THE UNIVERSITY OF BRITISH COLUMBIA

FACULTY OF GRADUATE STUDIES

PROGRAMME OF THE

FINAL ORAL EXAMINATION

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

of

GRAHAM KEITH ZELMER


TUESDAY, May 9, 1967, at 10:30 a.m.
In Room 104, Mathematics Building.

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Let \( f(x) = \sum_{k=0}^{\infty} f_k x^k \) be a function which is analytic at the origin with a radius of convergence \( R > 1 \) and with \( f(1) = 1 \).

Let \( f^n(z) = \sum_{k=0}^{\infty} f_{nk} z^k \) \( (f_{00} = 1; f_{0k} = 0, k > 0; f_{1k} = f_k) \). The matrix \( F = (f_{nk}) \) is known as a Sonnenschein summation matrix.

If \( f(z) \) is entire the summation method based on this matrix sums the geometric series in the domain \( D = \{ z : |f(z)| < 1 \} \).

In this thesis we consider the methods corresponding to the functions \( f(z) = (1-r) + rz \), \( f(z) = (1-r) + rz^2 \) and \( f(z) = (1-r)z + rz^3 \), \( 0 < r < 1 \). The summation method for \( f(z) = (1-r) + rz \) is the well-known Euler-Knopp or \((E,r)\) method.

Simple recursion relations are developed for the various rows of the matrices \( F \) obtained from the functions \( f(z) \) and the domain \( D \) is described in detail. More generally, the \((E,r,\alpha,\beta)\) methods are defined corresponding to the functions \( f(z) = (1-r)z^\alpha + rz^{\alpha+\beta} \), \( \alpha,\beta \) positive integers. Some theorems holding for the \((E,r)\) method \((= (E,r,0,1)\)
method) are generalized to this class.

Let \( a(z) = \sum_{k=0}^{\infty} a_k z^k \) be an arbitrary function, analytic at the origin. A well-known theorem in analytic function theory allows the construction of a domain \( D(a) \) in which a given summation method sums this series, from a knowledge of the domain \( D \) in which that method sums the geometric series. In Chapter II \( a(z) \) is taken as the solution to the two-body problem and the summation methods used are those corresponding to the three functions \( f(z) \) above. If the parameter \( r \) is chosen properly, the domain \( D(a) \) will contain the interval \( [0, \pi] \). The above three methods are shown to be progressively more effective in obtaining the solution on \( [0, \pi] \) and, in particular, are all much more effective than a technique used by V. A. Brumberg (1964).

In Chapter III these methods are applied with more limited success to the regularized three-body problem. An interesting numerical result is derived there concerning the width of the strip about the real axis in which the solution to the regularized three-body problem is analytic. Finally, a theorem is proved concerning the problem of the motion of a heavy rigid body
about a fixed point, showing that it can be treated with summation techniques in the same manner as the problems above.

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PUBLICATIONS

SUMMATION METHODS IN THE TWO- AND THREE-BODY PROBLEMS

by

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We accept this thesis as conforming to the required standard

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April, 1967
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ABSTRACT

Let \( f(z) \) be a complex-valued function of the complex variable \( z \) which is regular at the origin, has radius of convergence \( R > 1 \), and satisfies the condition \( f(1) = 1 \). If we write \( f(z) = \sum f_k z^k \) and \( f^n(z) = \sum f_{nk} z^k \), \( n=0,1,2,\ldots \), the matrix \( F = (f_{nk}) \) leads to a summation method generally known as a Sonnenschein method. The utility of these methods lies in the fact that much can be said about them simply from a knowledge of the function \( f(z) \). In the present work we are concerned with the three methods associated with the three functions \( f(z) = (1-r) + rz \), \( f(z) = (1-r) + rz^2 \) and \( f(z) = (1-r)z + rz^3 \) where \( r \neq 0 \) is an arbitrary complex parameter (generally, \( 0 < r < 1 \)). The method based on the function \( (1-r) + rz \) leads to the well-known Euler-Knopp method which has already been extensively studied. We show that there exist simple recursion relations between the various rows of the matrix \( F \) and we make a detailed study of the domain \( D_r \) on which these methods sum the geometric series \( \sum z^k \). A more general sub-class of Sonnenschein methods called the \((E,r,\alpha,\beta)\) methods is then defined and some of the well-known theorems applicable to the \((E,r)\) method are shown to hold for this sub-class.
The practical application of the above three methods to the two- and three-body problems of classical mechanics forms the major portions of Chapters II and III. Much use is made in these chapters of a theorem, stated in Chapter I, which allows us to construct a domain $D_r(a)$ on which one of the above methods sums an arbitrary function $a(z)$ regular at the origin.

On the limited intervals for which the above methods are applicable, it is shown that they provide effective methods for obtaining the solution to the two-and three-body problems. Comparison is always made with similar results obtained by V. A. Brumberg and it is shown that the methods used here have certain advantages over his.

The Sundman series for the three-body problem are also set up and utilized. Although the series are not very effective, the convergence is not as bad as is generally supposed. An interesting argument based on numerical evidence shows that the width of the strip about the real axis, in which the solution to the regularized three-body problem is known to be analytic, is not as narrow as Sundman's estimates give.

Finally, a theorem is proved for the problem of the motion of a heavy rigid body about a fixed point showing that this problem can be treated in the complex plane in
the same way as the two-body problem and regularized three-body problem.
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## BIBLIOGRAPHY
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Summation Methods in the Two-and Three-Body Problems

Introduction

The motivation for this thesis comes from a recent paper by V. A. Brumberg [1] in which he attempted to get solutions for the two-and three-body problems of classical mechanics valid for all real values of the independent parameter. Such a program however is extremely ambitious and, of course, subject to practical limitations. In order to apply Brumberg's technique it is necessary to know i) the Taylor series coefficients of the solution functions and ii) the entries $g_{nk}$ of an infinite matrix $G = (g_{nk})$ known as a summation matrix. With this information one is able to calculate a sequence of polynomials which, depending on the independent parameter used, converges to the solution for all real values of this parameter.

The idea behind the present work is to consider matrices $G$ which are more limited in nature, i.e. for which the sequence of polynomials above converges to the solution for all values of the parameter within a certain finite interval. In Chapter I it is shown that these matrices correspond to certain of a fairly large class of summation methods known as Sonnenschein methods. Associated with each such method is a function $f(z)$, regular at the origin, from which the corresponding matrix $G$ is derived. For the most part we shall take
\[ f(z) = (1-r) + rz \, , \, f(z) = (1-r) + rz^2 \, \text{or} \, f(z) = (1-r)z + rz^3 \]
where, in general, \( r \) is allowed to assume any complex value.

It is pointed out that the Sonnenschein method for
\[ f(z) = (1-r) + rz \] is the well-known Euler-Knopp or \((E,r)\) method while the methods corresponding to the other two functions \( f(z) \) do not seem to have been discussed anywhere in the literature to date.

Having outlined the methods in Chapter I, Chapters II and III are involved with their application to the two- and three-body problems respectively. It will be shown that on their limited interval of convergence these methods can be more effective than Brumberg's. Moreover, our methods are easier to apply since the matrix \( G \) is much simpler to evaluate. Finally, the practical limit to Brumberg's method does not seem to extend too far beyond our limited interval of convergence.
CHAPTER I

Sonnenschein Methods

1.1 Summability and Analytic Continuation

The notions of summability and the related concepts of generalized sum and generalized limit as well as the detailed treatment of many summation methods are expounded in Cooke [1], Dienes [1], Hardy [1] and Knopp [1] as well as in the works of many other authors. For completeness, let us briefly review some of the main ideas below.

If \( \{s_n\} \) is an arbitrary sequence and \( F = (f_{nk}) \) \((n,k=0,1,2,\ldots)\) an infinite matrix the sequence \( \{t_n\} \) known as the F-transform of \( \{s_n\} \) is formally defined by

\[
(1.1) \quad t_n = \sum f_{nk} s_k \quad (n=0,1,2,\ldots)
\]

(Assume the index \( k \) runs from 0 to \( \infty \) unless otherwise specified.) If \( \{t_n\} \) converges whenever \( \{s_n\} \) does, and to the same limit, and if \( t_n \rightarrow s \) for some divergent sequence \( \{s_n\} \), we can assign to this sequence the limit \( s \) called its generalized limit by \( F \) and we write \( s_n \rightarrow s(F) \). In such a circumstance the matrix \( F \) is said to be regular; well-known necessary and sufficient conditions on a matrix \( F \) in order that it be regular are...
1) \[ \sum |f_{nk}| < M \] for each \( n \), where \( M \) is independent of \( n \),

(I)

ii) \( f_{nk} \to 0 \) as \( n \to \infty \) for each \( k \), and

iii) \( \sum f_{nk} \to 1 \) as \( n \to \infty \).

Again, if \( \sum u_k \) is an arbitrary sequence and \( G = (g_{nk}) \) \((n,k=0,1,2,\ldots)\) an infinite matrix, the sequence \( \{t_n\} \) in this case is formally defined by

\[ t_n = \sum g_{nk} u_k \quad (n=0,1,2,\ldots) \tag{1.2} \]

If \( \{t_n\} \) converges whenever \( \sum u_k \) does having a limit equal to the sum of the series and if \( t_n \to s \) for some divergent series \( \sum u_k \), we can assign to this series the sum \( s \) called its generalized sum by \( G \) and write \( \sum u_k = s(G) \). As above, \( G \) is called regular. Necessary and sufficient conditions for a matrix of this type to be regular are

(I) \[ \sum |g_{nk} - g_{nk+1}| < M \] for each \( n \), where \( M \) is independent of \( n \),

(II)

ii) \( g_{nk} \to 1 \) as \( n \to \infty \) for each \( k \).
More fully the matrices $F$ and $G$ above are referred to as regular sequence-to-sequence and series-to-sequence summation matrices respectively and the two notions are not unrelated. Thus from any regular sequence-to-sequence matrix $F = (f_{nk})$ one can obtain a regular series-to-sequence matrix $G = (g_{nk})$ by putting

$$g_{nk} = \sum_{m=k}^{\infty} f_{nm},$$

while from any regular matrix $G = (g_{nk})$ the matrix $F = (f_{nk})$ with $f_{nk} = g_{nk} - g_{nk+1}$ will also be regular if, in addition, $\lim_{n \to \infty} \lim_{k \to \infty} g_{nk} = 0$. If $G$ provides a generalized sum for the series $\Sigma u_k$, $F$ will provide a generalized limit for the sequence $\{s_n\}$ of partial sums of this series and vice-versa.

The above ideas are utilized in analytic function theory as follows: Let $a(z)$ be an arbitrary function of the complex variable $z$ which is analytic at the origin so that

$$a(z) = \sum a_k z^k.$$  \hspace{1cm} (1.3)

If a regular series-to-sequence summation matrix $G$ is applied to series (1.3) (here $u_k = a_k z^k$) there results a sequence of functions $\{t_n(z)\}$ with
(1.4) \[ t_n(z) = \sum g_{nk} a_k z^k \quad (n=0,1,2,\ldots) \]

and, if the radius of convergence of (1.3) is \( R \), we have \( t_n(z) \to a(z) \) at least for all \( z \) satisfying \( |z| < R \). But in fact this limit may hold on a larger domain than that given by \( |z| < R \). Thus we are led to a representation of the function \( a(z) \) outside its circle of convergence, that is, we are in possession of a means of analytically continuing the function \( a(z) \).

It is worthwhile noting that very often the matrix \( G \) will be row-finite (each row has finitely many non-zero entries). In this case the expressions (1.4) are all polynomials, a result which is obviously desirable for practical purposes. We also note at this point that while most matrices \( G \) which are used in practice are regular, this is in general an unnecessary restriction. By this is implied that there do exist matrices \( G \) which are not regular but for which sequence (1.4) exists and converges to \( a(z) \) on a domain extending beyond the circle of convergence of the Taylor series.

In these cases however the sequence may fail to converge at points inside the circle of convergence.

To illustrate the above ideas consider the well-known Euler-Knopp, or \((E,r)\) summation method (Knopp [2], [3]; Agnew [1]). The corresponding matrices
\[ F(r) = (f_{nk}(r)) \text{ and } G(r) = (g_{nk}(r)) \text{ depend on a parameter } r \neq 0 \text{ and are defined by } \]

\[
(1.5) \quad f_{nk}(r) = \binom{n}{k} r^k (1-r)^{n-k} \quad (k=0,1,\ldots,n) \\
= 0 \quad (k > n)
\]

and

\[
(1.6) \quad g_{nk}(r) = \sum_{m=k}^{n} \binom{n}{m} r^m (1-r)^{n-m} \quad (k=0,1,\ldots,n) \\
= 0 \quad (k > n)
\]

The matrices are regular if and only if \( 0 < r \leq 1 \).

For \( r=1 \) \( F(1) \) is the identity matrix while \( G(1) \) is the lower semi-matrix

\[
(1.7) \quad \begin{pmatrix}
1 & 0 & 0 & \cdots \\
1 & 1 & 0 & \cdots \\
1 & 1 & 1 & \cdots
\end{pmatrix}
\]

and thus the method is equivalent to ordinary convergence (usually considered a regular method).

For \( a(z) \) let us consider the function \((1-z)^{-1}\) so that series (1.3) becomes the geometric series \( \sum z^k \). Then if \( G(r) \) is applied to this series (or \( F(r) \) to its sequence of partial sums) we get the sequence \( \{g_n(z)\} \) where
(1.9) \[ g_n(z) = \frac{1-z[(1-r)+rz]^{n}}{1-z} \]

which will converge at all points inside the circular domain

\[ D_r = \{ z : |z-(1-r^{-1})| < |r|^{-1} \} \]

to the function \((1-z)^{-1}\). Note that the boundary of \(D_r\) always passes through \(z=1\) and that for \(r \notin (0,1)\), \(D_r\) does not contain the unit disk and consequently \(G(r)\) (and \(F(r)\) as well) cannot be regular.

We conclude this section with a lemma and a theorem which relate the domain on which a given summation method with matrix \(G\) sums the geometric series to the domain on which it sums an arbitrary function \(a(z)\). Let the former domain be \(D\). Thus if \(G = (g_{nk})\) and if \(g_n(z)\) is defined by

(1.10) \[ g_n(z) = \sum g_{nk}z^k \quad (n=0,1,2,...) \]

we have \(g_n(z) \to (1-z)^{-1}\) as \(n \to \infty\) for each \(z \in D\).

