \equiv x$ for all $x \in X$.

4-2 The spaces $\mathcal{O}(Z, Y)$ and $\mathcal{O}(X)$ of bounded operators

The spaces $\mathcal{O}(Z, Y)$ and $\mathcal{O}(X)$ are linear spaces over the field of scalars F. Clearly the elements of $\mathcal{O}(Z, Y)$ or of $\mathcal{O}(X)$ are continuous mappings on their domains of definition, Z or X.

The norm on Z, defined by (4.1) has the following properties:

(i) In accordance with convention (ii), page 31, both the norms on X and on Y are expressed in terms of the same real number ρ (Definition 2-1.1). It follows that the norm of an operator in $\mathcal{O}(Z, Y)$ has a norm equal to zero or to ρ^{-n} for some integer n.

(ii) $\|\alpha A\|_Z = \|A\|_Z$ for all $A \in \mathcal{O}(Z, Y)$ and all $\alpha \in F$, $\alpha \neq 0$.

(iii) $\|A + B\|_Z \leq \operatorname{Max}\{\|A\|_Z, \|B\|_Z\}$ for all $A, B \in \mathcal{O}(Z, Y)$.

Indeed, for all $u, v \in Z$:

$$\begin{aligned}\|Au + Bu - Av - Bv\| &\leq \operatorname{Max}\{\|Au - Av\|, \|Bu - Bv\|\}\\ &\leq (\operatorname{Max}\{\|A\|_Z, \|B\|_Z\})\|u - v\|.\end{aligned}$$

(iv) $\|A + B\|_Z = \operatorname{Max}\{\|A\|_Z, \|B\|_Z\}$ whenever $\|A\|_Z \neq \|B\|_Z$.

To prove this, suppose without loss of generality that $\|A\|_Z > \|B\|_Z$. Then, for every $\varepsilon > 0$ such that

$$0 < \varepsilon < (\|A\|_Z - \|B\|_Z),$$

there exist $u = u(\varepsilon)$ and $v = v(\varepsilon)$, in Z, such that

$$\|Au - Av\| > (\|A\|_Z - \varepsilon)\|u - v\| > \|B\|_Z \cdot \|u - v\|$$
$$\geq \|Bu - Bv\|.$$

Thus,

$$\|Au + Bu - Av - Bv\| = \|Au - Av\|.$$

It follows that for every $\varepsilon > 0$,

$$\|A + B\|_Z > \|A\|_Z - \varepsilon,$$

$$\|A + B\|_Z = \|A\|_Z = \mathrm{Max}\{\|A\|_Z, \|B\|_Z\}.$$

These results lead to the following theorem on the structure of $\mathcal{O}(Z, Y)$, (and of $\mathcal{O}(X)$ when $Z = X = Y$).

Theorem 2.1. The space $\mathcal{O}(Z, Y)$, under the norm on Z defined by (4.1), is a V-space.

Proof: It follows from (i) - (iv) above that $\mathcal{O}(Z, Y)$ satisfies all the defining properties of V-spaces, except possibly for completeness.

To prove the completeness of $\mathcal{O}(Z, Y)$, consider a Cauchy sequence $\{A_n\}$ in $\mathcal{O}(Z, Y)$. Since (Th. 1-4.1)

$$\lim_{n\to\infty} \|A_{n+1} - A_n\|_Z = 0,$$

for any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $u, v \in Z$ and all $n > N(\varepsilon)$,

$$\|(A_{n+1}u - A_{n+1}v) - (A_nu - A_nv)\| < \varepsilon\|u - v\|.$$

Let us select an arbitrary point x_o of Z. For each $x \in Z$, the sequence $\{A_n x - A_n x_o\}$ is a Cauchy sequence in Y; since Y is complete, this sequence has a limit. Let A be an operator from Z to Y defined by

$$Ax = \lim_{n\to\infty} (A_n x - A_n x_o), \quad x \in Z.$$

We shall show that A is a limit of $\{A_n\}$. For u, v $\in$ Z, define

$$y_{n,p}(u, v) \equiv [(A_{n+p}u - A_{n+p}x_o) - (A_n u - A_n x_o)] - [(A_{n+p}v - A_{n+p}x_o) - (A_n v - A_n x_o)].$$

For any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all u, v $\in$ Z, all $n > N(\varepsilon)$ and all $p > 0$,

$$\|y_{n,p}(u, v)\| = \|(A_{n+p}u - A_n u) - (A_{n+p}v - A_n v)\| \leq \|A_{n+p} - A_n\|_Z \cdot \|u - v\| < \varepsilon\|u - v\|.$$

With n fixed, we have, for all u, v $\in$ Z:

$$\lim_{p\to\infty} y_{n,p}(u, v) \equiv (Au - A_n u) - (Av - A_n v)$$

Since $\lim_{p\to\infty} \|y_{n,p}(u, v)\| = \|\lim_{p\to\infty} y_{n,p}(u, v)\|$, we have, for $n > N(\varepsilon)$ and all u, v $\in$ Z:

$$\|(Au - A_n u) - (Av - A_n v)\| < \varepsilon\|u - v\|.$$

Hence

$$\lim_{n\to\infty} \|A - A_n\|_Z = 0$$

This shows that A is a limit of the sequence $\{A_n\}$. As a limit of a Cauchy sequence A is bounded on Z and

$$\|A\|_Z = \lim_{n\to\infty} \|A_n\|_Z$$

We note that the operator A defined above depends on the selected point x_0. Clearly two different selections of x_0 will in general generate two distinct limits for the sequence $\{A_n\}$; the norm of the difference between two such limits is obviously zero.

<u>Remark.</u> From (ii), (iii), (iv), pp. 80, 81, and the fact that the proof of the completeness of $\mathfrak{S}(Z, Y)$ requires the completeness of Y only, we can deduce that:

(i) If X and Y are (pseudo-) valued spaces, then $\mathfrak{S}(Z, Y)$ is a pseudo-valued space;

(ii) If X is a (pseudo-) valued space and Y is a strongly (pseudo-) valued space, then $\mathfrak{S}(Z, Y)$ is a strongly pseudo-valued space.

(iii) If Y is complete, then $\mathfrak{S}(Z, Y)$ is complete.

4-3 <u>The spaces $\mathfrak{J}(Z, Y)$ and $\mathfrak{J}(X)$ of bounded linear operators</u>

$\mathfrak{J}(Z, Y)$ is the set of bounded linear operators from Z to Y. To avoid meaningless or trivial statements we shall assume that Z properly lends itself to linearity arguments. This is achieved by requiring that Z be <u>linearly non-trivial</u>, defined as follows: A subset Z of a linear space is said to be linearly non-trivial if and only if there exist u, v $\in$ Z and α, β $\in$ F such that

$$z = \alpha u + \beta v \in Z, \quad z \neq u, \quad z \neq v \quad \text{and} \ \|z\| \neq 0.$$

Obviously, any non-trivial subspace of a V-space is linearly non-trivial.

<u>Theorem 3.1.</u> The space $\mathfrak{J}(Z, Y)$ is a V-space. The space $\mathfrak{J}(X)$ is a V-algebra.

<u>Proof</u>: Using continuity of the operators involved, it is easy to verify that $\mathfrak{J}(Z, Y)$ is a closed linear subspace of $\mathfrak{O}(Z, Y)$. In $\mathfrak{J}(X)$, the product of two linear operators is a linear operator. Then, the theorem is a corollary of Theorem 2.1.

The following theorems are analogous to theorems valid in topological normed linear spaces over the real or complex fields with their usual valuations. The proofs are similar to those of the corresponding theorems in [36], pp. 18, 85-86, and are omitted. We shall use the following definition:

<u>Definition 3.2.</u> Let $A \in \mathfrak{O}(Z, Y)$. An operator A^{-1} from $A(Z)$ to Z is called a <u>pseudo-inverse of A on A(Z)</u> if $A^{-1}(Az) = z$ for all $z \in Z$.

<u>Theorem 3.3.</u> If Z is a subspace of X, then $A \in \mathfrak{J}(Z, Y)$ is continuous either at every point of Z or at no point of Z.

<u>Theorem 3.4.</u> Let Z be a subspace of X and $A \in \mathfrak{J}(Z, Y)$.

(i) A pseudo-inverse of A on $A(Z)$, when it exists, is linear on $A(Z)$.

(ii) A admits a bounded pseudo-inverse on $A(Z)$ if and only if there exists a constant $m > 0$ such that $m\|z\| \leq \|Az\|$ for all

$z \in Z$.

A linear operator from a Banach space to another is bounded if and only if it is continuous ([36], p. 85). In V-spaces boundedness implies continuity but the converse is not true. (See Example 1, p. 86). A. F. Monna ([24], Part III, p. 1136) has proved that linear operators from a V-space to its field of scalars F, considered as a V-space over itself, are continuous if and only if they are bounded. The following theorem generalizes this result; the proof is modelled after that of Monna.

Theorem 3.5. Let $A \in \mathfrak{J}(Z, Y)$ and suppose that $A(Z)$ is a discrete and bounded subspace of Y. Then, A is bounded if and only if it is continuous.

Proof: Boundedness implies continuity.

To prove the converse, suppose that A is continuous. Since $A(Z)$ is a discrete subspace, there exists $\varepsilon > 0$ such that

$$y \in A(Z) \text{ and } \|y\| < \varepsilon \text{ imply } \|y\| = 0. \tag{4.4}$$

Since A is continuous, there exists $\delta(\varepsilon)$ such that

$$z \in Z \text{ and } \|z\| < \delta(\varepsilon) \text{ imply } \|Az\| < \varepsilon,$$

and therefore, by (4.4)

$$z \in Z \text{ and } \|z\| < \delta(\varepsilon) \text{ imply } \|Az\| = 0.$$

Since $A(Z)$ is bounded, there exists $M > 0$ such that $\|y\| \leq M$ for all $y \in A(Z)$.

For all $z \in Z$ such that $\|z\| \geq \delta(\varepsilon)$, we have

$$\|Az\| \leq M = \frac{M}{\delta(\varepsilon)} \cdot \delta(\varepsilon) \leq \frac{M}{\delta(\varepsilon)} \|z\|$$

Hence, $\|A\|_Z \leq \frac{M}{\delta(\varepsilon)}$, and A is bounded.

