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LOCAL RADICAL AND SEMI-SIMPLE CLASSES OF RINGS

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ABSTRACT

For any cardinal number $K \geq 2$ and any non-empty class of rings $\mathcal{R}$ we make the following definitions. The class $\mathcal{R}(K)$ is the class of all rings R such that every subring of R which is generated by a set of cardinality strictly less than K is in $\mathcal{R}$. The class $\mathcal{R}_{g(K)}$ is the class of all rings R such that every non-zero homomorphic image of R contains a non-zero subring in $\mathcal{R}$ which is generated by a set of cardinality strictly less than K.

Several properties of the classes $\mathcal{R}_{g(K)}$ and $\mathcal{R}(K)$ are determined. In particular, conditions are specified which imply that $\mathcal{R}(K)$ is a radical class or a semi-simple class. Necessary and sufficient conditions that the class $\mathcal{J}$ of all $\mathcal{R}_{g(K)}$ semi-simple rings be equal to $\mathcal{J}(K)$ are given.

The classes $\mathcal{R}(K)$ and $\mathcal{R}_{g(K)}$ when $K = 2$ or $K = \aleph_0$ are considered in detail for various classes $\mathcal{R}$ (including the cases when $\mathcal{R}$ is one of the well-known radical classes). In all cases when $\mathcal{R}$ is one of the well-known radical classes it is shown that $\mathcal{R}(2)$ and $\mathcal{R}(\aleph_0)$ are radical classes and whenever they contain all nilpotent rings they are shown to be special radical classes. Those radical classes $\mathcal{R}(2)$ which are contained in FC ($R \in FC$ if and only if for all $x \in R$, x is torsion) are characterized.

Let $\mathfrak{R}$ be any radical class. The largest radical class $\mathfrak{H}(\aleph_0)$ (if one exists) such that $\mathfrak{H}(\aleph_0)(R) \cap \mathfrak{R}(R) = (0)$ for all rings R is defined to be the local complement of $\mathfrak{R}$ and is denoted by $\overline{\mathfrak{R}}$. If $\mathfrak{R} = \mathfrak{R}(\aleph_0)$ then the local complement $\overline{\mathfrak{R}}$ exists and $\overline{\mathfrak{R}} = \overline{\mathfrak{R}}(2)$. The local complements of all radicals discussed are determined.

We are able to apply some of these results in order to classify those classes of rings which are both semi-simple and radical classes.

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INTRODUCTION

The purpose of this thesis is to investigate radical classes and semi-simple classes which are determined by conditions on finitely generated subrings. A class of rings $\mathcal{R}$ will be called a local class if a ring $R \in \mathcal{R}$ if and only if every finitely generated subring of R satisfies some special condition. Similarily, a class of rings $\mathcal{R}$ is an elementary class if a ring $R \in \mathcal{R}$ if and only if every subring of R generated by one element satisfies some special condition.

Let us consider some examples. Clearly the class of all commutative rings is a local class since a ring R is commutative if and only if every finitely generated subring of R is commutative. This class of rings is not an elementary class.

The class of all nil rings is an elementary class since a ring R is nil if and only if for all $x \in R$ the subring generated by x is nil.

We shall prove that the class of all Jacobson radical rings is not a local class. Notice that the definition of a Jacobson radical ring (for all $x \in R$ there is a $y \in R$ such that $x + y + xy = 0$) involves the existential quantification of a ring element. On the other hand the definition of a nil ring involves only the universal quantification of ring elements. This illustrates what we would naturally expect : that classes of rings which are defined by conditions involving only universal quantification of ring elements would be local classes whereas classes of rings which are defined by conditions involving existential quantification of ring elements would not be local classes. For example, we would expect that the class of all rings satisfying a given set of polynomial identities would be a local class.

In Chapter II we consider some general results about local classes. In particular, we specify several conditions under which local classes are radical classes or semi-simple classes.

The remainder of the thesis is devoted to a consideration of specific local radical classes and specific local semi-simple classes.

Any class of rings $\mathfrak{R}$ determines a local and an elementary class (the class of all rings such that every finitely generated subring (subring generated by one element) is in $\mathfrak{R}$). We consider the local and elementary classes determined by the well-known radical classes. All of these classes are radical classes and those which contain all nilpotent rings are shown to be special radical classes. In this and in other ways we arrive at several new radical classes. All those which are elementary classes and which are contained in FC (FC is the class of all rings R such that for all $x \in R$, x is torsion) can be characterized as "sums" of certain simple elementary radical classes. In some cases we are able to obtain structure theorems by assuming certain chain conditions.

Any class of rings which is closed under homomorphic images (actually a slightly weaker condition is sufficient here) determines an elementary and a local semi-simple class. These classes are investigated in Chapter IV.

Following Andrunakievic [2] we define local complementary radicals and determine the local complements of the radicals which are discussed.

Finally we are able to apply some of our results in order to classify all classes which are both semi-simple and radical classes.

CHAPTER I

PRELIMINARIES

1.1 RADICAL THEORY:

In this thesis we shall use the following notational conveniences;

(1) Let R be a ring:

(i) If $S \subseteq R$, $\langle S \rangle$ = the subring of R generated by the elements of S .

(ii) If $x_1, \dots, x_N \subseteq R$, $\langle x_1, \dots, x_N \rangle =$ $\langle \{x_1, \dots, x_N\} \rangle$.

(iii) If $S \subseteq R$, $(S)_R$ = ideal of R generated by the elements of S .

(iv) If $x_1, \dots, x_N \in R$; $(x_1, \dots, x_N)_R =$ $(\{x_1, \dots, x_N\})_R$.

(2) We shall write $I \triangleleft R$ for "I is an ideal of R."

(3) Classes of rings will usually be denoted by script letters and all classes of rings are assumed to be non-empty.

(4) If $\mathcal{R}$ and $\mathcal{H}$ are two classes of rings we shall write $\mathcal{H} \leq \mathcal{R}$ for "$\mathcal{H}$ is contained in $\mathcal{R}$".

(5) Two classes $\mathcal{R}$ and $\mathcal{H}$ are <u>unrelated</u> if neither $\mathcal{R} \leq \mathcal{H}$ nor $\mathcal{H} \leq \mathcal{R}$.

Let $\mathcal{R}$ be a class of rings. We list several conditions which $\mathcal{R}$ may satisfy:

(A) If $R \in \mathcal{R}$ and R' is a homomorphic image of R then $R' \in \mathcal{R}$.

(B) For any ring R there exists $\mathcal{R}(R) \triangleleft R$ such that $\mathcal{R}(R) \in \mathcal{R}$ and if $J \triangleleft R$ and $J \in \mathcal{R}$ then $J \subseteq \mathcal{R}(R)$.

(D) If every non-zero homomorphic image of a ring R contains a non-zero ideal in $\mathcal{R}$, then $R \in \mathcal{R}$.

(E) Every non-zero ideal of a ring in $\mathcal{R}$ can be homomorphically mapped onto a non-zero ring in $\mathcal{R}$.

(F) If every non-zero ideal of R can be homomorphically mapped onto a non-zero ring in $\mathcal{R}$ then $R \in \mathcal{R}$.

And if $\mathcal{R}$ satisfies (B), it may also satisfy:

(C) For any ring R, $\mathcal{R}(R/\mathcal{R}(R)) = (0)$.

1.1.1 <u>DEFINITION</u>:

(i) If $\mathcal{R}$ is any class of rings, and I is an ideal of a ring R such that $I \in \mathcal{R}$, then I is a <u>$\mathcal{R}$-ideal</u> of R.

(ii) A class of rings $\mathcal{R}$ is a <u>radical class</u> if and only if $\mathcal{R}$ satisfies conditions (A), (B) and (C).

(iii) If $\mathcal{R}$ is a radical class then R is <u>$\mathcal{R}$ semi-simple</u> (<u>$\mathcal{R}$ s.s.</u>) if and only if $\mathcal{R}(R) = (0)$. A class of rings $\mathcal{G}$ is a <u>semi-simple class</u> if and only if $\mathcal{G} =$ the class of all $\mathcal{H}$ s.s. rings for some radical class $\mathcal{H}$.

(iv) If $\mathcal{M}$ is a class of rings satisfying (E), $\mathcal{U}_{\mathcal{M}} =$ the class of all rings R which cannot be homomorphically mapped onto a non-zero ring in $\mathcal{M}$.

(v) A class of rings $\mathcal{R}$ is <u>hereditary</u> if whenever $I \triangleleft R \in \mathcal{R}$, $I \in \mathcal{R}$.

(vi) A class of rings $\mathcal{M}$ is a <u>special class</u> of rings if and only if:

(a) If $R \in \mathcal{M}$ then R is a prime ring.

(b) $\mathcal{M}$ is hereditary.

(c) If $R \in \mathcal{M}$ and R is an ideal of a ring K , then $K/(0:R) \in \mathcal{M}$ where $(0:R) = \{x \in K : xR = Rx = (0)\}$.

The above definitions and the following theorems can be found in <u>Rings and Radicals</u> by N. J. Divinsky [7].

1.1.2 <u>THEOREM</u>:

A class of rings $\mathcal{R}$ is a radical class if and only if $\mathcal{R}$ satisfies (A) and (D).

1.1.3 <u>THEOREM</u>:

If $\mathcal{M}$ is a class of rings which satisfies (E) then $\mathcal{U}_{\mathcal{M}}$ is a radical class. When $\mathcal{U}_{\mathcal{M}}$ is a radical class we will refer to $\mathcal{U}_{\mathcal{M}}$ as the <u>upper radical class</u> determined by $\mathcal{M}$.

1.1.4 <u>THEOREM</u>:

If $\mathcal{H}$ is a radical class then the class of all

$\mathcal{H}$ s.s. rings satisfies conditions (E) and (F). Conversely, if $\mathcal{M}$ is a class of rings satisfying conditions (E) and (F) then $\mathcal{M}$ = the class of all $\mathcal{U}_{\mathcal{M}}$ s.s. rings. Thus, G is a semi-simple class if and only if G satisfies conditions (E) and (F).

1.1.5 DEFINITION:

$\mathcal{H}$ is a special radical class if and only if $\mathcal{H} = \mathcal{U}_{\mathcal{M}}$ for some special class of rings $\mathcal{M}$.

1.1.6 THEOREM:

If $\mathcal{H}$ is a hereditary radical class and no nil-potent rings are in $\mathcal{H}$ than $\mathcal{H}$ is a special radical class if and only if $\mathcal{H} = \mathcal{U}_{\mathcal{M}}$ where $\mathcal{M}$ = the class of prime $\mathcal{H}$ s.s. rings.

1.1.7 THE LOWER RADICAL CONSTRUCTION.

If $\mathcal{H}$ is any class of rings, we define:

$\mathcal{H}_1$ = the class of all rings which are homomorphic images of rings in $\mathcal{H}$.

And for any ordinal number $\beta \geq 2$, if β is not a limit ordinal:

$\mathcal{H}_\beta$ = the class of all rings R such that every non-zero homomorphic image of R contains a non-zero ideal which is in $\mathcal{H}_{\beta-1}$.

If β is a limit ordinal:

$\mathcal{H}_\beta = \cup\{\mathcal{H}_\gamma : \gamma < \beta\}$.

Let $\overline{\mathcal{H}} = \cup\{\mathcal{H}_\beta : \beta$ is an ordinal number$\}$. Then $\overline{\mathcal{H}}$ is a radical class. We will refer to $\overline{\mathcal{H}}$ as the lower radical class determined by the class $\mathcal{H}$.

1.1.8 DEFINITION:

(i) Let R be a ring, I a subring of R and N a positive integer; I is an accessible subring of degree N of R if and only if there exists $I_1, \ldots, I_N \subseteq R$ such that $I = I_1 \lhd I_2 \lhd \ldots \lhd I_N \Delta R$.

(ii) A subring I of a ring R is an accessible subring of R if and only if I is an accessible subring of degree N of R for some positive integer N .

In order to establish the notation we list the following radical classes all but the first of which are discussed in Divinsky [7].

FF = the upper radical class determined by the class of all finite fields.

F = the upper radical class determined by the class of all fields.

$\mathcal{J}$ = the upper radical class determined by the class of full matrix rings over division rings.

$\mathcal{S}$ = the upper radical class determined by the class of all simple rings with unity.

$\mathcal{U}$ = the upper radical class determined by the class of all simple non-trivial rings.

J = the upper radical class determined by the class of all primitive rings.

β_φ = the upper radical class determined by the class of all subdirectly irreducible rings with idempotent hearts.

$\mathcal{N}_g$ = the upper radical class determined by the class of all rings without proper divisors of zero.

$\mathcal{N}$ = the class of all nil rings.

$\mathcal{L}$ = the class of all locally nilpotent rings.

β = the lower radical class determined by the class of all nilpotent rings.

$\mathfrak{D}$ = the lower radical class determined by the class of all nilpotent rings N such that $N = \mathcal{N}(R)$ for some ring R with D.C.C. on left ideals.

$\mathcal{J}$ = the lower radical class determined by the class of all zero simple rings.

1.2 RINGS WITHOUT NILPOTENT ELEMENTS.

Our purpose in this section is to establish:

1.2.1 THEOREM:

A ring R without nilpotent elements is isomorphic to a subdirect sum of rings without proper divisors of zero.

It will be convenient to first prove:

1.2.2 LEMMA:

If R has no nilpotent elements and $0 \neq x \in R$ then (i) $x_r = \{y \in R : xy = 0\} \lhd R$ and

$x_r = x_\ell = \{y \in R : yx = 0\}$,

(ii) $x \notin x_\ell$,

(iii) if $r \in R$ and $rx \in x_\ell$ then $r \in x_\ell$,

(iv) the factor ring R/x_ℓ has no nilpotent elements.

Proof:

Let R be a ring with no nilpotent elements and $0 \neq x \in R$. If $a \in R$ and $ax = 0$ then $(xa)^2 = 0$ so $xa = 0$. Similarily if $xa = 0$ then $ax = 0$. this establishes (i). Since $x^2 \neq 0$, (ii) is clear. If $a,b \in R$ and $ab^2 = 0$ then $(bab)^2 = 0$ so $bab = 0$, but then $(ab)^2 = 0$ so $ab = 0$. From this (iii) and (iv) follow immediately.

Q.E.D.

To prove the theorem it is sufficient to find, for each non-zero $x \in R$, an ideal $I(x)$ of R for which $R/I(x)$ has no proper divisors of zero and $x \notin I(x)$. Let $Z(x) = \{I \lhd R : x \notin I$, if $rx \in I$ then $r \in I$, and R/I has no nilpotent elements$\}$. By 1.2.2 $x_\ell \in Z(x)$ so $Z(x) \neq \emptyset$ and it is clear that the union of an ascending chain in $Z(x)$ is also in $Z(x)$. Thus we may choose, by Zorn's Lemma, $I(x)$ maximal in $Z(x)$.

If $a \in R$ and $a \notin I(x)$ let

$J = \{y \in R : ay \in I(x)\} \supseteq I(x)$. Then $J/I(x) = (a + I(x))_r$ in $R/I(x)$ and by 1.2.2(i) $(a + I(x))_\ell = (a + I(x))_r$ in $R/I(x)$. Since $a \notin I(x)$, $ax \notin I(x)$ so $x \notin J$. If $rx \in J$ then $arx \in I(x)$ so $ar \in I(x)$, hence $r \in J$. Finally by 1.2.2(iv) $R/J \cong R/I(x)/J/I(x)$ has no nilpotent elements, so $J \in Z(x)$. Hence $J = I(x)$ so $R/I(x)$ has no proper divisors of zero.

Q.E.D.

This result has also been proven by Andrunakievic and Rjabuhin [4] using an argument involving m-systems.

In terms of the radical class $\mathcal{N}_g$ of Andrunakievic [3] and Thierrin [14] we can restate 1.2.1 as follows: A ring R is $\mathcal{N}_g$ semi-simple if and only if R has no nilpotent elements.

CHAPTER II

K-CLASSES AND GENERALIZED K-CLASSES

2.1 K-CLASSES:

We begin with the following definitions.

2.1.1 DEFINITION:

A class of rings $\mathfrak{R}$ is strongly hereditary if and only if all subrings of rings in $\mathfrak{R}$ are in $\mathfrak{R}$.

2.1.2 DEFINITION:

For any cardinal number K,

(i) A subring S of a ring R is a K-subring of R if and only if there is a set $A \subseteq S$ such that $\langle A \rangle = S$ and the cardinality of A is $\lneq K$.
A ring R is a K-ring if R is a K-subring of R.

(ii) For any class of rings $\mathfrak{R}$, $\mathfrak{R}(K)$ is the class of all rings R such that every K-subring of R is in $\mathfrak{R}$.

(iii) A class of rings $\mathfrak{J}$ is a K-class if and only if there is a class of rings $\mathfrak{R}$ such that $\mathfrak{J} = \mathfrak{R}(K)$.

Some immediate consequences of these definitions are the following:

2.1.3 PROPOSITION:

Let $\mathfrak{H}$ and $\mathfrak{R}$ be classes of rings and K and Γ be cardinal numbers.

(i) $\mathfrak{H}(K)$ is strongly hereditary and $\mathfrak{H}(K) \leq \mathfrak{H}(K)(\Gamma)$.

(ii) If $\mathfrak{H} \leq \mathfrak{R}$ then $\mathfrak{H}(K) \leq \mathfrak{R}(K)$.

(iii) If $K \leq \Gamma$ then $\mathfrak{H}(\Gamma) \leq \mathfrak{H}(K) = \mathfrak{H}(K)(\Gamma)$.

(iv) $\mathfrak{H}$ is a K-class if and only if $\mathfrak{H} = \mathfrak{H}(K)$.

(v) If subdirect sums of rings in $\mathfrak{H}$ are in $\mathfrak{H}$ then subdirect sums of rings in $\mathfrak{H}(K)$ are in $\mathfrak{H}(K)$.

(vi) If $\mathfrak{H}$ satisfies condition (A), so does $\mathfrak{H}(K)$.

<u>Proof</u>:

(i) If S is a subring of a ring R then subrings of S are subrings of R . So if S is a subring of R and $R \in \mathfrak{H}(K)$, $S \in \mathfrak{H}(K)$; and in particular, all Γ-subrings of R are in $\mathfrak{H}(K)$ so $R \in \mathfrak{H}(K)(\Gamma)$.

(ii) Assume that $\mathfrak{H} \leq \mathfrak{R}$ and $R \in \mathfrak{H}(K)$. If S is a K-subring of R then $S \in \mathfrak{H} \leq \mathfrak{R}$ so $S \in \mathfrak{R}$. Hence, $R \in \mathfrak{R}(K)$.

(iii) Assume that $K \leq \Gamma$. If $R \in \mathfrak{H}(\Gamma)$ and S is a K-subring of R then S is also a Γ-subring of R since $K \leq \Gamma$. Thus $S \in \mathfrak{H}$ so $R \in \mathfrak{H}(K)$.

Let $R \in \mathfrak{H}(K)(\Gamma)$ and S be a K-subring of R . As above, S is a Γ-subring of R so $S \in \mathfrak{H}(K)$. Then $S \in \mathfrak{H}$ since S is a K-ring. Therefore, $\mathfrak{H}(K)(\Gamma) \leq \mathfrak{H}(K)$ and hence by (i) $\mathfrak{H}(K)(\Gamma) = \mathfrak{H}(K)$.

(iv) Assume that $\mathfrak{H}$ is a K-class. Then there is a class of rings $\mathfrak{R}$ such that $\mathfrak{H} = \mathfrak{R}(K)$. Therefore, $\mathfrak{H}(K) = \mathfrak{R}(K)(K)$. However, by (iii) $\mathfrak{R}(K)(K) = \mathfrak{R}(K) = \mathfrak{H}$, so $\aleph = \aleph(K)$.

(v) Suppose R is a ring with ideals $I_\alpha : \alpha \in \Lambda$ such that $\cap\{I_\alpha : \alpha \in \Lambda\} = (0)$ and $R/I_\alpha \in \aleph(K)$ for all $\alpha \in \Lambda$.

Let S be a K-subring of R . Then $\frac{S}{S\cap I_\alpha} \cong \frac{S+I_\alpha}{I_\alpha}$ is a K-subring of R/I_α , hence $\frac{S}{S\cap I_\alpha} \in \aleph$. Moreover, $\cap\{S \cap I_\alpha : \alpha \in \Lambda\} = (0)$ so by our assumption on $\aleph$, $S \in \aleph$. Therefore, $R \in \aleph(K)$.

(vi) Assume that $R \in \aleph(K)$ and the R' is a homomorphic image of R . Let S' be a K-subring of R' . Then there is a K-subring S or R such that S' is a homomorphic image of S . (If S' is generated by the cosets $\{x'_\alpha\}$ determined by $\{x_\alpha\}$ let S be the subring generated by $\{x_\alpha\}$.) Since $R \in \aleph(K)$, $S \in \aleph$ and since $\aleph$ satisfies (A), $S' \in \aleph$. Therefore, $R' \in \aleph(K)$.

Q.E.D.

In Proposition 2.1.3 (v) - (vi) we see that some conditions on $\aleph$ are inherited by $\aleph(K)$. Unfortunately, it is not true that $\aleph(K)$ must be a radical class whenever $\aleph$ is a radical class. The next theorem shows that this situation can not occur when $K > \aleph_0$.

2.1.4 <u>THEOREM</u>:

If K is a cardinal number $\gneq \aleph_0$ and $\aleph$ is a radical class then $\aleph(K)$ is a radical class.

<u>Proof</u>:

Assume that $K \gneq \aleph_0$ and $\aleph$ is a radical class.

First we show that the cardinality of K-subrings is $\lneq K$. If S is a K-subring of a ring R there is a set $A \subseteq S$ such that $\langle A \rangle = S$ and the cardinality of $A = \Gamma \lneq K$. Thus the cardinality of $S \leq \aleph_0 \cdot \Gamma \lneq K$.

Now suppose that A is a ring, $B \triangleleft A$ and that both A and A/B are in $\aleph(K)$. Let A' be a K-subring of A. The cardinality of $A' \cap B \leq$ the cardinality of $A' \lneq K$, so $A' \cap B$ is a K-subring of B. Since $B \in \aleph(K)$, $A' \cap B \in \aleph$; and since $\frac{A' + B}{B}$ is a K-subring of $A/B \in \aleph(K)$, $\frac{A}{A' \cap B} \cong \frac{A' + B}{B} \in \aleph$. Now $A' \in \aleph$ because $\aleph$ is a radical class. So we conclude that if A and A/B are in $\aleph(K)$ then $A \in \aleph(K)$. (*)

If I and J are $\aleph(K)$-ideals of a ring R then $\frac{I + J}{J} \cong \frac{I}{I \cap J} \in \aleph(K)$ by 2.1.3 (vi); and so by (*), $I + J \in \aleph(K)$. It follows that finite sums of $\aleph(K)$ - ideals are $\aleph(K)$ - ideals. (**)

As we have already noticed, $\aleph(K)$ satisfies condition (A) by 1.1.3 (vi).

Now we shall show that $\aleph(K)$ satisfies condition (B). Let R be any ring and let I = the sum of all $\aleph(K)$-ideals of R. If S is a K-subring of I and $x \in S$ then x is in a finite sum of $\aleph(K)$ - ideals of R, but by (**) this sum of ideals is an $\aleph(K)$ - ideal of R, so $S =$ $\Sigma\{S \cap J : J$ is an $\aleph(K)$ - ideal of $R\}$. Now the cardinality of $S \cap J \leq$ the cardinality of $S \lneq K$ so $S \cap J$ is a K-subring of J. Thus $S \cap J \in \aleph$, so, since $\aleph$ is a radical property and S is the sum of $\aleph$ - ideals, $S \in \aleph$. Therefore, I is $\aleph(K)$ and (B) is established.

The class $\aleph(K)$ satisfies (C) because of (*) so the proof is complete.

Q.E.D.

2.2 <u>LOCAL CLASSES</u>.

This section deals with general properties of $\aleph_0$-classes. Henceforth, $\aleph_0$-classes will be referred to as local classes and we shall write $\aleph^*$ for $\aleph(\aleph_0)$. A subring S of a ring R is $\aleph_0$-generated if and only if S is finitely generated as a ring.

2.2.1 <u>PROPOSITION</u>:

Let $\aleph$ be a class of rings. $\aleph^*$ is a radical class if and only if $\aleph^*$ satisfies condition (A) and for all rings R, if $I \triangleleft R$ such that $R/I \in \aleph^*$ and $I \in \aleph^*$ then $R \in \aleph^*$.

Proof:

One way is obvious since the two conditions hold for all radical classes.

Conversely, assume that the two conditions are satisfied by $\mathcal{H}^*$. Let R be a ring and let I be the sum of all $\mathcal{H}^*$-ideals of R . If S is a finitely generated subring of I then S is a subring of a finite sum of $\mathcal{H}^*$-ideals. Just as in 2.1.4 our conditions on $\mathcal{H}^*$ imply that finite sums of $\mathcal{H}^*$-ideals are $\mathcal{H}^*$-ideals. Hence $S \in \mathcal{H}$, so $I \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ satisfies condition (B). Again as in 2.1.4, $\mathcal{H}^*$ satisfies condition (C); so $\mathcal{H}^*$ is a radical class.

Q.E.D.

The following theorem provides a sufficient condition for concluding that certain classes are radical classes. This condition and the one given in 2.2.7 will be useful when we consider specific local classes in Chapter V.

2.2.2 THEOREM:

If $\mathcal{H}$ is a class of rings satisfying:

(i) $\mathcal{H}$ satisfies condition (A) .

(ii) If A is a ring, $B \triangleleft A$ and both B and A/B are in $\mathcal{H}$ then $A \in \mathcal{H}$.

(iii) $\mathcal{H}^* \leq \mathcal{H}$.

Then $\mathcal{H}^*$ is a radical class.

Proof:

We shall show that the conditions of 2.2.1 are satisfied.

Since $\mathcal{H}$ satisfies (A), by 2.1.3 (vi), $\mathcal{H}^*$ satisfies (A).

Suppose that A is a ring and $B \triangleleft A$ such that $A/B \in \mathcal{H}^*$ and $B \in \mathcal{H}^*$. Let A' be a finitely generated subring of A. Now $\frac{A' + B}{B} \cong \frac{A'}{A' \cap B} \in \mathcal{H}$ since $A/B \in \mathcal{H}^*$. Since $B \in \mathcal{H}^*$ and $\mathcal{H}^*$ is strongly hereditary, $A' \cap B \in \mathcal{H}^*$, so by (iii) $A' \cap B \in \mathcal{H}$. Therefore by (ii) $A' \in \mathcal{H}$ so $A \in \mathcal{H}^*$.

Q.E.D.

2.2.3 COROLLARY:

In the theorem, condition (iii) can be replaced by the condition that the union of a countable increasing sequence of finitely generated rings in $\mathcal{H}^*$ is in $\mathcal{H}^*$.

Proof:

In the proof of 2.2.2 condition (iii) was needed to insure that $A' \cap B \in \mathcal{H}$. Since $A' \cap B \subseteq A'$ and A' is countable it is clear that $A' \cap B$ is the union of a countable increasing sequence of finitely generated subrings. These subrings are $\mathcal{H}^*$ since $\mathcal{H}^*$ is strongly hereditary. Thus the proof of the corollary follows immediately.

Q.E.D.

2.2.4 <u>COROLLARY</u>:

If $\aleph$ is a radical class then condition (iii) is equivalent to the condition that for all rings A , if $(0) \neq A \in \aleph^*$ then $\aleph(A) \neq (0)$.

<u>Proof</u>:

Clearly condition (iii) implies that if A is a ring and $(0) \neq A \in \aleph^*$ then $\aleph(A) = A \neq (0)$.

Conversely, assume that for all non-zero rings A, if $A \in \aleph^*$ then $\aleph(A) \neq (0)$. Let $A \in \aleph^*$. Then $A/\aleph(A) \in \aleph^*$ since $\aleph^*$ satisfies condition (A). Thus $A/\aleph(A)$ must be (0) or else it would have a non-zero $\aleph$-ideal. Therefore $A \in \aleph$.

Q.E.D.

If we assume conditions (i) and (ii) of 2.2.2 the problem of showing that $\aleph^*$ is a radical class reduces to showing that if A' is a finitely generated ring and $B' \lhd A'$ such that $A'/B' \in \aleph^*$ and $B' \in \aleph^*$ then $B' \in \aleph$. In 2.2.2 - 2.2.4 we have accomplished this by requiring that $\aleph^* \leq \aleph$. Another possibility is to show that B' must be finitely generated as a ring (and hence in $\aleph$). We now turn our attention in this direction.

2.2.5 <u>DEFINITION</u>:

$\mathfrak{F}.\mathfrak{D}.$ is the class of all rings R which contain a finite set of elements $\{x_1, \ldots, x_N\}$ such that for all $y \in R$, $y = \sum_{i=1}^{N} a_i x_i$ where the a_i are integers depending on y.

The proof of the following lemma is based on a proof given by Jacobson (page 241)[11].

2.2.6 LEMMA:

If A is a finitely generated ring and $B \lhd A$ such that $A/B \in \mathfrak{F}.\mathfrak{D}$. then B is a finitely generated subring of A.

Proof:

Choose $x_1, \ldots, x_N$ in A such that for all $\overline{y} \in A/B$ there are integers $\alpha_1, \ldots, \alpha_N$ for which $\overline{y} = \alpha_1\overline{x}_1 + \ldots + \alpha_N\overline{x}_N$ and such that $\{x_1, \ldots, x_N\}$ contains a set of generators of A. Select integers γ_{ijK} and elements $b_{ij} \in B$ such that $x_i x_j = \sum_{K=1}^{N} \gamma_{ijK}\, x_K + b_{ij}$. Let $\overline{B}$ be the subring of B generated by the finite set $Y = \{b_{ij}, x_K b_{ij}, b_{ij} x_L, x_K b_{ij} x_L\}$ where all the subscripts vary from 1 to N.

First we shall show that if $b \in \overline{B}$ then $x_M b \in \overline{B}$ for $M = 1, \ldots, N$. Clearly it suffices to consider the cases in which b is a generator of $\overline{B}$. If $b = b_{ij} x_L$ or $b = b_{ij}$ the result is obvious. Suppose $b = x_L b_{ij}$. Then

$x_M b = (x_M x_L) b_{ij} = (\sum_{K=1}^{N} \gamma_{M,L,K}\, x_K + b_{ML}) b_{ij} \in \overline{B}$. And if $b = x_L b_{ij} x_H$ then $x_M b = (x_M x_L)(b_{ij} x_H) = (\sum_{K=1}^{N} \gamma_{M,L,K}\, x_K + b_{ML})(b_{ij} x_H) \in \overline{B}$. Similarily, if $b \in \overline{B}$ then $b x_M \in \overline{B}$ for $M = 1, \ldots, N$.

Next we shall show that if $a \in A$ there exist integers n_i such that $a = \sum_{i=1}^{N} n_i x_i + b$ where $b \in \overline{B}$. Since $\{x_1, \ldots, x_N\}$ contains a set of generators of A it is sufficient to consider the case when $a = x_{i_1} \ldots x_{i_K}$. By our definition of $\overline{B}$ it is clear that this is true when $K \leq 2$. Let $M > 2$ and suppose that the result holds for all $K < M$. Now, if $a = x_{i_1} \ldots x_{i_M} = (x_{i_1} \ldots x_{i_{M-1}}) x_{i_M} = (\sum_{j=1}^{N} n_i x_i + b) x_{i_M}$ where $b \in \overline{B}$, then $a = \sum_{j=1}^{N} n_j x_j x_{i_M} + b x_{i_M} = \sum_{j=1}^{N} \sum_{K=1}^{N} n_j \gamma_{j i_M K}\, x_K + \sum_{j=1}^{N} b_{j i_M} + b x_{i_M}$ and this is of the required form since $b x_{i_M} \in \overline{B}$.

Now consider $C = \{ \sum_{i=1}^{N} n_i x_i$ which are in $B : n_i$ are integers$\}$. C is a subgroup of the finitely generated additive group A^+ . Since C is a subgroup of a finitely generated abelian group, C is finitely generated as an additive group. Let $\{c_1, \ldots, c_M\}$ be a set of generators for C .

We now show that $X = Y \cup \{c_1, \ldots, c_M\}$ generates

B as a subring of A. Since $X \subseteq B$, $\langle X \rangle \subseteq B$. Suppose $z \in B$. Then $z \in A$ so $z = \sum_{i=1}^{N} n_i x_i + b$ where $b \in \overline{B} \subseteq \langle X \rangle$ and the n_i are integers; hence $z - b \in C$. That is, $\sum_{i=1}^{N} n_i x_i \in C \subseteq \langle X \rangle$ and $b \in \overline{B} \subseteq \langle X \rangle$. Therefore, $B \subseteq \langle X \rangle$ so $B = \langle X \rangle$.