Let us further assume in all that follows that \(D\) is star-like with respect to the origin. Consider the function \(a(z)\) given by (1.3) and construct for \(a(z)\) the domain \(D(a)\) as follows:
For each $\theta$, $0 \leq \theta < 2\pi$, let $r_\theta = \lim \inf \{r : r > 0, z = re^{i\theta} \text{ a singular point of } a(z)\}$. Take $r_\theta = \infty$ if $z = re^{i\theta}$ is always a regular point of $a(z)$. For all $\theta$ for which $r_\theta$ is finite define $D(\theta)$ by

$$D(\theta) = \{z : z = z^' e^{i\theta}, z^' \in D\}$$

and finally define $D(a)$ by

$$D(a) = \bigcap \{D(\theta) : 0 \leq \theta < 2\pi, r_\theta < \infty\}.$$

It is easy to check that $D(a)$ will also be star-like with respect to the origin. Moreover, if $D$ is the principal star of $(1-z)^{-1}$, $D(a)$ will be the principal star $A(a)$ of $a(z)$. (The principal star of $a(z)$ is that domain which remains when all the half-lines originating at a singular point of $a(z)$ and extending radially outward to infinity are deleted from the complex plane.)

We now have the following lemma (Hille [1, p. 70]):

**Lemma 1** With the notation as above let $\Delta$ be a compact subset of $D(a)$. Then there exists a "scroc" (simple closed rectifiable orientable curve) $C_\Delta$ in $A(a)$ containing 0 and $\Delta$ in its interior such
that for each \( z \in \Delta \) and \( \zeta \in C_\Delta \) the point \( z/\zeta \in D \). In fact \( K = \{ w : w = z/\zeta , z \in \Delta , \zeta \in C_\Delta \} \) is a compact subset of \( D \).

The following theorem is basic for the work that follows. It constitutes a generalization of the corresponding one found in Hardy [1, p. 190] while that found in Hille [1, p. 71] seems slightly in error.

**THEOREM 1** Suppose the matrix \( G \) sums the geometric series to the limit \( (1-z)^{-1} \) for every \( z \) in a domain \( D \) which is star-like with respect to the origin and that the convergence of the sequence (1.10) is uniform with respect to \( z \) on compact subsets of \( D \). Then \( G \) also sums the series (1.3) to the function \( a(z) \) in the star \( D(a) \), uniformly with respect to \( z \) on compact subsets.

**PROOF.** For any compact set \( \Delta \subset D(a) \) determine the "scroc" \( C_\Delta \) provided by Lemma 1. Since \( C_\Delta \subset A(a) \), Cauchy's integral formula gives

\[
(1.13) \quad a(z) = \frac{1}{2\pi i} \oint_{C_\Delta} \frac{a(\zeta)}{\zeta} \frac{1}{1-z/\zeta} d\zeta
\]

for each \( z \in \Delta \). Consider the function \( t_n(z) \) defined by
(1.14) \[ t_n(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{a(\zeta)}{\zeta} g_n(z/\zeta) d\zeta \quad (n=0,1,2,\ldots) \]

where \( g_n \) is given by (1.10). This integral exists since by the lemma \( z/\zeta \in D \) and \( C_\Delta \subset A(a) \). Now (1.10) is a Taylor series and since \( g_n(z) \to (1-z)^{-1} \) for \( z \in D \), its radius of convergence must be at least as great as \( \lim \sup \{|z| : z \in D\} \). Thus \( K \) in Lemma 1 lies interior to the circle of convergence of \( g_n(z/\zeta) \). Replacing \( g_n(z/\zeta) \) by series (1.10) in (1.14), interchanging summation and integral signs (which the above argument allows) and recalling that \( a_k = \frac{1}{2\pi i} \int_{\Gamma_n} a(\zeta) d\zeta \)

we finally obtain

\[ t_n(z) = \sum_{n=0}^{\infty} g_n(z/\zeta) a_k \]

i.e. we obtain sequence (1.4). However, from (1.13) and (1.14) we get the estimate

(1.15) \[ |a(z)-t_n(z)| \leq \frac{1}{2\pi} \int_{\Gamma_n} \frac{|a(\zeta)|}{|\zeta|} |(1-z/\zeta)^{-1}-g_n(z/\zeta)| d\zeta \]

\[ \leq M \max_{\zeta \in C_\Delta} |(1-z/\zeta)^{-1}-g_n(z/\zeta)| \]

since \( \zeta \) is bounded away from zero and \( a(\zeta) \) is bounded on \( C_\Delta \). But the points \( z/\zeta \) lie in the compact set \( K \)
and thus for sufficiently large $n$ the above maximum can be made arbitrarily small, which proves the theorem.

1.2 Sonnenschein Matrices and their Construction

If any summation method is to be used practically, that is for actual calculations in computing machines, it would be desirable if it possessed the following two properties. Firstly, one would hope that the individual rows of the matrix $F$ or $G$ used in the method could be easily calculated, either explicitly or recursively. Secondly, it would be desirable if, without too much difficulty, one could obtain the domain $D$ on which the method sums the geometric series. Fortunately these two properties generally hold for a class of methods known as Sonnenschein methods. The methods themselves were outlined by J. Sonnenschein [1] in his doctoral thesis in 1946 and were later detailed in his papers [2] and [3]. In his treatment of the subject Sonnenschein adopts a functional analytic point of view. Below we shall attempt to establish his ideas more specifically in the realm of analytic function theory and to develop some specific methods which will be applied in later chapters.

We begin by remarking that the application of a summation matrix $G$ to the geometric series or a matrix
F to its sequence of partial sums very often leads to a sequence \( \{g_n(z)\} \) where \( g_n(z) \) can be obtained in the closed form,

\[
g_n(z) = \frac{1-z f_n(z)}{1-z}.
\]

The simplest example of this is when \( F \) is the identity matrix or \( G \) is the matrix (1.7) so that \( g_n(z) \) is just the \( n \)-th partial sum of the geometric series, that is

\[
g_n(z) = \frac{1-z z^n}{1-z}.
\]

Further examples are given by

1) the Euler-Knopp or \( (E_r, r) \) methods (Section 1.1) where \( g_n(z) \) is given by (1.8),

\[
g_n(z) = 1 - z \int_0^1 [1 - (1-z)u]^n dq(u)
\]

with \( q(u) \in BV[0,1] \) and \( q(0) = 0, q(1) = 1 \) (note that when \( 0 < r < 1 \), i) is a particular case of ii) with \( q(u) = 0, 0 \leq u \leq r, q(u) = 1, r < u \leq 1 \),
iii) the Borel or (B) method [see Hardy [1, p. 79])

where

\( g_n(z) = \frac{1-ze^{-n(1-z)}}{1-z} \) ,

iv) and more recently, the Lototsky or (L) method (Agnew, [2]) where

\[ g_n(z) = \frac{1-z(z+1)\cdots(z+n-1)/n!}{1-z} \]

The obvious conclusion to be drawn in all these cases is that \( g_n(z) \to (1-z)^{-1} \) on \( D \) if and only if \( f_n(z) \to 0 \) on \( D \). Comparing (1.10) and (1.16) we obtain

\[ g_n(z) = \sum g_{nk}z^k = \frac{1-zf_n(z)}{1-z} \]

In any useful method of analytic continuation the domain \( D \) on which \( g_n(z) \to (1-z)^{-1} \) must extend beyond the unit circle. This implies that the series in (1.20)

must converge in some circular domain \(|z| < R, \, R > 1\)

which contains \( D \). But this in turn implies that \( f_n(z) \) must be analytic in the domain given by \(|z| < R\) and, in order that the expression on the right hand side of (1.20) does not have a pole at \( z = 1 \), that \( f_n(1) = 1 \) for all \( n \).
One of the simplest ways of obtaining a sequence of functions \( \{ f_n(z) \} \) satisfying the conditions of the last paragraph would be to start with a fixed function \( f(z) \) (which, loosely translating Sonnenschein, will be called a displacement function (fonction de glissement)) satisfying

\[\text{a)} \quad f(1) = 1 \]
\[\text{b)} \quad f(z) \text{ is analytic for } |z| < R, \quad R > 1.\]

Let \( D \) be defined by \( D = \{ z : |f(z)| < 1, |z| < R \} \) or, if \( f(z) \) is entire, by \( \{ z : |f(z)| < 1 \} \). Putting \( f_n(z) = f^n(z) \) we do in fact obtain a sequence of functions tending to zero on \( D \). It is to be noted that \( z = 1 \) is always a boundary point of \( D \) and by the maximum modulus principle, no deleted neighbourhood of \( z = 1 \) could lie in \( D \). To obtain the matrices \( F \) and \( G \) let us write

\[(1.21) \quad f^n(z) = \Sigma f_{nk} z^k, \quad |z| < R\]

and form the Sonnenschein matrix \( F = (f_{nk}) \). If \( F \) is applied to the partial sums of the geometric series we obtain

\[g_n(z) = \Sigma f_{nk} \frac{1-z^{k+1}}{1-z}
= \frac{1}{1-z} \Sigma f_{nk} - \frac{z}{1-z} \Sigma f_{nk} z^k = \frac{1-zf^n(z)}{1-z}\]
since \( f(1) = 1 \) by a). Consequently, \( F \) is indeed the type of matrix we wished to construct and the series-to-sequence matrix \( G = (g_{nk}) \) can be obtained from \( F \) by putting \( g_{nk} = \sum_{m=k}^{\infty} f_{nm} \). Notice at this point that examples i) and iii) above are examples of Sonnenschein methods with displacement functions corresponding to \( f(z) = (1-r) + rz \) and \( f(z) = e^{(1-z)} \) respectively.

Let the displacement function now be taken to be a polynomial

\[
(1.22) \quad p(z) = p_0 + p_1 z + \cdots + p_\lambda z^\lambda
\]

which we shall henceforth call a displacement polynomial. The condition \( f(1) = 1 \) here implies \( \sum_{i=0}^{\lambda} p_i = 1 \). \( R \) is now + \( \infty \) while \( D \) is the lemniscatic region given by \( \{ z : |p(z)| < 1 \} \) (which may not be star-like with respect to the origin). The matrix \( F = (f_{nk}) \) has the form

\[
(1.23) \quad \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
p_0 & p_1 & p_2 & \cdots & p_\lambda & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

Since \( f^n(z) = p^n(z) \) is a polynomial of degree \( n\lambda \) the \( n \)-th row of the matrix will have \( n\lambda + 1 \) entries which are in general non-zero. In fact it is not difficult
to see that the entry \( f_{nk} \) can be given explicitly by

\[
(1.24) \quad f_{nk} = \sum \frac{n!}{b_0!b_1! \cdots b_\lambda!} p_0^{b_0} p_1^{b_1} \cdots p_\lambda^{b_\lambda}, \quad (0 \leq k \leq n\lambda)
\]

\[
= 0 \quad \quad (k > n\lambda)
\]

where the above sum is taken over all permutations of the non-negative integers \( b_0, b_1, \ldots, b_\lambda \) satisfying

\[
\sum_{i=0}^{\lambda} b_i = n \quad \text{and} \quad \sum_{i=0}^{\lambda} i b_i = k.
\]

Use of (1.24) to compute the entries \( f_{nk} \) is, however, not practical. A more useful method of computing these entries is to obtain them recursively by noting that

\[
p^{n+1}(z) = p^n(z) \cdot p(z)
\]

so that

\[
(1.25) \quad f_{n+1,k} = \sum_{i=0}^{\lambda} p_i f_{n,k-i} \quad (0 \leq k \leq (n+1)\lambda)
\]

\[
= 0 \quad \quad (k > (n+1)\lambda).
\]

This formula remains valid for all \( n \) and \( k \) provided \( f_{nk} \) is taken to be zero if \( k \) is a negative integer.

The series-to-sequence matrix \( G = (g_{nk}) \) can now be obtained from the relationship

\[
g_{nk} = \sum_{m=k}^{\lambda} f_{nm}.
\]

We see \( G \) is also row-finite with the \( n \)-th row having in general \( n\lambda + 1 \) non-zero entries. If one uses the above sum to define \( g_{nk} \) for negative values of \( k \), one obtains
(1.26) \[ g_{nk} = 1 \quad (k < 0) \cdot \]

With this convention

\[
(1.27) \quad g_{n+1,k} = \sum_{m=k}^{(n+1)\lambda} f_{n+1,m} = \sum_{m=k}^{(n+1)\lambda} \sum_{i=0}^{(n+1)\lambda} p_i f_{n,m-i} \\
= \sum_{i=0}^{(n+1)\lambda} p_i f_{n,m-i} \\
= \sum_{i=0}^{(n+1)\lambda} p_i g_{n,k-i} \\
= 0 \quad (0 \leq k \leq (n+1)\lambda) \\
= 0 \quad (k > (n+1)\lambda) 
\]

which is a recursion relation identical with (1.25) for the \( f_{nk} \). We notice that \( g_{n0} = 1 \) for all \( n \) and to initiate the recursion relation

\[
(1.28) \quad g_{1k} = \sum_{i=k}^{\lambda} p_i \\
= 0 \quad (0 \leq k \leq \lambda) \\
= 0 \quad (k > \lambda) .
\]

### 1.3 Certain Explicit Methods

In this section we shall outline several Sonnenschein methods obtained by taking special displacement polynomials and we shall note the domain \( D_r \) involved in all cases.

1) As a first example let us reconsider the method with displacement polynomial
(1.29) \[ p(z) = (1-r) + rz \quad (r \neq 0) \]

In this case \( p_0 = 1-r \) and \( p_1 = r \) so that the recursion relation (1.27) becomes

(1.30) \[ g_{n+1,k} = (1-r)g_{nk} + rg_{nk-1} \quad (0 \leq k \leq n) \]
\[ = 0 \quad (k > n) \]

For \( 0 < r < 1 \) the matrix \( G \) is regular and the domain \( D_r \) is the circular region given by (1.9). It is interesting to observe that as \( r \to 0 \) the domain \( D_r \) approaches the left half-plane given by \( \text{Re}(z) < 1 \).

