ABELIAN VON NEUMANN ALGEBRAS

by

Charles R. Kerr

B.A., Washington State University, 1962

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

in the Department

of

Mathematics

We accept this thesis as conforming to the

required standard

THE UNIVERSITY OF BRITISH COLUMBIA

September 1965

This thesis carries out some of classical integration theory in the context of an operator algebra. The starting point is measure on the projections of an abelian von Neumann algebra. This yields an integral on the self-adjoint operators whose spectral projections lie in the algebra. For this integral a Radon-Nikodym theorem, as well as the usual convergence theorems, is proved.

The methods and results of this thesis generalize to non-commutative von Neumann Algebras [2, 3, 5].

(1) J. Dixmier Les Algèbres d'Opérateurs dans l'Espace Hilbertien. Paris, 1957.

(2) H.A. Dye The Radon-Nikodym theorem for finite rings of operators, Trans. Amer. Math. Soc., 72, 1952, 243-280.

(3) F.J. Murray and J. von Neumann, On Rings of Operators, Ann. Math. 37, 1936, 116-229.

(4) F. Riesz and B. v. Sz.-Nagy, Functional Analysis, New York, 1955.

(5) I.E. Segal A non-commutative extension of abstract integration, Ann. of Math. (2) 57, 1953, 401-457.

TABLE OF CONTENTS

INTRODUCTION

Let H be a complex Hilbert space. Let $L(H)$ be the set of bounded linear transformations on H. $L(H)$ is an algebra over the complex field. It has an involution $T \to T^*$, where T^* is the unique $L(H)$-member which satisfies

$$(Tx, y) = (x, T^*y)$$

for all x, y in H.

In what follows $L(H)$ will be topologized three ways.

(a) The weak topology on $L(H)$: In this topology each neighborhood of $A \in L(H)$ must contain the intersection of a finite family of sets of form

$$N(A,x,y,\epsilon) = \{T : T \in L(H),\ |(Ax,y)-(Tx,y)| < \epsilon\},$$

where $x,y \in H$ and $\epsilon > 0$.

(b) The strong topology on $L(H)$: Each neighborhood of $A \in L(H)$ contains the intersection of a finite family of sets of form

$$N(A,x,\epsilon) = \{T : T \in L(H),\ \|Tx-Ax\| < \epsilon\},$$

where $x \in H$ and $\epsilon > 0$.

(c) The uniform topology on $L(H)$: This is the metric topology induced by the operator norm.

These three topologies are comparable:

Weak $\subseteq$ Strong $\subseteq$ Uniform, so that if $S \subseteq L(H)$

$$\overline{S}\ (\text{weak}) \supseteq \overline{S}\ (\text{strong}) \supseteq \overline{S}\ (\text{uniform}).$$

Under each of these topologies $L(H)$ is a topological algebra, that is, the operations

$$(X,Y) \longrightarrow (X+Y)$$

$$(a,X) \longrightarrow aX$$

$$X \longrightarrow AX$$

$$X \longrightarrow XA$$

are continuous for A, X, Y in $L(H)$ and scalar a.

1. <u>Definition</u>. If $R \subset L(H)$ then R', the <u>commutant</u> of R, is the set of all operators T in $L(H)$ such that

$$ST = TS \qquad S^*T = TS^*$$

for all $S \in R$.

For any subset $R \subseteq L(H)$, R' is always an algebra: If A and B are in R', then for $S \in R$

$$(A+B)S = AS + BS = SA + SB = S(A+B)$$
$$(AB)S = A(SB) = S(AB).$$

Similarly, $(A+B)$ and AB commute with S^*. Thus $(A+B)$ and AB are also in R'.

Note that R' always contains the identity operator, and that T is in R' if and only if T^* is in R'.

Furthermore, R' is closed in $L(H)$ under any of the above topologies. To prove this note that for $S \in R$, $\{S\}'$ is merely the set on which the continuous functions

$$f(T) = ST - TS$$
$$g(T) = S^*T - TS^*$$

are equal to zero, and hence that $\{S\}'$ is the intersection of two closed sets. Finally

$$R' = \bigcap (S \in R) \quad \{S\}' ,$$

so that R' is a closed set.

2. Definition. A von Neumann algebra is a set $R \subseteq L(H)$ such that $R = (R')' = R''$.

Such a set is, in view of previous remarks, a strongly closed symmetric subalgebra of $L(H)$ which contains the identity operator. On the other hand, all such algebras are von Neumann algebras. This follows from the "von Neumann density theorem" which says that

$$R'' = \overline{R} \text{ (weak)} = \overline{R} \text{ (strong)}$$

for every symmetric subalgebra R with identity.

3. Theorem. If R is any von Neumann algebra and T is any bounded self-adjoint operator with resolution

$$\{E_a : -\infty < a < +\infty\},$$

then $T \in R$ if and only if $E_a \in R$ for all a.

Proof. If $T \in R$ then

$$\{T\} \subseteq R,$$
$$\{T\}' \supseteq R',$$

so that

$$\{T\}'' \subseteq R'' = R.$$

Since $E_a \in \{T\}''$, $E_a \in R$ for all a.

On the other hand,

$$T = \int_{-\infty}^{+\infty} a \, dE_a,$$

where the right-hand side is a uniform limit of a generalized sequence of linear combinations of E_a. Now

$$R \subset \overline{R}(\text{uniform}) \subset \overline{R}(\text{weak}) = R'' = R,$$

so that R is uniformly closed. Thus $\{E_a\} \subset R$ implies $T \in R$.

This proves theorem 3.

4. Definition. If R is any von Neumann algebra R^U is the set of unitary operators in R.

5. Theorem. If R is a von Neumann algebra, then the bounded operator T belongs to R if and only if T commutes with every unitary operator in R', in other words

$$R = ((R')^U)' \qquad \text{(a)}$$

Proof. Since $(R')^U \subseteq R'$

$$((R')^U)' \supseteq R'' = R \tag{b}$$

On the other hand, every operator in the uniformly closed symmetric algebra $\mathfrak{a}$ may be written as a linear combination of unitary operators in $\mathfrak{a}$.

To see this note first that if $A \in \mathfrak{a}$, then

$$H_1 = 1/2\ (A+A^*)$$

$$H_2 = i/2\ (A^*-A)$$

are self-adjoint operators in $\mathfrak{a}$ and

$$A = H_1 + iH_2$$

Next if H is self-adjoint in $\mathfrak{a}$ and $\|H\|<1$, then

$$U = \sqrt{1-H^2} + iH$$

is in $\mathfrak{a}$ (since $\mathfrak{a}$ is uniformly closed) and is unitary.

Thus if an operator T commutes with every unitary operator in $\mathfrak{a}$, it commutes with all other members of $\mathfrak{a}$, that is

$$(\mathfrak{a}^U)' \subseteq \mathfrak{a}'$$

and in particular

$$((R')^U)' \subseteq (R')' = R. \tag{c}$$

Combining (b) and (c) yields (a).

<u>An example of a von Neumann algebra.</u>

Let (X, μ) be a σ-finite measure space. $L_\infty(X, \mu)$ is the set of measurable functions which are bounded almost everywhere in X. $L_2(X, \mu)$ is a Hilbert space.

Given $t \in L_\infty$ there is a corresponding linear transformation T on L_2:

$$(Tg)(x) = t(x)g(x)$$

for $g \in L_2$.

Let M be the family of linear transformations so induced by L_∞.

(i) M is a family of bounded linear transformations of L_2 onto itself.

To prove this, let $g \in L_2$.

$$\int_X |Tg|^2 \, d\mu = \int_X |tg|^2 \, d\mu$$

$$\leq \|t\|_\infty^2 \int_X |g|^2 \, d\mu$$

(where $\|t\|_\infty = \inf \quad k : \mu \quad x : |t(x)| > k \quad = 0$)

$$= \|t\|_\infty^2 \, \|g\|^2 < \infty$$

Thus $Tg \in L_2$ and

$$\|Tg\| \leq \|t\|_\infty \, \|g\| ,$$

so that $T \in L(L_2)$.

This last inequality implies

$$\|T\| \leq \|t\|_\infty$$

To show that equality obtains, let $\epsilon > 0$ be given, then

$$\{x : |t(x)| > \|t\|_\infty - \epsilon\}$$

has positive measure, and since μ is σ-finite, this set has a subset S of <u>finite</u> positive measure. Let g be the characteristic function of the set S. Then $g \in L_2$. Moreover

$$\begin{aligned}
\|Tg\|^2 &= \int_X |tg|^2 \, d\mu \\
&= \int_{X \sim S} |tg|^2 \, d\mu + \int_S |tg|^2 \, d\mu \\
&= \int_S |t(x)|^2 \, d\mu > (\|t\|_\infty - \epsilon)^2 \, \mu S \\
&= (\|t\|_\infty - \epsilon)^2 \, \|g\|^2 .
\end{aligned}$$

Thus for all $\epsilon > 0$,

$$\|T\| \geq \|t\|_\infty - \epsilon \, .$$

Hence $\|T\| \geq \|t\|_\infty$. Combining this with the previous inequality yields

$$\|T\| = \|t\|_\infty \, .$$

This means that M and L_∞ are isometric copies of one another.

(ii) M is an abelian algebra.

To prove this let S and T be M-operators corresponding respectively to s and t in L_∞.

$$\begin{aligned}([S+T]g)(x) &= (Sg)(x) + (Tg)(x) \\ &= s(x)g(x) + t(x)g(x) \\ &= (s(x) + t(x))\, g(x)\end{aligned}$$

Thus $S+T$ corresponds to $s+t$ in L_∞, so $S+T \in M$.

If c is any complex number, then

$$\begin{aligned}([cS]g)(x) &= c(Sg)(x) \\ &= c\, s(x)\, g(x),\end{aligned}$$

whence cS corresponds to $c\, s(x) \in L_\infty$, so $cS \in M$.

Next,

$$\begin{aligned}(STg)\,(x) &= (S[Tg])\,(x) \\ &= s(x)\,[Tg]\,(x) \\ &= s(x)\, t(x)\, g(x).\end{aligned}$$

Since $st \in L_\infty$, $ST \in M$. From this it is obvious that

$$ST = TS.$$

Thus M is an abelian algebra: $M \subseteq M'$.

(iii) M is a symmetric algebra.

To calculate T^* note first that it is by definition the unique $L(L_2)$ operator such that

$$(Tf, g) = (f, T^*g)$$

for all $f, g \in L_2$, that is

$$\int_X tf\overline{g}\, d\mu = \int_X f(\overline{T^*g})\, d\mu$$

or

$$\int_X f[t\overline{g} - (\overline{T^*g})]\, d\mu = 0 \qquad \text{(a)}$$

for all $f, g \in L_2$. This means that

$$f(x)\ [t(x)\ \overline{g(x)} - (\overline{T^*g})\ (x)] = 0$$

almost everywhere for <u>all</u> $f, g \in L_2$, hence that

$$t(x)\ \overline{g(x)} - (\overline{T^*g})\ (x) = 0$$

almost everywhere for all $g \in L_2$. Thus

$$(\overline{T^*g})\ (x) = t(x)\ \overline{g(x)}$$

and

$$T^*g = \overline{t}g$$

in L_2. Thus T^* corresponds to $\overline{t} \in L_\infty$.

This shows T^* explicitly and proves that M is a symmetric subalgebra of $L(L_2)$.

(iv) $M = M'$

Since M is abelian, $M \subseteq M'$. To get the reverse inclusion let $A \in M'$. A commutes with all the projections in M. These projections are induced by the characteristic functions of measurable subsets of X. Thus

$$AE = AE$$

for all projections $E \in M$ implies that

$$(AEf)(x) = (EAf)(x) = e(x)(Af)(x),$$

where $f \in L_2$ and E corresponds to the characteristic function $e \in L_\infty$.

If now both e and f are characteristic functions of sets of finite measure then

$$e, f \in L_2 \cap L_\infty$$

and

$$(Aef)(x) = e(x)(Af)(x) = f(x)(Ae)(x)$$

Since (X, μ) is σ-finite,

$$X = \bigcup (i \epsilon \omega)\, E_i$$

where the E_i are disjoint and

$$\mu E_i < \infty$$

Thus, for f as above,

$$(Af)(x) = \sum (i \epsilon \omega)\, e_i(x)\ (Af)(x)$$

(where e_i is the characteristic function of E_i)

$$= \sum (Ae_i f)\ (x)$$
$$= \sum f(x)\ (Ae_i)\ (x)$$
$$= f(x) \sum (Ae_i)\ (x)$$

Let

$$a(x) = \sum (i\epsilon\omega)\ (Ae_i)\ (x)$$

Then

$$(Af)\ (x) = a(x)\ f(x), \qquad (*)$$

whenever f is the characteristic function of a set of finite measure.

This $a(x)$, being the point-wise limit of a sequence of measurable functions, is itself measurable.

Indeed, the function $a(x)$ is in $L_\infty(X,u)$, for let

$$S = \{x : |a(x)| > \|A\|\}.$$

Then S has a subset S' of finite measure. If $\mu S'$ is positive, and φ is the characteristic function of S'. then $\varphi \in L_2$ and

$$\|A\varphi\|^2 = \int_{S'} |a\varphi|^2\, d\mu$$
$$> \|A\|^2 \int_{S'} |\varphi|^2\, d\mu = \|A\|^2\ \|\varphi\|^2 .$$

That is,

$$\|A\varphi\| > \|A\|\ \|\varphi\|.$$

This is a contradiction, hence $\mu S' = 0$ and $\mu S = 0$. Thus $a(x) \in L_\infty$ and $\|a\|_\infty \leq \|A\|$.

The linearity of the operator A immediately extends (*) to the case where f is a summable simple function.

Since (X,μ) is σ-finite, the set of summable simple functions is dense in L_2. Hence given $f \in L_2$ and $\epsilon > 0$, there exists a summable simple function $s(x)$ such that

$$\|f - s\| < \frac{\epsilon}{2\|A\|}$$

Then

$$\begin{aligned} &\|Af - af\| \\ &\quad \leq \|Af - as\| + \|as - af\| \\ &\quad = \|Af - As\| + \|as - af\| \end{aligned}$$

(since $(As)(x) = a(x)s(x)$)

$$\begin{aligned} &\quad \leq \|A\| \; \|f-s\| + \|a\|_\infty \; \|f-s\| \\ &\quad \leq 2\|A\| \; \|f-s\| \; . \end{aligned}$$

Thus $\|Af - af\| < \epsilon$. But ϵ is arbitrary, hence (*) must hold almost everyhwere for every $f \in L_2$. Since $a \in L_\infty$, this means that

$$A \in M.$$

Thus $M = M'$.

Now, because $M = M'$, M is maximal abelian, i.e. M is properly contained in no symmetric subalgebra which is abelian.

If

$$M \subseteq N,$$

where N is abelian, then

$$N \subseteq N'$$

and

$$M' \supseteq N',$$

or combining

$$M = M' \supseteq N' \supseteq N \supseteq M,$$

so that

$$M = N.$$

More to the point is the observation that

$$M' = M$$

implies that

$$M'' = M'$$

so that

$$M = M' = M'',$$

that is, M is a von Neumann algebra.

6. Definition. Let R be a von Neumann algebra. Then R^P is the set of projections in R.

If R is any von Neumann algebra, then a measure can be defined on R^P.