Q.E.D.

2.2.7 <u>THEOREM</u>:

If $\aleph$ is a class of rings such that:

(i) $\aleph$ satisfies condition (A).

(ii) For all rings A, if $B \triangleleft A$ such that both A/B and B are in $\aleph$ then $A \in \aleph$.

(iii) If A is a finitely generated ring and $A \in \aleph$ then $A \in \mathfrak{F}.\mathfrak{D}$.

Then $\aleph^*$ is a radical class.

<u>Proof</u>:

We shall show that the conditions of 2.2.1 are satisfied.

By 2.1.3 (vi), $\aleph^*$ satisfies condition (A).

Suppose that $B \triangleleft A$ and both A/B and B are in $\aleph^*$. Let A' be a finitely generated subring of A. Then $\frac{A' + B}{B} \cong \frac{A'}{A' \cap B} \in \aleph$ and by (iii) $\frac{A'}{A' \cap B} \in \mathfrak{F}.\mathfrak{D}$. Lemma 2.2.6 implies that $A' \cap B$ is finitely generated as a subring of B, so since $B \in \mathcal{H}^*$, $A' \cap B \in \mathcal{H}$. Now by (ii) $A' \in \mathcal{H}$ so A is $\mathcal{H}^*$. This completes the proof.

Q.E.D.

2.2.8 <u>COROLLARY</u>:

If $\mathcal{H}$ is a radical class and $\mathcal{H}^* \leq \mathcal{F}.\mathcal{D}.$ then $\mathcal{H}^*$ is a radical class.

The next theorem provides sufficient conditions for $(\mathcal{U}_{\mathcal{M}})^*$ to be a radical class.

2.2.9 <u>THEOREM</u>:

If $\mathcal{M}$ is a class of rings satisfying condition (E) and $\mathcal{M}$ also satisfies the condition that if R is a finitely generated ring in $\mathcal{M}$ then every non-zero homomorphic image of R can be homomorphically mapped onto a non-zero ring in $\mathcal{M}$, then $(\mathcal{U}_{\mathcal{M}})^*$ is a radical class.

<u>Proof</u>:

Clearly $(\mathcal{U}_{\mathcal{M}})^*$ satisfies condition (A).

Suppose that B is an ideal of a ring A and that both A and A/B are in $(\mathcal{U}_{\mathcal{M}})^*$. Let A' be a finitely generated subring of A. If $A' \notin \mathcal{U}_{\mathcal{M}}$ then there is an ideal I of A' such that $A'/I \neq (0)$ and $A'/I \in \mathcal{M}$. We shall consider two possible cases and show that they both lead to a contradiction.

<u>Case 1</u>: $A' \cap B + I = A'$.

In this case $A'/I = \frac{A' \cap B + I}{I} \cong \frac{A' \cap B}{A' \cap B \cap I}$. Since $(\mathcal{U}_{\mathcal{M}})^*$ is strongly hereditary and $A \in (\mathcal{U}_{\mathcal{M}})^*$, $A' \cap B \in (\mathcal{U}_{\mathcal{M}})^*$. Then since $(\mathcal{U}_{\mathcal{M}})^*$ satisfies (A), the factor ring $\frac{A' \cap B}{A' \cap B \cap I} \in (\mathcal{U}_{\mathcal{M}})^*$. Thus $A'/I \in \mathcal{U}_{\mathcal{M}}$ since A'/I is finitely generated. This is a contradiction.

<u>Case 2</u>: $A' \cap B + I \neq A'$.

In this case $\frac{A' + B}{B} \cong \frac{A'}{A' \cap B}$ can be homomorphically mapped onto the non-zero ring $\frac{A'}{A' \cap B + I}$. Since $\frac{A'}{A' \cap B + I}$ is also a homomorphic image of $A'/I \in \mathcal{M}$; by assumption, $\frac{A'}{A' \cap B + I}$ can be homomorphically mapped onto a non-zero ring in $\mathcal{M}$. But this is a contradiction since $\frac{A'}{A' \cap B} \in \mathcal{U}_{\mathcal{M}}$.

Since both cases lead to a contradiction we conclude that $A' \in \mathcal{U}_{\mathcal{M}}$. Therefore, $A \in (\mathcal{U}_{\mathcal{M}})^*$ and so $(\mathcal{U}_{\mathcal{M}})^*$ is a radical class by 2.2.1.

Q.E.D.

2.2.10 <u>COROLLARY</u>:

If $\mathcal{M}$ satisfies condition (E) and (A) then $(\mathcal{U}_{\mathcal{M}})^*$ is a radical class. In particular, if $\mathcal{M}$ is a class of simple rings then $(\mathcal{U}_{\mathcal{M}})^*$ is a radical class.

2.2.11 COROLLARY:

If $\aleph$ is a radical class and no non-zero homomorphic image of an $\aleph$ s.s. ring is in $\aleph$ then $\aleph^* = (\mathcal{U}_{\aleph\text{ s.s.}})^*$ is a radical class.

The next lemma will be useful in showing that 2.1.4 is not true when $K = \aleph_0$. Recall that β, the Baer lower radical class, is the lower radical class determined by the class of all nilpotent rings.

2.2.12 LEMMA:

For any ring R, if $\beta(R) = (0)$ then,

(i) if $(0) \neq I$ is an accessible subring of R then there is an ideal J of R such that $(0) \neq J \subseteq I$ and $I^N \subseteq J$ for some positive integer N.

(ii) if $(0) \neq I$ is an accessible subring of R which is finitely generated as a subring of R then there exists a non-zero ideal J of R which is finitely generated as a subring of R and such that $J \subseteq I$.

Proof:

(i) We will prove this result by induction on N = the degree of I. If $N = 1$, I itself is an ideal of R. Suppose that the result holds for all subrings of degree less than N. Let $I = I_1 \triangleleft I_2 \triangleleft I_3 \triangleleft \dots \triangleleft I_N \triangleleft R$. Now $((I)_{I_3})^3 \subseteq I$ by Andrunakievic's Lemma (page 109, [7]). Moreover, since $\beta(R) = (0)$ and $((I)_{I_3})$ is an accessible subring of R it follows from Lemma 33 in Divinsky [7] that $((I)_{I_3})^3 \neq (0)$. Thus, by our induction hypothesis, there exists $(0) \neq J \triangleleft R$, $J \subseteq ((I)_{I_3})^3$ and $\left(((I)_{I_3})^3\right)^N \subseteq J$ for some integer N. Then $J \subseteq I$ and $I^{3N} \subseteq J$ so we are done.

(ii) Assume that $(0) \neq I$ is an accessible subring of R and that I is finitely generated as a subring of R. Then by (i) there exists $(0) \neq J \triangleleft R$ such that $J \subseteq I$ and $I^N \subseteq J$ for some integer N. Let $\{z_1, \ldots, z_K\}$ be a set of generators of I. Then $D = \{z_{i_1} \cdots z_{i_L} : L \leq N-1\}$ is a finite set such that if $\overline{w} \in I/J$ then there are integers α_i such that $\overline{w} = \Sigma\{\alpha_i \overline{d} : d \in D\}$. Therefore $I/J \in \mathfrak{F}.\mathfrak{D}.$ so by 2.2.6 J is a finitely generated subring of R. This completes the proof.

Q.E.D.

We now present an example of a radical class $\aleph$ such that $\aleph^*$ is not a radical class.

2.2.13 <u>EXAMPLE</u>:

Let K be the free ring generated by two non-commuting indeterminants x and y.

Define: $M = (2y)_K$

$$A = K/M$$

$$B = (2\overline{x})_A$$

$$\mathfrak{M} = \{I : I \text{ is an accessible subring of } A\}.$$

<u>LEMMA A</u>:

If $I \in \mathfrak{M}$, then I is commutative if and only if $I \subseteq B$.

<u>Proof</u>:

First we show that B is commutative. Since $2yx \in M$ and $2xy \in M$, $\overline{y} \cdot 2\overline{x} = 2\overline{yx} = \overline{0}$ in A and $2\overline{x} \cdot \overline{y} = \overline{0}$ in A. Hence, if $b \in B$, $b = \sum_{i=1}^{N} a_i \overline{x}^i$ where the a_i are even integers depending on b (notice that such a representation of b is unique). Clearly then, B is commutative.

To show the converse we begin with the following calculations, the purpose of which is to show that if $\overline{u} \notin B$ then $\overline{xu} \neq \overline{ux}$ or $\overline{yu} \neq \overline{uy}$. Suppose $0 \neq \overline{u} \in A$ and $\overline{u} \notin B$. Then $\overline{u} = \alpha_1 \overline{m}_1 + \dots + \alpha_K \overline{m}_K + b$ where $b \in B$, the α_i are odd integers and the m_i are distinct monomials (by a monomial we mean a product of x and y with <u>coefficient 1</u>). Moreover, we may assume that all the $\alpha_i = 1$. Since $\overline{x}b - b\overline{x} = \overline{0}$ and $\overline{y}b = 0 = b\overline{y}$ we have that $\overline{xu} - \overline{ux} =$ $\overline{x}(\overline{m}_1 + \dots + \overline{m}_K) - (\overline{m}_1 + \dots + \overline{m}_K)\overline{x}$ and $\overline{yu} - \overline{uy} =$ $\overline{y}(\overline{m}_1 + \dots + \overline{m}_K) - (\overline{m}_1 + \dots + \overline{m}_K)\overline{y}$. If $\overline{u} = \overline{x}$ or

$\bar{y}$ or $\bar{x} + \bar{y}$ then since $\overline{xy} \neq \overline{yx}$ we are finished. Otherwise we may assume (by rearranging the m_i if necessary) that $m_1 \in Kx$ or $m_1 \in Ky$. Suppose $m_1 \in K\bar{y}$. Now if $\overline{xu} - \overline{ux} = \bar{0}$ we have

$$xm_1 + \dots + xm_K = m_1x + \dots + m_Kx + \Delta \quad \text{where} \quad \Delta \in M \,. \quad (*)$$

Since $\Delta \in M = (2y)_K$, we can write $\Delta = \Sigma b_i n_i$ where the n_i are distinct monomials in K and the b_i are <u>even</u> integers. Now, $xm_1 + xm_j \neq 0$ for if so, $m_1 + m_j = 0$ which is not permitted. If $xm_1 = m_jx$ then $m_1 \in Kx$ which contradicts our assumption since K is free. But again, since K is free, xm_1 must be equal to one of the monomials on the right hand side of (*). Thus $xm_1 = n_j$ for some j. However, since an even number of the monomials n_j appear on the right hand side of (*), and since $n_j \neq n_i$ for $i \neq j$ we must have that $n_j = xm_i$ for $i \neq 1$ or $n_j = m_ix$ some i. Both of these situations lead us to one of the cases already considered, both of which lead to a contradiction. Therefore, $\overline{xu} - \overline{ux} \neq 0$. If we assume $m_1 \in K\bar{x}$, we similarily show that $\overline{yu} - \overline{uy} \neq \bar{0}$. Thus in any case either $\overline{xu} - \overline{ux} \neq \bar{0}$ or $\overline{yu} - \overline{uy} \neq \bar{0}$.

Now suppose I is an accessible subring of A and that I is commutative. If $I \not\subset B$ then by the above calculations there exists a $\bar{u} \in I$, $\bar{u} \notin B$, such that $\overline{xu} - \overline{ux} \neq \bar{0}$ or $\overline{yu} - \overline{uy} \neq \bar{0}$. Now $\overline{yu}^N \in I$ for some positive integer N so $\overline{yu}^N \cdot \overline{u} - \overline{u} \cdot \overline{yu}^N =$ $(\overline{yu} - \overline{uy})\overline{u}^N = \overline{0}$. Thus $(yu - uy)u^N \in 2K$. Since $\overline{u} \notin B$, $u \notin 2K$; therefore, $u^N \notin 2K$ so $yu - uy \in 2K$. Since $yu - uy$ is also in $(y)_K$ it follows that $yu - uy \in (2y)_K = M$. Since $\overline{yu} - \overline{uy} = \overline{0}$ we must have $\overline{xu} - \overline{ux} \neq \overline{0}$. As above it follows that $xu - ux \in 2K$. However, since x commutes with any monomial in u involves only x , $xu - ux \in (y)_K$. We conclude that $\overline{xu} - \overline{ux} = \overline{0}$. This is a contradiction so $I \subseteq B$.

Q.E.D.

<u>LEMMA B</u>:

No non-zero accessible subring of A which is contained in B is finitely generated as a subring of A .

<u>Proof</u>:

By Lemma 2.2.12(ii) it is sufficient to consider the case when $I \lhd A$ and $I \subseteq B$. Suppose $I \lhd A$, $I \subseteq B$ and $I = \langle b_1, \ldots, b_k \rangle \neq (0)$. We have seen that the b_i must be polynomials in $\overline{x}$ with even coefficients. Let $L = \max\{\text{degree } b_i : i = 1,\ldots,k\}$ and suppose the degree of $b_H = L$. Now, $b_H =$ $a_1\overline{x} + \ldots + a_L\overline{x}^L$ where the a_i are even integers. Let $w = \max\{n : 2^n \text{ divides } a_i \text{ all } i = 1, \ldots, L\}$. Define $b = x^{WL} b_H \in I$. The smallest power of x

which appears with a non-zero coefficient in b is $wL + 1$. Since $b \in \langle b_1, \ldots, b_K \rangle$, b must be a sum of products of the b_i's. Because the degree of $b_i \leq L$ for all i, each product which contributes to a non-zero coefficient of b must contain at least $w + 1$ terms and hence 2^{w+1} must divide all the coefficients of b. This is a contradiction because the coefficients of b are exactly the coefficients of b_H.

Hence I cannot be finitely generated.

Q.E.D.

LEMMA C:

No accessible non-zero subring of A contained in the ideal $(\overline{y})_A$ is finitely generated as a subring of A.

Proof:

We can apply Lemma 2.2.12(ii) again so we need only consider ideals of A.

Suppose that $I \lhd A$, $(0) \neq I \subseteq (\overline{y})_A$ and $I = \langle z_1, \ldots, z_K \rangle$ where each $z_i \neq \overline{0}$ and $z_i = \overline{m}_{i1} + \ldots + \overline{m}_{iK_i}$ for non-zero monomials m_{ij}. Consider all the m_{ij} and let d be the largest integer h such that some $m_{ij} \in x^h K$. It follows that if

$\overline{w} \in \langle z_1, \ldots, z_K \rangle$ then $\max\{h : w \in x^h K\} \leq d$. But if $\overline{w} \in I$ and $\overline{w} \neq 0$ then $\overline{x}^{d+1}\overline{w} \in I$ and $\overline{x}^{d+1}\overline{w} \neq \overline{0}$. This is a contradiction so I cannot be finitely generated as a subring of A.

Q.E.D.

We are now ready to prove:

<u>THEOREM</u>:

$\mathcal{U}_{\mathcal{M}}$ is a radical class but $(\mathcal{U}_{\mathcal{M}})^*$ is not a radical class.

<u>Proof</u>:

The class $\mathcal{M}$ satisfies property (E) so $\mathcal{U}_{\mathcal{M}}$ is a radical class and $R \in \mathcal{U}_{\mathcal{M}}$ if and only if R can not be homomorphically mapped onto a non-zero ring in $\mathcal{M}$.

In order to show that $(\mathcal{U}_{\mathcal{M}})^*$ is not a radical class we will show that $A/B \in (\mathcal{U}_{\mathcal{M}})^*$, $B \in (\mathcal{U}_{\mathcal{M}})^*$ but $A \notin (\mathcal{U}_{\mathcal{M}})^*$.

Since A is finitely generated and $A \in \mathcal{M}$ it is clear that $A \notin (\mathcal{U}_{\mathcal{M}})^*$.

To see that $A/B \in (\mathcal{U}_{\mathcal{M}})^*$, suppose that R' is a finitely generated subring of A/B. If $R' \notin \mathcal{U}_{\mathcal{M}}$ then R' can be homomorphically mapped onto $R'/L \cong I \in \mathcal{M}$ and $I \neq (0)$. Since A/B is a ring of characteristic 2, I must be of characteristic 2. Clearly $(y)_A$ contains all accessible subrings of A which are of characteristic 2 so $I \subseteq (y)_A$. Since I is a homomorphic image of R', I must be finitely generated as a ring. This contradicts Lemma C. Therefore, $R' \in \mathcal{U}_{\mathcal{M}}$ so $A/B \in (\mathcal{U}_{\mathcal{M}})^*$.

To see that $B \in (\mathcal{U}_{\mathcal{M}})^*$, suppose R' is a finitely generated subring of B. Then R' is commutative so if $R' \notin \mathcal{U}_{\mathcal{M}}$, it can be homomorphically mapped onto a non-zero $I \in \mathcal{M}$ and I must be commutative and finitely generated as a ring. By Lemma A, $I \subseteq B$ and by Lemma B this implies that I is not finitely generated as a ring. This is a contradiction, so $R' \in \mathcal{U}_{\mathcal{M}}$. Hence $B \in (\mathcal{U}_{\mathcal{M}})^*$.

This completes the proof.

Q.E.D.

Notice that since A is generated by only two elements, $A \notin \mathcal{U}_{\mathcal{M}}(K)$ for any cardinal K such that $2 \lneqq K \leq \aleph_0$. Thus, although $\mathcal{U}_{\mathcal{M}}$ is a radical class, $\mathcal{U}_{\mathcal{M}}(K)$ is not a radical class for any cardinal K such that $2 \lneqq K \leq \aleph_0$.

2.3 ELEMENTARY CLASSES.

The problems involved in dealing with classes $\aleph(K)$ where $2 < K < \aleph_0$ are similar to those involved in dealing with local classes. For this reason we will not deal specifically with such K-classes but will pass on to the properties of 2-classes. We shall write $\aleph'$ for $\aleph(2)$ and refer to 2-classes as elementary classes.

A subring is a 2-subring if and only if it is generated, as a ring, by one element. Such rings are all homomorphic images of the ring $\mathfrak{Z}[X]$ of all polynomials over the integers in one variable X and with zero constant coefficient. Hence all rings generated by one element are of the form $\mathfrak{Z}[X]/I$ where $I = (f_1(X), \ldots, f_K(X))$ for some finite set of elements $\{f_1(X), \ldots, f_K(X)\} \subseteq \mathfrak{Z}[X]$.

The proofs of the following three results are similar to the proofs of 2.2.1, 2.2.2 and 2.2.9 respectively.

2.3.1 PROPOSITION:

Let $\aleph$ be a class of rings. $\aleph'$ is a radical class if and only if $\aleph'$ satisfies condition (A) and for all rings R , if $I \triangleleft R$ such that $R/I \in \aleph'$ and $I \in \aleph'$ then $R \in \aleph'$.

2.3.2 THEOREM:

If $\aleph$ is a class of rings satisfying:

(i) $\aleph$ satisfies condition (A).

(ii) If A is a ring, $B \triangleleft A$ and both B and A/B are in $\aleph$ then $A \in \aleph$.

(iii) $\aleph' \leq \aleph$.

Then $\mathcal{H}'$ is a radical class.

2.3.3 THEOREM:

Let $\mathcal{M}$ be a class of rings satisfying condition (E). If all non-zero homomorphic images of rings in $\mathcal{M}$ which are generated by one element can be homomorphically mapped onto non-zero rings in $\mathcal{M}$, then $(\mathcal{U}_{\mathcal{M}})'$ is a radical class.

In [13] Rjabuhin discusses elementary radical classes. If A is a ring, $I \lhd A$ and $a \in A$ define $(I*a) = \{f(X) \in \mathcal{P}[X] : f(a) \in I\}$. Let $\mathfrak{R}$ be any set of ideals of $\mathcal{P}[X]$ satisfying:

(i) if $A \supseteq B \in \mathfrak{R}$ and $A \lhd \mathcal{P}[X]$ then $A \in \mathfrak{R}$.

(ii) if $B \in \mathfrak{R}$ and $f(X) \in \mathcal{P}[X]$ then $(B*f) \in \mathfrak{R}$.

(iii) if $A \supseteq B$, $A \in \mathfrak{R}$ and for all $f(X) \in A$, $(B*f) \in \mathfrak{R}$, then $B \in \mathfrak{R}$.

Rjabuhin calls such a set of ideals an r-set; and defines for any r-set $\mathfrak{R}$, the class of rings $\mathcal{H}(\mathfrak{R})$ to be all rings R such that $\{((0)*a) : a \in R\} \subseteq \mathfrak{R}$. He proves that if $\mathfrak{R}$ is an r-set then $\mathcal{H}(\mathfrak{R})$ is an elementary radical class (Rjabuhin calls such classes semi-strictly hereditary radicals) and that if $\mathcal{H}$ is any elementary radical class then there is an r-set $\mathfrak{R}$ such that $\mathcal{H} = \mathcal{H}(\mathfrak{R})$. We shall give an outline of the proof.

Suppose that $\mathfrak{R}$ is an r-set, then (i) implies that $\mathcal{H}(\mathcal{R})$ has property (A). Suppose $B \lhd A$ and both A/B and B are in $\mathcal{H}(\mathcal{R})$. Let $a \in A$, then $((0)*a) \subseteq (B*a) = ((\overline{0})*\overline{a}) \in \mathcal{R}$ where $\overline{a} = a + B$ in A/B. Now (iii) implies that $((0)*a) \in \mathcal{R}$ for if $f(X) \in ((\overline{0})*\overline{a})$, then $\big(((0)*a)*f\big) = ((0)*f(a)) \in \mathcal{R}$ since $f(a) \in B$. Thus $A \in \mathcal{H}(\mathcal{R})$. By the very definition of $\mathcal{H}(\mathcal{R})$, $\mathcal{H}(\mathcal{R})$ is an elementary class so by our Proposition 2.3.1, $\mathcal{H}(\mathcal{R})$ is a radical class. Notice that (ii) implies that if $I \in \mathcal{R}$ then $\mathcal{P}[X]/I \in \mathcal{H}(\mathcal{R})$.

Conversely, if $\mathcal{H}'$ is a radical class let $\mathcal{S} = \{I \lhd \mathcal{P}[X] : \mathcal{P}[X]/I \in \mathcal{H}'\}$. Since $\mathcal{H}'$ satisfies (A), $\mathcal{S}$ satisfies (i); and since $\mathcal{H}'$ is strongly hereditary, $\mathcal{S}$ satisfies (ii). Suppose $A \supseteq B$ and $A \in \mathcal{S}$ and for all $f(X) \in A$, $(B*f) \in \mathcal{R}$. This implies that $\dfrac{\mathcal{P}[X]}{A} \in \mathcal{H}'$ and $A/B \in \mathcal{H}'$ so since $\mathcal{H}'$ is a radical property, $\mathcal{P}[X]/B \in \mathcal{H}'$. Hence $B \in \mathcal{S}$ so $\mathcal{S}$ satisfies (iii). Therefore $\mathcal{S}$ is an r-set and clearly $\mathcal{H}' = \mathcal{H}(\mathcal{S})$.

2.3.4 <u>THEOREM</u>:

Let $\mathcal{H}'$ be an elementary class which is a radical class and which contains the Baer lower radical β. A ring R is $\mathcal{H}'$ s.s. if and only if R is isomorphic to a subdirect sum of prime $\mathcal{H}'$ s.s. rings. Hence $\mathcal{H}'$ is a special radical class.

Proof:

Subdirect sums of semi-simple rings are always semi-simple so we need only show that if R is $\aleph'$ s.s. then R is isomorphic to a subdirect sum of prime $\aleph'$s.s. rings. So it is sufficient to find, for each non-zero element x of an $\aleph'$ s.s. ring R, an ideal I of R such that $x \notin I$ and R/I is a prime $\aleph'$s.s. ring.

Let R be $\aleph'$s.s. and $0 \neq x \in R$. Then there is a $y \in (x)_R$ such that $\langle y \rangle \notin \aleph$ and hence $\langle y \rangle \notin \aleph'$. Let $Z = \{I \triangleleft R : \frac{\langle y \rangle + I}{I} \notin \aleph'\}$. Then $Z \neq \emptyset$ since $(0) \in Z$. Suppose $N_\alpha : \alpha \in \Lambda$ is an ascending chain of ideals in Z. Then $N_\alpha \cap \langle y \rangle : \alpha \in \Lambda$ is an ascending chain of ideals in $\langle y \rangle$. Since $\langle y \rangle$ has A.C.C. there is a $\gamma \in \Lambda$ such that $N_\alpha \cap \langle y \rangle \subseteq N_\gamma \cap \langle y \rangle$ for all $\alpha \in \Lambda$. Let $N = \cup\{N_\alpha : \alpha \in \Lambda\}$. Then $N \cap \langle y \rangle = N_\gamma \cap \langle y \rangle$ so $\frac{\langle y \rangle + N}{N} \cong \frac{\langle y \rangle}{\langle y \rangle \cap N} = \frac{\langle y \rangle}{\langle y \rangle \cap N_\gamma} \cong \frac{\langle y \rangle + N_\gamma}{N_\gamma} \notin \aleph'$. Therefore $N \in Z$. Thus we may choose, by Zorn's Lemma, I maximal in Z.

Since $y \notin I$ and $y \in (x)$, $x \notin I$.

To see that R/I is $\aleph'$s.s., suppose that $J \triangleleft R$ and $J \supsetneqq I$. Then $\frac{\langle y \rangle + J}{J} \cong \frac{\langle y \rangle}{\langle y \rangle \cap J} \in \aleph'$. If $\frac{\langle y \rangle \cap J}{\langle y \rangle \cap I} \in \aleph'$ then $\frac{\langle y \rangle}{\langle y \rangle \cap I} \in \aleph'$. But $\frac{\langle y \rangle}{\langle y \rangle \cap I} \notin \aleph'$ so $\frac{\langle y \rangle \cap J + I}{I} \cong \frac{\langle y \rangle \cap J}{\langle y \rangle \cap I} \notin \aleph'$. Thus $J/I \notin \aleph'$ since $\aleph'$ is strongly hereditary and $\frac{\langle y \rangle \cap J + I}{I} \subseteq \frac{J}{I}$. Therefore R/I is $\aleph'$s.s..

Next we shall show that R/I is a prime ring. If J_1 and J_2 are ideals of R which properly contain I then $J_1 \cap J_2 \supsetneqq I$. For if $J_1 \cap J_2 = I$ then $\frac{\langle y\rangle+I}{I}$ is a subdirect sum of $R_1 = \frac{\langle y\rangle+J_1}{J_1}$ and $R_2 = \frac{\langle y\rangle+J_2}{J_2}$ both of which are in $\aleph'$; hence, the (external) direct sum of R_1 and R_2 is in $\aleph'$ and since $\aleph'$ is strongly hereditary and the subdirect sum is a subring of the direct sum, $\frac{\langle y\rangle+I}{I} \in \aleph'$. Now if $J_1 \cdot J_2 \subseteq I$ then $(J_1 \cap J_2)^2 \subseteq I$. But then $\frac{J_1 \cap J_2}{I} \in \beta \leq \aleph'$ which contradicts our previous conclusion that R/I is $\aleph'$s.s. Therefore, $J_1 \cdot J_2 \nsubseteq I$ so R/I is a prime ring.

This completes the proof.

Q.E.D.

In Theorem 2.1.4 we proved that if $\aleph$ is a radical class and K is a cardinal number such that $K \gneqq \aleph_0$ then $\aleph(K)$ is a radical class. In 2.2.13 we presented an example of a radical class $\aleph$ such that $\aleph(K)$ was not a radical class for all cardinals K such that $2 \lneqq K \leq \aleph_0$.

The analogous question concerning elementary classes is unsolved. That is, we do not know whether or not $\aleph' = \aleph(2)$ must be a radical class whenever $\aleph$ is a radical class.

2.4 GENERALIZED K-CLASSES:

In the past three sections we have been concerned with the question, "When are K-classes radical classes?" The purpose of this section is to consider the question, "When are K-classes semi-simple classes?" We begin by describing a class of radicals which contains all radicals $\mathcal{U}_{\mathcal{H}}$ where $\mathcal{H}$ is a semi-simple K-class.

2.4.1 DEFINITION:

Let $\mathcal{H}$ be a class of rings and K a cardinal number which is ≥ 2 :

(i) $\mathcal{H}_{g(K)}$ is the class of all rings R such that every non-zero homomorphic image of R contains a non-zero K-subring in $\mathcal{H}$.

(ii) $\mathcal{J}$ is a generalized K-class if and only if $\mathcal{J} = \mathcal{H}_{g(K)}$ for some class of rings $\mathcal{H}$.

If $\mathcal{H}$ is a semi-simple class then there is some radical class $\mathcal{S}$ such that $\mathcal{H}$ is the class of all $\mathcal{S}$ semi-simple rings. The radical class $\mathcal{S}$ is exactly the upper radical class determined by $\mathcal{H}$; that is, $\mathcal{S} = \mathcal{U}_{\mathcal{H}}$. In 2.3.4 we shall prove that if $\mathcal{H}$ is also a K-class then $\mathcal{U}_{\mathcal{H}}$ is a generalized K-class.

For any class of rings $\mathcal{R}$ and any cardinal number $K \geq 2$ we may form the classes $\mathcal{R}(K)$ and $\mathcal{R}_{g(K)}$. The class $\mathcal{R}(K)$ is always a K-class but need not be a radical class (even when $\mathcal{R}$ is itself a radical class). The preceding three sections of this chapter have been largely concerned with conditions on $\mathcal{R}$ which imply that $\mathcal{R}(K)$ is a radical class.

As we shall prove below, $\mathcal{R}_{g(K)}$ is always a radical class. However the class of $\mathcal{R}_{g(K)}$ semi-simple rings need not be a K-class. Much of this section will be concerned with conditions on $\mathcal{R}$ which guarantee that the class of $\mathcal{R}_{g(K)}$ semi-simple rings is a K-class.

In the following proposition we list some basic properties of generalized K-classes and point out some relationships between K-classes and generalized K-classes. In 2.4.2(ix) we prove that if $\mathcal{H}$ is a generalized K-class then $\mathcal{H}(K')$ is a radical class for any cardinal $K' \geq 2$.

2.4.2 <u>PROPOSITION</u>:

Let $\mathcal{H}$ and $\mathcal{R}$ be classes of rings and K and Γ be cardinal number which are ≥ 2.

(i) $\mathcal{H}_{g(K)}$ is a radical class.

(ii) If $\mathcal{H} \leq \mathcal{R}$ then $\mathcal{H}_{g(K)} \leq \mathcal{R}_{g(K)}$.

(iii) If $K \leq \Gamma$ then $\mathcal{H}_{g(K)} \leq \mathcal{H}_{g(\Gamma)}$.

(iv) $(\mathcal{H}_{g(K)})_{g(\Gamma)} \leq \mathcal{H}_{g(K)}$.

(v) If $\mathcal{H} \leq \mathcal{R} \leq \mathcal{H}_{g(K)}$ then $\mathcal{R}_{g(K)} = \mathcal{H}_{g(K)}$.

(vi) $(\mathcal{H}_{g(K)})_{g(K)} = \mathcal{H}_{g(K)}$ if and only if $\mathcal{H}_{g(K)} = \mathcal{J}_{g(K)}$ for some class of rings $\mathcal{J}$ which satisfies (A).

(vii) $(\aleph(\Gamma))_{g(K)} = (\aleph(\Gamma))_{g(2)} \leq \aleph_{g(2)}$ and if $\aleph \leq \aleph(\Gamma)$, $(\aleph(\Gamma))_{g(K)} = \aleph_{g(2)}$.

(viii) $(\aleph_{g(K)})(\Gamma) \leq (\aleph_{g(K)})_{g(2)} \leq \aleph_{g(K)}$.

(ix) $(\aleph_{g(K)})(\Gamma) = ((\aleph_{g(K)})_{g(2)})(\Gamma)$ is a radical class.

Proof:

(i) From the definition it is clear that $\aleph_{g(K)}$ satisfies condition (A). We will show that $\aleph_{g(K)}$ also satisfies condition (D). Suppose that every non-zero homomorphic image of a ring R contains a non-zero $\aleph_{g(K)}$-ideal. Let R' be a non-zero homomorphic image of R . Then the $\aleph_{g(K)}$-ideal of R' contains a non-zero K-subring which is in $\aleph$. Of course, this subring is also a K-subring of R' . Therefore $R \in \aleph_{g(K)}$. Now, by 1.1.2, $\aleph_{g(K)}$ is a radical class.