2) Consider now a displacement polynomial of the form

(1.31) \[ p(z) = (1-r) + rz^2 \quad (r \neq 0) \]

Here \( p_0 = 1-r \), \( p_1 = 0 \), and \( p_2 = r \) so that the recursion relation becomes

(1.32) \[ g_{n+1,k} = (1-r)g_{nk} + rg_{nk-2} \quad (0 \leq k \leq 2(n+1)) \]
\[ = 0 \quad (k > 2(n+1)) \]

If we take \( 0 < r < 1 \), then \( p(z) \) has roots at the points \( \pm i\mu \) where \( \mu = [(1-r)/r]^{1/2} \). Since \( p(1) = p(-1) = 1 \) both \( z = \pm 1 \) lie on the boundary of \( D_r \). When \( z \) is pure imaginary, \( p(z) \) is real.
and \( z = \pm i \xi_L \), and we find \( p(z) = -1 \) for 
\[ \xi_L = \left\{ \frac{(2-r)}{r} \right\}^{1/2} = (\mu^2 + 1)^{1/2} > \mu. \]

Thus \( \pm i \xi_L \) are also points on the boundary of \( D_r \). If we put 
\[ z = \xi \cdot e^{i\varphi}, \]

it is an easy matter to calculate that points on the boundary of \( D_r \) satisfy the polar equation

\[(1.33) \quad \xi^2 = (\mu^4 \cos^2 2\varphi + 2\mu^2 + 1)^{1/2} - \mu^2 \cos 2\varphi. \]

Hence for a given \( \varphi \) there is only one positive solution for \( \xi \) and thus \( D_r \) is star-like with respect to the origin. We note \( (1.33) \) gives \( \xi = \pm \xi_L \) for \( \varphi = \pm \pi/2 \).

We note also that as \( r \to 0 \) \( \mu \) and \( \xi_L \to c. \) In an attempt to find the limiting domain as \( r \to 0 \) let us restrict \( \varphi \) to the interval \( 0 < \varphi < \pi/2 \) and let us take \( \mu \) large. For \( 0 < \varphi < \pi/4 \), \( (1.33) \) can be written as

\[ \xi^2 = \mu^2 \cos 2\varphi \left[ (1 + \frac{2}{\mu^2 \cos^2 2\varphi} + \frac{1}{4 \mu^2 \cos^2 2\varphi})^{1/2} - 1 \right] \]

or, after applying the binomial theorem, as

\[(1.33) \quad \xi^2 = \sec 2\varphi + o(1), \quad \mu \to c. \]

Thus for \( 0 < \varphi < \pi/4 \), \( \xi \to \sqrt{\sec 2\varphi} \) as \( \mu \to c. \) \( r \to 0 \).

On the other hand, for \( \pi/4 < \varphi < \pi/2 \), \( (1.33) \) gives
\[ \xi^2 = -\mu^2 \cos 2\varphi \left( 1 + \frac{2}{\mu \cos^2 2\varphi} + \frac{1}{\mu^2 \cos^2 2\varphi} \right)^{1/2} + 1 \]

or

\[ (1.33)' \quad \xi^2 = -\mu^2 \cos 2\varphi + O(1), \quad \mu \to 0 \]

and therefore \( \xi \to 0 \). The limiting domain is shown cross-hatched in Fig. 1 as well as the domains \( D_r \) for \( \mu = 1 \) and \( \mu = 2 \). The summation method based on this displacement polynomial will be of use in latter work.

3) Continuing in this direction we find that another displacement polynomial leading to a useful method is the polynomial

\[ (1.34) \quad p(z) = (1-r)z + rz^3 \quad (r \neq 0). \]

For \( 0 < r < 1 \) this polynomial has roots at \( z = 0 \) and \( z = \pm i\mu \) where \( \mu = \left( (1-r)/r \right)^{1/2} \) as above.

The recursion formula is now

\[ (1.35) \quad g_{n+1,k} = (1-r)g_{n,k-1} + rg_{n,k-3} \quad (0 \leq k \leq 3(n+1)) \]

\[ = 0 \quad (k > 3(n+1)). \]

The points \( z = \pm 1 \) are again on the boundary of \( D_r \) and if \( z = \xi e^{i\varphi} \) the polar equation of this boundary is given by
Figure 1. Limiting domain for $p(z) = (1-r) + rz^2$ and the domains $D_\mu$ for $\mu = 1$ and $\mu = 2$. 

\[ z^2 = \sec 2\phi \]

\[ x^2 - y^2 = 1 \]
(1.36) \[ \xi^2 (\xi^4 + 2\xi^2 \mu^2 \cos 2\varphi + \mu^4) = (1+\mu^2)^2 \]

which for \( \varphi = \pi/2 \) becomes

(1.37) \[ \xi(\xi^2 - \mu^2) = \pm(1+\mu^2) \]

It suffices to consider the roots of this equation taking the positive sign since the roots for the negative sign are the negative of these. Hence the equation becomes

(1.38) \[ \xi^3 - \xi \mu^2 - (1+\mu^2) = 0 \]

which has one root at \( \xi_L > \mu \) while the cubic has a relative maximum at \( \xi = -\mu \sqrt{3} \) and a value \( 2\mu^3 / 3\sqrt{3} - \mu^2 - 1 \) at this value. This value is positive for all values of \( \mu \) greater than \( \mu_L = 2.910684 \). Thus equation (1.38) has only one real root provided \( \mu < \mu_L \), that is, provided \( r > r_L = (1+\mu_L)^{-2} = .065387 \). The shape of the domain \( D_r \) is as follows: For small values of \( \mu \) it is similar to that in Fig. 1 with \( \mu = 1 \). As \( \mu \) increases, the domain gets pinched in above and below the real axis, while for still larger values of \( \mu \) the domain \( D_r \) breaks into three distinct ovals (Cassini ovals). When \( \mu = \mu_L \) we have the limiting case where the three ovals just meet. While the domain \( D_r \) is not star-like with respect to the origin even
for values of $\mu < \mu_0$, it is true however that the
largest sub-domain of $D_r$ which is star-like with
respect to the origin will contain that segment of
the imaginary axis from $-i\xi_L$ to $+i\xi_L$ provided
$\mu < \mu_0$.

4) Consider finally displacement polynomials of
the form

$$p_\lambda(z) = 1 - r(1-z)^\lambda$$

where $r$ is any complex number $\neq 0$. Let us make the
transformation $w = 1 - z$ so that $p_\lambda(z)$ becomes

$$q_\lambda(w) = p_\lambda(1-w) = 1 - rw^\lambda.$$  

If we now put $w = re^{i\nu}$ and let $E = \{w : |q_\lambda(w)| < 1\}$,
it is not difficult to show that points on the boundary
of $E$ satisfy the polar equation

$$r^\lambda = \frac{2}{r} \cos \lambda \nu.$$  

Specifically, the curve (1.41) is a lemniscate which
has $\lambda$ loops if $\lambda$ is an even integer and $2\lambda$ loops
if $\lambda$ is an odd integer. Thus if we define $D_r$ by
referring to (1.40) we find \(|p_\lambda(z)| < 1\) for all \(z \in D_r\).

By (1.42) \(D_r\) is simply the lemniscate given by (1.41) rotated through an angle \(\pi\) and translated to the point \(z = 1\) from the origin. For \(\lambda > 1\) (\(\lambda = 1\) reduces to the \((E,r)\) method) it is clear that \(D_r\) never contains the unit disk and consequently the matrices \(F\) and \(G\) derived from this displacement polynomial can never be regular.

1.4 The Methods \((E,r,\alpha,\beta)\)

It is possible to obtain a generalization of the \((E,r)\) method and parts of some of Agnew's proofs [1] relating to it by considering the sub-class of Sonnenschein methods for which the displacement polynomial is given by

\[
p(z) = (1-r)z^\alpha + rz^{\alpha+\beta}, \quad \alpha, \beta \text{ positive integers.}
\]

Let us denote by \((E,r,\alpha,\beta)\) the method obtained from (1.43). Thus, in this context, the \((E,r)\) method is equivalent to \((E,r,0,1)\) while our two previous methods corresponding to \(p(z) = (1-r)z^2 + rz^3\) and \(p(z) = (1-r)z + rz^2\) become \((E,r,0,2)\) and \((E,r,1,2)\) respectively. Note that \((E,r,\alpha,0) = (E,0,\alpha,\beta)\). It is easy to see that the matrix \(F = (f_{nk})\) obtained from (1.43) will be given by
From (1.44) we see that

\[(1.45) \quad \sum |f_{nk}| = (|r| + |1-r|)^n \quad (n=0,1,2,\ldots)\]

and that this sum is bounded if and only if \(0 \leq r \leq 1\).

However, for \(\alpha = 0\) and \(r = 0\), (1.43) becomes \(p(z) = 1\) and for such a displacement function the method is certainly not regular. It is now only a simple matter of checking i), ii) and iii) of Section 1.1 to obtain

**THEOREM 2** For \(\alpha = 0\), the \((E,r,\alpha,\beta)\) method is regular if and only if \(0 < r \leq 1\) and for \(\alpha > 0\), if and only if \(0 \leq r \leq 1\).

Let us point out that Sonnenschein [3] has shown that sufficient conditions for the regularity of any Sonnenschein method are that \(f(z)\) satisfy \(\alpha)\) and \(\beta)\) (see Section 1.2) and
Thus for any \((E,r,\alpha,\beta)\) method (aside from the case \(\alpha=0\), \(r=0\)), the conditions are necessary as well.

The next two theorems follow from equivalent ones by Agnew with only a change of subscript and consequently the proofs will not be given in detail. The product of two summation methods can be defined as the method obtained by applying the methods in succession; in terms of matrices it corresponds to the usual matrix product of the two matrices. If such a product is denoted by "\(*\)" we can prove

\textbf{THEOREM 3} \quad (E, r)*(E, s, \alpha, \beta) = (E, rs, \alpha, \beta).

For an arbitrary sequence \(\{s_n\}\), if \(\{t_n\}\) denotes its transform by \((E, r, \alpha, \beta)\) then, using (1.44),

\begin{equation}
(1.46) \quad t_n = \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k s_{\beta k+\alpha n}.
\end{equation}

Assuming \(r \neq 0\) application of \((E, r^{-1})\) to \(\{t_n\}\) yields the sequence \(\{\sigma_n\}\) with

\begin{equation}
(1.47) \quad \sigma_n = \sum_{k=0}^{n} \binom{n}{k} (1-r^{-1})^{n-k} r^{-k} t_n.
\end{equation}

By theorem 3, \(\{\sigma_n\}\) can be obtained by applying \((E, 1, \alpha, \beta)\) to \(\{s_n\}\); but this yields
For any positive integer \( \gamma \), let us say a method permits omission of \( \gamma \) elements if summability of 
\[ s_0, s_1, s_2, \ldots \] implies summability of 
\[ s_\gamma, s_{\gamma+1}, s_{\gamma+2}, \ldots \]
to the same value and permits adjunction of \( \gamma \) elements if summability of 
\[ c_1, c_2, \ldots, c_\gamma, s_0, s_1, \ldots \] implies summability of 
\[ c_1, c_2, \ldots, c_\gamma, \] to the same value where \( c_1, c_2, \ldots, c_\gamma \) are arbitrary constants. (For Agnew's definitions \( \gamma = 1 \)).

**THEOREM 4** If \( \beta = 0 \), \( \alpha \neq 0 \), the \((E, r, \alpha, \beta)\) method permits omission and adjunction of \( \alpha \) elements. 
If \( \alpha = 0 \), \( \beta \neq 0 \) the \((E, r, \alpha, \beta)\) permits omission of \( \beta \) elements if \( r \neq 0 \) and permits adjunction of \( \beta \) elements if and only if \( |1-r| < 1 \).

The first part is immediate for then \( p(z) = z^\alpha \) from (1.43). If \( \{s_n\} \) is an arbitrary sequence its transform under the \((E, r, \alpha, 0)\) method is then \( \{t_n\} \) where \( t_n = s_\alpha n \). The omission of \( \alpha \) elements produces the sequence \( \{t'_n\} \) where \( t'_n = s_\alpha(n+1) \) while the adjunction of \( \alpha \) elements \( c_1, c_2, \ldots, c_\alpha \) produces the sequence \( \{t''_n\} \) where \( t''_0 = c_1 \), \( t''_n = s_\alpha(n-1) \), \( n=1, 2, \ldots \) Thus \( t_n = t'_{n-1} = t''_{n+1} \) and the proof holds.
The second part follows the corresponding theorem by Agnew. It is essential in this proof that one be able to obtain $s_{\beta k}$ in terms of $t_0, t_1, \ldots, t_k$; this follows from (1.48).
CHAPTER II

Application to the Two-Body Problem

2.1 The Two-Body Problem

In the classical two-body problem of celestial mechanics we are interested in the motion of two particles about one another when each is influenced by the gravitational field of the other and by no other forces. Usually the equations of motion are set up for the relative motion of one of the particles about the other. Since such motion is known to take place in a fixed plane, it can be suitably described in relation to a set of rectangular Cartesian axes situated in this plane with the origin at one of the two particles. If proper initial conditions exist, the second particle will then move in an elliptic path with the origin (the first particle) at one of the foci.

The standard equations of motion for the elliptic two-body problem have been reduced to the following (see Danby [1, Chapter 6])

\begin{align}
\tag{2.1} 
& x = a \cos E - ae, \\
& y = a\sqrt{1-e^2} \sin E \\
\tag{2.2} 
& M = E - e \sin E = n(t-T)
\end{align}

where the symbols used are
\textbf{x, y} rectangular coordinates of the moving particle,
\textbf{a} length of semi-major axis of ellipse,
\textbf{e} eccentricity of the ellipse,
\textbf{E} eccentric anomaly,
\textbf{M} mean anomaly,
\textbf{n} mean motion,
\textbf{t} time,
\textbf{T} time of perihelion passage.

The left hand equality in (2.2) gives the well-known Kepler equation. Since \( a \) occurs as a multiplicative constant in (2.1) there will be no loss in generality if we take it equal to 1 hereafter. The solution is obtained when x and y are known functions of \( M \) (thus by (2.2) known functions of \( t \)). F. R. Moulton\cite{1} has shown that these solutions possess singularities in the complex plane at the points

\begin{equation}
M = 2n\pi + \Omega \quad \quad (n=0, \pm 1, \pm 2, \ldots)
\end{equation}

where \( \Omega \) is given in terms of the eccentricity by

\begin{equation}
\Omega(e) = -\sqrt{1-e^2} + \ln[(1+\sqrt{1-e^2})/e] .
\end{equation}

It is not hard to verify that \( \Omega(0) = +\infty \) and \( \Omega(1) = 0 \) and that \( \Omega(e) \) is a monotone decreasing function of
e on the unit interval. Table I gives the values of \( \Omega(e) \) for \( e = .05 (.05) 1.00 \) (i.e. as \( e \) varies from .05 to 1.00 in steps of .05).