7. Definition. m is a measure on R^P if m is an extended real function on R^P such that

(i) $m\,E \geq 0$ for all $E \in R^P$

(ii) If $\{E_i\}$ is any family of mutually orthogonal R^P-members,

$$m \sum_i E_i = \sum_i m\,E_i .$$

(Theorem 27 will show that

$$\sum_i E_i = \sup \{E_i\} \in R^P).$$

8. Definition. A measure m is finite if $m\,I < \infty$ and semi-finite if for every non-zero $E \in R^P$ there is $F \in R^P$ such that

$$0 < F \leq E \quad \text{and} \quad m\,F < \infty.$$

9. Definition. If m and n are two measures on R^P, then n is absolutely continuous with respect to m if $nE = 0$ for all $E \in R^P$ such that $m\,E = 0$.

In what follows a semi-finite measure m on R^P - R is abelian - is extended to an integral on the set of all self-adjoint operators (not necessarily bounded) whose spectral resolutions lie in R^P. This integration theory is developed far enough to prove a Radon-Nikodym theorem for semi-finite measures.

WEAK COMPACTNESS OF UNIFORMLY CLOSED BALL

10. Theorem. The uniformly closed ball

$$\mathcal{B} = \{A : A \in L(H) \text{ and } \|A\| \leq 1\}$$

is compact in the weak topology of $L(H)$.

Proof begins with a lemma.

Lemma. In its weak topology $L(H)$ is homeomorphic to a subset of the product space

$$\mathcal{P} = \prod \{C_{(x,y)} : (x,y) \in H \times H\},$$

where $C_{(x,y)}$ is a copy of the complex plane.

Proof. The homeomorphism is the evaluation map e

$$e : L(H) \longrightarrow \mathcal{P}$$

$$P_{(x,y)}\, e(A) = (Ax,y),$$

where $P_{(x,y)}$ is the projection of $\mathcal{P}$ onto $C_{(x,y)}$ and $A \in L(H)$.

This map is one-to-one: If A and B in $L(H)$ are such that $e(A) = e(B)$, then

$$(Ax,y) = P_{(x,y)}e(A) = P_{(x,y)}e(B) = (Bx,y)$$

for all x and y in H, that is $A = B$.

Both e and e^{-1} are continuous: Let $\mathfrak{F}$ be a finite subset of $H \times H$, let $\epsilon > 0$, let $A \in L(H)$, let $S(c,\epsilon)$ be the sphere of radius ϵ about the complex number c, and let

$$Z_1 = \{T : T\epsilon L(H) \text{ and } |(Ax,y) - (Tx,y)| < E \text{ for all } (x,y) \in \mathfrak{F}\}$$

$$Z_2 = \bigcap((x,y)\epsilon\mathfrak{F})\ P^{-1}_{(x,y)}\ [S(P_{(x,y)}e(A),\ \epsilon)].$$

The sets of form $Z_1(A,\mathfrak{F},\epsilon)$ form a neighbourhood basis for A in the weak topology of $L(H)$. The sets of the form

$$e\ [L(H)] \cap Z_2(A,\mathfrak{F},\epsilon)$$

form a neighbourhood basis for $e(A)$ in the product topology of $\mathcal{P}$ relativized to $e[L(H)]$.

Now, since $P_{(x,y)}e(A) = (Ax,y)$,

$$e[Z_1] = \{e(T) : T\epsilon L(H) \text{ and } |P_{(x,y)}e(A) - P_{(x,y)}e(T)| < \epsilon \text{ for all } (x,y) \in \mathfrak{F}\}$$

$$= e[L(H)] \cap \{w : w \in \mathcal{P} \text{ and } |P_{(x,y)}e(A) - P_{(x,y)}(w)| < \epsilon \text{ for all } (x,y) \in \mathfrak{F}\}$$

$$= e[L(H)] \cap Z_2.$$

That is

$$e[Z_1] = e[L(H)] \cap Z_2,$$

so that e and e^{-1} are continuous. This proves the lemma.

This lemma implies that $\mathcal{B}$ is compact if and only if $e[\mathcal{B}]$ is compact.

For x and y in H let

$$\mathcal{C}(x,y) = \{z : z \text{ is a complex number } |z| \leq \|x\| \; \|y\|\}.$$

$\mathcal{C}(x,y)$ is a closed and bounded subset of the complex plane, and is thus compact for every x and y. Hence by the Tihonov theorem the product space

$$\mathcal{X} = \prod \{\mathcal{C}(x,y) : (x,y) \in H \times H\}$$

is also compact.

Now $e[\mathcal{B}] \subseteq \mathcal{X}$, since for $A \in \mathcal{B}$

$$|P_{(x,y)} e(A)| = |(Ax,y)| \leq \|Ax\| \; \|y\|$$

$$\leq \|A\| \; \|x\| \; \|y\| \leq \|x\| \; \|y\|$$

Thus the theorem will be proven if it is shown that $e[\mathcal{B}]$ is a closed set in the compact space $\mathcal{X}$.

First let x, y, z be in H and let a,b be complex numbers. Let

$$X_1(x,y,z)$$

$$= \{w : w \in \mathcal{X}, \; P_{(x+y,z)}(w) = P_{(x,z)}(w) + P_{(y,z)}(w)\}$$

$$X_2(x,y,z)$$

$$= \{w : w \in \mathcal{X}, \; P_{(x,yz)}(w) = P_{(x,y)}(w) + P_{(x,z)}(w)\}$$

$$X_3(x,y,a,b)$$

$$= \{w : w \in \mathcal{X},\ P_{(ax,bx)}(w) = a\bar{b}\, P_{(x,y)}(w)\}$$

The projection $P_{(u,v)}$ of $\mathcal{X}$ onto $\mathcal{C}(x,y)$ is continuous. Hence $X_1(x,y,z)$, being the set on which two continuous functions agree, is closed for all x, y, z. Similarly X_2 and X_3 are closed in $\mathcal{X}$.

Note next that for any A in $\mathcal{B}$,

$$\begin{aligned} P_{(x+y,z)}e(A) &= (A(x+y), z) \\ &= (Ax,z) + (Ay,z) \\ &= P_{(x,z)}e(A) + P_{(y,z)}e(A), \end{aligned}$$

so that $e(A) \in X_1(x,y,z)$ for all x,y,z. Similarly $e(A)$ is in $X_2(x,y,z)$. Also

$$\begin{aligned} P_{(ax,by)}e(A) &= (Aax,by) \\ &= a\bar{b}\,(Ax,y) \\ &= a\bar{b}\, P_{(x,y)}e(A), \end{aligned}$$

so that $e(A) \in X_3\,(x,y,a,b)$ for every x,y and a,b. Thus

$$e[\mathcal{B}] \subseteq \mathcal{S}$$

where

$$\mathcal{S} = \bigcap\{X_1 : x,y,z\} \cap \bigcap\{X_2 : x,y,z\} \cap \bigcap\{X_3 : x,y,a,b\}$$

which is a closed set.

Actually $e[\mathcal{B}]$ exhausts $\mathcal{S}$, for if $\varphi \in \mathcal{S}$, then

$$P_{(x+y,z)}(\varphi) = P_{(x,z)}(\varphi) + P_{(y,z)}(\varphi)$$

$$P_{(x,y+z)}(\varphi) = P_{(x,y)}(\varphi) + P_{(x,z)}(\varphi)$$

$$P_{(ax,by)}(\varphi) = a\bar{b}\, P_{(x,y)}(\varphi),$$

that is, φ determines a bilinear functional

$$f(x,y) = P_{(x,y)}(\varphi)$$

on H. Moreover

$$|f(x,y)| = |P_{(x,y)}(\varphi)| \leq \|x\|\ \|y\|,$$

since $P_{(x,y)}(\varphi) \in \mathcal{C}(x,y)$. Hence f is a bounded bilinear functional on H and $\|f\| \leq 1$.

This means that by the Riesz representation theorem for bounded bilinear functionals there exists an operator $F \in L(H)$ such that

$$f(x,y) = (Fx,y)$$

and

$$\|F\| = \|f\| \leq 1.$$

But this implies that

$$P_{(x,y)}(\varphi) = f(x,y) = (Fx,y) = P_{(x,y)}\ e(F)$$

for all x,y, so that $\varphi = e(F)$ and $\varphi \in e[\mathfrak{B}]$. Thus $\mathcal{S} \subseteq e[\mathfrak{B}]$ and so

$$e[\mathfrak{B}] = \mathcal{S}.$$

As stated before, the compactness of $\mathfrak{B}$ now follows from the fact that $e[\mathfrak{B}]$ is a closed subset of the compact space $\mathcal{X}$. This proves theorem 10.

BOUNDED SELF-ADJOINT OPERATORS IN A VON NEUMANN ALGEBRA

11. Definition. If R is a von Neumann algebra, R^S is the set of self-adjoint operators in R.

With this definition theorem 3 rephrases as "$T \in R^S$ if and only if the spectral resolution of T is in R^P."

Any family of bounded self-adjoint operators is partially ordered by a relation $\leq$ defined as follows,

$$S \leq T \text{ if and only if } (Sx,x) \leq (Tx,x) \text{ for all } x \text{ in } H.$$

In most cases this $\leq$ does not furnish a linear ordering; for example, the operators E and $I-E$, for a projection E, are not comparable.

12. Definition. If $\mathfrak{F} \subseteq L(H)$ is a family of self-adjoint operators, then whenever it exists,

$$\sup \mathfrak{F}$$

is the smallest self-adjoint operator which majorizes every $\mathfrak{F}$-member. If R is a von Neumann algebra then

$$\sup_R \mathfrak{F},$$

whenever it exists, is the smallest R^S-operator majorizing every member of $\mathfrak{F} \subseteq R^S$. The operators

$$\inf \mathfrak{F}\ , \quad \inf_R \mathfrak{F}$$

are analogously defined.

If S and T are in R^S then $S-T$ is in R^S also. If F_a is the spectral resolution of $S-T$, then by theorem 3 $F_a \in R^P$. If R is abelian, then S,T commute with $S-T$, and, consequently, F_a commutes with both S and T. This permits the following definition

13. Definition. If $S,T \in R^S$ for abelian R, let F_a be the resolution of $S-T$. Then

$$(S \cup T) = T\,F_0 + S(I-F_0)$$

$$(S \cap T) = T(I-F_0) + SF_0$$

Obviously $(S\cup T)$ and $(S\cap T)$ are in R. Furthermore

$$\begin{aligned}(S \cup T)^* &= F_0{}^*T^* + (I-F_0)^*S^* \\ &= F_0T + (I-F_0)S \\ &= TF_0 + S(I-F_0) \\ &= (S \cup T)\end{aligned}$$

so that $(S \cup T) \in R^S$. In the same way $(S \cap T) \in R^S$

14. Theorem. Under the assumptions of definition 13,

$$(S \cup T) = \sup_R \{S,T\}\ .$$

Proof. Since R is abelian,

$$\{F_0\}' \supseteq R' \supseteq R.$$

The following proof shows that $(S \cup T)$ is the smallest operator in $\{F_0\}'$ which majorizes both S and T.

For any $x \in H$,

$$(S \cup T\, x,x) - (T\, x,x)$$

$$= (TF_0x,x) + (S(I-F_0)x,x)$$

$$- (TF_0x,x) - (T(I-F_0)x,x)$$

$$= ((S-T)(I-F_0)x,x) \geq 0$$

by definition of F_0. Thus

$$(S \cup T) \geq T.$$

Similarly

$$(S \cup T) \geq S.$$

Now let $A \in \{F_0\}'$ be an operator such that

$$A \geq S \qquad A \geq T\,.$$

Then

$$A = F_0\, AF_0 + (I-F_0)AF_0$$

$$+ F_0A(I-F_0) + (I-F_0)\, A(I-F_0)$$

$$= F_0\, AF_0 + (I-F_0)\, A(I-F_0)$$

(since $A \in \{F_0\}'$). Since

$$(S \cup T) = TF_0 + S(I-F_0)$$

$$= F_0TF_0 + (I-F_0)\ S(I-F_0),$$

for any $x \in H$

$$(Ax,x) - (S\cup Tx,x)$$

$$= (F_0AF_0x,x) + (I-F_0)A(I-F_0)x,x)$$

$$- (F_0TF_0x,x) + ((I-F_0)S(I-F_0)x,x)$$

$$= ((A-T)F_0x,F_0x) + ((A-S)(I-F_0)x,\ (I-F_0)x).$$

This last line is non-negative, since A majorizes S and T.

This proves theorem 14.

In the same way,

15. Corollary

$$(S \cap T) = \inf_R \{S,T\}$$

16. Theorem. If R is any von Neumann algebra (not necessarily abelian) and $\mathfrak{F} \subseteq R^S$ is directed upward and bounded above by the self-adjoint operator S_o, then $\sup \mathfrak{F}$ exists and belongs to R^S.

Proof. For all $F \in \mathfrak{F}$ let $W(F)$ be the weak closure of the set

$$V(F) = \{T : T \in \mathfrak{F} \text{ and } T \geq F\}$$

Choose $F_0 \in \mathfrak{F}$ and let

$$\mathfrak{F}_0 = V(F_0) = \{T : T \in \mathfrak{F} \text{ and } T \geq F_0\}$$

Observe that if $\{F_i : i \in n\}$ is any finite $\mathfrak{F}_0$-subset, then

$$\bigcap \{V(F_i) : i \in n\} \neq \emptyset .$$

To prove this note that since $\mathfrak{F}$ is directed upward there exists $T_1 \in \mathfrak{F}$ such that

$$T_1 \geq F_0 \qquad T_1 \geq F_1 .$$

For $k < n$, there exists $T_{k+1} \in \mathfrak{F}$ such that

$$T_{k+1} \geq F_{k+1} \qquad T_{k+1} \geq T_k .$$

Obviously $T_n \geq F_i$ for all $i \in n$, so

$$T_n \in \bigcap \{V(F_i) : i \in n\}$$

A fortiori

$$\bigcap \{W(F_i) : i \in n\} \neq \emptyset$$

for all finite families

$$\{F_i : i \in n\} \subseteq \mathfrak{F}_0$$

This means that

$$W(F) : F \in \mathfrak{F}_0$$

has the finite intersection property in

$$\mathcal{B} = \left\{ T : T \in L(H),\ \|T\| \leq \max \left\{ \|F_0\|,\ \|S_o\| \right\} \right\} .$$

Since $\mathcal{B}$ is weakly compact

$$\bigcap \left\{ W(F) : F \in \mathfrak{F}_0 \right\} = \emptyset$$

If A is in this intersection, then since A is in the weak closure of $V(F) \subseteq \mathfrak{F} \subseteq R^S$.

(i) A is self-adjoint

(ii) $A \geq F$ for all $F \in \mathfrak{F}_0$ and hence $A \geq F$ for all $F \in \mathfrak{F}_0$

(iii) $A \in R$

Actually A is the only operator in the intersection and is moreover the supremum of $\mathfrak{F}$. For if T is <u>any</u> self-adjoint operator which majorizes every $\mathfrak{F}$-member, then in particular,

$$T \geq S$$

for all $S \in W(F)$ for all $F \in \mathfrak{F}$. Hence $T \geq A$.

This proves theorem 16.

17. <u>Corollary</u>. If R is any von Neumann algebra, and $\mathfrak{F} \subseteq R^S$ is directed downward and bounded below by the self-adjoint operator S_o, then $(\inf \mathfrak{F})$ exists and belong to R^S.