We conclude this section with two examples. The first is an example of a continuous unbounded linear operator from a V-space to itself; the second shows that the Uniform Boundedness Principle ([36], p. 204; [7], p. 66) does not hold in V-spaces, i.e. a family of bounded linear operators on a V-space which is pointwise bounded is not necessarily uniformly bounded.

Example 1. Let X be a V-space over the real numbers, with a countable distinguished basis $H = \{h_o, h_1, h_2, \cdots\}$ such that, for some integer k,

$$\|h_n\| = \rho^{-k-n}, \quad n = 0, 1, 2, \cdots .$$

(The space $\mathcal{P}_k$ of 3-4 is such a space.)

Every non-trivial element of X has an expansion in terms of H:

$$x = \sum_{n=N}^{\infty} (\alpha_n h_{2n} + \beta_n h_{2n+1}), \quad \alpha_n, \beta_n \in R, \tag{4.5}$$

where $N \geq 0$ and $|\alpha_N| + |\beta_N| \neq 0$.

Let A be an operator from X to itself defined by

$$Ax = \begin{cases} \Theta & \text{if } \|x\| = 0, \\ \sum_{n=N}^{\infty} (\alpha_n + \beta_n) h_n & \text{if } x \text{ is given by (4.5).} \end{cases}$$

Clearly A is linear. A is unbounded since

$$\|Ah_{2n}\| = \rho^n \|h_{2n}\| \quad \text{for all } n = 0, 1, 2, \cdots .$$

Yet, given any integer $n \geq 0$,

$$A(S(\theta, \rho^{-k-2n})) \subset S(\theta, \rho^{-k-n}).$$

This shows that A is continuous at θ and, from Theorem 3.3, that A is continuous on all of X.

Example 2. Let X be as in Example 1. Every non-trivial element x of X has an expansion in terms of H:

$$(4.6) \qquad x = \sum_{n=N}^{\infty} \alpha_n h_n, \quad \alpha_N \neq 0, \quad N \geq 0.$$

For each non-negative integer p, let A_p be an operator from X to itself defined by

$$A_p x = \begin{cases} \theta & \text{if } \|x\| = 0, \\ \sum_{n=N}^{\infty} \alpha_n h_{n-p} & \text{if } x \text{ is given by } (4.6),\ N \geq p, \\ \sum_{n=N}^{p-1} \alpha_n h_o + \sum_{n=p}^{\infty} \alpha_n h_{n-p} & \text{if } x \text{ is given by } (4.6),\ N < p; \end{cases}$$

i.e.: the image of h_n is h_o if $n \leq p$ and is h_{n-p} if $n \geq p$.

The linearity of A_p is easily verified. We have

$$\|A_p x\| = \rho^{-k-N+p} = \rho^p \|x\| \quad \text{if } N \geq p \text{ in } (4.6),$$

$$\|A_p x\| = \rho^{-k} \leq \rho^p \|x\| \text{ if } N < p \text{ in } (4.6).$$

Hence: $\|A_p\| = \rho^p$, $p = 0, 1, 2, \cdots$.

This shows that the family of linear operators $\{A_p\}$ is a family of bounded linear operators which is not uniformly bounded since $\lim_{p\to\infty} \rho^p = \infty$.

Yet, the family $\{A_p\}$ is point-wise bounded since $\|A_p x\| \leq \rho^{-k}$ for all $x \in X$. We have shown that the Uniform Boundedness Principle does not hold in V-spaces. ([36], p. 204.)

4-4 Characterization of bounded linear operators

In this section X and Y are V-spaces, Z is a linear subspace of X which is not necessarily closed, H is a distinguished basis of Z.

With each element $h \in H$, let there be associated an element $Ah \in Y$ such that for some $M \geq 0$,

$$\text{(4.7)} \qquad \begin{aligned} \|Ah\| &\leq M\|h\| \quad \text{for all } h \in H, \\ \|Ah_o\| &= M\|h_o\| \quad \text{for some } h_o \in H. \end{aligned}$$

Each $z \in Z$ is a sum of an expansion in terms of H:

$$\text{(4.8)} \qquad z = \alpha_1 h_1 + \alpha_2 h_2 + \cdots, \quad \alpha_i \in F, \ \alpha_i \neq 0, \ \|h_i\| \geq \|h_{i+1}\|.$$

If $\{h_1, h_2, \cdots\}$ is infinite, then $\lim_{n\to\infty} \|h_n\| = 0$ and, by (4.7), $\lim_{n\to\infty} \|Ah_n\| = 0$.

We extend the definition of A to all of Z, by setting

$$\text{(4.9)} \qquad Az = \alpha_1 Ah_1 + \alpha_2 Ah_2 + \cdots, \quad z \text{ given by (4.8)}.$$

Since Y is complete, this series converges and Az is defined, up to addition of trivial elements.

Clearly, A is a linear operator from Z to Y. It is also bounded since, by Lemma 1-7.5,

$$\|Az\| \leq \sup_n\{\|Ah_n\|\} \leq M\cdot\sup_n\{\|h_n\|\} = M\|h_1\| = M\|z\|.$$

In view of (4.7), $\|A\|_Z = M$.

We have constructed an element of $\mathcal{J}(Z, Y)$. It is important to note that the values of A on H were completely arbitrary, except for conditions (4.7).

Now, suppose that B is a continuous linear operator and that

$$Bh - Ah \in [\theta] \quad \text{for all } h \in H.$$

Then

$$Bx - Ax \in [\theta] \quad \text{for all } x \in (H);$$

(recall that (H) is the set of all finite linear combinations of elements of H).

Since (H) is dense in Z and B - A is continuous, we must have

$$Bz - Az \in [\theta] \quad \text{for all } z \in Z.$$

This result can be stated as follows:

Theorem 4.1. Let X and Y be a V-space and let Z be a linear subspace (not necessarily closed) of X.

(i) An element A of $\mathcal{J}(Z, Y)$ is determined, up to addition of trivial elements, by its values on a distinguished basis H of Z, and

$$\|A\|_Z = \inf\{M \geq 0 : \|Ah\| \leq M\|h\| \text{ for all } h \in H\}.$$

(ii) If a single valued mapping A is arbitrarily defined on H except for the requirement that

$$\sup_{h \in H} \frac{\|Ah\|}{\|h\|} \tag{4.10}$$

be finite, then A can be extended by linearity to all of Z and $\|A\|_Z$ is equal to (4.10). Furthermore, if B is a continuous linear operator from Z to Y and $\|Bh - Ah\| = 0$ for all $h \in H$, then $B \in \mathcal{J}(Z, Y)$ and $\|B - A\|_Z = 0$.

<u>Application 1.</u> The important feature of the assertion of part (ii) of the above theorem is that, provided (4.10) is finite, the values of A on the elements of H are arbitrary.

The same is not true in an infinite dimensional Hilbert space X in which $H = \{h_1, h_2, h_3, \cdots\}$ would represent a countable complete orthonormal basis. As two examples, consider A and B defined on H by

$$Ah_n = h_1 \quad \text{for all } n, \tag{4.11}$$

$$Bh_n = h_{2^i - 1} \quad \text{for } 2^{i-1} \leq n \leq 2^i - 1. \tag{4.12}$$

Then

$$\sup_{h \in H} \frac{\|Ah\|}{\|h\|} = \sup_{h \in H} \frac{\|Bh\|}{\|h\|} = 1 < \infty.$$

However, neither A nor B can be extended by linearity to all of X: they are not defined at the point $S = \sum_{n=1}^{\infty} \frac{1}{n} h_n$.

If we suppose now that X is a V-space with distinguished basis $H = \{h_1, h_2, \cdots\}$ and $\lim_{n\to\infty} \|h_n\| = 0$, then the mapping A of (4.11) is not acceptable under Theorem 4.1 since $\sup_{h\in H} \frac{\|Ah\|}{\|h\|} = \infty$; the mapping B can be extended into an element of $\mathcal{J}(X)$ since $\sup_{h\in H} \frac{\|Bh\|}{\|h\|} = 1$.

Application 2. Theorem 4.1 finds an application in a paper of H. F. Davis [4]. We present the problem of [4] in our own terminology. The notation is that of Section 4, Chapter 3.

Let $X = Y = \mathcal{R}$. Let Z be the open subspace of $\mathcal{R}$ for which the set $\Phi_o \cup E$ (see (3.16)) is a Hamel basis.

Let A' be a single valued mapping defined on $\Phi_o \cup E$ by

$$A'\varphi_n \equiv \beta(n)\varphi_n, \quad n = 0, 1, 2, \cdots,$$

$$A'e_\alpha \equiv f_\alpha, \qquad -\infty < \alpha < \infty, \quad \alpha \neq 0,$$

where $\beta(n)$ is a scalar and f_α is an element of $\mathcal{R}$.

Since every element of Z is a unique finite linear combination of the elements of $\Phi_o \cup E$, A' can be uniquely extended by linearity to all of Z. Let B denote this extension.

The main theorem of Davis [4] asserts that a necessary and sufficient condition for B to be continuous on Z is that, in the Φ-asymptotic norm on $\mathcal{R}$:

$$(4.13) \qquad f_\alpha = \sum_{n=0}^{\infty} \frac{\alpha^n \beta(n)}{n!} \varphi_n, \quad \text{for all } \alpha \neq 0.$$

The result is obtained from Theorem 4.1 through the following argument:

Φ_o is a distinguished basis of $Z \subset \mathcal{P}_o$ (see Section 3-4). The mapping A' is defined on Φ_o in such a way that $\sup_{n \geq 0} \frac{\| A' \varphi_n \|}{\| \varphi_n \|} = 1$. Thus, by Theorem 4.1(ii), there exists a "unique" linear operator A from Z to $\mathcal{R}$ which agrees with A' on Φ_o; this extension A is such that $\| A \| = 1$ and since

$$e_\alpha = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \varphi_n,$$

we must have

$$Ae_\alpha = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} A' \varphi_n = \sum_{n=0}^{\infty} \frac{\alpha^n \beta(n)}{n!} \varphi_n .$$

Hence, the above linear operator B is continuous on Z if and only if $\| Bz - Az \| = 0$ for all $z \in Z$; i.e. B is continuous if and only if $f_\alpha = Be_\alpha = Ae_\alpha$, so that (4.13) is satisfied.