Let the solutions of (2.1) and (2.2) be given by

\[
(2.5) \quad x = x(M), \quad y = y(M).
\]

Consideration of (2.3) shows that the singularities of these functions nearest the origin are those at \( \pm i \Omega(e) \) so that the Taylor series for these functions, written as

\[
(2.6) \quad x(M) = \sum x_k M^k, \quad y(M) = \sum y_k M^k,
\]

converge if \( |M| < \Omega(e) \). If we are given a series-to-sequence summation matrix \( G = (g_{nk}) \) which sums the geometric series to \( (1-z)^{-1} \) on \( D \), using (2.3) again we can form the domains \( D(x) \) and \( D(y) \) for the functions (2.5) according to the method outlined in Section 1.1 (here \( D(x) = D(y) \)). Referring to Theorem 1, the functions \( x_n(M) \) and \( y_n(M) \) defined by

\[
(2.7) \quad x_n(M) = \sum g_{nk} x_k M^k, \quad y_n(M) = \sum g_{nk} y_k M^k,
\]

will then converge to \( x(M) \) and \( y(M) \) respectively at all points \( M \) inside \( D(x) = D(y) \). In order to evaluate the functions occurring in (2.7) it is evident that we will need to know the Taylor series coefficients.
### TABLE I

Values of \( \Omega(e) \) for \( e = 0.05(0.05)1.00 \)

<table>
<thead>
<tr>
<th>( e )</th>
<th>( \Omega(e) )</th>
<th>( e )</th>
<th>( \Omega(e) )</th>
<th>( e )</th>
<th>( \Omega(e) )</th>
<th>( e )</th>
<th>( \Omega(e) )</th>
</tr>
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<tbody>
<tr>
<td>0.05</td>
<td>2.6895</td>
<td>0.30</td>
<td>0.9199</td>
<td>0.55</td>
<td>0.3698</td>
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### TABLE II

Degree of Polynomials in the \((E,r)\) Method Giving About 8 Decimal Places of Accuracy for \( e = 0.05, 0.25, 0.50, 0.75, 0.95 \)

\( r/r_L = 0.35(0.05)0.65, \ M' = 0.50 \) and \( \Omega \).

<table>
<thead>
<tr>
<th>( r/r_L )</th>
<th>( M' = 0.35 )</th>
<th>( M' = 0.40 )</th>
<th>( M' = 0.45 )</th>
<th>( M' = 0.50 )</th>
<th>( M' = 0.55 )</th>
<th>( M' = 0.60 )</th>
<th>( M' = 0.65 )</th>
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<tbody>
<tr>
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<td>31</td>
<td>25</td>
<td>23</td>
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<td>( \Omega = 0.50 )</td>
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<td>23</td>
<td>22</td>
<td>20</td>
<td>19</td>
<td>17</td>
</tr>
</tbody>
</table>
in (2.6). For ease of computation a weight factor \( \chi \) is introduced so that \( x_k = \chi^k x_k^* \) and \( y_k = \chi^k y_k^* \). The series (2.6) can now be written

\[
(2.6)' \quad x(M) = \Sigma x_k^*(\chi M)^k, \quad y(M) = \Sigma y_k^*(\chi M)^k.
\]

From (2.1) and (2.2) it is an easy matter to obtain the differential equations

\[
(1-e^2-ex) \frac{dx}{dM} = \frac{-y}{\sqrt{1-e^2}} \quad (2.8)
\]
\[
(1-e^2-ex) \frac{dy}{dM} = \sqrt{1-e^2} (e+x)
\]

and the initial conditions \((M = 0)\)

\[
(2.9) \quad x(0) = x_0^* = 1 - e, \quad y(0) = y_0^* = 0.
\]

When \((2.6)'\) is substituted into \((2.8)\) we obtain the recurrence relations

\[
(1-e)x_n^* = \frac{1}{\chi} \left( \frac{1+e}{1-e} \right)^{1/2} x_1^*,
\]

\[
(1-e)x_n^* = -\sqrt{1-e^2} y_{n-1}^* + e \chi \sum_{k=1}^{n-1} (n-k)x_k^* x_{n-k}^*, \quad (n=2,3,4,\ldots)
\]

\[
(1-e)ny_n^* = \sqrt{1-e^2} x_{n-1}^* + e \chi \sum_{k=1}^{n-1} (n-k)x_k^* y_{n-k}^*.
\]
In this chapter the weight factor $\chi$ is always taken equal to $\Omega^{-1}(e)$. We note that with the weight factor $\chi$ involved the expressions (2.7) now become

$$(2.7)' \quad x_n(M) = \sum g_{nk} x_k^*(\chi M)^k, \quad y_n(M) = \sum g_{nk} y_k^*(\chi M)^k.$$ 

### 2.2 The $(E,r)$ Method in the Two-Body Problem; Brumberg's Results

The matrix $G(r) = (g_{nk}(r))$ for the $(E,r)$ method was discussed in Section 1.3 and a recursive formula for the entries $g_{nk}(r)$ used in the expressions (2.7)' was given there by (1.30). It remains therefore to determine some suitable value for the parameter $r$ and to construct the domain $D_r(x) = D_r(y)$ which is dependent on this parameter and in which the functions $x_n(M,r)$ and $y_n(M,r)$ in (2.7)' converge. Hereafter, we take $0 < r < \infty$. Since the functions $x(M)$ and $y(M)$ are periodic in $M$ of period $2\pi$ and since $x(\pi+M) = x(\pi-M)$, $y(\pi+M) = -y(\pi-M)$ (see (2.1) and (2.2)), let us consider the problem of determining $r$ so that the interval $[0,M']$, $M' < \pi$, lies inside $D_r(x)$. Referring to the construction of $D_r(x)$ as outlined in Section 1.1, for every singular point in (2.3) of the form $2n\pi \pm i\Omega$, $n \neq 0$, the corresponding domains $D(\theta)$ contain $[0,2\pi]$, and hence $[0,M']$. On the
other hand, the domains \( D(\pi/2) \) and \( D(3\pi/2) \) corresponding to the singularities \( i\Omega \) and \(-i\Omega\) respectively will contain \([0, M']\) if and only if \( r \) is chosen so that (see Figure 2)

\[
\Omega(\frac{2}{r} - 1)^{1/2} > M',
\]

i.e., if and only if

\[
0 < r < r_L = \frac{2}{1 + (M'/\Omega)^2}.
\]

We note in passing that if \( M' = \Omega \), then \( r_L = 1 \) and if \( M' > \Omega \), then \( r_L < 1 \). Consequently, in choosing the parameter \( r \) we require \( 0 < r < r_L \). However, if \( r \) is taken too small, while it is true that the expressions (2.7)' still converge on \([0, M']\), the effectiveness of this convergence will be reduced. This will be pointed out in the following paragraphs where we shall obtain the estimate \( r = .5r_L \) for the most suitable value for \( r \).

The basis for this estimate is as follows: Let \( K \) be any compact subset of the \( z \)-plane which can be contained in some circular domain \( D_{r_o} \), \( 0 < r_o < \infty \), corresponding to a given \((E, r_o)\) method. Then for \( 0 < r < r_o \), \( K \subset D_r \). If \( d_r(z) \) represents the
Figure 2 The domain $D(\pi/2)$.

Figure 3 The domain $K$. 
distance from the center of the circle \( D_r \) to an arbitrary point \( z \), define \( d_r(K) \) by

\[
(2.12) \quad d_r(K) = \lim \sup_{z \in K} d_r(z).
\]

Finally, if this quantity is divided by the radius of the circle we obtain the ratio

\[
(2.13) \quad \xi(r,K) = r d_r(K)
\]

which is a measure of the relative distance of the compact set \( K \) from the center of the circle \( D_r \). Let \( \{g_n(z,r)\} \) represent the sequence of polynomials which arises from the \((E,r)\) method and converges to \((1-z)^{-1}\) in \( D_r \). Then it is easy to check that the polynomial \( g_n(z,r) \) can be obtained by constructing the Taylor series for \((1-z)^{-1}\) about the point \( 1 - \frac{1}{r} \), i.e. about the center of \( D_r \), truncating this series after \( n \) terms, expanding the expressions \( \{z-(1-1/r)\}^k \), \( k=1,2,\ldots,n \), by the binomial theorem, and rearranging the resulting double series in powers of \( z \). Thus the sequence \( \{g_n(z,r)\} \) is a sequence of polynomials arising from a truncated Taylor series with circle of convergence \( D_r \) and \( K \) lies interior to this circle.
Consequently, the best $r$ value should be that for which the relative distance from $K$ to the center of the circle is a minimum, i.e. for which $\xi(r,K)$ is a minimum.

Returning to the two-body problem with a given $(E,r)$ method, we must first reconsider (1.15) of Chapter I where the function $a(z)$ is now replaced by the function $x(M)$ of the two-body problem, the functions $a_n(z)$ by $x_n(M,r)$ and $g_n(z/\zeta)$ by $g_n(M/\zeta,r)$. The compact set $\Delta$ is now taken to be the line sequent $[0,M']$. This equation can now be written

\begin{equation}
|x(M)-x_n(M,r)| = \frac{1}{2\pi} \left| \int_{C_\Delta} \frac{x(\zeta)}{\zeta} \left\{ (1-M/\zeta)^{-1} - g_n(M/\zeta,r) \right\} d\zeta \right|
\end{equation}

where $C_\Delta$ lies in the principal star of $x(M)$ and contains the origin and $\Delta$. The point set $K(C_\Delta)$ defined by

$$K(C_\Delta) = \{ z : z = M/\zeta, M \in \Delta, \zeta \in C_\Delta \}$$

is a compact subset of the $z$-plane and $K(C_\Delta) \subset D_r$ provided $0 < r < r_L$. Furthermore, it is not difficult to prove that once $C_\Delta$ is determined any other simple closed rectifiable orientable curve lying in the principal star of $x(M)$ and containing $C_\Delta$ in its interior can
replace $C$ in Lemma 1. Moreover, the difference in (2.14) is not affected by using this larger curve. However, as the curve is taken larger and larger the point set $K(C^\Delta)$ shrinks onto the point set

\[(2.15) \quad K = \{ z : z = M/\zeta, M \in \Delta, \zeta \text{ a singular point of } x(M) \} . \]

Using (2.3) we see that $K$ consists of all radial line segments issuing from the origin and ending at a point $M'/(2\pi r + i\Omega)$, $n = 0, +1, +2, \ldots$ (see Figure 3). We note further that $K$ is dependent only on $M'$ and the singularities of $x(M)$ and not on $C^\Delta$. Let us now assume that, as a function of $r$, the left-hand side of (2.14) attains its minimum value whenever the bracketed quantity inside the integral on the right-hand side does, $M/\zeta$ being taken in $K$. From the last paragraph, this will occur when $\xi(r, K)$ is a minimum.

It is not difficult to show that, except for very small values of $\epsilon (< .0001)$, the quantity $d_r(K)$ in (2.12) is obtained by taking the maximum of the distances from the center of the circle $D_r$ to either the point $\frac{M'}{-i\Omega} = \frac{\lambda M'}{-i\pi}$ or the point $\frac{M'}{2\pi - i\Omega} = \frac{M'}{\pi}(-\frac{2\lambda^2}{1+4\lambda^2} + \frac{\lambda}{1+4\lambda^2})$ where $\lambda = \frac{\pi}{\Omega}$. The perpendicular bisector of
the line joining these two points intersects the real
z-axis at \(-\frac{M'}{\pi} \lambda^2\). Thus

\[ d_r(K) = \{ (1 - \frac{1}{r})^2 + \left( \frac{M'}{\pi} \right)^2 \}^{1/2}, \quad -\frac{M'}{\pi} \lambda^2 \leq 1 - \frac{1}{r} \leq 0 \]

, i.e. \( \frac{1}{1 + \frac{M'}{\pi} \lambda^2} \leq r \leq 1 \),

\[ = \{ (1 - \frac{1}{r} - \frac{M'}{\pi} \frac{2\lambda^2}{1+4\lambda^2})^2 + \left( \frac{M'}{\pi} \lambda^2 \right)^2 (1+4\lambda^2)^2 \}^{1/2}, \]

\[ -\infty < 1 - \frac{1}{r} \leq -\frac{M'}{\pi} \lambda^2 \]

, i.e. \( 0 < r \leq \frac{1}{1 + \frac{M'}{\pi} \lambda^2} \).

If we now form \( \xi(r,K) = rd_r(K) \) we can use elementary
calculus to show that, since \( 0 < M' \leq \pi \), a minimum
for \( \xi(r,K) \) occurs when

(2.16) \[ r = \frac{1}{1 + (M'/\Omega)^2} = .5r_L. \]

The validity of this estimate is illustrated very
well in Table II. In the calculation for this table
\( M' = .5\Omega(e) \) and \( \Omega(e) \) so that \( r_L = 1.6 \) and \( 1.0 \)
respectively. The quantities \( x_n(M',r) \) and \( y_n(M',r) \)
were calculated recursively for e-values \( .05, .25, .50, .75 \) and \( .95 \) and for \( r/r_L \) ratios \( .35(.05).65 \).
until an n-value giving around 8 decimal places of accuracy was obtained. This n-value (the degree of the polynomials $x_n(M',r)$ and $y_n(M',r)$) was then listed in Table II where it is noted that it is minimal for the $r/r_L$ ratio 0.5.

If $M' = \Omega$, then (2.11) gives $r_L = 1$ and, according to the last paragraph, the best $r$-value is 0.5. With $r$ set equal to 0.5 the quantities $x(M)$ and $y(M)$ were calculated to about 8 decimal places of accuracy for $e = 0.05(0.05)0.95$ and $M = 0.1\Omega(0.1\Omega)\Omega$ and these values were listed in Table III. The corresponding degree of the polynomials $x_n(M,r)$ and $y_n(M,r)$ necessary to obtain this accuracy was noted as well, and these results were compiled in Table IV.