Proof. The set

$$-\mathfrak{F} = \{-F : F \in \mathfrak{F}\}$$

is directed upward and bounded above by $-S_o$. Hence 17 follows from 16.

18. Definition. For sequences of self-adjoint operators,

$$S_n \uparrow$$

will mean that

$$S_n \leq S_{n+1}$$

and

$$S_n \uparrow S \quad \text{(strong)}$$

will mean that

$$S_n \uparrow$$

and

$$S = \lim \text{(strong) } S_n.$$

The expressions

$$S_n \uparrow S\text{(weak)} \qquad S_n \downarrow S\text{(uniform)}$$

are defined analogously.

19. Theorem. For monotone sequences, strong and weak convergence are equivalent: if $\{T_n\}$ is a monotone sequence,

then

$$T = \lim(\text{weak})\ T_n$$

if and only if

$$T = \lim(\text{strong})\ T_n$$

<u>Proof</u>. Since the strong topology includes the weak topology

$$T = \lim(\text{strong})\ T_n$$

implies

$$T = \lim(\text{weak})\ T_n$$

For the converse, suppose

$$T_n \uparrow T\ (\text{weak}).$$

Since

$$T_0 \leq T_n \leq T,$$

$$\|T_n\| \leq K,$$

where

$$K = \max \{\|T_0\|,\ \|T\|\}$$

Now for any x in H

$$\|(T-T_n)x\|^4 = |((T-T_n)x,\ (T-T_n)x)|^2$$
$$\leq ((T-T_n)^2x,\ (T-T_n)x)((T-T_n)x,x)$$

(since $T-T_n \geq 0$)

$$= ((T-T_n)^3x,x)((T-T_n)x,x)$$

$$\leq \|(T-T_n)^3x\| \; \|x\| \; ((T-T_n)x,x)$$

$$\leq \|T-T_n\|^3 \; \|x\|^2 \; ((T-T_n)x,x)$$

$$\leq (2K)^3 \; \|x\|^2 \; ((T-T_n)x,x)$$

(since $\|T-T_n\| \leq \|T\| + \|T_n\| \leq 2K$)

Since

$$T = \lim \text{ (weak) } T_n,$$

there exists $N(\epsilon,x)$ such that $n \geq N$ implies

$$((T-T_n)x,x) < \frac{\epsilon^4}{8K^3\|x\|^2},$$

and hence that

$$\|(T-T_n)x\| < \epsilon.$$

Thus

$$T_n \uparrow T \text{ (strong)}$$

and 19 is proven.

Remark. In view of 19

$$S_n \uparrow S \qquad\qquad T_n \downarrow T$$

will be written with the understanding that the convergence is both weak and strong.

20. Theorem. If R is abelian with S_n, S, T in R^S, then if

$$S_n \uparrow S,$$

then also

$$(S_n \cup T) \uparrow (S \cup T)$$

$$(S_n \cap T) \uparrow (S \cap T)$$

Proof depends on two lemmas:

21. Lemma. In any R with $S_n, S \in R^S$, if $S_n \uparrow S$ then

$$S = \sup \{S_n\} .$$

Proof. $\{S_n\}$ is directed upward and bounded above. Hence

$$C = \sup \{S_n\}$$

exists and belongs to R^S by 16. Hence

$$S \geq C.$$

If equality does not hold here then there must exist $x \in H$ such that

$$(Sx,x) > (Cx,x),$$

and, since

$$S_n \uparrow S,$$

there exists N such that $n \geq N$ implies

$$(Sx,x) \geq (S_n x,x) > (Cx,x),$$

contradicting the fact that $C \geq S_n$ for all n. Hence

$$S = C = \sup \{S_n\}$$

This proves 21.

22. Lemma. In any R, if $\{S_n\} \subseteq R^S$ is bounded above and $S_n \uparrow$, then (by 16)

$$S = \sup \{S_n\}$$

exists and belongs to R^S. Moreover,

$$S_n \uparrow S$$

Proof. Observe that

$$\{S_n\} \subseteq \mathcal{B} = \{A : A \in L(H), \|A\| \leq \max \{\|S_0\|, \|S\|\}\}.$$

Since $\mathcal{B}$ is weakly compact, $\{S_n\}$ has a subsequence $\{S_{n(k)}\}$ such that

$$S_{n(k)} \uparrow C$$

for some $C \in \mathcal{B}$. Since $\{S_{n(k)}\} \subseteq R^S$, $C \in R^S$ also.

Now for any $x \in H$,

$$(S_{n(k)}x,x) \uparrow (Cx,x) .$$

Hence also

$$(S_n x,x) \uparrow (Cx,x)$$

Since all the operators involved are self-adjoint

$$S_n \uparrow C.$$

By 21 this implies that

$$C = \sup \{S_n\} = S$$

Thus

$$S_n \uparrow S$$

and 22 is proven.

Now to complete the proof of 20.

Since

$$(S \cup T) \geq S \geq S_n$$

and

$$(S \cup T) \geq T,$$

obviously

$$(S \cup T) \geq (S_n \cup T) \qquad \text{(a)}$$

Furthermore,

$$(S_{n+1} \cup T) \geq S_{n+1} \geq S_n$$

and

$$(S_{n+1} \cup T) \geq T,$$

so that

$$(S_{n+1} \cup T) \geq (S_n \cup T) \ .$$

That is

$$(S_n \cup T) \uparrow \qquad \text{(b)}$$

By (a) and (b) and 16, there exists $C \in R^S$ such that

$$C = \sup \left\{ (S_n \cup T) \right\}$$

Now

$$(S \cup T) \geq (S_n \cup T)$$

implies that

$$(S \cup T) \geq C \qquad \text{(c)}$$

By 21, the fact that

$$S_n \uparrow S$$

implies that

$$S = \sup \ \{S_n\} \ ,$$

hence that

$$C \geq S,$$

since

$$C \geq (S_n \cup T) \geq S_n$$

for all n. Moreover,

$$C \geq (S_n \cup T) \geq T,$$

so that by 14, since $C \in R^S$,

$$C \geq (S \cup T) \tag{d}$$

Combining (c) and (d) yields

$$C = (S \cup T),$$

that is,

$$(S \cup T) = \sup \left\{(S_n \cup T)\right\}.$$

Thus by 22

$$(S_n \cup T) \uparrow (S \cup T)$$

To prove the second assertion of 20 note that

$$(S_n \cap T) \leq S_n \leq S_{n+1}$$

and

$$(S_n \cap T) \leq T,$$

so that by 15, since $(S_n \cap T) \in R^S$,

$$(S_n \cap T) \leq (S_{n+1} \cap T).$$

Thus

$$(S_n \cap T) \uparrow .$$

That this sequence tends weakly to $(S \cap T)$ follows from the formula

$$\begin{aligned}(S+T) &= SF_0 + S(I-F_0) + TF_0 + T(I-F_0) \\ &= SF_0 + T(I-F_0) + TF_0 + S(I-F_0) \\ &= (S \cap T) + (S \cup T).\end{aligned}$$

This proves 20.

SIMPLE FUNCTIONS IN R^S.

23. Definition. R^{SP} is the set of all bounded operators of the form

$$S = \sum_{i=1}^{p} a_i E_i ,$$

where p is a positive integer, a_i is real, and $\{E_i\}$ is a family of R^P-members such that

$$\sum_{i=1}^{p} E_i = I$$

(this implies that the E_i are mutually orthogonal).

24. Theorem. If $T \in R^S$ then there is a sequence $\{T_n\} \subseteq R^{SP}$ such that

$$T_n \uparrow T \text{ (uniform)}.$$

Proof. Choose $d > 0$ and let P be any finite point set partitioning the interval

$$[-d-\|T\|, \|T\|].$$

$$P : -d-\|T\| = a_o < a_1 < \ldots < a_{p-1} < a_p = \|T\|$$

If E_a is the spectral resolution of T, then $E_a \in R^P$ for all real a, and

$$a_{i-1}(E_{a_i} - E_{a_{i-1}}) \leq T(E_{a_i} - E_{a_{i-1}}) \leq a_i(E_{a_i} - E_{a_{i-1}})$$

or, letting $E_i = E_{a_i} - E_{a_{i-1}}$.

$$a_{i-1} E_i \leq T E_i \leq a_i E_i ,$$

so that

$$\begin{aligned} L(P) &= \sum_{i=1}^{p} a_{i-1} E_i \\ &\leq \sum_{i=1}^{p} T E_i = T \\ &\leq \sum_{i=1}^{p} a_i E_i = U(P) , \end{aligned}$$

that is,

$$L(P) \leq T \leq U(P) .$$

Now

$$\begin{aligned} U(P) - L(P) &= \sum_{i=1}^{p} (a_i - a_{i-1}) E_i \\ &\leq \max \{a_i - a_{i-1}\} \sum_{i=1}^{p} E_i \\ &= (\max P)\, I , \end{aligned}$$

so that

$$0 \leq T - L(P) \leq (\max P)\, I$$

and hence,

$$\|T - L(P)\| \leq \max P .$$

Thus if a sequence $\{P_n\}$ of partitions of $[-d-\|T\|, \|T\|]$ is chosen so that

$$\lim (\max P_n) = 0,$$

then

$$\lim \|T - L(P_n)\| = 0.$$

Hence

$$T = \lim \text{ (uniform) } L(P_n).$$

Now $\{L(P_n)\} \subseteq R^{SP}$, and its convergence to T may be made monotone by requiring that

$$P_n \subseteq P_{n+1} ;$$

for, let

$$P' = \{a'\} \cup P,$$

where $a' \notin P$, $a_{j-1} < a' < a_j$, say. Let

$$L(P-j) = \sum_{i=1}^{p} (i \neq j) \; a_{i-1} E_i .$$

Then

$$\begin{aligned} L(P) &= L(P-j) + a_{j-1} E_j \\ &= L(P-j) + a_{j-1} (E_{a_j} - E_{a_{j-1}}) \\ &= L(P-j) + a_{j-1} (E_{a_j} - E_{a'}) \\ &\qquad + a_{j-1} (E_{a'} - E_{a_{j-1}}) \end{aligned}$$

$$\leq L(P-j) + a' (E_{a_j} - E_{a'}) + a_{j-1} (E_{a'} - E_{a_{j-1}})$$

$$= L(P'),$$

that is,

$$L(P) \leq L(P') \leq T.$$

This proves 24.

POSITIVE AND NEGATIVE PARTS OF AN R^S-OPERATOR

Recall that if T is any bounded self-adjoint operator with spectral resolution E_a, then

$$T = T^+ - T^-,$$

where

$$T^+ = T(I-E_0) \qquad T^- = -TE_0$$

are positive operators which commute with T and with each other, and for which

$$T^+T^- = 0.$$

The following theorem will be important to the extension of a measure from R^P to R^S:

25. Theorem. If T is any self-adjoint operator and

$$T = A-B,$$

where A and B are positive operators that commute with T, then

$$A = T^+ + P$$

$$B = T^- + P,$$

where P is a positive operator.

Proof. Since

$$A - B = T^{+} - T^{-}$$

implies

$$A - T^{+} = B - T^{-},$$

it remains only to show that

$$P = A - T^{+}$$

is positive.

If E_a is the resolution of T, then for any $x \in H$

$$(Px,x) = (A-T^{+}x,x) = (A-T^{+}x,E_0x) + (A-T^{+}x,I-E_0x)$$

$$= ((A-T^{+})E_0x,E_0x) + ((A-T^{+})(I-E_0)x, (I-E_0)x)$$

(since E_0 commutes with $A-T^{+}$)

$$= (AE_0x,E_0x) + ((A-T^{+})(I-E_0)x, (I-E_0)x)$$

(since $T^{+}E_0 = T(I-E_0)E_0 = 0$)

$$= (AE_0x,E_0x) + ((A-T)(I-E_0)x, (I-E_0)x)$$

(since $T^{+} = T(I-E_0)$)

$$= (AE_0x,E_0x) + ((A- [A-B])(I-E_0)x, (I-E_0)x)$$

(since $T = A-B$)

$$= (AE_0x,E_0x) + (B(I-E_0)x, (I-E_0)x).$$

This last line is non-negative since A and B are positive.

Thus $P \geq 0$ and 25 is proven.

Important Remark. If $T \in R^S$, then so also are T^+, T^-. If A, B are in R^S, so is P.

SUPREMA AND INFIMA IN R^P

Let R be any von Neumann algebra. If attention is restricted to subsets of R^P, then 16 can be improved: subsets of R^P need not be directed to have suprema and infima. Moreover, the supremum (or infimum) of definition 12 turns out to be the usual supremum (infimum) of a collection of projections.

26. Theorem. If R is any von Neumann algebra and

$$\mathcal{E} \subseteq R^P$$

$$P = \mathrm{pr} \bigcap \{\mathrm{rng}\ E : E \in \mathcal{E}\},$$

then $P \in R^P$ and

$$P = \inf \mathcal{E}$$

Proof. To show that

$$P = \inf \mathcal{E},$$

suppose that S is a self-adjoint operator for which

$$S \leq E \qquad \text{for all } E \in \mathcal{E} .$$

Assume, moreover, that there exists $x \in H$ with $\|x\| = 1$ such that

$$(Sx, x) > (Px, x).$$

Now

$$(Sx,x) \leq 1$$

and

$$(Px,x) = 1 \quad \text{or} \quad (Px,x) = 0.$$

Thus it must be that

$$1 \geq (Sx,x) > (Px,x) = 0$$

But $(Px,x) = 0$ implies that

$$x \notin \text{rng } E_o$$

for some $E_o \in \mathcal{E}$, and hence that

$$0 = (E_o x,x) \geq (Sx,x) > 0.$$

This is a contradiction. Hence

$$(Sx,x) \leq (Px,x)$$

for all $x \in H$, that is

$$S \leq P.$$

Since

$$P \leq E$$

for all $E \in \mathcal{E}$, this means that

$$P = \inf \mathcal{E}$$

To show that $P \in R$, let U be a unitary operator in R'.

Then

$$UE = EU \qquad U^*EU = E \qquad UEU^* = E \tag{a}$$

for all $E \in R^P$.

If F is any projection such that $F \leq E$ for all $E \in \mathcal{E}$, then for all $x \in H$,

$$\begin{aligned}(U^*FUx,x) &= (FUx,Ux)\\ &\leq (EUx,Ux)\\ &= (U^*EUx,x)\\ &= (Ex,x)\ ,\end{aligned}$$

by (a), so that

$$U^*FU \leq E$$

for all $E \in \mathcal{E}$. Similarly,

$$UFU^* \leq E$$

for all $E \in \mathcal{E}$. Thus, in particular,

$$UPU^* \leq E \qquad U^*PU \leq E$$

for all $E \in \mathcal{E}$.

Now $P = \inf \mathcal{E}$, thus

$$UPU^* \leq P \tag{b}$$

$$U^*PU \leq P \tag{c}$$

Now by (c)

$$(U^*PUx,x) \leq (Px,x)$$

for all $x \in H$, hence for $x = U^*y$, where $y \in H$:

$$(U^*PU(U^*y), (U^*y)) \leq (PU^*y,U^*y),$$

$$(UU^*PUU^*y,y) \leq (UPU^*y,y) ,$$

whence

$$(Py,y) \leq (UPU^*y,y)$$

for all $y \in H$. Thus

$$P \leq UPU^* \tag{d}$$

Combining (b) and (d) yields

$$P = UPU^*$$

or

$$PU = UP.$$

Since U is otherwise arbitrary, this means that P commutes with every unitary operator in R'. By theorem 5, $P \in R$.