(ii) Suppose that $\aleph \leq \mathcal{R}$ and $R \in \aleph_{g(K)}$. Let R' be a non-zero homomorphic image of R . Then R' contains a non-zero K-subring which is in $\aleph$ and hence in $\mathcal{R}$. Therefore $R \in \mathcal{R}_{g(K)}$.

(iii) Suppose that $K \leq \Gamma$ and $R \in \aleph_{g(K)}$. Let R' be a non-zero homomorphic image of R . Then R' contains a non-zero K-subring which is in $\aleph$. Since $K \leq \Gamma$, this subring is a Γ-subring so $R \in \aleph_{g(\Gamma)}$.

(iv) Suppose $R \in (\aleph_{g(K)})_{g(\Gamma)}$ and let R' be a non-zero homomorphic image of R . Then there is a non-zero Γ-subring $S \subseteq R'$ such that $S \in \aleph_{g(K)}$. Since S

is a non-zero homomorphic image of itself S contains a non-zero K-subring which is in $\mathfrak{M}$. Hence $R \in \mathfrak{M}_{g(K)}$.

(v) Suppose $\mathfrak{M} \leq \mathfrak{R} \leq \mathfrak{M}_{g(K)}$. By (ii), $\mathfrak{M}_{g(K)} \leq \mathfrak{R}_{g(K)} \leq (\mathfrak{M}_{g(K)})_{g(K)}$ and by (iv) $(\mathfrak{M}_{g(K)})_{g(K)} \leq \mathfrak{M}_{g(K)}$. Thus $\mathfrak{M}_{g(K)} \leq \mathfrak{R}_{g(K)} \leq \mathfrak{M}_{g(K)}$. And so $\mathfrak{R}_{g(K)} = \mathfrak{M}_{g(K)}$.

(vi) Assume that $(\mathfrak{M}_{g(K)})_{g(K)} = \mathfrak{M}_{g(K)}$. Then since $\mathfrak{M}_{g(K)}$ satisfies condition (A) we are finished. Conversely, assume that $\mathfrak{M}_{g(K)} = \mathfrak{T}_{g(K)}$ where $\mathfrak{T}$ satisfies condition (A). Suppose $R \in \mathfrak{M}_{g(K)}$ and let R' be a non-zero homomorphic image of R . Then there is a non-zero K-subring S of R' such that $S \in \mathfrak{T}$. Since $\mathfrak{T}$ satisfies condition (A) and S is a K-subring, $S \in \mathfrak{T}_{g(K)} = \mathfrak{M}_{g(K)}$. Therefore $R \in (\mathfrak{M}_{g(K)})_{g(K)}$. Therefore $\mathfrak{M}_{g(K)} \leq (\mathfrak{M}_{g(K)})_{g(K)}$ so by (iv), $\mathfrak{M}_{g(K)} = (\mathfrak{M}_{g(K)})_{g(K)}$.

(vii) Since $K \geq 2$, by (iii) we have that $(\mathfrak{M}(\Gamma))_{g(K)} \geq (\mathfrak{M}(\Gamma))_{g(2)}$. Suppose that $R \in (\mathfrak{M}(\Gamma))_{g(K)}$ and let R' be a non-zero homomorphic image of R . Then R' contains a non-zero K-subring S such that $S \in \mathfrak{M}(\Gamma)$. Hence every 2-subring of R' which is contained in S is in $\mathfrak{M}(\Gamma) \cap \mathfrak{M}$ so certainly $R \in (\mathfrak{M}(\Gamma))_{g(2)} \cap \mathfrak{M}_{g(2)}$.

If $\mathfrak{M} \leq \mathfrak{M}(\Gamma)$ then by (ii) $\mathfrak{M}_{g(2)} \leq (\mathfrak{M}(\Gamma))_{g(2)}$

so $(\mathcal{H}(\Gamma))_{g(K)} = \mathcal{H}_{g(2)}$.

(viii) Suppose that $R \in (\mathcal{H}_{g(K)})(\Gamma)$ and let R' be a non-zero homomorphic image of R . Let S' be any 2-subring of R' . Then S' is a homomorphic image of 2-subring S of R . Since $R \in (\mathcal{H}_{g(K)})(\Gamma)$, $S \in \mathcal{H}_{g(K)}$ so $S' \in \mathcal{H}_{g(K)}$. Therefore $R \in (\mathcal{H}_{g(K)})_{g(2)}$.

By (iv), $(\mathcal{H}_{g(K)})_{g(2)} \leq \mathcal{H}_{g(K)}$.

(ix) By (viii) $(\mathcal{H}_{g(K)})(\Gamma) \leq (\mathcal{H}_{g(K)})_{g(2)}$ so by 2.1.3 (iii) and (ii), $(\mathcal{H}_{g(K)})(\Gamma) = (\mathcal{H}_{g(K)})(\Gamma)(\Gamma) \leq ((\mathcal{H}_{g(K)})_{g(2)})(\Gamma)$. By (iv), $(\mathcal{H}_{g(K)})_{g(2)} \leq \mathcal{H}_{g(K)}$ so using 2.1.3(ii) again we conclude that $(\mathcal{H}_{g(K)})(\Gamma) = ((\mathcal{H}_{g(K)})_{g(2)})(\Gamma)$.

Since $\mathcal{H}_{g(K)}$ is a radical class, if $\Gamma \gneq \aleph_0$, then $(\mathcal{H}_{g(K)})(\Gamma)$ is a radical class by 2.1.4.

By (viii) we know that $(\mathcal{H}_{g(K)})(\Gamma) \leq \mathcal{H}_{g(K)}$ so when $\Gamma \leq \aleph_0$ arguments exactly paralleling 2.2.1 and 2.2.2 can be used to show that $(\mathcal{H}_{g(K)})(\Gamma)$ is a radical class.

Q.E.D.

In Proposition 2.4.3 below we shall prove that if $\mathcal{G}$ is a semi-simple K-class then $\mathcal{U}_{\mathcal{G}}$ is a generalized K-class. Unfortunately the converse is false since there are generalized K-classes $\mathcal{H}_{g(K)}$ for which the class of $\mathcal{H}_{g(K)}$

semi-simple rings is not a K-class. For example, let $K = 2$ and $\mathcal{H}$ be the class of all rings R such that the additive group R^+ is torsion free (that is, $R \in \mathcal{H}$ if and only if for all $x \in R$ if $hx = 0$ for some positive integer h then $x = 0$). Let Q = the ring of rational numbers. If $x \in Q$ then every non-zero ideal I of $\langle x \rangle$ can be homomorphically mapped to a non-zero ring of finite characteristic (that is, a ring R such that $nR = (0)$ for some non-zero integer n). Therefore, for all $x \in Q$, $\langle x \rangle$ is $\mathcal{H}_{g(2)}$ semi-simple. On the other hand $Q \in \mathcal{H}_{g(2)}$. So the class of $\mathcal{H}_{g(2)}$ semi-simple rings is not a 2-class.

2.4.3 PROPOSITION:

Let $K \geq 2$ be a cardinal number. If $\mathcal{G}$ is a semi-simple class and $\mathcal{G} = \mathcal{G}(K)$ then $\mathcal{U}_{\mathcal{G}} = \mathcal{R}_{g(K)} = \mathcal{H}_{g(K)}$ where $\mathcal{R}$ = the class of all rings which are not in $\mathcal{G}$. $\mathcal{H}$ = the class of all rings R such that $\mathcal{R}_{g(K)}(R) \neq (0)$. Moreover, the class $\mathcal{H}$ satisfies the following condition: "If $(0) \neq T \in \mathcal{H}$ and T is a K-ring then $\mathcal{H}_{g(K)}(T) \neq (0)$."

Proof:

We shall begin by showing that $\mathcal{U}_{\mathcal{G}} = \mathcal{R}_{g(K)}$.

Suppose that $R \in \mathcal{U}_{\mathcal{G}}$. Let R' be a non-zero homomorphic image of R. Then $R' \notin \mathcal{G} = \mathcal{G}(K)$ so there is a K-subring $T \neq (0)$ of R' such that $T \notin \mathcal{G}$. Therefore

$R \in \mathcal{R}_{g(K)}$.

Suppose $R \in \mathcal{R}_{g(K)}$. Let R' be a non-zero homomorphic image of R . Then there is a K-subring T of R' such that $T \in \mathcal{R}$. Then $T \notin \mathcal{G} = \mathcal{G}(K)$ so $R' \notin \mathcal{G}$. Therefore, no non-zero homomorphic image of R is in $\mathcal{G}$, so $R \in \mathcal{U}_{\mathcal{G}}$. Then $\mathcal{U}_{\mathcal{G}} = \mathcal{R}_{g(K)}$.

Now we shall prove that $\mathcal{R}_{g(K)} = \mathcal{H}_{g(K)}$.

Suppose $R \in \mathcal{R}_{g(K)}$. Let R' be a non-zero homomorphic image of R . Then there is a K-subring T of R' which is in $\mathcal{R}$, that is, T is not $\mathcal{U}_{\mathcal{G}}$s.s. Since $\mathcal{U}_{\mathcal{G}} = \mathcal{R}_{g(K)}$, T is not $\mathcal{R}_{g(K)}$s.s. So $T \in \mathcal{H}$. Therefore, $R \in \mathcal{H}_{g(K)}$.

Since $\mathcal{R}_{g(K)} = \mathcal{U}_{\mathcal{G}}$, $\mathcal{H} \leq \mathcal{R}$. Thus by 1.4.2(ii), $\mathcal{H}_{g(K)} \leq \mathcal{R}_{g(K)}$. Therefore $\mathcal{H}_{g(K)} = \mathcal{R}_{g(K)}$.

To see that the class $\mathcal{H}$ satisfies the desired condition, suppose that $(0) \neq T \in \mathcal{H}$ and that T is a K-ring. Since $T \in \mathcal{H}$, $\mathcal{R}_{g(K)}(T) \neq (0)$. So, because $\mathcal{R}_{g(K)} = \mathcal{H}_{g(K)}$, $\mathcal{H}_{g(K)}(T) \neq (0)$. This completes the proof.

Q.E.D.

The condition of the proceeding proposition will be useful in some of the following results. In order to make it easy to refer to this condition (and to another condition closely associated with it) we make the following definition.

2.4.4 DEFINITION:

For each cardinal number $K \geq 2$, a class of rings $\mathcal{H}$ may satisfy either of the following:

Condition $r(K)$: If $(0) \neq T \in \mathcal{H}$ and T is a K-ring then $\mathcal{H}_{g(K)}(T) \neq (0)$.

Condition $s(K)$: If T is a K-ring and all non-zero ideals of T can be homomorphically mapped onto non-zero rings in $\mathcal{H}$, then $T \in \mathcal{H}$.

In order to prove a converse of Proposition 2.4.3, it is necessary to know something about the status of ideals which are generated by K-subrings in $\mathcal{H}_{g(K)}$ (and by ideals of such subrings). So we make the following defintion.

2.4.5 DEFINITION:

A cardinal number K is absorbent if and only if whenever T is a non-zero K-subring of R and $(0) \neq I \triangleleft T$ such that $I \in \mathcal{H}_{g(K)}$ then $(I)_R \in \mathcal{H}_{g(K)}$, for all rings R and all classes of rings $\mathcal{H}$.

We are mainly interested in the situation when $K = 2$ and when $K = \aleph_0$. In Chapter IV we shall prove that all cardinals which are $\leq \aleph_0$ are absorbent. An interesting question is whether or not all cardinal numbers are absorbent.

2.4.6 PROPOSITION:

If K is an absorbent cardinal ≥ 2 and $\mathcal{S}$ is any class of rings then $\mathcal{C} \leq \mathcal{C}(K)$ where $\mathcal{C}$ = the class of $\mathcal{S}_{g(K)}$ semi-simple rings.

Proof:

Assume $R \in \mathcal{C}$. Let T be a K-subring of R. If $T \notin \mathcal{C}$ then $(0) \neq \mathcal{S}_{g(K)}(T) = I$. Since K is absorbent $(I)_R \in \mathcal{S}_{g(K)}$ so R is not $\mathcal{S}_{g(K)}$s.s. This is a contradiction, so $T \in \mathcal{C}$. Hence $R \in \mathcal{C}(K)$.

Q.E.D.

2.4.7 THEOREM:

Let K be an absorbent cardinal number ≥ 2. If $\mathcal{C}$ is a semi-simple class, then:

$\mathcal{C}$ is a K-class if and only if $\begin{cases} \mathcal{C} = \text{the class of } \mathcal{H}_{g(K)} \\ \text{semi-simple rings for some} \\ \text{class of rings } \mathcal{H} \text{ satisfying} \\ \text{Condition } r(K). \end{cases}$

Proof:

Let K be an absorbent cardinal number ≥ 2 and let $\mathcal{C}$ be a semi-simple class.

Assume that $\mathcal{C}$ is a K-class. Then by 2.4.3 $\mathcal{C}$ = the class of $\mathcal{H}_{g(K)}$s.s. rings where $\mathcal{H}$ satisfies Condition $r(K)$.

Conversely, assume $\mathcal{G}$ = the class of $\mathcal{H}_{g(K)}$s.s. rings where $\mathcal{H}$ satisfies Condtion $r(K)$. Then by 2.4.6 $\mathcal{G} \leq \mathcal{G}(K)$. We need only show that $\mathcal{G}(K) \leq \mathcal{G}$. Let $R \in \mathcal{G}(K)$ and let $(0) \neq I \triangleleft R$. If $I \in \mathcal{H}_{g(K)}$ then there is a K-subring T of I such that $T \in \mathcal{H}$. Since $\mathcal{H}$ satisfies Condition $r(K)$, $\mathcal{H}_{g(K)}(T) \neq (0)$ so $T \notin \mathcal{G}$. This is a contradiction since we assumed that $R \in \mathcal{G}(K)$. Therefore $I \notin \mathcal{H}_{g(K)}$ so $R \in \mathcal{G}$.

Q.E.D.

2.4.8 COROLLARY:

Let K be an absorbent cardinal. If $\mathcal{S}$ is a class of rings satisfying condition (A) then the class of $\mathcal{S}_{g(K)}$s.s. rings is a K-class.

Proof:

The class $\mathcal{S}$ satisfies Condition $r(K)$. In fact, since $\mathcal{S}$ satisfies (A) , if $T \in \mathcal{S}$ and T is a K-ring then $T \in \mathcal{S}_{g(K)}$.

Q.E.D.

For absorbent cardinal numbers K , Theorem 2.4.7 answers the question, "When is a semi-simple class a K-class?" The next theorem answers the question, "When is a K-class a semi-simple class?"

2.4.9 <u>THEOREM</u>:

Let K be an absorbent cardinal ≥ 2. If $\mathcal{H}$ is a K-class then:

> $\mathcal{H}$ is a semi-simple class <u>if and only if</u> $\mathcal{H}$ satisfies Condition $s(K)$.

<u>Proof</u>:

Let K be an absorbent cardinal ≥ 2 and $\mathcal{H}$ a K-class.

Assume $\mathcal{H}$ is a semi-simple class. Then $\mathcal{H}$ satisfies condition (F) so certainly $\mathcal{H}$ satisfies Condition $s(K)$.

Conversely, assume that $\mathcal{H}$ satisfies Condition $s(K)$. Since $\mathcal{H} = \mathcal{H}(K)$, $\mathcal{H}$ is strongly hereditary so certainly $\mathcal{H}$ satisfies condition (E). To show that $\mathcal{H}$ satisfies condition (F), suppose R is a ring and every non-zero ideal of R can be homomorphically mapped onto a non-zero ring in $\mathcal{H}$. If $R \notin \mathcal{H} = \mathcal{H}(K)$ then there is a K-subring T of R such that $T \notin \mathcal{H}$. Since $\mathcal{H}$ satisfies Condition $s(K)$, there is a non-zero ideal I of T such that no non-zero homomorphic image of I is in $\mathcal{H}$. Because $\mathcal{H} = \mathcal{H}(K)$ this implies that every non-zero homomorphic image of I contains a K-subring which is not in $\mathcal{H}$. Thus I is $\mathcal{R}_{g(K)}$ where $\mathcal{R}$ = the class of all rings which are not in $\mathcal{H}$. Now, since K is absorbent, $(I)_R \in \mathcal{R}_{g(K)}$. But this contradicts our assumption that every non-zero ideal of R can be homomorphically mapped onto a non-zero ring in $\aleph(K) = \aleph$. Therefore $R \in \aleph$. Thus, $\aleph$ satisfies both condition (E) and (F), so $\aleph$ is a semi-simple class.

Q.E.D.

Usually, when we are considering a generalized K-class $\aleph_{g(K)}$, the class $\aleph$ satisfies condition (A) and hence $\aleph_{g(K)} = (\aleph_{g(K)})_{g(K)}$. In 2.4.8 we saw that this implies that the class of $\aleph_{g(K)}$s.s. rings is a K-class (provided that K is an absorbent cardinal). The following definitions will be useful in investigating generalized K-classes $\aleph_{g(K)}$ for which $(\aleph_{g(K)})_{g(K)} = \aleph_{g(K)}$.

2.4.10 DEFINITION:

For each cardinal number K , a class of rings may satisfy either of the following:

Condition $\overline{r}(K)$: If $(0) \neq T \in \aleph$ and T is a K-ring then there is a non-zero K-subring L of T such that $L \in \aleph_{g(K)}$.

Condition $\overline{s}(K)$: If T is a K-ring and all non-zero K-subrings of T can be homomorphically mapped onto non-zero rings in $\aleph$, then $T \in \aleph$.

Conditions $\overline{r}(K)$ and $\overline{s}(K)$ seem to be slightly stronger than Conditions $r(K)$ and $s(K)$ respectively. In fact, if K is an absorbent cardinal then Condition $\overline{r}(K)$ implies Condition $r(K)$. To see this one need only notice that in this case if L is a K-subring of a ring R and $L \in \aleph_{g(K)}$ then the ideal $(L)_R \in \aleph_{g(K)}$.

The relationship between Condition $\overline{s}(K)$ and $s(K)$ is not so clear. However, in 2.4.12 we are able to prove that if K is an absorbent cardinal and $\aleph \leq \aleph(K)$ then $\aleph$ satisfies Condition $s(K)$ if $\aleph$ satisfies Condition $\overline{s}(K)$.

Corresponding to 2.4.7 we prove:

2.4.11 THEOREM:

Let K be an absorbent cardinal number ≥ 2. If $\mathcal{G}$ is a semi-simple class, then:

$$\left.\begin{array}{l}\mathcal{G} \text{ is a K-class and} \\ (\mathcal{U}_{\mathcal{G}})_{g(K)} = \mathcal{U}_{\mathcal{G}}\end{array}\right\} \text{ if and only if } \left\{\begin{array}{l}\mathcal{G} = \text{the class of} \\ \aleph_{g(K)}\text{s.s. rings for some} \\ \text{class of rings } \aleph \\ \text{satisfying Condition} \\ \overline{r}(K)\ .\end{array}\right.$$

Proof:

Let K be an absorbent cardinal ≥ 2 and let $\mathcal{G}$

be a semi-simple class.

Assume $\mathfrak{C}$ is a K-class and $(\mathcal{U}_{\mathfrak{C}})_{g(K)} = \mathcal{U}_{\mathfrak{C}}$. Then by 2.4.3, $\mathcal{U}_{\mathfrak{C}} = \mathfrak{R}_{g(K)}$ where $\mathfrak{R}$ = the class of all rings which are not in $\mathfrak{C}$. Now, $(\mathfrak{R}_{g(K)})_{g(K)} = \mathfrak{R}_{g(K)}$ so by 2.4.2 (vi), $\mathfrak{R}_{g(K)} = \mathfrak{J}_{g(K)}$ for some class of rings $\mathfrak{J}$ satisfying condition (A). Therefore, $\mathcal{U}_{\mathfrak{C}} = \mathfrak{J}_{g(K)}$ and, since $\mathfrak{J}$ satisfies condition (A), $\mathfrak{J}$ certainly satisfies Condition $\overline{r}(K)$.

Conversely, assume that $\mathfrak{C}$ = the class of $\mathfrak{H}_{g(K)}$s.s. rings and that $\mathfrak{H}$ satisfies Condition $\overline{r}(K)$. First we shall show that $\mathfrak{C}$ is a K-class. By Proposition 2.4.6 $\mathfrak{C} \leq \mathfrak{C}(K)$. Suppose that $R \notin \mathfrak{C}$. Then $\mathfrak{H}_{g(K)}(R) \neq (0)$ so there is a non-zero K-subring T of $\mathfrak{H}_{g(K)}(R)$ such that $T \in \mathfrak{H}$. Since $\mathfrak{H}$ satisfies Condition $\overline{r}(K)$ there is a K-subring L of T (and hence of R) such that $L \in \mathfrak{H}_{g(K)}$. Thus, $L \notin \mathfrak{C}$ so $R \notin \mathfrak{C}(K)$. So we conclude that $\mathfrak{C}(K) \leq \mathfrak{C}$. Thus $\mathfrak{C} = \mathfrak{C}(K)$.

Next we must show that $(\mathcal{U}_{\mathfrak{C}})_{g(K)} = \mathcal{U}_{\mathfrak{C}}$. Now $\mathcal{U}_{\mathfrak{C}} = \mathfrak{H}_{g(K)}$, so by 2.4.2 (iv) it is sufficient to show that $\mathfrak{H}_{g(K)} \leq (\mathfrak{H}_{g(K)})_{g(K)}$. Suppose $R \in \mathfrak{H}_{g(K)}$ and let R' be a non-zero homomorphic image of R. Then there is a non-zero K-subring T' of R' such that $T' \in \mathfrak{H}$. Since $\mathfrak{H}$ satisfies Condition $\overline{r}(K)$, T' contains a non-zero K-subring L' such that $L' \in \mathfrak{H}_{g(K)}$. Therefore $R \in (\mathfrak{H}_{g(K)})_{g(K)}$. This completes the proof.

Q.E.D.

Before proving a theorem which corresponds to Theorem 2.4.9, we notice the following relationship between Conditions $s(K)$ and $\overline{s}(K)$.

2.4.12 <u>PROPOSITION</u>:

Let K be an absorbent cardinal ≥ 2 , and let $\mathfrak{H}$ be a class of rings such that $\mathfrak{H} \leq \mathfrak{H}(K)$. If $\mathfrak{H}$ satisfies Condition $\overline{s}(K)$ then $\mathfrak{H}$ satisfies Condition $s(K)$.

<u>Proof</u>:

Let K be an absorbent cardinal ≥ 2 and let $\mathfrak{H}$ be a class of rings such that $\mathfrak{H} \leq \mathfrak{H}(K)$.

Assume that $\mathfrak{H}$ satisfies Condition $\overline{s}(K)$. Let $\mathfrak{R}$ = the class of all rings which are not in $\mathfrak{H}$. If T is a K-ring and $T \notin \mathfrak{H}$, then by Condition $\overline{s}(K)$, there is a non-zero K-subring L of T such that L cannot be homomorphically mapped onto a ring in $\mathfrak{H}$. Thus $L \in \mathfrak{R}_{g(K)}$. Now, since K is absorbent, $(L)_T \in \mathfrak{R}_{g(K)}$. Thus $(L)_T$ cannot be homomorphically mapped onto a non-zero $\mathfrak{H}(K)$ ring. Since $\mathfrak{H} \leq \mathfrak{H}(K)$, this establishes the result that $\mathfrak{H}$ must satisfy Condition $s(K)$.

Q.E.D.

Notice that, by 2.4.6, this implies that if the class of $\mathcal{S}_{g(K)}$s.s. rings satisfies Condition $\overline{s}(K)$ then this class satisfies Condition $s(K)$. (Provided, of course, that K is absorbent).

Now, corresponding to 2.4.9 we prove:

2.4.13 THEOREM:

Let K be an absorbent cardinal ≥ 2 . If $\mathfrak{C}$ is a K-class then:

$$\left.\begin{array}{l}\mathfrak{C} \text{ is a semi-simple class} \\ \text{and } (\mathcal{U}_{\mathfrak{C}})_{g(K)} = \mathcal{U}_{\mathfrak{C}}\end{array}\right\} \underline{\text{if and only if}} \left\{\begin{array}{l}\mathfrak{C} \text{ satisfies} \\ \text{Condition } \overline{s}(K)\end{array}\right. .$$

Proof:

Let K be an absorbent cardinal ≥ 2 and let $\mathfrak{C}$ be a K-class.

Assume that $\mathfrak{C}$ is a semi-simple class and that $(\mathcal{U}_{\mathfrak{C}})_{g(K)} = \mathcal{U}_{\mathfrak{C}}$. By 2.4.11, $\mathfrak{C}$ = the class of $\mathcal{H}_{g(K)}$s.s. rings for some class $\mathcal{H}$ which satisfies Condition $\overline{r}(K)$. Suppose that T is a K-ring and $T \notin \mathfrak{C}$. Then $\mathcal{H}_{g(K)}(T) \neq (0)$; so, since $\mathcal{H}$ satisfies Condition $\overline{r}(K)$, there is a non-zero K-subring L of $\mathcal{H}_{g(K)}(T)$ such that $L \in \mathcal{H}_{g(K)}$. Thus not all non-zero K-subrings of T can be homomorphically mapped to non-zero rings in $\mathfrak{C}(K) = \mathfrak{C}$. Therefore, $\mathfrak{C}$ satisfies Condition $\overline{s}(K)$.

Conversely, assume that $\mathfrak{G}$ satisfies Condition $\bar{s}(K)$. Then, by 2.4.12, $\mathfrak{G}$ satisfies Condition $s(K)$, so by 2.4.9, $\mathfrak{G}$ is a semi-simple class. Now, by 2.4.7, $\mathfrak{G}$ = the class of $\mathfrak{H}_{g(K)}$ s.s. rings for some class $\mathfrak{H}$ which satisfies $r(K)$. Now, $\mathcal{U}_{\mathfrak{G}} = \mathfrak{H}_{g(K)}$ and by 2.4.2(iv), $(\mathfrak{H}_{g(K)})_{g(K)} \leq \mathfrak{H}_{g(K)}$. To complete the proof we need only show that $\mathfrak{H}_{g(K)} \leq (\mathfrak{H}_{g(K)})_{g(K)}$. Suppose $R \in \mathfrak{H}_{g(K)}$ and R' is a non-zero homomorphic image of R. Now, $R' \notin \mathfrak{G} = \mathfrak{G}(K)$ so there is a K-subring T of R' such that $T \notin \mathfrak{G}$. Since $\mathfrak{G}$ satisfies Condition $\bar{s}(K)$, there is a K-subring L of T such that no non-zero homomorphic image of L is in $\mathfrak{G}$. Thus $L \in \mathfrak{H}_{g(K)}$. Therefore, $R \in (\mathfrak{H}_{g(K)})_{g(K)}$.

Q.E.D.

2.4.14 <u>COROLLARY</u>:

Let K be an absorbent cardinal ≥ 2, and let $\mathfrak{H}$ be any class of rings. There is a class of rings $\mathfrak{R}$ which satisfies condition (A) and such that $\mathfrak{H}$ = the class of $\mathfrak{R}_{g(K)}$ s.s. rings if and only if $\mathfrak{H}$ satisfies Condition $\bar{s}(K)$ and $\mathfrak{H}(K) = \mathfrak{H}$.

<u>Proof</u>:

Since any class of rings which satisfies condition (A) also satisfies Condition $\bar{r}(K)$, the corollary follows immediately from 2.4.11 and 2.4.13.

Q.E.D.

We conclude this chapter with a result concerning those generalized K-classes $\mathcal{H}_{g(K)}$ for which $\mathcal{H}$ is a radical class.

2.4.15 <u>THEOREM</u>:

Let K be an absorbent cardinal ≥ 2. If $\mathcal{H}$ is a radical class and either:

(i) $\mathcal{H} \leq \mathcal{H}(K)$ or

(ii) if $R \in \mathcal{H}$ then $\mathcal{H}_{g(K)}(R) \neq (0)$

then a ring R is $\mathcal{H}_{g(K)}$s.s. if and only if $R \in \mathcal{S}(K)$ where $\mathcal{S}$ = the class of $\mathcal{H}$ s.s. rings.

<u>Proof</u>:

Let K be an absorbent cardinal ≥ 2 and let $\mathcal{H}$ be a radical class. Let $\mathcal{S}$ = the class of $\mathcal{H}$ s.s. rings.

(a) Suppose that $\mathcal{H} \leq \mathcal{H}(K)$.

Assume that R is a $\mathcal{H}_{g(K)}$s.s. ring. If $R \notin \mathcal{S}(K)$ then there is a non-zero K-subring T of R such that $I = \mathcal{H}(T) \neq (0)$. Since $\mathcal{H} \leq \mathcal{H}(K)$, $I \in \mathcal{H}(K)$. So I (and hence, T) contains a non-zero K-subring $L \in \mathcal{H}$. Since $\mathcal{H}$ satisfies condition (A), $L \in \mathcal{H}_{g(K)}$ and since K is absorbent, $(L)_R \in \mathcal{H}_{g(K)}$. This contradicts our assumption, so we must have that $R \in \mathcal{S}(K)$.

Conversely, assume that $R \in \mathcal{S}(K)$. Then no K-subring of R is in $\mathfrak{H}$ so clearly $\mathfrak{H}_{g(K)}(R) = (0)$.

(b) Suppose that whenever $R \in \mathfrak{H}$, $\mathfrak{H}_{g(K)}(R) \neq (0)$.

Assume that R is $\mathfrak{H}_{g(K)}$s.s. If $R \notin \mathcal{S}(K)$ then there is a K-subring T of R such that $\mathfrak{H}(T) = I \neq (0)$. Then by (ii), $\mathfrak{H}_{g(K)}(I) \neq (0)$. So there is a K-subring L of I such that $L \in \mathfrak{H}$. Just as in (a), this contradicts our assumption. Therefore, $R \in \mathcal{S}(K)$.

Conversely, assume that $R \in \mathcal{S}(K)$. Then no K-subring of R is in $\mathfrak{H}$ so clearly $\mathfrak{H}_{g(K)}(R) = (0)$.

Q.E.D.

CHAPTER III

ELEMENTARY RADICAL CLASSES

3.1 THE ELEMENTARY RADICAL CLASSES $\mathcal{L}$, $\mathcal{L}_R$, AND FC.

We shall begin our study of elementary radical classes with a discussion of $\mathcal{L}$ and $\mathcal{L}_R$.

3.1.1 DEFINITION:

(i) $\mathcal{L}$ is the class of all rings R such that for each $x \in R$, $a_N x^N + \dots + a_1 x = 0$ for some integers N , $a_N, \dots, a_1$ (not all the a_i's are 0) which depend on x.

(ii) $\mathcal{L}_R$ is the class of all rings R such that for each $x \in R$, $x^K + a_{K-1}x^{K-1} + \dots + a_1 x = 0$ for some integers $a_{K-1}, \dots, a_1$ which depend on x .

If $R \in \mathcal{L}$ and R' is a homomorphic image of R then clearly $R' \in \mathcal{L}$. Suppose that A is a ring and $B \triangleleft A$ such that $B \in \mathcal{L}$ and $A/B \in \mathcal{L}$. Let $x \in A$. Then

$$a_N x^N + \dots + a_1 x = b \in B$$

for some integers $a_N, \dots, a_1$ (not all of which are zero). Since $B \in \mathcal{L}$, there are integers $c_K, \dots, c_1$, not all zero, such that,

$$\sum_{j=1}^{K} c_j \cdot b^j = 0$$

thus $\sum_{j=1}^{K} c_j (\sum_{i=1}^{N} a_i \cdot x^i)^j = \sum_{H=1}^{K+N} \sum_{j+i=H} c_j a_i x^H = 0$. This implies that $A \in \mathcal{L}$. By the definition it is clear that $\mathcal{L}$ is an elementary class so by 2.3.1 $\mathcal{L}$ is a radical class. A similar argument shows that $\mathcal{L}_R$ is also a radical class. Both $\mathcal{L}$ and $\mathcal{L}_R$ are clearly elementary classes which contain the class of all nilpotent rings (in fact, one easily sees that $\beta < \mathcal{N} \leq \mathcal{L}_R \leq \mathcal{L}$). Combining the above remarks with 2.3.4 we have:

3.1.2 PROPOSITION:

$\mathcal{L}$ and $\mathcal{L}_R$ are special elementary radical classes.

3.1.3 LEMMA:

If $I \lhd \mathcal{P}[X]$ and $I \neq (0)$ then $\mathcal{P}[X]/I \in \mathcal{L}$.

