THE EXACT THEORY OF LINEAR CYCLOTRON INSTABILITIES APPLIED TO HYDROMAGNETIC EMISSIONS IN THE MAGNETOSPHERE

by

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ABSTRACT

The complex dispersion relation which describes transverse plasma waves propagating in a cold gyrotropic ambient plasma parallel to the background magnetic field as they interact with charged particle streams is derived by solving the linearized collisionless Boltzmann equation simultaneously with Maxwell's equations using the Fourier-Laplace transform method. The wave frequency is allowed to be complex with a positive imaginary part corresponding to a growing instability. The real and imaginary parts of the dispersion relation yield two separate equations. Under several assumptions, the equations can be simplified to yield an expression for the imaginary part of the frequency (the growth rate) and an equation relating the real wave frequency and the wave number.

The theory is then applied to the magnetosphere by choosing a dipole model for the earth's magnetic field and a suitable distribution function for the particles. The specific case of waves of the ion resonance mode interacting with mono-energetic, contra-streaming protons is considered in detail, and the results of this calculation are used in explaining hydromagnetic (hm) emissions. In particular, it is suggested that the high frequency cutoff is a result of the pitch angle distribution of the particle stream.

Computer calculations are done in order to display the general results of the theory. Specifically, when low energy protons (10 - 20 kev), trapped on a field line with an L value of 5.6 are considered, it is found that the region of instability occurs near the geomagnetic equator, and that the growth rate is a sharply peaked function of the frequency.

TABLE OF CONTENTS

FIGURES

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CHAPTER I

INTRODUCTION

General Discussion

Since the earth's upper atmosphere contains a significant number of charged particles, a physical study of that region involves the concepts of magnetohydrodynamics and plasma physics. The theory of plasma waves has been used to explain such phenomena as atmospheric whistlers and geomagnetic micropulsations. In such studies, the geomagnetic field is fundamental.

Atmospheric whistlers are electromagnetic waves which occur in the frequency range 300 - 30,000 cps and propagate in the electron resonance mode (fast mode) which has an upper frequency limit at the electron cyclotron frequency - ω_e. They originate in lightning flashes (Helliwell and Morgan, 1959) and bounce between the northern and southern hemispheres along paths which approximately follow the magnetic field lines (Helliwell, 1965).

An analogous left-hand circularly polarized wave exists in the ion resonance mode (slow mode) for frequencies below the ion gyrofrequency ω_i. The dispersion relation for these two types of waves propagating parallel to the background magnetic field $\underline{B}_0$ in a cold, ambient plasma can be written (Aström, 1950)

$$\omega^2 - c^2 k^2 = \frac{\Omega_{pe}^2 \omega}{\omega \mp \omega_e} + \frac{\Omega_{pi}^2 \omega}{\omega \mp \omega_i} \qquad (1\text{-}1)$$

for a plasma with one singly-charged, ionic component; ω is the wave frequency, k is the wave number and c is the speed of light. In this equation, both ω and k are real quantities. Ω_{pe} and Ω_{pi} are the electron and ion plasma frequencies respectively, and are defined by

$$\Omega_{pe}^2 = \frac{4\pi N_p q^2}{m_e}$$

$$\Omega_{pi}^2 = \frac{4\pi N_p q^2}{m_i} \tag{1-2}$$

where N_p is the electron number density of the plasma, q is the charge on an electron or a proton and q is negative or positive respectively, and m_e and m_i are the electron and ion masses respectively. The Gaussian system of units is used throughout the thesis. In equation 1-1, the upper sign is used for the ion resonance mode and the lower sign for the electron resonance mode.

It is possible that waves of the ion resonance mode are directly involved with the production of micropulsations in the pc 1 frequency range 0.2 - 5 cps. Tepley and Wentworth (1962) were the first to present the dynamic spectra (frequency-time plots) of such micropulsations. Those which showed a distinct fine structure consisting of repetitive rising tones which often overlapped were called hydromagnetic emissions, or briefly, hm emissions. They have also presented a theory which accounted for this fine structure (Wentworth and Tepley, 1962).

Jacobs and Watanabe (1965) have described the history of the research done on hm emissions and they emphasize the following points. At hydromagnetic frequencies, waves of the ion resonance mode tend to be guided by the magnetic field to a much greater extent than waves of the electron resonance mode (Jacobs and Watanabe, 1964). The dispersion of 'hm whistlers' or 'micropulsation whistlers' yields a theoretical spectrum which agrees approximately with the observed characteristics of the structured hm emissions. The hm whistler signals differ from those of atmospheric whistlers in that the signal intensity does not constantly decrease after the first bounce but often grows before decaying (Tepley and Wentworth, 1964).

The idea developed in this thesis is that the waves gain energy through a cyclotron instability process involving low energy protons which are trapped in the magnetosphere. The process is exactly analogous to the instability found by Bell and Buneman (1964) for electrons interacting with waves of the whistler mode. It is not a single particle effect (cyclotron radiation) but a plasma instability involving the transfer of some of the transverse kinetic energy of the particles to electromagnetic energy in the wave (Brice, 1964; Neufeld and Wright, 1965a).

In order to have an instability at all, an initial wave disturbance must exist so that the wave-particle interaction can take place. The actual source of this initial, small 'seed' wave is not known at present. The problem has been discussed by Jacobs and Watanabe (1965) and Obayashi (1965). In the present discussion, the existence of perturbing hm whistler waves is assumed.

Cyclotron Resonance

It is assumed that the streaming particles have an initial transverse component of velocity. In order to determine whether the wave grows or is damped, the velocity distribution function for the particles must be specified. It has often been noted that growing instabilities require an anisotropic distribution (Stix, 1962; Montgomery and Tidman, 1964; Cornwall, 1965).

It can be seen intuitively that a 'resonance' might occur if a particle is gyrating with the same sense of rotation as the wave's polarization, and if the particle sees a wave frequency equal to its own cyclotron frequency. In a laboratory reference frame, the resonant frequency is different from the cyclotron frequency because of the Doppler shift arising from the particle's longitudinal velocity u.

For the case of protons and a left-hand polarized wave, the resonance conditions mentioned above are satisfied with a positive real frequency ω given by

$$\omega - ku = \omega_i \tag{1-3}$$

In the magnetosphere, $\omega - \omega_i < 0$ (Booker, 1962). It can then be seen that the product ku must be negative, viz., the wave and particles must travel in opposite directions. However, it must be noted that protons can interact with waves of the whistler mode because of the anomalous Doppler effect (Brice, 1964). When a particle travels faster than the wave and in the same direction, it sees a reversal of the wave's polarization. Jacobs and Watanabe (1965) have discussed the different possibilities leading to cyclotron instabilities.

Thesis Outline

In Chapter II, a general linear analysis of the problem is carried out starting from Maxwell's equations and the collisionless Boltzmann equation. The Fourier-Laplace transform method is used, the general procedure being similar to that outlined by Stix (1962) for longitudinal plasma oscillations. This method was suggested by Watanabe (1965a), and the results of these calculations agree with those of Cornwall (1965).

Chapter III involves the application of the general results to the magnetosphere. The proton streams are assumed to be monoenergetic. The pitch angle distribution function is chosen to satisfy a differential equation which is valid for particles trapped in a strong, steady, magnetic field in a tube of flux which has a small normal cross-section (Watanabe, 1964).

In Chapter IV, the results of numerical calculations made on a computer are presented. In order to carry out the calculations, several assumptions are made: the earth's field is assumed to be a centered dipole field having a value of 0.3 Gauss on the earth's surface at the geomagnetic equator; the Smith model (Smith, 1961) of the equatorial electron density, which is valid only for distances up to four earth radii from the earth's surface, is assumed to hold in all regions of the magnetosphere. These two assumptions limit the exactness of the results. A discussion of the variation of electron density along field lines has been given by Carpenter and Smith (1964). Watanabe (1965c) has indicated how information about the distribution of electrons at altitudes greater than about four earth radii may be obtained.

The final chapter summarizes several relevant papers which deal with hm emissions and cyclotron instabilities in the magnetosphere and discusses the limitations of the thesis.

CHAPTER II

MATHEMATICAL ANALYSIS

To describe the interaction between the waves and the particle stream, one must determine the evolution in time of the particle distribution function. Knowing the initial conditions, the electromagnetic fields in the plasma can then be determined. It is assumed that the distribution function $f(\underline{v},\underline{r},t)$ satisfies the collisionless Boltzmann equation, and if the plasma is cold, then the fields produced by the particle density fluctuations (due to thermal motions) are negligible compared to the fields of the wave and the equation can be written

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{r}} + \frac{q}{m}\left[\underline{E} + \frac{\underline{v}}{c}\times(\underline{B} + \underline{B}_0)\right]\cdot\frac{\partial f}{\partial \underline{v}} = 0 \tag{2-1}$$

where t and $\underline{r}$ are the time and space coordinates respectively, $\underline{v}$ is the particle velocity, m is the particle mass and $\underline{E}$ and $\underline{B}$ are the electric field strength and the magnetic flux density, respectively, of the wave. $\partial/\partial\underline{r}$ represents the spatial gradient and $\partial/\partial\underline{v}$ the gradient in velocity space. The Maxwell equations used are

$$\text{curl } \underline{B} = \frac{4\pi}{c}\underline{j} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t} \tag{2-2}$$

$$\text{curl } \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \tag{2-3}$$

where

$$\underline{j} = \sum_{\text{comp}} q \int d\underline{v}\ \underline{v}f \tag{2-4}$$

defines the current density. The summation is taken over all the components of the plasma.

Equation 2-1 is expanded by assuming that the fields $\underline{B}$ and $\underline{E}$ are first order quantities ($\underline{B}_0$ is zeroth order) and that f can be written

$$f(\underline{v},\underline{r},t) = f_0(\underline{v}) + f_1(\underline{v},\underline{r},t) \tag{2-5}$$

where f_1 is a first-order perturbation on f_0. The background field $\underline{B}_0$ is taken to be in the positive z direction. Only transverse waves are considered and the spatially varying quantities are assumed to depend only on the coordinate z, and not on x and y. Neglecting terms of second order in equation 2-1, the zeroth and first-order equations are found to be

$$\frac{\partial f_0}{\partial t} + \underline{v}\cdot\frac{\partial f_0}{\partial \underline{r}} + \frac{q}{mc}(\underline{v}\times\underline{B}_0)\cdot\frac{\partial f_0}{\partial \underline{v}} = 0 \tag{2-6}$$

and

$$\frac{\partial f_1}{\partial t} + \underline{v}\cdot\frac{\partial f_1}{\partial \underline{r}} + \frac{q}{mc}(\underline{v}\times\underline{B}_0)\cdot\frac{\partial f_1}{\partial \underline{v}} + \frac{q}{m}\left[\underline{E} + \frac{1}{c}(\underline{v}\times\underline{B})\right]\cdot\frac{\partial f_0}{\partial \underline{v}} = 0 \tag{2-7}$$

Introducing the cylindrical coordinates (u,w,ϕ) in velocity space, the last term in equation 2-6 can be written

$$\frac{q}{mc}(\underline{v}\times\underline{B}_0)\cdot\frac{\partial f_0}{\partial \underline{v}} = -\omega_c\frac{\partial f_0}{\partial \phi} \tag{2-8}$$

where

$$\omega_c = \frac{qB_0}{mc} \tag{2-9}$$

is the particle gyrofrequency which can be positive or negative. It is assumed that

$$\frac{\partial f_0}{\partial t} = \frac{\partial f_0}{\partial \underline{r}} = \frac{\partial f_0}{\partial \phi} = 0 \tag{2-10}$$

so that equation 2-6 is satisfied.

Using a Fourier transform in space and a Laplace transform in time and using two component equations obtained from equations 2-2 and 2-3, the transformed equation 2-7 can be solved for $f_1(\underline{v},k,\omega)$ (Appendix I). In this way, equation 2-7 is handled as an initial value problem, where the particle distribution function at time t = 0 must be specified. This method was first used by Landau (1946) in discussing the longitudinal vibrations of an electronic plasma. If the initial distribution can be written

$$f_1(\underline{v},k,0) = \sum_{m=-\infty}^{+\infty} f_1^{(m)} e^{im\phi} \tag{2-11}$$

then $f_1(\underline{v},\underline{k},\omega)$ is given by

$$\begin{aligned} f_1(\underline{v},k,\omega) = {} & \frac{e^{i\phi}}{\omega - ku + \omega_c}\left\{\frac{q}{2mc}\left[\left(\frac{\omega}{k} - u\right)\frac{\partial f_0}{\partial w} + w\frac{\partial f_0}{\partial u}\right]\left[B_x(\omega,k) - iB_y(\omega,k)\right]\right. \\ & \left. - \frac{i}{2}\frac{q}{mck}\frac{\partial f_0}{\partial w}\left[B_x(0,k) - iB_y(0,k)\right]\right\} \\ & - \frac{e^{-i\phi}}{\omega - ku - \omega_c}\left\{\frac{q}{2mc}\left[\left(\frac{\omega}{k} - u\right)\frac{\partial f_0}{\partial w} + w\frac{\partial f_0}{\partial u}\right]\left[B_x(\omega,k) + iB_y(\omega,k)\right]\right. \\ & \left. - \frac{i}{2}\frac{q}{mck}\frac{\partial f_0}{\partial w}\left[B_x(0,k) + iB_y(0,k)\right]\right\} \\ & + \sum_{m=-\infty}^{+\infty}\frac{i f_1^{(m)}}{\omega - ku + m\omega_c} e^{im\phi} \end{aligned} \tag{2-12}$$

Using equation 2-4, simple algebra gives

$$j_x \pm i j_y = \sum_{comp} q\int d\underline{v}\, e^{\pm i\phi} w f = \sum_{comp} q \int d\underline{v}\, e^{\pm i\phi} w f_1 \tag{2-13}$$

since $f_0(\underline{v})$ is constant with respect to ϕ. Using equation 2-12, and the other two component Maxwell equations after transformation (Appendix I), it can be shown (Appendix II) that

$$B_x(\omega,k) \pm iB_y(\omega,k) = \frac{iA\left[B_x(0,k) \pm iB_y(0,k)\right] \mp ck\left[E_x(0,k) \pm iE_y(0,k)\right] \pm iD}{\omega^2 - c^2k^2 + \sum_{comp}\frac{4\pi q^2}{m}\int d\underline{v}\,\frac{\frac{1}{2}(\omega - ku)w\frac{\partial f_0}{\partial w} + \frac{1}{2}kw^2\frac{\partial f_0}{\partial u}}{\omega - ku \mp \omega_c}} \tag{2-14}$$

where

$$A = \omega + \sum_{comp} \frac{4\pi q^2}{m} \int d\underline{v} \, \frac{\frac{1}{2} w \frac{\partial f_o}{\partial w}}{\omega - ku}$$

$$D = \sum_{comp} 4\pi q c k \int d\underline{v} \, \frac{w f_1^{(\mp 1)}}{\omega - ku \mp \omega_c} \qquad (2\text{-}15)$$

The upper and lower signs correspond to left and right polarized waves respectively (Appendix IV).