The reader will notice that the definition of continuity which we use and the definition of "asymptotic continuity" given by Davis ([4], p. 91) are different. Keeping to our own terminology, Davis calls an operator A "asymptotically continuous" if $\| x \| < \rho^n$ implies $\| Ax \| < \rho^n$. This is equivalent to saying that A is continuous if and only if $\| A \|_Z \leq 1$. However, in the example above $\| A \|_Z = 1$ and the two definitions of continuity lead to identical results.

This definition of "asymptotic continuity" is too restrictive. An operator which is asymptotically continuous is also continuous in the topological sense but the converse is not true.

The desirability of removing such restrictions is comparable to the desirability of accepting asymptotic expansions which are not of the Poincaré type (see Section 3-3,e).

Application 3. Define the linear operator L from the space $\mathcal{K}_0$ to the space $\mathcal{K}_0'$ of Section 3-5 by:

$$(Lx)(z,\ w) = \int_0^\infty \int_0^\infty e^{-zu-wv} x(u,\ v)\,du\ dv.$$

Since

$$Lx_{mn} = m!\,n!\,y_{m+1,n+1}, \quad m,\ n \geq 0,$$

and

$$\| Lx_{mn} \| = \rho^{-2} \| x_{mn} \|, \quad \text{for all } m,\ n \geq 0,$$

L can be extended (by Theorem 4.1) to all of $\mathcal{K}_0$ and $\| L \| = \rho^{-2}$.

If, in the asymptotic norm on $\mathcal{K}_0$, the function x admits the expansion

$$x = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \alpha_{n,j} x_{n-j,j}\ ,$$

then, in the asymptotic norm on $\mathcal{K}_0'$,

$$Lx = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \alpha_{n,j} (n-j)!\,j!\,y_{n-j+1,j+1}\ .$$

This result provides means to obtain asymptotic expansions of Laplace transforms in two variables. See [6]. For example, since $J_0(\sqrt{uv})$ has the asymptotic expansion:

$$J_0(\sqrt{uv}) = \sum_{n=0}^{\infty} \frac{u^n v^n}{2^{2n}(n!)^2}\ ,$$

$$LJ_o(\sqrt{uv}) = \frac{1}{zw} \sum_{n=0}^{\infty} \frac{1}{(4zw)^n} = \frac{4}{4zw-1} .$$

(Compare with [6], p. 100.)

4-5 Inverses and spectra in $\mathfrak{J}(X)$

The V-space $\mathfrak{J}(X)$ is a V-algebra.

In accordance with Definition 2-6.2, a pseudo-identity in $\mathfrak{J}(X)$ is a linear operator I' such that $\| I' - I \| = 0$.

The definition of a pseudo-inverse A^{-1} of A on its range $A(X)$ has been given (Definition 3.2). The operator A^{-1} belongs to $\mathfrak{J}(X)$ if and only if it is bounded and defined on all of X. Therefore, A is (pseudo-) regular in the sense of Definition 2-6.3 if and only if it admits a bounded (pseudo-) inverse and $A(X) \equiv X$. In such a case, we shall say that A admits a (pseudo-) inverse A^{-1}, without any mention of the range of A.

Let $A_\lambda \equiv A - \lambda I$. By Definition 2-6.7, λ belongs to the spectrum $\sigma(A)$ of A if and only if A_λ is singular.

Theorems 6.6, 6.8 and 6.9 of Chapter 2 apply to the V-algebra $\mathfrak{J}(X)$, (with $x \in X$ replaced by $A \in \mathfrak{J}(X)$ and e replaced by I). The formulation of these three theorems for bounded linear operators on a V-space should be compared with similar theorems for bounded linear operators on Banach spaces: see [36], Theorems 4.1-C and 4.1-D, page 164, and Theorem 5.1-A, page 256.

Note: As in Theorem 5.1-A of [36], we can add to the statement of Theorem 2-6.9 the following precision: Let $A \in \mathfrak{J}(X)$, $\|A\| > 1$; if for some scalar $\mu \in F$, A_μ has a pseudo-inverse A_μ^{-1} on its range $A_\mu(X)$ and $\|A_\mu^{-1}\| < 1$, then, for all $\lambda \in F$, A_λ has a pseudo-inverse on its range $A_\lambda(X)$ and the topological closure of the range of A_λ is not a proper subset of the topological closure of the range of A_μ.

The proof is identical to that given in [36], p. 256. Our modification of Riesz's Lemma (Theorem 1-6.1) must be used.

4-6 Complete spectral decompositions

The scalar λ is called an eigenvalue of $A \in \mathfrak{J}(X)$ if for some $x_\lambda \in X$, $\|x_\lambda\| \neq 0$, $Ax_\lambda = \lambda x_\lambda$. The point x_λ is called an eigenelement associated with λ. The set

$$X_\lambda = \{x \in X : Ax = \lambda x\}$$

is a closed subspace of X and is called the eigenspace associated with λ.

Definition 6.1. An operator $A \in \mathfrak{J}(X)$ is said to have the complete spectral decomposition $\{(\lambda_i, h_i) : i \in J\}$ if for each i in the index set J, $Ah_i = \lambda_i h_i$, not all λ_i are equal to 0 and the set of eigenelements $H = \{h_i : i \in J\}$ is a distinguished basis of X.

Theorem 6.2. If A has a complete spectral decomposition $\{(\lambda_i, h_i) : i \in J\}$, then:

(i) $\|A\| = 1$;

(ii) For all $\lambda \notin \{\lambda_i : i \in J\}$, A_λ is pseudo-regular and $\|A_\lambda\| = \|A_\lambda^{-1}\| = 1$.

(iii) If $\lambda_i \neq 0$ for each $i \in J$, then A is an isometry on X, i.e. $\|Ax\| = \|x\|$ for all $x \in X$.

Proof: (i) The operator A satisfies (4.7) with $M = 1$. Thus, by Theorem 4.1, $\|A\| = 1$.

(ii) Let x be an arbitrary point in X. It admits a non-increasing expansion in terms of H:

$$(4.14) \qquad x = \sum_{n=0}^{\infty} \alpha_n h_n, \quad \alpha_n \in F, \quad \alpha_o \neq 0, \quad h_n \in H.$$

By Lemma 1-7.5, $\|x\| = \|h_o\|$.

Then,

$$(4.15) \qquad A_\lambda x = \sum_{n=0}^{\infty} \alpha_n (\lambda_n - \lambda) h_n.$$

If $\lambda \notin \{\lambda_i : i \in J\}$, $\|A_\lambda x\| = \|h_o\| = \|x\|$. It follows from Theorem 4.1, that the operator A_λ^{-1}, defined on H by

$$A_\lambda^{-1} h_i = \frac{1}{\lambda_i - \lambda} h_i, \quad h_i \in H,$$

is a pseudo-inverse of A_λ. If x is given by (4.14),

$$A_\lambda^{-1} x = \sum_{n=0}^{\infty} \alpha_n \frac{1}{\lambda_n - \lambda} h_n.$$

Thus, $\|A_\lambda^{-1} x\| = \|h_o\| = \|x\|$.

(iii) follows from (4.14) and (4.15) with $\lambda = 0$, $\lambda_n \neq 0$ for each $n \in J$.

<u>Corollary 6.3.</u> If $A \in \mathcal{J}(X)$ admits a complete spectral decomposition, then the cardinality of the set of its eigenvalues cannot exceed the dimension of the space.

<u>Proof</u>: Let $\{(\lambda_i, h_i) : i \in J\}$ be a complete spectral decomposition of A. If the cardinality of the set of eigenvalues exceeds the dimension of the space, i.e. the cardinality of the distinguished basis $\{h_i : i \in J\}$, there exists an eigenvalue λ which does not belong to $\{\lambda_i : i \in J\}$. Since λ is an eigenvalue, A_λ is singular. This contradicts (ii) of Theorem 6.2.

In the following Lemma 6.4 and Theorem 6.5, the assumptions and notations are as follows:

$A \in \mathcal{J}(X)$ admits a complete spectral decomposition $\{(\lambda_i, h_i) : i \in J\}$. $H = \{h_i : i \in J\}$.
For an arbitrary scalar λ,

$$J_\lambda = \{i \in J : \lambda_i = \lambda\}, \quad H_\lambda = \{h_i \in H : \lambda_i = \lambda\}.$$

X_λ denotes the closed subspace generated by H_λ. Clearly, if λ is not an eigenvalue, by Theorem 6.2(ii), $\lambda \neq \lambda_i$ for each $i \in J$ and, therefore, J_λ, H_λ and X_λ are empty. If λ is an eigenvalue, then $\lambda = \lambda_i$ for some $i \in J$ and X_λ is the non-empty eigenspace associated with λ.

Let P_λ denote a linear operator from X to X, defined on H by

$$P_\lambda h = \begin{cases} h & \text{if } h \in H_\lambda, \\ \theta & \text{if } h \in H \setminus H_\lambda. \end{cases}$$

By Theorem 4.1, $\| P_\lambda \| = 1$ if X_λ is not empty and $\| P_\lambda \| = 0$ if X_λ is empty.

Lemma 6.4. For all $x \in X$ and all scalars λ:

$$\| x - P_\lambda x \| = \| Ax - \lambda x \|.$$

Proof: Given $x \in X$, x admits an expansion of the form

$$x = \sum_{h_i \in H_\lambda} (x, h_i)_H \, h_i + \sum_{h_i \in H \setminus H_\lambda} (x, h_i)_H \, h_i$$

(For notation, see p. 28). Thus,

$$\| x - P_\lambda x \| = \Big\| \sum_{h_i \in H \setminus H_\lambda} (x, h_i)_H \, h_i \Big\|$$

$$\| Ax - \lambda x \| = \Big\| \sum_{h_i \in H \setminus H_\lambda} (\lambda_i - \lambda)(x, h_i)_H \, h_i \Big\|$$

By Lemma 1-7.5,

$$\| x - P_\lambda x \| = \| Ax - \lambda x \| = \begin{cases} 0 & \text{if } (x, h_i)_H = 0 \text{ for all } h_i \in H \setminus H_\lambda, \\ \sup\limits_{\substack{h_i \in H \setminus H_\lambda \\ (x, h_i) \neq 0}} \| h_i \| & \text{otherwise} \end{cases}$$

Lemma 6.4 is the equivalent, in V-spaces, of a theorem of C. A. Swanson, valid for Hilbert spaces: Theorem 1 of [34], Theorem 2 of [35]. This lemma is used to prove the following comparison theorem:

Theorem 6.5. Let $B \in \mathcal{J}(X)$ and suppose that λ is an eigenvalue of B, with the associated eigenspace Y_λ. If $\| B - A \| < 1$, then:

(i) λ is also an eigenvalue of A,

(ii) the dimension of Y_λ is less than or equal to the dimension of X_λ.