Proof:

Let $(0) \neq I \lhd \mathcal{P}[X]$ and $f(X) \in \mathcal{P}[X]$. Choose $g(X) \neq 0$, $g(X) \in I$. Let $\alpha_1, \ldots, \alpha_K$ be the roots of $g(X)$. Since the α_i are all algebraic numbers, so are $\gamma_i = f(\alpha_i)$. Thus there are non-zero polynomials $h_i(X) \in \mathcal{P}[X]$ such that $h_i(\gamma_i) = 0$. Let $h(X) = \prod_{i=1}^{K} h_i(X)$, then there are integers $a_1, \ldots, a_M$ such that $h(X) = a_1 X + \ldots + a_M X^M$. Consider $h(f(X)) = \ell(X) \in \mathcal{P}[X]$. Now $\ell(\alpha_i) = 0$ for all roots $\alpha_1, \ldots, \alpha_K$ of $g(X)$, so there is a polynomial $d(X)$

with rational coefficients such that $\ell(X) = d(X)\cdot g(X)$. Since the coefficients of $d(X)$ are rational there is an integer n such that $n\cdot d(X)$ has integer coefficients. Now $n\cdot h(f(X)) = na_1 f(X) + \ldots + na_M(f(X))^M = nd(X)\cdot g(X) \in I$. Thus, $na_1\overline{f(X)} + \ldots + na_M(\overline{f(X)})^M = \overline{0}$ in $\mathcal{P}[X]/I$. Therefore, $\mathcal{P}[X]/I \in \mathcal{L}$.

Q.E.D.

The elementary results about algebraic numbers used in 3.1.3 can be found in Chapter 9 of Niven and Zuckerman [12]. Lemma 3.1.3 implies that $\mathcal{L}$ is the largest elementary radical class (except for the class of all rings). In the following proposition we collect some information about the relationship between $\mathcal{L}$, $\mathcal{L}_R$ and the well-known radical classes listed in Chapter I.

3.1.4 PROPOSITION:

(i) If $\mathcal{H}'$ is an elementary radical class and $\mathcal{H}'$ is not the class of all rings then $\mathcal{H}' \leq \mathcal{L}$.

(ii) $\mathcal{L} \gneqq \mathcal{L}_R \gneqq \mathcal{N}$.

(iii) Neither $\mathcal{L}$ nor $\mathcal{L}_R$ is related to $\mathcal{N}_g$ or J .

Proof:

(i) Assume that $\mathcal{H}'$ is an elementary class and $\mathcal{H}'$ is not the class of all rings. Then $\mathcal{P}[X] \notin \mathcal{H}'$; for if $\mathcal{P}[X] \in \mathcal{H}'$ then every ring generated by one element (being a homomorphic image of $\mathcal{P}[X]$) is in $\aleph'$, so $\aleph'$ is the class of all rings. Let $R \in \aleph'$. If $x \in R$ then $\langle x \rangle \cong \mathcal{P}[X]/I \in \aleph'$. Since $\mathcal{P}[X] \notin \aleph'$, $I \neq (0)$ so by 3.1.3 $\langle x \rangle \in \mathcal{L}$. Therefore $R \in \mathcal{L}$, so $\aleph' \leq \mathcal{L}$.

(ii) If R is a nil ring and $x \in R$ then $x^{n(x)} = 0$ for some non-zero integer $n(x)$. Certainly then, $\mathcal{N} \leq \mathcal{L}_R \leq \mathcal{L}$. However, all finite fields are in $\mathcal{L}_R$ and $\mathcal{L}$ but not in $\mathcal{N}$. Hence $\mathcal{N} \lneqq \mathcal{L}_R$ and $\mathcal{N} \lneqq \mathcal{L}$. The ring $\mathcal{P}[X]/(2X) \notin \mathcal{L}_R$ but is in $\mathcal{L}$ so $\mathcal{L}_R \lneqq \mathcal{L}$.

(iii) Clearly $\mathcal{L} \not\leq \mathcal{N}_g$, $\mathcal{L}_R \not\leq \mathcal{N}_g$, $\mathcal{L} \not\leq J$ and $\mathcal{L}_R \not\leq J$. For example, consider any finite field F. $F \in \mathcal{L}$ and $F \in \mathcal{L}_R$ but $F \notin \mathcal{N}_g$ and $F \notin J$.

To see that $J \not\leq \mathcal{L}$ consider the ring R of all formal power series in one indeterminate x over the rational field Q. Let R' be the ideal of R consisting of all $\sum_{i=0}^{\infty} a_i x^i \in R$ such that $a_o = 0$. It is well-known (and easy to prove) that $R' \in J$. However, if $a_1, \ldots, a_K$ are integers, $a_1 x + \ldots + a_K x^K \neq 0$ unless $a_1 = \ldots = a_K = 0$. Thus, $R' \notin \mathcal{L}$. This example shows that $J \not\leq \mathcal{L}_R$ and $J \not\leq \mathcal{L}$.

Let $Q(x)$ be the field of all rational functions in an indeterminate x over the rational field Q. Then

$R = (Q(x))_2$, the ring of 2×2 matrices over (x) , is in $\mathcal{N}_g$. However, $R \notin \mathcal{L}_R$ and $R \notin \mathcal{L}$, because if $a_1, \ldots, a_K$ are integers such that

$$a_1\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + \ldots + a_K\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}^K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then $a_1x + \ldots + a_Kx^K = 0$ in $Q(x)$ so $a_1 = \ldots = a_K = 0$.

Q.E.D.

We now turn to a brief consideration of the radical class FC .

3.1.5 <u>DEFINITION</u>:

(i) Let p be a prime number. FC_p is the class of all rings R such that for each $x \in R$ there is a positive integer $\alpha(x)$ such that $p^{\alpha(x)}.x = 0$.

(ii) FC is the class of all rings R such that for each $x \in R$ there is a non-zero integer $n(x)$ such that $n(x)\cdot x = 0$.

Let p be a prime. Clearly FC_p satisfies condition (A). Suppose that $B \lhd A$ and both B and A/B are in FC_p . Let $x \in A$, then $p^\alpha x \in B$ for some integer α . But since $p^\alpha x \in B \in FC_p$, $p^\beta(p^\alpha x) = 0$ for some integer β . Therefore $p^{\beta+\alpha}x = 0$ so $A \in FC_p$. Now, it is clear from the definition that FC_p is an elementary class so by 2.3.1, FC_p is an elementary radical class. A similar argument shows that FC is an elementary radical class.

3.1.6 PROPOSITION:

(i) FC is an elementary radical class.

(ii) For each prime p, FC_p is an elementary radical class.

(iii) For all rings R, $FC(R) = \bigoplus\{FC_p(R) : p \text{ is a prime}\}$.

Proof:

We have already seen that the classes mentioned in (i) and (ii) are elementary radical classes.

To see that (iii) is true, notice that $R \in FC$ if and only if the additive group R^+ is torsion, and $R \in FC_p$ if and only if R^+ is a p-group. Part (iii) of the proposition now follows from a well-known result about torsion groups.

Q.E.D.

3.2 THE ELEMENTARY RADICAL CLASSES $\mathcal{J}'$ AND $\mathcal{D}'$.

Recall that $\mathcal{D}$ is the lower radical class determined by all nilpotent rings N such that $N = \mathcal{N}(R)$ for some ring R with D.C.C. on left ideals. This class contains $\mathcal{J}$ which is the lower radical class determined by the class of all zero simple rings.

In this section we shall consider the classes $\mathcal{J}'$ and $\mathcal{D}'$. We begin with the following result about the class $\mathcal{J}$.

3.2.1 PROPOSITION:

$$\mathcal{J} = \beta \cap FC .$$

Proof:

Let $\mathcal{R}$ = the class of zero simple rings. If $S \in \mathcal{R}$ then the additive group S^+ is a simple abelian group; so, S^+ is the cyclic group of p elements for some prime p. Therefore, $\mathcal{R} \leq FC$ and so $\mathcal{J} \leq FC$ because $\mathcal{J}$ is the lower radical class determined by $\mathcal{R}$. Since $\mathcal{J} \leq \beta$, $\mathcal{J} \leq \beta \cap FC$.

Suppose that $\beta \cap FC \nleq \mathcal{J}$. Then there is a ring R which is in $\beta \cap FC$ but is $\mathcal{J}$ semi-simple. Since $R \in \beta$ there is a non-zero ideal I of R such that $I^2 = (0)$. Suppose I' is a non-zero homomorphic image of I . Since $R \in FC$, $I' \in FC$. Thus, there is a prime p and an element $x' \in I'$ such that $x' \neq 0$ but $px' = 0$. Since $(I')^2 = (0)$, $\langle x' \rangle \triangleleft I'$, and $\langle x' \rangle$ is isomorphic to the zero ring on the cyclic group of p-elements. But then $\langle x' \rangle \in \mathcal{R}$ so $I \in \mathcal{R}_2$ (see 1.1.7) and hence $I \in \mathcal{J}$. This is a contradiction. Therefore $\beta \cap FC \leq \mathcal{J}$, so $\mathcal{J} = \beta \cap FC$.

Q.E.D.

3.2.2 THEOREM:

$$\mathfrak{D}' = \mathcal{J}' = \mathfrak{N} \cap FC .$$

<u>Proof</u>:

First we shall show that $\mathcal{S}' = \mathcal{N} \cap FC$. Notice that $\beta' \leq \mathcal{N}' = \mathcal{N}$ since $\beta \leq \mathcal{N}$; and that $\mathcal{N}' \leq \beta'$, for if $\langle x \rangle \in \mathcal{N}$ then $\langle x \rangle$ is nilpotent so $\langle x \rangle \in \beta$. Thus $\beta' = \mathcal{N}'$.

Since, $\mathcal{S} = \beta \cap FC$ (by 3.2.1), $\mathcal{S} \leq \beta$ and $\mathcal{S} \leq FC$. Therefore, $\mathcal{S}' \leq \beta' = \mathcal{N}$ and $\mathcal{S}' \leq FC' = FC$ so $\mathcal{S}' \leq \mathcal{N} \cap FC$.

Suppose $R \in \mathcal{N} \cap FC$, and let $x \in R$. Then $\langle x \rangle$ is finite since there are positive integers K and N such that $x^K = 0$ and $Nx = 0$. Thus $\langle x \rangle$ has D.C.C. and so, since $\langle x \rangle \in \mathcal{N}$, $\langle x \rangle \in \mathcal{S}$ (see Lemma 28 in Divinsky [7]). Therefore, $\mathcal{N} \cap FC \leq \mathcal{S}'$ so $\mathcal{S}' = \mathcal{N} \cap FC$.

Since $\mathcal{S} \leq \mathcal{D}$, $\mathcal{S}' \leq \mathcal{D}'$. Suppose $\langle x \rangle \in \mathcal{D}'$ but $\langle x \rangle \notin \mathcal{S}'$. The ring $\langle x \rangle$ must be nil since $\mathcal{D}' \leq \mathcal{N}' = \mathcal{N}$, so $\langle x \rangle \notin FC$. Let $\langle \bar{x} \rangle = \langle x \rangle / FC\langle x \rangle$. First we shall prove that the ring $\langle \bar{x} \rangle / \langle \bar{x} \rangle^2 \notin FC$. Suppose $n\bar{x} \in \langle \bar{x} \rangle^2$. Then there are integers $a_2, \ldots, a_M$ such that $n\bar{x} = \sum_{i=2}^{M} a_i \bar{x}^i$. By successive substitutions for $n\bar{x}$ we see that $n^K\bar{x} \in \langle \bar{x} \rangle^{K+1}$ for all positive integers K . Since $\langle \bar{x} \rangle$ is nilpotent, $n^K\bar{x} = 0$ for some integer K , and since $\langle \bar{x} \rangle$ is a ring of characteristic 0 this implies that $\bar{x} = 0$. This is a contradiction on since $\langle x \rangle \notin FC$. Therefore $\langle \bar{x} \rangle / \langle \bar{x} \rangle^2 \notin FC$ so $\langle x \rangle$ can be homomorphically mapped to a zero ring of characteristic zero. Since $\langle x \rangle$ is generated by one element this ring must be isomorphic to C^{∞} = the zero ring on the infinite cyclic group. Since $\langle x \rangle \in \mathfrak{D}$, $C^{\infty} \in \mathfrak{D}$. But $C^{\infty} \notin \mathfrak{D}$ (see Theorem 14, Divinsky [7]) so this is a contradiction. Hence $\mathfrak{D}' \leq \mathcal{J}'$ so $\mathfrak{D}' = \mathcal{J}'$ and the proof is complete.

Q.E.D.

3.3 CLASSES $\mathfrak{X}$ FOR WHICH $\mathfrak{X}' = \mathcal{N}$.

In [9] Goldman defines a Hilbert ring to be a commutative ring R with identity such that $J(R') = \mathcal{N}(R')$ for all homomorphic images R' of R. He proves that if R is a Hilbert ring then so is the polynomial ring $R[X]$.

Since the ring of integers is clearly a Hilbert ring the following proposition is a special case of Goldman's theorem.

3.3.1 PROPOSITION:

If R is a ring generated by one element then $J(R) = \mathcal{N}(R)$.

3.3.2 THEOREM:

If $\beta \leq \mathfrak{X} \leq FF$ then $\mathfrak{X}' = \mathcal{N}$.

Proof:

Suppose that $\mathfrak{X}$ is a class of rings such that $\beta \leq \mathfrak{X} \leq FF$.

Since $\beta \leq \aleph$, $\mathcal{N} = \beta' \leq \aleph'$.

Assume that $FF' \not\leq \mathcal{N}$. Then there is a ring $\langle x \rangle \in FF'$ such that $\mathcal{N}(\langle x \rangle) = (0)$. By 3.3.1 $J(\langle x \rangle) = (0)$ and so, since $\langle x \rangle$ is commutative, by Lemma 87 in Divinsky [7] $F(\langle x \rangle) = (0)$. It follows that $\langle x \rangle$ is a subdirect sum of fields each of which is generated by one element. To reach a contradiction it is sufficient to show that any field which is generated as a ring by one element must be finite.

Let $\langle y \rangle$ be a field and suppose $\langle y \rangle$ is of characteristic zero. Then $\langle y \rangle$ contains a copy of the rationals. Choose $f(y) = a_1 y + \ldots + a_k y^k$ of minimal degree so that $N = f(y)$ is a non-zero integer in $\langle y \rangle$. Let p be a prime which does not divide any of $a_1, \ldots, a_k$ and let $1/p = c_1 y + \ldots + c_n y^n$. By continued substitutions for $a_k y^{k+1} = Ny - a_1 y^2 - \ldots - a_{k-1} y^k$ we see that for some positive integer L, $a_k^L/p = b_1 y + \ldots + b_k y^k$. But then $0 = Na_k^L - Na_k^L = (a_1 a_k^L - pNb_1)y + \ldots + (a_k^{L+1} - pNb_k)y^k$. Since p does not divide any of the a_i not all of the coefficients in this expression are 0. Since $\langle y \rangle$ has no proper divisors of zero it follows that there are integers $d_1, \ldots, d_\ell$ with $d_1 \neq 0$ such that $d_1 y + \ldots + d_\ell y^\ell = 0$ and $\ell \leq k$. But then $d_1 = -d_2 y - \ldots - d_\ell y^{\ell-1}$ which contradicts the minimality of k.

Thus $\langle y \rangle$ is of finite characteristic and since

$\langle y \rangle$ must be algebraic it follows that $\langle y \rangle$ is finite.

This is a contradiction and so $FF' \leq \mathcal{N}$.

Q.E.D.

This theorem may be paraphrased in the following way, "A ring R is nil if and only if no subring of R which is generated by one element can be homomorphically mapped onto a finite field".

3.4 ELEMENTARY RADICAL CLASSES WHICH ARE $\leq$ FC .

We will begin this section with a discussion of the elementary radical class $\mathcal{E}'$. This radical class is unrelated to all of the well-known radical classes listed in Chapter 1. In fact, all $\mathcal{E}'$ rings are $\mathcal{N}$ semi-simple. This radical plays a central role in our discussions concerning radical classes which contain only $\mathcal{N}$ semi-simple rings.

3.4.1 DEFINITION:

$\mathcal{E}$ is the class of all idempotent rings (that is, all rings R such that $R = R^2$).

Let R be a ring and $x \in R$. Clearly $\langle x \rangle = \langle x \rangle^2$ if and only if $x \in \langle x \rangle^2$ and hence if and only if there are integers $a_2, \ldots, a_k$ such that $x = \sum_{i=2}^{K} a_i x^i$. Using this characterization it is clear that homomorphic images of $\mathcal{E}'$-rings are in $\mathcal{E}'$ and that if A is a ring with an ideal B such that both

A/B and B are in $\mathcal{E}'$ then $A \in \mathcal{E}'$. Therefore, by 2.3.1, $\mathcal{E}'$ is an elementary radical class.

3.4.2 PROPOSITION:

A non-zero $\mathcal{E}'$-ring without proper divisors of zero is an algebraic field of prime characteristic.

Proof:

Let R be a non-zero $\mathcal{E}'$-ring without proper divisors of zero. If $0 \neq x \in R$ then there are integers $a_2, \ldots, a_K$ such that $x = \sum_{i=2}^{K} a_i x^i$, hence $e_x = \sum_{i=2}^{K} a_i x^{i-1}$ is an identity for $\langle x \rangle$. Let $w \in R$. Then $x(e_x w - w) = (we_x - x)w = 0$, so $e_x w = w$. Similarily, $we_x = w$ so e_x is an identity for R . If $0 \neq v \in R$ then $e_v \in \langle v \rangle = \langle v \rangle^2$ so $e_v \in \langle v \rangle \cdot v \subseteq Rv$. Therefore $R = Rv$ for all non-zero $v \in R$; so, since $R \neq (0)$, R is a division ring.

Let e be the identity of R . Then $\langle e \rangle \not\cong$ the ring of integers since $\langle 2e \rangle = \langle 2e \rangle^2 = \langle 4e \rangle$. Therefore the characteristic of R is a prime. Since $e = e_w \in \langle w \rangle$ for all non-zero $w \in R$, R is algebraic. Therefore, by Theorem 2 on page 183 of Jacobson [11], R is a field.

Q.E.D.

3.4.3 <u>COROLLARY</u>:

If $(0) \neq R \in \mathcal{E}'$ then R is isomorphic to a sub-direct sum of algebraic fields of prime characteristic. So, in particular, R is commutative.

<u>Proof</u>:

Let $(0) \neq R \in \mathcal{E}'$ and $x \in R$. If $x^N = 0$ then $\langle x \rangle = \langle x \rangle^2 = \ldots = \langle x \rangle^N = (0)$ so $x = 0$. Hence $\mathcal{E}'$ rings have no non-zero nilpotent elements so the corollary follows from 1.2.1 and 3.4.2.

Q.E.D.

From 2.1.3(iii) we know that $\mathcal{E}^* \leq \mathcal{E}' = (\mathcal{E}')^*$. The following theorem provides a characterization of $\mathcal{E}'$ which makes it clear that in fact $\mathcal{E}^* = \mathcal{E}'$.

3.4.4 <u>THEOREM</u>:

A ring $R \in \mathcal{E}'$ if and only if every non-zero finitely generated subring of R is isomorphic to a finite direct sum of finite fields.

<u>Proof</u>:

Assume that $R \in \mathcal{E}'$ and let R' be a non-zero finitely generated subring of R. Then $R' \in \mathcal{E}'$ so by 3.4.3 R' is commutative. Since R' is finitely generated and commutative we may conclude, from the Hilbert Basis

Theorem, that R' satisfies A.C.C. If P' is a prime ideal of R' and $P' \neq R'$ then P' is a maximal ideal because by 3.4.2 R'/P' is a field. Since R' is finitely generated, commutative, and $\langle g\rangle$ has an identity for each generator g of R' ; R' has an identity (if, for $i = 1, 2,$ e_i is an identity for $\langle g_i\rangle$ then $e_1 + e_2 - e_1e_2$ is an identity for $\langle g_1, g_2\rangle$). Now, by Theorem 2, page 203 of Zariski and Samuel [15], R' satisfies D.C.C. Then R' is a commutative Wedderburn ring so R' is isomorphic to a finite direct sum of fields. These fields must be finite, since they are finitely generated and by 3.4.2 they are algebraic of prime characteristic.

The converse is obvious; in fact, if $x \in R'$ and R' is isomorphic to a finite direct sum of finite fields then there is an integer $n(x) \geq 2$ such that $x^{n(x)} = x$.

Q.E.D.

3.4.5 <u>COROLLARY</u>:

A ring $R \in \mathcal{E}'$ if and only if for each $x \in R$ there is an integer $n(x) \geq 2$ such that $x^{n(x)} = x$.

3.4.6 <u>COROLLARY</u>:

A ring $R \in \mathcal{E}'$ if and only if for all $0 \neq x \in R$, $\langle x\rangle$ is isomorphic to a finite direct sum of finite fields.

If $(0) \neq R \in \mathcal{E}'$ and R has D.C.C. then R is a commutative Wedderburn ring so R is isomorphic to a finite direct sum of fields in $\mathcal{E}'$. In the next theorem we see that a condition which is apparently weaker than D.C.C. is sufficient to obtain this result.

3.4.7 THEOREM:

If $(0) \neq R \in \mathcal{E}'$ and R satisfies A.C.C. on annihilators then R is isomorphic to a finite direct sum of fields in $\mathcal{E}'$ (that is, algebraic fields of prime characteristic).

Proof:

Let R be a non-zero $\mathcal{E}'$-ring which satisfies A.C.C. on annihilators. By 3.4.3 R is commutative. Since A.C.C. on left annihilators is equivalent to D.C.C. on right annihilators, R satisfies D.C.C. on annihilators.

Assume $R = A_1 \oplus \ldots \oplus A_K \oplus B_{K+1}$ where the A_i are fields in $\mathcal{E}'$. If B_{K+1} has no proper divisors of zero then, by 3.4.2, B_{K+1} is a field in $\mathcal{E}'$. If there are non-zero elements $b_1, b_2 \in B_{K+1}$ such that $b_1 \cdot b_2 = 0$ then B_{K+1} contains the annihilator

$$\begin{aligned}\overline{A}_{K+1} &= (0 : A_1 \oplus \ldots \oplus A_K \oplus b_1R) \\ &= \text{the annihilator of } A_1 \oplus \ldots \oplus a_K \oplus b_1R \neq (0)\end{aligned}$$

and $\overline{A}_{K+1} \subseteq B_{K+1}$. Choose $A_{K+1} = (0 : C_{K+1})$ to be a minimal non-zero annihilator contained in B_{K+1} . Now, if there are non-zero elements $x,y \in A_{K+1}$ such that $xy = 0$ then $D = (0 : C_{K+1} + yR)$ is a non-zero annihilator $(x \in D)$ such that $D \subsetneqq A_{K+1}$ $(y \notin D)$. This contradicts the minimality of A_{K+1} . Therefore A_{K+1} has no proper divisors of zero so A_{K+1} is a field in $\mathcal{E}'$ by 3.4.2. Since A_{K+1} has an identity, A_{K+1} is a direct summand of B_{K+1} . That is, there is an ideal B_{K+2} of B_{K+1} such that $B_{K+1} = A_{K+1} \oplus B_{K+2}$. Therefore $B_{K+2} \triangleleft R$ and $R = A_1 \oplus \cdots \oplus A_{K+1} \oplus B_{K+2}$. Notice that this proof is valid when $K = 0$, and since $(0) \neq R$ we can begin the above process.

Since R satisfies A.C.C. on annihilators, the process above must stop. That is, for some n , B_n has no proper divisors of zero and hence is a field in $\mathcal{E}'$. This completes the proof.

Q.E.D.

This completes our investigation of the elementary radical class $\mathcal{E}'$.

We now present a classification of all elementary radicals which are $\leq FC$.

3.4.8 <u>DEFINITION</u>:

Define $\mathcal{H}_p$ to be $\mathcal{H} \cap FC_p$ where $\mathcal{H}$ is any class of rings and p is a prime number.

3.4.9 PROPOSITION:

If $\aleph'$ is an elementary radical class and R is a ring then $(\aleph' \cap FC)(R) = \bigoplus\{\aleph'_p(R) : p$ is a prime$\}$.

Proof:

Let $\aleph'$ be an elementary radical class and let R be a ring. Since intersections of radical classes are radical classes, $\aleph'_p(R)$ and $(\aleph' \cap FC)(R)$ are defined.

From 3.1.6(iii) we know that for any ring A, $FC(A) = \bigoplus\{FC_p(A) : p$ is a prime$\}$. So $(\aleph' \cap FC)(R) = \bigoplus\{FC_p((\aleph' \cap FC)(R)) : p$ is a prime$\}$. Now, $FC_p((\aleph' \cap FC)(R)) \in \aleph'$ since $\aleph'$ is hereditary; therefore, $FC_p((\aleph' \cap FC)(R)) \subseteq \aleph'_p(R)$. Since $\aleph'_p(R) \subseteq (\aleph' \cap FC)(R)$ and $\aleph'_p(R) \in FC_p$, $\aleph'_p(R) \subseteq FC_p((\aleph' \cap FC)(R))$. Thus, $\aleph'_p(R) = FC_p((\aleph' \cap FC)(R))$. This completes the proof.

Q.E.D.

3.4.10 DEFINITION:

A set of positive integers is a C.U.D. set of integers if and only if whenever $n \in S$ and k is a positive integer which divides n , $k \in S$.

Suppose that S is a C.U.D. (closed under divisors) set of integers and p is any prime number. Then for each

$n \in S$ we consider the finite field F_{p^n} = the field of p^n elements. Since S is a C.U.D. set, if $n \in S$ and k divides n then $k \in S$; hence, the set $\{F_{p^n} : n \in S\}$ of all such fields is strongly hereditary because a non-zero subring of the field F_{p^n} is the field F_{p^k} for some k which divides n .

Let $\mathcal{R}$ be the set of all possible finite direct sums of the fields $F_{p^n} : n \in S$. It is, in fact, the class $\mathcal{R}'$ which we are defining in the first part of the following definition. Notice that the set of fields $\{F_{p^n} : n \in S\} \subseteq \mathcal{R}'$ so as the C.U.D. set S changes so does $\mathcal{R}'$. The dependence of $\mathcal{R}'$ on the prime p is obvious.

3.4.11 DEFINITION:

(i) $\mathcal{J}_p(S)$ is the class of all rings R with the property that for all non-zero $x \in R$, $\langle x \rangle$ is isomorphic to a finite direct sum of fields taken from $\{F_{p^n} : n \in S\}$ where p is a prime number and S is a C.U.D. set of integers.

(ii) $\mathcal{J}_p\mathcal{N}(S)$ is the class of all rings R with the property that $R \in FC_p$ and for all $x \in R$ the factor ring $\langle x \rangle / \mathcal{N}(\langle x \rangle)$ is in $\mathcal{J}_p(S)$.

It is clear from the above definition that $\mathcal{J}_p(S) \leq \mathcal{J}_p\mathcal{N}(S)$. In the following proposition we prove that these classes are radical classes.

3.4.12 PROPOSITION:

If S is a C.U.D. set of positive integers and p is a prime number then $\mathfrak{I}_p(S)$ and $\mathfrak{I}_p\mathcal{N}(S)$ are elementary radical classes.

Proof:

Let S be a C.U.D. set of positive integers and let p be a prime number.

From the definition it is clear that $\mathfrak{I}_p(S)$ is an elementary class. It is also clear that $\mathfrak{I}_p(S)$ satisfies condition (A). Suppose that B is an ideal of a ring A and both B and A/B are in $\mathfrak{I}_p(S)$. By 3.4.6, both B and A/B are in $\mathcal{E}'$ so $A \in \mathcal{E}'$. Let $0 \neq x \in A$. Then by 3.4.6 again, $\langle x\rangle$ is isomorphic to a finite direct sum of fields. Therefore $\langle x\rangle \cap B$ is a direct summand of $\langle x\rangle$ so $\langle x\rangle \cong (\langle x\rangle/\langle x\rangle \cap B) \oplus (\langle x\rangle \cap B)$. Now because A/B and B are in $\mathfrak{I}_p(S)$, the fields in question must be of the form F_{p^α} where $\alpha \in S$. Therefore, $A \in \mathfrak{I}_p(S)$; so, by 2.3.1, $\mathfrak{I}_p(S)$ is a radical class.

Suppose $R \in \mathfrak{I}_p\mathcal{N}(S)$ and that R' is a homomorphic image of R. Let $x' \in R'$. Then there is an $x \in R$ such that $\langle x'\rangle$ is a homomorphic image of $\langle x\rangle$. So $\langle x'\rangle/\mathcal{N}(\langle x'\rangle)$ is a homomorphic image of $\langle x\rangle/\mathcal{N}(\langle x\rangle)$, hence $\langle x'\rangle/\mathcal{N}(\langle x'\rangle) \in \mathfrak{I}_p(S)$. Since $R \in FC_p$, $R' \in FC_p$. Therefore $R' \in \mathfrak{I}_p\mathcal{N}(S)$ so $\mathfrak{I}_p\mathcal{N}(S)$ satisfies condition (A).

Suppose that B is an ideal of a ring A and that both A/B and B are in $\mathfrak{I}_p\mathcal{N}(S)$. Both A/B and B are in FC_p so $A \in FC_p$.

Let $x \in A$ and let $\langle\bar{x}\rangle = \langle x\rangle/\langle x\rangle \cap B$. Then $\langle\bar{x}\rangle/\mathcal{N}(\langle\bar{x}\rangle) \in \mathfrak{I}_p(S)$ so $\langle\bar{x}\rangle/\mathcal{N}(\langle\bar{x}\rangle)$ is finite. Thus, by 2.2.6, $\mathcal{N}(\langle\bar{x}\rangle)$ is finitely generated as a ring and so; since $\mathcal{N}(\langle\bar{x}\rangle)$ is also nilpotent and in FC_p , $\mathcal{N}(\langle\bar{x}\rangle)$ is finite. Therefore $\langle\bar{x}\rangle$ must be finite. Now, by 2.2.6 again, $\langle x\rangle \cap B$ is finitely generated as a ring. Since $\langle x\rangle \cap B/\mathcal{N}(\langle x\rangle \cap B)$ has no non-zero nilpotent elements, it must be in $\mathfrak{I}_p(S)$. Thus by 3.4.6 $\langle x\rangle \cap B/\mathcal{N}(\langle x\rangle \cap B) \in \mathcal{E}'$ so by 3.4.4 $\langle x\rangle \cap B/\mathcal{N}(\langle x\rangle \cap B)$ is finite. Now, just as above, $\langle x\rangle \cap B$ is finite. Therefore, $\langle x\rangle$ is finite.

Let $N = \mathcal{N}(\langle x\rangle)$. Since $\langle x\rangle$ is finite and commutative, $\langle x\rangle/N$ is a finite direct sum of fields. Thus $\dfrac{\langle x\rangle \cap B + N}{N}$ is a direct summand of $\langle x\rangle/N$. Let $\dfrac{\langle x\rangle}{N} = \dfrac{L}{N} \oplus \dfrac{\langle x\rangle \cap B + N}{N}$. Since $N \cap (\langle x\rangle \cap B) = \mathcal{N}(\langle x\rangle \cap B)$, $\dfrac{\langle x\rangle \cap B + N}{N} \cong \dfrac{\langle x\rangle \cap B}{\mathcal{N}(\langle x\rangle \cap B)}$. Thus, $\dfrac{\langle x\rangle \cap B + N}{N} \in \mathfrak{I}_p(S)$.

Now, $\dfrac{L}{N} \cong \dfrac{\langle x\rangle}{N} \Big/ \dfrac{\langle x\rangle \cap B + N}{N} \cong \dfrac{\langle x\rangle}{\langle x\rangle \cap B + N}$. Thus L/N is a homomorphic image of $\langle\bar{x}\rangle$ which has no non-zero nilpotent elements. Hence L/N is a homomorphic image of $\langle\bar{x}\rangle/\mathcal{N}(\langle\bar{x}\rangle) \in \mathfrak{I}_p(S)$. Thus $L/N \in \mathfrak{I}_p(S)$. Therefore $\langle x\rangle/N \in \mathfrak{I}_p(S)$ so $A \in \mathfrak{I}_p\mathcal{N}(S)$.

From the definition it is clear that $\mathfrak{I}_p\mathcal{N}(S)$ is an elementary class so by 2.3.1 $\mathfrak{I}_p\mathcal{N}(S)$ is a radical class.

Q.E.D.

The radical classes $\mathfrak{I}_p(S)$, $\mathfrak{I}_p\mathcal{N}(S)$ and FC_p will be our basic building blocks for describing all elementary radicals which are $\leq FC$. We begin with the following result.