This proves 26.

27. Theorem. If R is any von Neumann algebra and

$$\mathcal{E} \subseteq R^P$$

$$Q = \text{pr } [\bigcup \{\text{rng } E : E \in \mathcal{E}\}]$$

(that is, Q is the projection onto the closed subspace generated by this set union), then $Q \in R^P$ and

$$Q = \sup \mathcal{E}$$

<u>Proof</u>. $Q = I - \mathrm{pr}[\bigcup \{\mathrm{rng}\ E : E \in \mathcal{E}\}]^{\perp}$

$$= I - \mathrm{pr} \bigcap \{(\mathrm{rng} E)^{\perp} : E \in \mathcal{E}\}$$

$$= I - \mathrm{pr} \bigcap \{\mathrm{rng}\ (I-E) : E \in \mathcal{E}\}$$

$$= I - \inf \{I - E : E \in \mathcal{E}\}$$

by 26. Since

$$P = \inf \{I-E : E \in \mathcal{E}\} \in R^P,$$

$Q \in R^P$ also.

If now $S \geq E$ for all $E \in \mathcal{E}$, then

$$I - S \leq I-E$$

for all $E \in \mathcal{E}$, hence by 26,

$$I - S \leq P$$

and

$$S = I - (I-S) \geq I-P = Q .$$

so that

$$Q = \sup \mathcal{E} .$$

This proves 27.

Remark. Now that 27 is proven, definition 7 of a measure on R^p is complete.

R^N - OPERATORS

28. Definition. Let R^N be the set of all self-adjoint operators (including unbounded ones) whose spectral resolution lies in R^P.

In light of theorem 3

$$R^S = L(H) \cap R^N.$$

Proof of the following theorem may be found in Functional Analysis by Reisz and Sz.-Nagy (Ungar, New York, 1955), page 314.

Von Neumann's Theorem.

Given

(a) a sequence $\{E_i : i \in \omega\}$ of projections such that

$$\sum \{E_i : i \in \omega\} = I$$

and

(b) a sequence $\{A_i : i \in \omega\}$ of bounded self-adjoint operators such that

$$A_i E_i = E_i A_i E_i$$

for all i, then there exists a unique self-adjoint operator A (which may not be bounded) such that

$$AE_i = E_i AE_i = E_i A_i E_i = A_i E_i$$

for all i. Moreover

$$D(A) = \left\{ x : x \in H, \sum_i \|A_i E_i x\|^2 < \infty \right\}$$

and for $x \in D(A)$

$$Ax = \sum_i A_i E_i x.$$

29. Theorem. Let R be an abelian von Neumann algebra. If the hypotheses of von Neumann's theorem are strengthened:

$$\{E_i\} \subseteq R^P, \qquad \{A_i\} \subseteq R^S,$$

then $A \in R^N$.

Hence if A is any self-adjoint operator for which

$$E_i A \subseteq AE_i \in R^S$$

for all $i \in \omega$, where

$$\{E_i\} \subseteq R^P, \quad \sum_i E_i = I,$$

then $A \in R^N$.

Proof. The proof is taken up mainly with deriving the spectral resolution of A from those of the A_i.

Let $F_a(i)$ be the spectral resolution of A_i for each i. Since R is abelian,

$$A_i E_i = E_i A_i$$

for all i, and

$$F_a(i) E_i = E_i F_a(i)$$

for all a and i.

Let

$$F_a = \sup \left\{ F_a(i)E_i : i \in \omega \right\},$$

which is a projection of R by 27. Since the E_i are mutually orthogonal,

$$F_a = \sum_i F_a(i)\, E_i$$

whence obviously $F_a \leq F_b$ if $a \leq b$.

To show that F_a yields a resolution of the identity, it must be shown how the properties

(i) $\lim\limits_{b \to a+} (\text{strong})\ F_b = F_a$

(ii) $\lim\limits_{a \to +\infty} (\text{strong})\ F_a = I$

(iii) $\lim\limits_{a \to -\infty} (\text{strong})\ F_a = 0$

are inherited from the properties of the $F_a(i)$.

To prove (i), let

$$x_n = \sum (i \epsilon n)\, E_i x$$

for $x \in H$. Then

$$\lim x_n = x.$$

Let $\epsilon > 0$ be given. Then for $a < b$,

$$\|F_a x_n - F_b x_n\|^2$$

$$= \sum (i \epsilon \omega)\ \|(F_a(i) - F_b(i))\, E_i x_n\|^2$$

$$= \sum (i \in n) \ \|(F_a(i) - F_b(i)) \ E_i x_n\|^2$$

For each $i \in n$ there is an $a_i > a$ such that if

$$a_i > b \geq a$$

then

$$\|(F_a(i) - F_b(i)) \ E_i x_n\| < \frac{\epsilon}{\sqrt{n}}$$

Hence if

$$a \leq b < \min \ \{a_i : i \in n\},$$

then

$$\sum (i \in n) \ \|(F_a(i) - F_b(i)) \ E_i x_n\|^2 < \epsilon^2$$

that is,

$$\|F_a x_n - F_b x_n\| < \epsilon.$$

Consider now the following:

$$\begin{aligned} | \ \|F_a x - F_b x\| - \|F_a x_n - F_b x_n\| \ | &\leq \|F_a(x-x_n) - F_b(x-x_n)\| \\ &\leq \|F_b - F_b\| \ \|x-x_n\| \end{aligned}$$

Thus, since

$$0 \leq F_b - F_a \leq I,$$

this yields

$$| \ \|F_a x - F_b x\| - \|F_a x_n - F_b x_n\| \ | \leq \|x - x_n\|$$

Now for $n \geq N_\epsilon$,

$$\|x - x_n\| < \epsilon/2,$$

so that

$$0 \leq \|(F_a - F_b)x\| \leq \epsilon/2 + \|(F_a - F_b)x_n\|$$

for $n \geq N_\epsilon$.

But there is also an $a(n)$ such that

$$a \leq b < a(n)$$

implies that

$$\|(F_b - F_a)x_n\| < \epsilon/2.$$

Thus if

$$a \leq b < a(n),$$

$$\|(F_b - F_a)x\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since ϵ is arbitrary, this proves (i). Properties (ii) and (iii) follow in the same manner.

Thus

$$\left\{F_a : -\infty < a < +\infty\right\}$$

is a resolution of the identity.

In order to show that F_a is the spectral resolution of the operator A it remains to show that

(iv) $F_a A \subseteq AF_a$ for all a

(v) $AF_a \leq a F_a$ and $A(I-F_a) \geq a(I-F_a)$ on $D(A)$.

To prove (iv) note that from the uniqueness guarantee in von Neumann's theorem, if $x \in D(A)$, then

$$Ax = \sum (i\epsilon\omega)\ A_i E_i x$$

Hence

$$F_a Ax = F_a \sum_i E_i A_i E_i x$$

$$= \sum_i F_a(i)\ E_i A_i E_i x$$

$$= \sum_i E_i F_a(i)\ A_i E_i x$$

(since $E_i F_a(i) = F_a(i)\ E_i$)

$$= \sum_i E_i A_i F_a(i)\ E_i x$$

(since $A_i F_a(i) = F_a(i)\ A_i$)

$$= AF_a x.$$

This proves (iv).

To prove the first part of (v), let $x \in D(A)$. Then

$$(AF_a x, x) = \sum_i (A_i F_a(i)\ E_i x,\ E_i x)$$

$$\leq \sum_i a(F_a(i)\ E_i x,\ E_i x)$$

$$= a\ (F_a x, x)$$

This proves the first assertion; proof of the second is analogous.

Now the self-adjoint operator A has a resolution F_a'.

In virtue of (iv)

$$F_a F_b' = F_b' F_a$$

for all real a and b. Hence

$$F_a(I-F_a') \qquad F_a'(I-F_a)$$

are projections.

To show that these are zero, suppose that

$$x \in D(A) \cap \operatorname{rng} F_a\,(I-F_a').$$

Then since $x \in \operatorname{rng} F_a$,

$$((A-aI)x,x) \leq 0$$

by (vi). That is

$$\int_{-\infty}^{+\infty} (b-a)d\,(F_b'x,x) \leq 0.$$

This means that $(F_b'x,x)$ is constant for $b > a$. Thus since F_b' is right continuous

$$(F_b'x,x) = (F_a'x,x)$$

for $b > a$.

But $x \in \operatorname{rng}(I-F_a')$ also, so that

$$(F_b'x,x) = 0$$

for $b \leq a$. Hence

$$(F_b'x,x) = 0$$

for <u>all</u> b. Thus $x=0$ and

$$F_a(I-F_a') = 0.$$

Similarly

$$F_a'\,(I-F_a) = 0.$$

Hence $F_a \leq F_a'$ and $F_a' \leq F_a$, so that $F_a = F_a'$ for all a. That is, F_a is the spectral resolution of A.

This proves that $A \in R^N$.

The second part of theorem 29 follows from the first: Since

$$E_iA \subseteq AE_i \in R^S$$

the first part of 29 says that there is an operator $A' \in R^N$ such that

$$E_iA' \subseteq A'E_i = AE_i$$

By the uniqueness guarantee in von Neumann's theorem, $A=A'$. Hence $A \in R^N$.

This completes the proof of 29.

30. <u>Theorem</u>. If T is a linear transformation with domain and range in H such that

$$E_iT \subseteq TE_i \in R^S$$

$$(TE_i)^* = TE_i$$

for all $i \epsilon \omega$, where R is an abelian von Neumann algebra and

$$\{E_i : i \epsilon \omega\} \subseteq R^P, \sum_i E_i = I,$$

then there exists a unique $T' \in R^N$ such that $T' \supseteq T$.

Proof. By von Neumann's theorem there exists unique T' which is self-adjoint and satisfies

$$E_i T' \subseteq T' E_i = TE_i$$

for all i.

$$D(T') = \{x : x \in H, \sum_i \|TE_i x\|^2 < \infty\}$$

and if $x \in D(T')$

$$T'x = \sum_i TE_i \, x$$

By 29, $T' \in R^N$.

To show that T' extends T, let $x \in D(T)$. Then

$$\begin{aligned} Tx &= (\sum_i E_i) \, Tx \\ &= \sum_i E_i Tx \\ &= \sum_i TE_i x \qquad \text{(a)} \end{aligned}$$

(since $E_i T \subseteq TE_i$)

Now since

$$TE_i = E_i TE_i ,$$

the terms of the sum (a) are mutually orthogonal. Thus from (a)

$$\|Tx\|^2 = \sum_i \|TE_i x\|^2,$$

that is, $x \in D(T')$.

Now (a) also implies that $Tx = T'x$. Therefore

$$T \subseteq T'.$$

T' is the only R^N - operator which extends T, for if $T'' \in R^N$ and $T \subseteq T''$, then

$$TE_i \subseteq T''E_i$$

for all i. Since $TE_i \in R^S$, it is defined on all of H. Hence actually,

$$TE_i = T''E_i,$$

and moreover

$$TE_i = T''E_i = T'E.$$

But this, together with the uniqueness guarantee in von Neumann's theorem, implies that

$$T'' = T'$$

This proves theorem 30.

31. Lemma. If

$$\{E_i : i \in \mathcal{A}\} \qquad \{F_j : j \in \mathcal{B}\}$$

are families of projections such that

$$\sum_i E_i = \sum_j F_j = I$$

and

$$E_i F_j = F_j E_i,$$

then also,

$$\sum_{ij} E_i F_j = I.$$

Proof. Given $x \in H$ and $\epsilon > 0$ there must exist a finite subset $\mathcal{A}' \subseteq \mathcal{A}$ such that

$$|y-x| < \epsilon/2,$$

where

$$y = \sum (i \in \mathcal{A}') \; E_i x.$$

There is also a finite subset $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$|y - \sum (j \in \mathcal{B}') \; F_j y| < \epsilon/2$$

Thus

$$\begin{aligned} &|x - \sum (i \in \mathcal{A}', \; j \in \mathcal{B}') \; E_i F_j x| \\ &\quad = |x - \sum (j \in \mathcal{B}') \; F_j y| \\ &\quad \leq |x-y| + |y - \sum (j \in \mathcal{B}') \; F_j y| \\ &\quad < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Hence

$$\sum_{ij} E_i F_j = I$$

and the lemma is proven.

32. Lemma. Let E_a, F_a be two resolutions of the identity in R^P, where R is abelian. If

$$G_a = E_a F_a,$$

then G_a is also a resolution of the identity in R^P.

Proof. Since R is abelian, $G_a \in R^P$.

If $a \leq b$,

$$\begin{aligned} G_a G_b = G_b G_a &= E_b F_b E_a F_a \\ &= E_a E_b F_a F_b \\ &= E_a F_a \\ &= G_a, \end{aligned}$$

so that

$$G_a \leq G_b.$$

Moreover, if again $a \leq b$,

$$\begin{aligned} &\|G_b x - G_a x\| \\ &\quad = \|(E_b F_b - E_a F_a)x\| \\ &\quad \leq \|(E_a F_a - E_a F_b)x\| + \|(E_a F_b - E_b F_b)x\| \\ &\quad \leq \|(F_a - F_b)x\| + \|(E_a - E_b)x\|, \end{aligned}$$

for any $x \in H$. Thus G_a inherits the right continuity of E_a and F_a.

The properties

$$\lim_{a \to -\infty} (\text{strong}) G_a = 0$$

and

$$\lim_{a \to +\infty} (\text{strong}) G_a = I$$

can be demonstrated in similar or simpler fashions.

This proves 32.

33. Theorem. If R is abelian and $A, B \in R^N$, then there are unique R^N-operators S, T, N such that

$$S \supseteq A + B$$
$$T \supseteq A - B$$
$$N \supseteq AB \quad \text{and} \quad N \supseteq BA.$$

Proof. Let A and B have spectral resolutions E_a and F_a respectively. Let

$$E(i) = E_i - E_{i-1}, \qquad F(j) = F_j - F_{j-1}$$

where i and j are integers (negative too!). By lemma 31, since R is abelian, $\sum_{ij} E(i)\, F(j) = I$.

Now for any i and j

$$E(i)\, F(j)\, (A-B)$$
$$= E(i)\, F(j)\, A - E(i)\, F(j)\, B$$
$$= F(j)\, E(i)\, A - E(i)\, F(j)\, B$$

(since R is abelian)

$$\subseteq F(j)\, (AE(i)) - E(i)\, (BF(j))$$

(since $E(i)$ and $F(j)$ are spectral projections)

$$= (AE(i))\ F(j) \ - \ (BF(j))\ E(i) \qquad \text{(a)}$$

(since $AE(i)$ and $BF(j) \in R^S$, which is abelian)

$$= AE(i)\ F(j) \ - \ BE(i)\ F(j)$$

$$= (A - B)\ E(i)\ F(j).$$

Since line (a) is an R^S-operator,

$$E(i)\ F(j)\ (A-B) \subseteq (A-B)\ E(i)\ F(j) \in R^S.$$

Since the family of $E(i)\ F(j)$ is countable, and R is abelian, theorem 30 applies, so that there exists a unique $T \in R^N$ such that

$$T \supseteq A-B.$$

The assertion about $(A+B)$ follows immediately.