In principle, $\underline{B}(t,z)$ can now be found by applying the inverse transformations to equation 2-14. This means that the response of the plasma system to an initial perturbation of particle distributions by a particle beam can be found. It is this result that justifies the use of the transform method, but in order to make the problem feasible mathematically, it is not solved in general. In the Laplace transformation, the parameter is allowed to be complex, with the restriction that its imaginary part be positive. Later, ω is identified as the wave frequency. The inverse transformation must be carried out along a path which lies in the upper half ω-plane above the singularities of $\underline{B}(\omega,k)$. But physically, negative imaginary parts for ω should be allowed. The procedure followed in overcoming this difficulty involves the analytic continuation of a singular integral and has been discussed by Stix (1962). Using the Cauchy Principal Value ($\mathcal{P}$), it is found (Appendix V) that the dispersion relation is given by

$$\omega^2 - c^2 k^2 + 4\pi\omega \sum_{comp} \frac{q}{m} \int d\underline{w} \left\{ \mathcal{P} \int du \frac{w G(f_o)}{\omega - ku \mp \omega_c} - \frac{i}{|k|} \left[w G(f_o) \right]_{u=V} \right\} = 0 \qquad (2\text{-}16)$$

where

$$G(f_o) = \frac{1}{2}\left[\left(1 - \frac{ku}{\omega}\right) \frac{\partial f_o}{\partial w} + \frac{kw}{\omega} \frac{\partial f_o}{\partial u} \right]$$

and

$$V = \frac{\omega \mp \omega_c}{k}$$

Equation 2-16 is valid for both positive and negative imaginary parts of ω. The presence of the singularity in the inverse transformation results in the last term in equation 2-16 being evaluated under the condition

$$\omega - ku \mp \omega_c = 0 \tag{2-17}$$

This is how the cyclotron resonance condition enters the problem mathematically.

Equation 2-16 can be simplified by specifying the cold, background part of f_0 as f_B by writing

$$f_0 = f_B + f_s \tag{2-18}$$

with

$$f_B = \frac{N_p}{2\pi} \frac{\delta(w)}{w} \delta(u) \tag{2-19}$$

where N_p is the number density of the background plasma and δ represents the Dirac delta function. The distribution function f_B is normalized to N_p and f_s represents the streaming particle distribution function. The $\partial f_B/\partial u$ term in the principal value integral vanishes under the integral over $\underline{w}$. The $\partial f_B/\partial w$ term can be simplified using integration by parts so that

$$\int d\underline{w}\; P\int_{-\infty}^{+\infty} du \frac{wG(f_B)}{\omega - ku \mp \omega_c} = -\frac{1}{\omega}\int d\underline{w}\; P\int_{-\infty}^{+\infty} du \frac{(\omega - ku) f_B}{\omega - ku \mp \omega_c} = -\frac{N_p}{\omega \mp \omega_c} \tag{2-20}$$

and since $\left[G(f_B)\right]_{u=V} = 0$, equation 2-16 becomes

$$\omega^2 - c^2k^2 - \sum_{comp} \frac{\Omega_p^2 \omega}{\omega \mp \omega_c} + 4\pi\omega \sum_{comp} \frac{q^2}{m} \int d\underline{w}\; P\int du \frac{wG(f_s)}{\omega - ku \mp \omega_c}$$

$$- i\frac{4\pi^2\omega}{|k|} \sum_{comp} \frac{q^2}{m} \int d\underline{w} \left[wG(f_s)\right]_{u=V} = 0 \tag{2-21}$$

where $\Omega_p^2 = 4\pi N_p q^2/m_e$ is the electron plasma frequency of the background plasma and is taken as the total plasma frequency since N_S is assumed to be much smaller than N_P. If ω is written

$$\omega = \omega_R + i\omega_I \tag{2-22}$$

it is also assumed that

$$|\omega_I| \ll \omega_R \tag{2-23}$$

This condition means that the instability grows or decays very little during a time interval corresponding to the period of the wave. Assuming that N_S and ω_I are first order quantities compared to N_P and ω_R, equation 2-21 can be simplified (Appendix V) and setting the real and imaginary components separately equal to zero gives

$$\omega_R^2 - c^2 k^2 - \sum_{comp} \frac{\Omega_p^2 \omega_R}{\omega_R \mp \omega_c} = 0 \tag{2-24}$$

and

$$\omega_I = \frac{\frac{4\pi^2 \omega_R}{|k|} \sum_{comp} \frac{q^2}{m} \int d\underline{w} \left[wG(f_s) \right]_{u=V_R}}{2\omega_R + \sum_{comp} \frac{\pm \Omega_p^2 \omega_c}{(\omega_R \mp \omega_c)^2}} \tag{2-25}$$

where

$$V_R = \frac{\omega_R \mp \omega_c}{k}$$

Equation 2-24 is the real dispersion equation which relates ω_R and k, and equation 2-25 is the expression for the growth rate of the instability.

CHAPTER III

APPLICATION OF THE GENERAL RESULTS TO THE MAGNETOSPHERE

Discussion

The results of the previous chapter have been derived for the case of plane waves infinite in extent propagating parallel to a homogeneous background magnetic field which extends over all space. In applying these results to the magnetosphere, it is assumed that the region of interaction is small enough that the geomagnetic field can be considered homogeneous there, but large enough that the hm waves are well approximated by plane waves. This problem has been mentioned by Hruska (1966).

In order to calculate ω_I using equation 2-25, an explicit expression for f_s must be determined in a meaningful way. Although much has been learned experimentally about particles contained in the van Allen belts, almost nothing is known about the distribution of low energy protons at higher altitudes. Davis and Williamson (1962) have reported data obtained from the satellite 'Explorer 12' and Cornwall (1965) suggested these protons might be important in cyclotron emissions as well as constituting a ring current. Most of the results concerned protons in the energy range 50 kev - 5 mev. Hoffman and Bracken (1965) have given a more complete report of the same data. Some of these results will be quoted later.

Two distribution functions are now considered. The shifted, anisotropic Maxwellian distribution is used as an example since it has been used several times before (Sudan, 1963; Guthart, 1964; Hultqvist, 1965; Hruska, 1966). The second distribution chosen is discussed in detail below.

The Shifted, Anisotropic, Maxwellian Distribution

This type of distribution represents a particle stream whose spread of random thermal velocities perpendicular to the background field is different than the spread parallel to it, and there is an organized, uniform velocity parallel to the field. In this case, the distribution function is written

$$f_s = N_s \frac{\beta_\parallel}{\pi^{1/2}} \frac{\beta_\perp^2}{\pi} e^{-\beta_\parallel^2 (u-u_0)^2} e^{-\beta_\perp^2 w^2} \tag{3-1}$$

where

$$\beta_\parallel^2 = \frac{m}{2K_B T_\parallel} \qquad \beta_\perp^2 = \frac{m}{2K_B T_\perp} \tag{3-2}$$

f_s has been normalized to N_s . In this case,

$$G(f_s) = -\left\{\left[1 - \frac{ku}{\omega_R}\right]\beta_\perp^2 w + \frac{kw}{\omega_R}\left[\beta_\parallel^2 (u-u_0)\right]\right\} f_s \tag{3-3}$$

Setting $u = V_R$, and integrating over w and ϕ,

$$\int_0^{2\pi}\int_0^\infty w\,dw\,d\phi \left[wG(f_s)\right]_{u=V_R} = -\frac{\pi A}{\beta_\perp^4} e^{-\beta_\parallel^2 (V_R-u_0)^2}\left[\beta_\perp^2\left(1-\frac{kV_R}{\omega_R}\right) + \frac{k}{\omega_R}\beta_\parallel^2 (V_R-u_0)\right] \tag{3-4}$$

and so equation 2-25 gives

$$\omega_I = \frac{\dfrac{4\pi^2\omega_R}{|k|}\displaystyle\sum_{comp}\frac{q^2}{m}\frac{\pi A}{\beta_\perp^4} e^{-\beta_\parallel^2 (V_R-u_0)^2}\left[\beta_\perp^2\left(1-\frac{kV_R}{\omega_R}\right) + \frac{k}{\omega_R}\beta_\parallel^2 (V_R-u_0)\right]}{2\omega_R + \displaystyle\sum_{comp}\frac{\pm\Omega_p^2\,\omega_c}{(\omega_R \mp \omega_c)^2}} \tag{3-5}$$

and

$$A = N_s \frac{\beta_\parallel \beta_\perp^2}{\pi^{1/2}\pi}$$

If only one type of particle is streaming, then the summation can be removed and the condition for positive ω_I is

$$\frac{T_\perp}{T_\parallel} > \frac{\mp\omega_c}{\omega_R \mp \omega_c - ku_o} \tag{3-6}$$

This result is known (Stix, 1962).

A Monoenergetic Pitch Angle Distribution

Using a theoretical approach, Watanabe (1964) has obtained a differential equation which governs the distribution function of particles trapped in a 'strong' magnetic field, viz., one for which the scale of spatial variations is much larger than the gyroradius of the particle. If the field is steady in time,

$$\frac{\partial f}{\partial t} + v\cos\psi\frac{\partial f}{\partial l} + \frac{1}{2}v\sin\psi\,\frac{1}{B}\frac{dB}{dl}\frac{\partial f}{\partial \psi} = 0 \tag{3-7}$$

where ψ is the local pitch angle of a particle, l is distance measured along a field line, and B is the local magnetic field strength. This equation is valid only in the one-dimensional case, when the particles are confined to a tube of flux for which the linear dimensions of any normal cross-section are much smaller than the scale length of the trapping region. A particular solution is given by

$$f = \mathcal{C}\left(\frac{\sin^2\psi}{B}\right)^\alpha \tag{3-8}$$

where α is an arbitrary constant and $\mathcal{C}$ is constant with respect to l, ψ, and t and contains the normalization factor. In a 'strong' field, with no perturbing wave, $\sin^2\psi/B$ is a constant of the motion since it is proportional to the magnetic moment of a particle, the first adiabatic invariant (Chandrasekhar, 1960; Alfven and Falthammar, 1963). The result that f_s depends on the adiabatic invariant is to be expected (Cornwall, 1965). The discussion in the remainder of the thesis concerns only mono-energetic protons. The assumption that the particles are monoenergetic is not too restrictive and helps to simplify the mathematics. Monoenergetic electrons have been considered previously (Wentworth and Tepley, 1962). In the numerical calculations which are done later, the particle energy is varied as a parameter. The distribution function is written

$$f_s = A\,\delta(v - v_0)\,\frac{\sin^{\gamma}\psi}{B^{\gamma/2}} \tag{3-9}$$

where γ is the 'pitch angle distribution parameter'. It is assumed that the number density of the streaming particles is known at some point in the magnetosphere, that is for some value of the main field B^*, for instance at the equator. The constant 'A' is determined by normalizing f_s to N^* at this point, and it is necessary that $\gamma > -2$ so that the integral does not diverge. In this case,

$$A = \frac{N_s^* B^{*\frac{\gamma}{2}}}{2\pi\sqrt{\pi}\, V_0^2} \cdot \frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{p+1}{2}\right)} \tag{3-10}$$

where $p = \gamma + 1$ and Γ represents the gamma function. The integral over the pitch angle ψ, is taken from 0 to π because the particles are supposed to stream in the positive and the negative z directions, although for a given wave at any point, only one-half the particles can participate in the cyclotron interaction.

Using equations 2-25 and 3-9, ω_I is calculated (Appendix VI) and is found to be non-zero only when $|V_R| < V_0$. In this case,

$$\omega_I = \frac{\dfrac{4\pi^2\omega_R}{|k|}\displaystyle\sum_{comp}\frac{q^2}{m}\frac{\pi A}{B^{\gamma/2}}\,\frac{(V_0^2 - V_R^2)^{\gamma/2}}{V_0^{\gamma-1}}\left[\gamma\left(\pm\frac{\omega_c}{\omega_R} - 1\right) - 2\right]}{2\omega_R + \displaystyle\sum_{comp}\frac{\pm\,\Omega_p^2\,\omega_c}{(\omega_R \mp \omega_c)^2}} \tag{3-15}$$

The following important qualitative results can be obtained from this expression.

1. The growth rate is directly proportional to the density of streaming particles since $A \sim N_S$.

2. The only factor which can be negative is

$$\gamma\left(\pm \frac{\omega_c}{\omega_R} - 1\right) - 2$$

   a) The '-2' term represents a constant damping factor which originates in the expression $-2\int_0^\infty dw\, w f_S$.

   b) The quantity $\pm\, \omega_c/\omega_R - 1$ is positive for both the electron-whistler interaction and the proton-hm whistler interaction. Therefore, for wave growth in either case, γ must be at least positive. In fact, γ must satisfy

$$\gamma > \frac{2}{\pm\frac{\omega_c}{\omega_R} - 1} \qquad (3\text{-}16)$$

   If ω_c is taken as the equatorial value, then this condition allows wave growth at any point on that field line provided $v_o^2 > V_R^2$.

   c) Suppose a particle stream trapped in the magnetosphere can be described by a specific value of γ . It follows that there is an upper frequency limit for waves that will be amplified. This maximum frequency ω_U is given by

$$\omega_U = \frac{\pm\omega_c}{1 + \frac{2}{\gamma}} \qquad (3\text{-}17)$$

   Any waves with frequencies higher than this will be damped, and using the value of ω_c at the equatorial plane will indicate approximately the maximum frequency of any amplified waves. If the mechanism for wave amplification suggested here is correct, then the existence of a maximum frequency gives a method of determining γ, provided the guiding magnetic field line can be determined. This problem is discussed qualitatively later.

3. At a given point in space and for a given particle energy, as waves of lower frequencies are considered, $V_R^2 \rightarrow v_o^2$, and the waves are not amplified. This fact may be used to explain the observed minimum frequency of hm emissions.
4. For a given particle energy and wave frequency, V_R can vary only if ω_c varies. V_R^2 approaches v_o^2 as ω_c gets larger and this occurs as the region under consideration moves down the field line away from the equatorial region. Past a certain point, V_R^2 will always be greater than v_o^2 and instability can no longer take place so intuitively it seems that the unstable region tends to be situated near the equatorial plane. The numerical calculations show that this interpretation is valid.