3.4.13 <u>PROPOSITION</u>:

If $\aleph'$ is an elementary radical class and p is a prime then $\aleph'_p \cap \mathcal{N} = \{(0)\}$ or $\mathcal{N} \cap FC_p \leq \aleph'_p$.

<u>Proof</u>:

Let $\aleph'$ be an elementary radical class. If $\aleph'_p \cap \mathcal{N} \neq \{(0)\}$ then there is a non-zero ring $R \in \aleph'_p \cap \mathcal{N}$. Since $R \in FC_p$ and $R \in \mathcal{N}$ there is an $x \in R$, $x \neq 0$ such that $x^2 = 0$ and $px = 0$. Thus $\langle x\rangle \cong C_p$ = the zero ring on the cyclic group of p elements, so $C_p \in \aleph'_p$.

Let $A \in \mathcal{N} \cap FC_p$. Suppose $A \notin \aleph'_p$. Then $\overline{A} = A/\aleph'_p(A) \neq (0)$. Since $\overline{A} \notin \aleph'_p$ there is an $x \in \overline{A}$ such that $\langle x\rangle \notin \aleph'_p$. Let $(0) \neq \langle\overline{x}\rangle = \langle x\rangle/\aleph'_p(\langle x\rangle)$. Now, $\langle\overline{x}\rangle \in \mathcal{N} \cap FC_p$ so there is a $y \in \langle\overline{x}\rangle$ such that $py = 0$ and $y^2 = 0$ but $y \neq 0$. Let $Y = (y)_{\langle\overline{x}\rangle}$. Then $Y^2 = (0)$ and $pY = 0$. So if $w \in Y$, $\langle w\rangle \triangleleft Y$ and $\langle w\rangle \cong C_p \in \aleph'_p$.

Therefore $Y \in \mathcal{H}'_p$. This is a contradiction. Hence $A \in \mathcal{H}'_p$ so $\mathcal{N} \cap FC_p \leq \mathcal{H}'_p$.

Q.E.D.

Suppose that $\mathcal{H}_\alpha : \alpha \in \Lambda$ is a collection of radical classes such that $\mathcal{H}_\beta(R) \cap \sum_{\substack{\alpha \in \Lambda \\ \alpha \neq \beta}} \mathcal{H}_\alpha(R) = (0)$ for all $\beta \in \Lambda$ and all rings R . Then for any ring R we can form the direct sum $\bigoplus \{\mathcal{H}_\alpha(R) : \alpha \in \Lambda\}$. In such a situation we shall denote by $\bigoplus \{\mathcal{H}_\alpha : \alpha \in \Lambda\}$ the class of all rings R for which $R = \bigoplus \{\mathcal{H}_\alpha(R) : \alpha \in \Lambda\}$.

In terms of this notation Proposition 3.4.9 tells us that $\mathcal{H}' \cap FC = \bigoplus \{\mathcal{H}'_p : p$ is a prime$\}$ whenever $\mathcal{H}'$ is an elementary radical class (recall that $\mathcal{H}'_p = FC_p \cap \mathcal{H}'$) .

It follows that if $\mathcal{H}' \leq FC$ then $\mathcal{H}' = \bigoplus \{\mathcal{H}'_p : p$ is a prime$\}$. In the following theorem we shall prove that each $\mathcal{H}'_p$ must equal either FC_p or $\mathcal{J}_p\mathcal{N}(S)$ or $\mathcal{J}_p(S)$ for some C.U.D. set of integers S . And conversely, any "direct sum" of such elementary radical classes is again an elementary radical class.

In other words, every elementary radical class which is contained in FC is a "direct sum" of these simple radical classes (FC_p , $\mathcal{J}_p\mathcal{N}(S)$, $\mathcal{J}_p(S)$) and all "direct sums" of such classes are elementary radical classes.

3.4.14 THEOREM:

For each prime p, $\mathcal{H}'_p = FC_p$ or $\mathcal{J}_p \mathcal{N}(S_p)$ or $\mathcal{J}_p(S_p)$ for some C.U.D. set of integers S_p if $\mathcal{H}'$ is an elementary radical class. Also $\mathcal{H}' \cap FC = \bigoplus\{\mathcal{H}'_p : p \text{ is a prime}\}$.

Conversely, $\mathcal{H}' = \bigoplus\{\mathcal{H}_{[p]} : p \text{ is a prime}\}$ is an elementary radical class if for each prime p, $\mathcal{H}_{[p]} = FC_p$ or $\mathcal{J}_p \mathcal{N}(S_p)$ or $\mathcal{J}_p(S_p)$ for some C.U.D. set of integers S_p. Moreover, $\mathcal{H}_p = \mathcal{H}_{[p]}$ for all primes p.

Proof:

Let $\mathcal{H}'$ be an elementary radical class.

Define $S_p = \{n : F_p n \in \mathcal{H}'\}$. Then S_p is a C.U.D. set of positive integers since $\mathcal{H}'$ is strongly hereditary. We must show that $\mathcal{H}'_p = FC_p \cap \mathcal{H}'$ is FC_p or $\mathcal{J}_p(S_p)$ or $\mathcal{J}_p \mathcal{N}(S_p)$.

If $\mathcal{H}'_p = FC_p$ we are done so suppose $\mathcal{H}'_p \neq FC_p$. We will consider the two cases of 3.4.13.

If $\mathcal{H}'_p \cap \mathcal{N} = \{(0)\}$ we will show that $\mathcal{H}'_p = \mathcal{J}_p(S_p)$.

Suppose that $R \in \mathcal{H}'_p = FC_p \cap \mathcal{H}'$ and that x is a non-zero element of R. Then $\mathcal{N}(\langle x\rangle) = (0)$ since $\mathcal{H}'_p \cap \mathcal{N} = \{(0)\}$. Let P be a prime ideal of $\langle x\rangle$. Then $\langle x\rangle/P$ must have characteristic p so either $\langle x\rangle/P$ is finite or $\langle x\rangle/P \cong F_p[X]$ where X is an indeterminate. If $\langle x\rangle/P \cong F_p[X]$ then every ring of characteristic p is in $\mathcal{H}'$ (since they are all homomorphic images of $F_p[X]$).

But then $FC_p \leq \mathfrak{H}'$ because if A is a ring and $B \triangleleft A$ such that $A/B \in \mathfrak{H}'$ and $B \in \mathfrak{H}'$ then $A \in \mathfrak{H}'$. Hence $\mathfrak{H}_p' = FC_p$. This is contrary to our supposition that $\mathfrak{H}_p' \neq FC_p$. Hence $\langle x \rangle / P$ is finite. Then $\langle x \rangle / P$ is a Wedderburn ring without zero divisors so if $P \neq \langle x \rangle$ then $\langle x \rangle / P$ is a field and hence P is a maximal ideal of $\langle x \rangle$. Since $\langle x \rangle$ satisfies A.C.C. and all prime ideals are maximal, $\langle x \rangle$ satisfies D.C.C. (Theorem 2, page 203 of Zariski and Samuel [15]). Therefore $\langle x \rangle$ is a commutative Wedderburn ring so $\langle x \rangle \cong F_p \alpha_1 \oplus \ldots \oplus F_p \alpha_K$. Then $F_p \alpha_i \in \mathfrak{H}'$ so the $\alpha_i \in S_p$. Therefore $R \in \mathfrak{J}_p(S_p)$.

Conversely, if $R \in \mathfrak{J}_p(S_p)$ then for all $x \in R$, $\langle x \rangle$ is a finite direct sum of fields in $\mathfrak{H}_p'$ so $\langle x \rangle \in \mathfrak{H}_p'$. Therefore $R \in \mathfrak{H}_p'$, so $\mathfrak{H}_p' = \mathfrak{J}_p(S_p)$.

If $\mathcal{N} \cap FC_p \leq \mathfrak{H}_p'$ we will show that $\mathfrak{H}_p' = \mathfrak{J}_p \mathcal{N}(S_p)$.

Suppose $R \in \mathfrak{H}_p' = FC_p \cap \mathfrak{H}'$ and $x \in R$. Then as above $\langle x \rangle / \mathcal{N}(\langle x \rangle) \in \mathfrak{J}_p(S_p)$. Since $R \in \mathfrak{H}_p'$, $R \in FC_p$ so $R \in \mathfrak{J}_p \mathcal{N}(S_p)$.

Conversely, if $R \in \mathfrak{J}_p \mathcal{N}(S_p)$ then for all $x \in R$, $\langle x \rangle / \mathcal{N}(\langle x \rangle)$ is a finite direct sum of rings in $\mathfrak{H}_p'$ so $\langle x \rangle / \mathcal{N}(\langle x \rangle) \in \mathfrak{H}_p'$. Since $R \in \mathfrak{J}_p \mathcal{N}(S_p)$, $R \in FC_p$ so $\mathcal{N}(\langle x \rangle) \in FC_p \cap \mathcal{N} \leq \mathfrak{H}_p'$. Therefore $\langle x \rangle \in \mathfrak{H}_p'$ so $R \in \mathfrak{H}_p'$. Hence $\mathfrak{H}_p = \mathfrak{J}_p \mathcal{N}(S_p)$.

From 3.4.9 we know that

$\aleph' \cap FC = \bigoplus\{\aleph'_p : p$ is a prime$\}$.

We shall now prove the converse. Assume that $\aleph = \bigoplus\{\aleph_{[p]} : p$ is a prime$\}$ where $\aleph_{[p]} = FC_p$ or $\mathcal{J}_p\mathcal{N}(S_p)$ or $\mathcal{J}_p(S_p)$ for some C.U.D. set of integers S_p for each prime p . Since $\aleph_{[p]} \leq FC_p \leq FC$ for all primes p , $\aleph \leq FC$.

First we shall prove that $\aleph = \aleph'$. Suppose $R \in \aleph$ and let $x \in R$. Since $R \in \aleph$, $R = \bigoplus\{\aleph_{[p]}(R) : p$ is a prime$\}$. Therefore $x = x_1 + \ldots + x_n$ where $x_i \in \aleph_{[p_i]}(R)$ and $p_i^{\alpha_i}x_i = 0$ for some integers $\alpha_i \geq 1$. Moreover, if $i \neq j$, $p_i \neq p_j$. Thus, for each i , there is an integer d_i such that $d_ix = x_i$. Since $(x_i)_{\langle x\rangle} \in FC_{p_i}$,

$(x_i)_{\langle x\rangle} \cap \sum_{j\neq i}(x_j)_{\langle x\rangle} = (0)$. Therefore,

$\langle x\rangle = (x_1)_{\langle x\rangle} + \ldots + (x_n)_{\langle x\rangle}$. Since $\aleph_{[p_i]}$ is strongly hereditary, $(x_i)_{\langle x\rangle} \in \aleph_{[p_i]}$. Therefore, $\langle x\rangle \in \aleph$; so $R \in \aleph'$.

Suppose $R \in \aleph'$. Then $R \in FC$ so $R = \bigoplus\{FC_p(R) : p$ is a prime$\}$. We shall show that $FC_p(R) = \aleph_{[p]}(R)$. Clearly $\aleph_{[p]}(R) \subseteq FC_p(R)$. Let $x \in FC_p(R)$. Since $R \in \aleph'$, $\langle x\rangle \in \aleph$. Therefore, $\langle x\rangle = \bigoplus\{\aleph_{[p]}(\langle x\rangle) : p$ is a prime$\}$. But $\langle x\rangle \in FC_p$ so

$\mathcal{H}_{[q]}(\langle x\rangle) = (0)$ for all $q \neq p$. Thus, $\langle x\rangle = \mathcal{H}_{[p]}(\langle x\rangle)$ so $\langle x\rangle \in \mathcal{H}_{[p]}$. Since $\mathcal{H}_{[p]} = \mathcal{H}'_{[p]}$, $FC_p(R) \in \mathcal{H}_{[p]}$. Therefore, $FC_p(R) = \mathcal{H}_{[p]}(R)$ so $R \in \mathcal{H}$. Hence $\mathcal{H} = \mathcal{H}'$ is an elementary class. Notice that we have shown that $\mathcal{H}_p = FC_p \cap \mathcal{H} \subseteq \mathcal{H}_{[p]}$ so clearly $\mathcal{H}_p = \mathcal{H}_{[p]}$.

Suppose $R \in \mathcal{H}$ and let $R' = R/I$ be a homomorphic image of R. Then $\mathcal{H}_{[p]}(R') \supseteq \mathcal{H}_{[p]}(R) + I/I$ so clearly $R' = \bigoplus\{\mathcal{H}_{[p]}(R') : p \text{ is a prime}\}$. Therefore $R' \in \mathcal{H}$ so $\mathcal{H}$ satisfies condition (A).

Suppose A is a ring and $B \lhd A$ such that A/B and $B \in \mathcal{H}$. Since both A/B and B are in FC, $A \in FC$. Therefore, by 3.1.6(iii)

$$A = \bigoplus\{FC_p(A) : p \text{ is a prime}\}. \tag{*}$$

Since $B \in \mathcal{H}$, $FC_p(B) = \mathcal{H}_{[p]}(B)$ and since both FC_p and $\mathcal{H}_{[p]}$ are hereditary (see Theorem 48 in Divinsky [7]),

$$\mathcal{H}_{[p]}(A) \cap B = \mathcal{H}_{[p]}(B) = FC_p(B) = FC_p(A) \cap B. \tag{**}$$

Now

$(FC_p(A)/FC_p(A) \cap B) \cong (FC_p(A) + B/B) \subseteq FC_p(A/B) \in \mathcal{H}_{[p]}$ since $A/B \in \mathcal{H}$. By (**), $FC_p(A) \cap B \in \mathcal{H}_{[p]}$, so since $\mathcal{H}_{[p]}$ is a radical class, $FC_p(A) \in \mathcal{H}_{[p]}$. Clearly $\mathcal{H}_{[p]}(A) \subseteq FC_p(A)$

so $\aleph_{[p]}(A) = FC_p(A)$. Now (*) implies that $A \in \aleph$.

Therefore, by 2.3.1, $\aleph$ is an elementary radical class.

Q.E.D.

The following list provides a representation of each of the elementary radical classes $\aleph \leq FC$ which we have already discussed. Let Z^+ be the set of all positive integers.

$$\mathcal{E}' = \bigoplus \{ \mathfrak{J}_p(Z^+) : p \text{ is a prime} \} .$$

$$\mathcal{S}' = \mathfrak{D}' = \mathcal{N} \cap FC = \bigoplus \{ \mathfrak{J}_p \mathcal{N}(\emptyset) : p \text{ is a prime} \} .$$

$$\mathcal{L}_R \cap FC_p = \mathfrak{J}_p \mathcal{N}(Z^+) .$$

$$\mathcal{N} \cap FC_p = \mathfrak{J}_p \mathcal{N}(\emptyset) .$$

$$\mathcal{L}_R \cap FC = \bigoplus \{ \mathfrak{J}_p \mathcal{N}(Z^+) : p \text{ is a prime} \} .$$

The relationships between the elementary radical classes which we have discussed can be illustrated by the following diagrams.

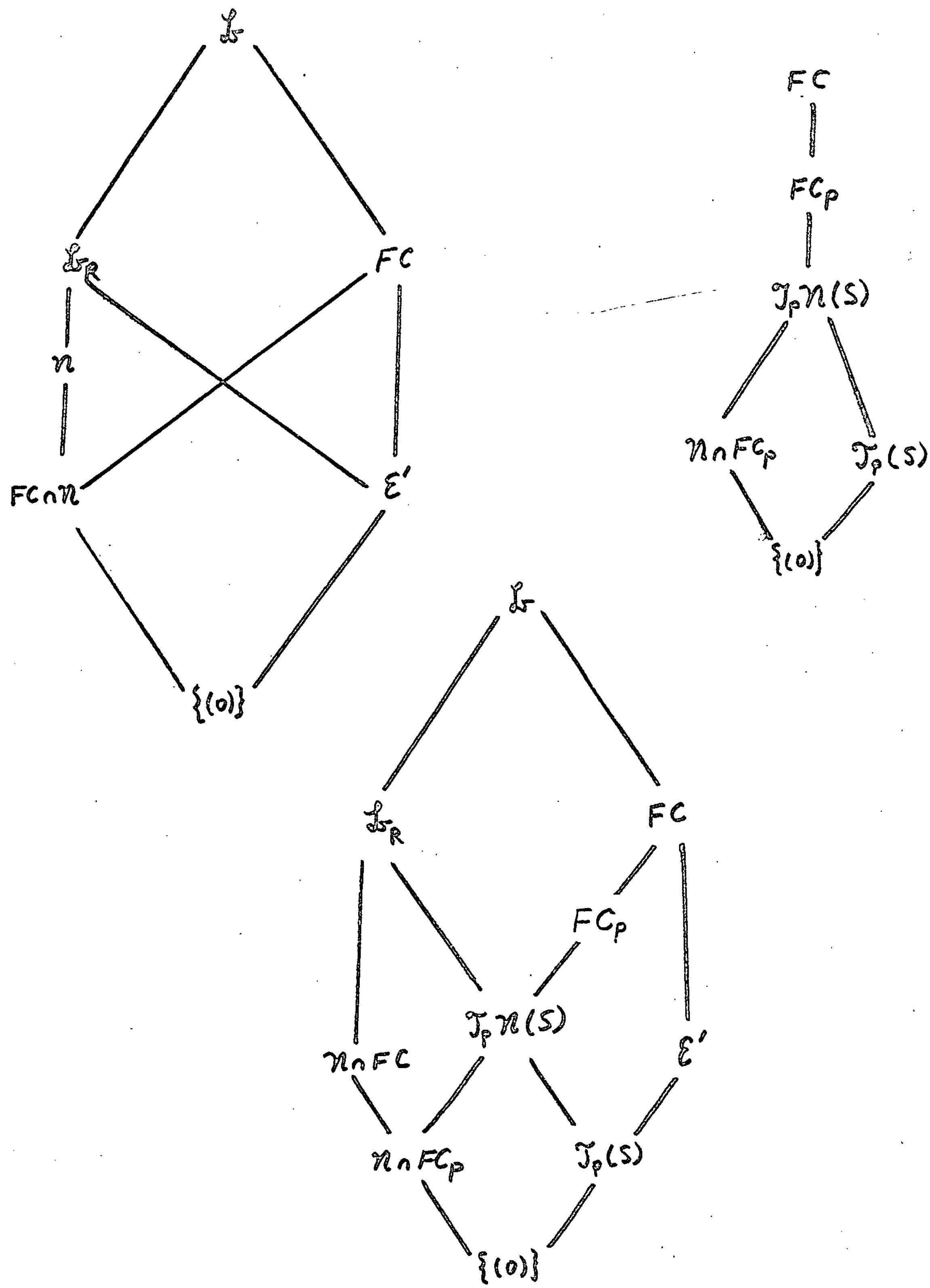


ILLUSTRATION 1

CHAPTER IV

GENERALIZED ELEMENTARY AND LOCAL RADICAL CLASSES

4.1 ABSORBENT CARDINAL NUMBERS.

In this first section of Chapter IV we shall prove that 2 and $\aleph_0$ are absorbent cardinals. In fact, we shall show that any cardinal K such that $2 \leq K \leq \aleph_0$ is absorbent. We begin with the following two lemmas.

4.1.1 LEMMA:

If $R \neq (0)$ and R is a zero ring $(R^2 = (0))$ of characteristic p for some prime p then R can be homomorphically mapped onto Cp = the zero ring on the cyclic group of p elements.

Proof:

Let R be a ring such that $R \neq (0)$, $R^2 = (0)$ and $pR = (0)$. Choose $x \in R$, $x \neq 0$. By Zorn's Lemma choose K maximal in $Z = \{I \triangleleft R : x \notin I\}$. Suppose $0 \neq w \in R$ and $w \notin K$. Then, $x \in (w)_R + K$ so there is a non-zero integer n such that $x = nw + y$ when $y \in K$. Since $pR = (0) \subseteq K$ and $x \notin K$, p does not divide n. Therefore, there are integers r and s such that $rp + sn = 1$; so, $w = (rp + sn)w = snw = sx - sy$. Thus, $\overline{w} \in \langle\overline{x}\rangle \subseteq R/K$ so $R/K = \langle\overline{x}\rangle \cong Cp$.

Q.E.D.

Let p be a prime number. The ring p^{∞} is discussed in Rings and Radicals, Divinsky [7]. We may think of this ring as the set of all rational numbers of the form $\frac{a}{p^n}$ where p does not divide a. The addition is modulo 1 and the multiplication is trivial. All ideals of p^{∞} are isomorphic to Cp^n = the zero ring on the cyclic group of p^n elements for some n, and all non-zero homomorphic images of p^{∞} are isomorphic to p^{∞}.

4.1.2 LEMMA:

If R is a zero ring and there is an $x \in R$ such that $x \neq 0$ but $px = 0$ for some prime p then R can be homomorphically mapped onto Cp or onto p^{∞}.

Proof:

Let R be a ring such that $R^2 = (0)$ and let $x \in R$ such that $x \neq 0$ but $px = 0$. By Zorn's Lemma choose I maximal in $Z = \{I \triangleleft R : x \notin I\}$.

If $p(R/I) \neq R/I$ then R/I can be homomorphically mapped onto a zero ring of characteristic p so by Lemma 4.1.1 R can be homomorphically mapped onto Cp.

Suppose $p(R/I) = R/I$. Let $w \in R$ and $w \notin I$. Then $x \in (w)_R + I$ so there is a non-zero integer n such that $x - nw \in I$. Therefore $pnw \in I$ so $R/I \in FC$. Now $FC_p(R/I)$ is a direct summand of R/I and hence a homomorphic image of R/I . Therefore, $FC_p(R/I) \cong R/K$ for some ideal K of R and since $\overline{x} \in FC_p(R/I)$, $x \notin K$. By the maximality of I , $FC_p(R/I) = R/I$.

Let $\overline{w} \in R/I$ and n be a positive integer. Write $n = p^{\alpha}n'$ where p does not divide n' . Since $p(R/I) = R/I$, $p^{\alpha}(R/I) = R/I$ so there is an element $\overline{v} \in R/I$ such that $p^{\alpha}\overline{v} = \overline{w}$. Now, $R/I \in FC_p$ so there is an integer k such that $p^k\overline{w} = \overline{0}$. Since p does not divide n' there are integers r and s such that $rp^k + sn' = 1$. Then $\overline{w} = (rp^k + sn')\overline{w} = sn'\overline{w} = sn'p^{\alpha}\overline{v} = n'p^{\alpha}(s\overline{v}) = n(s\overline{v})$. Therefore, the additive group R/I^+ is divisible. Since $(R/I)^2 = (0)$ the ideals of R/I are just the subgroups of the abelian group R/I^+ . Therefore, by the theorem for divisible torsion groups (see for instance, Fuchs [8]), R/I is isomorphic to a direct sum of copies of p^{∞} . Therefore R/I can be homomorphically mapped onto the ring p^{∞} .

Q.E.D.

We are now ready to prove the theorem.

4.1.3 <u>THEOREM</u>:

If K is a cardinal number and $2 \leq K \leq \aleph_0$ then K is absorbent.

<u>Proof</u>:

Let K be a cardinal, $2 \leq K \leq \aleph_0$. Suppose that

$\mathcal{S}$ is a class of rings and $I \in \mathcal{S}_{g(K)}$ where $I \triangleleft R'$ and R' is a K-subring of the ring R . Let $M = (I)_R = I + IR + RI + RIR$.

Suppose that $M \notin \mathcal{S}_{g(K)}$. Then there is a non-zero homomorphic image $M' \cong M/K'$ of M such that no K-subring of M' is in $\mathcal{S}$. Since $I \in \mathcal{S}_{g(K)}$, $I \subseteq K'$. Therefore, by Andrunakievic's Lemma (Lemma 61 in Divinsky [7]) $M^3 \subseteq (I)_M \subseteq K'$. We shall consider two cases.

Case 1: There is a $z \in M$ such that $C^\infty \cong \langle \bar{z} \rangle \subseteq M/K'$. First we shall prove that $I/I^2 \notin FC$. Suppose $I/I^2 \in FC$. Let $x \in I$, then $mx = \sum_{i=1}^{N} u_i v_i$ where u_i , $v_i \in I$ and m is a non-zero integer. Now, there is a non-zero integer k such that $ku_i \in I^2$ for $i = 1, \ldots, n$. Therefore $(km)x \in I^3$, so $I/I^3 \in FC$. But then if $w \in M = (I)_R$, $nw \in M^3 \subseteq K'$ for some integer $n \neq 0$. This is impossible because $\bar{z} \in M/K'$ and $\langle \bar{z} \rangle \notin FC$. Therefore $I/I^2 \notin FC$ so I/I^2 can be homomorphically mapped onto a non-zero ring L such that $L^2 = (0)$ and L has characteristic 0 (factor out $FC(I/I^2)$).

Choose $x \in L$, $x \neq 0$ and by Zorn's Lemma choose H maximal in the class $Z = \{J \triangleleft L$: if n is a non-zero integer then $nx \notin J\}$. Then $FC(L/H) = (0)$, for if $y \in L$, $y \notin H$ but $ny \in H$ for some non-zero integer n then by the maximality of H , $kx \in (y)_L + H$ for some non-zero integer k so $(nk)x \in H$ which contradicts the way in which H was chosen.

Let S be a K-subring of L/H. Since $K \leq \aleph_0$, S is finitely generated. Because L/H is a zero ring the ideals of S are just the subgroups of the additive group S^+. Since S is finitely generated we may apply the fundamental theorem for finitely generated abelian groups to see that S is isomorphic to a finite direct sum of copies of C^∞.

If u and v are two non-zero elements of L/H there are non-zero integers k and n such that $kx \in (u)_L + H$ and $nx \in (v)_L + H$. Hence there are non-zero integers r and s such that $kx - ru \in H$ and $nx - sv \in H$. This insures that the direct sum in the preceding paragraph is of length 1 ; that is, $S \cong C^\infty$.

Now, since $I \in \mathcal{S}_{g(K)}$, some K-subring of L/H (which is a homomorphic image of I) is in $\mathcal{S}$. Since all K-subrings of L/H are isomorphic to C^∞, $C^\infty \in \mathcal{S}$. This is a contradiction since we assumed that M/K' did not contain a K-subring.

<u>Case 2</u>: There is a $z \in M$ such that $Cp \cong \langle\bar{z}\rangle \subseteq M/K'$ for some prime p.

If $p(I/I^2) \neq I/I^2$ then I/I^2 can be homomorphically mapped onto a zero ring $L \neq (0)$ of characteristic p. By Lemma 4.1.1, I/I^2 can be homomorphically mapped onto Cp. So, since $I \in \mathcal{S}_{g(K)}$, $Cp \in \mathcal{S}$. This is a contradiction because we have assumed that M/K' has no K-subrings in $\mathcal{S}$.

We must conclude then, that $p(I/I^2) = I/I^2$. Since $z \in M = (I)_R$ there are integers m_i and elements $x_i \in I$ and $r_i, s_i, \bar{r}_i, \bar{s}_i \in R$ such that

$$z = \sum_{i=1}^{L} r_i x_i s_i + \bar{r}_i x_i + x_i \bar{s}_i + m_i x_i \ . \qquad (*)$$

Now, $p(I/I^2) = I/I^2$, so $pI + I^2 = I$. Hence $I^2 = I(pI + I^2) = pI^2 + I^3$ so $I = pI + I^2 = PI + pI^2 + I^3 = pI + I^3$. Thus $p^n(I/I^3) = I/I^3$ for all positive integers n . For each $n \geq 1$ choose $x_{i_n} \in I$ such that $x_i - p^n x_{i_n} \in I^3$.

Let $z_n = \sum_{i=1}^{L} r_i x_{i_n} s_i + \bar{r}_i x_{i_n} + x_{i_n} \bar{s}_i + m_i x_{i_n}$. Then $z - p^n z_n \in M^3 \subseteq K'$ so $\bar{z} = p^n \bar{z}_n \neq \bar{0}$ in M/K' . Moreover, since $p\bar{z} = 0$, $p^{n+1}\bar{z}_n = \bar{0}$ so $\langle \bar{z}_n \rangle \cong Cp^{n+1}$ in M/K' .

Now, suppose that for all $x \in I$, $x \in (px)_I + I^2$. Then if $x \in I$, there is an integer n such that $x - pnx \in I^2$ so $(1 - pn)x \in I^2$. Therefore $I/I^2 \in FC$.

Suppose $FC_p(I/I^2) = (0)$. Let $x \in I$. Then there is a non-zero integer r such that p does not divide r and $rx = \sum_{i=1}^{H} u_i v_i \in I^2$. Now there is a non-zero integer s which is not divisible by p such that $su_i \in I^2$ for $i = 1, \ldots, H$. Thus, $srx \in I^3$ and p does not divide

sr . Therefore, we may choose a non-zero integer k which is not divisible by p and such that $kx_i \in I^3$ for $i = 1, \ldots, L$. Now from (*), $kz \in (I^3)_R \subseteq M^3 \subseteq K'$. This contradicts our assumption that $\langle \bar{z} \rangle \cong Cp$.

Therefore $FC_p(I/I^2) \neq (0)$ so by Lemma 4.1.2 I/I^2 can be homomorphically mapped onto Cp or onto p^∞ .

On the other hand, if there is an $x \in I$ such that $x \notin (px)_I + I^2$ then I/I^2 can be homomorphically mapped onto $I/((px)_I + I^2)$ which by Lemma 4.1.2 can be homomorphically mapped onto Cp or onto p^∞ .

Thus our assumption in Case 2 leads to the conclusion that I can be homomorphically mapped to Cp or to p^∞ . We have seen that the conclusion that I can be homomorphically mapped to Cp leads to a contraduction.

Now, if I can be homomorphically mapped onto p^∞, $p^\infty \in \mathcal{S}_{g(K)}$. Since all K-subrings are finitely generated, $Cp^n \in \mathcal{S}$ for some positive integer n . But then $\langle \bar{z}_{n-1} \rangle \in \mathcal{S}$ if $n \geq 2$ and $\langle \bar{z} \rangle \in \mathcal{S}$ if $n = 1$. In any case this contradicts our assumption that M/K' contains no K-subring in $\mathcal{S}$.

Either Case 1 or Case 2 must occur since $(M/K')^3 = (0)$. Both cases lead to a contradiction so we conclude that $M \in \mathcal{S}_{g(K)}$. Therefore K is an absorbent cardinal.

Q.E.D.

4.2 GENERALIZED RADICAL CLASSES WHICH ARE $\subsetneq \mathcal{N}_g$.

We shall refer to generalized 2-classes as generalized elementary classes. Generalized $\aleph_0$-classes will be referred to as generalized local classes. We shall write $\mathcal{N}_g$ for $\mathcal{N}_{g(\aleph_0)}$ and $\mathcal{N}_{g_1}$ for $\mathcal{N}_{g(2)}$. The following proposition shows that this will not conflict with our notation for the generalized nil radical class of Andrunakievic and Thierrin.

4.2.1 PROPOSITION:

$$\mathcal{N}_g = \mathcal{N}_{g(2)} = \mathcal{N}_{g(\aleph_0)} .$$

Proof:

Assume that $R \in \mathcal{N}_g$. Let R' be a non-zero homomorphic image of R . Then $R' \in \mathcal{N}_g$ so by 1.2.1 there is a non-zero nilpotent element in R' . Therefore $R \in \mathcal{N}_{g(2)}$.

If $R \in \mathcal{N}_{g(2)}$ then every non-zero homomorphic image of R contains a nil subring so clearly no non-zero homomorphic image of R is $\mathcal{N}_g$s.s. Therefore, $R \in \mathcal{N}_g$.

By 2.4.2(vii), $\mathcal{N}_{g(2)} = \mathcal{N}_{g(\aleph_0)}$ so
$\mathcal{N}_g = \mathcal{N}_{g(2)} = \mathcal{N}_{g(\aleph_0)}$.

Q.E.D.

4.2.2 THEOREM:

If $\mathcal{N}$ is any class of rings such that $\beta \leq \mathcal{N} \leq FF$

then $\aleph_{g_1} = \mathcal{N}_g$.

Proof:

Assume that $\aleph$ is a class of rings and $\beta \leq \aleph \leq FF$. Since $\mathcal{N}_g = \mathcal{N}_{g_1}$, by 2.4.2(ii) we need only show that $\beta_{g_1} \geq \mathcal{N}_g$ and $FF_{g_1} \leq \mathcal{N}_g$.