The case of AB and BA is similar, but it requires some additional remarks:

$$E(i)\ F(j)\ AB$$

$$= F(j)\ E(i)\ AB$$

$$\subseteq F(j)\ (AE(i))\ B$$

(since E_a is the spectral resolution of A)

$$= (AE(i))\ F(j)\ B$$

(since $AE(i)$ and $F(j)$ are in R, which is abelian)

$$\subseteq (AE(i))\ (BF(j)) \qquad \text{(b)}$$

(since $F(j)$ is a spectral projection of B)

$$= A\ (BF(j))\ E(i)$$

(since both $BF(j)$ and $E(i)$ are in R, which is abelian)

$$= AB\, E(i)\, F(j).$$

Line (b) is a product of R^S operators, hence

$$E(i)\, F(j)\, AB \subseteq (AE(i))\,(BF(j)) = ABE(i)\, F(j) \in R^S.$$

By symmetry

$$E(i)\, F(j)\, BA \subseteq (BF(j))\,(AE(i)) = BAE(i)\, F(j) \in R^S$$

Thus by 30 there exists unique N, N' in R^N such that

$$N \supseteq AB \qquad N' \supseteq BA.$$

However, since R is abelian

$$\begin{aligned} ABE(i)\, F(j) &= (AE(i))\,(BF(j)) \\ &= (BF(j))\,(AE(i)) = BAE(i)\, F(j) \end{aligned}$$

in R^S. Hence $N = N'$.

This completes the proof of 33.

Because the S, T, N of theorem 33 are unique, the following definitions are possible.

34. <u>Definition</u>. In 33

$$S = A \dot{+} B \qquad T = A \dot{-} B \qquad N = A \circ B.$$

Note that 33 implies that for $A, B \in R^N$,

$$A \circ B = B \circ A$$

and that

$$A \dot{+} B = B \dot{+} A$$

and

$$A \overset{\circ}{-} A = 0.$$

Moreover, if $C \in R^N$ also, then

$$(A+B)C = AC + BC$$

and

$$AC + BC \subseteq (A \circ C) + (B \circ C)$$
$$\subseteq (A \circ C) \overset{\circ}{+} (B \circ C)$$

while on the other hand

$$(A+B)C \subseteq (A \overset{\circ}{+} B)C \subseteq (A \overset{\circ}{+} B) \circ C.$$

Since these extensions are unique

$$(A \overset{\circ}{+} B) \circ C = (A \circ C) \overset{\circ}{+} (B \circ C).$$

Thus R^N is a commutative algebra under the operations $\overset{\circ}{+}$ and $\circ$.

35. Corollary. If $A-B \geq 0$ on $D(A-B)$, then $A \overset{\circ}{-} B \geq 0$ also.

Proof. Carrying on with the notation of 33, by 30

$$D(A \overset{\circ}{-} B) = \left\{ x : x \in H, \sum_{ij} \|(A-B)E(i)\, F(j)x\|^2 < \infty \right\}$$

and for $x \in D(A \overset{\circ}{-} B)$

$$(A \overset{\circ}{-} B)\, x = \sum_{ij} (A-B)\, E(i)\, F(j)\, x,$$

whence it's obvious that $A-B \geq 0$ implies $A \overset{\circ}{-} B \geq 0$.

36. Corollary. If $A \in R^N$ and $B \in R^S$, then

$$A \circ B = B \circ A = AB,$$

and so

$$BA \subseteq AB \in R^N$$

Proof. Let $N = A \circ B = B \circ A$. Then

$$N \supseteq BA.$$

Since $B \in R^S$,

$$D(BA) = D(A)$$

so that BA is densely defined and the operator $(BA)^*$ exists.

Hence

$$N = N^* \subseteq (BA)^* = A^*B^* = AB \subseteq N$$

whence

$$N = AB$$

This proves 36.

POSITIVE AND NEGATIVE PARTS OF AN R^N-OPERATOR

Let N be an operator in R^N with spectral resolution G_a. Then

$$
\begin{aligned}
N &= N[(I-G_0) + G_0] \\
&\supseteq N(I-G_0) + NG_0 \\
&\supseteq (I-G_0)N + G_0N \\
&= [(I-G_0) + G_0]N = N
\end{aligned}
$$

Thus

$$N = N(I-G_0) - (-NG_0)$$

Let

$$N^+ = N(I-G_0) \qquad N^- = -NG_0 .$$

Then both N^+, N^- are positive, and by 36, both are in R^N. Hence

$$N = N^+ \overset{\circ}{-} N^-$$

as well as

$$N = N^+ - N^-$$

37. Theorem. If

$$N = A \overset{\circ}{-} B,$$

where A,B are positive in R^N, then there exists a unique positive operator P in R^N such that

$$A = P \overset{\circ}{+} N^+ \qquad B = P \overset{\circ}{+} N^-$$

Proof. If A,B have the respective resolutions E_a,F_a,

$$E(i) = E_i - E_{i-1} \qquad i \in \omega$$

and

$$F(j) = F_j - F_{j-1} \qquad j \in \omega,$$

then

$$\sum_{ij} E(i)\ F(j) = I,$$

since A and B are positive.

Note first that

$$\begin{aligned}(A \mathring{-} B)\ E(i)\ F(j) &= (A-B)\ E(i)\ F(j) \\ &= AE(i)\ F(j) - BE(i)\ F(j),\end{aligned}$$

where the latter is in R^S . Also

$$\begin{aligned}(A \mathring{-} B)\ E(i)\ F(j) &= (N^+ - N^-)\ E(i)\ F(j) \\ &= N^+E(i)\ F(j) - N^-E(i)\ F(j).\end{aligned}$$

Thus

$$N^+E(i)\ F(j) - N^-E(i)\ F(j) \in R^S.$$

This means that the self-adjoint operators $N^+E(i)\ F(j)$, $N^-E(i)\ F(j)$ are defined everywhere. Hence they are bounded and belong to R^S.

Thus

$$AE(i)F(j) - BE(i)F(j) = N^+E(i)F(j) - N^-E(i)F(j),$$

where all terms are in R^S.

$$AE(i)F(j) - N^+E(i)F(j) = BE(i)F(j) - N^-E(i)F(j)$$

$$(A-N^+)\ E(i)\ F(j) = (B-N^-)\ E(i)\ F(j).$$

Thus

$$A \dot{-} N^{+} = B \dot{-} N^{-}$$

Now let

$$P = B \dot{-} N^{-}.$$

Then for any i, j,

$$PE(i)\ F(j) = (B-N^{-})\ E(i)\ F(j)$$

$$= (B-(-NG_0))\ E(i)\ F(j)$$

(since G_a is the spectral resolution of N, and $N^{-}=-NG_0$)

$$= \left[B + (A-B)G_0\right]\ E(i)\ F(j)$$

$$= [B(I-G_0) + AG_0]\ E(i)\ F(j)$$

Thus if $x \in \text{rng}\ E(i)\ F(j)$ (then $x \in \text{rng}\ P$, $\text{rng}\ A$, $\text{rng}\ B$),

$$(Px,x) = (B(I-G_0)x,x) + (AG_0x,x)$$

$$= ((I-G_0)\ Bx,x) + (G_0Ax,x)$$

(by 36 $G_0A \subseteq AG_0$, etc.)

$$= ((I-G_0)Bx, (I-G_0)x) + (G_0Ax,G_0x)$$

$$= (B(I-G_0)x, (I-G_0)x) + (AG_0x, G_0x).$$

This last is non-negative. Hence

$$P = A \dot{-} N^{+} = B \dot{-} N^{-} \geq 0$$

and

$$A = N^{+} \dot{+} P \qquad B = N^{-} \dot{+} P$$

as required. This proves 37.

MEASURE THEOREMS

38. Theorem. [A covering theorem for semi-finite measures]. Let R be any von Neumann algebra. Let m be a semi-finite measure on R^P (definitions 7 and 8), and let E be a non-zero projection in R^P. Then there exists a family $\{E_i\} \subseteq R^P$ such that

(i) the E_i are mutually orthogonal

(ii) $mE_i < \infty$ for all E_i

(iii) $\sum_i E_i = E$

Proof. Let $\mathcal{K}$ be the set of all subsets $\mathcal{B} \subseteq R^P$ such that

(i) $\mathcal{B}$-members are mutually orthogonal

(ii) if $F \in \mathcal{B}$, then $0 < F \leq E$ and $mF < \infty$

$\mathcal{K}$ is partially ordered by set inclusion, and every linearly ordered $\mathcal{K}$-subset has an upper bound in $\mathcal{K}$ (namely, its union). Hence by Zorn's Lemma, $\mathcal{K}$ has a maximal element $\underline{\mathcal{B}}$. Since $\underline{\mathcal{B}} \subseteq R^P$, $\sup \underline{\mathcal{B}} \in R^P$ and $E - \sup \underline{\mathcal{B}} \in R^P$, by 27. Now

$$E - \sup \underline{\mathcal{B}} = 0,$$

for otherwise, since m is semi-finite, there is $F \in R^P$ such that

$$0 < F \leq E - \sup \underline{\mathcal{B}}$$

and $mF < \infty$. That is, the $\mathcal{K}$-member $\underline{\mathcal{B}} \cup \{F\}$ properly contains $\underline{\mathcal{B}}$. But this contradicts the maximality of $\underline{\mathcal{B}}$. Thus

$$\sup \underline{\mathfrak{B}} = E$$

and $\underline{\mathfrak{B}}$ furnishes a family of the required sort. QED.

39. Theorem. If R is any von Neumann algebra and E_n, $E \in R^P$, where

$$E_n \uparrow E ,$$

then

$$mE_n \uparrow mE$$

Proof. By lemma 21,

$$E = \sup \{E_n\} .$$

Now

$$\{E_{n+1} - E_n : n \in \omega\}$$

is a family of mutually orthogonal R^P-members. Hence

$$\begin{aligned} E &= \sup \{E_n\} \\ &= E_0 + \sup \{E_{n+1} - E_n : n \in \omega\} \\ &= E_0 + \sum_n (E_{n+1} - E_n) \end{aligned}$$

so that

$$\begin{aligned} mE &= mE_0 + \sum_n m(E_{n+1} - E_n) \\ &= mE_0 + \sum_n (mE_{n+1} - mE_n) \\ &= mE_0 + \lim (mE_n - mE_0) \\ &= \lim mE_n . \end{aligned}$$

Finally, since m is monotone,

$$mE_n \uparrow mE \ .$$

40. Corollary. If

$$E_n \downarrow E$$

and $mE_0 < \infty$, then

$$mE_n \downarrow mE$$

Proof. If $E_n \downarrow E$, then

$$(E_0 - E_n) \uparrow (E_0 - E)$$

so that

$$m(E_0 - E_n) \uparrow m(E_0 - E)$$

by 39.

EXTENSION OF m TO R^{SP}

4. Definition. If S is a positive R^{SP}-operator,

$$S = \sum_{i=1}^{p} a_i E_i$$

then define

$$mS = \sum_{i=1}^{p} a_i \, mE_i$$

(which is non-negative and finite or infinite according as

$$\sum (a_i \neq 0) \, mE_i$$

is finite or infinite).

S is summable if $mS < \infty$.

If S is an arbitrary R^{SP}-operator, then

$$S = S^+ - S^-,$$

where S^+, S^- are positive R^{SP}-operators.

S is integrable if at least one of S^+, S^- is summable.

If S is integrable, define

$$mS = mS^+ - mS^-$$

(this is well defined since one of mS^+, mS^- is finite).

S is summable if $|mS| < \infty$, that is, if both S^+, S^- are summable.

Let R^{SS} be the set of all summable R^{SP}-operators.

Remarks. An R^{SP}-operator is summable if and only if its support has finite measure.

If $S \in R^{SP}$ is integrable,

$$mS = \sum_{i=1}^{p} a_i \, mE_i \; .$$

Henceforth R is assumed to be an abelian von Neumann algebra.

42. Theorem. If $S \in R^{SP}$, then its integrability is independent of its representation in R^{SP}.

If S is integrable in R^{SP}, then its integral is independent of its representation in R^{SP}.

Proof. It is enough to prove the second assertion for positive R^{SP}-operators.

Let $S, T \in R^{SP}$ such that

$$S = T \geq 0,$$

where

$$S = \sum_{i=1}^{p} a_i \, E_i \qquad\qquad T = \sum_{j=1}^{q} b_j \, F_j \; .$$

Note first that since $S = T$,

$$\sum (a_i \neq 0) \, E_i = \sum (b_j \neq 0) \, F_j$$

so that mS, mT are finite or infinite simultaneously. This proves the assertion for the case that $S \notin R^{SS}$.

If mS, mT are finite

$$mS - mT = \sum_i a_i \, mE_i - \sum_j b_j \, mF_j$$

$$= \sum_{ij} a_i \, mE_iF_j - \sum_{ij} b_j \, mE_iF_j$$

(since $\sum_i E_i = \sum_j F_j = I$ and R is abelian)

$$= \sum_{ij} (a_i - b_j) \, mE_iF_j$$

If $mE_iF_j > 0$, then $E_iF_j \neq 0$, so that $a_i = b_j$. Thus if $mE_iF_j \neq 0$, so also

$$a_i - b_j = 0,$$

and hence

$$mS - mT = 0$$

This proves 42.

43. <u>Theorem</u>. If $S,T \in R^{SS}$, then

$$S+T \in R^{SS} \qquad aS \in R^{SS}$$

for all real a. Moreover

$$m(S+T) = mS+mT \qquad maS = amS.$$

The second assertion about $(S+T)$ holds if S,T are positive and one of mS,mT is infinite.

<u>Proof</u>. The assertions about aS are obvious, the others tedious.

If S and T are represented as in 42, then

$$S+T = \sum_{ij} (a_i+b_j) \, E_iF_j,$$

since

$$\sum_i E_i = \sum_j F_j = I.$$

Note that $E_iF_j \in R^P$, since R is abelian.

Now $(a_i+b_j) \neq 0$ only if $a_i \neq 0$ or $b_j \neq 0$. If $a_i \neq 0$, then $mE_i < \infty$ and

$$mE_i F_j \leq mE_i < \infty.$$

Similarly $b_j \neq 0$ implies that $mE_i F_j < \infty$. Therefore

$$\sum (a_i+b_j \neq 0) \; mE_iF_j < \infty$$

and $S+T \in R^{SS}$.

By definition

$$\begin{aligned}
m(S+T) &= \sum_{ij} (a_i+b_j) \; mE_iF_j \\
&= \sum_i a_i \sum_j mE_iF_j + \sum_j b_j \sum_i mE_iF_j \\
&= \sum_i a_i \; mE_i \; (\sum_j F_j) \\
&\qquad + \sum_j b_j \; mF_j \; (\sum_i Ei) \\
&= \sum_i a_i \; mE_i + \sum_j b_j \; mF_j \\
&= mS + mT
\end{aligned}$$

This proves 43.

44. <u>Theorem.</u> R^{SS} is closed under the R^S lattice operations (R is abelian!).

Proof. For S,T in R^{SS}, recall that

$$(S\cup T) = TF_0 + S(I-F_0)$$

$$(S\cap T) = T(I-F_0) + SF_0 ,$$

where F_a is the spectral resolution of $S-T$. Obviously all of the terms on the right are in R^{SS}, hence by 43, $(S\cup T)$ and $(S\cap T)$ belong to R^{SS}. QED.