The existence of maximum and minimum frequencies as discussed above can be thought of as roughly defining a band width for the emissions

The suggestion that the instability tends to occur in the equatorial region is due to Watanabe (1965b) and has been mentioned by Jacobs and Watanabe (1965). The same idea has been put forth by Tepley and Wentworth (1964) for different reasons. They suggest that streaming protons in the magnetosphere can sometimes be superluminous with respect to hm waves and that since the particles move faster near the equatorial plane, it is in these regions that the proton cyclotron radiation is subject to the anomalous Doppler shift and is likely to be most intense. They suggest that on each pass through the equatorial region, the same process occurs. In order that the wave be reinforced each time, they suggest that the bounce periods of the wave packet and particles be approximately equal so that the particles pass through the wave packet at the equator each time. It is not obvious that by the time the particles return to the equatorial plane, they will still be in phase with the wave which they initially generated. Tepley and Wentworth also had to assume that the particle stream was coherent to begin with in order to obtain a significant amount of radiation in the first emission. Obayashi (1965) has discussed this point. Besides this, there is no 'a priori' reason for the two travel time periods to be the same. But the most important fault in the theory is the suggestion that such a superluminous particle stream can interact in a collective manner with the hm waves. Since the particles see anomalously Doppler shifted waves, the sense of the waves' polarization is opposite to that of the gyration of the particles (Brice, 1964) and cyclotron resonance cannot occur. Other different attempts have been made to determine likely regions of wave growth in the magnetosphere and several of these are discussed in Chapter V.

It was suggested above that γ must exceed a minimum value before wave growth can occur. The energy which is gained by the wave in the amplification process comes from particle kinetic energy. It is possible that although some energy is transferred from the particles, it is not enough to balance the constant damping which is present and the wave decays. It is seen then, that the transverse component of velocity of the particles cannot be arbitrarily small (Neufeld and Wright, 1965b); Obayashi, 1965).

Since the pitch angle distribution is given by $\sin^{\gamma}\psi$, when γ is positive, more particles have large pitch angles than small (fig. 1). Increasing γ from zero effectively increases the average transverse kinetic energy of the particles while decreasing the average longitudinal kinetic energy. When the particles lose transverse energy to the wave in a growing instability, there is a general reduction of pitch angles and some particles may be lost because they have pitch angles which are inside the 'loss-cone' (Cornwall, 1965; Brice, 1964).

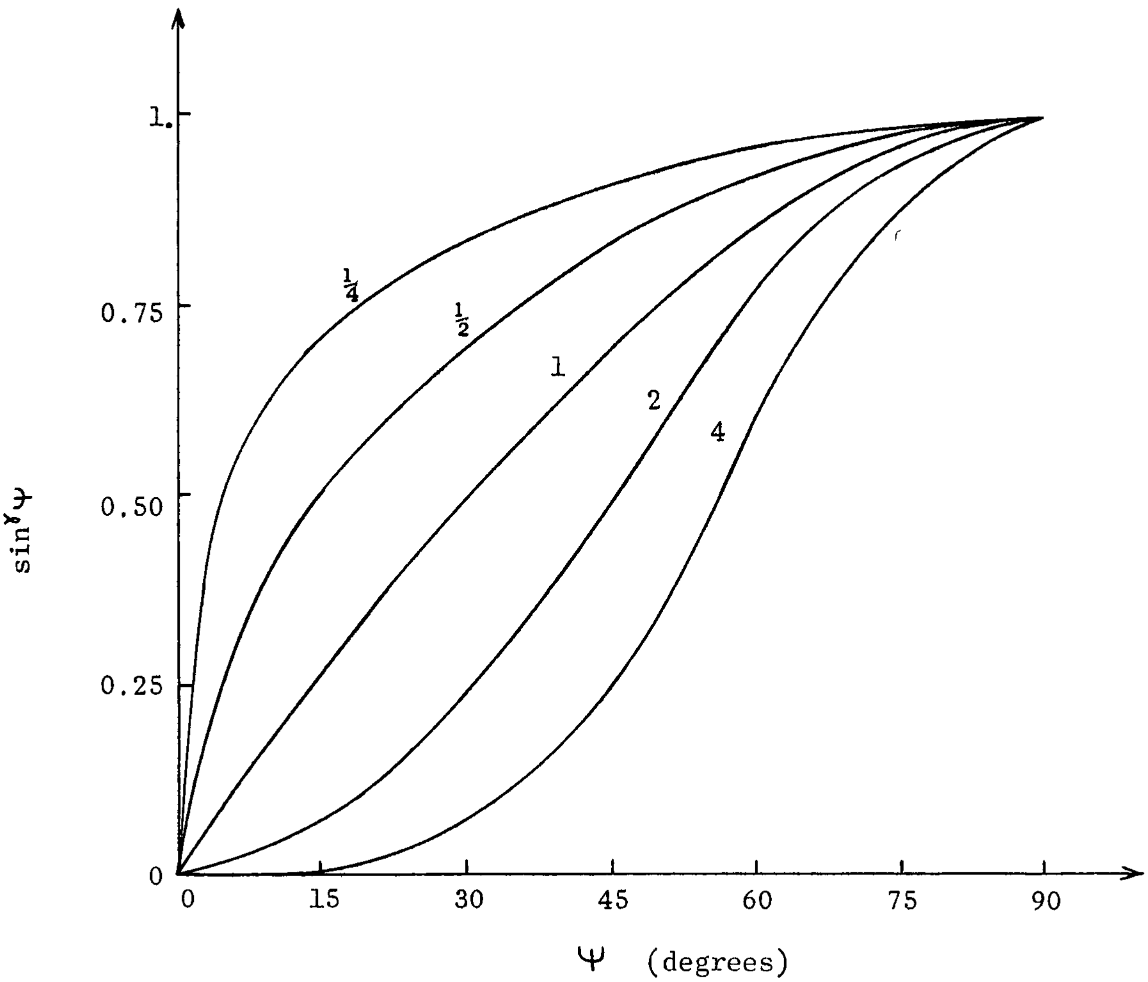

Fig. 1 Family of curves of $\sin^\gamma\psi$. Values of the parameter γ are written beside the corresponding curve in the diagram.

CHAPTER IV

NUMERICAL CALCULATIONS

Normalization of the Equations

It is assumed that only protons are contained in the particle stream. The background plasma contains therefore, more electrons than protons by a small amount in order to preserve overall charge neutrality. Since f_S is non-zero only for protons, the summation over components in the numerator of the expression for the growth rate of the instability is not required.

It is often convenient to write the important equations obtained in a study in normalized form involving dimensionless variables so that the general results can be seen without employing numerical values which are valid for a specific case only.

In equation 3-15, the term $2\omega_R$ in the denominator originates in the displacement current term in Maxwell's equations and at hm frequencies it can be neglected (Jacobs and Watanabe, 1965). Eliminating this term allows the equation to be put into dimensionless form with the help of the following relationships.

$$\omega_I' = \frac{\omega_I}{\omega_i} \qquad K_p = \frac{N_s}{N_p} \qquad \omega_R' = \frac{\omega_R}{\omega_i}$$
$$U_s = \frac{V_0}{V_A} \qquad \omega_{ci} = \omega_i \qquad \omega_{ce} = -\omega_e \qquad M = \frac{m_p}{m_e} \tag{4-1}$$
$$\Omega_{pe}^2 = \frac{c^2}{V_A^2}\omega_{ce}\omega_{ci} \qquad \Omega_{pi}^2 = \frac{c^2}{V_A^2}\omega_{ci}^2$$

where

$$V_A = \frac{B_0}{(4\pi\rho)^{1/2}} \qquad \rho = \sum_{comp} Nm \simeq N_p m_i \tag{4-2}$$

It is then convenient to write

$$U_i = \frac{V_R}{V_A} = \frac{\omega_R/\omega_i \mp 1}{V_A k/\omega_i} = \frac{\omega' \mp 1}{k'} \tag{4-3}$$

by defining

$$k' = \frac{V_A k}{\omega_i} \tag{4-4}$$

If the $2\omega_R$ term in equation 3-15 is neglected, then by dividing both sides of the equation by ω_i , it is found that

$$\frac{\omega_I}{\omega_i} = \omega_I' = \frac{\frac{1}{2}\pi^{1/2} K_p \frac{\Gamma\left(\frac{\gamma+3}{2}\right)}{\Gamma(\gamma/2+1)} \cdot \frac{\omega'}{|k'|} \cdot \frac{(U_s^2 - U_i^2)^{\gamma/2}}{U_s^{\gamma+1}} \left[\gamma\left(\pm\frac{1}{\omega'} - 1\right) - 2\right]}{\frac{\pm 1}{(\omega' \mp 1)^2} + \frac{\mp M^2}{(\omega' \pm M)^2}} \tag{4-5}$$

In a similar manner, the real dispersion relation (equation 2-24) can be put into normal form by neglecting the ω_R^2 term for the same reasons as above, and then it is written

$$k'^2 = \frac{\omega'}{\pm 1 - \omega'} - M \frac{\omega'}{\pm M + \omega'} \tag{4-6}$$

Parameter Values

The first requirement of any theory of hm emissions is that the emitted frequency be in the Pc 1 range from 0.2 to 5 cps. If a proton's velocity and pitch angle are known, then the resonant frequency for that particle can be determined from the resonance condition

$$\omega_R + k|u| = \omega_i \tag{4-7}$$

If the distribution function for the particles is anisotropic such that there are more particles with parallel components of velocity slightly less than $|u|$ than there are particles with components slightly greater, then energy will be transferred to the wave. Transverse Landau damping of the wave occurs if the particle distribution is isotropic (Scarf, 1962; Stix, 1962). If ω_i is specified, and the emission is to be of a certain frequency, then the value for $|u|$ can be calculated from equation 4-7, and a lower bound for the energies of the particles involved can be calculated.

In order to specify ω_i , the earth's main magnetic field is assumed to be a centered dipole field with a value $B_o' = 0 \cdot 3$ on the surface of the earth at the geomagnetic equator. If λ is the geomagnetic latitude and L is the McIlwain coordinate (McIlwain, 1961) in this case applied to a dipole field, then the total field strength at a point with coordinates (L,λ) is given by

$$B_o = \frac{B_o'}{L^3} \frac{(1 + 3\sin^2\lambda)^{1/2}}{\cos^6\lambda} \qquad (4\text{-}8)$$

L represents the distance, measured in units of earth radii, that a given field line in the equatorial plane lies from the centre of the earth. For a dipole field

$$L = \frac{1}{\cos^2\lambda_o} \qquad (4\text{-}9)$$

where λ_o is the geomagnetic latitude at the point where the relevant line of force intersects the earth's surface. It is recognized that a dipole representation of the earth's main field is not perfect because of the compression on the daytime side but it is a good approximation and very easy to describe mathematically.

Under the dipole model, ω_i is inversely proportional to L^3. At very low frequencies, the wave's phase velocity is very nearly the Alfven velocity, V_A , and using equation 4-7, the frequency of emission can be approximated by

$$\omega_R = \frac{\omega_i}{1 + |u|/V_A} \tag{4-10}$$

Using the density model of Smith (1961), the local electron number density is linearly proportional to the gyrofrequency, or

$$N_p \sim B \sim \frac{1}{L^3} \tag{4-11}$$

It is assumed that this model holds not only in the equatorial plane below L = 5, but that it is valid along field lines away from the equatorial plane and at altitudes which correspond to L values greater than about 5 (Brice, 1964; Carpenter and Smith, 1964). Using equation 4-10, it can be seen that

$$\omega_R \sim \frac{1}{L^3(1+|u|\cdot L^{3/2})} \tag{4-12}$$

and it can be seen that for a given emitted frequency, the position in the magnetosphere at which the interaction takes place strongly determines the energy range of the particles involved.

Cornwall (1965) has suggested that since the data from the Explorer 12 satellite, first reported by Davis and Williamson, (1963) and later in more detail by Hoffman and Bracken (1965), indicate a large flux of protons with energies of the order of hundreds of kev at $L \simeq 3.5$, these protons may be very important in emission processes. The energy range is the right order for resonance in the Pc 1 range in a dipole field.

Cornwall (1965) also mentions that 10 - 20 kev protons at $L \simeq 5.6$ have been suggested as the energy source for the emissions. Hoffman and Bracken (1965) have reported the presence of protons in the region of the magnetosphere between these two extremes, with the flux of low energy particles increasing with increasing altitude. If these energies are in the correct range, then resonance could occur on any line of force having an L value between about 3.5 and 5.6. Obayashi (1965) indicates that all hm emissions should occur in the region between L = 4.0 and L = 5.6. However, out of nine examples, Watanabe (1965c) found no L values for the guiding line of force below 4.98 for a dipole field. For the distorted dipole field which he used, this value becomes 4.75.

Taking into account the outline of hm emissions given in Chapter I, the repetition of rising tones separated by a constant time interval is interpreted as an hm wave packet bouncing between ionospheric reflections in the northern and southern hemispheres, being guided by the geomagnetic field lines. It is suggested that the wave is strengthened by the cyclotron interaction with the proton stream each time it traverses the field line. The bounce period of hm waves has been calculated theoretically by Jacobs and Watanabe (1965) as a function of the frequency and the L value, and it involves calculating an integral numerically. Using their table, the bounce periods for a wave with a frequency of 1.3 cps for L = 5.6 and L = 3.5 are found to be approximately 280 sec. and 60 sec. respectively. Tepley and Wentworth (1964) mention that the repetition period of the rising tones in hm emissions can vary from one to five minutes so that these values are not outstanding. This result is physically reasonable since the dispersion relation indicates that the phase velocities of ion resonance mode waves tend toward zero as ω_R approaches ω_i, and so, for a given frequency range, the wave goes slower at higher altitudes since the cyclotron frequency decreases. At the same time, the path which the wave follows is longer at higher altitudes.

Such differences of repetition period between different emissions is noticeable even by making very rough measurements on different dynamic spectra. In the example presented by Cornwall (1965), the period measured over the interval between 7 min. and 14 min. is approximately 84 sec. with a mean deviation of about 10 sec. In the sonogram given by Jacobs and Watanabe (1965), the period measured between $13\frac{1}{2}$ hr. and 14 hr. is never less than 130 sec. The difference between these two spectra is measureable. This type of measurement has been made by Watanabe (1965c).

The frequencies of the emission in the example above in which the bounce period is 130 sec. are low, around 0.3 cps. If this event is to have occurred on the line of force given by L = 3.5, then at the least, the protons would have had to have energies of about 30 mev. At this energy, the protons are relativistic and such particles are not mentioned in Cornwall's presentation (Cornwall, 1965).

Besides the particle energy, the streaming particle density and the pitch angle parameter must be chosen. Hoffman and Bracken (1965) fitted their data to a pitch angle distribution and found that the best fit was made when they considered two ranges of pitch angles separately, 0° to 30° and 30° to 90°. The parameter values which gave a good fit ranged from 1 to 4 over the region L = 2 to L = 7. Each calculation here is done for $\gamma = 2$. The ratio N_s/N_p is taken to be 1 at the equator so that relative sizes can be seen from the results, but the calculated growth rates are too large. Jacobs and Watanabe (1965) assumed the ratio to be 10^{-5}. The correct ratio probably varies from case to case. One limiting factor in the choice of N_s is that the growth rate must be very much smaller than the real frequency.