We note from this table that there is an improvement in the rate of convergence for larger values of $e$. It is clear that the degrees of the polynomials as listed in Table IV could be lowered if the $r$-value was adjusted to each value of $M$ used above instead of being fixed at 0.5. This essentially involves a recalculation of the matrix $G(r) = (g_{nk}(r))$ for each $M$ value. The above program was instituted, however, and, for each $M$ value, $r$ was put equal to $0.5r_L$ where $r_L$ is given by (2.11). The values obtained for $x(M)$ and
$y(M)$ were identical with those in Table III up to 7 decimal places and often in the 8-th. The corresponding polynomial degrees were entered in Table V where a comparison shows them to be much lower than those in Table IV for small values of $M$ and, of course, equal for $M = \Omega$. Moreover, because of the simplicity of calculation of the matrix $G(r)$ and the reduced degrees of the polynomials the computer time in both cases was roughly the same.

For the sake of comparison let us consider the methods and results of V. A. Brumberg [1] pertaining to the two-body problem. His first method is based on a summation matrix $G = (g_{nk})$ which sums the geometric series in its principal star. While such a method has obvious advantages over the methods used in this chapter when applied to the three-body problem (see Section 5.1), it is not as effective when applied to the two-body problem. As an example we note from Brumberg's tables that in order to obtain the values of $x(M)$ and $y(M)$ for $M = \Omega$ and $e = .05(.05).95$ with the same accuracy as ours above, his polynomials are roughly of degree 150 which exceeds the degree of our polynomials by 3 to 5 times (see Table IV, also Tables VII and IX to come). We note in passing that his values for $x(\Omega)$ and $y(\Omega)$ agree with ours at least to 7 decimal places and often in the 8-th.
### TABLE III (First part)

Values of $x(M)$ and $y(M)$ from the $(E,r)$ Method for $\epsilon = 0.05(0.05).95$ and $M = 1.0(1.0).100$

<table>
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<tr>
<th>$M/\epsilon$</th>
<th>$0.05$</th>
<th>$0.10$</th>
<th>$0.15$</th>
<th>$0.20$</th>
<th>$0.25$</th>
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<td>$1\Omega$</td>
<td>0.91024775E00*</td>
<td>0.87549766E00</td>
<td>0.83246209E00</td>
<td>0.78659904E00</td>
<td>0.73938225E00</td>
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<td>$2\Omega$</td>
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<td>0.50791608E00</td>
</tr>
<tr>
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<td>-0.14255914E00</td>
<td>0.17215206E00</td>
<td>0.3110718QE00</td>
<td>0.38143954E00</td>
<td>0.41453135E00</td>
</tr>
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<tr>
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</table>

* $E + nm$ is equivalent to multiplication by $10^{+nm}$.
### TABLE III (Second part)

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<th>$y$-values</th>
</tr>
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TABLE III (Third part)
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### TABLE IV

Degrees of Polynomials in \((E,r)\) Method Giving About 8 Dec. Places of Accuracy for \(e = .05(.05)95, M = .10(.10)\), \(r = .5\)

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### TABLE V

Degrees of Polynomials in \((E,r)\) Method giving About 8 Dec. Places of Accuracy for \(e = .05(.05)95, M = .1 (.1)\), \(r = .5r_L\)

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A second more effective method investigated by Brumberg involves the so-called Sundman series. In this method the strip of width 2Ω about the real M-axis is mapped conformally onto the unit disk in the θ-plane under the Poincare transformation

\[
\theta = \frac{\exp(\pi M/2\Omega) - 1}{\exp(\pi M/2\Omega) + 1}
\]

in which the real M-axis is mapped onto the real θ-interval (-1,+1). The solutions of the two-body problem as functions of θ on (-1,+1) are then given directly by their Taylor series. A comparison of the effectiveness of this method with the (E,r) method as illustrated in Table V shows them to be approximately equivalent, the (E,r) method being somewhat more effective for higher values of e. Of course, with the (E,r) method there is the disadvantage of having to calculate the matrix G(r), but, on the other hand, it gives the solution directly in terms of M.

2.3 Application of the Methods (E,r,0,2) and (E,r,1,2)

Let us first consider the (E,r,0,2) method based on the displacement polynomial \( p(z) = (1-r) + rz^2 \) where the matrix \( G(r) = (g_{nk}(r)) \) is obtained with the aid of the recursion formula (1.32). The domain \( D_r \)
is illustrated in Figure 1 for \( \mu = 1 \) and \( \mu = 2 \) where \( \mu = \{(1-r)/r\}^{1/2} \). As in the \((E,r)\) method it is clear that one need only determine \( r \) so that the domains \( D^{(\pi/2)} \) and \( D^{(3\pi/2)} \) contain \([0,M']\), \( M' \leq \pi \), in order for this interval to lie in \( D_r(x) = D_r(y) \). We note that for this method \( D^{(\pi/2)} = D^{(3\pi/2)} \). From Section 1.3, the above will hold provided

\[
M' < \Omega = \Omega(2r^2+1)^{1/2}
\]

or, since \( \Omega = (2r^2+1)^{1/2} = \{(2-r)/r\}^{1/2} \), provided

(2.18)

\[
0 < r \leq r_L = \frac{2}{1 + (M/\Omega)^2}
\]

where we notice that \( r_L \) is given by the same expression as in the \((E,r)\) method. Unfortunately, the problem of determining the best \( r \)-value between these bounds seems rather difficult. Indeed, as opposed to the \((E,r)\) method, the best \( r/r_L \) ratio varies with the choice of \( M' \). This is pointed out in Table VI which is equivalent to Table II for the \((E,r)\) method. In the former table the values of \( x(M) \) and \( y(M) \) were calculated to around 8 decimal places of accuracy for \( M = .5\Omega \) and \( M = \Omega \), \( e = .05, .25, .50, .75 \) and \( .95 \) and \( r/r_L = .55(.05).85 \). Table VI clearly shows that
for $M = .5\Omega$ the best $r/r_L$ ratio is .60 while for $M = \Omega$ it becomes .70. The table also shows on comparison with Table II that the dependence of the rate of convergence on the parameter $r$ is more crucial with the present method.

There is however a geometrically intuitive reason for supposing that this method will provide better convergence than the $(E,r)$ method. Note that the condition $[0,M'] \subset D_r(x)$ is equivalent to the condition $K \subset D_r$ where $K$ is the point set given by (2.15) and illustrated in Figure 3. From Figure 1, we see that, if $0 < r < r_L$, the domain $D_r$ of the present method fits symmetrically about the point set $K$ which is not the case in the $(E,r)$ method. Consequently, $K$ lies closer to the "center" of $D_r$. On the strength of this argument, the values of $x(M)$ and $y(M)$ were calculated with the same accuracy as in the $(E,r)$ method for $e = .05(.05).95$ and $M = .1\Omega(.1\Omega)\Omega$. In these calculations, since $M = \Omega$ and, therefore, $r_L = 1$, the value of $r$ was fixed at .70 according to the results of Table IV. The actual values for $x(M)$ and $y(M)$ agreed with those in Table III; the degrees of the polynomials necessary to obtain these values were listed in Table VII. These results are very interesting in that they show the rate of convergence
TABLE VI
Degree of Polynomials Giving About 8 Dec. Places of Accuracy in 
(E, r, 0, 2) Method for \( e = 0.05, 0.25, 0.50, 0.75, 0.95 \), \( r/r_L = 0.55(0.05) \), \(0.85\) and \( M = 0.50 \) and \( \Omega \).

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TABLE VII
Degree of Polynomials Giving About 8 Dec. Places of Accuracy in 
(E, r, 0, 2) Method for \( e = 0.05(0.05)0.95, M = 0.10(0.10)\Omega \) and \( r = 0.7 \).

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for a fixed \( r \) to be almost independent of \( M \) in \([0,\Omega]\). Indeed, for \( e = .60 \) and .65 the degree of the polynomial is 30 for each \( M \)-value. Comparing this table with Table IV for the \((E,r)\) method we note that for smaller values of \( M \) the \((E,r)\) method is more effective while for larger values of \( M \), and in particular for \( M = \Omega \), the latter method is.

Finally, let us consider the \((E,r,1,2)\) method based on the displacement polynomial \( p(z) = (1-r)z + rz^3 \) whose matrix is determined from (1.35). For such a polynomial, \( r_L \) is given as the solution of \( M = \Omega \) where \( \xi \) satisfies the cubic equation (1.37) with \( \mu = \{(1-r_L)/r_L\}^{1/2} \).

Consequently, \( r_L \) does not have the simple form of the previous two methods. In order to determine a best \( r \)-value for \( M' = \Omega \), \( x(\Omega) \) and \( y(\Omega) \) were calculated with the usual accuracy for \( e = .05, .25, .50, .75, .95 \) and \( r = .45(.05).85 \). The degrees of the polynomials necessary to obtain these results are listed in Table VIII from which we see \( r = .60 \) is the best value. With \( r \) fixed at .60, Table XI was then prepared giving the degrees of the polynomials \( x_n(M,r) \) and \( y_n(M,r) \) necessary to obtain the usual accuracy for \( e = .05(.05).95 \) and \( M = .1\Omega(.1\Omega)\Omega \). Comparing this table with Table VII we note a further general improvement in the rate of
### TABLE VIII

Degree of Polynomials Giving About 8 Dec. Places of Accuracy in (E,r,1,2) Method for e=.05,.25,.50,.75,.95, M=.45(.05).85 and M=Ω

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### TABLE IX

Degrees of Polynomials Giving About 8 Dec. Places of Accuracy in (E,r,1,2) Method for e=.05,.10,.15,.20,.25,.30,.35,.40,.45,.50,.55,.60,.65,.70,.75,.80,.85,.90,1.00, M=.1Ω(1Ω)Ω and r=.60

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convergence in the present method over the \((E,r,0,2)\) method.

In summary, we can conclude that the methods \((E,r)\), \((E,r,0,2)\) and \((E,r,1,2)\) become progressively more effective in obtaining the solution to the two-body problem on the interval \([0,\Omega]\) in the sense that the maximum degree of polynomial necessary to obtain a fixed accuracy is progressively reduced. However, the methods themselves become progressively more complicated and, in particular, the best value of the parameter \(r\) becomes harder to evaluate. Since in the \((E,r)\) method this value can be computed to be 
\[
.5r_L = 1/(1+(M/\Omega)^2)
\]
for any \(M \in [0,\pi]\), this method is particularly useful. The usefulness of the "stability" of the \(r/r_L\) ratio is further illustrated in the next section.

2.4 Analytic Continuation to \([0,\pi]\)

Ideally, as far as the two-body problem is concerned one would like a method which would sum to the solutions on \([0,\pi]\) for all values of \(e\); as pointed out in Section 2.1 symmetry and periodicity would then give their values for all values of \(M\). From Table I we note that for all values of \(e\) treated thus far, \(\Omega(e) < \pi\). Consequently, in order to obtain the same accuracy on \([0,\pi]\) as on \([0,\Omega]\) one would expect the
degrees of the polynomials $x_n(M,r)$ and $y_n(M,r)$ to increase. Moreover, since $\Omega(e) \to 0$ as $e \to 1$, the degree of these polynomials will also increase as $e$ increases.

Table X illustrates the results obtained in attempting to apply the above three methods to obtain the solutions at $M = \pi$. From (2.1) and (2.2) it is easy to see that the exact values of $x(M)$ and $y(M)$ at $M = \pi$ are $-(1+e)$ and 0 respectively. For Table X these results were achieved to at least 8 decimal places of accuracy. The table itself lists the degrees of the polynomials necessary to obtain this accuracy for various $r/r_L$ ratios for the $(E,r)$ and $(E,r,0,2)$ methods and for various $r$-values for the $(E,r,1,2)$ method. However, the polynomials $x_n(M,r)$ and $y_n(M,r)$ were arbitrarily limited to degree 150 and in the cases where this degree was reached before a sufficient accuracy was obtained a dash is registered in the table. In no case were results obtained for $e > .15$. From the table we note that the best $r/r_L$ ratio for the $(E,r)$ method is .55 which still agrees quite well with our estimate of .5. Note that for $e = .05$ and $e = .10$ the maximum degree is again progressively lowered with each method. However, for $e = .15$ the $(E,r)$ method alone gave a result within the allotted range. This may
be due to the fact that not enough $r/r_L$ ratios or $r$ values were tested for the other two methods.

In concluding this chapter let us note that the limitation above to polynomials of degree 150 is completely arbitrary. The only practical consideration in this direction is the time used on the computer which is increased if the degree is larger. Moreover, the time taken to evaluate the $n$-th polynomial varies with $n$ and therefore it grows disproportionately large as $n$ increases. Aside from any academic interest in the above methods, it is for this reason that greater effectiveness is desired.
## TABLE X

Degree of Polynomials Giving 8 Dec. Places of Accuracy for Various $r/r_L$ Ratios and $r$ Values, $e=.05,.10,.15$ and $M=\pi$

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CHAPTER III

Application to the Three-Body Problem

3.1 The Three-Body Problem; Brumbergs Method

The three-body problem is concerned with the motion of three particles with masses $m_0$, $m_1$ and $m_2$ attracting one another according to Newton's law of gravitation. A suitable origin for an inertial reference frame in which to describe this motion can be taken at the center of mass of the three particles for, in any inertial frame, this center of mass moves with a constant velocity. Often however the equations are not set up using the coordinates of the particles as the dependent variables, but rather their relative coordinates. Thus if the vector distances between the particles are given by $\mathbf{r}_0 = m_1 m_2$, $\mathbf{r}_1 = m_2 m_0$ and $\mathbf{r}_2 = m_0 m_1$, the equations of motion in relative coordinates can be written

$$\mathbf{r}_i = -fM \frac{\mathbf{r}_i}{r_i^3} + fm_j (\frac{\mathbf{r}_j}{r_j^3} + \frac{\mathbf{r}_1}{r_1^3} + \frac{\mathbf{r}_2}{r_2^3}) \quad (i=0, 1, 2)$$

where $f$ is the gravitational constant, $M$ is the sum of the masses and $r_i = |\mathbf{r}_i|$ ($\cdot = \frac{d}{dt}$). Given $\mathbf{r}_1(0)$ and $\dot{\mathbf{r}}_1(0)$, the solution of the three-body problem would then entail solving the system of
differential equations (3.1) subject to these initial conditions.