45. Theorem. If S and T are integrable in R^{SP}, then $S \leq T$ implies

$$mS \leq mT$$

Proof. If $0 \leq S \leq T$, then (using the representations in 42)

$$\sum (a_i \neq 0)\ E_i \leq \sum (b_j \neq 0)\ F_j .$$

Thus $mS = \infty$ implies that $mT = \infty$.

If $mS < \infty$ then as before $mT - mS = \sum_{ij} (b_j - a_i) mE_i F_j$. Now if $mE_i F_j \neq 0$, it must be that $E_i F_j \neq 0$, and

$$b_j - a_i \geq 0,$$

since $S \leq T$. Thus $mT - mS \geq 0$ and so

$$mS \leq mT.$$

If S and T are arbitrary R^{SP}-operators which are integrable,

$$S \leq T$$

implies that

$$S^+ \leq T^+ \qquad T^- \leq S^-,$$

whence from above

$$mS^+ \leq mT^+ \qquad mT^- \leq mS^-,$$

so that

$$mS = mS^+ - mS^- \leq mT^+ - mT^- = mT.$$

This proves 45.

46. Theorem. If $\{T_n\} \subseteq R^{SS}$ and

$$T_n \downarrow 0$$

then

$$mT_n \downarrow 0$$

Remark. This theorem is not true in R^{SP}. For let

$$T_n = \frac{1}{n} I \qquad n = 1, 2, \ldots .$$

Then $T_n \downarrow 0$ (uniform), hence also $T_n \downarrow 0$. But $mT_n = \infty$ for all n if $mI = \infty$.

Proof. Let

$$T_n = \sum_{i=1}^{M(n)} b_{in} F_{in} .$$

For any real number c, let

$$E(n,c) = \sum (b_{in} > c) F_{in} .$$

$E(n,c)$ is thus the projection onto the subspace where $T_n > c$. Note that

$$E(n,c) \geq E(n,c+\epsilon)$$

for $\epsilon > 0$. Consequently

$$E(n,0) \geq E(n,c)$$

and

$$mE(n,c) < \infty .$$

Moreover, since $T_n \downarrow$,

$$E(n,c) \geq E(n+1,c).$$

Let

$$C(n) = \max \{b_{in} : \text{fixed } n\} .$$

Then $C(n) \downarrow$ since $T_n \downarrow$.

The pièce de résistance of this proof is the fact that for $c > 0$

$$\lim_n m\, E(n,c) = 0.$$

To prove this, suppose contrarily that

$$\lim_n m\, E(n,c) > 0.$$

Since $mE(n,c) < \infty$ and $E(n,c) \downarrow$, 40 applies and

$$\lim mE(n,c) = m(\lim[\text{strong}]\ E(n,c))$$

Let

$$E = \inf \{ E(n,c) : \text{fixed } c \} .$$

Then $E \in R^P$ by 26, and by 22

$$E = \lim\ (\text{strong})\ E(n,c).$$

Therefore

$$mE = m \lim \text{(strong)} E(n,c)$$

$$= \lim m\, E(n,c) > 0,$$

so that $E \neq 0$. Thus there is non-zero x in $\operatorname{rng} E$. Now

$$x \in \operatorname{rng} E(n,c)$$

for all $n \in \omega$. Thus

$$(T_n x, x) > c\|x\|^2 > 0$$

for all n, contradicting the fact that $T_n \downarrow 0$. Hence

$$\lim m\, E(n,c) = 0$$

for $c > 0$.

To complete the proof, for all non-negative integers n

$$E(0,0) \geq E(n,c)$$

for $c > 0$.

$$\begin{aligned} T_n &= E(0,0)\, T_n \\ &= E(n,c)\, T_n + (E(0,0) - E(n,c))\, T_n \\ &\leq C(n)\, E(n,c) + c(E(0,0) - E(n,c)) \\ &\leq C(0)\, E(n,c) + cE(0,0)\, . \qquad [*] \end{aligned}$$

Now, given $\epsilon > 0$, choose c so that

$$0 < c < \frac{\epsilon}{2mE(0,0)} ,$$

then choose N so that $n > N$ implies

$$mE(n,c) < \epsilon/2C(0).$$

Then from [*], $n > N$ implies

$$mT_n \leq \epsilon/2 + \epsilon/2 = \epsilon,$$

that is,

$$\lim mT_n = 0.$$

By 45, this convergence is monotone. QED.

In the next section. m will be extended to R^S. The following theorem hints how this will be done.

47. Theorem. If R is abelian, $S \in R^{SP}$, $\{S_n\} \subseteq R^{SP}$, $S_n \geq 0$ and

$$S_n \uparrow S,$$

then

$$mS_n \uparrow mS.$$

Proof. Since

$$0 \leq S_n \leq S$$

for all n,

$$0 \leq mS_n \leq mS_{n+1} \leq mS$$

by 45, and

$$\lim mS_n \leq mS$$

Thus if $\lim mS_n = \infty$, $mS = \infty$.

For the converse of this, let $E(n,c)$ and $E(c)$ be the respective projections on which S_n and S are greater than

$c \geq 0$. Note that

$$E(n,c) \leq E(n+1,c) \leq E(c) \qquad (a)$$

If $c < \max S$, the largest of the non-zero coefficients of S (assume $S \neq 0$!) and

$$x \in \operatorname{rng} E(c) \qquad \|x\| = 1,$$

then

$$(Sx,x) > c$$

and since

$$S_n \uparrow S,$$

there exists an integer $Q(c,x)$ such that $n > Q$ implies

$$(Sx,x) \geq (S_n x,x) > c$$

so that

$$x \in \operatorname{rng} E(n,c)$$

for $n \geq Q$, that is, using (a)

$$E(n,c) \uparrow E(c).$$

By lemma 39

$$mE(n,c) \uparrow mE(c) \qquad (b)$$

where $c < \max S$.

Now if $c \geq 0$

$$S_n \geq E(n,c) \; S_n \geq c\, E(n,c).$$

Thus

$$mS_n \geq c\; mE(n,c) \qquad (c)$$

for all $c \geq 0$.

If

$$0 < c < \min S \tag{d}$$

then

$$E(c) = E(0)$$

and so, if $mS = \infty$ and

$$mE(c) = \infty,$$

subject to (d), then by (b)

$$\lim mE(n,c) = \infty,$$

and by (c)

$$\lim m\, S_n = \infty.$$

Thus $mS = \infty$ if and only if $\lim mS_n = \infty$, and the theorem is proven in this case.

The above also shows that

$$mS < \infty \qquad\qquad \lim m\, S_n < \infty$$

are simultaneously true. In this case all the operators involved are in R^{SS}.

Since

$$S_n \uparrow S,$$

also

$$S - S_n \downarrow 0$$

and by lemma 46

$$m(S-S_n) \downarrow 0.$$

But

$$m(S-S_n) = mS - mS_n$$

by 43. Hence

$$mS_n \uparrow mS.$$

48. <u>Lemma</u>. If S is a positive R^{SP}-operator for which $mS = +\infty$, then there exists $\{S_n\} \subseteq R^{SS}$ such that

$$S_n \leq S, \qquad S_n \uparrow, \qquad m\, S_n \uparrow \infty$$

<u>Proof</u>. Let

$$S = \sum_{i=1}^{n} a_i\, E_i\ .$$

Then $mS = \infty$ if and only if $mE_i = \infty$ for some i such that $a_i \neq 0$.

By 38

$$E_i = \sum (j \epsilon J)\ F_j\ ,$$

where the F_j are mutually orthogonal and are of finite measure. Choose a nest of finite J-subsets $F(n)$:

$$F(n) \subseteq F(n+1) \subseteq J$$

for $n \epsilon \omega$, such that

$$\sum (j \epsilon F(n))\ m\, F_j > \frac{n}{a_i}\ .$$

Then the sequence $\{S_n\}$,

$$S_n = a_i \sum (j \epsilon F(n))\ F_j\ ,$$

has the required properties.

This proves 48.

49. <u>Independence Theorem</u>. If $\{S_n\}$ and $\{T_n\}$ are sequences in R^{SP} such that

$$S_n \uparrow S \qquad T_n \uparrow T,$$

then $T \leq S$ implies

$$\lim mT_n \leq \lim mS_n \ .$$

<u>Proof</u>. By 20

$$(T_m \cap S_n) \ \uparrow_n \ (T_m \cap S) \ ,$$

and since

$$S \geq T \geq T_n \ ,$$

then

$$T_m \cap S = T_m$$

so that

$$(T_m \cap S_n) \uparrow T_m$$

and by 47

$$m(T_m \cap S_n) \ \uparrow_n \ mT_n \ .$$

Thus for arbitrary $m \in \omega$ and $\mathcal{M} < mT_m$, there exists $N(m, \mathcal{M})$ such that if $n \geq N$

$$\mathcal{M} < M(T_m \cap S_n) \ .$$

But

$$(T_m \cap S_n) \leq S_n \ ,$$

so

$$\mathcal{M} < m(T_m \cap S_n) \leq mS_n$$

for $n \geq N$. Thus

$$\mathfrak{M} \leq \lim mS_n$$

for all $\mathfrak{M} < mT_m$, hence

$$mT_m \leq \lim mS_n \ .$$

Since m is arbitrary

$$\lim mT_m \leq \lim mS_n .$$

This proves 49.

<u>Remark</u>. If, as above,

$$S_n \uparrow S \qquad\qquad T_n \uparrow S$$

then

$$\lim mS_n = \lim mT_n .$$

EXTENDING m TO R^S

50. Definition. Let T be a positive R^S-operator. By 24 there is a sequence $\{T_n\} \subseteq R^{SP}$ such that

$$T_n \uparrow T$$

Define

$$mT = \lim mT_n.$$

(Theorem 49 shows that mT does not depend on the choice of sequence $\{T_n\}$. Theorem 47 shows that this is a consistent extension of definitions already made.)

T is summable if $mT < \infty$.

51. Lemma. If S is a positive R^S-operator for which $mS = +\infty$, then there exists $\{S_n\} \subseteq R^{SS}$ such that

$$S_n \leq S, \qquad S_n\uparrow \quad , \qquad mS_n \uparrow \infty$$

Proof. By 24 there exists a sequence $\{T_n\} \subseteq R^{SP}$ such that

$$T_n \uparrow S,$$

and by definition

$$mT_n \uparrow mS = \infty .$$

Either $\{T_n\} \subseteq R^{SS}$, and the theorem is proven, or there exists $T_N \not\in R^{SS}$. Then $T_n \not\in R^{SS}$ for all $n \geq N$. By 48 there exists a sequence $\{S_n\} \subseteq R^{SS}$ such that

$$S_n \leq T_N \leq S, \quad S_n \uparrow \ , \quad mS_n \uparrow \infty.$$

This proves 51.

52. <u>Corollary</u>. If T is a positive R^S operator, then there exists $\{T_n\} \subseteq R^{SS}$ such that

$$0 \leq T_n \leq T \qquad T_n \uparrow$$

$$\lim mT_n = mT.$$

53. <u>Theorem</u>. If S and T are R^S operators for which

$$0 \leq T \leq S,$$

then

$$mT \leq mS.$$

Hence if S is summable, so is T.

<u>Proof</u>. By 24 there exist $\{S_n\}$ and $\{T_n\}$, sequences in R^{SP}, such that

$$S_n \uparrow S \qquad T_n \uparrow T$$

Theorem 53 thus follows immediately from 49.

54. <u>Theorem.</u> If T is a positive R^S-operator and

$$\mathfrak{a} = \{H : H \in R^S,\ 0 \leq H \leq T\}$$

$$\mathfrak{B} = \{H : H \in R^{SP},\ 0 \leq H \leq T\}$$

$$\mathfrak{K} = \{H : H \in R^{SS},\ 0 \leq H \leq T\}$$

then

$$\begin{aligned} mT &= \sup \{mH : H \in \mathfrak{a}\} \\ &= \sup \{mH : H \in \mathfrak{B}\} \\ &= \sup \{mH : H \in \mathfrak{K}\}. \end{aligned}$$

Proof. By 52, $\mathcal{H} \neq \emptyset$. Obviously

$$\mathcal{H} \subseteq \mathcal{B} \subseteq \mathcal{G}.$$

Thus by 53

$$\begin{aligned} mT &\geq \sup \{mH : H \in \mathcal{G}\} \\ &\geq \sup \{mH : H \in \mathcal{B}\} \\ &\geq \sup \{mH : H \in \mathcal{H}\} \\ &\geq \lim mT_n = mT, \end{aligned}$$

where $\{T_n\} \subseteq R^{SS}$ is the sequence whose existence is guaranteed by 52.

This proves 54.

The extension of m to R^S-operators of arbitrary sign is formally the same as the analogous extension in definition 41. Without further ado the terms "integrable" and "summable" will be applied to R^S-operators.

The following is a lemma toward a proof that m is linear on the summable R^S-operators.

55. Lemma. Let A and B be positive R^S-operators.

(a) $m(A+B) = mA + mB$

(b) If one of A, B is summable then $(A-B)$ is integrable and $m(A-B) = mA - mB$.

Hence if A and B are summable, so are $(A+B)$ and $(A-B)$.

Proof. Since A, B are positive, there exist sequences $\{A_n\}$, $\{B_n\}$ in R^{SP} such that

$$A_n \uparrow A \qquad B_n \uparrow B$$

and

$$mA = \lim mA_n \qquad mB = \lim mB_n.$$

Now

$$(A_n + B_n) \uparrow (A + B),$$

and since $m(A+B)$ is independent of the sequence (A_n+B_n), the assertion (a) now follows easily:

$$\begin{aligned} m(A+B) &= \lim m(A_n + B_n) \\ &= \lim (mA_n + mB_n) \\ &\qquad \text{(by 43)} \\ &= \lim mA_n + \lim mB_n \\ &= mA + mB, \end{aligned}$$

so that (a) is proven.

To prove (b), note that by 25

$$A = (A-B)^+ + P \qquad B = (A-B)^- + P$$

for $P \geq 0$, whence

$$A \geq (A-B)^+ \geq 0 \qquad B \geq (A-B)^- \geq 0$$

$$A \geq P \geq 0 \qquad B \geq P \geq 0,$$

so that by 53, since one of A,B is summable, so is P and so is one of $(A-B)^+$, $(A-B)^-$. Thus $(A-B)$ is integrable.

Now

$$
\begin{aligned}
mA \; - \; mB & \\
&= m[(A-B)^{+}+P] - m[(A-B)^{-}+P] \\
&= m(A-B)^{+}+mP - m(A-B)^{-}-mP \\
&\qquad \text{(by part (a))} \\
&= m(A-B)^{+} - m(A-B)^{-} \\
&= m(A-B).
\end{aligned}
$$

This proves 55.

56. Theorem. If S and T are summable operators, then $(S+T)$ is summable and

$$m(S+T) = mS + mT$$

Proof. If S and T are summable, then so are

$$S^{+},\ S^{-},\ T^{+},\ T^{-}\ .$$

Since these are postive

$$S^{+} + T^{+} \qquad\qquad S^{-} + T^{-}$$

are summable by 55 (a). Thus $(S+T)$ can be expressed as the difference of two positive summable operators:

$$
\begin{aligned}
S+T &= (S^{+} - S^{-}) + (T^{+} - T^{-}) \\
&= (S^{+} + T^{+}) - (S^{-} + T^{-}).
\end{aligned}
$$

By 55 (b), $S+T$ is thus also summable and

$$m(S+T) = m(S^{+}+ T^{+}) - m(S^{-} + T^{-})$$

$$= mS^+ + mT^+ - (mS^- + mT^-)$$
(by 54 (a))

$$= (mS^+ - mS^-) + (mT^+ - mT^-)$$

$$= mS + mT.$$

This proves 56.