It was mentioned above that Watanabe (1965c) found no emissions taking place on a line of force with an L value less than 4.98 in nine examples considered. Cornwall (1965) suggests that the emissions would not take place at lower altitudes if the ratio N_s/N_p is too small and this might occur because the background plasma density is large.

All the calculations have been done using an L value of 5.6. For each ω_I, the quantity $v_0^2 - V_R^2$ is calculated and if $v_0^2 \leqslant V_R^2$, ω_I is set equal to zero. The wave frequencies have been varied from $\omega_i(eq)/20$ to $19\,\omega_i(eq)/20$ and at each frequency, the growth rate is calculated for twenty-nine values of the geomagnetic latitude from 0° to 29° in half-degree steps.

The Smith model of electron density (Smith, 1961) can be written

$$N_p = 12,000 \frac{\omega_{ce}}{2\pi} m^{-3} \tag{4-13}$$

This model has been used by Brice (1964) in the form

$$\Omega_{pe} = (2\pi)^{1/2} \cdot 1,000\, \omega_{ce}^{1/2} \tag{4-14}$$

In the calculations, equation 4-13 is written

$$N_p = 2\pi \cdot 5,530\, B_0\, cm^{-3} \tag{4-15}$$

and using this value for the density, the Alfven velocity can be calculated for the magnetosphere. Values corresponding to particle energies in the range approximately 10 - 20 kev can then be assigned to U_s. For L = 5.6, the value of V_A at the equatorial plane is 4.95×10^7 cm-sec^{-1}. A proton energy of 10 kev corresponds to a velocity of 1.38×10^8 cm-sec^{-1} so in this case, $U_s(eq) = 2.79$. Since the velocity increases as the square root of the energy, for 20 kev, $U_s(eq) = \sqrt{2} \cdot (2.79) = 3.95$. In the programme, v_0 is given the following seven values; $2.8V_A(eq)$, $3.0V_A(eq)$, ... , $4.0V_A(eq)$.

It should be noted that equation 4-13 describes N_p empirically and is an average value. Since N_p can vary by as much as a factor of 2 at different times, the calculated results are not exact and differences exist between one specific example of hm emissions and another. The results will serve as an indication of general effects which result from the mechanism which has been considered.

The calculations have been done at the University of British Columbia Computing Centre on an I.B.M. 7040 computer using Fortran IV language.

Results

The largest growth rate at any point was found to occur at the equatorial plane ($\lambda = 0$) at a frequency of 0.65 cps. Figure 2-a indicates how the growth rate varies with frequency and particle energy at the equatorial plane. The lower cutoff is very sharp and the peak itself is narrow. Figures 2-b and 2-c show the same type of plot for $\lambda = 10^\circ$ and $\lambda = 20^\circ$ respectively. It can be seen that as λ increases, the frequency band of amplification moves toward higher frequencies for a given v_o. This is to be expected and results from the requirement that

$$v_o^2 > \frac{(\omega_R - \omega_i)^2}{k^2}$$

which means that as ω_i increases with λ, ω_R must increase in order to restrict the size of V_R^2.

This effect can be seen in another way by observing how ω_I changes with λ for several frequencies. It is found that the growth rate is practically zero for all frequencies less than or equal to

0.2 ω_i (eq). Remembering that the largest growth rate occurs near ω_R = 0.25 ω_i (eq), figure 3-a gives the results for ω_R = 0.3 ω_i (eq). At the higher energies, it is noticeable that the growth rate maximum occurs near $\lambda \simeq 10^\circ$. In figures 3-b and 3-c, this effect is much more noticeable and as the wave frequency is increased still further (figures 3-d and 3-e), the wave is damped slightly near the equator and amplified in the region around 25° geomagnetic latitude. This damping effect occurs near the equator because at that point on the field line, the ratio ω_i/ω_R is smallest and if γ does not happen to be large enough, the expression $\gamma(\omega_i/\omega_R - 1) - 2$ can easily be negative.

For energies in the range 10 - 20 kev, the sharp low frequency cutoff occurs at 0.2 ω_i(eq) and it is important to note that the maximum proton energy determines this cutoff point. The proton energy also is very important in determining the size of the growth rate.

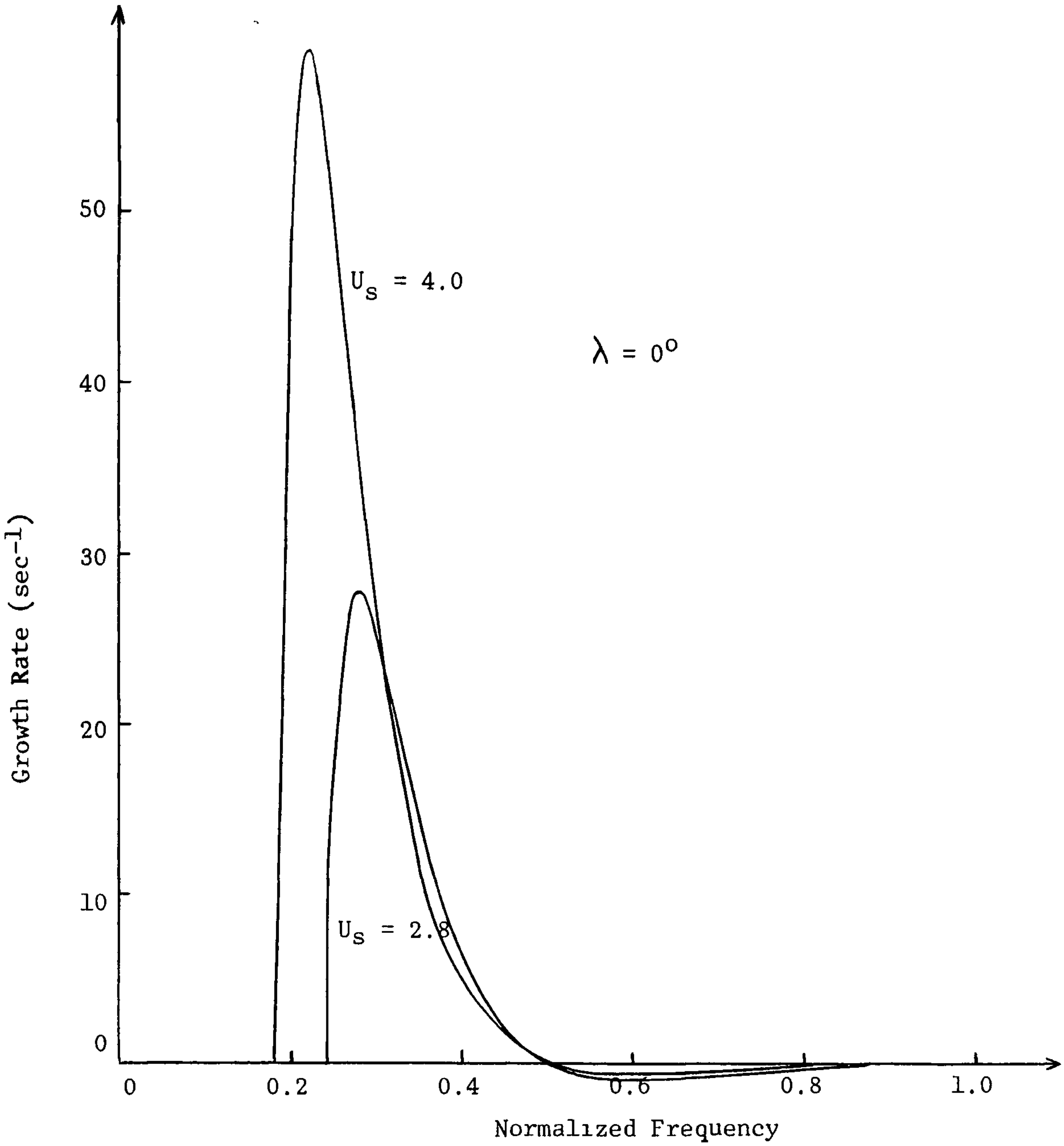

Fig. 2-a Growth rate as a function of frequency at $\lambda = 0^{\circ}$ for two particle energies. The frequency is normalized to the equatorial cyclotron frequency, 16.4 sec^{-1}.

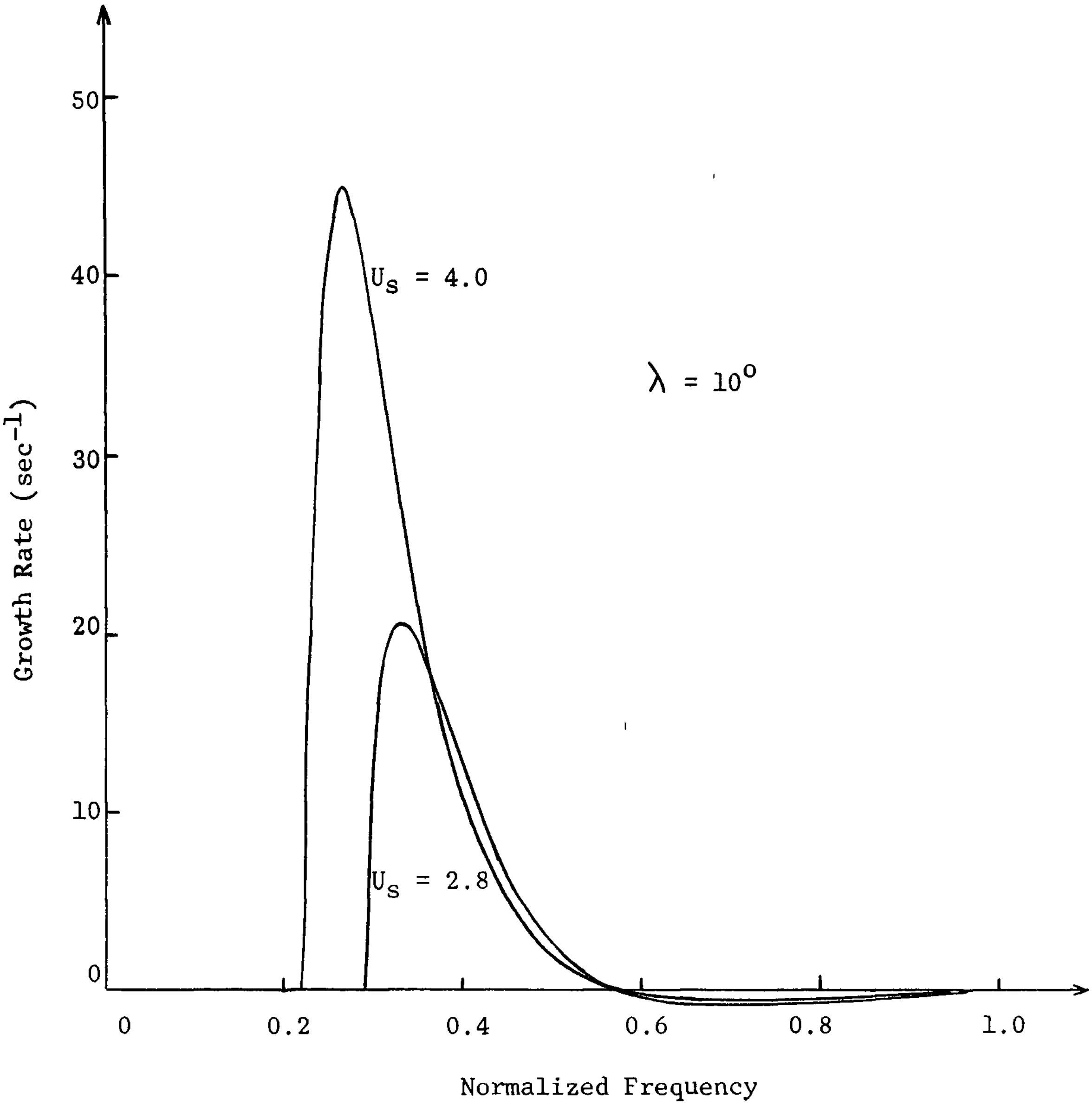

Fig. 2-b Growth rate as a function of frequency at $\lambda = 10^o$ for two particle energies. The frequency is normalized to the equatorial cyclotron frequency, 16.4 sec^{-1}.

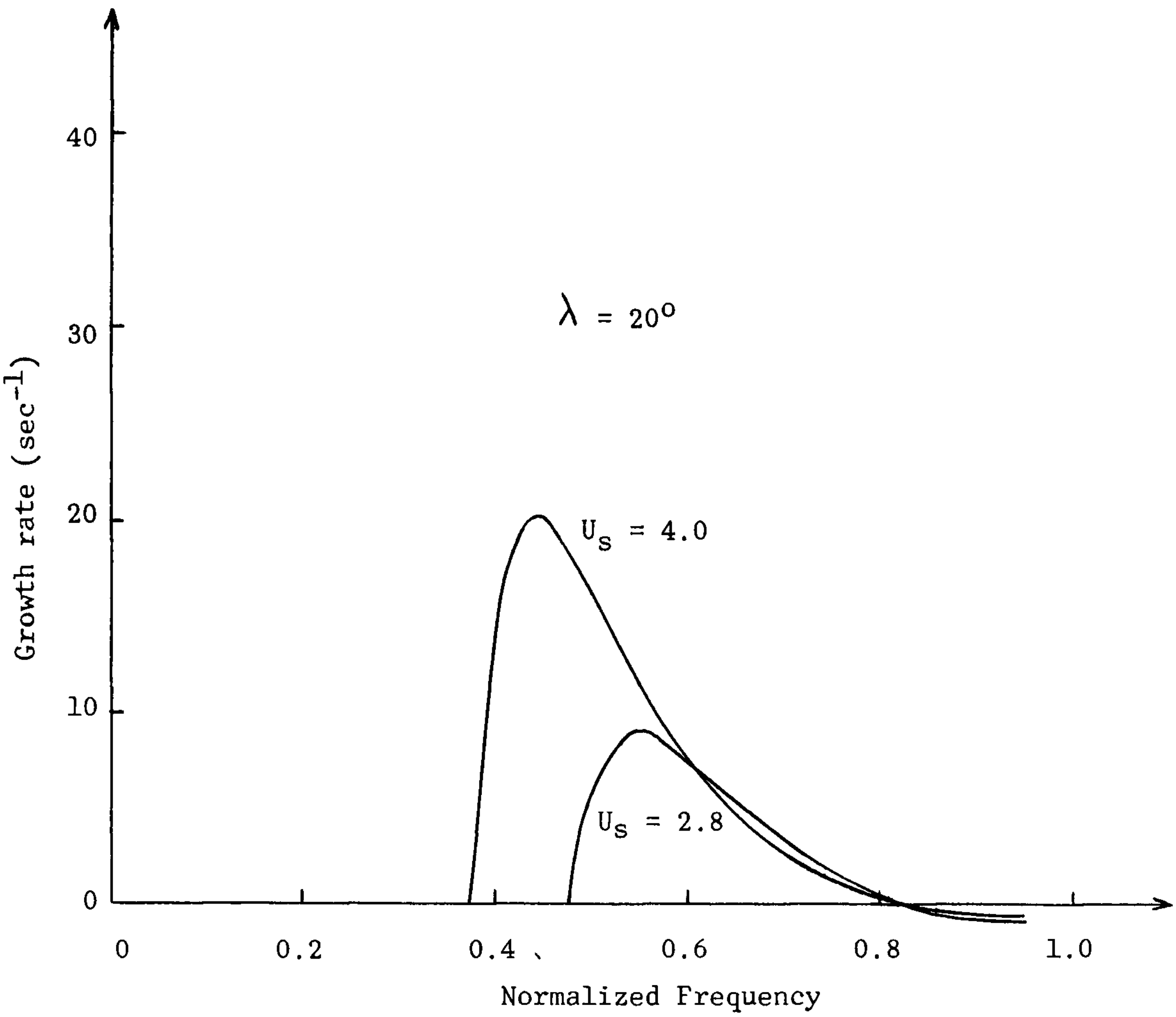

Fig. 2-c Growth rate as a function of frequency for two particle energies for $\lambda = 20^o$. The frequency is normalized to the equatorial cyclotron frequency, 16.4 sec^{-1}.