In an exhaustive paper on the subject K. Sundman [1] showed that the only singularities in the three-body problem for real time occur when two or all three particles collide. He also proved that if the angular momentum (which is constant during the motion) is not zero, triple collisions cannot take place. Finally he showed that by introducing a new regularizing parameter \( w \) in place of the parameter \( t \), the motion can be analytically continued past any double collision which might occur and consequently that the relative coordinates \( \vec{r}_i, i=0,1,2 \) and the time \( t \) are analytic in a strip of width \( 2\Omega \) about the real \( w \)-axis. The constant \( \Omega \) depends on the initial conditions and the masses of the particles. The choice of \( w \) itself may vary; a usual choice is given by

\[
(3.2) \quad \frac{dw}{dt} = U, \quad w = 0 \text{ when } t = 0,
\]

where \( U \) is the force function

\[
(3.3) \quad U = f\left(\frac{m_1 m_2}{r_0} + \frac{m_2 m_0}{r_1} + \frac{m_0 m_1}{r_2}\right).
\]

If we introduce at this point a weight factor \( \chi \) so that instead of (3.2) we use
(3.2)' \quad \text{d}w = \chi \text{v} \text{U} \text{d}t, \\

then (3.1) can be transformed into

\begin{equation}
(3.4) \quad \bar{r}_1'' + \frac{1}{U} u' \bar{r}_1' = \frac{1}{2U^2} \left[ -rM \bar{r}_1 + rM_1 \left( \bar{r}_2 + \bar{r}_1 + \bar{r}_2 \right) \right] \\
(i=0,1,2)
\end{equation}

where \( \cdot' = \frac{d}{dw} \).

In order to apply any summation methods it is necessary to find the Taylor series coefficients of the vector functions \( \bar{r}_1(w) \). This problem was neatly solved by Steffanson [1] as follows: Let \( \Delta_1 \) and \( \sigma_1 \) be defined by

\begin{equation}
(3.5) \quad \Delta_1 = r_1^2, \quad \sigma_1 = r_1^{-3} \quad (i=0,1,2).
\end{equation}

Then system (3.4) can be reduced to the following system:

\begin{align*}
\Delta_1 &= r_1^2, \quad (i=0,1,2) \\
2\Delta_1 \sigma_1' + 3\sigma_1 \Delta_1' &= 0 \quad (i=0,1,2)
\end{align*}

\begin{equation}
(3.6) \quad \bar{v} \bar{r}_1'' + \frac{1}{2} \bar{v}' \bar{r}_1' = -\frac{rM}{\chi} \sigma_1 \bar{r}_1 + \frac{rM_1}{\chi} (\sigma_0 \bar{r}_0 + \sigma_1 \bar{r}_1 + \sigma_2 \bar{r}_2), \\
(U = f(\bar{m}_1 \bar{m}_2 \sigma_0 \Delta_0 + \bar{m}_2 \sigma_1 \Delta_1 + \bar{m}_1 \sigma_2 \Delta_2), \\
V = U^2, \\
\chi Ut' &= 1.
\end{equation}
The importance of this system is that it is of second degree only in the variables concerned. Consequently, if one assumes power series solutions for these variables of the form

\[
\bar{r}_i = \sum \bar{r}_{ik} \omega^k, \quad \Delta_i = \sum \Delta_{ik} \omega^k, \quad \sigma_i = \sum \sigma_{ik} \omega^k \quad (i=0,1,2)
\]

\[
U = \sum u_k \omega^k, \quad V = \sum \sigma_k \omega^k, \quad t = \sum t_k \omega^k \quad (t_0=0)
\]

the first coefficients are given by

\[
\bar{r}_{10} = \bar{r}_1(0), \quad \Delta_{10} = |\bar{r}_1(0)|^2, \quad \sigma_{10} = |\bar{r}_1(0)|^{-3} \quad (i=0,1,2)
\]

\[
u_0 = f(m_1m_2\sigma_{o0}\Delta_{o0} + m_2m_1\sigma_{10}\Delta_{10} + m_1m_2\sigma_{20}\Delta_{20}),
\]

\[
v_0 = u_o^2, \quad \bar{r}_{11} = \frac{\bar{r}_1(0)}{\chi u_o}, \quad t_1 = \frac{1}{\chi u_o}.
\]

The substitution of (3.7) into (3.6) gives the recurrence relations

\[
\Delta_{ik} = \sum_{j=0}^{k} \bar{r}_{ij} \bar{r}_{ik-j},
\]

\[
\sigma_{ik} = \sum_{j=0}^{k-1} \frac{1}{2k\Delta_{10}} \sigma_{ij} \Delta_{ik-j} (3k-j) \sigma_{ij} \Delta_{ik-j}
\]

\[
u_k = f \sum_{j=0}^{k} (m_1m_2\sigma_{o0}\Delta_{ok-j} + m_2m_1\sigma_{10}\Delta_{1k-j} + m_1m_2\sigma_{20}\Delta_{2k-j}),
\]

\[
v_k = \sum_{j=0}^{k} u_j u_{k-j},
\]

\[
\bar{r}_{ik+1} = \frac{1}{k(k+1)v_o} \sum_{j=0}^{k-1} \left[ -\frac{1}{2}(j+1)(k+1)v_{k-j} \bar{r}_{ij+1} - \frac{fM}{\chi^2} \sigma_{ij} \bar{r}_{ik-j-1} \right]
\]
Using the above ideas V. A. Brumberg [1] outlined a method which can theoretically be used to obtain the solution of the three-body problem at each point of the real $\omega$-axis, at least in those cases where the total angular momentum is not zero. It was noted in Section 2.2 that this method is not the most effective when applied to the two-body problem. Brumberg's basic idea is to construct a matrix $G = (g_{nk})$ which sums the geometric series in its principal star. According to Section 1.1 such a method could then be used to sum the functions $\bar{r}_i = (x_i, y_i, z_i)$, $i = 0, 1, 2$, and $t$ of the three-body problem in their principal stars which, in the regularized problem, contain the entire real $\omega$-axis. In applying his method to the three-body problem Brumberg gives results corresponding to three particular problems which were worked out numerically by Zumkley [1], Strömgren [1] and Burrau [1]. In each of these examples the choice of the weight factor $\chi$ in (3.2)' is practically important for it determines the rate of growth of the coefficients obtained recursively from (3.9); if this rate is too high or too low one is
limited to the number of coefficients the computer will handle. Brumberg uses $\chi^{-1} = 3.25, 16.25 \text{ and } 40$ respectively in the above three examples and gives in these examples the results of employing his method for $\omega = 1$.

3.2 Limitations and Results for Sonnenschein Methods

Because of the simplicity of the methods used in Chapter II and their greater effectiveness in the two-body problem as opposed to Brumberg's method outlined above it would be of interest to try and apply them to the three-body problem. Of course one could never get a solution valid for all real values of $\omega$ because in all cases the domains $D_r$ are finite, and consequently so are the domains $D_r(x_1), D_r(y_1)$, etc. where $\overline{r}_1 = (x_1, y_1, z_1), \ i=0,1,2,$ and the domain $D_r(t)$. What then is the largest interval on which convergence will occur?

To solve this problem let us assume that every point of the form $\omega_0 \pm i\Omega$, $-\infty < \omega_0 < +\infty$ is a singular point of the three-body problem where $\Omega$ is the constant referred to earlier in this section, and let $(-W(r), W(r))$ denote the largest obtainable interval for a specific method and a fixed value of $r$. 
Considering first the \((E,r)\) method one finds that according to the construction of Section 1.1 the interval \((-W(r), W(r))\) must be interior to every domain \(D^{(\theta)}\), \(-\pi < \theta \leq \pi\) where we have put \(\omega_o + i\Omega = re^{i\theta}\). Here \(D^{(\theta)}\) is the interior of the circle passing through \(re^{i\theta}\) and centered at 
\[\Omega\left(\frac{1}{r} - 1\right)(\cot \theta + 1)\] (see Figure 4). It is clear that we need only consider \(0 < \theta < \pi/2\). If the above circle intersects the positive real \(\omega\)-axis at \(W(r,\theta)\) then from the equation of this circle we obtain

\[(3.11) \quad W(r,\theta) = \Omega\left[\frac{1}{r^2} \csc^2 \theta - (\frac{1}{r} - 1)^2\right]^{1/2} - (\frac{1}{r} - 1)\cot \theta].\]

Minimizing \(W(r,\theta)\) as a function of \(\theta\) we find

\[(3.12) \quad \cot \theta = 1 - r.\]

Consequently, as \(r\) varies from 1 to 0, \(\theta\) varies from \(\pi/2\) to \(\pi/4\). Further, \((3.11)\) and \((3.12)\) combine to yield

\[(3.13) \quad W(r) = (2-r)\Omega\]

and therefore as \(r \to 0\), \(W(r) \to 2\Omega\). Thus in the regularized three-body problem the \((E,r)\) method can only be used with confidence on the interval \((-2\Omega, 2\Omega)\).
Figure 4  Domain $D(\theta)$

Figure 5
If the same notation is adopted for the \((E,r,0,2)\) method, use of equation (1.33) easily gives

\begin{equation}
W(r,\theta)^2 = \Omega^2 \csc^2 \theta \left[ (\mu^4 \cos^2 2\theta + 2\mu^2 + 1)^{1/2} - \mu^2 \cos 2\theta \right]
\end{equation}

where as usual \(\mu = (\frac{1}{r}-1)^{1/2}\). Putting \(A(\mu,\theta) = [W(r,\theta)/\Omega]^2\) (3.14) becomes

\begin{equation}
\sin^4 \theta A^2(\mu,\theta) + 2\mu^2 \sin^2 \theta (1-2 \sin^2 \theta) A(\mu,\theta) - (2\mu^2 + 1) = 0.
\end{equation}

Noting that \(A(\mu,\theta)\) takes on a minimum value whenever \(W(r,\theta)\) does, differentiation of (3.15) to determine this minimum gives

\[\cos \theta = 0, \text{ or } A(\mu,\theta) = \mu^2 (4 - \csc^2 \theta).\]

But for \(\theta = \pi/2\) (3.14) yields \(W(r,\pi/2) = \Omega(2\mu^2 + 1)^{1/2}\) which tends to infinity with \(\mu\) (therefore as \(r \to 0\)).

Using the definition of \(A(\mu,\theta)\) and the second condition above, (3.14) now gives

\begin{equation}
\sin \theta = \frac{1+\mu^2}{2\mu^2}
\end{equation}

and therefore \(\theta \to \pi/6^+\) as \(\mu \to \infty\). If \(\mu\) is large (3.14) gives

\[W(r) \approx \sqrt{2} \Omega \mu [(1+\frac{8}{\mu^2} + \frac{4}{\mu^4})^{1/2} - 1]^{1/2}\]
or, using the binomial expansion twice,

\[ W(r) = 2\sqrt{2} \Omega (1+o(1)), \quad r \to 0. \]

Consequently as \( r \to 0 \), \( W(r) \to 2\sqrt{2} \Omega \) and the present method is restricted to the interval \((-2\sqrt{2}\Omega, 2\sqrt{2}\Omega)\).

The above estimates can be improved if the radius of convergence \( R \) of the series in (3.7) for \( \bar{r}_1 \) and \( t \) are taken into account (\( R \) may be taken as the radius of a specific one of these series or perhaps as the minimum of the 10 possible values). Suppose \( R > \Omega \) and that \( \overline{\theta} = \sin^{-1}(\Omega/R) \) (see Figure 5). For the \((E,r)\) method, since \( \theta \to \pi/4^+ \) as \( r \to 0 \) in the above analysis, we see that for \( \pi/4 \leq \overline{\theta} \leq \pi/2 \) the interval \((-2\Omega, 2\Omega)\) remains the limiting interval. If on the other hand \( 0 < \overline{\theta} < \pi/4 \), the value of \( W(r) \) will be obtained solely from the construction of \( D(\overline{\theta}) \), that is \( W(r) = W(r, \overline{\theta}) \). Since \( \csc \theta = R/\Omega \) and \( \cot \overline{\theta} = (R^2-\Omega^2)^{1/2}/\Omega \), (3.13) gives

\[ W(r) = \frac{R}{[1-(\Omega/R)^2]^{1/2}} + o(1), \quad r \to 0 \]

so that as \( r \to 0 \), \( W(r) \to R/[1-(\Omega/R)^2]^{1/2} \). We note for \( \overline{\theta} = \pi/4 \), \( R = \sqrt{2}\Omega \) and (3.18) gives a limiting value of \( 2\Omega \) as before, while for \( \overline{\theta} = 0 \), \( \Omega = 0 \) and \( W(r) \to R \) as expected.
For the \((E,r,0,2)\) method a similar argument to that above shows that the limiting interval remains unchanged for \(\pi/6 \leq \bar{\theta} \leq \pi/2\), while for \(0 < \bar{\theta} < \pi/6\), (3.14) gives

\[
W(r) = W(r,\bar{\theta}) = \frac{R}{[1-2(\Omega/R)^2]^{1/2}} + o(1), \quad r \to 0
\]

and therefore \(W(r) \to R/[1-2(\Omega/R)^2]^{1/2}\) as \(r \to 0\).

Again, if \(\bar{\theta} = \pi/6\), \(R = 2\Omega\) and \(W(r) \to 2/2\Omega\) while for \(\bar{\theta} = 0\) \(\Omega = 0\) and \(W(r) \to R\). We note that no estimates similar to the above were attempted for the \((E,r,1,2)\) method.