57. <u>Theorem</u>. If S is an integrable R^S-operator and a is any real number, then aS is also integrable and

$$m(aS) = a\, mS$$

58. <u>Theorem</u>. If S and T are summable, and $S \geq T$, then

$$mS \geq mT$$

<u>Proof</u>. The standard argument. Since $S-T \geq 0$, $m(S-T) \geq 0$, hence

$$mS = m(S-T) + mT \geq mT.$$

59. <u>Lebesgue's Monotone Convergence Theorem</u>.

If $\{T_n\}$ is a sequence of positive summable R^S-operators such that

$$T_n \uparrow T,$$

then

$$mT = \lim mT_n.$$

Proof. For each n there exists $\{S_n{}^m\} \subseteq R^{SS}$ such that

$$S_n{}^m \uparrow_m T_n$$

and

$$\lim_m mS_n{}^m = mT_n < \infty.$$

Let

$$U_m = \sup \{S_q{}^m : q \leq m\},$$

where the sup is taken in the sense of 13. Then $U_n \in R^{SS}$ by 44. Moreover

$$U_n \uparrow,$$

for

$$U_{m+1} = \sup \{S_q{}^{m+1} : q \leq m+1\}$$

$$\geq \sup \{S_q{}^m : q \leq m+1\}$$

(since $S_1{}^{m+1} \geq S_1{}^m, \ldots, S^{m+1}_{m+1} \geq S^m_{m+1}$)

$$\geq \sup \{S_q{}^m : q \leq m\}.$$

$$= U_m .$$

Now

$$S_q{}^m \leq U_m \leq T_m \qquad (*)$$

for $q \leq m$. Hence (strong limits)

$$\lim_m S_n{}^m = T_n \leq \lim U_m \leq \lim T_m = T$$

so that

$$T_n \leq \lim U_m \leq T$$

for all n, or

$$T = \lim T_n \leq \lim U_m \leq T$$

so that

$$T = \lim U_m$$

and hence $U_m \uparrow T$. Since $U_m \in R^{SS}$

$$mT = \lim mU_n \qquad (**)$$

Now by (*)

$$m\, S_q^m \leq mU_m \leq mT_m$$

for $q \leq m$. Hence

$$\lim_m S_q^m = mT_q \leq \lim mU_m \leq \lim mT_n$$

and

$$mT_q \leq \lim mU_n \leq \lim mT_n$$

whence

$$mT = \lim mU_n = \lim mT_n$$

by (**). Thus

$$mT = \lim mT_n.$$

This proves 59.

The following theorem is an analog of the definition

$$\int_0^\infty f(x)\, dx = \lim_{R\to\infty} \int_0^R f(x)\, dx$$

for the Riemann integral.

60. <u>Theorem</u>. If T is any integrable R^S-operator and G_a is any resolution of the indentity in R^P, then

$$mT = \lim\ (a \to \infty)\ mT\ G_a$$

Proof. First observe that if $T \geq 0$, then the limit must exist and

$$mT \geq \lim\ (a \to \infty)\ mT\ G_a$$
$$\geq \lim\ mT\ G_n \qquad (*)$$

If $T \geq 0$ there exists $\{T_n\} \subseteq R^{SP}$ such that

$$0 \leq T_n \leq T \qquad T_n \uparrow T$$
$$mT_n \uparrow mT.$$

Now $T_nG_n \in R^{SP}$ for all n. Moreover

$$T_n\ G_n \uparrow .$$

Now for any $x \in H$

$$\|(T_nG_n - T)x\|$$
$$\leq \|(T_nG_n - TG_n)x\| + \|(TG_n - T)x\|$$
$$\leq \|G_n\|\ \|(T_n - T)x\| + \|T\|\ \|(I - G_n)x\|$$
$$= \|(T_n - T)x\| + \|T\|\ \|(I - G_n)x\|,$$

whence

$$T_nG_n \uparrow T,$$

since

$$T_n \uparrow T \qquad G_n \uparrow I.$$

Since mT is independent of the sequence

$$mT_nG_n \uparrow mT.$$

Now the fact that

$$mT_n\, G_n \leq mT\, G_n$$

and (*) proves the theorem for $T \geq 0$.

For arbitrary integrable T in R^S,

$$\begin{aligned} mT &= mT^+ - mT^- \\ &= \lim\,(a\to\infty)\; mT^+G_a - \lim(a\to\infty)\; mT^-G_a \\ &= \lim\,(a\to\infty)\; (mT^+G_a - mT^-G_a) \\ &= \lim\,(a\to\infty)\; (m(T^+ - T^-)\, G_a) \\ &= \lim\,(a\to\infty)\; mT\, G_a \quad . \end{aligned}$$

This proves 60.

MEASURE AND POSITIVE R^N-OPERATORS

If N is a positive R^N-operator with resolution E_n, then $NE_a \in R^S$ for all a. Moreover

$$NE_a \leq NE_b$$

for $a \leq b$, so that mNE_a is an increasing function of a. This justifies the following.

61. <u>Definition</u>. If N is positive in R^N,

$$mN = \lim\ (a \to \infty)\ mNE_a$$

N is <u>summable</u> if $mN < \infty$.

62. <u>Theorem</u>. If N is a positive R^N-operator and G_a is any resolution of the identity in R^P, then

$$mN = \lim\ (a \to \infty)\ mN\, G_a\ .$$

<u>Proof</u>. By 36, $NG_a \in R^N$ for all a. The proof must show that

$$\lim(a \to \infty)\ mNG_a$$

exists and equals mN.

Let E_a be the resolution of N and let

$$\{H_c(a) : -\infty < c < \infty\}$$

be the resolution of the R^N-operator NG_a.

Then

$$NG_a\, H_c(a) \in R^S$$

for all a and c, and for fixed a, $mNG_a\, H_c(a)$ is an increasing function of c,

$$mNG_a = \lim(c \to \infty)\; mNG_a\, H_c(a)$$

so that

$$mNG_a \geq m\, NG_a\, H_c(a) \qquad \text{(a)}$$

for all c.

Furthermore, for all b and c,

$$NG_a\, H_c(a)\, E_b \in R^S$$

and

$$NG_a\, H_c(a) \geq NG_a\, H_c(a)\, E_b,$$

so that

$$mNG_a\, H_c(a) \geq mNG_a\, H_c(a)\, E_b.$$

Combining this with (a)

$$mNG_a \geq mNG_a\, H_c(a)\, E_b \qquad \text{(b)}$$

for all b and c.

Now

$$NG_a\, H_c(a)\, E_b = NE_b\, G_a\, H_c(a),$$

$NE_b\, G_a \in R^S$, and by 60

$$mNE_b\, G_a = \lim\,(c \to \infty)\; mNE_b\, G_a\, H_c(a)$$

Hence by (b)

$$mNG_a \geq mNE_b\, G_a \qquad \text{(c)}$$

for all b.

To obtain a further inequality on mNG_a, note that

$$NG_a\ H_c(a)\ E_b \leq NE_b$$

in R^S for all b and c. Again by 60, since $NG_a\ H_c(a) \in R^S$ for all c,

$$mNG_a\ H_c(a) = \lim(b\to\infty)\ mNG_a\ H_c(a)\ E_b$$
$$\leq \lim(b\to\infty)\ mNE_b = mN$$

for all c. Thus

$$mNG_a = \lim(c\to\infty)\ mNG_a\ H_c(a) \leq mN \qquad \text{(d)}$$

for all a.

It remains to apply 60 once more:

$$mN = \lim(b\to\infty)\ mNE_b = \lim(b\to\infty)\ \lim(a\to\infty)\ mNE_b\ G_a\ .$$

This means that given $\mathcal{M} < mN$, there exist a', b' such that

$$mN \geq mNE_{b'}\ G_{a'} > \mathcal{M} \qquad \text{(e)}$$

For $a \geq a'$

$$mNE_{b'}\ G_a \geq mNE_{b'}\ G_{a'}\ ,$$

and by (c)

$$mNG_a \geq mNE_{b'}\ G_a\ .$$

Thus for $a \geq a'$, (e) becomes

$$mN \geq mNG_a \geq mNE_{b'}\ G_a \geq mNE_{b'}\ G_{a'} > \mathcal{M}.$$

Since $\mathcal{M}$ is arbitrary, this implies that

$$\lim\ (a\to\infty)\ mNG_a$$

exists, and

$$mN = \lim\ (a\to\infty)\ mNG_a\ .$$

This proves 62.

63. Theorem. If A and B are positive R^N-operators

$$m(A \dotplus B) = mA + mB$$

Proof. Let E_a, F_a be the respective spectral resolutions of A and B. Then

$$G_a = E_a F_a$$

is a resolution of the identity in R^P (by 32).

Now

$$A \dotplus B \supseteq A + B$$

so that

$$(A \dotplus B)\ G_a \supseteq (A + B)\ G_a = AG_a + BG_a.$$

Since the right-hand side is in R^S, it is defined everywhere in H for all a, hence

$$(A \dotplus B)\ G_a = AG_a + BG_a$$

in R^S, and

$$m(A \dotplus B)\ G_a = mAG_a + mBG_a.$$

Letting a tend to ∞ and applying 62 yields

$$m(A \dotplus B) = mA + mB\ .$$

This proves 63.

MEASURE AND R^N-OPERATORS OF ARBITRARY SIGN

64. Definition. If $N \in R^N$ and one of N^+, N^- is summable, then N is integrable and

$$mN = mN^+ - mN^-$$

N is summable if $|mN| < \infty$

65. Theorem. If A and B are positive R^N-operators, one of which is summable, then $(A \overset{\circ}{-} B)$ is integrable and

$$m(A \overset{\circ}{-} B) = mA - mB$$

Proof. Let $N = A \overset{\circ}{-} B$, then by 37 there is a positive R^N-operator P such that

$$A = N^+ \overset{\circ}{+} P \qquad B = N^- \overset{\circ}{+} P \,.$$

By 63,

$$mA = mN^+ + mP$$

$$mB = mN^- + mP.$$

And since N^+, N^-, P are positive

$$mA \geq mP \geq 0 \qquad mB \geq mP \geq 0$$

$$mA \geq mN^+ \geq 0 \qquad mB \geq mN^- \geq 0,$$

so that, since one of A,B is summable, P and one of N^+,N^- is summable.

Thus N is integrable and

$$mN = mN^+ - mN^-$$

$$= mN^+ + mP - mN^- - mP$$

(since mP is finite)

$$= m(N^+ \overset{\circ}{+} P) - m(N^- \overset{\circ}{+} P)$$

(by 63)

$$= mA - mB \ .$$

This proves 65.

ABSOLUTE CONTINUITY

Let now R be an abelian von Neumann algebra.

Let m and n be semi-finite measures (definitions 7 and 8) such that n is absolutely continuous with respect to m (definition 9). The remainder of this opus will be taken up with proving a Radon-Nikodym theorem for this situation. Approximately: there exists an operator N in R^N such that

$$nT = m(N \circ T)$$

for all T in R^N.

66. Definition. For any real number

$$G(a) = \{E : E \in R^P,\ a\, mE \geq nE\}$$

67. Definition. Let $E \in G(a)$. E is a-good if $F \in G(a)$ whenever $0 < F \leq E$ and $F \in R^P$.

68. Lemma. If E is a non-zero $G(a)$-member, then there exists $F \in R^P$ such that

$$0 < F \leq E$$

and F is a-good.

Proof. Let $\mathcal{K}$ be the set of all families $\mathcal{B} \subseteq R^P$ such that

(i) $\mathcal{B}$-members are mutually orthogonal

(ii) if $K \in \mathfrak{B}$, then

$$0 < K \leq E$$

and

$$a\, mK < nK.$$

$\mathfrak{K}$ is partially ordered by set inclusion. Obviously every nest in $\mathfrak{K}$ has an upper bound in $\mathfrak{K}$. Thus $\mathfrak{K}$ contains a maximal element $\underline{\mathfrak{B}} \subseteq R^P$.

Now

$$a\, m \sup \underline{\mathfrak{B}} = a\, m \sum (K \in \mathfrak{B})\, K$$

(since $\underline{\mathfrak{B}}$-members are mutually orthogonal and $\sup \underline{\mathfrak{B}} \in R^P$ by 27)

$$= \sum (K \in \underline{\mathfrak{B}})\, a\, mK$$

$$< \sum (K \in \underline{\mathfrak{B}})\, nK$$

$$= n \sum (K \in \underline{\mathfrak{B}})\, K$$

$$= n \sup \underline{\mathfrak{B}}$$

that is,

$$a\, m \sup \underline{\mathfrak{B}} < n \sup \underline{\mathfrak{B}}$$

Now

$$\sup \underline{\mathfrak{B}} \leq E,$$

since $E \in G(a)$, therefore,

$$\sup \underline{\mathfrak{B}} < E.$$

The non-zero R^P-member

$$F = E - \sup \underline{\mathfrak{B}}$$

has the desired properties. For if $K \epsilon R^P$ and

$$0 < K \leq F \qquad a\, mK < nK,$$

then

$$\underline{\mathfrak{B}} \cup \{K\} \ \epsilon\ \mathcal{K}$$

and

$$\underline{\mathfrak{B}} \subsetneqq \underline{\mathfrak{B}} \cup \{K\},$$

contradicting the maximality of $\underline{\mathfrak{B}}$.

This proves 68.

The set $G(a)$ is non-empty for all real a (it contains at least the zero projection). Thus the following defintion is justified.

69. <u>Definition</u>. $\mathcal{E} \subseteq R^P$ is a maximal family of mutually orthogonal a-good projections.

By 27, $\sup \mathcal{E} \ \epsilon\ R^P$

70. <u>Lemma</u>. $\sup \mathcal{E} \ \epsilon\ G(a)$

<u>Proof</u>. $a\ m \sup \mathcal{E} \ = a\ m \sum (E \epsilon \quad)E$
(since $\mathcal{E}$ – members are mutually ortogonal)

$$= \sum (E \epsilon \mathcal{E})\ amE$$

$$\geq \sum (E \epsilon \mathcal{E})\ nE$$

$$= n \sup \mathcal{E} \ .$$

71. <u>Lemma</u>. $\sup \mathcal{E}$ is a-good.

<u>Proof</u>. Let F be an R^P-member such that

$$0 < F \leq \sup \mathcal{E} .$$

To show $F \in G(a)$,

$$\begin{aligned} amF &= a\, m\, (F \sup \mathcal{E}) \\ &= a\, m\, (F \textstyle\sum (E \in \mathcal{E}) E) \\ &= a\, m \textstyle\sum (E \in \mathcal{E})\, FE \end{aligned}$$

(where FE is a projection, since R is abelian)

$$\begin{aligned} &= \textstyle\sum (E \in \mathcal{E})\, am\, FE \\ &\geq \textstyle\sum (E \in \mathcal{E})\, n\, FE \end{aligned}$$

(since $FE \leq E$ and E is a-good)

$$\begin{aligned} &= n \textstyle\sum (E \in \mathcal{E})\, FE \\ &= n\, (F \sup \mathcal{E}) = nF \end{aligned}$$

Thus $\sup \mathcal{E}$ is a-good.

72. <u>Definition</u>. $E_a = \sup \mathcal{E}$.