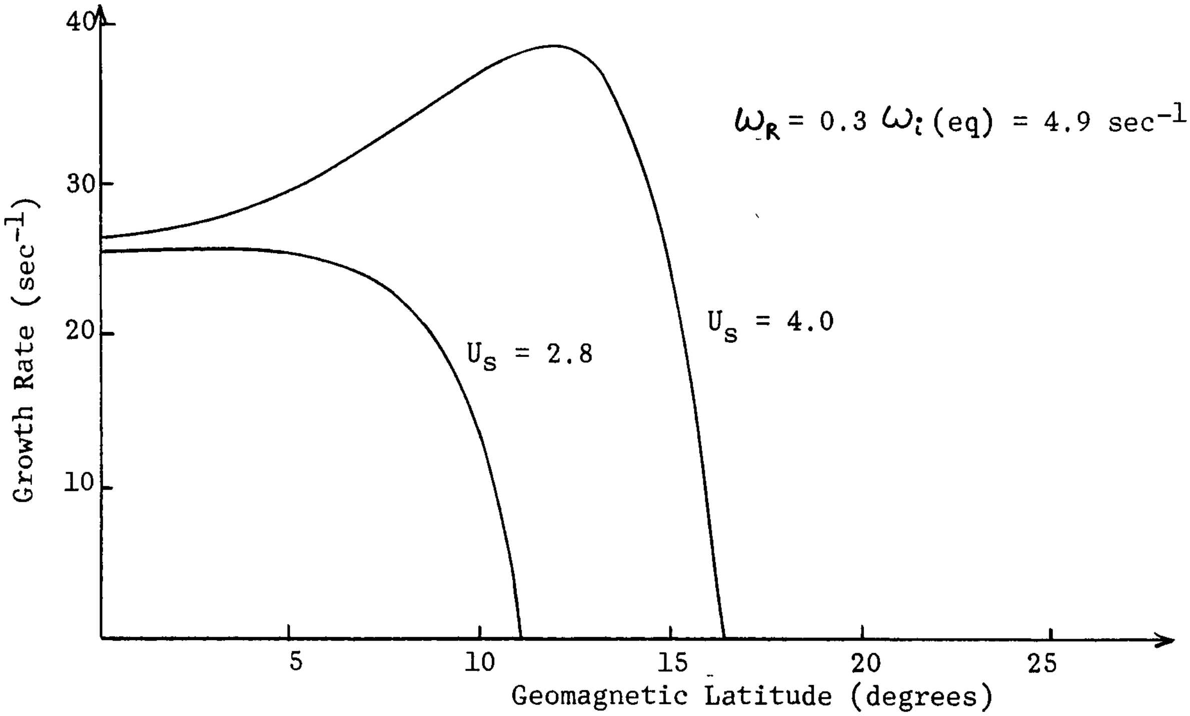

Fig. 3-a Variation of ω_I with λ for $\omega_R = 4.9\ sec^{-1}$.

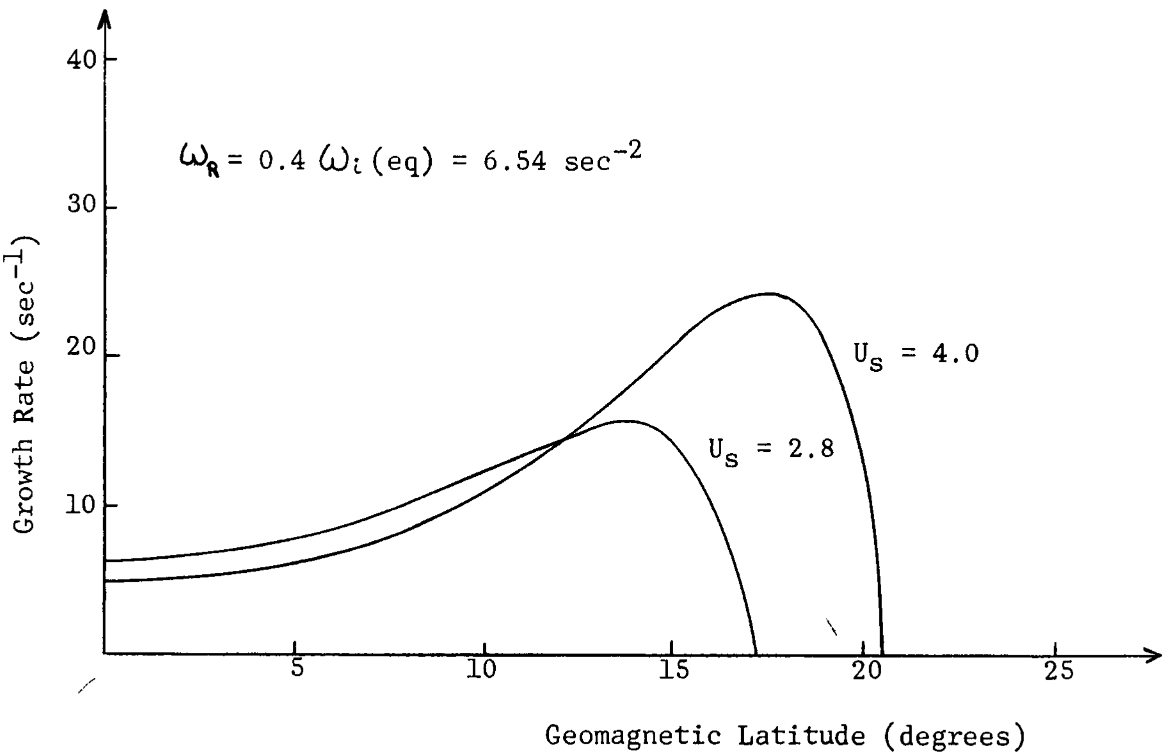

Fig. 3-b Variation of ω_I with λ for $\omega_R = 6.54\ sec^{-1}$.

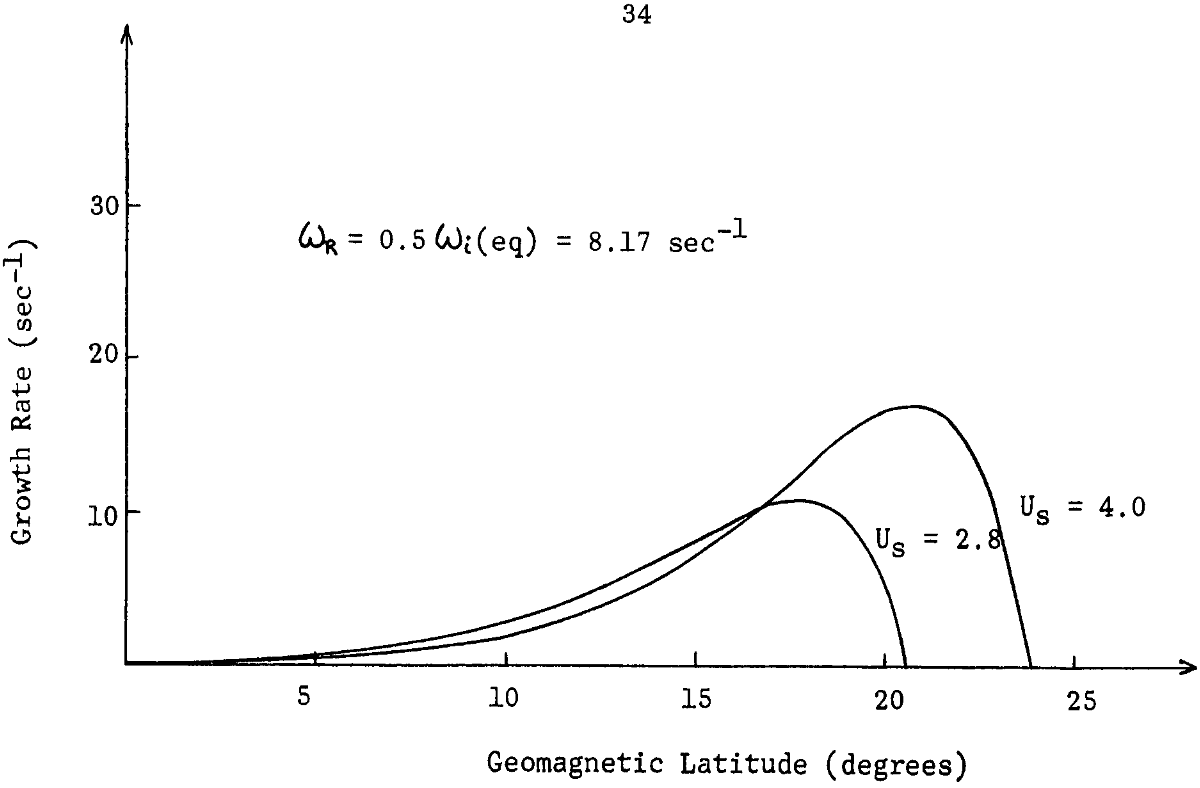

Fig. 3-c Variation of ω_I with λ for $\omega_R = 8.17\ sec^{-1}$

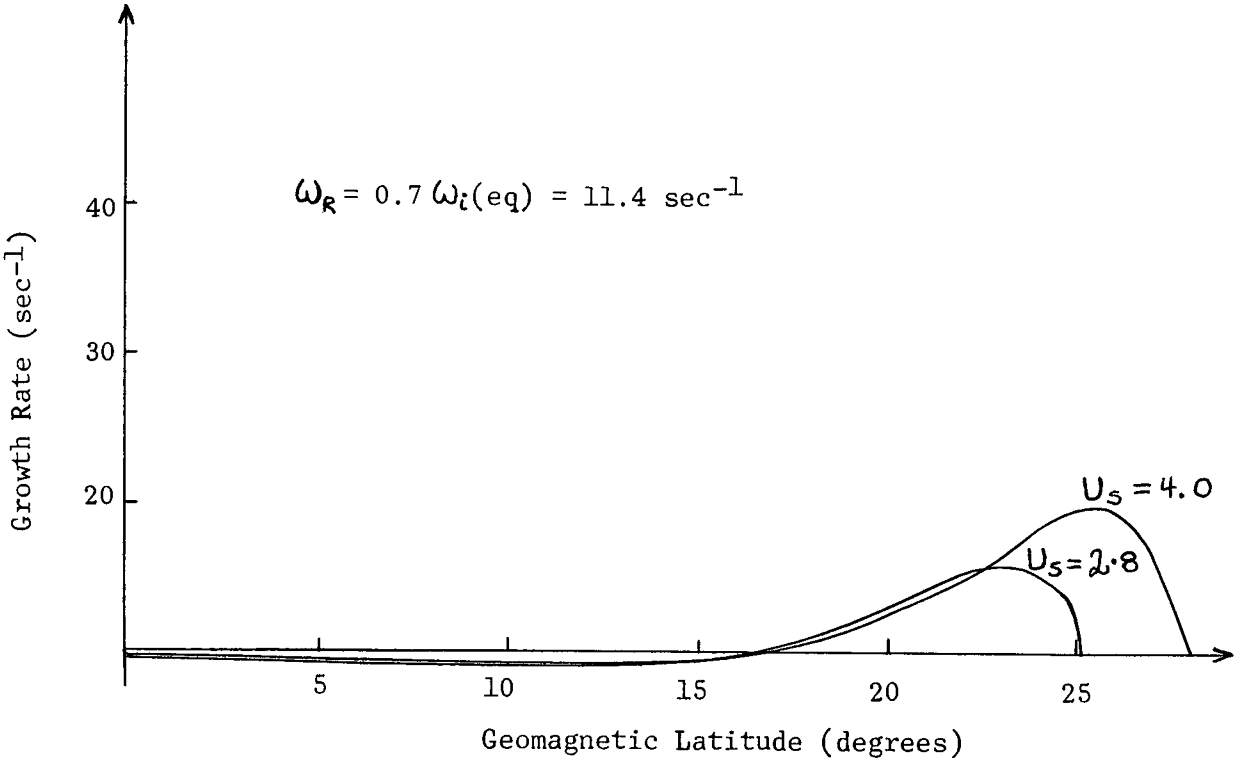

Fig. 3-d Variation of ω_I with λ for $\omega_R = 11.4\ sec^{-1}$.

CHAPTER V

SUMMARY

Discussion

It is important to remember that the analysis is valid only to first order. The growth of waves can be indicated but soon after it begins, the linear theory becomes invalid and nothing further can be said about the behaviour of the system. It might happen that when the particles have lost a sufficient amount of their energy, many are dumped into the ionosphere so that the wave growth becomes negligible compared to its attenuation. In this case, the thermal background plasma may be an important damping agent, but it was assumed to have zero temperature in the above analysis. Kennel and Petschek (1966) have considered the stability of trapped particles in detail and Cornwall (1966) and Watanabe (1966) have discussed some non-linear aspects of the problem. It is conceivable that N_s might sometimes be large enough that the ratio N_s/N_p is of the order of unity and this would invalidate the linear theory.

It is also known that the earth's field is not accurately represented by a dipole but this representation makes the analysis much simpler and general results can still be obtained.

Considering only monoenergetic particle streams is an over-simplification, although the variation of ω_I with energy has been calculated numerically. Hoffman and Bracken (1965) indicate that a doubly sloped exponential energy spectrum describes well the distribution of proton fluxes which they observed. If their detailed observations were taken at lower energies, then the introduction of an energy distribution of the form e^{-E/E_0} where E_0 is empirically determined would make the results more quantitative.