Proof: Let H_λ' be a distinguished basis for Y_λ. By Lemma 6.4, we have for each $h' \in H_\lambda'$:

$\|h' - P_\lambda h'\| = \|Ah' - \lambda h'\| = \|Ah' - Bh'\| \leq \|h'\|$.

Therefore,

$$\|P_\lambda h'\| = \|h'\| \neq 0.$$

Hence, X_λ is non-trivial and (i) is proved.

By Theorem 2-2.4 (Paley-Wiener Theorem), the set $P_\lambda H'$ is a distinguished subset of X_λ and, by Corollary 2-2.3, the cardinality of a distinguished basis of X_λ is greater than or equal to the cardinality of $P_\lambda H'$. (ii) follows.

The following corollaries are immediate:

Corollary 6.6. If both A and B admit complete spectral decompositions and $\|B - A\| < 1$, then

(i) A and B have the same eigenvalues,

(ii) for each eigenvalue λ, the associated eigenspaces for A and for B have the same dimensions.

Corollary 6.7. Suppose that A admits a complete spectral decomposition and $\|B - A\| < 1$. If λ is an eigenvalue of A but is not an eigenvalue of B, then B does not admit a complete spectral decomposition.

Example 1. This example shows that the converse of Theorem 6.5(i) is not true, i.e. if A has a complete spectral decomposition and $\|B - A\| < 1$, an eigenvalue of A is not necessarily an eigenvalue of B.

Let $H = \{h_i : i = 0, 1, 2, \ldots\}$ be a distinguished basis of a V-space X, with $\|h_i\| > \|h_{i+1}\|$ for all $i \geq 0$. Define A and B by their values on H (Theorem 4.1):

$$Ah_i = h_i \text{ if } i \text{ is even}, \; Ah_i = \theta \text{ if } i \text{ is odd},$$

$$Bh_i = h_i \text{ if } i \text{ is even}, \; Bh_i = h_{i+2} \text{ if } i \text{ is odd}.$$

A admits a complete spectral decomposition and, by Theorem 6.2, its only eigenvalues are 0 and 1.

Let $x \in X$. Then for some $N = N(x)$:

$$x = \sum_{i \geq N} (\alpha_i h_{2i} + \beta_i h_{2i+1}), \; \alpha_i \in F, \; \beta_i \in F, \; |\alpha_N| + |\beta_N| \neq 0,$$

$$\|x\| > \|h_{2N+2}\|,$$

$$Ax = \alpha_N h_{2N} + \sum_{i \geq N+1} \alpha_i h_{2i},$$

$$Bx = \alpha_N h_{2N} + \sum_{i \geq N+1} \alpha_i h_{2i} + \sum_{i \geq N} \beta_i h_{2i+3},$$

$$\|Ax - Bx\| = \Big\| \sum_{i \geq N} \beta_i h_{2i+3} \Big\| \leq \|h_{2N+2}\|.$$

Thus, $\|A - B\| < 1$.

In accordance with Theorem 6.5(i), the eigenvalue 1 of B is also an eigenvalue of A. It is easily verified that the eigenvalue 0 of A is not an eigenvalue of B. It follows from

Corollary 6.7 that B does not have a complete spectral decomposition.

Example 2. The following are linear operators on the space $\mathcal{P}_k$ of 3-4, for $k \geq 0$.

(i) $$(\mathcal{L} x)(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-\frac{t}{\lambda}} x(t)dt.$$

$\mathcal{L}$ admits the complete spectral decomposition $\{(n!, \varphi_n); n = k, k+1, k+2, \cdots\}$ since $\mathcal{L}\varphi_n = n!\varphi_n$.

$\mathcal{L}$ has arbitrarily large eigenvalues and was studied by T. E. Hull [15].

Compare $\mathcal{L}$ with the Laplace Transform ([8], Vol. I):

$$x(\lambda) \to \int_0^\infty e^{-t\lambda} x(\lambda)d\lambda.$$

(ii) For $\mu > 0$,

$$(\mathcal{M}_\mu x)(\lambda) = \int_0^\lambda (\lambda - t)^{\mu-1} \frac{x(t)}{t^\mu} dt.$$

$\mathcal{M}_\mu$ admits the complete spectral decomposition

$$\left\{ \left(\frac{\Gamma(\mu)\Gamma(n+1)}{\Gamma(\mu+n+1)}, \varphi_n \right); n = k, k+1, k+2, \cdots \right\}.$$

Compare $\mathcal{M}_\mu$ with the Riemann-Liouville fractional integral ([8], Vol. II):

$$x(\lambda) \to \left[\Gamma(\mu)\right]^{-1} \int_0^t (t - \lambda)^{\mu-1} x(\lambda)d\lambda.$$

(iii) For $K \geq 1$,

$$(\mathcal{C}x)(\lambda) = \int_0^{\lambda} J_0(\lambda - t)\, \frac{x(t)}{t}\, dt.$$

$\mathcal{C}$ admits the complete spectral decomposition $\{(\frac{1}{n}, J_n) : n = K, K+1, K+2, \dots\}$. Concerning this convolution product, see, for example, Mikusiński [23], pp. 174-178 and p. 456.

Since none of the above operators $\mathcal{L}$, $\mathcal{M}_\mu$, $\mathcal{C}$ has eigenvalue 0, they are isometries:

$$\|\mathcal{L}x\| = \|\mathcal{M}_\mu x\| = \|\mathcal{C}x\| = \|x\| \text{ for all } x \in \mathcal{P}_k.$$

From Theorem 6.2, they have no other eigenvalues than those given in their respective spectral decompositions above.

4-7 Note on projections

A. F. Monna [24], [25] has introduced a notion of projection in non-Archimedean normed linear spaces. In the special case of V-spaces we have the following:

Definition 7.1. Let Y be a closed subspace of a V-space X. An operator $P \in \mathfrak{J}(X)$ is called a projection on Y if for all $x \in X$, $Px \in Y$ and

(4.16) $\|x - Px\| \leq \|x - y\|$ for all $y \in Y$.

Theorems on projections and comparisons with projections in Hilbert space theory [7], [36] will be found in Monna [24], Part IV and [25], Part I.

The existence and non-uniqueness of projections on a given subspace Y of X were proved by Monna. The proofs of Monna do not involve explicitly the use of distinguished bases. We give here an alternate and simple proof.

Let $H(Y)$ be a distinguished basis of Y and H be an arbitrary extension of $H(Y)$ to all of X. Denote by Z the closed subspace generated by $H \setminus H(Y)$.

Define the linear operator P on X by its values on H (Theorem 4.1):

$$Ph \equiv h \text{ if } h \in H(Y), \quad Ph \equiv \Theta \text{ if } h \in H \setminus H(Y).$$

By Theorem 2-4.4 and Corollary 2-3.4, the spaces Y and Z are distinguished complements of one another. Therefore for each $x \in X$, there exist $y_x \in Y$ and $z_z \in Z$ such that $x = y_x + z_x$. Since the restriction of P to Y is the identity mapping and its restriction to Z is the 0-operator:

$$Px = Py_x + Pz_x = y_x \in Y,$$

and

$$\| x - Px \| = \| x - y_x \| = \| z_x \|.$$

For an arbitrary $y \in Y$, $y - y_x \in Y$ and, since Y and Z are distinguished subsets of X:

$$\| x - y \| = \| (y_x - y) + z_x \| = \mathrm{Max}\{\| y - y_x \|, \| z_x \|\}$$

Hence (4.16) is satisfied. This proves that P is a projection on Y.

The non-uniqueness of the projections on Y is a consequence of the non-uniqueness of the extensions H of H(Y).

Remark: The operator P_λ of Lemma 6.4 is a projection on X_λ (see page 98).

CHAPTER 5

Solution of Equations

5-1 Introduction

The problem studied by C. A. Swanson and M. Schulzer in [32] and [33] is that of the existence and the approximation of a class of equations in Banach spaces.

In this Chapter we generalize Theorems 4 and 5 of [33] to arbitrary V-algebras and V-spaces. The hypotheses of [33] are slightly weakened.

5-2 Equations in V-algebras

In this Section, X is a V-algebra.

We consider two points, $x, y \in X$ which have the following finite or infinite expansions:

$$x = x_o + x_1 + x_2 + \cdots,$$
$$y = y_o + y_1 + y_2 + \cdots,$$

and we assume that x_o admits a pseudo-inverse x_o^{-1} such that

$$\|x - x_o\| < \|x_o^{-1}\|^{-1}. \tag{5.1}$$

It follows from Theorem 2-6.6 that:

Theorem 2.1. The element x admits a pseudo-inverse x^{-1} and the equation $xw = y$ admits a pseudo-solution $z = x^{-1}y$ (i.e. $xz = y$).

The problem is to make use of the known expansions of x and y to obtain approximations to z and x^{-1}, as defined in the above theorem. The sequences $\{z_n\}$ and $\{u_n\}$ defined by

(5.2) $$z_o = x_o^{-1} y_o, \quad z_n = x_o^{-1}\Big(\sum_{i=0}^{n} y_i - \sum_{i=1}^{n} x_i z_{n-i} \Big),$$

(5.3) $$u_o = x_o^{-1}, \quad u_n = x_o^{-1}\Big(e - \sum_{i=1}^{n} x_i u_{n-i} \Big),$$

will be shown to approximate z and x^{-1}, respectively, provided the rates of convergence of the series $\sum x_n$ and $\sum y_n$ satisfy certain conditions.

More precisely, we shall consider two sets of assumptions on the rates of convergence of the series $\sum x_n$ and $\sum y_n$ and, under these assumptions we shall obtain upper bounds for the values of $\|z - z_n\|$ and $\|x^{-1} - u_n\|$.