Since $G(a)$ is never empty, E_a exists for all real a. By 27, 70 and 71, $E_a \in R^P$, $E_a \in G(a)$, and E_a is a-good.

73. <u>Lemma</u>. If $F \in R^P$ and

$$0 < F \leq I - E_a$$

then $F \notin G(a)$, that is, $amF < nF$.

<u>Proof</u>. Suppose on the contrary that $F \in G(a)$. By 68, there is $F' \in R^P$ such that F' is a-good and

$$0 < F' \leq F.$$

Now

$$0 < F' \leq I-E_a$$

so that F' is orthogonal to every E in $\mathcal{E}$. This means that

$$\mathcal{E} \cup \{F'\}$$

is a family of mutually orthogonal a-good elements which properly contains the maximal family $\mathcal{E}$, a contradiction. Thus $F' \notin G(a)$ and 73 is proven.

74. <u>Summary</u> of the properties of the projection E_a.

For every real number a there exists a projection E_a such that

(i) $E_a \in R^P$

(ii) If $F \in R^P$ and $0 \leq F \leq E_a$,

$$amF \geq nF$$

(iii) If $F \in R^P$ and $0 < F \leq I-E_a$,

$$amF < nF$$

(iv) E_a is the unique such R^P-projection.

<u>Proof</u> of (iv): Let $E \in R^P$ satisfy (ii) and (iii). Since R is abelian,

$$P = E(I-E_a) \qquad Q = E_a(I-E)$$

are projections in R^P. If $E \neq E_a$, at least one of P, Q is non-zero.

If $P \neq 0$, then since $P \leq E$, (ii) implies that

$$amP \geq nP.$$

But since $P \leq I - E_a$, (iii) implies that

$$amP < nP$$

A similar contradiction arises if $Q \neq 0$. This proves (iv).

The next few lemmas will show that

$$\{E_a : -\infty < a < \infty\}$$

is almost a resolution of the identity in R^P.

75. Lemma. E_a is a monotone increasing function of a: if $a \leq b$, then $E_a \leq E_b$.

Proof. Since R is abelian

$$E_a E_b = E_b E_a,$$

so it remains to show

$$E_a E_b = E_a.$$

Now

$$P = E_a - E_a E_b = E_a(I - E_b) \geq 0 .$$

If $P > 0$, then

$$0 < P \leq E_a$$

and

$$0 < P \leq I - E_b,$$

so that by (ii) and (iii) of 74

$$amP \geq nP \qquad bmP < nP$$

or

$$bmP < nP \leq amP,$$

which is a contradiction since $a \leq b$. Thus $P = 0$ and 75 is proven.

76. Lemma.

$$\lim \text{ (strong) } (a \to +\infty)\ E_a = I$$

Proof. Let

$$E = \sup \{E_a : a \text{ real}\}$$

then also by 75,

$$E = \sup \{E_p : p \in \omega\}$$

Again by 75,

$$E_p \uparrow,$$

so that by 22,

$$E = \lim \text{ (strong) } E_p.$$

and similarly

$$E = \lim \text{ (strong) } E_{a(p)}$$

where $\{a(p)\}$ is any real sequence such that $a(p) \uparrow \infty$. Thus

$$E = \lim \text{ (strong) } (a \to \infty)\ E_a$$

The second part of the proof is to show that $E = I$. If $I - E \neq 0$, then, since n is a semi-finite measure, there exists $P \in R^P$ such that

$$0 < P \leq I-E \qquad nP < \infty.$$

Now

$$0 < P \leq I-E \leq I-E_a$$

for all real a. Thus by (iii) of 74

$$a\, mP < nP$$

for all real a. Thus

$$mP = 0 \qquad nP > 0.$$

But n is absolutely continuous with respect to m, so that $mP = 0$ implies $nP = 0$. This means that

$$nP = 0 \qquad nP > 0.$$

Thus the assumption $I-E \neq 0$ has lead to a contradiction. This proves 76.

77. Lemma. E_a is strongly right continuous:

$$\lim\ (\text{strong})\ (b \to a+)\ E_b = E_a$$

Proof. This proof is exactly like that of 76.

Let

$$E = \inf \{E_b : b > a\} \in R^P.$$

Then, since E_b is a monotone function of b,

$$E = \inf \{E_{a(p)} : p \in \omega\}$$

where $\{a(p)\}$ is any real sequence such that $a(p) \downarrow a$. By 22

$$E_{a(p)} \downarrow E$$

for all such sequences, hence

$$E = \lim \text{ (strong) } (b \to a+) \; E_b$$

Obviously $E \geq E_a$.

If

$$P = E - E_a \neq 0,$$

then

$$0 < P \leq I - E_a$$

and

$$0 < P \leq E_b$$

for $b > a$. Hence by 74

$$amP < nP$$

and

$$bmP \geq nP$$

for $b > a$, so that

$$amP \geq nP$$

also. This is a contradiction.

Hence $P = E - E_a = 0$ and 77 is proven.

If $\{E_a : -\infty < a < \infty\}$ is a resolution of the identity then

$$\lim \text{ (strong) } (a \to -\infty) \; E_a = 0.$$

As things now stand, this is <u>not</u> true: let $E \in G(a)$ for $a < 0$. Then

$$amE \geq nE$$

or

$$0 \geq amE \geq nE \geq 0$$

whence

$$amE = nE = 0,$$

or, since $a < 0$,

$$mE = nE = 0.$$

Thus by 77

$$\lim \text{(strong)} (a \to -\infty) E_a = F$$

where F is a projection of m-measure zero.

The troublesome possibility that $F \neq 0$ can be eliminated by <u>defining</u> $E_a = 0$ for $a < 0$. This preserves the right continuity and monotonicity of E_a. So <u>now</u> E_a is a resolution of the identity.

78. <u>Definition</u>. N is the unique positive R^N-operator which has E_a as a spectral resolution.

79. <u>Theorem</u>. If $E \in R^P$ and $mE < \infty$, then

$$nE = \int_0^{+\infty} a \, d(mE\, E_a)$$

<u>Proof</u>. Since mEE_a is a finite monotone function of a,

$$s(Q) = \int_0^Q a \, d(mEE_a)$$

exists and is finite for all integers $Q > 0$. Moreover, since

$$s(Q) \uparrow,$$

$$\int_0^\infty = \int_0^\infty a \; d(mEE_a)$$

exists, and, given $\mathcal{M}$ such that

$$0 < \mathcal{M} < \int_0^\infty ,$$

there exists a Q such that

$$\mathcal{M} < s(Q) \leq \int_0^\infty$$

Partition the interval $[0,Q]$

$$0 = a_o < a_1 < \ldots < a_p = Q.$$

Let

$$E(i) = E_{a_i} - E_{a_{i-1}}$$

and form the corresponding upper and lower sums:

$$U = \sum_{i=1}^{p} a_i \; mEE(i)$$

$$L = \sum_{i=1}^{p} a_{i-1} \; mEE(i)$$

Since

$$s(Q) = \sup \{L\} = \inf \{U\} ,$$

the partition can be made so that

$$\mathcal{M} < L \leq s(Q) \leq U \leq \int_0^\infty$$

Now

$$a_{i-1}\ mEE(i) < nEE(i) \leq a_i\ mEE(i)$$

Thus

$$\mathcal{M} < L = \sum_i a_{i-1}\ mEE(i)$$

$$< \sum_i nEE(i)$$

$$= n\ E(E_Q - E_0)$$

$$= nEE_Q$$

(since $nE_0 = 0$)

$$\leq \sum_i a_i\ mEE(i)$$

$$= U \leq \int_0^\infty$$

That is, given $\mathcal{M} < \int_0^\infty$ there exists Q such that

$$\mathcal{M} < nEE_Q \leq \int_0^\infty$$

Since

$$EE_Q \uparrow E,$$

by 39

$$nEE_Q \uparrow nE,$$

so that

$$nE = \int_0^\infty a\ d(nEE_a)$$

This proves 79.

To get on with the true business, recall that N is the positive R^N-operator with spectral resolution E_a (defintion 78).

Let

$$E(a,b) = E_b - E_a \quad \text{for} \quad a < b.$$

Then

$$N(a,b) = NE(a,b) \in R^S$$

and for any $T \in R^N$

$$T(a,b) = T \circ E(a,b) = TE(a,b) \in R^N$$

by 36.

80. Lemma. If T is a positive R^N-operator, then for $a < b$

$$amT(a,b) \leq mT \circ N(a,b) \leq bmT(a,b).$$

Proof. Let T have the resolution F_c in R^P, then

$$T_c = TF_c$$

is a positive R^S-operator for all c. Hence

$$T_c \circ N(a,b) = T_cN(a,b) = N(a,b)T_c \in R^S.$$

If now $x \in H$ and c is fixed

$$(T_c \circ N(a,b)\ x,x)$$

$$= (T_c^{1/2} T_c^{1/2}\ N(a,b)\ x,x)$$

($T_c^{1/2}$ exists in R^S, since T_c is postive in R^S)

$$= (T_c^{1/2}\ N(a,b)\ x,\ T_c^{1/2}x)$$

$$= (N(a,b)\ T_c^{1/2}x,\ T_c^{1/2}x)$$

Now

$$a(E(a,b)\, T_c^{1/2}x,\ T_c^{1/2}x)$$

$$\leq (N(a,b)\, T_c^{1/2}x,\ T_c^{1/2}x)$$

$$\leq b(E(a,b)\, T_c^{1/2}x,\ T_c^{1/2}x)$$

Thus

$$aT_c(a,b) \leq T_c \circ N(a,b) \leq b\, T_c(a,b)$$

in R^S.

If $a < b \leq 0$, then all the above terms are zero.

If $a \leq 0 < b$, then

$$0 \leq T_c \circ N(a,b) \leq b\, T_c(a,b)$$

in R^S. Hence by 53

$$mT_c \circ N(a,b) \leq mbT_c(a,b) = bmT_c(a,b)$$

If $0 < a \leq b$

$$0 \leq a\, T_c(a,b) \leq T_c \circ N(a,b) \leq b\, T_c(a,b)$$

in R^S and again by 53,

$$maT_c(a,b) \leq mT_c \circ N(a,b) \leq mbT_c(a,b)$$

Hence

$$amT_c(a,b) \leq mT_c \circ N(a,b) \leq bmT_c(a,b)$$

holds for all c and $a < b$. In particular this inequality holds in the limit as $c \to +\infty$. Thus by 62, since $T_c = TF_c$,

$$amT\,(a,b) \leq mT \circ N(a,b) \leq bmT(a,b).$$

This proves 80.

81. <u>Lemma.</u> If T is a positive R^N-operator, then for $a < b$

$$amT(a,b) \leq nT(a,b) \leq bmT(a,b)$$

<u>Proof</u>. Let T_c be as in the preceeding lemma. Fix the number c.

Now there exists a sequence $\{S_p\} \subseteq R^{SP}$ such that

$$S_p \geq 0 \qquad S_p \uparrow T_c(a,b)$$

and by definition

$$mS_p \uparrow mT_c(a,b) \qquad nS_p \uparrow nT_c(a,b)$$

Let

$$S_p = \sum_{i=1}^{q} a_i \; G_i$$

be an arbitrary member of this sequence. Then

$$S_p = S_p \; E(a,b)$$

and

$$\begin{aligned} bmS_p &= b \; m \; E(a,b) \sum_i a_i \; G_i \\ &= b \; m \sum_i a_i \; G_i \; E(a,b) \\ &= b \sum_i a_i \; mG_i \; E(a,b) \\ &\geq \sum_i a_i \; nG_i \; E(a,b) = nS_p \\ &> a \sum_i a_i \; mG_i \; E(a,b) \\ &= a \; m \; S_p \end{aligned}$$

so that

$$amS_p \leq nS_p \leq bmS_p \ .$$

Passing to the limit in p,

$$a\, m\, T_c(a,b) \leq n\, T_c(a,b) \leq b\, mT_c(a,b)$$

for all c and $a < b$. Employing 62 again and letting $c \to +\infty$

$$a\, m\, T(a,b) \leq n\, T(a,b) \leq b\, m\, T(a,b)$$

for $a < b$.

This proves 81.

82. Lemma. If T is a positive R^N operator, then

$$mT \circ N(a,b) = nT(a,b)$$

for $a < b$.

Proof. Let P denote a partition of the interval $[a,b]$

$$P : a = a_o < a_1 < \dots < a_{p-1} < a_p = b$$

For $i \epsilon p$ let

$$E_i = E(a_i,\ a_{i+1}) \qquad N_i = N(a_i,\ a_{i+1}).$$

From 80 and 81

$$a_i\, mTE_i \leq \left\{ \begin{matrix} mT \circ N_i \\ \\ nTE_i \end{matrix} \right\} \leq a_{i+1}\, mTE_i$$

for all $i \epsilon p$.

Examining upper and lower sums,

$$L(P) = \sum_{i\epsilon p} a_i \, mTE_i$$

$$\leq \left\{ \begin{array}{ll} \sum_i m\, T \circ N_i & = m\, T \bullet N(a,b) \\ \sum_i n\, TE_i & = n\, T(a,b) \end{array} \right\}$$

$$\leq \sum_i a_{i+1}\, m\, TE_i = U(P).$$

Now if $m\,T(a,b) < \infty$, then

$$U(P) - L(P) = \sum_i (a_{i+1} - a_i)\, m\, TE_i$$

$$\leq (\max P) \sum_i m\, TE_i$$

$$\left(\text{where} \quad \max P = \max \left\{ a_{i+1} - a_i : i\epsilon p \right\} \right)$$

$$= (\max P)\, m\, T(a,b)$$

Thus

$$(\max P)\, mT(a,b)$$

$$\geq U(P) - L(P)$$

$$\geq |mT \bullet N(a,b) - nT(a,b)|$$

If $mT(a,b) = 0$, the lemma is proven. If $mT(a,b) > 0$, then, given $\epsilon > 0$, there exists a partition P such that

$$\max P < \epsilon / mT(a,b)$$

so that

$$|mT \bullet N(a,b) - nT(a,b)| < \epsilon$$

for arbitrary $\epsilon > 0$. Hence

$$mT \circ N(a,b) = nT(a,b)$$

In the case that $mT(a,b) = +\infty$, the assertion is obvious from 80 and 81.

This proves 82.

83. Theorem. If T is a positive R^N-operator

$$mT \circ N = nT$$

Proof. From 82,

$$mT \circ N(0,a) = nT(0,a)$$

or

$$m(T \circ N)E_a = nT \circ E_a$$

By 62, letting $a \to +\infty$, 83 is proven.

84. Theorem. If $T \in R^N$ is n-integrable, then $T \circ N$ is m- integrable and

$$mT \circ N = nT$$

Proof. If T is n-integrable, then

$$nT = nT^+ - nT^-$$

where at least one of

$$nT^+ = mT^+ \circ N \quad nT^- = mT^- \circ N$$

(by 83) is finite.

Now

$$(T \circ N) = (T^{+} \dot{-} T^{-}) \circ N$$

$$= (T^{+} \circ N) \dot{-} (T^{-} \circ N)$$

where one of $(T^{+} \circ N)$, $(T^{-} \circ N)$ is m-summable. Therefore, by 65, $T \circ N$ is m-integrable and

$$m(T \circ N) = m(T^{+} \circ N) - m(T^{-} \circ N)$$

$$= nT^{+} - nT^{-} = nT$$

This proves 84.

Theorem 84 is the promised Radon-Nikodym theorem.