In order to determine more exactly the actual growth of a wave of some fixed frequency, the wave amplitude must be integrated over the region in which ω_I is non-zero. Jacobs and Watanabe (1965) have used Sturrock's analysis of growing waves (Sturrock, 1961) and shown that the mechanism which has been considered here gives rise to a non-convective instability (the point where the instability occurs initially remains fixed in space, although the disturbance can spread out around it) and they have discussed briefly the problem of how such a disturbance might come to be observed on the earth's surface.

In Chapter III, the theory presented by Tepley and Wentworth (1964) has been discussed and references to the papers by Cornwall (1965) and Obayashi (1965) have been made in several places.

Gendrin (1965) and Hruska (1966) have considered the problem of cyclotron emissions in the magnetosphere. Both authors finally consider only the (L, e) and (R, p) interactions where 'R' and 'L' refer to left and right-hand polarized waves and 'e' and 'p' refer to electrons and protons respectively.

Gendrin comments briefly on instabilities which arise when the transverse velocity components of the particles are important but suggests implicitly that this would never occur except near the mirror points. This assumption does not seem reasonable. He suggests that hm emissions occur when super-luminal protons interact with R waves and describes the process of repeated emissions as Tepley and Wentworth (1964) do. This idea has been criticized above.

Hruska (1966) considers a plasma instability by considering the net transfer of energy between waves and particles. He chooses a shifted

Maxwellian distribution for the streaming particles but gives no reason for this choice. Since he takes the temperature distribution to be isotropic, the contra-streaming (L,p) and (R,e) interactions do not give rise to a growing instability and he does not discuss them any further.

Conclusions

It is suggested that hm emissions result from hm wave packets propagating along the earth's magnetic field lines guided between ionospheric reflections in the northern and southern hemispheres. The ion resonance mode of wave is considered because it is guided by the earth's field at hydromagnetic frequencies much more than the electron resonance mode of wave and because the dispersion characteristics of the ion resonance mode are the same as the observed spectra of structured hm emissions.

Since the signals sometimes increase in intensity in time before dying out, it is suggested that the wave packets gain energy via a cyclotron instability process as they interact with low-energy protons which are trapped in the magnetosphere. An expression for the growth of the waves was developed starting from Maxwell's equations and the collisionless Boltzmann equation. Choosing a pitch-angle distribution function containing the factor $\sin^{\gamma}\psi/B^{\gamma/2}$ results in an upper cutoff frequency of $\omega_i/(1 + 2/\gamma)$ at any given point. It is not known whether this effect is more important than the damping which results from the thermal background plasma.

Computer calculations (for $\gamma = 2$) indicate three important features of the theory. First, the growth rate is a sharply peaked function of the frequency. Second, changes in the proton energy greatly influence the magnitude of the growth rate as well as the frequency of the steep lower cutoff. This suggestion was also made by Obayashi (1965). Finally, the largest growth rates for the instability are found to occur near the equatorial plane, although at some frequencies, there are two regions of largest growth, each slightly removed from the equator by ten or twenty degrees.

The observed latitude dependence of hm emissions may be explained by the fact that none of the L waves in the hm packet can have a frequency above the ion cyclotron frequency at the geomagnetic equator and that it decreases as the latitude of the point where the line of force intersects the earth's surface increases. Another important consideration is the effect of the ionosphere on the wave as it travels from the lower regions of the magnetosphere to the observation point on the earth's surface. Ionospheric wave guiding may restrict the wave packet frequencies because of a latitude variation of the duct characteristics.

If measurements on wave propagation above the ionosphere could be made to determine polarizations and if detailed records of low-energy proton fluxes (1 - 100 kev) could be obtained, many uncertainties in the theories of hm emissions would be eliminated.

APPENDIX I

Transformation of Equations and Solution for $f_1(\underline{v},k,\omega)$.

Let G(t,z) represent any of the quantities which are to be transformed. If G(t,z) is well-behaved, then the Fourier-Laplace transform of G exists and is defined by

$$G(\omega,k) = \int_0^{\infty} dt \int_{-\infty}^{+\infty} dz\, G(t,z)\, e^{-i(kz-\omega t)} \tag{A1-1}$$

For the Fourier transform to exist it is sufficient that G(t,z) be of bounded variation and absolutely integrable, i.e.,

$$\int_{-\infty}^{+\infty} |G(t,z)|\, dz < \infty \tag{A1-2}$$

and it is implied that $G(t,z)\to 0$ as $z\to\pm\infty$. In order to assure existence of the Laplace transform, it is convenient to assume (Stix, 1962; Sokolnikoff and Redheffer, 1958) that for some choice of the constants M and μ,

$$|G(t,k)| \leqslant M e^{\mu t} \tag{A1-3}$$

and $\mathrm{Im}\,\omega > \mu$.

Applying the transformations defined above to the following component Maxwell equations

$$-\frac{\partial B_y}{\partial z} = \frac{4\pi}{c} j_x + \frac{1}{c}\frac{\partial E_x}{\partial t} \tag{A1-4}$$

$$\frac{\partial B_x}{\partial z} = \frac{4\pi}{c} j_y + \frac{1}{c}\frac{\partial E_y}{\partial t} \tag{A1-5}$$

$$\frac{\partial E_y}{\partial z} = \frac{1}{c}\frac{\partial B_x}{\partial t} \tag{A1-6}$$

$$\frac{\partial E_x}{\partial z} = -\frac{1}{c}\frac{\partial B_y}{\partial t} \tag{A1-7}$$

and using integration by parts and the conditions outlined above, it is found that

$$-ik B_y(\omega,k) = \frac{4\pi}{c} \gamma_x(\omega,k) - \frac{1}{c}\left[i\omega E_x(\omega,k) + E_x(0,k)\right] \tag{A1-8}$$

$$ik B_x(\omega,k) = \frac{4\pi}{c} \gamma_y(\omega,k) - \frac{1}{c}\left[i\omega E_y(\omega,k) + E_y(0,k)\right] \tag{A1-9}$$

$$ik E_y(\omega,k) = -\frac{1}{c}\left[i\omega B_x(\omega,k) + B_x(0,k)\right] \tag{A1-10}$$

$$ik E_x(\omega,k) = \frac{1}{c}\left[i\omega B_y(\omega,k) + B_y(0,k)\right] \tag{A1-11}$$

The first-order Boltzmann equation which was derived in Chapter II (equation 2-7) can be written

$$\begin{aligned} \frac{\partial f_1}{\partial t} + u\frac{\partial f_1}{\partial z} - \omega_c \frac{\partial f_1}{\partial \phi} + \frac{q}{m}\left(E_x - \frac{1}{c} u B_y\right)\frac{\partial f_0}{\partial v_x} & \\ + \frac{q}{m}\left(E_y + \frac{1}{c} u B_x\right)\frac{\partial f_0}{\partial v_y} & \\ + \frac{q}{mc}\left(v_x B_y - v_y B_x\right)\frac{\partial f_0}{\partial u} &= 0 \end{aligned} \tag{A1-12}$$

Applying the combined transformation defined by equation A1-1 to this equation and eliminating $E_x(\omega,k)$ and $E_y(\omega,k)$ from the resulting equation using equations A1-10 and A1-11, it is found that

$$\begin{aligned} -\omega_c \frac{\partial}{\partial \phi} f_1(\underline{v},k,\omega) - i(\omega - ku) f_1(\underline{v},k,\omega) - f_1(\underline{v},k,0) & \\ + \frac{q}{mc}\left[\left(\frac{\omega}{k} - u\right) B_y(\omega,k) - \frac{i}{k} B_y(0,k)\right]\frac{\partial f_0}{\partial v_x} & \\ - \frac{q}{mc}\left[\left(\frac{\omega}{k} - u\right) B_x(\omega,k) - \frac{i}{k} B_x(0,k)\right]\frac{\partial f_0}{\partial v_y} & \\ + \frac{q}{mc}\left[v_x B_y(\omega,k) - v_y B_x(\omega,k)\right]\frac{\partial f_0}{\partial u} &= 0 \end{aligned} \tag{A1-13}$$

is the differential equation governing the transformed distribution function $f_1(\underline{v},k,\omega)$.

It is now convenient to transform coordinates from (v_x, v_y) to (w,ϕ) using the following relations

$$\frac{\partial f_0}{\partial v_x} = \frac{\partial f_0}{\partial w}\cdot\frac{\partial w}{\partial v_x} + \frac{\partial f_0}{\partial \phi}\cdot\frac{\partial \phi}{\partial v_x} \tag{A1-14}$$

$$\frac{\partial f_0}{\partial v_y} = \frac{\partial f_0}{\partial w}\cdot\frac{\partial w}{\partial v_y} + \frac{\partial f_0}{\partial \phi}\cdot\frac{\partial \phi}{\partial v_y} \tag{A1-15}$$

and it was assumed above that $\partial f_0/\partial\phi$ vanishes. Since $w^2 = v_x^2 + v_y^2$, it is easy to show that

$$\frac{\partial w}{\partial v_x} = \frac{v_x}{w} = \cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2} \tag{A1-16}$$

$$\frac{\partial w}{\partial v_y} = \frac{v_y}{w} = \sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i} \tag{A1-17}$$

By writing $f_1(\underline{v},k,0)$ as a Fourier series in ϕ,

$$f_1(\underline{v},k,0) = \sum_{m=-\infty}^{+\infty} f_1^{(m)} e^{im\phi} \tag{A1-18}$$

equation A1-13 can finally be written

$$\begin{aligned}
&\frac{\partial}{\partial\phi} f_1(\underline{v},k,\omega) + i\,\frac{\omega - ku}{\omega_c} f_1(\underline{v},k,\omega) = \\
&+\frac{1}{2}\frac{iq}{mc\,\omega_c}\left[\left(\frac{\omega}{k} - u\right)\frac{\partial f_0}{\partial w} + w\frac{\partial f_0}{\partial u}\right]e^{i\phi}\left[B_x(\omega,k) - iB_y(\omega,k)\right] \\
&-\frac{1}{2}\frac{iq}{mc\,\omega_c}\left[\left(\frac{\omega}{k} - u\right)\frac{\partial f_0}{\partial w} + w\frac{\partial f_0}{\partial u}\right]e^{-i\phi}\left[B_x(\omega,k) + iB_y(\omega,k)\right] \\
&+\frac{1}{2}\frac{q}{mc\,\omega_c}\cdot\frac{1}{k}\frac{\partial f_0}{\partial w}e^{i\phi}\left[B_x(0,k) - iB_y(0,k)\right] \\
&-\frac{1}{2}\frac{q}{mc\,\omega_c}\;\frac{1}{k}\frac{\partial f_0}{\partial w}e^{-i\phi}\left[B_x(0,k) + iB_y(0,k)\right] \\
&-\frac{1}{\omega_c}\sum_{m=-\infty}^{+\infty} f_1^{(m)} e^{im\phi}
\end{aligned} \tag{A1-19}$$

This equation is linear, non-homogeneous, first-order with constant coefficients and can be solved using standard methods. Coddington (1961) uses the notation

$$y' + ay = b(x) \tag{A1-20}$$

where a is a constant and b is a continuous function on an interval I. All solutions must have the form

$$y = \Phi(x) = e^{-ax}\int_{x_0}^{x} e^{at}\, b(t)\, dt + c\, e^{-ax} \tag{A1-21}$$

where x_0 is in I and c is any constant. Note that if the anti-derivative of the integrand is evaluated at x_0, then this 'constant' can be grouped with 'c' in multiplying e^{-ax} and the integral evaluated at the upper limit of 'x' is just the indefinite integral. The new constant is then chosen to be zero, and the result is the same. In this way, the term involving the indefinite integral represents the particular solution of the inhomogeneous equation and the term containing the redefined constant is the general solution of the homogeneous equation. The redefined constant is chosen to be zero because $f_1(\underline{v},z,t)$ must be zero when the wave and particle stream perturbations are removed, so the solution to the homogeneous equation must be dropped from the general solution of the inhomogeneous equation. The resulting solution for $f_1(\underline{v}, k,\omega)$ is given by

$$\begin{aligned} f_1(\underline{v},k,\omega) = {} & \frac{e^{i\phi}}{\omega - ku + \omega_c}\left\{\frac{1}{2}\frac{q}{mc}\left[\left(\frac{\omega}{k} - u\right)\frac{\partial f_0}{\partial w} + w\frac{\partial f_0}{\partial u}\right]\left[B_x(\omega,k) - i B_y(\omega,k)\right]\right. \\ & \left. - \frac{i}{2}\frac{q}{mc}\frac{1}{k}\frac{\partial f_0}{\partial w}\left[B_x(0,k) - i B_y(0,k)\right]\right\} \\ & - \frac{e^{-i\phi}}{\omega - ku - \omega_c}\left\{\frac{1}{2}\frac{q}{mc}\left[\left(\frac{\omega}{k} - u\right)\frac{\partial f_0}{\partial w} + w\frac{\partial f_0}{\partial u}\right]\left[B_x(\omega,k) + i B_y(\omega,k)\right]\right. \\ & \left. - \frac{i}{2}\frac{q}{mc}\frac{1}{k}\frac{\partial f_0}{\partial w}\left[B_x(0,k) + i B_y(0,k)\right]\right\} \\ & + \sum_{m=-\infty}^{+\infty}\frac{i f_1^{(m)}}{\omega - ku + m\omega_c}\, e^{im\phi} \end{aligned} \tag{A1-22}$$

APPENDIX II

Determination of the Transformed Magnetic Fields

Applying the Fourier-Laplace transform defined in Appendix I to equation 2-13 gives

$$j_x(\omega,k) \pm i j_y(\omega,k) = \sum_{comp} q \int d\underline{v}\, e^{\pm i\phi}\, w f_1(\underline{v},k,\omega) \tag{A2-1}$$

and using the expression for $f_1(\underline{v}, k,\omega)$ found in the first appendix, equation A2-1 gives the current density (transformed) in terms of the transformed magnetic fields. An expression for the current density can also be found from the transformed Maxwell equations. By eliminating $E_x(\omega,k)$ and $E_y(\omega,k)$ from the four transformed equations in appendix I, it is found that

$$i(\omega^2 - c^2 k^2) B_x(\omega,k) = -4\pi c k j_y(\omega,k) - \omega B_x(0,k) + ckE_y(0,k) \tag{A2-2}$$

and

$$i(\omega^2 - c^2 k^2) B_y(\omega,k) = 4\pi c k j_x(\omega,k) - \omega B_y(0,k) - ckE_x(0,k) \tag{A2-3}$$

and hence

$$j_x \pm i j_y = \frac{1}{4\pi c k}\Big\{\pm(\omega^2 - c^2k^2)\big[B_x(\omega,k) \pm i B_y(\omega,k)\big] \mp i\omega\big[B_x(0,k) \pm i B_y(0,k)\big] + ck\big[E_x(0,k) \pm i E_y(0,k)\big]\Big\} \tag{A2-4}$$