In the first case we assume that

(5.4a) $$\|x_n\| \leq \rho^{-n} \|x_o^{-1}\|^{-1} \qquad \text{for } n \geq 1,$$

(5.4b) $$\|y_n\| \leq \rho^{-n} \|y_o\| \qquad \text{for } n \geq 1.$$

In the second case, our assumptions are that

(5.5a) $$\|x_o\| \geq \|x_1\| \geq \|x_2\| \geq \cdots,$$

(5.5b) $$\|x_n\| \cdot \|x_m\| \leq \|x_{m+n}\| \cdot \|x_o^{-1}\|^{-1} \quad \text{for all } n, m \geq 1 \text{ such that } \|x_{m+n}\| \neq 0,$$

(5.5c) $$\|y_o\| \geq \|y_n\| \quad \text{for all } n \geq 1,$$

(5.5d) $$\|y_n\| \leq \|x_{n-1}\| \cdot \|x_o^{-1}\| \cdot \text{Max}\{\|x_o\|, \|y_o\|\} \quad \text{for all } n \geq 1.$$

The interest of the second case lies in its applicability in V-algebras which admit distinguished bases with many elements having the same norm (e.g. the V-algebra $\mathcal{K}_o$ of 3-5). In such cases, the norms of the terms in the expansions of x or y will not necessarily decrease as rapidly as required by (5.4), and to sum up the terms having the same norms may be inconvenient or difficult.

Theorem 2.2. (i) If (5.4a) and (5.4b) hold, then the sequence $\{z_n\}$ defined by (5.2) converges to z and

(5.6) $$\| z - z_n \| \leq \rho^{-n} \| x_o^{-1} \| \| y_o \| \quad \text{for all } n = 0, 1, 2, \cdots .$$

(ii) If (5.5a), (5.5b), (5.5c) and (5.5d) hold, then

(5.7) $$\| z - z_n \| \leq \| x_n \| \| x_o^{-1} \|^2 \operatorname{Max}\{\| x_o \|, \| y_o \|\} \quad \text{for all n such that } \| x_n \| \neq 0;$$

if for all integers n, $\| x_n \| \neq 0$, then $\{z_n\}$ converges to z.

Before proving the theorem, we note that if $y_o \equiv e$ and $y_n \equiv \theta$ for all $n \geq 1$, then (5.4b), (5.5c) and (5.5d) are satisfied and, hence, the following corollary is deduced from Theorem 2.2:

Corollary 2.3. (i) If (5.4a) holds, then the sequence $\{u_n\}$ defined by (5.3) converges to x^{-1} and

$$\| x^{-1} - u_n \| \leq \rho^{-n} \| x_o^{-1} \| \quad \text{for all } n = 0, 1, 2, \cdots .$$

(ii) If (5.5a) and (5.5b) hold, then

$$\|x^{-1} - u_n\| \le \|x_n\|\|x_o^{-1}\|^2 \operatorname{Max}\{\|x_o\|, 1\}$$

for all n such that $\|x_n\| \ne 0$;

if for all integers n, $\|x_n\| \ne 0$, then $\{u_n\}$ converges to x^{-1}.

Proof of Theorem 2.2. One verifies directly that

$$z = x_o^{-1}[y-(x-x_o)z] = x_o^{-1}\Big[\sum_{i\ge 0} y_i - \Big(\sum_{i\ge 1} x_i\Big) z\Big] .$$

Thus,

$$\|z-z_n\| = \|x_o^{-1}\Big(\sum_{i\ge n+1} y_i\Big) - x_o^{-1}\Big(\sum_{i\ge n+1} x_i\Big)z - $$
$$- x_o^{-1}\Big(\sum_{i=1}^{n} x_i(z-z_{n-i})\Big)\| ,$$

and

(5.8) $\|z - z_n\| \le \operatorname{Max}\{\alpha_n, \beta_n, \gamma_n\}$, where

$$\alpha_n = \|x_o^{-1}\|\cdot\|\sum_{i\ge n+1} y_i\|,$$

$$\beta_n = \|x_o^{-1}\|\cdot\|\sum_{i\ge n+1} x_i\|\cdot\|z\|,$$

$$\gamma_n = \|x_o^{-1}\|\cdot\|\sum_{i=1}^{n} x_i(z - z_{n-i})\|.$$

Both (5.4b) and (5.5c) imply $\|y\| \le \|y_o\|$; hence, from Theorem 2-6.6(ii) and the relation $1 \le \|x_o\|\|x_o^{-1}\|$:

(5.9) $\|z\| \le \|x^{-1}\|\|y\| \le \|x_o^{-1}\|\|y_o\|,$

$$\|z-z_o\| \le \|z-x_o^{-1}y_o\| \le \|x_o^{-1}\|\cdot\|y_o\| \le \|x_o\|\|x_o^{-1}\|^2 \operatorname{Max}\{\|x_o\|, \|y_o\|\}.$$

This shows that both (5.6) and (5.7) are satisfied for $n = 0$. We complete the proof by induction, and for each set of assumptions separately.

(i) In the first case, suppose that (5.6) is satisfied for $n = 0, 1, 2, \cdots, m-1$.

From (5.4b) and Theorem 1-4.2(ii):

$$\alpha_m \leq \|x_o^{-1}\| \|y_{m+1}\| \leq \rho^{-m-1} \|x_o^{-1}\| \|y_o\|;$$

from (5.4a), Theorem 1-4.2(ii) and (5.9):

$$\beta_m \leq \|x_o^{-1}\| \|x_{m+1}\| \|z\| \leq \rho^{-m-1} \|x_o^{-1}\| \|y_o\|;$$

from (5.2a) and the induction hypothesis:

$$\gamma_m \leq \|x_o^{-1}\| \operatorname*{Max}_{1 \leq i \leq m} \{\|x_i\| \|z-z_{m-i}\|\} \leq \rho^{-m} \|x_o^{-1}\| \|y_o\|.$$

It follows from (5.8) and these three inequalities that (5.6) holds for $n = m$ and hence for all n.

The convergence of ρ^{-n} to 0 implies that $\lim_{n \to \infty} \|z-z_n\| = 0$ and, consequently, $\{z_n\}$ converges to z.

(ii) In the second case, we note that (5.5a) implies that when $\|x_m\| \neq 0$, then $\|x_n\| \neq 0$ for $n = 0, 1, \cdots, m-1$. Suppose that (5.7) holds for $n = 0, 1, \cdots, m-1$. Then an argument similar to that conducted in the first case shows that (5.7) holds also for $n = m$.

If for each integer n, $\|x_n\| \neq 0$, the convergence of the series $\sum x_n$ implies, as in the first case, the convergence of $\{z_n\}$ to z.

Application. Let Z be an arbitrary V-space. Let $\{A_n : n = 0, 1, 2, \cdots\}$ be a sequence of linear operators in the V-algebra $\mathcal{J}(Z)$.

Assume that, in the norm of $\mathcal{J}(Z)$,

$$A = A_o + A_1 + A_2 + \cdots .$$

Assume also that A_o is pseudo-regular, with pseudo-inverse A_o^{-1}, and that

(5.10) $$\| A - A_o \| < \| A_o^{-1} \|^{-1} ,$$

(5.11) $$\| A_n \| \leq \rho^{-n} \| A_o^{-1} \|^{-1} .$$

Under these assumptions the equation

(5.12) $$Az = w, \qquad z, w \in Z$$

has a solution z for each $w \in Z$.

Indeed, from Corollary 2.3, A has a pseudo-inverse A^{-1}, so that $z = A^{-1}w$ is a solution of (5.12).

Furthermore, it follows from (5.3) that A^{-1} is a limit of the sequence $\{B_n\}$:

$$B_o = A_o^{-1}, \quad B_n = A_o^{-1} \left(I - \sum_{i=1}^{n} A_i B_{n-i} \right)$$

and, by the linearity of the operators A_o^{-1}, z is a limit of the sequence $\{z_n\}$:

$$z_o = A_o^{-1} w, \quad w_n = B_n w = A_o^{-1} \left(w - \sum_{i=1}^{n} A_i w_{n-i} \right) .$$

(Compare this result with the results of Section 4 below.)

5-3 The equation $Ax = y$

In this section, X and Y are V-spaces, $A \in \mathcal{O}(Z, Y)$.

Definition 3.1. Let $y \in Y$ and $D \subset X$.

(i) The equation $Ax = y$ is said to have the pseudo-solution z in D if $z \in D$ and $Az = y$.

(ii) The equation $Ax = y$ is said to have a unique pseudo-solution in D if it has at least one pseudo-solution z in D and if $z' = z$ for all pseudo-solutions in D.

We consider the linear operator $A_o \in \mathcal{J}(X, Y)$ and assume that A_o has a bounded pseudo-inverse A_o^{-1} on its range $A_o(X)$. The operator A_o^{-1} is linear. (See Theorem 4-3.4).

Theorem 3.2. Let $y_o \in A_o(X)$ and $u = A_o^{-1}y_o$. If there exists a ball $D = S'(u, r)$, $r > 0$, such that $K = \| A_o^{-1} \|_{A_o(D)}$ and such that the following conditions (5.13) and (5.14) are satisfied:

$$(5.13) \qquad \| A - A_o \|_D < K^{-1};$$

$$(5.14) \qquad (A - A_o)x \in S'(\theta, rK^{-1}) \subset A_o(X) \quad \text{for all } x \in D;$$

then, for all $y \in A_o(D)$ the equation $Ax = y$ has a unique pseudo-solution z in D. Furthermore, the sequence $\{z_n\}$ defined by

$$(5.15) \qquad z_o = A_o^{-1}y, \quad z_n = A_o^{-1}y - A_o^{-1}(A - A_o)z_{n-1},$$

converges to z.

<u>Proof:</u> Let $y' = y - y_o$. Since $y \in A_o(D)$, $A_o^{-1}y \in D$ and

(5.16) $\quad \|A_o^{-1}y'\| = \|A_o^{-1}y - u\| \leq r.$

Since, from (5.14), $(A - A_o)x \in A_o(X)$ for all $x \in D$, the equation

(5.17) $$Ax = y = y_o + y'$$

is equivalent, for $x \in D$, to the equation

(5.18) $$Lx = x, \quad \text{where}$$

$$Lx = u + A_o^{-1}y' - A_o^{-1}(A - A_o)x, \quad x \in D.$$

From (5.14)

$$\|A_o^{-1}(A - A_o)x\| \leq K \cdot rK^{-1} = r \quad \text{for all } x \in D.$$

From this inequality and (5.16), it follows that

$$\|Lx - u\| = \|A_o^{-1}y' - A_o^{-1}(A - A_o)x\| \leq r \quad \text{for all } x \in D.$$

Thus, L maps D into itself.