Since a ring generated by one element is in β if and only if it is in $\mathcal{N}$ it is clear that $\beta_{g_1} = \mathcal{N}_{g_1} = \mathcal{N}_g$.

Let R be a ring such that $R \notin \mathcal{N}_g$. Then there is a non-zero homomorphic image R' of R such that R' has no non-zero nilpotent elements. Thus, for all $0 \neq x \in R'$, $\langle x \rangle$ is $\mathcal{N}$ semi-simple. In Theorem 3.3.2 we proved that such a ring $\langle x \rangle$ could be homomorphically mapped onto a finite field. Hence, no non-zero subring $\langle x \rangle$ of R' is in FF , so $R \notin FF_{g_1}$. Therefore, $FF_{g_1} \leq \mathcal{N}_g$.

Q.E.D.

If $R \in \mathcal{N}_g$ then every non-zero homomorphic image of R contains a subring $\langle x \rangle$ such that $\langle x \rangle$ is nilpotent. Thus, $R \in \beta_{g_1} \leq \beta_g$ (by 2.4.2(iii)). Now, by 2.4.2(ii) we must have $\mathcal{N}_g = \mathcal{L}_g = \beta_g$.

Using 2.4.2(ii) again we see that $\mathcal{N}_g \leq J_g \leq \mathcal{B}_g \leq \mathfrak{J}_g \leq F_g \leq FF_g$. Almost all questions concerning these radical classes are open. We do not even know which of the above inclusions are strict. Notice however, that $FF_g \leq FF$ since subrings of finite fields are finite fields.

The next proposition is concerned with the generalized classes associated with $\mathcal{L}$ and $\mathcal{L}_R$. Notice that by 2.4.2(vii) $\mathcal{L}_g = \mathcal{L}_{g_1}$ and $(\mathcal{L}_R)_g = (\mathcal{L}_R)_{g_1}$.

4.2.3 <u>PROPOSITION</u>:

(i) $R \in \mathcal{L}_g$ if and only if $R/(\mathcal{L}_R)_g(R) \in FC$.

(ii) $\mathcal{L}_g \gneqq (\mathcal{L}_R)_g \gneqq \mathcal{N}_g$.

(iii) R is $\mathcal{L}_g$ semi-simple if and only if for all $x \in R$, $\langle x\rangle \cong \mathcal{P}[X]$.

<u>Proof</u>:

(i) Assume that $R \in \mathcal{L}_g$. If $R/(\mathcal{L}_R)_g(R) \notin FC$ let $I \triangleleft R$ such that $I \supseteq (\mathcal{L}_R)_g(R)$ and $I/(\mathcal{L}_R)_g(R) = FC(R/(\mathcal{L}_R)_g(R))$. Since $R/I \in \mathcal{L}_g$ there is an $x \in R$ such that $x \notin I$ and $a_nx^n + \ldots + a_1x \in I$ for some integers $a_1, \ldots, a_n$. Then there is a non-zero integer m such that
$ma_nx^n + \ldots + ma_1x \in (\mathcal{L}_R)_g(R)$. Let $b_i = ma_i(ma_n)^{n-i-1}$ and let $y = ma_nx$. Then

$$y^n + b_{n-1}y^{n-1} + \ldots + b_1y \in (\mathcal{L}_R)_g(R) . \qquad (\dagger)$$

Since $x \notin I$, $y \notin I$ so certainly $y \notin (\mathcal{L}_R)_g(R)$. We will prove that $\langle\bar{y}\rangle = \dfrac{\langle y\rangle + (\mathcal{L}_R)_g(R)}{(\mathcal{L}_R)_g(R)}$ is in $(\mathcal{L}_R)g$.

By (†), the additive group $\langle\bar{y}\rangle^+$ is finitely generated so if $w \in \langle\bar{y}\rangle$, $\langle w\rangle^+$ is also a finitely generated abelian group. Thus there are polynomials $f_1(w), \ldots, f_K(w)$ which generated $\langle w\rangle^+$. Choose an integer h which is larger than the degree of each f_i. Then $w^h = \sum_{i=1}^{L} b_i f_i(w)$ for some integers $b_1, \ldots, b_2$. Therefore, $\langle\bar{y}\rangle \in \mathcal{L}_R$.

Since $\mathcal{L}_R$ satisfies condition (A), $\langle\bar{y}\rangle \in (\mathcal{L}_R)_g$. Now since $\aleph_0$ is absorbent (see 4.1.3) the non-zero ideal of $R/(\mathcal{L}_R)_g(R)$ which is generated by $\langle\bar{y}\rangle$ is in $(\mathcal{L}_R)_g$. This is a contradiction. Hence $R/(\mathcal{L}_R)_g(R) \in FC$.

Both FC and $(\mathcal{L}_R)_g$ are $\leq \mathcal{L}_g$ so the converse is obvious.

(ii) Clearly $\mathcal{L}_g \geq (\mathcal{L}_R)_g \geq \mathcal{N}_g$. The ring $F_p[X] \in \mathcal{L}_g$ but is not in $(\mathcal{L}_R)_g$ so $\mathcal{L}_g \neq (\mathcal{L}_R)_g$. No finite field is in $\mathcal{N}_g$ but they are all in $(\mathcal{L}_R)_g$ so $(\mathcal{L}_R)_g \neq \mathcal{N}_g$.

(iii) This follows immediately from 2.4.15 and 3.1.3.

Q.E.D.

We shall now consider the classes $((\mathcal{L}_R)_g)'$ and

$(\mathcal{L}_g)'$. By 2.4.2(ix) both classes are radical classes and clearly $((\mathcal{L}_R)_g)' = ((\mathcal{L}_R)_g)^*$ and $(\mathcal{L}_g)' = (\mathcal{L}_g)^*$. Since $\mathcal{L}$ and $\mathcal{L}_R$ are elementary radical classes, $(\mathcal{L}_g)' \geq \mathcal{L}$ and $((\mathcal{L}_R)_g)' \geq \mathcal{L}_R$.

If $R \in (\mathcal{L}_g)'$ and $x \in R$ then $\langle x\rangle \in \mathcal{L}_g$ so $\langle x\rangle \not\cong \mathcal{P}[X]$, thus $\langle x\rangle \in \mathcal{L}$ so $R \in \mathcal{L}$. Therefore, $(\mathcal{L}_g)' = \mathcal{L}$.

Now $\mathcal{L}_R \leq (\mathcal{L}_R)_g \leq \mathcal{L}_g$ so $\mathcal{L}_R \leq ((\mathcal{L}_R)_g)' \leq \mathcal{L}$.

4.2.4 PROPOSITION:

(1) $FC \cap ((\mathcal{L}_R)_g)' \leq \mathcal{L}_R$.

(2) $R \in ((\mathcal{L}_R)_g)'$ if and only if for all $x \in R$, $\langle x\rangle/\mathcal{L}_R(\langle x\rangle)$ has characteristic 0 and is in $\mathcal{L}$.

Proof:

(1) Suppose $R \in FC \cap ((\mathcal{L}_R)_g)'$. Then $R = \bigoplus\{FC_p(R) : p$ is a prime$\}$ so since $\mathcal{L}_R$ is a radical class it is sufficient to show that $FC_p \cap ((\mathcal{L}_R)_g)' \leq \mathcal{L}_R$ for each prime p . Let $A \in FC_p \cap ((\mathcal{L}_R)_g)'$ and let x be a non-zero element of A . Then there are integers $a_1, \ldots, a_\ell$ and a $y = a_1x + \ldots + a_\ell x^\ell \notin p\langle x\rangle$ such that $y^K + b_{K-1}y^{K-1} + \ldots + b_1y \in p\langle x\rangle$ for some integers $b_{K-1}, \ldots, b_1$. We may assume that p does not divide a_ℓ . Thus $x_1 = x^{K\ell} + c_{K\ell-1}x^{K\ell-1} + \ldots + c_1x \in p\langle x\rangle$

for some integers $c_1, \ldots, c_{K\ell-1}$. By repeating the above argument we see that for each integer $n \geq 1$ there is a monic polynomial in x which is in $p^n\langle x\rangle$. Since $p^\alpha x = 0$ for some integer α , we conclude that $A \in \mathcal{L}_R$.

(2) Assume that $R \in ((\mathcal{L}_R)_g)'$ and let $x \in R$. Then since $\langle x\rangle/\mathcal{L}_R(\langle x\rangle)$ is $\mathcal{L}_R$ semi-simple, by part (1) $\langle x\rangle/\mathcal{L}_R(\langle x\rangle)$ must be of characteristic 0 . Since $((\mathcal{L}_R)_g)' \leq \mathcal{L}$, $\langle x\rangle \in \mathcal{L}$.

Conversely, assume that for all $x \in R$, $\langle x\rangle/\mathcal{L}_R(\langle x\rangle) \in \mathcal{L}$ and has characteristic 0 . Let $x \in R$ such that $\langle\overline{x}\rangle = \langle x\rangle/\mathcal{L}_R(\langle x\rangle) \neq (0)$. Then $\langle\overline{x}\rangle \in \mathcal{L}$ so $a_K\overline{x}^K + \ldots + a_1\overline{x} = 0$ for integers $a_K, \ldots, a_1$. Since $\langle\overline{x}\rangle$ has characteristic 0 we may assume that $K \geq 2$ and that the greatest common divisor of the a_i is 1 . Now if $y = a_K\overline{x}$,

$$y^K + a_{K-1}y^{K-1} + (a_Ka_{K-2})y^{K-2} + \ldots + (a_K{}^{K-2}a_1)y = 0 .$$

This guarantees that $\langle y\rangle \in (\mathcal{L}_R)_g$. Therefore, since $\mathcal{H}_0$ is absorbent, $(y)_{\langle\overline{x}\rangle} \in (\mathcal{L}_R)_g$. Now, since $a_K\overline{x} = y \in (y)_{\langle\overline{x}\rangle}$ and the greatest common divisor of the a_i is 1 , the ring $\langle\overline{x}\rangle/(y)_{\langle\overline{x}\rangle}$ is finite and hence in $(\mathcal{L}_R)_g$. Thus, $\langle\overline{x}\rangle \in (\mathcal{L}_R)_g$. Therefore, $\langle x\rangle \in (\mathcal{L}_R)_g$ so $R \in ((\mathcal{L}_R)_g)'$.

Q.E.D.

It follows immediately that $\mathcal{L}_R \subsetneqq ((\mathcal{L}_R)_g)' \subsetneqq \mathcal{L}_g$. By 2.4.2(vii) $(((\mathcal{L}_R)_g)')_g = (\mathcal{L}_R)_g$.

4.3 GENERALIZED RADICAL CLASSES WHICH ARE $\leq$ FC .

4.3.1 LEMMA:

Let p be a prime. Then $(FC_p)_{g(K)} = FC_p$ for all cardinal numbers $K \geq 2$.

Proof:

Let p be a prime and K be a cardinal ≥ 2 .

Since FC_p satisfies condition (A) and FC_p is strongly hereditary, $FC_p \leq (FC_p)_{g(K)}$.

Suppose $R \in (FC_p)_{g(K)}$. Let $\overline{R} = R/FC_p(R)$. If $\overline{R} \neq (0)$ then since $R \in (FC_p)_{g(K)}$ there is a non-zero K-subring S of $\overline{R}$ such that $S \in FC_p$. But then $(S)_{\overline{R}} \in FC_p$ which is a contradiction since $\overline{R}$ is FC_p semi-simple. Hence $\overline{R} = (0)$ so $R \in FC_p$.

Q.E.D.

4.3.2 PROPOSITION:

Let $\aleph$ be an elementary radical class $\leq$ FC and let $\aleph = \bigoplus\{\aleph_p : p \in S\}$ be the representation of $\aleph$ given in Theorem 3.4.14. Then for any cardinal $K \geq 2$, $R \in \aleph_{g(K)}$

if and only if $R = \bigoplus\{(\aleph_p)_{g(K)}(R) : p \in S\}$. Thus we may denote $\aleph_{g(K)}$ by $\bigoplus\{(\aleph_p)_{g(K)} : p \in S\}$.

Proof:

Let S be a set of primes, $\aleph = \bigoplus\{\aleph_p : p \in S\}$, and let K be a cardinal ≥ 2 . Since $\aleph_p = \aleph \cap FC_p \leq FC_p$, $(\aleph_p)_{g(K)} \leq (FC_p)_{g(K)} = FC_p$. Thus, for any ring R , the sum of ideals $(\aleph_p)_{g(K)}(R)$ is direct.

Suppose $R \in \aleph_{g(K)}$. Then $R \in FC_{g(K)}$ and just as in the proof of 4.3.1, $R \in FC$. Hence by 3.1.6 $R = \bigoplus\{FC_p(R) : p$ is a prime$\}$. Now, for any prime p , $FC_p(R)$ is a homomorphic image of R so since $R \in \aleph_{g(K)}$, $FC_p(R) = (0)$ if $p \notin S$. Thus, $R = \bigoplus\{FC_p(R) : p \in S\}$.

Let $p \in S$ and let $\overline{R}$ be a non-zero homomorphic image of $FC_p(R)$. Then $\overline{R}$ is a homomorphic image of $R \in \aleph_{g(K)}$ so there is a non-zero K-subring $H \subseteq \overline{R}$ such that $H \in \aleph$. Now $H \in \aleph \cap FC_p = \aleph_p$ so $FC_p(R) \in (\aleph_p)_{g(K)}$. By 4.3.1 $(\aleph_p)_{g(K)} \leq FC_p$ so $FC_p(R) = (\aleph_p)_{g(K)}(R)$. Therefore, $R = \bigoplus\{(\aleph_p)_{g(K)}(R) : p \in S\}$.

Conversely, suppose that $R = \bigoplus\{(\aleph_p)_{g(K)}(R) : p \in S\}$. Since $\aleph_p \leq \aleph$ for all $p \in S$ it is clear that $R \in \aleph_{g(K)}$.

This completes the proof.

Q.E.D.

Combining 4.3.1 and 4.3.2 we see that if S is any set of primes then $\mathfrak{N} = \bigoplus\{FC_p : p \in S\}$ is a generalized K-class for all cardinals $K \geq 2$. In fact, $\mathfrak{N}_{g(K)} = \mathfrak{N}$. Now $\mathfrak{N}$ is also a K-class. In the next theorem we shall show that these are the only classes which are both K-classes and generalized K-classes (except; of course, for the class of all rings).

4.3.3 <u>THEOREM</u>:

Let K be a cardinal number ≥ 2 and let $\mathfrak{R}$ be a class of rings which does not contain all rings. Then $\mathfrak{R}$ is a strongly hereditary generalized K-class if and only if $\mathfrak{R} = \bigoplus\{FC_p : p \in S\}$ for some set of prime numbers S.

<u>Proof</u>:

Assume that S is a set of prime numbers and $\mathfrak{R} = \bigoplus\{FC_p : p \in S\}$. By 3.4.14 $\mathfrak{R}$ is an elementary radical class so certainly $\mathfrak{R}$ is strongly hereditary. From 4.3.1 and 4.3.2 $\mathfrak{R}$ is a generalized K-class.

Conversely, assume that $\mathfrak{R}$ is a strongly hereditary generalized K-class. Then there is a class of rings $\mathfrak{J}$ such that $\mathfrak{R} = \mathfrak{J}_{g(K)}$. Since $\mathfrak{R}$ is strongly hereditary $\mathfrak{R} \leq \mathfrak{R}(2)$ and by 2.4.2(viii), $(\mathfrak{J}_{g(K)})(2) \leq (\mathfrak{J}_{g(K)})_{g(2)} \leq \mathfrak{J}_{g(K)}$. Thus $\mathfrak{R} \leq \mathfrak{R}(2) = (\mathfrak{J}_{g(K)})(2) \leq (\mathfrak{J}_{g(K)})_{g(2)} \leq \mathfrak{J}_{g(K)} = \mathfrak{R}$. Hence $\mathfrak{R} = \mathfrak{R}(2) = (\mathfrak{J}_{g(K)})_{g(2)} = \mathfrak{R}_{g(2)}$. Now we shall prove that if

$\mathcal{R} \not\leq FC$ then $\mathcal{R}$ is the class of all rings. Suppose $\mathcal{R} \not\leq FC$, then there is a ring $R \in \mathcal{R}$ such that $FC(R) = (0)$. First we shall prove that $C^\infty \in \mathcal{R}$.

Let $0 \neq x \in R$. Then $\langle x\rangle \in \mathcal{R}$ so $\langle x\rangle/\langle x\rangle^2 \in \mathcal{R}$. If $\langle x\rangle/\langle x\rangle^2 \not\cong C^\infty$ then $nx \in \langle x\rangle^2$ for some integer n. Thus, there are integers $a_2, \ldots, a_K$ such that $nx = a_2x^2 + \ldots + a_Kx^K$. Let $y = a_2x + \ldots + a_Kx^{K-1}$. Then $nx = yx$ so $\langle y\rangle \cong n\cdot Z$ = the ideal of the integers generated by n. Since $\mathcal{R} = \mathcal{R}(2)$, $\langle y\rangle \in \mathcal{R}$. Now consider $Y = (\langle y\rangle)_2$ = the ring of 2x2 matrices with entries from $\langle y\rangle$. Then every non-zero homomorphic image of Y contains a subring generated by one element which is isomorphic to a homomorphic image of $\langle y\rangle$. Therefore $Y \in \mathcal{R}_{g(2)} = \mathcal{R}$. Hence, $C^\infty \cong \langle\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}\rangle \in \mathcal{R}(2) = \mathcal{R}$. So in any case $C^\infty \in \mathcal{R}$.

Since $C^\infty \in \mathcal{R}$ and β = the lower radical class determined by $\{C^\infty\}$, $\beta \leq \mathcal{R}$. Thus $\mathcal{N} = \beta(2) \leq \mathcal{R}(2) = \mathcal{R}$. Let $Q(X)$ be the field of rational functions in an indeterminate X over Q = the field of rational numbers. Then $(Q(X))_2$ is a simple ring with non-zero nilpotent elements. Thus $(Q(X))_2 \in \mathcal{R}_{g(2)} = \mathcal{R}$. But then, since $\mathcal{R} = \mathcal{R}(2)$, $\mathcal{P}[X] \cong \langle\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\rangle \in \mathcal{R}$. Then all rings generated by one element are in $\mathcal{R}$ since $\mathcal{R}$ satisfies condition (A). Therefore $\mathcal{R} = \mathcal{R}_{g(2)}$ = the class of all rings.

Since $\mathcal{R}$ is not the class of all rings, $\mathcal{R} \leq FC$.

Let $S = \{p : \mathfrak{R}_p \neq \{(0)\}\}$. Then, by Theorem 3.4.14, $\mathfrak{R} = \bigoplus\{\mathfrak{R}_p : p \in S\}$. We will show that $Fp[X]$ = the ring of polynomials in an indeterminate X over Fp is in $\mathfrak{R}_p$ and hence $\mathfrak{R}_p = FC_p$ for all $p \in S$.

Let $(0) \neq \langle x \rangle \in \mathfrak{R}_p$. Then if $\langle x \rangle / \langle x \rangle^2$ has a non-zero nilpotent element, Cp = the zero ring on the group of p-elements is in $\mathfrak{R}_p$. If $\langle x \rangle / \langle x \rangle^2$ has no non-zero nilpotent elements then $\langle x \rangle$ has an identity so $Zp \in \mathfrak{R}_p$. In this case $(Zp)_2 \in \mathfrak{R}_p \leq \mathfrak{R}$ so since $\mathfrak{R}(2) = \mathfrak{R}$, $Cp \in \mathfrak{R}_p$. So in any case $Cp \in \mathfrak{R}_p$. Now $(Fp[X])_2 \in \mathfrak{R}_{g(K)} = \mathfrak{R}$ so $Fp[X] \in \mathfrak{R}(2) = \mathfrak{R}$. Therefore every ring of characteristic p is in $\mathfrak{R}_p$. Now if B is an ideal of a ring A and both A/B and B are in $\mathfrak{R}_p$ then $A \in \mathfrak{R}_p$. Hence $FC_p \leq \mathfrak{R}_p$.

Therefore, $\mathfrak{R}_p = FC_p$ for all primes $p \in S$. This completes the proof.

Q.E.D.

We now turn to a consideration of some classes of rings $\mathfrak{H}$ such that $\mathfrak{H}_g = \mathfrak{N}_g \cap FC$.

4.3.4 <u>PROPOSITION</u>:

If $R \in \mathfrak{D}$ and R is finitely generated then $R \in \mathcal{S}$.

Proof:

Let R be a finitely generated ring in $\mathfrak{D}$. If $R \in FC$ then $R \in \beta \cap FC$ so $R \in \mathfrak{J}$ by 3.2.1.

Suppose that $R \notin FC$. Let $\overline{R} = R/FC(R)$. Since $\overline{R} \in \mathfrak{D} \leq \mathfrak{L}$, there is a positive integer n such that $\overline{R}^n \neq (0)$ but $\overline{R}^{n+1} = (0)$. Since $\overline{R} \in \mathfrak{D}$, there is a ring A with D.C.C. on left ideals such that $\overline{R} = \mathfrak{N}(A)$. Now $\overline{R}^n$ is finitely generated as a subring of $\overline{R}$ by 2.2.6. Moreover, $\overline{R}^n$ has characteristic 0 so from the Fundamental Theorem for finitely generated abelian groups $\overline{R}$ is isomorphic to a finite direct sum of copies of C^∞ . But then $2\overline{R}^n \supsetneqq 2^2\overline{R}^n \supsetneqq \ldots \supsetneqq 2^k\overline{R}^n \supsetneqq \ldots$ and since $2^k\overline{R}^n \triangleleft A$ for all positive integers k this contradicts the D.C.C. condition for A .

Thus we must have $R \in FC$ so $R \in \mathfrak{J}$.

Q.E.D.

Since $\mathfrak{J} \leq \mathfrak{D}$ it follows immediately from 4.3.4 that $\mathfrak{D}_{g_1} = \mathfrak{J}_{g_1}$ and $\mathfrak{D}_g = \mathfrak{J}_g$.

Let R be a finitely generated ring in $\mathfrak{D}$ and let $x \in R$. Now, $R \in \mathfrak{J} = \beta \cap FC$ so R is nilpotent and in FC. Therefore $\langle x \rangle \in \beta \cap FC = \mathfrak{J}$ so $\langle x \rangle \in \mathfrak{D}$. It follows that $\mathfrak{D}_g = \mathfrak{D}_{g_1}$.

Notice that Proposition 4.3.4 also implies that $\mathfrak{D}^* = \mathfrak{J}^*$ and $\mathfrak{D}' = \mathfrak{J}'$.

4.3.5 THEOREM:

$\mathfrak{J}_g = \mathfrak{J}_{g_1} = \mathcal{N}_g \cap FC$ if $\mathfrak{J}$ is equal to any of the classes $\mathfrak{D}$, $\mathfrak{J}$, $\mathfrak{D}^*$, $\mathfrak{J}^*$, $\mathfrak{D}'$ or $\mathfrak{J}'$.

Proof:

We have already noticed that $\mathfrak{D}_g = \mathfrak{J}_g = \mathfrak{D}_{g_1} = \mathfrak{J}_{g_1}$. Since $\mathfrak{J} \leq FC$, $\mathfrak{J}_g \leq FC$ and since $\mathfrak{J} \leq \beta$, $\mathfrak{J}_g \leq \beta_g = \mathcal{N}_g$. Thus $\mathfrak{J}_g \leq \mathcal{N}_g \cap FC$.

Let $R \in \mathcal{N}_g \cap FC$ and suppose that $\overline{R}$ is a non-zero homomorphic image of R. Then there is an $0 \neq x \in \overline{R}$ such that $\langle x\rangle \in FC$ and x is nilpotent. Thus $\langle x\rangle \in \mathfrak{J}$ (and $\langle x\rangle \in \mathfrak{J}^*$). Thus $\mathcal{N}_g \cap FC \leq \mathfrak{J}_g$ so $\mathfrak{J}_g = \mathcal{N}_g \cap FC$.

By 2.1.3(iii) $\mathfrak{J}^* \leq \mathfrak{J}'$. Then by 2.4.2(ii), $(\mathfrak{J}^*)_{g_1} \leq (\mathfrak{J}')_{g_1}$ and $(\mathfrak{J}^*)_g \leq (\mathfrak{J}')_g$ and by 2.4.2(iii), $(\mathfrak{J}^*)_{g_1} \leq (\mathfrak{J}^*)_g$ and $(\mathfrak{J}')_{g_1} \leq (\mathfrak{J}')_g$. Notice that in the above paragraph we actually proved that $\mathcal{N}_g \cap FC \leq (\mathfrak{J}^*)_{g_1}$. Now combining these inclusions we have

$\mathcal{N}_g \cap FC \leq (\mathfrak{J}^*)_{g_1} \leq (\mathfrak{J}^*)_g \leq (\mathfrak{J}')_g$ and

$\mathcal{N}_g \cap FC \leq (\mathfrak{J}^*)_{g_1} \leq (\mathfrak{J}')_{g_1} \leq (\mathfrak{J}')_g$. Since $\mathfrak{D}^* = \mathfrak{J}^*$ and $\mathfrak{D}' = \mathfrak{J}'$ we need only show that $(\mathfrak{J}')_g \leq \mathcal{N}_g \cap FC$ to complete the proof.

Since $\mathfrak{J} \leq FC$, $(\mathfrak{J}')_g \leq (FC')_g = FC$ and since $\mathfrak{J} \leq \mathcal{N}$, $\mathfrak{J}' \leq \mathcal{N}$ so $(\mathfrak{J}')_g \leq \mathcal{N}_g$. Therefore

$(\mathcal{S}')_g \leq \mathcal{N}_g \cap FC$.

Q.E.D.

In view of 3.4.14 and 4.3.2 to determine $\mathcal{H}_g$ for any elementary radical $\mathcal{H}$ which is $\leq FC$ it is sufficient to consider $\mathfrak{J}_g$ where $\mathfrak{J}$ is one of the basic building blocks of 3.4.14.

4.3.6 PROPOSITION:

(1) $(\mathfrak{J}_p(S))_g = (\{F_p\})_g$ if $S \neq \emptyset$.

(2) $(\mathfrak{J}_p \mathcal{N}(S))_g = FC_p \cap \mathcal{N}_g$ if $S = \emptyset$.

(3) $(\mathfrak{J}_p \mathcal{N}(S))_g = FC_p \cap (\mathcal{L}_R)_g$ if $S \neq \emptyset$.

Proof:

Notice that since $\mathfrak{J}_p(S)$ and $\mathfrak{J}_p \mathcal{N}(S)$ are elementary classes $(\mathfrak{J}_p(S))_{g_1} = (\mathfrak{J}_p(S))_g$ and $(\mathfrak{J}_p \mathcal{N}(S))_{g_1} = (\mathfrak{J}_p \mathcal{N}(S))_g$.

(1) Suppose that S is a non-zero C.U.D. set of positive integers. Since every non-zero finitely generated ring in $\mathfrak{J}_p(S)$ contains a finite field of characteristic p it is clear that every non-zero homomorphic image of a ring which is in $(\mathfrak{J}_p(S))_g$ contains a non-zero idempotent e such that $pe = 0$. Thus $(\mathfrak{J}_p(S))_g \leq (\{F_p\})_g$.

Conversely, suppose that every non-zero homomorphic image of R contains an idempotent $e \neq 0$, such that $pe = 0$. Since $S \neq \emptyset$ and S is a C.U.D. set, $1 \in S$. Hence $\langle e \rangle \cong Fp \in \mathfrak{I}_p(S)$ so $R \in (\mathfrak{I}_p(S))_g$. This completes the proof.

(2) Since $\mathfrak{I}_p(\emptyset) = \{(0)\}$, $\mathfrak{I}_p \mathcal{N}(\emptyset) = FC_p \cap \mathcal{N}$ and clearly $(FC_p \cap \mathcal{N})_g = FC_p \cap \mathcal{N}_g$.

(3) Suppose that S is a non-zero C.U.D. set of positive integers. Now $\mathfrak{I}_p \mathcal{N}(S) \leq \mathfrak{I}_p \mathcal{N}(Z^+) = FC_p \cap \mathcal{L}_R$ so clearly $(\mathfrak{I}_p \mathcal{N}(S))_g \leq FC_p \cap (\mathcal{L}_R)_g$.

If $R \in FC_p \cap (\mathcal{L}_R)_g$ then every non-zero homomorphic image contains a non-zero element x such that $p^\alpha x = 0$ for some integer $\alpha \geq 1$ and $x^N = \sum_{i=1}^{N-1} a_i x^i$ for some integers $a_1, \ldots, a_{N-1}$. If x is not nil then $\langle x \rangle / \mathcal{N}(\langle x \rangle)$ is a commutative Wedderburn ring. Let $y \in \langle x \rangle$ such that $\overline{y} = y + (\langle x \rangle)$ is the identity of $\langle x \rangle / \mathcal{N}(\langle x \rangle)$. Then $\langle y \rangle / \mathcal{N}(\langle y \rangle) \cong Fp \in \mathfrak{I}_p(S)$, thus $\langle y \rangle \in \mathfrak{I}_p \mathcal{N}(S)$. So in any case, $R \in (\mathfrak{I}_p \mathcal{N}(s))_g$.

Therefore, $(\mathfrak{I}_p \mathcal{N}(S))_g = FC_p \cap (\mathcal{L}_R)_g$.

Q.E.D.

4.4 THE GENERALIZED RADICAL CLASS $\mathcal{E}_{g_1}$.

A ring $R \in \mathcal{E}$ if and only if $R^2 = R$. Hence $R \in \mathcal{E}$ if and only if for each $x \in R$ there are elements $u_i , v_i \in R$ such that $x = \sum_{i=1}^{N} u_i v_i$. Using this characterization of $\mathcal{E}$ one easily sees that $\mathcal{E}$ is a radical class.

Let $R \in \mathcal{E}_g$. Since R/R^2 cannot contain a non-zero idempotent subring, $R/R^2 = (0)$. Thus $R \in \mathcal{E}$. By 2.4.2(iii), $\mathcal{E}_{g_1} \leq \mathcal{E}_g$ so $\mathcal{E}_{g_1} \leq \mathcal{E}_g \leq \mathcal{E}$.

Let R be the subring of the ring of real numbers generated by the set of positive real numbers $\{+(2)^{1/2^n} : n \geq 1\}$. Clearly $R^2 = R$ since $[(2)^{1/2^{n+1}}]^2 = 2^{1/2^n}$. However, if R' is a finitely generated subring of R then $R' \neq (R')^2$. This follows from a lemma on page 215 of Zariski and Samuel [15] which implies that if $(0) \neq I = I^2$ and I is finitely generated as an ideal of a commutative ring then I has an identity.

Thus $\mathcal{E}_g \lneq \mathcal{E}$. The result from Zariski and Samuel implies that $\mathcal{E}_{g_1} = \mathcal{E}_g$ for commutative rings but we do not know if $\mathcal{E}_{g_1}(R) = \mathcal{E}_g(R)$ for all rings R .

Notice that $R \in \mathcal{E}_{g_1}$ if and only if every non-zero homomorphic image of R contains an idempotent $e \neq 0$, and that R is $\mathcal{E}_{g_1}$ semi-simple if and only if R has no idempotent $e \neq 0$.

Clearly $\mathcal{E}' \leq \mathcal{E}_{g_1}$ so $(\mathcal{E}_{g_1})' = (\mathcal{E}_g)' = \mathcal{E}'$. Of course, $(\mathcal{E}')_g = (\mathcal{E}')_{g_1} \leq FC \cap \mathcal{E}_{g_1}$. Since the ring of integers moduls 4 is not in $(\mathcal{E}')_g$, $(\mathcal{E}')_g \lneq FC \cap \mathcal{E}_{g_1}$.

If R is a commutative ring and $(0) \neq I$ is a finitely generated ideal of R which is in $\mathcal{E}_{g_1} \leq \mathcal{E}$ then $I^2 = I$ so I has an identity. Thus I is a direct summand of R . Therefore, for a commutative ring R with A.C.C. $\mathcal{E}_{g_1}(R)$ has an identity and is a direct summand of R . If we do not insist that R be commutative we obtain the following weaker result.

4.4.1 <u>THEOREM</u>:

If $(0) \neq R \in \mathcal{E}_{g_1}$ and R has A.C.C. on one-sided ideals then there is an $e \in R$ such that $e^2 = e \neq 0$, $R = ReR$ and $\bar{e}$ is an identity for $R/\mathcal{N}(R) \neq (0)$.