Noting that when the solution for $f_1(\underline{v},k,\omega)$ is substituted into equation A2-1, the integral involving $f_1^{(m)} e^{\pm i m\phi}$ is non-zero only when $m = \pm 1$, it is found that

$$j_x(\omega,k) \pm i j_y(\omega,k) = \sum_{comp} \mp \frac{1}{2} \frac{q^2}{mc} \int d\underline{v} \, \frac{\left[\left(\frac{\omega}{k}-u\right) w \frac{\partial f_o}{\partial w} + w^2 \frac{\partial f_o}{\partial u}\right]}{\omega - ku \mp \omega_c} \cdot \left[B_x(\omega,k) \pm i B_y(\omega,k)\right]$$

$$+ \sum_{comp} \pm \frac{i}{2} \frac{q^2}{mc} \frac{1}{k} \int d\underline{v} \, \frac{w \frac{\partial f_o}{\partial w}}{\omega - ku \mp \omega_c} \cdot \left[B_x(0,k) \pm i B_y(0,k)\right]$$

$$+ \sum_{comp} iq \int d\underline{v} \, \frac{w f_1^{(\mp 1)}}{\omega - ku \mp \omega_c} \qquad \text{(A2-5)}$$

Finally, using equations A2-4 and A2-5, it is found that

$$B_x(\omega,k) \pm i B_y(\omega,k) = \frac{i\left(\omega + \sum_{comp} \frac{4\pi q^2}{m} \int d\underline{v} \, \frac{\frac{1}{2} w \frac{\partial f_o}{\partial w}}{\omega - ku \mp \omega_c}\right)\left[B_x(0,k) \pm i B_y(0,k)\right]}{\omega^2 - c^2 k^2 + \sum_{comp} \frac{4\pi q^2}{m} I^*}$$

$$+ \frac{\mp ck\left[E_x(0,k) \pm i E_y(0,k)\right] \pm \sum_{comp} 4\pi q c k \int d\underline{v} \, \frac{w f_1^{(\mp 1)}}{\omega - ku \mp \omega_c}}{\omega^2 - c^2 k^2 + \sum_{comp} \frac{4\pi q^2}{m} I^*} \qquad \text{(A2-6)}$$

where

$$I^* = \int d\underline{v} \, \frac{\frac{1}{2}(\omega - ku) w \frac{\partial f_o}{\partial w} + \frac{1}{2} k w^2 \frac{\partial f_o}{\partial u}}{\omega - ku \mp \omega_c} \qquad \text{(A2-7)}$$

APPENDIX III

Analytic Continuation of the Integrals

The outline of the procedure as given by Stix (1962) will not be copied in detail. There, the case of longitudinal plasma oscillations is considered as an example. The differences for the case of transverse waves interacting with a particle stream are noted. The problem is simplified by calculating the asymptotic value of $\underline{B}(t,k)$ as $t \to \infty$.

The expression for the magnetic fields given in equation 2-14 is valid for $\mathrm{Im}\,\omega > \mu$, where μ was defined in Chapter I in connection with the definition of the Laplace transform. The analytic continuation of equation 2-14 must be determined.

It is assumed that $B_x(0,k)$, $B_y(0,k)$, $E_x(0,k)$, $E_y(o,k)$, $\partial f_o/\partial w$, $\partial f_o/\partial u$, and $f_1^{(\pm 1)}$ are all entire functions of u. One must then consider integrals of the form

$$I(V) = -\frac{1}{k}\int_{-\infty}^{+\infty} \frac{F(u)}{u - V}\, du \qquad \text{(A3-1)}$$

with

$$V = \frac{\omega \mp \omega_c}{k} \qquad \text{(A3-2)}$$

where the integral is to be taken along the real axis. F(u) is assumed to be an entire function of u. If one is considering u as a complex variable, then the path of integration in equation A3-1 can be changed in accordance with complex variable theory.

There are four cases, depending on the signs of k and ω_I.

Case 1. $k>0, \omega_I > 0$.

In this case, $\mathrm{Im}(V) > 0$. The path of integration can be raised (fig. 10) from the real axis above the singularity using the residue theorem.

$$I(V)(\text{path } 1) = I(V)(\text{path } 2) - \frac{2\pi i}{k} F(V) \tag{A3-3}$$

Case 2. $k>0, \omega_I < 0$.

In this case, $\mathrm{Im}(V) < 0$ and the analytic continuation is obtained by deforming the path of integration down from the real axis (fig. 11) so that

$$I(V)(\text{path } 1) = I(V)(\text{path } 2) - \frac{2\pi i}{k} F(V) \tag{A3-4}$$

and path 2 can easily be chosen to be the real axis.

Case 3. $k<0, \omega_I > 0$.

In this case, $\mathrm{Im}(V)<0$. The path is taken along the real axis and it is trivial to write (fig. 12)

$$I(V)(\text{path } 1) = I(V)(\text{path } 2) \tag{A3-5}$$

Case 4. $k<0, \omega_I < 0$.

$\mathrm{Im}(V) > 0$ and the continuation by contour deformation is analogous to case 2. From figure 13, it can be seen that

$$I(V)(\text{path } 1) = I(V)(\text{path } 2) \tag{A3-6}$$

Using the residue theorem

$$I(V)(\text{real axis}) - I(V)(\text{path } 2) = -\frac{2\pi i}{k} F(V) \tag{A3-7}$$

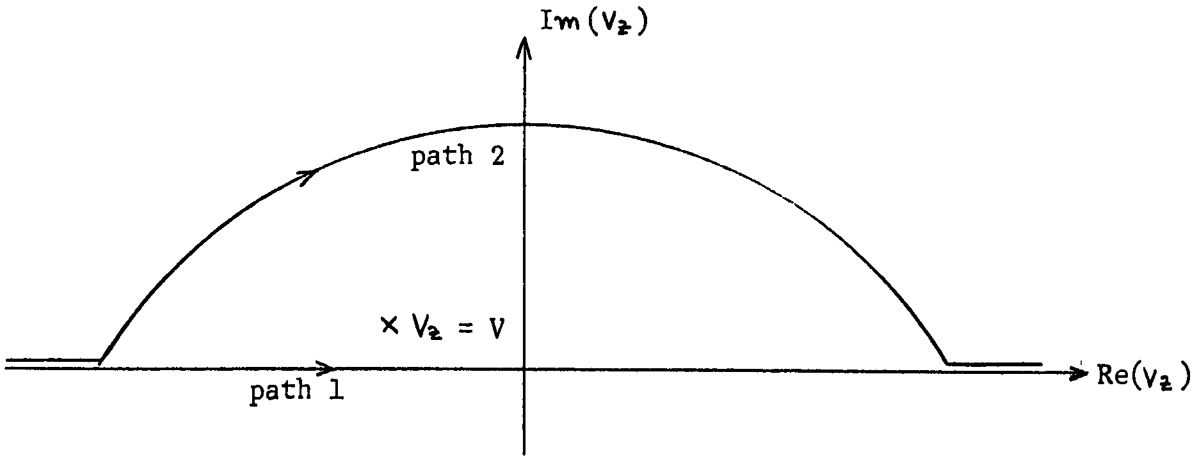

Fig. 4 Contours for integration when $k > 0$ and $\omega_I > 0$

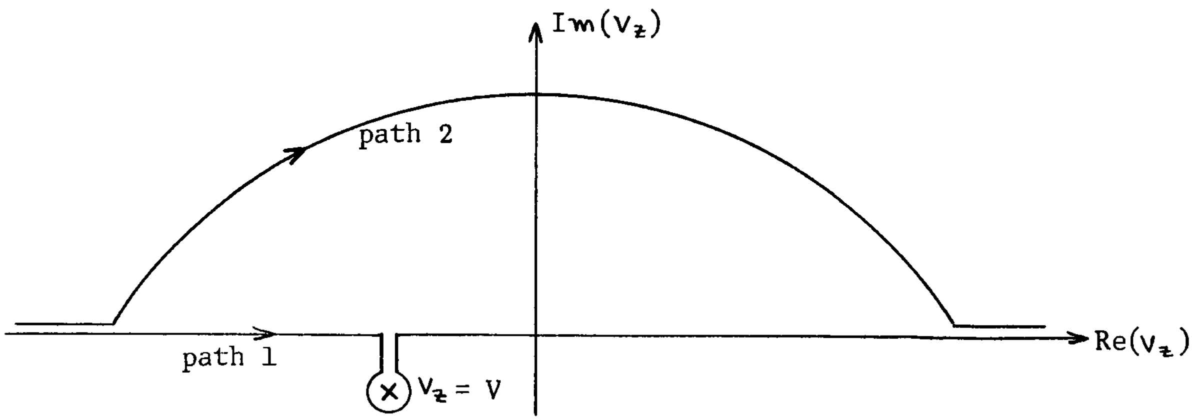

Fig. 5 Contours for integration when $k > 0$ and $\omega_I < 0$

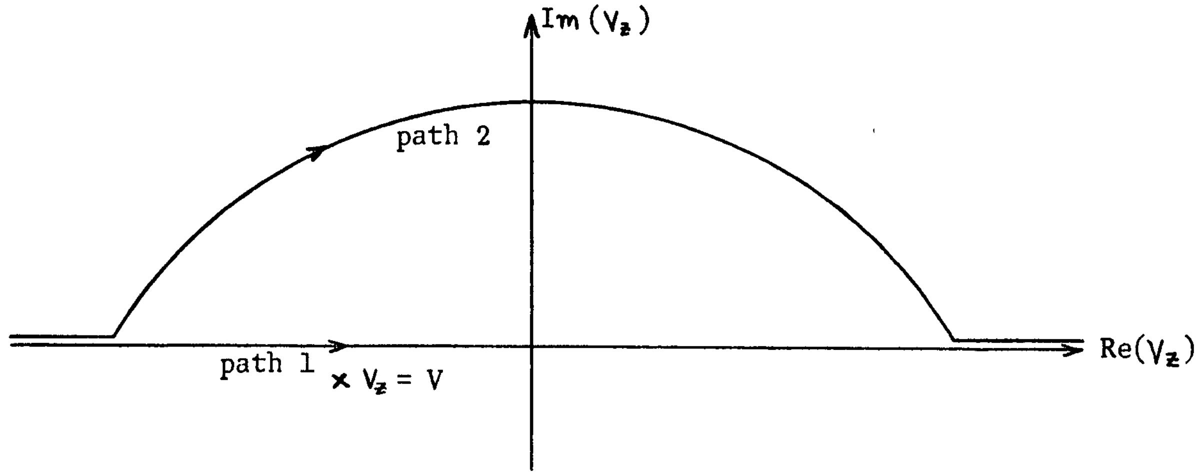

Fig. 6 Contours for integration when $k < 0$ and $\omega_I > 0$.

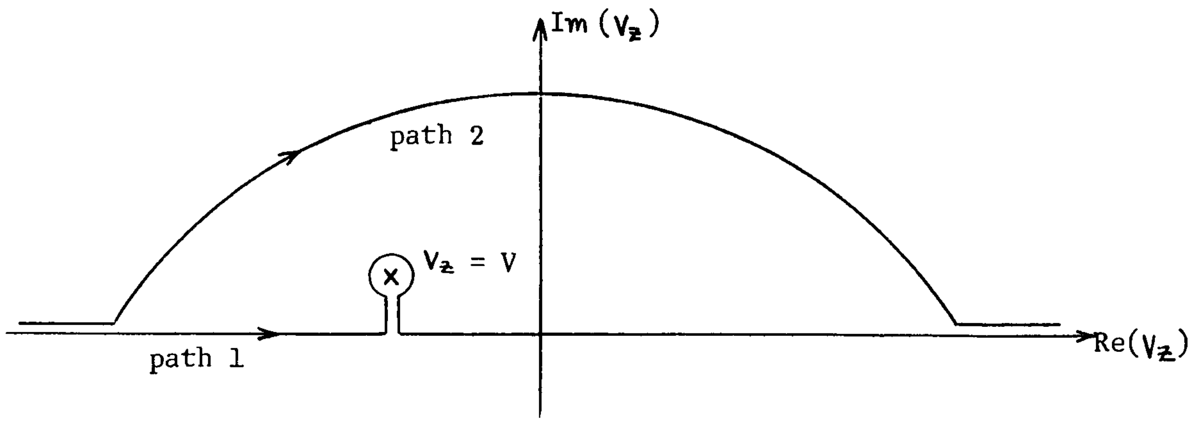

Fig. 7 Contours for integration when $k < 0$ and $\omega_I < 0$

and so

$$I(V)(\text{path }1) = I(V)(\text{real axis}) - \frac{2\pi i}{|k|}F(V) \qquad \text{(A3-8)}$$

since k<0 in this case.

It can be seen from the above discussion that the integrals over 'u' for $\omega_I > 0$ are taken strictly as they appear, along the real u axis (cases 1 and 3). For $\omega_I < 0$, (cases 2 and 4), again the integrals are taken along the real axis but there is an additional residue term in each of the two cases.

Following Stix (1962), it is seen that both the numerator and denominator of $B_x(\omega,k) \pm iB_y(\omega,k)$ are analytic functions of u in the whole plane (entire functions) and so the poles of $\underline{B}(\omega,k)$ come only from the zeroes of the denominator. There are two equations, one for each of positive and negative ω_I.

$$\omega^2 - c^2k^2 + \sum_{comp} \frac{4\pi q^2}{m}\int d\underline{v}\, \frac{\frac{1}{2}(\omega - ku)w\frac{\partial f_0}{\partial w} + \frac{1}{2}kw^2\frac{\partial f_0}{\partial u}}{\omega - ku \mp \omega_c} = 0 \qquad \text{(A3-9)}$$

for $\omega_I > 0$, and

$$\omega^2 - c^2k^2 + \sum_{comp} \frac{4\pi q^2}{m}\left\{\int d\underline{v}\, \frac{\frac{1}{2}(\omega - ku)w\frac{\partial f_0}{\partial w} + \frac{1}{2}kw^2\frac{\partial f_0}{\partial u}}{\omega - ku \mp \omega_c}\right.$$

$$\left. + \int d\underline{w}\left[-\frac{2\pi i}{|k|}\left(\frac{1}{2}(\omega - ku)w\frac{\partial f_0}{\partial w} + \frac{1}{2}kw^2\frac{\partial f_0}{\partial u}\right)_{u=V}\right]\right\} = 0 \qquad \text{(A3-10)}$$

for $\omega_I < 0$.