From (5.13), we have, for all x_1, $x_2 \in D$:

$$\|Lx_1 - Lx_2\| = \|A_o^{-1}(A - A_o)x_1 - A_o^{-1}(A - A_o)x_2\|$$

$$\leq K \cdot \|A - A_o\|_D \|x_1 - x_2\| < \|x_1 - x_2\|.$$

Since 0 is the only accumulation point of the norm range of a V-space, it follows that

$$\|L\|_D \leq \rho^{-1} < 1.$$

The contraction mapping principle ([19], Vol. I, p. 43) can be applied to L on the closed sphere D, to conclude that the equation (5.17) and, hence, the equation (5.18) have a unique pseudo-solution z in D.

The contraction mapping principle also asserts that the sequence $\{z_n\}$ defined by

$$z_o = A_o^{-1}y, \qquad z_n = Lz_{n-1}$$

converges to the pseudo-solution z.

Since $K = \|A_o^{-1}\|_{A_o(D)} \leq \|A_o^{-1}\|_{A_o(X)}$ and $\|A-A_o\|_D \leq \|A-A_o\|_X$, we see that the theorem holds if, in (5.13), $\|A - A_o\|_D$ is replaced by $\|A - A_o\|_X$ and/or if, in one or both of (5.13) and (5.14), K is replaced by $\|A_o^{-1}\|_{A_o(X)}$.

This theorem extends Theorem 4 of [33] (Th. 7.1 of [32]) to arbitrary V-spaces.

<u>Application</u>. For some integer $k \geq 1$, let $X = Y = \mathcal{P}_k$, where $\mathcal{P}_k$ is defined in Section 3-4.

We consider an operator $F \in \mathcal{O}(\mathcal{P}_k)$ such that

$$0 < \|I - F\| < 1. \tag{5.19}$$

(Examples of such operators are $F = F_n$ where $F_n x = x + x^n$, $n = 2, 3, \cdots$, or $F_n x = x(1 + \varphi_n)$, $n = 1, 2, \cdots$.)

Let $\mathcal{L}$ be the operator defined in Section 4-6, page 101; namely:

$$(\mathcal{L}x)(\lambda) = \int_o^\infty \frac{1}{\lambda} e^{-\frac{t}{\lambda}} x(t)dt.$$

Consider the equation

$$(5.20) \qquad y + \mathcal{L}Fx = \alpha x,$$

(i.e.: $y(\lambda) + \int_0^\infty \frac{1}{\lambda} e^{-\frac{t}{\lambda}} F(x(t))\,dt = \alpha x(\lambda)$), where $y \in \mathcal{P}_k$ and α is a real number.

We shall apply Theorem 3.2 to prove that (5.20) has a unique pseudo-solution in $\mathcal{P}_k$ when

$$(5.21) \qquad \alpha \neq n! \quad \text{for each integer } n \geq k.$$

Define, for $x \in \mathcal{P}_k$:

$$Ax = \alpha x - \mathcal{L}Fx = (\alpha I - \mathcal{L}F)x$$

$$A_o x = \alpha x - \mathcal{L}x = (\alpha I - \mathcal{L})x$$

The equation (5.20) is equivalent to the equation

$$(5.22) \qquad Ax = y$$

It follows from the results of page 101, 4-6, that if $\alpha \neq n!$ for each integer $n \geq k$, $A_o = \alpha I - \mathcal{L}$ is pseudo-regular and that its pseudo-inverse A_o^{-1} is defined on all of $\mathcal{P}_k$, with $\| A_o^{-1} \| = \| A_o \| = 1$.

To apply Theorem 3.2, select $y_o = u = 0$ and $r = \rho^{-k}$. Then, $D = \mathcal{P}_k$.

Since $A - A_o = \mathcal{L}(I - F)$ and $\| \mathcal{L} \| = 1$, we have from (5.19)

$$\| A - A_o \| \leq \| \mathcal{L} \| \| I - F \| < 1 = \| A_o^{-1} \|^{-1},$$

and hence, (5.13) is satisfied.

Clearly (5.14) is also satisfied since $(A - A_o)$ maps $\mathcal{P}_k$ into itself and since $\| A_o^{-1} \| = 1$.

The conclusion is that (5.22) and (5.20) have a unique pseudo-solution z in $\mathcal{P}_k$ when (5.21) holds. Furthermore, z is a limit of the sequence $\{z_n\}$ defined by

$$z_o = A_o^{-1}y, \qquad z_n = A_o^{-1}y - A_o^{-1}\mathcal{L}(I - F)z_{n-1}.$$

Other examples of applications of Theorem 3.2 will be found in [32].

5-4 The equation Ax = y involving expansions of A and y.

As in the previous Section, X and Y are V-spaces, $A \in \mathcal{O}(X, Y)$ and we consider the equation $Ax = y$. However, we now suppose that A and y are known from their finite or infinite expansions

$$A = A_o + A_1 + A_2 + \cdots$$
$$y = y_o + y_1 + y_2 + \cdots$$

We assume that $A_n \in \mathcal{O}(X, Y)$ for n = 1, 2, $\cdots$; that $A_o \in \mathcal{J}(X, Y)$ and that $y_o \in A_o(X)$. We also assume that A_o has a pseudo-inverse A_o^{-1} on its range $A_o(X)$. Let $u = A_o^{-1}y_o$.

Suppose that there exists a ball $D = S'(u, r)$, $r > 0$, such that $K = \| A_o^{-1} \|_D$ and such that the following conditions (5.23) - (5.26) are satisfied:

(5.23) $\qquad \| A - A_o \|_D < K^{-1}$;

(5.24) $\qquad \| A_n x \| \leq rK^{-1} \min\{1, \| A_n \|_D\}$ for all $n \geq 1$ and all $x \in D$;

(5.25) $\|y_n\| \leq rK^{-1}$ for all $n \geq 1$;

(5.26) In Y, the ball $S'(\theta, rK^{-1})$ is contained in $A_o(X)$.

<u>Theorem 4.1.</u> Under the conditions above, the equation $Ax = y$ has a unique pseudo-solution z in D.

<u>Proof</u>: The convergence on D of the series $\sum_{n\geq 0} A_n$ implies that $\lim_{n\to\infty} \|A_n\|_D = 0$. Therefore, from (5.24), $\lim_{n\to\infty} \|A_n x\| = 0$ for all $x \in D$ and, hence, the series $\sum_{n\geq 1} A_n x$ is convergent on D. Then, it follows, also from (5.24) that

(5.27) $\|(A - A_o)x\| = \|\sum_{n\geq 1} A_n x\| \leq rK^{-1}$ for all $x \in D$.

A consequence of (5.25) and (5.26) is that $y_n \in A_o(X)$ for all $n \geq 1$ and that

$$\|A_o^{-1} y_n\| \leq K \cdot rK^{-1} = r.$$

From the linearity of A_o we conclude that $y \in A_o(X)$ and the last inequality gives

$$\|A_o^{-1} y - u\| = \|A_o^{-1}(y - y_o)\| = \|\sum_{n\geq 1} A_o^{-1} y_n\| \leq r.$$

Hence:

(5.28) $y \in A_o(D)$.

The relations (5.23), (5.24), (5.27) and (5.28) establish the applicability of Theorem 3.2. Thus, $Ax = y$ has a unique pseudo-solution z in D.

As in Theorem 2.2, we now seek an approximation to the pseudo-solution z. We consider the sequence $\{z_n\}$ defined by

$$(5.29) \qquad z_o = A_o^{-1} y_o, \quad z_n = A_o^{-1}\left(\sum_{i=0}^{n} y_i - \sum_{i=1}^{n} A_i z_{n-i} \right).$$

The existence of this sequence is guaranteed by the following lemma.

<u>Lemma 4.2.</u> Let $u_n = \sum_{i=0}^{n} y_i - \sum_{i=1}^{n} A_i z_{n-i}$, $n = 1, 2, \cdots$. The domain of A_o^{-1} contains all u_n, $n = 1, 2, \cdots$ and $z_n \in D$ for all $n = 0, 1, 2, \cdots$.

<u>Proof</u>: Clearly $z_o \in D$. Suppose that $z_i \in D$ for $i = 0, 1, 2, \cdots, n-1$. Then from (5.24)

$$\| A_i z_{n-i} \| \leq rK^{-1} \quad \text{for } i = 1, 2, \cdots, n,$$

and (5.26) implies that $A_i z_{n-i} \in A_o(X)$ for $i = 1, 2, \cdots, n$.

It was just shown that $y_i \in A_o(X)$ for all $i \geq 0$. Hence, $u_i \in A_o(X)$ for $i = n$. This induction shows that $u_i \in A_o(X)$ for all $i \geq 1$, provided $z_i \in D$ for all $i \geq 0$.

By induction, $z_i \in D$ for all $i \geq 0$, since by (5.25) and the above inequality:

$$\| A_o^{-1} u_n - A_o^{-1} y_o \| \leq K \| \sum_{i=1}^{n} (y_i - A_i z_{n-i}) \| \leq K \cdot rK^{-1} = r.$$

In Theorem 4.3 we show that z_n is an approximation to z and we give an upper bound for $\| z - z_n \|$. As in Theorem 2.2, the degree of this approximation depends on the rates of convergence of the series $\sum_{n\geq 0} A_n$ and $\sum_{n\geq 0} y_n$. In that respect, we make two distinct sets of additional assumptions on A_n and y_n. First:

(5.30a) $\quad \|A_n\|_D \leq \rho^{-n}K^{-1}$ for $n \geq 1$,

(5.30b) $\quad \|y_n\| \leq r\rho^{-n+1}K^{-1}$ for $n \geq 1$;

secondly:

(5.31a) $\quad \|A_o\|_D \geq \|A_1\|_D \geq \|A_2\|_D \geq \cdots$,

(5.31b) $\quad \|A_m\|_D\|A_n\|_D \leq K^{-1}\|A_{m+n}\|_D$ for all $m \geq 1$ and $n \geq 1$ such that $\|A_{m+n}\| \neq 0$,

(5.31c) $\quad \|y_o\| \geq \|y_n\|$ for all $n \geq 1$,

(5.31d) $\quad \|y_n\| \leq rK^{-1} \min\{1, \|A_{n-1}\|_D\}$ for all $n \geq 1$.