<u>Proof</u>:

Let $(0) \neq R \in \mathcal{E}_{g_1}$ and suppose that R has A.C.C. on left ideals. Then $(0) \neq \bar{R} = R/\mathcal{N}(R)$ is semi-prime and so by Goldie's Theorem (Theorem 29 in Divinsky [7]) $\bar{R}$ has a left quotient ring Q which is a finite direct sum of matrix rings over division rings. Clearly all idempotents of Q are in the centre of Q so all idempotents of $\overline{R}$ are in the centre of $\overline{R}$.

Since $R \in \mathcal{E}_{g_1}$ there is a non-zero idempotent in $\overline{R}$. Choose $\overline{R}f$ maximal in the set $\{\overline{R}e : e$ is an idempotent of $\overline{R}\}$. Since $f \in$ centre of $\overline{R}$, $\overline{R}f \triangleleft \overline{R}$. If $\overline{R}f \neq \overline{R}$ then there is an element $w \in \overline{R}$ such that $\overline{w}$ is a non-zero idempotent in $\overline{R}/\overline{R}f$. One easily checks that $f' = f + w - fw$ is an idempotent and since $ff' = f$, $\overline{R}f \subseteq \overline{R}f'$. By the maximality of $\overline{R}f$, $\overline{R}f = \overline{R}f'$. This implies that $w \in \overline{R}f$ which is a contradiction. Therefore, $\overline{R}f = \overline{R}$ so f is an identity for $\overline{R}$. Then by Lemma 1.12 in Herstein [10] there is an idempotent $e \in R$ such that $e + \mathcal{N}(R) = f$.

Since $\overline{R}f = \overline{R}$, $ReR + N = R$. Therefore R/ReR is nil and hence does not contain a non-zero idempotent. Thus $R = ReR$ because $R \in \mathcal{E}_{g_1}$.

The proof goes through in exactly the same way if we assume that R has A.C.C. on right ideals.

Q.E.D.

The following proposition (together with Theorem 2.4.13) implies that if $\mathcal{H}$ is an elementary semi-simple class and $\mathcal{H}$ contains all $\mathcal{E}_{g_1}$ semi-simple rings then $\mathcal{H} =$ the class of $\mathcal{J}_{g_1}$ semi-simple rings for some class $\mathcal{J}$ which satisfies condition (A).

4.4.2 <u>PROPOSITION</u>:

If $\aleph$ is an elementary class of rings which contains all $\mathcal{E}_{g_1}$ semi-simple rings then $\aleph$ satisfies condition s(2) if and only if $\aleph$ satisfies condition $\overline{s}(2)$.

<u>Proof</u>:

Let $\aleph$ be an elementary class of rings which contains all $\mathcal{E}_{g_1}$ s.s. rings.

If $\aleph$ satisfies condition $\overline{s}(2)$ then by 2.4.12 $\aleph$ satisfies condition s(2).

Suppose $\aleph$ satisfies condition s(2). By Theorem 2.4.9 $\aleph$ is a semi-simple class. If $\langle x\rangle \notin \aleph$ then $\mathcal{U}_{\aleph}(\langle x\rangle) = I \neq (0)$. Since $\aleph$ contains all $\mathcal{E}_{g_1}$ s.s. rings, $\mathcal{U}_{\aleph} \leq \mathcal{E}_{g_1}$. Therefore $I \in \mathcal{E}_{g_1}$ so by virtue of the remarks at the beginning of this section I has an identity and so there is an ideal J of $\langle x\rangle$ such that $I \oplus J = \langle x\rangle$. Therefore I is generated by one element as a subring of $\langle x\rangle$ and I cannot be homomorphically mapped onto a non-zero ring in $\aleph$.

We have proven the contrapositive of condition $\overline{s}(2)$. This completes the proof.

Q.E.D.

In the following proposition we show that if $\aleph$ is an elementary semi-simple class which is not of the type with which 4.4.2 is concerned then $\mathcal{H}$ must be contained in the class of $\mathcal{N}_g \cap FC_p$ semi-simple rings for some prime p.

4.4.3 PROPOSITION:

If $\mathfrak{J}$ be a class of rings which satisfies condition r(2) then $\mathfrak{J}_{g_1} \geq \mathcal{N}_{g_1} \cap FC_p$ for some prime p <u>or</u> $\mathfrak{J}_{g_1} \leq \mathcal{E}_{g_1}$.

<u>Proof</u>:

Let $\mathfrak{J}$ be a class of rings which satisfies condition r(2).

Suppose that for all primes p , $\mathfrak{J}_{g_1} \not\geq \mathcal{N}_g \cap FC_p$.

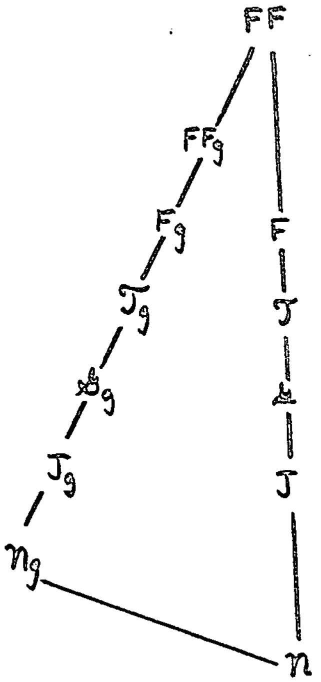
Then for each prime p there is a ring $Rp \in \mathcal{N}_g \cap FC_p$ which is $\mathfrak{J}_{g_1}$ s.s.. Since $\mathfrak{J}$ satisfies condition r(2), by Theorem 2.4.7, the class of $\mathfrak{J}_{g_1}$ s.s. rings is an elementary class. Therefore, for each prime p there is a ring $\langle x_p \rangle \subseteq Rp$ such that $\langle x_p \rangle$ is $\mathfrak{J}_{g_1}$ s.s. and $\langle x_p \rangle \cong Cp$. Since subdirect sums of semi-simple rings are semi-simple, C^∞ is $\mathfrak{J}_{g_1}$ s.s. Now every zero ring on a cyclic group is $\mathfrak{J}_{g_1}$ s.s. so since the class of $\mathfrak{J}_{g_1}$ s.s. rings is elementary and satisfies condition (F) all nil rings are $\mathfrak{J}_{g_1}$ s.s.

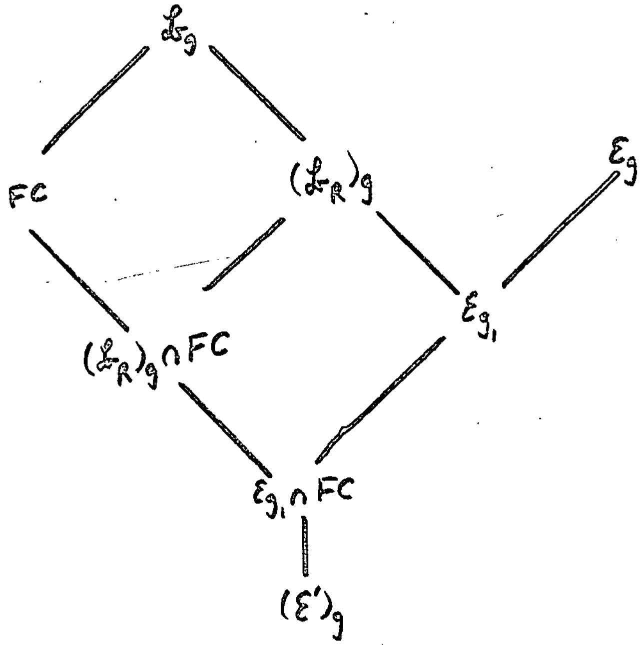
If $R \notin \mathcal{E}_{g_1}$ then there is a non-zero homomorphic image $\overline{R}$ of R such that there are no non-zero idempotents in $\overline{R}$. Let $0 \neq x \in \overline{R}$ and $(0) \neq I \triangleleft \langle x \rangle$. Since there are no non-zero idempotents in $\langle x \rangle$, $I \neq I^2$. Thus I can be homomorphically mapped onto the non-zero $\mathfrak{J}_{g_1}$ s.s. ring I/I^2. Therefore $\langle x \rangle$ is $\mathfrak{J}_{g_1}$ s.s. Since the class of $\mathfrak{J}_{g_1}$ s.s. rings is elementary, $\overline{R}$ is $\mathfrak{J}_{g_1}$ s.s. so $R \notin \mathfrak{J}_{g_1}$.

Therefore $\mathfrak{J}_{g_1} \leq \mathcal{E}_{g_1}$. This completes the proof.

Q.E.D.

The relationships between the generalized radical classes which we have discussed can be illustrated in the following diagrams.





The following diagram remains valid when FC is everywhere replaced by FC_p .

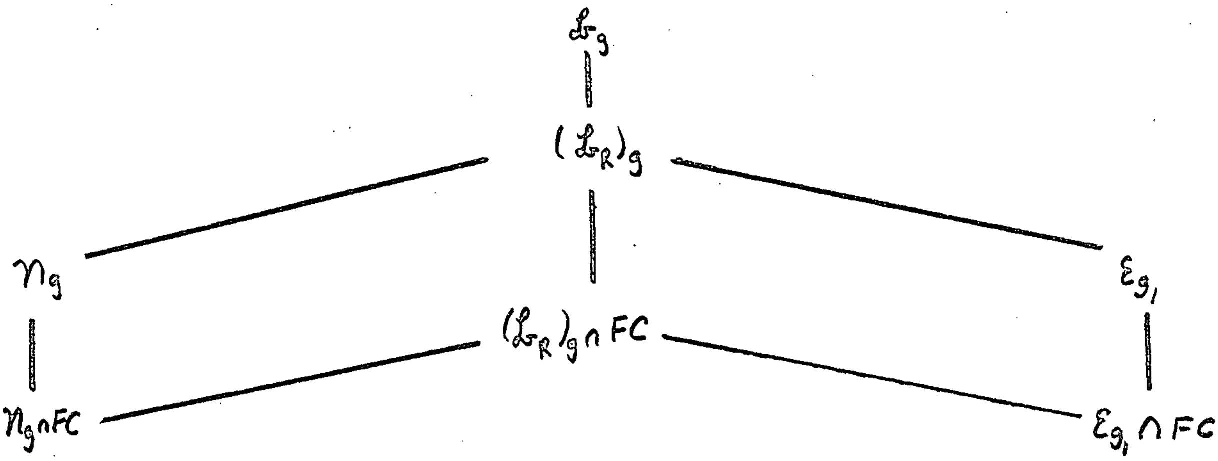


ILLUSTRATION 2

CHAPTER V

LOCAL RADICAL CLASSES

5.1 THE RADICAL CLASSES $\mathfrak{F}.\mathfrak{D}.*$ AND $\mathcal{L}$.

The class $\mathfrak{F}.\mathfrak{D}.*$ is a local radical class by Theorem 2.2.7.

If $R \in \mathcal{L}$ then $R \in \mathfrak{F}.\mathfrak{D}.*$ for if $R' = \langle x_1, \ldots, x_N \rangle$ and $R'^K = (0)$ then $R' = \{\sum_i \alpha_i d_i : d_i \in D$ and the α_i are integers$\}$ where $D = \{x_{i_1} \ldots x_{i_L} : L \leq K - 1\}$. However, finite fields are in $\mathfrak{F}.\mathfrak{D}.*$ but not in $\mathcal{L}$. Therefore, $\mathfrak{F}.\mathfrak{D}.* \gneq \mathcal{L}$.

Since $\mathcal{L} \lneq \mathcal{N}$ the following proposition implies that $\mathfrak{F}.\mathfrak{D}.* \not\geq \mathcal{N}$. Hence $\mathfrak{F}.\mathfrak{D}.*$ is unrelated to $\mathcal{N}$.

5.1.1 PROPOSITION:

$$\mathcal{L} = \mathfrak{F}.\mathfrak{D}.* \cap \mathcal{N} .$$

Proof:

Since $\mathcal{L} \leq \mathfrak{F}.\mathfrak{D}.*$ and $\mathcal{L} \leq \mathcal{N}$ it is clear that $\mathcal{L} \leq \mathfrak{F}.\mathfrak{D}.* \cap \mathcal{N}$.

Assume $R \in \mathfrak{F}.\mathfrak{D}.* \cap \mathcal{N}$. Let R' be a finitely generated subring of R . Then since $R' \in \mathfrak{F}.\mathfrak{D}.*$, R' satisfies A.C.C. on one-sided ideals (in fact, the additive group R'^+ has A.C.C. on subgroups). Now $R' \in \mathcal{N}$ and

$\mathcal{N} = \beta$ for rings which satisfy A.C.C. (Theorem 16 in Divinsky [7]) so $R' \in \beta \leq \mathcal{L}$. Thus R' is nilpotent so $R \in \mathcal{L}$.

Q.E.D.

Since $\beta \leq \mathcal{L}$, $\beta^* \leq \mathcal{L}^* = \mathcal{L}$ and since all nilpotent rings are in β , $\mathcal{L} \leq \beta^*$. Therefore, $\mathcal{L} = \mathcal{L}^* = \beta^*$.

5.1.2 <u>THEOREM</u>:

If $\aleph^*$ is a local radical class and $\beta \leq \aleph^* \leq \mathcal{F}.\mathcal{D}.^*$ then a ring R is $\aleph^*$ semi-simple if and only if R is isomorphic to a subdirect sum of prime $\aleph^*$ semi-simple rings.

<u>Proof</u>:

Let $\aleph^*$ be a local radical class such that $\beta \leq \aleph^* \leq \mathcal{F}.\mathcal{D}.^*$.

Since subdirect sums of semi-simple rings are semi-simple one direction is clear.

Conversely, suppose that R is $\aleph^*$ semi-simple. It is sufficient to find, for each non-zero $x \in R$ an ideal $I(x)$ such that $x \notin I(x)$ and $R/I(x)$ is a prime $\aleph^*$ semi-simple ring.

Let $0 \neq x \in R$. Since $(x)_R \notin \aleph^*$ there is a finitely generated subring $R' \subseteq (x)_R$ such that $R' \notin \aleph^*$. Let $Z(x) = \{I \triangleleft R : R'/R' \cap I \notin \aleph^*\}$. Let $J_\alpha : \alpha \in \Lambda$ be an ascending chain in $Z(x)$ and let $J = \cup\{J_\alpha : \alpha \in \Lambda\}$. If

$J \notin Z(x)$ then $R'/R' \cap J \in \mathfrak{N}^* \leq \mathfrak{F}.\mathfrak{D}.^*$ so $R'/R' \cap J \in \mathfrak{F}.\mathfrak{D}.$ Now by 2.2.6 $R' \cap J$ is finitely generated as a ring so $R' \cap J \subseteq R' \cap J_\alpha$ for some $\alpha \in \Lambda$. Thus $R' \cap J = R' \cap J_\alpha$ which is a contradiction since $J_\alpha \in Z(x)$. Therefore $J \in Z(x)$ so by Zorn's Lemma we may choose $I(x)$ maximal in $Z(x)$.

First we shall prove that $R/I(x)$ is $\mathfrak{N}^*$ semi-simple. Let $K \triangleleft R$ such that $K \supsetneqq I(x)$. Then $R'/R' \cap K \in \mathfrak{N}^*$.

Now, $\dfrac{R'}{R' \cap K} \cong \dfrac{R'/(R' \cap I(x))}{(R' \cap K)/(R' \cap I(x))}$. Thus if

$\dfrac{R' \cap K}{R' \cap I(x)} \in \mathfrak{N}^*$ then $\dfrac{R'}{R' \cap I(x)} \in \mathfrak{N}^*$. Therefore

$\dfrac{R' \cap K}{R' \cap I(x)} \cong \dfrac{R' \cap K + I(x)}{I(x)} \notin \mathfrak{N}^*$ and since $\mathfrak{N}^*$ is strongly hereditary, $K/I(x) \notin \mathfrak{N}^*$. Hence $R/I(x)$ is $\mathfrak{N}^*$ semi-simple.

In order to prove that $R/I(x)$ is prime we begin by showing that if K and H are ideals of R such that $K \supsetneqq I(x)$ and $H \supsetneqq I(x)$ then $K \cap H \supsetneqq I(x)$. Suppose this is not true; that is, suppose $K \cap H = I(x)$. Then the ring

$\dfrac{R' \cap K}{R' \cap I(x)} = \dfrac{R' \cap K}{R' \cap H \cap K} \cong \dfrac{R' \cap K + R' \cap H}{R' \cap H}$ is a subring of

$\dfrac{R'}{R' \cap H} \in \mathfrak{N}^*$ so $\dfrac{R' \cap K}{R' \cap I(x)} \in \mathfrak{N}^*$. Now

$\dfrac{R'/(R' \cap I(x))}{(R' \cap K)/(R' \cap I(x))} \cong \dfrac{R'}{R' \cap K} \in \mathfrak{N}^*$ so $\dfrac{R'}{R' \cap I(x)} \in \mathfrak{N}^*$.

This is a contradiction. Therefore $K \cap I \supsetneqq I(x)$.

Now we can prove that $R/I(x)$ is a prime ring. Suppose that K and H are as above and that $KH \subseteq I(x)$. Then $K \cap H \supsetneq I(x)$ and $(K \cap H)^2 \subseteq I(x)$. Then $\beta(R/I(x)) \neq (0)$. This is a contradiction since $R/I(x)$ is $\mathfrak{H}^*$ semi-simple and $\beta \leq \mathfrak{H}^*$. Therefore $R/I(x)$ is prime.

Since $R'/R' \cap I(x) \notin \mathfrak{H}^*$, $R' \nsubseteq I(x)$ so $x \notin I(x)$. This completes the proof.

Q.E.D.

In view of 1.1.6 this theorem implies that all local radical classes $\mathfrak{H}^*$ which are between β and $\mathfrak{J}.\mathfrak{D}.*$ are special radical classes. In particular, $\mathfrak{J}.\mathfrak{D}.*$ is a special radical class. Theorem 5.1.2 provides an alternate proof for Theorem 52 ($\mathcal{L}$ is a special radical class) in Divinsky [7].

We conclude this section by considering the generalized and elementary classes related to $\mathcal{L}$ and $\mathfrak{J}.\mathfrak{D}.*$. By 2.4.2(vii) $\mathcal{L}_g = \mathcal{L}_{g_1}$ and $(\mathfrak{J}.\mathfrak{D}.*)_g = (\mathfrak{J}.\mathfrak{D}.*)_{g_1}$.

We have already seen that $\mathcal{L}_g = \mathcal{N}_g$ and $\mathcal{L}' = \mathcal{N}$ (Theorems 4.2.2 and 3.3.4 respectively).

Since a ring $\langle x \rangle \in \mathfrak{J}.\mathfrak{D}.$ if and only if there are integers $a_1, \ldots, a_{n-1}$ such that $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x = 0$ it is clear that

$(\mathfrak{F}.\mathfrak{D}.*)_g = (\mathcal{L}_R)_g$ and $(\mathfrak{F}.\mathfrak{D}.*)' = \mathcal{L}_R = \mathfrak{F}.\mathfrak{D}.'$.

5.2 THE LOCAL RADICAL CLASSES $\mathcal{J}*$, $\mathfrak{D}*$ AND FI* .

Let FI be the class of all finite rings. We shall begin this section with a discussion of those rings R such that every finitely generated subring of R is finite. In the following proposition we collect several elementary properties of this class of rings.

5.2.1 PROPOSITION:

(1) FI* is a radical class.

(2) $\mathcal{J} \subsetneqq \mathrm{FI}* \subsetneqq \mathfrak{F}.\mathfrak{D}.*$.

(3) $\varepsilon' \subsetneqq \mathrm{FI}* \subsetneqq \mathrm{FC}$.

(4) $\mathrm{FI}* = \mathfrak{F}.\mathfrak{D}.* \cap \mathrm{FC}$.

(5) $\mathfrak{D}$ and FI* are unrelated.

(6) FF and FI* are unrelated.

Proof:

(1) Clearly the conditions of Theorem 2.2.7 are satisfied so FI* is a local radical class.

(2) Since $\mathcal{J} = \beta \cap \mathrm{FC}$ any finitely generated $\mathcal{J}$ ring must be finite and of course any finite ring is in $\mathfrak{F}.\mathfrak{D}$. Therefore, $\mathcal{J} \leq \mathrm{FI} \leq \mathfrak{F}.\mathfrak{D}.*$. Since the ring of integers is in $\mathfrak{F}.\mathfrak{D}.*$, $\mathrm{FI}* \subsetneqq \mathfrak{F}.\mathfrak{D}.*$. The ring $F_p \in \mathrm{FI}*$ but $F_p \notin \mathcal{J}$ so $\mathcal{J} \subsetneqq \mathrm{FI}$.

(3) By 3.4.4 $\mathcal{E}' \leq FI$ and since any finite nilpotent ring is in FI^*, $\mathcal{E}' \lneq FI$. Clearly $FI^* \lneq FC$.

(4) This is clear since a ring in $\mathfrak{J}.\mathfrak{D}. \cap FC$ must be finite.

(5) Since $F_p \in FI^*$, $FI^* \nleq \mathfrak{D}$. Example 6 in <u>Rings and Radicals</u> Divinsky [7] shows that the zero ring on the additive group of rational numbers is in $\mathfrak{D}$ so $\mathfrak{D} \nleq FI^*$.

(6) Since finite fields are in FI^*, $FI^* \nleq FF$. Clearly $FF \nleq FI^*$ since $\mathfrak{D} \leq FF$ and $\mathfrak{D} \nleq FI^*$.

Q.E.D.

The following theorem provides an interesting characterization of FI^*.

5.2.2 <u>THEOREM</u>:

$R \in FI^*$ if and only if every finitely generated subring of R satisfies D.C.C. on left ideals.

<u>Proof</u>:

Since all finite rings satisfy D.C.C. one direction is clear.

Conversely, assume that every finitely generated subring of the ring R satisfies D.C.C. on left ideals.

Let R' be a finitely generated subring of R and let $N' = \mathcal{N}(R')$. The ring R'/N' is isomorphic to a finite direct sum of matrix rings over division rings. Let $R'/N' \cong (D_1)_{n_1} \oplus \cdots \oplus (D_k)_{n_k}$. Then each D_i is finitely generated as a ring and if A_i is a finitely generated subring of D_i then A_i satisfies D.C.C. Let $x_i \in D_i$. Then $\langle x_i \rangle^n = \langle x_i \rangle^{n+1}$ for some integer $n \geq 1$ so $x_i^n = a_{n+1}x_i^{n+1} + \ldots + a_i x_i^L$ for some integers $a_{n+1}, \ldots, a_L$. Since D_i has no proper divisors of zero $x_i = a_{n+1}x_i^2 + \ldots + a_L x_i^{L-n+1}$. Therefore $D_i \in \mathcal{E}'$ so by 3.4.4, D_i is a finite field. Thus R'/N' is finite and so by 2.2.6 N' is finitely generated as a ring. Since N' satisfies D.C.C. on left ideals $N' \in \mathcal{J}$ by Lemma 28 in Divinsky [7]. By 3.2.1 $\mathcal{J} = \beta \cap FC$ so N' is finitely generated, nilpotent and of finite characteristic. Thus, N' is finite so R' must be finite.

Therefore $R \in FI^*$.

Q.E.D.

We now turn to a consideration of the local classes $\mathcal{J}^*$ and $\mathcal{D}^*$. By 4.3.4 $\mathcal{J}^* = \mathcal{D}^*$.

5.2.3 <u>PROPOSITION:</u>

$$\mathcal{J}^* = \mathcal{L} \cap FC = \mathcal{L} \cap FI^* = \mathcal{N} \cap FI^* .$$

Proof:

Since $\mathcal{J} \leq FC$ and $\mathcal{J} \leq \mathcal{L}$, $\mathcal{J}^* \leq \mathcal{L} \cap FC$. If $R \in \mathcal{L} \cap FC$ then every finitely generated subring of R is nilpotent and of finite characteristic and hence is in $\beta \cap FC = \mathcal{J}$; thus $R \in \mathcal{J}^*$. Therefore, $\mathcal{J}^* = \mathcal{L} \cap FC$.

Since $FI^* \leq FC$, $\mathcal{L} \cap FI^* \leq \mathcal{L} \cap FC = \mathcal{J}^*$. Now a finitely generated nilpotent ring of finite characteristic is finite so $\mathcal{J}^* \leq FI^*$. Thus $\mathcal{J}^* = \mathcal{L} \cap FI^*$.

As above $\mathcal{J}^* \leq \mathcal{N} \cap FI^*$. Since a finite nil ring is nilpotent and in FC, $\mathcal{N} \cap FI^* \leq \mathcal{J}^*$.

This completes the proof.

Q.E.D.

The following is a slight modification of an example given by Baer [6]. We present this example to show that $\mathcal{L} \cap FC \not\leq \beta$ and hence that $\mathcal{J}^* \not\leq \beta$.

5.2.4 EXAMPLE:

For each integer k let $G_k = \{0, a(k)\} \cong$ the additive group of 2 elements.

For each integer $i \geq 1$ let $T_i \in \mathrm{Hom}(\sum_{-\infty}^{\infty} G_k, \sum_{-\infty}^{\infty} G_k)$ such that

$$T_i(a(k)) = \begin{cases} 0 & \text{if } k \equiv 0 \mod 2^i . \\ a(k-1) & \text{if } k \not\equiv 0 \mod 2^i . \end{cases}$$

Let R be the subring of $\mathrm{Hom}(\sum_{-\infty}^{\infty} G_k, \sum_{-\infty}^{\infty} G_k)$ which is generated by the set $\{T_i : i \geq 1\}$.

(1) First we shall prove that $\beta(R) = (0)$.

Let $(0) \neq I \lhd R$ and let $0 \neq X \in I$. Let h be any integer ≥ 1. We shall prove that $I^{2^h} \neq (0)$.

Now X is a sum of monomials in the T_i's so we may write $X = V_\ell + \ldots + V_k$ where $\ell \leq k$, $V_\ell \neq 0$, $V_k \neq 0$ and V_i is a sum of monomials of length $i+1$. Let $V_k = T_{m_{1,0}} \ldots T_{m_{1,k}} + \ldots + T_{m_{n,0}} \ldots T_{m_{n,k}}$.

Since $V_k \neq 0$ there is an integer t such that $V_k(a(t)) \neq 0$. (We may choose $t > 0$ since $t \equiv 0 \bmod 2^i$ if and only if $-t \equiv 0 \bmod 2^i$).

Choose integers r and s such that $k+1 < 2^r$ and $m_{i,j} \leq r$ for all i and j such that $1 \leq i \leq n$ and $0 \leq j \leq k$, and $2^{r+1} \cdot 2^h + t < 2^s$.

We wish to show that $(T_s^{2^r-k-1} \cdot X)^{2^h}(a(t + 2^h \cdot 2^r)) \neq 0$. To begin we consider $(T_s^{2^r-k-1} V_k)(a(t + 2^h \cdot 2^r))$.

Consider $V_k(a(t + \varphi \cdot 2^r))$ where φ is an integer such that $1 \leq \varphi \leq 2^h$. Suppose $t + \varphi 2^r - j \equiv 0 \bmod 2^{m_{i,k-j}}$ for some i and j such that $1 \leq i \leq n$ and $0 \leq j \leq k$. Since $r \geq m_{i,k-j}$, $\varphi 2^r \equiv 0 \bmod 2^{m_{i,k-j}}$. Thus $t - j \equiv 0 \bmod 2^{m_{i,k-j}}$. This implies that $T_{m_{i,0}} \ldots T_{m_{i,k}}$ maps $a(t + \varphi \cdot 2^r)$ onto

0 if and only if it maps $a(t)$ onto 0. Since $V_k(a(t)) \neq 0$ an odd number of the monomials must map $a(t)$ onto $a(t-k-1)$ so $V_k(a(t + \varphi\cdot 2^r)) = a(t + \varphi\cdot 2^r - k - 1)$.

Let us now consider $T_s^{2^r-k-1}(a(t + \varphi\cdot 2^r - k - 1))$. Suppose ℓ is an integer such that $1 \leq \ell \leq 2^r - k - 1$. Now $t + \varphi\cdot 2^r < 2^s$ so $t + \varphi\cdot 2^r - k - 1 < 2^s$. Since $\ell \leq 2^r - k - 1$, $k + 1 + \ell \leq 2^r$ so

$$t + \varphi\cdot 2^r - k - 1 - \ell \geq t + \varphi\cdot 2^r - 2^r = t + (\varphi - 1)2^r > 0.$$

Therefore, $0 < t + \varphi\cdot 2^r - k - 1 - \ell < 2^s$ so $T_s^{2^r-k-1}(a(t + \varphi\cdot 2^r - k - 1)) \neq 0$ so

$$\begin{aligned} T_s^{2^r-k-1}&(a(t + \varphi\cdot 2^r - k - 1)) \\ &= a(t + \varphi\cdot 2^r - k - 1 - (2^r - k - 1)) \\ &= a(t + (\varphi - 1)2^r) . \end{aligned}$$

It follows that

$$(T_s^{2^r-k-1}\cdot V_k)^{2^h}(a(t + 2^h\cdot 2^r)) = a(t) \neq 0 .$$

Now if $\ell = k$ then $X = V_k$ so

$0 \neq (T_s^{2^r-k-1}V_k)^{2^h} \in I^{2^h}$.

If $\ell < k$, $(T_s^{2^r-k-1}.X)^{2^h} = Z + (T_s^{2^r-k-1}.V_k)^{2^h}$ where $Z(a(t + 2^h.2^r))$ is either 0 or a sum of $a(\gamma_i)$'s where each $\gamma_i > t$. (Since the length of a monomial appearing in Z is strictly less than

$((k+1)+2^r-k-1)2^h = 2^r 2^h$ they cannot "move" $a(t+2^h \cdot 2^r)$ as "far down" as $a(t))$.

Therefore $0 \neq (T_s^{2^r-k-1} \cdot X)^{2^h} \in I^{2^h}$. So in any case $I^{2^h} \neq (0)$. Therefore no non-zero ideal of R is nilpotent so $\beta(R) = (0)$.

(2) We shall now prove that $R \in \mathcal{L}$. Let $R' = \langle X_1, \ldots, X_n \rangle$ be a finitely generated subring of R. Let $m = \max\{s : T_s$ occurs in some $X_i\}$. Choose $h > 2^m$. Now if $Z \in (R')^h$ then each monomial in Z is of length at least h. Since $h > 2^m$, 2^m must divide one of k, $k-1$, ..., $k-h$ for all integers k. Therefore $Z(a(k)) = 0$ for all integers k so $Z = 0$. Thus $(R')^h = (0)$.

(3) Therefore $R \in \mathcal{L} \cap FC$ (since $2R = (0)$) so R is β semi-simple and in $\mathcal{L} \cap FC$. Since $\mathcal{L} \cap FC = \mathcal{J}^*$, $R \in \mathcal{J}^*$. Therefore $\mathcal{J}^* \nleq \beta$.

From Proposition 5.2.1 (4) we know that $\mathfrak{D} \nleq FI^*$ so since $\mathcal{J}^* \leq FI^*$, $\mathfrak{D} \nleq \mathcal{J}^*$. Combining this with 5.2.4 we see that $\mathfrak{D}$ and $\mathcal{J}^*$ are unrelated. The classes $\mathcal{J}^*$ and β are also unrelated since it is clear that $\beta \nleq \mathcal{J}^*$.

The relations between these radicals classes can be illustrated by the following diagram.