Bearing these limitations in mind the above methods were applied to the special case of three-body problem considered by Zumkley and the results were listed in Table XI and XII. In Table XI the degrees of the polynomials giving six decimal places of accuracy are listed for \(\omega = .1(.1) 1.2\) and for various methods. Those spaces with a dash indicate insufficient accuracy with polynomials of degree 150. For the \((E,r)\), \((E,r,0,2)\) and \((E,r,1,2)\) methods, although several \(r\) values were tried, the values \(r = .5\) and \(r = .2\) give representative results. We note that for small values of \(\omega\) the Taylor series is still the most effective but that the range of \(\omega\) values for which it provides convergence is the smallest. Also, for
TABLE XI

Degree of Polynomials for Various Methods in the Three-Body Problem Zumkley's Example. Parameter \( w, x^{-1} \approx 3.25 \), Accuracy Approx. 6 Dec.Places

<table>
<thead>
<tr>
<th>( w )</th>
<th>Taylor Series</th>
<th>( r = 5 )</th>
<th>( r = 2 )</th>
<th>( r = 5 )</th>
<th>( r = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>6</td>
<td>19</td>
<td>51</td>
<td>38</td>
<td>104</td>
</tr>
<tr>
<td>.2</td>
<td>8</td>
<td>21</td>
<td>56</td>
<td>40</td>
<td>110</td>
</tr>
<tr>
<td>.3</td>
<td>11</td>
<td>22</td>
<td>59</td>
<td>40</td>
<td>112</td>
</tr>
<tr>
<td>.4</td>
<td>14</td>
<td>24</td>
<td>64</td>
<td>42</td>
<td>114</td>
</tr>
<tr>
<td>.5</td>
<td>19</td>
<td>27</td>
<td>68</td>
<td>42</td>
<td>116</td>
</tr>
<tr>
<td>.6</td>
<td>25</td>
<td>29</td>
<td>74</td>
<td>44</td>
<td>118</td>
</tr>
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<td>.7</td>
<td>39</td>
<td>31</td>
<td>80</td>
<td>44</td>
<td>122</td>
</tr>
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<td>.8</td>
<td>77</td>
<td>34</td>
<td>84</td>
<td>44</td>
<td>124</td>
</tr>
<tr>
<td>.9</td>
<td>-</td>
<td>36</td>
<td>93</td>
<td>46</td>
<td>124</td>
</tr>
<tr>
<td>1.0</td>
<td>-</td>
<td>40</td>
<td>100</td>
<td>52</td>
<td>128</td>
</tr>
<tr>
<td>1.1</td>
<td>-</td>
<td>42</td>
<td>107</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1.2</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

TABLE XII

Values of the Time and the Coordinates in the Three-Body Problem Zumkley's Example, for \( w = 1 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>Zumkley Values Interpolated</th>
<th>Brumbergs Results</th>
<th>((E, r)) Method ( r = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>1.810016</td>
<td>1.810016</td>
<td>1.810016</td>
</tr>
<tr>
<td>( x_0 )</td>
<td>0.616</td>
<td>0.616707</td>
<td>0.616708</td>
</tr>
<tr>
<td>( y_0 )</td>
<td>3.305</td>
<td>3.306862</td>
<td>3.306862</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>-1.124</td>
<td>-1.123587</td>
<td>-1.123587</td>
</tr>
<tr>
<td>( y_1 )</td>
<td>-2.114</td>
<td>-2.112465</td>
<td>-2.112464</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>0.507</td>
<td>0.506880</td>
<td>0.506879</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>-1.191</td>
<td>-1.194397</td>
<td>-1.194398</td>
</tr>
</tbody>
</table>
the other methods the table does not show a gain in the length of the interval of convergence for smaller values of \( r \). It should be pointed out in this regard that in the \((E,r)\) method the results were printed out for polynomials of degree 149 and 150 at \( \omega = 1.2 \). For \( r = 0.5 \) the values were clearly diverging while for \( r = 0.2 \) they were converging but agreed only to four decimal places. Let us finally note that, subject to the above restriction on the degree of the polynomials, the \((E,r)\) method provides convergence on the largest interval while the \((E,r,1,2)\) method is most effective on \([0,1]\). As a check on the accuracy of the above methods the computed values of the time \( t \) and the relative coordinates \( \overline{r}_i = (x_i, y_i, z_i) \), \( i=0,1,2 \), are given in Table XII for \( \omega = 1 \). Here we are using Zumkleys example where the masses and initial conditions are

\[
\begin{align*}
m_0 &= m_1 = m_2 = 1 \\
\overline{r}_0(0) &= (2.5, 0, 0), \quad \dot{\overline{r}}_0(0) = (0, 2.5, 0), \\
\overline{r}_1(0) &= (-1.5, 0, 0), \quad \dot{\overline{r}}_1(0) = (0, -1.0, 0), \\
\overline{r}_2(0) &= (-1.0, 0, 0), \quad \dot{\overline{r}}_2(0) = (0, -1.5, 0).
\end{align*}
\]

(3.20)

We note the almost exact agreement with Brumbergs results. The same procedure was then applied to
Strömgren's example in which the masses and initial conditions are

\begin{align*}
    m_0 &= m_1 = 1, \quad m_2 = 2 \\
    \vec{r}_0(0) &= (-10,0,0) \quad \dot{\vec{r}}_0(0) = (0,\sqrt{3/10}, 0) \\
    \vec{r}_1(0) &= (-7,0,0) \quad \dot{\vec{r}}_1(0) = (0,\sqrt{3/7}, 0) \\
    \vec{r}_2(0) &= (17,0,0) \quad \dot{\vec{r}}_2(0) = (0,\sqrt{3/10+\sqrt{3/7}}, 0)
\end{align*}

The results closely followed the pattern in Zumkley's example above.

3.3 The Parameters \( \tau \) and \( \theta \)

One could try to avoid the limitations posed in the last section in one of two possible ways. Firstly one could look for displacement polynomials for which \( W(r) \to \infty \) as \( r \to 0 \). The solution to this problem, however, seems remote (if indeed one exists). A more plausible method would be to try to find a suitable conformal mapping taking the real axis into itself or a portion of itself, and transforming the above strip of width \( 2\Omega \) into a domain more suitably adaptable to the present methods. In the case where the real axis is mapped into itself let us note that if any point on the boundary of the above strip is mapped into a point not on the imaginary axis, the \((E,r)\) method is again limited since the line passing through this
latter point and normal to the radial line intersects the real axis. This point of intersection will be an upper or lower bound for $W(r)$.

Consider therefore the mapping

\begin{equation}
(3.22) \quad \tau = \sinh \left( \frac{\mu w}{2\Omega} \right), \quad w = \frac{2\Omega}{\pi} \ln(\tau + (1 + \tau^2)^{1/2})
\end{equation}

which maps the strip of width $2\Omega$ in the $\omega$-plane conformally into the entire complex $\tau$-plane with the exception of the imaginary line segments $[i, i\infty)$ and $[-i, -i\infty)$. The real axis is mapped into itself and the points $\pm i\Omega$ are mapped into $\mp i$. From (3.2) and (3.22) the above transformation can be represented directly by

\begin{equation}
(3.23) \quad d\tau = \frac{\pi}{2\Omega} (1 + \tau^2)^{1/2} U \, dt, \quad \tau = 0 \text{ when } t = 0.
\end{equation}

We note that for this new variable $\tau$ the domains $D_\tau(x_\tau), D_\tau(y_\tau), \text{ etc.,}$ are all given by $D(\pi/2) \cap D(3\pi/2)$ for any of our three methods. Consequently, $W(r) = (2/r - 1)^{1/2}$ for the $(E,r)$ and $(E,r,0,2)$ methods where $W(r)$ has the same meaning with respect to the variable $\tau$ as with the variable $\omega$. Thus $W(r) \to \infty$ as $r \to 0$ for the $(E,r)$ and $(E,r,0,2)$ methods. However, from the arguments in Section 1.3, it is easy to see that $W(r)$ is bounded for the $(E,r,1,2)$ method.
Without actually writing down the system of equations for the parameter \( \tau \) corresponding to (3.6) for \( \omega \) and the recursion formulas for the Taylor series coefficients corresponding to (3.9), let us only remark that the above steps were carried out and that the solutions for \( t \) and \( \bar{r}_i \) (\( i=0,1,2 \)) for Zumkley's example were found. The method, however, turned out to be of little value since the degrees of the polynomials requiring a certain accuracy were very high. Such poor results can be attributed to two causes. Firstly, the exponential nature of the hyperbolic sine function in (3.22) requires large values of \( \tau \) corresponding to only moderate values of \( \omega \). Secondly, if \( \Omega \) is small the above problem is aggravated. Let us note that for Zumkley's initial conditions (3.20) the estimate of \( \Omega \) given in Sundman's paper is indeed very small; in fact \( \Omega \approx 1.1 \times 10^{-5} \). (However, it should be pointed out in this instance that Sundman uses a somewhat different function \( \omega \).) The smallness of this estimate has caused Belorizky [1] to despair using the Sundman series for any practical solution of the three-body problem in the same way that Brumberg used such series in the two-body problem (see Section 2.4). On the other hand, if Sundman's estimates are poor, it may be possible to show that \( \Omega \) is not small and that the Sundman series are usable.
Let us therefore reconsider the parameter $\gamma$ of (2.18) redefined here in terms of the parameter $\omega$ by

$$w = \frac{2\omega}{\pi} \ln \left( \frac{1+\theta}{1-\theta} \right), \quad \theta = \frac{\exp(\pi\omega/2\Omega)-1}{\exp(\pi\omega/2\Omega)+1},$$

keeping in mind that, as opposed to the two-body problem, we do not have an exact knowledge of $\Omega$. Recall that the transformation (3.24) defines a conformal mapping of the strip of width $2\Omega$ in the $w$-plane onto the unit disk in the $\theta$-plane taking the real $w$-axis into the $\theta$-interval $(-1,1)$. Combining (3.2) and (3.24) the parameter $\omega$ can be eliminated yielding the transformation

$$d\theta = \frac{\pi}{4\Omega} (1-\theta^2) \, dt, \quad \theta = 0 \text{ when } t = 0.$$ 

Let us now introduce an arbitrary weight factor $\chi$ and use in place of (3.25) the transformation

$$(3.25)' \quad d\theta = \frac{\pi \chi}{4\Omega} (1-\theta^2) \, dt, \quad \theta = 0 \text{ when } t = 0.$$ 

If we put

$$\kappa = \frac{\pi \chi}{4\Omega}$$

(3.25)' can be rewritten in the form
\( (3.25)' \quad d\theta = \chi (1-\theta^2)U \, dt \), \quad \theta = 0 \text{ when } t=0. \\

Since \( \chi \) is arbitrary, (3.26) shows \( \chi \) is also arbitrary and hence the transformation \( (3.25)'' \) is independent of \( \Omega \). On the other hand, the radius of convergence of the resultant series in terms of the variable \( \theta \) of \( (3.25)' \) is no longer 1 but \( \chi \). But from (3.26) \( \chi = \frac{4\sqrt{3}}{\pi}K \) and since \( \Omega \) is unknown so is \( \chi \).

However, using transformation \( (3.25)'' \), (3.6) can be rewritten as

\[
\begin{align*}
\Delta_i &= r_i^2, \quad (i=0,1,2) \\
2\Delta_i \sigma_i^* + 3\sigma_i \Delta_i^* &= 0, \quad (i=0,1,2) \\
(1-\theta^2)^2 v_r^{**} + \left[ \frac{(1-\theta^2)^2}{2} v^* - 2\theta (1-\theta^2) v \right] r_i^{**} &= \\
(3.27) \quad &= -\frac{f_{\Omega} \chi^2 \sigma_i r_i^* + f_{m_1} \chi^2 (\sigma_o r_o + m_2 \sigma_1 r_1 + \sigma_2 r_2)}{(1-\theta^2) U t^* = \chi}, \\
V &= U^2, \\
(1-\theta^2) U t^* &= \chi,
\end{align*}
\]

where \( * = \frac{d}{d\theta} \). We note again that this system is of second degree in the dependent variables and their derivatives. Proceeding therefore as in Section 3.1 we consider power series solutions of the form.
(3.28) \[ \bar{r}_i = \sum r_{ik} \theta^k, \quad \Delta_i = \sum \Delta_{ik} \theta^k, \quad \sigma_i = \sum \sigma_{ik} \theta^k \] (i=0,1,2)\\
U = \sum u_k \theta^k, \quad V = \sum v_k \theta^k, \quad t = \sum t_k \theta^k \quad (t_o = 0).

From the initial conditions we obtain the initial values

\[ \bar{r}_{i0} = \bar{r}_i(0), \quad \Delta_{i0} = |\bar{r}_i(0)|^2, \quad \sigma_{i0} = |\bar{r}_i(0)|^{-3} (i=0,1,2), \]

(3.29) \[ u_o = f(m_1 m_2 \sigma_{00} \Delta_{00} + m_1 m_0 \sigma_{01} \Delta_{01} + m_1 m_0 \sigma_{20} \Delta_{20}), \]
\[ v_o = u_o^2, \quad \bar{r}_{i1} = \frac{y}{u_o} \bar{r}_i(0), \quad t_1 = \frac{y}{u_o}. \]

Finally, substituting the power series (3.28) into (3.27) we obtain the recursion relations

\[ \Delta_{ik} = \sum_{j=0}^{k} r_{ij} r_{ik-j}, \]
\[ \sigma_{ik} = \frac{-1}{2k \Delta_{i0}} \sum_{j=0}^{k-1} (3k-j) \sigma_{ij} \Delta_{ik-j}, \]
\[ u_k = \sum_{j=0}^{k} f(m_1 m_2 \sigma_{o0} \Delta_{ok-j} + m_1 m_0 \sigma_{1j} \Delta_{1k-j} + m_1 m_0 \sigma_{2j} \Delta_{2k-j}), \]
\[ v_k = \sum_{j=0}^{k} u_j u_{k-j}, \]
\[ \bar{r}_{i(k+1)} = \frac{1}{(k+1)^2 y_v} [2(k-3)(-v_4 + 2v_2 - v_1) \bar{r}_{i3} + \]
\[ + (k-2)(k-3)(v_3 + 2v_1) \bar{r}_{i2} + (k-1)^2 (v_2 + 2v_1) \bar{r}_{i1} + \]
\[ + \sum_{j=0}^{k-5} (j+2)(j+1)(-v_{k-j-1} + 2v_{k-j-3} - v_{k-j}) \bar{r}_{i(j+2)} + \]
\[ + \sum_{j=0}^{k-5} (j+2)(j+1)(-v_{k-j-1} + 2v_{k-j-3} - v_{k-j}) \bar{r}_{i(j+2)} + \]
\[
\begin{align*}
&k-5 \\
&+ \sum_{j=0}^{k-1} (j+1)(k-j)(-\frac{1}{2} v_{k-j} + v_{k-j-2} - \frac{1}{2} v_{k-j-4}) \tilde{r}_{i,j+1} + f M r^2 \sum_{j=0}^{k-1} \sigma_{ik-j-1} \tilde{r}_{i,j+1} \\
&+ f M r^2 \sum_{j=0}^{k-1} (\sigma_{ok-j-1} \tilde{r}_{o,j+1} + \sigma_{1k-j-1} \tilde{r}_{1,j+1} + \sigma_{2k-j-1} \tilde{r}_{2,j+1})
\end{align*}
\]

\[
t_{k+1} = \frac{-k u_1}{(k+1) u_0} t_k + \frac{1}{(k+1) u_0} \sum_{j=0}^{k-2} (j+1)(u_{k-j-2} - u_{k-j}) t_{j+1}
\]

where \( \tilde{r}_{i,t} \) and \( \sum_{j=0}^{t} (\cdot) \) are zero if \( t < 0 \).