Assumptions (5.30b) and (5.31d) imply (5.25).

<u>Theorem 4.3.</u> (i) If (5.30a) and (5.30b) hold, then the sequence $\{z_n\}$ defined by (5.29) converges to z and

(5.32) $\quad \|z - z_n\| \leq r\rho^{-n} \operatorname{Max}\{1, K^{-1}\}$ for $n = 0, 1, 2, \cdots$.

(ii) If (5.31a), (5.31b), (5.31c) and (5.31d) hold, then

(5.33) $\quad \|z - z_n\| \leq r\|A_n\|_D \operatorname{Max}\{1, K^{-1}\}$ for all n such that $\|A_n\|_D \neq 0$;

if for each integer $n \geq 0$, $\|A_n\|_D \neq 0$, then the sequence $\{z_n\}$ converges to z.

Proof: From (5.27), it follows that $(A - A_o)z \in A_o(X)$. Hence $[y - (A - A_o)z]$ belongs to the domain of A_o^{-1}. It may be verified directly that

$$(5.34) \qquad z = A_o^{-1}[y - (A - A_o)z] = A_o^{-1}\Big(\sum_{n\geq 0} y_n - \sum_{n\geq 1} A_n z\Big).$$

From the definition of $\{z_n\}$ and the linearity of A_o^{-1}, we have, for $n = 1, 2, 3, \cdots$

$$\|z - z_n\| = \Big\|A_o^{-1}\Big(\sum_{i\geq n+1} y_i\Big) - A_o^{-1}\Big(\sum_{i\geq n+1} A_i z\Big) - A_o^{-1}\Big(\sum_{i=1}^{n} (A_i z - A_i z_{n-i})\Big)\Big\|.$$

Hence,

$$\|z - z_n\| \leq \mathrm{Max}\{\alpha_n, \beta_n, \gamma_n\} \quad \text{for } n = 1, 2, \cdots,$$

where

$$\alpha_n = \|A_o^{-1}\|_D \cdot \Big\|\sum_{i\geq n+1} y_i\Big\|,$$

$$\beta_n = \|A_o^{-1}\|_D \cdot \Big\|\sum_{i\geq n+1} A_i z\Big\|,$$

and, since $z \in D$ and $z_i \in D$ for all $i \geq 0$ (Lemma 4.2),

$$\gamma_n = \|A_o^{-1}\|_D \cdot \underset{1\leq i\leq n}{\mathrm{Max}} \{\|A_i\|_D \cdot \|z - z_{n-i}\|\}.$$

Since $1 \leq \|A_o\|_D \cdot \|A_o^{-1}\|_D$,

$$\|z - z_o\| \leq r \leq r\,\mathrm{Max}\{1, K^{-1}\},$$

$$\|z - z_o\| \leq r \leq r\|A_o\|_D\,\mathrm{Max}\{1, K^{-1}\}.$$

So, (5.32) and (5.33) are both satisfied for $n = 0$.

The rest of the proof is conducted, for each set of assumptions (5.30) and (5.31), by induction, and exactly as in Theorem 2.2. We omit this later part of the proof.

It is easily verified that Theorems 4.1, 4.3 and Lemma 4.2 hold if in the hypotheses (5.23), (5.24), (5.30a), (5.31a), (5.31b), (5.31d) and the estimate (5.33), we change all norms on D ($\|A_n\|_D$) to norms on X ($\|A_n\|_X$).

If we assume that $X = Y$ and that all operators A_n are linear, and that the above change to norms on X is made, the results of Theorem 4.1 and 4.3 are refinements of those of the Application of Theorem 2.2, page 110.

Applications can be found in [32].

CHAPTER 6

CONTINUOUS LINEAR FUNCTIONALS

6-1 Dual space

In this Chapter, X is a V-space over the field of scalars F. F is a V-space over itself and is given the discrete topology induced by its trivial valuation.*

The term "functional on X" will be used to denote an operator from X to F.

Definition 1.1. The space $X^* = \mathcal{J}(X, F)$ of bounded linear functionals on X is called the dual space of X.

Theorem 1.2. (i) X^* is a V-space.

(ii) Every continuous linear functional on X is bounded and belongs to X^*.

(iii) For each $x \in X$ and $f \in X^*$ there exists $r > 0$ such that $f(S(x, r)) = f(x)$.

Proof: (i) and (ii) are special cases of Theorem 4-3.1 and Theorem 4-3.5, respectively. (iii) follows from the continuity of f and the discreteness of F.

A direct proof of the validity of the Hahn-Banach Theorem (Th. 1.3(i) below; [36], p. 186) in V-spaces has been given by A. F. Monna ([24], Part III, pp. 1137-1138). A. W. Ingleton

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*Since $[0] = \{0\}$, the symbols "$\equiv$" and "$=$" have the same meaning in the V-space F. (See page 45.)

[17] constructed a proof based on the notion of spherical completeness (see 2-5, (ii)). Another proof is due to I. S. Cohen [13], p. 696. Monna has also proved (same reference) the existence of the linear functionals referred to in (ii) of the following theorem.

Theorem 1.3. (i) Let Z be a subspace of X. To each linear functional $f_1 \in Z^*$ there corresponds at least one linear functional $f_2 \in X^*$ such that

(6.1) $$\|f_2\|_X = \|f_1\|_Z \text{ and } f_2(x) \equiv f_1(x) \text{ for all } x \in Z.$$

(ii) For $x_o \in H$, $\|x_o\| \neq 0$ and every scalar $\alpha \in F$, $\alpha \neq 0$, there exists $f \in X^*$ such that

$$f(x_o) \equiv \alpha \text{ and } \|f\| = \|x_o\|^{-1}.$$

Proof: See the references quoted above.

We give a new proof of (i), using Theorem 4-4.1. Let H' be a distinguished basis of Z and H be an arbitrary extension of H' to all of X (see Definition 2-4.3). On H, define

$$f_2(h) \equiv \begin{cases} f_1(h) & \text{for } h \in H' \\ 0 & \text{for } h \in H \setminus H' \end{cases}$$

It follows from Theorem 4-4.1 that f_2 is determined on X by its values on H and that (6.1) is satisfied.

To prove (ii), define $f_1(x_o) \equiv \alpha$ and extend f_1 by linearity to the subspace $[x_o]$. Then $\|f_1\|_{[x_o]} = \|x_o\|^{-1}$. The conclusion follows from (i).

<u>Theorem 1.4.</u> One of X and X* is a bounded space if and only if the other one is a discrete space.

<u>Proof:</u> It follows from Theorem 1.3(ii) that if X is not discrete, i.e. if there are points in X with arbitrarily small non-zero norms, then X* is unbounded. The same theorem implies that if X is unbounded there exist linear functionals of arbitrarily small non-zero norms.

Suppose that X* is unbounded. Then, for any integer $K > 0$ there exists $f \in X^*$ with $\|f\| > K$. Since there must be a point $x \in X$ for which

$$\|f(x)\| = 1 = \|f\|\|x\|,$$

there must be non-trivial points of X with norms less than K^{-1}. Hence, X is not discrete.

Finally, suppose that X is bounded, i.e. for some $M > 0$, $\|x\| \leq M < \infty$ for all $x \in X$. For all $f \in X^*$, $\|f\| \neq 0$, we have

$$\|f(x)\| = 1 \leq \|f\| \cdot \|x\| \quad \text{for all } x \text{ such that } f(x) \neq 0.$$

Thus, $\|f\| \geq \frac{1}{M}$ and X* is discrete.

6-2 <u>The * norm on (H)</u>

Let $H = \{h_i : i \in J\}$, where J is some index set, be a distinguished basis of X. (H) denotes the set of all finite linear combinations of elements of H.

In this section we shall define a new norm on the elements of (H). In the next section we shall use this new norm to establish the relationship between X* and (H).

The symbol "$(x, h)_H$" was introduced on page 28.

Definition 2.1. For $x \in X$,

(i) $J(x) = \{i \in J : (x, h_i)_H \neq 0\}$;

(ii) $\varpi(x)$ is defined by the relation: $\|x\| = \rho^{-\varpi(x)}$;

(iii) $\ell(x) = \sup_{i \in J(x)} \{\varpi(h_i)\}$, $\ell(\theta) = (-\infty)$.

For $x \in X$, $J(x)$ is countable and $x = \sum_{i \in J(x)} (x, h_i)_H h_i$.

For $y \in (H)$, $J(y)$ is finite. It is easily verified that for all $y, z \in (H)$:

$$(6.2)\quad \begin{aligned} &\text{(i)} \quad \ell(\alpha y) = \ell(y) \quad \text{for all } \alpha \in F,\ \alpha \neq 0; \\ &\text{(ii)} \quad \ell(x + z) \begin{cases} \leq \operatorname{Max}\{\ell(y), \ell(z)\} \\ = \operatorname{Max}\{\ell(y), \ell(z)\} \text{ whenever } \ell(y) \neq \ell(z). \end{cases} \end{aligned}$$

The two sets of integers $\{\varpi(h_i) : h_i \in H\}$ and $\{\ell(h_i) : h_i \in H\}$ are identical since for each $h_i \in H$, $\varpi(h_i) = \ell(h_i)$. The set $\{\varpi(h_i) : h_i \in H\}$ is bounded above if and only if X is a discrete space; it is bounded below if and only if X is bounded in its norm.

Definition 2.2. The function which assigns to each point y of (H) the non-negative real number

$$(6.3)\qquad \|y\|^* = \rho^{\ell(y)}$$

will be called the * norm on (H).

Theorem 2.3. (i) Under the *norm (H) has all the defining properties of a V-space, except possibly when X is unbounded, in which case (H) may not be complete.

(ii) One of the spaces X and (H), under the *norm, is bounded if and only if the other is discrete.

(iii) The set H is a distinguished Hamel basis of (H) under the *norm.

<u>Proof:</u> Except for the completeness requirement, (i) is easily proved from (6.2) and (6.3).

(ii) follows from the remark preceding Definition 2.2, and the fact that the set $\{\ell(h_i) : h_i \in H\}$ is bounded above if and only if (H) is bounded under the *norm; - is bounded below if and only if (H) is a discrete space under the *norm.

If X is bounded, the completeness of (H) follows from its discreteness.