Using the Cauchy principal value ($\mathcal{P}$) notation, these two equations can be combined into one equation which is valid for both positive and negative values of the imaginary part of ω .

$$\omega^2 - c^2k^2 + 4\pi\omega\sum_{comp}\frac{q^2}{m}\int d\underline{w}\left\{\mathcal{P}\int du\,\frac{wG(f_0)}{\omega - ku \mp \omega_c} - \frac{\pi i}{|k|}\left[wG(f_0)\right]_{u=V}\right\} = 0 \qquad \text{(A3-11)}$$

where the functional form of G is given by

$$G(f_o) = \frac{1}{2}\left[\left(1 - \frac{ku}{\omega}\right)\frac{\partial f_o}{\partial w} + \frac{kw}{\omega}\frac{\partial f_o}{\partial u}\right] \tag{A3-12}$$

APPENDIX IV

Wave Polarization

Consider the case of pure wave propagation when no particle stream perturbation is present. In this case, $D \equiv 0$, since $f_1^{(\pm 1)} = 0$. Consider waves of the form

$$
\begin{aligned}
\underline{B} &= \underline{B}^{+} e^{i(kz-\omega t)} \\
\underline{E} &= \underline{E}^{+} e^{i(kz-\omega t)}
\end{aligned}
\tag{A4-1}
$$

with only x and y components present. Left hand polarization is defined by writing

$$B_x(t,z) + i B_y(t,z) = 0 \tag{A4-2}$$

It then follows from Maxwell's equations that

$$E_x(t,z) + i E_y(t,z) = 0 \tag{A4-3}$$

The Fourier-Laplace transformation of these two equations is straight forward so that at t = 0, the condition

$$B_x(0,k) \pm i B_y(0,k) = 0 \tag{A4-4}$$

implies that

$$E_x(0,k) \pm i E_y(0,k) = 0 \tag{A4-5}$$

The total field is written as the sum of the separate fields of two waves of opposite polarization.

$$\underline{B}(\omega,k) = \underline{B}^{(1)}(\omega,k) + \underline{B}^{(2)}(\omega,k) \tag{A4-6}$$

$$\underline{E}(\omega,k) = \underline{E}^{(1)}(\omega,k) + \underline{E}^{(2)}(\omega,k) \tag{A4-7}$$

where the superscript (1) indicates the left-hand mode and (2) the right-hand mode. Using equations A4-6 and A4-7, it is found that

$$\underline{B}(0,k) = \underline{B}^{(1)}(0,k) + \underline{B}^{(2)}(0,k) \tag{A4-8}$$

and

$$\underline{E}(0,k) = \underline{E}^{(1)}(0,k) + \underline{E}^{(2)}(0,k) \tag{A4-9}$$

Substitution of the above values of $B_x(\omega,k)$, $B_y(\omega,k)$, $B_x(0,k)$, $B_y(0,k)$, $E_x(0,k)$ and $E_y(0,k)$ into equation 2-14 and choosing, say, the upper sign everywhere, yields an equation involving only the superscript (1) because those quantities involving the superscript (2) all vanish because of the definition of the right and left modes of polarization. When the lower sign is chosen, the resulting equation contains only the superscript (2). Therefore, in equation 2-14, the upper sign corresponds to left-hand polarization and the lower sign corresponds to right-hand polarization.

APPENDIX V

Simplifying the General Dispersion Relation

Equation 2-21 is

$$\omega^2 - c^2 k^2 - \sum_{comp} \frac{\Omega_p^2 \omega}{\omega \mp \omega_c} + 4\pi\omega \sum_{comp} \frac{q^2}{m} \int d\underline{w}\, P \int du\, \frac{wG(f_s)}{\omega - ku \mp \omega_c}$$

$$- i\,\frac{4\pi^2\omega}{|k|} \sum_{comp} \frac{q^2}{m} \int d\underline{w} \left[wG(f_s)\right]_{u=V} = 0 \qquad \text{(A5-1)}$$

The first and third terms can be approximated using the fact that

$$\omega^2 \simeq \omega_R^2 + 2\,\omega_R \omega_I \qquad \text{(A5-2)}$$

and

$$\frac{\omega}{\omega \mp \omega_c} \simeq \frac{\omega_R}{\omega_R \mp \omega_c} + \frac{\pm \omega_c}{(\omega_R \mp \omega_c)^2} \cdot i\omega_I \qquad \text{(A5-3)}$$

The fourth term in equation A5-1 contains the product of $G(f_s)$ and ω and the product $\omega_I N_s$ is second order. At the same time, two of the three terms in $G(f_s)$ contain a multiplying factor $1/\omega$, and when this factor is written in the normal form of a complex number with a real denominator,

$$\frac{N_s}{\omega} = \frac{N_s}{\omega_R + i\omega_I} \simeq \frac{(\omega_R - i\omega_I)N_s}{\omega_R^2 + \omega_I^2} \simeq \frac{\omega_R N_s}{\omega_R^2} = \frac{N_s}{\omega_R} \qquad \text{(A5-4)}$$

Exactly the same procedure can be carried out for the denominator of the integrand of the fourth term, with the result that

$$\frac{wG(f_s)}{\omega - ku \mp \omega_c} \simeq \frac{w\,G(f_s)}{\omega_R - ku \mp \omega_c} \qquad \text{(A5-5)}$$

In the last term, the multiplying ω can be replaced by ω_R. One further simplification can be made in the last term. Expanding the numerator of the integrand in a Taylor series about u = V_R, it is found that

$$[wG(f_s)]_{u=V} = [wG(f_s)]_{u=V_R} + (V - V_R)\left\{\frac{d}{du}[wG(f_s)]\right\}_{u=V_R} + \cdots$$

$$\simeq [wG(f_s)]_{u=V_R} \tag{A5-6}$$

since V - $V_R \sim \omega_I$ and $G(f_s)$ contains the first order quantity N_s.

Using all the above approximations, equation A5-1 can be written

$$\omega_R^2 - c^2k^2 - \sum_{comp} \frac{\Omega_p^2 \omega_R}{\omega_R \mp \omega_c} + 4\pi\omega_R \sum_{comp} \frac{q^2}{m} P\int d\underline{v}\, \frac{wG(f_s)}{\omega_R - ku \mp \omega_c}$$

$$+ i\left\{\omega_I\left[2\omega_R + \sum_{comp} \frac{\pm\Omega_p^2\, \omega_c}{(\omega_R \mp \omega_c)^2}\right] - \frac{4\pi^2\omega_R}{|k|} \sum_{comp} \frac{q^2}{m} \int d\underline{w}\, [wG(f_s)]_{u=V_R}\right\} = 0 \tag{A5-7}$$

Setting the imaginary part of this equation equal to zero gives equation 2-25. A second equation results when the real part is set equal to zero. The fourth term can be neglected since it is of first order while the third term is zeroth order. This approximation yields the dispersion relation given in Chapter I.

APPENDIX VI

Coordinate Transformation and Calculation of ω_I

In cylindrical coordinates (u,w)

$$G(f_s) = \frac{1}{2}\left[\left(1-\frac{ku}{\omega}\right)\frac{\partial f_s}{\partial w} + \frac{kw}{\omega}\frac{\partial f_s}{\partial u}\right] \tag{A6-1}$$

Equation 3-9 gives f_s in spherical coordinates (v,ψ) as

$$f_s = A\,\delta(v-v_0)\,\frac{\sin^\gamma\psi}{B^{\gamma/2}} \tag{A6-2}$$

It is convenient to transform $G(f_s)$ using the chain rules

$$\frac{\partial f_s}{\partial w} = \frac{\partial f_s}{\partial v}\cdot\frac{\partial v}{\partial w} + \frac{\partial f_s}{\partial \psi}\cdot\frac{\partial \psi}{\partial w} \tag{A6-3a}$$

$$\frac{\partial f_s}{\partial u} = \frac{\partial f_s}{\partial v}\frac{\partial v}{\partial u} + \frac{\partial f_s}{\partial \psi}\frac{\partial \psi}{\partial u} \tag{A6-3b}$$

and the transformation

$$v = +\left(w^2+u^2\right)^{1/2} \tag{A6-4a}$$

$$\psi = \tan^{-1}\frac{w}{u} \tag{A6-4b}$$

Equations A1-3a and A1-3b then become

$$\frac{\partial f_s}{\partial w} = A\,\delta'(v-v_0)\,\frac{\sin^\gamma\psi}{B^{\gamma/2}}\,\frac{w}{v} + \gamma A\,\delta(v-v_0)\,\frac{\sin^{\gamma-1}\psi}{B^{\gamma/2}}\,\cos\psi\,\frac{1/u}{1+(w/u)^2} \tag{A6-5a}$$

$$\frac{\partial f_s}{\partial u} = A\,\delta'(v-v_0)\,\frac{\sin^\gamma\psi}{B^{\gamma/2}}\,\frac{u}{v} + \gamma A\,\delta(v-v_0)\,\frac{\sin^{\gamma-1}\psi}{B^{\gamma/2}}\,\cos\psi\,\frac{(-w/u^2)}{1+(w/u)^2} \tag{A6-5b}$$

and u and w can be eliminated using the inverse transformation.

However, the problem can be simplified because the integral in the numerator of equation 2-25 must be evaluated at $u = V_R$. It will be written

$$\int d\underline{w}\left[wG(f_s)\right]_{u=V_R} = 2\pi\int w\,dw\left[wG(f_s)\right]_{u=V_R} \tag{A6-6}$$

In equations A1-5a and A1-5b, u and w are eliminated by putting $u = V_R$, and $w = +\,(v^2 - V_R^2)^{\frac{1}{2}}$. This result for w implies that

$$w\,dw = v\,dv \tag{A6-7}$$

and

$$\sin\psi = \frac{w}{v} = \frac{(v^2 - V_R^2)^{1/2}}{v} \tag{A6-8}$$

Using these results, it then follows by algebra that

$$\int d\underline{w}\left[wG(f_s)\right]_{u=V_R} = 2\pi\int_{|V_R|}^{\infty} v\,dv\left[wG(f_s)\right]_{u=V_R} =$$

$$2\pi\int_{|V_R|}^{\infty} v\,dv\; w\frac{A}{2B^{\gamma/2}}\left\{\left(1-\frac{kV_R}{\omega_R}\right)\left[\delta'(v-v_0)\frac{(v^2-V_R^2)^{\frac{\gamma+1}{2}}}{v^{\gamma+1}} + \gamma\,\delta(v-v_0)(v^2-V_R^2)^{\frac{\gamma-1}{2}}\frac{V_R}{v^{\gamma+1}}\frac{V_R}{v}\right]\right.$$

$$\left. + \frac{k(v^2-V_R^2)^{1/2}}{\omega_R}\left[\delta'(v-v_0)\frac{V_R(v^2-V_R^2)^{\gamma/2}}{v^{\gamma+1}} - \gamma\,\delta(v-v_0)\frac{(v^2-V_R^2)^{\frac{\gamma-1}{2}}}{v^{\gamma+1}}(v^2-V_R^2)^{1/2}\frac{V_R}{v}\right]\right\} =$$

$$2\pi\int_{|V_R|}^{\infty} dv\frac{A}{2B^{\gamma/2}}\left[\delta'(v-v_0)\frac{(v^2-V_R^2)^{\frac{\gamma}{2}+1}}{v^{\gamma}} + \left(1-\frac{kV_R}{\omega_R}\right)\gamma\,\delta(v-v_0)\frac{V_R^2}{v^{\gamma+1}}(v^2-V_R^2)^{\gamma/2}\right.$$

$$\left. -\gamma\frac{kV_R}{\omega_R}\,\delta(v-v_0)\frac{(v^2-V_R^2)^{\frac{\gamma}{2}+1}}{v^{\gamma+1}}\right] \tag{A6-9}$$

with

$$\frac{kV_R}{\omega_R} = 1 \mp \frac{\omega_c}{\omega_R}$$

γ has been restricted to values greater than -2. If $|V_R| > v_0$, then the integral is zero. If $|V_R| = v_0$, the integral may be different from zero because of the presence of the delta functions. In this case, γ must be further restricted in order to avoid division by zero in factors containing the expression $v^2 - V_R^2$. From the third term in the integrand, it can be seen that $\gamma \geqslant -2$, and this is weaker than the original restriction. The second term requires that $\gamma \geqslant 0$. In the first term, the presence of the delta-function derivative means that the function which multiplies it in the integrand must be first differentiated and then evaluated at $v = v_o$. This calculation leads to an expression containing terms with the factors $(v^2 - V_R^2)^{\gamma/2}$ and $(v^2 - V_R^2)^{\frac{\gamma}{2}+1}$. It is seen, then, that for $|V_R| = v_o$, γ must be at least zero, and in this case, whether γ is zero or positive, the integral is zero.

This result is not unexpected since $|V_R| = v_o$ represents the resonance condition for the case of protons interacting with contra-streaming left-hand polarized waves when the proton beam is 'linear', that is, when the beam has no transverse kinetic energy. Neufeld and Wright (1963) have shown that for this case, a contra-streaming instability does not exist.

It is found then, that for $|V_R| < v_o$ and $\gamma > -2$,

$$2\pi\int v\,dv\left[wG(f_s)\right]_{u=V_R} =$$

$$2\pi\int dv\frac{A}{2B^{\gamma/2}}\left\{\frac{(v^2-V_R^2)^{\gamma/2}}{v^\gamma}\left[\left(\frac{\gamma}{2}+1\right)2v - \gamma(v^2-V_R^2)\right] \pm \gamma\frac{\omega_c}{\omega_R}\frac{V_R^2(v^2-V_R^2)^{\gamma/2}}{v^{\gamma+1}}\right.$$

$$\left. + \gamma\left(-1 \pm \frac{\omega_c}{\omega_R}\right)\frac{(v^2-V_R^2)^{\gamma/2}(v^2-V_R^2)}{v^{\gamma+1}}\right\}\delta(v-v_o) =$$

$$\frac{\pi A}{B^{\gamma/2}}\left[\frac{(v_o^2-V_R^2)^{\gamma/2}}{v_o^{\gamma-1}}\left(\pm\gamma\frac{\omega_c}{\omega_R} - 2 - \gamma\right)\right] \qquad \text{(A6-10)}$$

Substituting this result in equation 2-25, it is immediately found that

$$\omega_I = \frac{\frac{4\pi^2\omega_R}{|k|}\sum_{comp}\frac{q^2}{m}\frac{N_s}{2\sqrt{\pi}}\frac{\Gamma(\frac{\rho}{2}+1)}{\Gamma(\frac{\rho+1}{2})}\frac{(v_o^2-V_R^2)^{\gamma/2}}{v_o^{\gamma+1}}\left[\gamma\left(\pm\frac{\omega_c}{\omega_R}-1\right)-2\right]}{2\omega_R + \sum_{comp}\frac{\pm\Omega_p^2\,\omega_c}{(\omega_R\mp\omega_c)^2}} \qquad \text{(A6-11)}$$

where N_s is defined by

$$N_s = N_s^* \left(\frac{B^*}{B}\right)^{\frac{\gamma}{2}} \tag{A6-12}$$

and represents the streaming particle density at a point where the magnetic field has the value B.

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