Using the recursion formulas (3.30) the Sundman series were calculated for Zumkley's and Strömgren's cases corresponding to the initial conditions (3.20) and (3.21) respectively. (We note that Burrau's example mentioned earlier is never used since in his case the angular momentum is zero and thus the possibility of triple collisions is not eliminated.) In both these examples \( \kappa \) was taken as 1 and with this choice there was no difficulty in obtaining as many Taylor series coefficients as we wished. The results of our calculations are shown in Table XIII. To begin with we note that the method is not as effective as Brumberg's method discussed in Section 3.1. For example, with polynomials of degree 150 Brumberg reaches a \( t \)-value of about 1.81 in Zumkley's example and 33.29 in Strömgren's example. From Table XIII we see these values are respectively about 2 and 10 times our values for the same degree of polynomial. On the other hand it should be noted
### TABLE XIII (First Part)

Values of the Sundman Series for Zumkley's Example and Strömgren's Example of the Three Body Problem. \( \theta = 0.1(0.1)9, \ 0.91(0.01)95, \ X = 1 \), Accuracy of 6 Dec. Places.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>Deg of Poly.</th>
<th>( t )</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( x_1 )</th>
<th>( y_1 )</th>
<th>( x_2 )</th>
<th>( y_2 )</th>
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<tbody>
<tr>
<td>0.1</td>
<td>6</td>
<td>0.048558</td>
<td>2.497921</td>
<td>0.121354</td>
<td>-1.499942</td>
<td>-0.048572</td>
<td>-0.997979</td>
<td>-0.072782</td>
</tr>
<tr>
<td>0.2</td>
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<td>0.098167</td>
<td>2.491515</td>
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<tr>
<td>0.3</td>
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<tr>
<td>0.4</td>
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### TABLE XIII (Second Part)

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<tr>
<th>θ</th>
<th>Deg of Poly.</th>
<th>t</th>
<th>x₀</th>
<th>y₀</th>
<th>x₁</th>
<th>y₁</th>
<th>x₂</th>
<th>y₂</th>
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<tbody>
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<td>-0.100923</td>
<td>-6.999072</td>
<td>-0.120620</td>
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<tr>
<td>.2</td>
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<td>-9.999095</td>
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<td>-6.996210</td>
<td>-0.243689</td>
<td>16.995306</td>
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<tr>
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<td>.91</td>
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<td>-6.785784</td>
<td>-1.819122</td>
<td>16.734280</td>
<td>3.357836</td>
</tr>
</tbody>
</table>
that the use of Sundman series requires no summation
matrix G since we are evaluating the Taylor series
directly. A more interesting result is the following:

We note from the above argument that the radius of
convergence of the Sundman series for the parameter θ
of (3.25)" is χ. The numerical evidence of Table XIII
suggests that χ ≈ 1 in both Zumkley's and Strömgren's
example. Since θ was chosen equal to 1 in both
these examples, (3.26) gives the rather surprising estimate

\( \Omega \approx \frac{\pi}{4} \)

which is in marked contrast to the values usually
computed from Sundman's estimate.

Let us now note the following interesting possibility.
On the basis of the result in the last paragraph, the
poor results obtained with the transformation (3.22)
can then be attributed solely to the exponential nature
of the sinh function. Essentially, the transformation
(3.22) takes the lines \( w_o + i\Omega, \infty < w_o < +\infty \), of
the \( w \)-plane and folds them onto the half-lines \([1,i\infty)\)
and \([-1,-i\infty)\) in the \( \tau \)-plane. It was pointed out
earlier that such a "severe" mapping is necessary for
the \((E,r)\) method if \( W(r) \) is to tend to infinity
as \( r \to 0 \). However, such is not the case for the
\((E,r,0,2)\) method. Let us assume it is possible to
find a conformal mapping

(3.32) \[ z = h(w) \]

which takes the lines \( w_0 \pm i\Omega \) of the \( w \)-plane into two curves which lie above and below the two lines passing through the origin and inclined at \( 45^\circ \) to the positive and negative \( z \)-axes. Let us further assume that the strip of width \( 2\Omega \) is mapped into the region between the two curves and that the real \( w \)-axis is mapped onto the real \( z \)-axis. If we now consider the limiting shape of \( D_r \) for the \((E,r,0,2)\) method as illustrated in Figure 1 and refer to the method of constructing \( D_r(x_1), D_r(y_1), \) etc., as outlined in Section 1.1, it is not hard to see that in this case \( W(r) \to \infty \) as \( r \to 0 \). Unfortunately, the author has not yet been able to find a suitable mapping (3.32).

In concluding this chapter, let us make the general remark that the methods used in this work do provide effective means of obtaining the solution to the two- and three-body problems within a limited interval. Moreover, within this interval they are more effective than Brumberg's method as well as being much simpler to use. Furthermore, it is not clear from Brumberg's paper [1] that the practical limit to the interval of convergence on which his method will work extends much
beyond ours. It should be pointed out, however, that our methods require some knowledge of the distribution of the singular points of the solution while in Brumberg's method one need only know whether or not the real axis lies in the principal star of the solution function. Also, the entries in the matrix $G$ of Brumberg's method are of a universal character and thus, once calculated, they can be stored and used for any future problem.

One could possibly broaden and improve our results by considering other displacement polynomials; perhaps polynomials leading to methods outside the class $(E,r,\alpha,\beta)$. In addition one might be able to obtain good results by an appropriate combination of summation techniques and conformal mappings. Other problems might also be tried; for instance let us consider the problem of the motion of a heavy rigid body about a fixed point. The equations of motion for this problem are (see Leimanis, [1, Chapter I])

\[
\begin{align*}
\dot{A} &= (B-C)q r + mg(\beta z_o - \gamma y_o), \\
\dot{B} &= (C-A) r p + mg(\gamma x_o - \alpha z_o), \\
\dot{C} &= (A-B) p q + mg(\alpha y_o - \beta x_o),
\end{align*}
\]

\[
\begin{align*}
\alpha &= \beta r - \gamma q, \\
\beta &= \gamma p - \alpha r, \\
\gamma &= \alpha q - \beta p
\end{align*}
\]
where $A$, $B$, $C$, $m$, $x_o$, $y_o$ and $z_o$ are certain real positive constants, $p$, $q$ and $r$ are the components of the angular velocity vector with respect to a conveniently chosen body-centered set of axes, and $\alpha$, $\beta$ and $\gamma$ are the direction cosines of the fixed $OZ$ axis with respect to these axes. Such a system admits the integrals

\begin{equation}
(3.34) \quad A p^2 + B q^2 + C r^2 = 2(h_1 - x_o \alpha - y_o \beta - z_o \gamma)
\end{equation}

and

\begin{equation}
(3.35) \quad \alpha^2 + \beta^2 + \gamma^2 = 1
\end{equation}

where $h_1$ is also a constant. From (3.34) and (3.35) we immediately obtain

\begin{equation}
(3.36) \quad |\alpha|, |\beta|, |\gamma| \leq 1
\end{equation}

and

\begin{equation}
(3.37) \quad |p|, |q|, |r| \leq \frac{2}{K'}(|h_1| + \sqrt{3}k) = K
\end{equation}

where

\[ K' = \min(A, B, C) > 0, \quad k = \max(|x_o|, |y_o|, |z_o|). \]

Consider now the following theorem due to Cauchy on the existence of solutions of differential equations (see Picard, [1, Chapter XI]).
THEOREM 5  Let  \( Q_j(q_1, q_2, \ldots, q_n) \) be functions which do not contain the time \( t \) explicitly and which are developable in powers of \( q_i - q_{10} \) whenever

\[
|q_i - q_{10}| < q'_i \quad (i=1,2,\ldots,n)
\]

where the \( q'_i \) are certain positive constants. Assume there exists positive finite constants \( Q'_j \) such that

\[
|Q_j(q_1, q_2, \ldots, q_n)| < Q'_j \quad (j=1,2,\ldots,n)
\]

whenever the variables \( q_i \) satisfy (3.38).

Under these conditions the system of differential equations

\[
q_j = Q_j(q_1, q_2, \ldots, q_n) \quad (j=1,2,\ldots,n)
\]

admits one and only one solution which has the property that \( q_i \) tends to \( q_{10} \) as \( t \) tends to a finite value \( t_o \). In this solution, the functions \( q_i \) can be developed in power series in \( t - t_0 \) provided

\[
|t-t_0| \leq T
\]

where

\[
T = \min\left(\frac{q'_1}{Q'_1}, \frac{q'_2}{Q'_2}, \ldots, \frac{q'_n}{Q'_n}\right).
\]
Moreover, if \( t \) satisfies (3.41) the functions \( q_i \) satisfy (3.38).

Let us now write the differential equation (3.33) in the form

\[
\begin{align*}
\dot{p} &= \left( \frac{B-C}{A} \right) qr + \frac{mg}{A}(\beta z_o - \alpha y_o) = Q_1(p, q, r, \alpha, \beta, \gamma), \\
\dot{q} &= \left( \frac{C-A}{B} \right) rp + \frac{mg}{B}(\gamma x_o - \alpha z_o) = Q_2(p, q, r, \alpha, \beta, \gamma), \\
\dot{r} &= \left( \frac{A-B}{C} \right) pq + \frac{mg}{C}(\alpha y_o - \beta x_o) = Q_3(p, q, r, \alpha, \beta, \gamma), \\
\dot{\alpha} &= \beta r - \gamma q = Q_4(p, q, r, \alpha, \beta, \gamma), \\
\dot{\beta} &= \gamma p - \alpha r = Q_5(p, q, r, \alpha, \beta, \gamma), \\
\dot{\gamma} &= \alpha q - \beta p = Q_6(p, q, r, \alpha, \beta, \gamma). \\
\end{align*}
\]

(3.33)'

Assume we are given real constants \( p_0, q_0, r_0, \alpha_0, \beta_0 \) and \( \gamma_0 \) with \( \alpha_0, \beta_0 \) and \( \gamma_0 \) satisfying (3.34). Let us now choose arbitrary positive constants \( a \) and \( b \) and assume

\[
\begin{align*}
|p - p_0|, |q - q_0|, |r - r_0| &< a \\
|\alpha - \alpha_0|, |\beta - \beta_0|, |\gamma - \gamma_0| &< b.
\end{align*}
\]

(3.43)

Since \( p_0, q_0 \) and \( r_0 \) satisfy (3.34) and \( \alpha_0, \beta_0 \) and \( \gamma_0 \) (3.35) we obtain

\[
\begin{align*}
|p|, |q|, |r| &< K + a \\
|\alpha|, |\beta|, |\gamma| &< 1 + b
\end{align*}
\]

(3.44)
and thus, from (3.33)',

\[ |Q_1|, |Q_2|, |Q_3| < L(K+a)^2 + 2tk(1+b), \]

\[ |Q_4|, |Q_5|, |Q_6| < 2(K+a)(1+b) \]

(3.45)

where

\[ L = \max\left(\frac{|B-C|}{A}, \frac{|C-A|}{B}, \frac{|A-B|}{C}\right), t = \max\left(\frac{mg}{A}, \frac{mg}{B}, \frac{mg}{C}\right). \]

If we now define \( T \) by

\[ T = T(a,b) = \min\left(\frac{a}{L(K+a)^2 + 2tk(1+b)}, \frac{b}{2(K+a)(1+b)}\right) \]

(3.47)

we can prove the following

**THEOREM 6** If at some real instant \( t_0 \) we are given the real constants \( p_0, q_0, r_0, \alpha_0, \beta_0 \) and \( \gamma_0 \) satisfying (3.35), then the solutions \( p(t), q(t), r(t), \alpha(t), \beta(t) \) and \( \gamma(t) \) of equations (3.33) satisfying \( p(0) = p_0, q(0) = q_0 \), etc. are analytic functions of the complex variable \( t \) in a strip of width \( 2T \) about the real \( t \)-axis.

**Proof:** Since for arbitrary positive numbers \( a \) and \( b \) in (3.43) the conditions of Theorem 5 are satisfied, we are therefore guaranteed the existence of solutions which are analytic in the disk \( |t-t_0| \leq T \) where \( T \) is
given by (3.47). Let \( t' \) be any value of \( t \) satisfying 
\[ t_0 < t' < t_0 + T. \]
The functions \( p(t), q(t), \) etc. will be analytic at \( t' \) and their values there will satisfy (3.36) and (3.37) since for real \( t \) the functions are real valued. Taking their values at \( t' \) as initial values and using the same \( a \) and \( b \) as in (3.43), the estimates (3.44) and (3.45) are still valid and thus \( T \) remains unchanged. Applying Theorem 5 again we are able to conclude that the functions \( p(t), q(t), \) etc. are analytic in the disk \( |t-t'| < T. \)
It follows that for any real value \( t^* \) of \( t \) the functions are analytic in the disk \( |t-t^*| < T \) and thus they are analytic in the strip of width \( 2T \) about the real \( t \)-axis.

Recall that \( a \) and \( b \) used in (3.43) are arbitrary positive real numbers. Let us therefore define \( \Omega \) by

\[
\Omega = \max_{a>0, b>0} T(a, b).
\]

Determination of the maximum involves the solution of a quartic equation in \( b \) which reduces to a quadratic when \( L = 2 \). Theorem 6 can now be restated with \( T \) replaced by \( \Omega \).
Thus we see that, from the point of view of the
distribution of the singularities, the problem of the
motion of a heavy rigid body about a fixed point bears
some resemblance to that of the regularized three-body
problem. Consequently, the methods which we have
used in this work and that also of Brumberg's can be
applied to this problem as well.
BIBLIOGRAPHY


<table>
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<tr>
<th>Name</th>
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