(iii) follows from the fact that for all $y \in$ (H) such that $\|y\|^* \neq 0$:

$$\|y\|^* = \rho^{\ell(y)} = \rho^{\underset{i\in J(y)}{\mathrm{Max}}\{\omega(h_i)\}} = \underset{i\in J(y)}{\mathrm{Max}}\{\rho^{\omega(h_i)}\}$$

$$= \underset{i\in J(y)}{\mathrm{Max}}\{\rho^{\ell(h_i)}\} = \underset{i\in J(y)}{\mathrm{Max}}\{\|h_i\|^*\}.$$

6-3 <u>H-inner product and representation theorems</u>

To the notations, definitions and hypotheses of the previous section, we add the assumption that the field of scalars, F, is the field of the real or complex numbers. $\bar{\alpha}$ denotes the complex conjugate of $\alpha \in F$.

Definition 3.1. (i) $J(x, y) = J(x) \cap J(y)$.

(ii) The scalar valued function, defined on $X \times (H)$ by

$$<x, y>_H = \begin{cases} 0 & \text{if } J(x, y) = \emptyset, \\ \sum_{i \in J(x,y)} (x,h_i)_H \cdot (y,h_i)_H & \text{if } J(x, y) \neq \emptyset, \end{cases}$$

$x \in X$, $y \in (H)$, will be called the H-inner product on X.

The following properties of the H-inner product are easily verified: For all $u, v \in X$, all $y, z \in (H)$ and all $\alpha, \beta \in F$:

$$<y, z>_H = <z, y>_H;$$

$$<\alpha u, \beta y>_H = \alpha\beta <u, y>_H;$$

$$<u+v, y+z>_H = <u, y>_H + <v, y>_H + <u, z>_H + <v, z>_H.$$

The analogy with the usual inner product is evident ([7], p. 242; [19], Part II, p. 80; [36], p. 106). An important difference is that the H-inner product depends on H. Indeed, given two distinct distinguished bases H_1 and H_2 of X, if $y \in (H_1)$ and $y \notin (H_2)$, then, for all $x \in X$, $<x, y>_{H_1}$ is defined but $<x, y>_{H_2}$ is not; if $y \in (H_1) \cap (H_2)$, then there may exist $x \in X$ such that $<x, y>_{H_1} \neq <x, y>_{H_2}$. To pursue the analogy, we shall establish a relationship between the H-inner product and the bounded linear functionals on X.

An isomorphism between two V-spaces X and Y is a one-to-one continuous linear operator from all of X to all of Y. An isometric isomorphism φ is an isomorphism such that $\|\varphi(x)\| = \|x\|$ for all $x \in X$ ([7], p. 65).

Theorem 3.2. There exists an isometric isomorphism φ_H between (H) with its *norm and a subspace of X^*; for all $y \in (H)$, $\varphi_H(y) \equiv f_y$ is such that

(6.4) $$f_y(x) = \langle x, y\rangle_H \quad \text{for all } x \in X.$$

Furthermore, the set $\varphi_H(H)$ is a distinguished Hamel basis for the subspace $\varphi_H((H))$ of X^*; for $f_y \in \varphi_H((H))$,

(6.5) $$f_y \equiv \sum_{i \in J(y)} (y, h_i)_H f_{h_i}$$

in the norm of X^*.

Proof: For each fixed $y \in (H)$ it is easy to verify that the mapping defined by (6.4) is a linear functional on X. Let φ_H be the operator on (H) defined by $\varphi_H(y) \equiv f_y$. φ_H is linear since the H-inner product is linear in y.

a) If $J(x, y) = \emptyset$, then $\| f_y(x) \| = 0$.

b) If $J(x, y) \neq \emptyset$, then

$$\varpi(x) \leq \varpi(h_i) \text{ for all } i \in J(x),$$
$$\varpi(h_i) \leq \ell(y) \quad \text{for all } i \in J(y).$$

Therefore, $\varpi(x) \leq \ell(y)$ and

$$\| f_y(x) \| = 1 \leq \rho^{\ell(y)} \cdot \rho^{-\varpi(x)} = \| y \|^* \cdot \| x \|.$$

c) Since $\ell(y)$ is finite, there exists $i \in J$ such that $\ell(y) = \ell(h_i) = \varpi(h_i)$, and

$$\| f_y(h_i) \| = 1 = \rho^{\ell(y)} \rho^{-\varpi(h_i)} = \| y \|^* \| h_i \|.$$

d) If $y, z \in (H)$ and $y \neq z$, there exists $j \in J$ such that

$$(y, h_j)_H \neq (z, h_j)_H \text{ and, hence } \langle h_j, y\rangle_H \neq \langle h_j, z\rangle.$$

Thus, $\varphi_H(y) \neq \varphi_H(z)$.

a), b), c) show that φ_H is an isometric, and therefore continuous, operator from (H) with its *norm to X*. d) shows that φ_H is one-to-one.

The latter part of the theorem follows from the linearity of φ_H and Theorem 2.3(iii).

Definition 3.3. A subset A of a V-space is called locally finite if for every integer n, there is at most a finite number of elements of A with norms equal to ρ^n.

Lemma 3.4. Let $f \in X^*$. If the subset H' of H on which f is non-zero is bounded and locally finite, then

(i) H' is a finite set;

(ii) there exists $y_f \in (H)$ such that $\varphi_H(y_f) \equiv f$.

Proof: (i) Since f is a continuous linear mapping into the discrete space F, there exists an integer m such that $f(x) \neq 0$ implies $\|x\| \geq \rho^m$. Thus, H' is bounded below, bounded above and locally finite; hence it is finite.

(ii) Let

$$y_f \equiv \sum_{h' \in H'} f(h')h'. \tag{6.6}$$

For all $h \in H$,

$$f(h) = \langle h, y_f \rangle_H = \varphi_H(y_f)(h).$$

From Theorem 4-4.1, it follows that $\varphi_H(y_f) \equiv f$.

Theorem 3.5. The operator φ_H is an isometric isomorphism between (H) with its *norm and X* if and only if X is bounded and H is locally finite.

Proof: If X is bounded and H is locally finite, every subset H' of H satisfies the hypotheses of Lemma 3.4. Therefore, φ_H maps (H) onto X*.

For the converse, suppose that X is unbounded or that H is not locally finite. Then, for some integer n there exists an infinite subset H' of H such that for all $h' \in H'$:

$$\|h'\| \geq \rho^n, \text{ when X is unbounded,}$$

$$\text{or} \quad \|h'\| = \rho^n, \text{ when H is not locally finite.}$$

From Theorem 4-4.1, there exists $f \in X^*$ such that

$$f(h') = 1 \text{ for all } h' \in H', \; f(h) = 0 \text{ for all } h \in H \setminus H'.$$

Should there exist $y_f \in (H)$ such that $\varphi_H(y_f) \equiv f$, y_f would have to have the infinite expansion (6.6). This is impossible.

Corollary 3.6. If X is unbounded and H is locally finite, then φ_H is an isometric isomorphism between (H) with its *norm and the subspace of X* formed by the continuous linear functionals which vanish outside of a bounded subset of X.

Corollary 3.7. If X is bounded and admits a locally finite distinguished basis, then X and X* have the same dimension.

The proof follows from Theorems 3.2 and 3.5.

Examples: The spaces $\mathcal{P}_k$ of 3-4 and Q_k of 3-5 are bounded and admit locally finite distinguished bases. Thus, the spaces $\mathcal{P}_k^*$ and Q_k^* are equivalent to the spaces of polynomials (6.7) and (6.8) respectively:

$$(6.7)\quad \begin{cases} \xi(\lambda) \equiv 0 \qquad , & \|\xi\|^* = 0, \\ \xi(\lambda) = \sum_{i=p}^{n} \alpha_i \lambda^i, & \|\xi\|^* = \rho^n, \quad \alpha_n \neq 0, \ k \leq p \leq n. \end{cases}$$

$$(6.8)\quad \begin{cases} \xi(u, v) = 0 & \|\xi\|^* = 0, \\ \xi(u,v) = \sum_{i=p}^{n} \sum_{j=0}^{i} \alpha_{ij} u^{i-j} v^j, & \\ & \|\xi\| = \rho^n, \ \sum_{j=0}^{n} |\alpha_{nj}| \neq 0, \ k \leq p \leq n. \end{cases}$$

According to (6.5), a continuous linear functional on $\mathcal{P}_k$ is a finite linear combination of the functionals f_{φ_n} :

$$f_{\varphi_n}(x) = \text{coefficient of } \lambda^n \text{ in the expansion of } x(\lambda) \text{ in powers of } \lambda.$$

This result was proved directly by H. F. Davis [4], p. 91, for the space $\mathcal{P}_o$.

It was shown in 3-4 that $\mathcal{P}_o$ admits as distinguished bases the sets Φ_o and J of (3.16), Ch. 3. Consider the continuous linear functional f defined on Φ_o by

$$f(\varphi_n) = \begin{cases} \alpha_n \neq 0 & \text{for } 0 \leq n \leq N, \\ 0 & \text{for } n > N. \end{cases}$$

The isometric isomorphism Ψ_{Φ_o} between (Φ_o) and $\mathcal{P}_o^*$ is such that

$$\Psi_{\Phi_o}^{-1}(f) = \sum_{n=0}^{N} \alpha_n \varphi_n.$$

The isometric isomorphism Ψ_J between (J) and $\mathcal{P}_o^*$ is such that

$$\Psi_J^{-1}(f) = \sum_{n=0}^{N} \beta_n J_n,$$

where the coefficients β_n, determined from the power series expansions of the J_n's ([8]), are the solutions of the system:

$$\sum_{i=0}^{p} (-1)^{p-i} \frac{\beta_{2i}}{2^{2p}(p-i)!(p+i)!} = \alpha_{2p}, \quad p = 0, 1, \cdots, [\tfrac{n}{2}],$$

$$\sum_{i=0}^{p} (-1)^{p-i} \frac{\beta_{2i+1}}{2^{2p+1}(p-i)!(p+i+1)!} = \alpha_{2p+1},$$
$$p = 0, 1, \cdots, [\tfrac{n-1}{2}].$$

Clearly, in $\mathcal{P}_o$, $\Psi_{\Phi_o}^{-1}(f) \neq \Psi_J^{-1}(f)$. This inequality reflects the dependence of the H-inner product on the distinguished basis H.

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