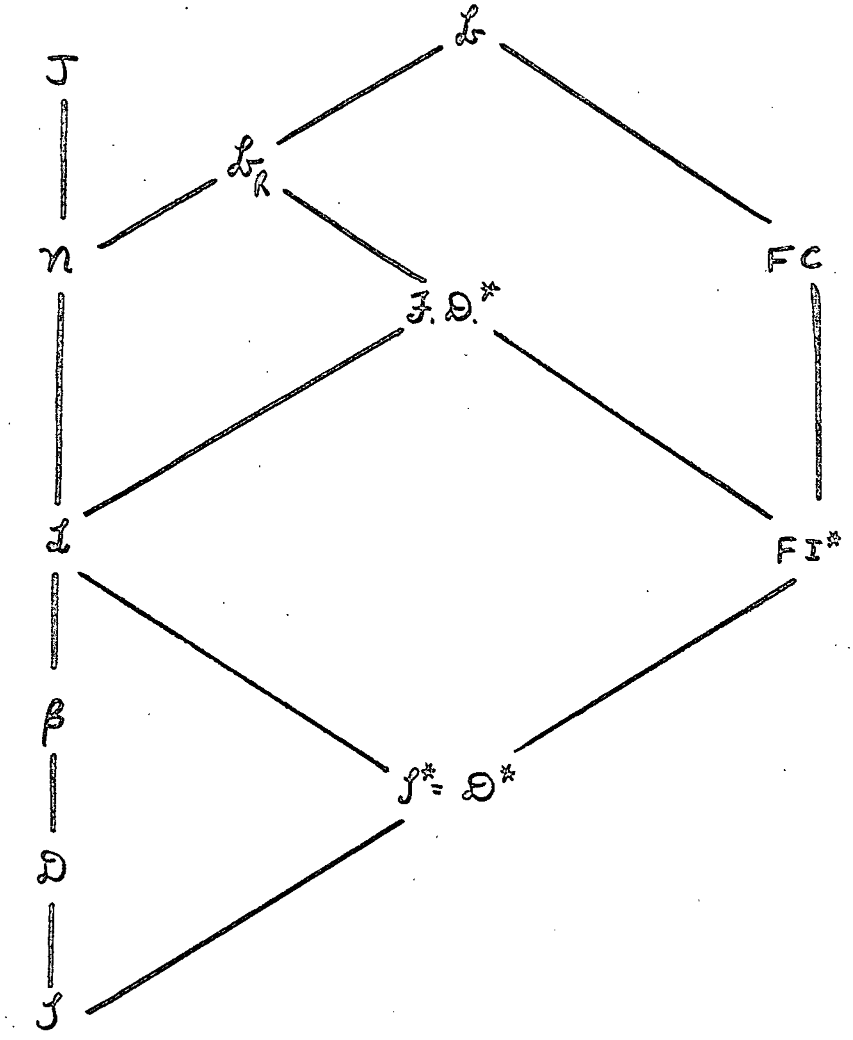


ILLUSTRATION 3

Let R be a non-zero $\mathcal{J}^*$ semi-simple ring with D.C.C. on left ideals. Then $R/\mathcal{N}(R)$ is a finite direct sum of matrix rings over division rings. Since R satisfies D.C.C. on left ideals $\mathcal{L}(R) = \mathcal{N}(R)$. Thus $\mathcal{N}(R) \cap FC(R) = \mathcal{L}(R) \cap FC(R) = \mathcal{J}^*(R) = (0)$ so $FC(R)$ is isomorphic to an ideal of $R/\mathcal{N}(R)$ and hence is a finite direct sum of matrix rings over division rings of finite characteristic. Suppose that $x \in R$ such that $0 \neq \bar{x} \in FC(R/\mathcal{N}(R))$, then the ideal generated by x contains an identity $\bar{e}$ and $\bar{e} \in FC(R/\mathcal{N}(R))$. Therefore $n\bar{e} = \bar{0}$ for some integer $n \neq 0$ so $(ne)^k = 0$ for some positive integer k, this implies that $e^k \in FC(R)$ so $\bar{e} = \bar{e}^k \in (FC(R) + \mathcal{N}(R))/\mathcal{N}(R)$. Therefore, $FC(R/\mathcal{N}(R)) = (FC(R) + \mathcal{N}(R))/\mathcal{N}(R)$.

Now $\mathcal{J} \leq \mathcal{J}^*$ so by Lemma 28 in Divinsky [7] if $R \in \mathcal{N}$ then $R \in \mathcal{J}^*$. Since $R \notin \mathcal{J}^*$, $\mathcal{N}(R) \neq R$. This completes the proof of the following theorem.

5.2.5 <u>THEOREM</u>:

If $(0) \neq R$ is a $\mathcal{J}^*$ semi-simple ring with D.C.C. on left ideals then $R \notin \mathcal{N}$, $FC(R)$ is a finite direct sum of matrix rings over division rings of finite characteristic and $R/(FC(R) + \mathcal{N}(R))$ is a finite direct sum of matrix rings over division rings of characteristic 0.

Notice that if R is not only $\mathcal{J}^*$ semi-simple but also FI* semi-simple then all the division rings are infinite. If $(R/\mathcal{N}(R)) \in FC$ then $\mathcal{N}(R)$ is a direct summand of R ; in fact, $R = \mathcal{N}(R) \oplus FC(R)$.

From Theorem 13 in Divinsky [7] we conclude that $\mathcal{J}^*(R)$ may not be equal to $\mathcal{N}(R)$ since $\mathcal{D} \not\leq \mathcal{J}^*$. However, if $R \in FC$ then $\mathcal{J}^*(R) = \mathcal{N}(R)$ since $\mathcal{J}^*(R) = \mathcal{L}(R) \cap FC(R) = \mathcal{L}(R) = \mathcal{N}(R)$. Of course, as we noticed above, $R \in \mathcal{J}^*$ if and only if $R \in \mathcal{N}$.

Suppose now that R is a non-zero ring with A.C.C. on left ideals. Then $\beta(R) = \mathcal{L}(R) = \mathcal{N}(R)$ so $\mathcal{J}(R) = \mathcal{J}^*(R) = \mathcal{J}'(R)$ since $\mathcal{J} = \beta \cap FC$, $\mathcal{J}^* = \mathcal{L} \cap FC$ and $\mathcal{J}' = \mathcal{N} \cap FC$.

Unfortunately we cannot use Goldie's Theorem to obtain a result similar to 5.2.5. First of all, if R is $\mathcal{J}^*$ semi-simple, R may be in β (for example, $C^\infty \in \beta$ but C^∞ is $\mathcal{J}^*$ semi-simple). Even if $R \notin \beta$, $FC(R/\beta(R))$ may not be the same as $(FC(R) + \beta(R))/\beta(R)$. To see this consider the ring $R = \mathcal{Q}[X]/(2X^2)_{\mathcal{Q}[X]}$. Then $FC(R) = (X^2)_R$ and $\mathcal{N}(R) = \beta(R) = (2X)_R$. Clearly $FC(R) \cap \mathcal{N}(R) = (0)$ and R satisfies A.C.C. but $R/\mathcal{N}(R) \in FC$.

However, if $R \in FC$ then clearly $\beta(R) = \mathcal{J}^*(R)$.

A ring $\langle x \rangle$ is finite if and only if $\langle x \rangle \in \mathcal{S}_R \cap FC$ so it follows that $(FI^*)' = \mathcal{S}_R \cap FC$ and

$(FI^*)_g = (FI^*)_{g_1} = (\mathcal{L}_R)_g \cap FC$. Since $\mathcal{S}^* = \mathcal{L} \cap FC$ it is clear the $(\mathcal{S}^*)' = \mathcal{N} \cap FC$ and $(\mathcal{S}^*)_g = \mathcal{N}_g \cap FC$.

From 2.1.3 (iii), $\mathcal{S}^* \leq \mathcal{S}'$ and clearly $\mathcal{S}' \leq \mathcal{S}_g \leq FC$ so of course $\mathcal{S}^* = \mathcal{L} \cap \mathcal{S}' = \mathcal{L} \cap \mathcal{S}_g$.

The relationships between these radical classes can be illustrated by the following diagram.

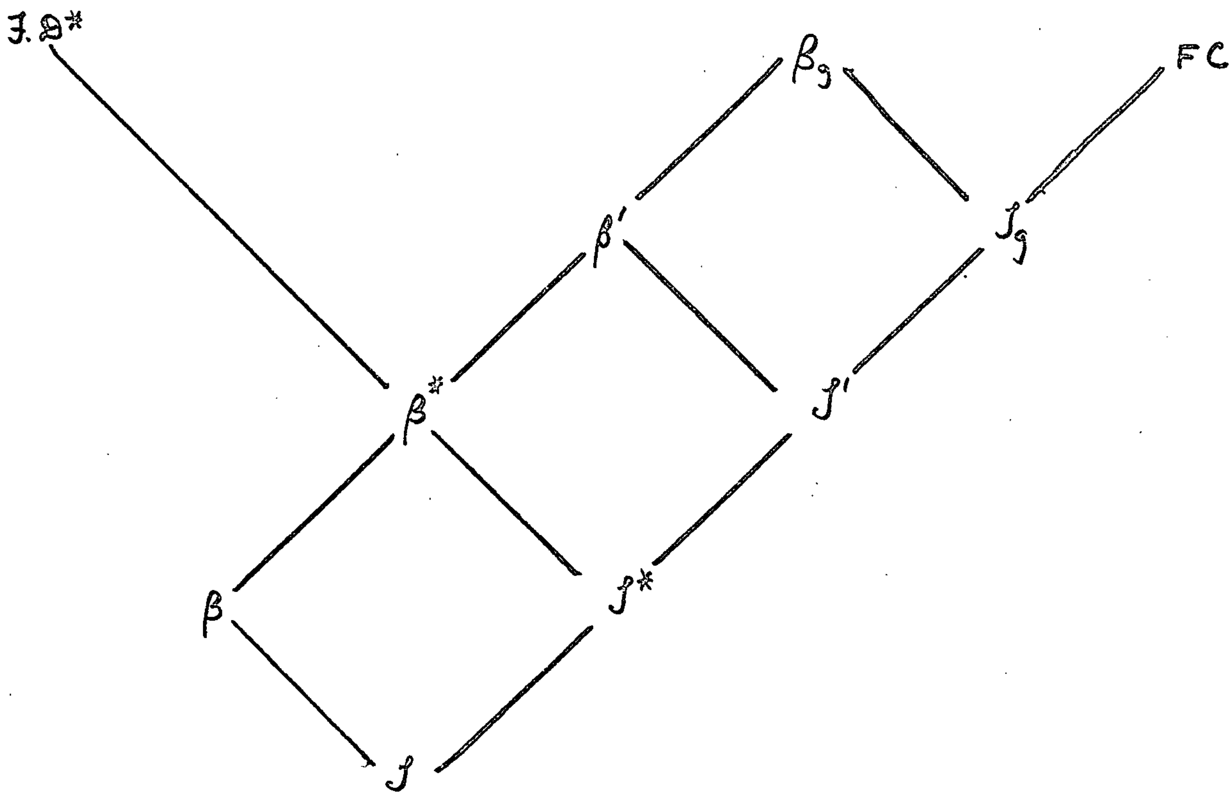


ILLUSTRATION 4

Recall that $\beta^* = \mathcal{L}$, $\beta' = \mathcal{N}$ and $\beta_g = \mathcal{N}_g$.

5.3 LOCAL RADICAL CLASSES $\mathcal{H}$ FOR WHICH $\mathcal{L} \subsetneq \mathcal{H} \subsetneq \mathcal{N}$.

We have already seen that $\beta^* = \mathcal{L}^* = \mathcal{L}$. For any class $\mathcal{H}$, $\mathcal{H}^* \leq \mathcal{H}'$ so from Theorem 3.3.2 we may conclude that $\mathcal{H}^* = \mathcal{N}$ for all classes of rings $\mathcal{H}$ such that $\mathcal{N} \leq \mathcal{H} \leq FF$. Since $\mathcal{N} \leq \mathcal{N}_g$ and $FF_g \leq FF$ it follows that $(\mathcal{H}_g)^* = (\mathcal{H}_{g_1})^* = (\mathcal{H}_g)' = (\mathcal{H}_{g_1})' = \mathcal{N}$ for all classes of rings $\mathcal{H}$ such that $\mathcal{N} \leq \mathcal{H} \leq FF$.

Recall that $\mathcal{U}$ is the upper radical class determined by the class of all simple idempotent rings. Clearly $\beta \leq \mathcal{U} \leq F$ so $\mathcal{U}' = \mathcal{N}$. Therefore $\mathcal{U}^* \leq \mathcal{U}' = \mathcal{N}$. To prove that $\mathcal{U}^* = \mathcal{N}$ it is sufficient to show that a finitely generated nil ring cannot be homomorphically mapped onto a simple idempotent ring. In fact we can prove the following.

5.3.1 PROPOSITION:

A non-zero finitely generated nil ring is not idempotent.

Proof:

Let $(0) \neq R = \langle x_1, \ldots, x_n \rangle$ be a finitely generated nil ring.

Suppose that $R = R^2 = Rx_1 + \ldots + R_{x_n}$. Choose a minimal subset $\{x_{i_1}, \ldots, x_{i_k}\}$ of $\{x_1, \ldots, x_n\}$ such that $R = Rx_{i_1} + \ldots + Rx_{i_k}$.

Since $x_{i_1} \in R$ there are elements $r_1, \ldots, r_k \in R$ such that

$$x_{i_1} = r_1 x_{i_1} + \ldots + r_k x_{i_k} \tag{*}$$

Since R is nil $r_1{}^m \cdot x_{i_1} = 0 \cdot x_{i_1} = 0 \in Rx_{i_2} + \ldots + Rx_{i_k}$ for some integer $m \geq 1$. Let ℓ be the smallest integer ≥ 1 such that $x_1{}^{\ell} x_{i_1} \in Rx_{i_2} + \ldots + Rx_{i_k}$. Then from (*), $r_1^{\ell-1} x_{i_1} = r_1^{\ell} x_{i_1} + r_1^{\ell-1} r_2 x_{i_2} + \ldots + r_1^{\ell-1} r_k x_{i_k} \in Rx_{i_2} + \ldots + Rx_{i_k}$. Since ℓ is minimal, $\ell = 1$ so $x_{i_1} \in Rx_{i_2} + \ldots + Rx_{i_k}$. This contradicts the minimality of k.

Since we have reached a contradiction we may conclude that $R \neq R^2$.

Q.E.D.

If for all finitely generated subrings R' of R, R' cannot be homomorphically mapped onto a non-zero idempotent ring then certainly $R \in FF' = \mathcal{N}$. It follows then that $R \in \mathcal{N}$ if and only if no finitely generated subring of R can be homomorphically mapped onto a non-zero idempotent ring.

Recall that β_φ is the upper radical class determined by the class of all subdirectly irreducible rings with idempotent hearts. The following lemma will enable us to prove that $\beta_\varphi{}^*$ is a radical class.

5.3.2 LEMMA:

If S is a non-zero simple idempotent ring then there is a finitely generated subring S' of S which can be homomorphically mapped onto a non-zero subdirectly irreducible ring with an idempotent heart.

Proof:

Let $(0) \neq S = S^2$ be a simple ring. By Theorem 55 in Divinsky [7] $S \notin \mathcal{L}$ and so by Theorem 53 in Divinsky [7] there is a $x \in S$ such that $x^4 \neq 0$. Then $(0) \neq Sx^2S \lhd S$ so $S = Sx^2S$. Thus there are elements $r_1, \ldots, r_k$ and $s_1, \ldots, s_k$ in S such that

$$x = \sum_{i=1}^{k} r_i x^2 s_i \ . \qquad (*)$$

Let S' be the subring of S which is generated by the set $\{x, r_1, \ldots, r_k, s_1, \ldots, s_k\}$. Choose I maximal in $Z = \{J \lhd S' : x \notin J\}$. Then S'/I is subdirectly irreducible with heart $H = ((x) + I)/I$. If $x^2 \in I$ then by $(*)$ $x \in I$ so $x^2 \notin I$. Therefore $H^2 \neq (0)$ so $H^2 = H$.

Q.E.D.

An interesting conclusion that follows from this lemma is that if there is a simple idempotent nil ring then there is a simple idempotent nil ring which is the heart of a finitely generated nil ring.

5.3.3 PROPOSITION:

(1) $\beta_\varphi^* \leq \beta_\varphi$ and so β_φ^* is a radical class.

(2) $\mathcal{L} \leq \beta_\varphi^* \leq \mathcal{N}$.

(3) $\beta_\varphi^* \neq \mathcal{N}$ if and only if there is a non-zero simple idempotent nil ring.

Proof:

(1) Let $R \in \beta_\varphi^*$. If $R \notin \beta_\varphi$ then R can be homomorphically mapped onto a subdirectly irreducible ring with a simple idempotent heart S. But then by 5.3.2 some finitely generated subring of R is not in β_φ. This is a contradiction so $R \in \beta_\varphi$. By Theorem 2.2.2 β_φ^* is a local radical class.

(2) By Theorem 55 in Divinsky [7] no simple idempotent ring is in $\mathcal{L}$. Therefore $\mathcal{L} \leq \beta_\varphi^*$. Clearly $\beta_\varphi \leq F$ so $\beta_\varphi^* \leq F^* = \mathcal{N}$.

(3) If $\beta_\varphi^* \lneq \mathcal{N}$ then there is a nil ring which can be homomorphically mapped onto a subdirectly irreducible ring with an idempotent hear $H \neq (0)$. Clearly H is a simple idempotent nil ring. Conversely, any simple idempotent nil ring $S \neq (0)$ is β_φ semi-simple (and hence β_φ^* semi-simple since $\beta_\varphi^* \leq \beta_\varphi$) but is in $\mathcal{N}$.

Q.E.D.

In view of 1.1.6 the following theorem implies that $\beta_\varphi{}^*$ is a special radical class.

5.3.4 THEOREM:

A ring R is $\beta_\varphi{}^*$ semi-simple if and only if R is isomorphic to a subdirect sum of prime $\beta_\varphi{}^*$ semi-simple rings.

Proof:

Since subdirect sums of semi-simple rings are semi-simple one direction is clear.

Conversely, let R be a $\beta_\varphi{}^*$ semi-simple ring. It is sufficient to prove that for all non-zero $x \in R$ there is an ideal $I(x)$ such that $x \notin I(x)$ and $R/I(x)$ is a prime $\beta_\varphi{}^*$ semi-simple ring.

Let $0 \neq x \in R$. Since $(x)_R \notin \beta_\varphi{}^*$ there is a finitely generated subring R' of $(x)_R$ and an ideal I' of R' such that R'/I' contains a non-zero simple idempotent heart S'/I'. Notice that since S'/I' is simple if $J \triangleleft R$ and $I' + (R' \cap J) \not\supseteq S'$ then $R' \cap J \subseteq I'$. (*)

Let $Z = \{J \triangleleft R : I' + (R' \cap J) \not\supseteq S'\}$. Let $J_\alpha : \alpha \in \Lambda$ be an ascending chain of ideals in Z and let $J = \cup\{J_\alpha : \alpha \in \Lambda\}$. By (*), $R' \cap J_\alpha \subseteq I'$ for all $\alpha \in \Lambda$. Then $R' \cap J \subseteq I'$ hence $I' + R' \cap J = I' \not\supseteq S'$ so $J \in Z$. Therefore by Zorn's Lemma we may choose $I(x)$ maximal in Z.

First we shall prove that $R/I(x)$ is $\beta_\varphi{}^*$ semi-simple. Let $L/I(x)$ be a non-zero ideal of $R/I(x)$. Since $L \supsetneqq I(x)$, $I' + R' \cap L \supseteq S'$. Now
$\frac{R' \cap L + I(x)}{I(x)} \cong \frac{R' \cap L}{R' \cap I(x)}$ and $\frac{R' \cap L}{R' \cap I(x)}$ can be homomorphically mapped onto $\frac{R' \cap L + I'}{I'}$ since by (*)
$R' \cap I(x) \subseteq I'$. Now $\frac{S'}{I'} \subseteq \frac{R' \cap L + I'}{I'}$ and so $\frac{R' \cap + L + I'}{I'}$
is subdirectly irreducible with idempotent heart S'/I'. Then $(R' \cap L + I')/I'$ is not in β_φ so since $\beta_\varphi{}^* \leq \beta_\varphi$,
$(R' \cap L + I')/I' \notin \beta_\varphi{}^*$. Thus $\frac{R' \cap L + I(x)}{I(x)}$ and hence
$L/I(x)$ is not in $\beta_\varphi{}^*$. Therefore $R/I(x)$ is $\beta_\varphi{}^*$ semi-simple.

Now we shall prove that $R/I(x)$ is a prime ring. Suppose that $L/I(x)$ and $H/I(x)$ are non-zero ideals of $R/I(x)$ and $LH \subseteq I(x)$. By the maximality of $I(x)$,
$I' + (R' \cap L) \supseteq S'$ and $I' + (R' \cap H) \supseteq S'$. Therefore
$$\frac{S'}{I'} = \left(\frac{S'}{I'}\right)^2 = \frac{S'^2 + I'}{I'} \subseteq \frac{(R' \cap L + I\ )(R' \cap H + I')}{I'} \subseteq \frac{I(x) \cap R'}{I'}.$$
This implies that $S' \subseteq I' + (I(x) \cap R')$ which is a contradiction since $I(x) \in Z$.

Therefore $R/I(x)$ is a prime $\beta_\varphi{}^*$ semi-simple ring and since $R' \nsubseteq I(x)$, $x \notin I(x)$.

This completes the proof.

Q.E.D.

5.4 <u>LOCAL COMPLEMENTARY RADICAL CLASSES</u>.

Let $\mathcal{S}$ be a radical class. If there is a radical class $\mathcal{H}$ such that:

(1) $\mathcal{H}(A) \cap \mathcal{S}(A) = (0)$ for all rings A .

(2) If $\mathcal{J}$ is a radical class such that $\mathcal{J}(A) \cap \mathcal{S}(A) = (0)$ for all rings A then $\mathcal{J} \leq \mathcal{H}$.

then Andrunakievic [2] defines $\mathcal{H}$ to be the <u>complement</u> of $\mathcal{S}$. We shall denote $\mathcal{H}$ by CRH($\mathcal{S}$) . Notice that for some radical classes $\mathcal{S}$, CRH($\mathcal{S}$) may not exist.

In [2] Andrunakievic proves the following theorem.

5.4.1 <u>THEOREM</u>:

If $\mathcal{H}$ is a hereditary radical class then CRH($\mathcal{H}$) exists and

(i) CRH($\mathcal{H}$) = the upper radical class determined by the class of all subdirectly irreducible rings with hearts in $\mathcal{H}$.

(ii) $R \in$ CRH($\mathcal{H}$) if and only if every homomorphic image of R is $\mathcal{H}$ semi-simple (such rings are called strongly $\mathcal{H}$ semi-simple).

5.4.2 <u>DEFINITION</u>:

Let $\mathcal{S}$ be a radical class. If there is a local radical class $\mathcal{H}$ such that:

(i) $\mathcal{H}(A) \cap \mathcal{S}(A) = (0)$ for all rings A .

(ii) If $\mathfrak{I}$ is a local radical class and $\mathfrak{I}(A) \cap \mathcal{S}(A) = (0)$ for all rings A then $\mathfrak{I} \leq \mathfrak{H}$.

then $\mathfrak{H}$ is the <u>local complement</u> of $\mathcal{S}$.

We will denote the local complement of $\mathcal{S}$ by $\overline{\mathcal{S}}$.

5.4.3 <u>THEOREM</u>:

If $\mathfrak{H}^*$ is a local radical class then $\overline{\mathfrak{H}}^*$ exists and $\overline{\mathfrak{H}}^* = CRH(\mathfrak{H}^*)^*$.

<u>Proof</u>:

Notice that $\mathfrak{H}^*$ is hereditary so $CRH(\mathfrak{H}^*)$ exists. We shall prove that $CRH(\mathfrak{H}^*)^*$ is a radical class which satisfies conditions (i) and (ii) of 5.4.2.

(1) Since $CRH(\mathfrak{H}^*)$ satisfies condition (A) so does $CRH(\mathfrak{H}^*)^*$.

Suppose that B is an ideal of a ring A and that both A/B and B are in $CRH(\mathfrak{H}^*)^*$. Let A' be a finitely generated subring of A . If A' is not strongly $\mathfrak{H}^*$ semi-simple then A' can be homomorphically mapped onto a non-zero ring which is not $\mathfrak{H}^*$ semi-simple. Thus there is a finitely generated subring L' of A' Such that L' can be homomorphically mapped onto $(0) \neq (L'/K') \in \mathfrak{H}^*$. Since $A/B \in CRH(\mathfrak{H}^*)^*$, $\frac{L' + B}{B} \in CRH(\mathfrak{H}^*)$. Thus $\frac{L' + B}{B} \cong \frac{L'}{L' \cap B}$ is strongly $\mathfrak{H}^*$ semi-simple. It follows that $(L' \cap B) + K' = L'$ (since $L'/L' \cap B$ can be homomorphically mapped onto

$L'/(L' \cap B + K))$. Therefore $L' \cap B$ can be homomorphically mapped onto $\frac{L' \cap B + K'}{K'} = L'/K' \in \mathcal{H}^*$. This is a contradiction since $L' \cap B \subseteq B$ and hence is in $CRH(\mathcal{H}^*)^*$ (so no finitely generated subring of $L' \cap B$ can be homomorphically mapped onto a non-zero ring in $\mathcal{H}^*$).

Therefore every finitely generated subring of A is strongly $\mathcal{H}^*$ semi-simple so by 5.4.1 (ii) $A \in CRH(\mathcal{H}^*)^*$. Then by Proposition 2.2.1 $CRH(\mathcal{H}^*)^*$ is a local radical class.

(2) Let A be a ring and let $I = \mathcal{H}^*(A) \cap CRH(\mathcal{H}^*)^*(A)$. Let R' be a finitely generated subring of I . Then $R' \in \mathcal{H}^*$ and $R' \in CRH(\mathcal{H}^*)$ so $R' = \mathcal{H}^*(R') \cap CRH(\mathcal{H}^*)(R')$. Therefore $R' = (0)$ since $CRH(\mathcal{H}^*)$ is the compliment of $\mathcal{H}^*$. Hence $I = (0)$ so condition (i) of 5.4.2 is satisfied.

(3) Suppose that $\mathcal{J}$ is a local radical class and $\mathcal{J}(A) \cap \mathcal{H}^*(A) = (0)$ for all rings A . Then $\mathcal{J} \leq CRH(\mathcal{H}^*)$ since $CRH(\mathcal{H}^*)$ is the compliment of $\mathcal{H}^*$. But then $\mathcal{J} = \mathcal{J}^* \leq CRH(\mathcal{H}^*)^*$ so condition (ii) of 5.4.2 is satisfied.

Therefore $\overline{\mathcal{H}}^* = CRH(\mathcal{H}^*)^*$.

Q.E.D.

If follows that if $\mathcal{H}^*$ is a local radical class then $R \in \overline{\mathcal{H}}^*$ if and only if every finitely generated subring of R is strongly $\mathcal{H}^*$ semi-simple.

The following theorem shows that there is no need to define elementary complements of local radical classes.

5.4.4 <u>THEOREM</u>:

If $\mathcal{H}^*$ is a local radical class then $(\overline{\mathcal{H}^*})' = \overline{\mathcal{H}^*}$.

<u>Proof</u>:

Let $\mathcal{H}^*$ be a local radical class.

Since $\overline{\mathcal{H}^*}$ is a local class, $(\overline{\mathcal{H}^*}) \leq (\overline{\mathcal{H}^*})'$.

Let $R \in (\overline{\mathcal{H}^*})'$ and let R' be a finitely generated subring of R . Let R'/I' be a homomorphic image of R' and let $J'/I' = \mathcal{H}^*(R'/I')$. If $x \in J'/I'$ then $\langle x \rangle \in \mathcal{H}^*$ and since $R \in (\overline{\mathcal{H}^*})'$ $\langle x \rangle \in \overline{\mathcal{H}^*}$. Thus $\langle x \rangle = (0)$ so $J'/I' = (0)$. Therefore, R' is strongly $\mathcal{H}^*$ semi-simple so $R \in \overline{\mathcal{H}^*}$.

Hence, $\overline{\mathcal{H}^*} = (\overline{\mathcal{H}^*})'$ is an elementary class.

Q.E.D.

Notice that if $\mathcal{H} \geq \mathcal{R}$ then $CRH(\mathcal{H}) \leq CRH(\mathcal{R})$ if they both exist and $\overline{\mathcal{H}} \leq \overline{\mathcal{R}}$ if both of these classes exist.

Let $\mathcal{H}^*$ be a local radical class and let R be a ring which is not in $\overline{\overline{\mathcal{H}^*}}$. Then some finitely generated subring R' of R is not strongly $\overline{\mathcal{H}^*}$ semi-simple. Thus R' can be homomorphically mapped onto a ring R'' such that

$\overline{\mathfrak{H}}^*(R'') \neq (0)$. Clearly then $\mathfrak{H}^*(R'') \neq R''$ since $\overline{\mathfrak{H}}^*(R) \cap \mathfrak{H}^*(R) = (0)$. Therefore, $R \notin \mathfrak{H}^*$. It follows that $\mathfrak{H}^* \leq \overline{\overline{\mathfrak{H}}}^*$.

Suppose that $\mathfrak{R} \leq \mathfrak{H} \leq \mathfrak{J}$ and all three classes are radical classes. Also assume that $\overline{\mathfrak{J}}$ and $\overline{\mathfrak{R}}$ exist and that $\overline{\mathfrak{J}} = \overline{\mathfrak{R}}$. Then $\overline{\mathfrak{J}}(R) \cap \mathfrak{H}(R) \subseteq \overline{\mathfrak{J}}(R) \cap \mathfrak{J}(R) = (0)$ for all rings R . If $\mathcal{S}$ is a local radical class and $\mathcal{S}(R) \cap \mathfrak{H}(R) = (0)$ for all rings R then $\mathcal{S}(R) \cap \mathfrak{R}(R) \subseteq \mathcal{S}(R) \cap \mathfrak{H}(R) = (0)$ for all rings R so $\mathcal{S} \leq \overline{\mathfrak{R}} = \overline{\mathfrak{J}}$. Therefore $\overline{\mathfrak{H}}$ exists and $\overline{\mathfrak{H}} = \overline{\mathfrak{J}} = \overline{\mathfrak{R}}$.

We shall now investigate the local complements of the radical classes we have been discussing.

5.4.5 <u>PROPOSITION</u>:

$\overline{FI}^* = \{(0)\}$

<u>Proof</u>:

We need only show that if $\mathfrak{H}^*$ is a local radical class and $\mathfrak{H}^* \neq \{(0)\}$ then $FI^*(A) \cap \mathfrak{H}^*(A) \neq (0)$ for some ring A .

Suppose that $0 \neq R \in \mathfrak{H}^*$ and that $\mathfrak{H}^*$ is a local radical class. Let $0 \neq x \in R$. Then $\langle x\rangle \in \mathfrak{H}^*$ and so is $\langle x\rangle/\langle x\rangle^2$. If $\langle x\rangle \neq \langle x\rangle^2$ then $\langle x\rangle/\langle x\rangle^2$ can be homomorphically mapped onto a finite ring. If $\langle x\rangle = \langle x\rangle^2$ for all non-zero $x \in R$, then $R \in \mathcal{E}'$ so every finitely generated subring is finite. In either case we see that there is a finite ring in $\mathcal{H}^*$.

This completes the proof.

Q.E.D.

Of course the class $\{(0)\}$ is the local complement of the class of all rings. It then follows from 5.4.5 that if $\mathcal{H}$ is a radical class and $\mathcal{H} \geq FI^*$ then $\overline{\mathcal{H}}$ exists and $\overline{\mathcal{H}} = \{(0)\}$. In particular then,

$$\overline{FI^*} = \overline{FC} = \overline{\mathfrak{F}.\mathfrak{D}.^*} = \overline{\mathcal{L}_R} = \overline{\mathcal{L}} = \overline{(\mathcal{L}_R)_g} = \overline{\mathcal{L}_g} = \{(0)\} .$$

5.4.6 PROPOSITION:

$\overline{\mathcal{J}}$, $\overline{F}$ and $\overline{FF_g}$ exist and $\overline{\mathcal{J}} = \overline{F} = \overline{FF_g} = \mathcal{E}'$.

Proof:

Let R be a ring and let $I = \mathcal{E}'(R) \cap FF_g(R)$. If $I \neq (0)$ then since $I \in \mathcal{E}'$ by 3.4.3 I can be homomorphically mapped onto an algebraic field K of prime characteristic. Since $I \triangleleft FF_g(R)$, $FF_g(R)$ can be homomorphically mapped onto K (see Theorem 46 in Divinsky [7]). Since all finitely generated subrings of K are finite fields this is a contradiction. Therefore $I = (0)$ so $\mathcal{E}' \cap FF_g = \{(0)\}$. Similarily one shows that $\mathcal{E}' \cap F = \{(0)\}$. Since $\mathcal{J} \leq FF_g$, $\mathcal{E}' \cap \mathcal{J} = \{(0)\}$.

Suppose that $\mathcal{H}$ is a local radical class and

$\mathcal{H} \not\leq \mathcal{E}'$. Then there is a ring $\langle x\rangle \in \mathcal{H}$ such that $\langle x\rangle \neq \langle x\rangle^2$. Thus $\langle x\rangle$ can be homomorphically mapped onto the trivial ring $\langle x\rangle/\langle x\rangle^2$ and so $\langle x\rangle$ can be homomorphically mapped onto a simple zero ring. Thus $\mathcal{H} \cap \mathcal{J} \neq \{(0)\}$. It follows that $\mathcal{H} \cap FF_g \neq \{(0)\}$ and $\mathcal{H} \cap F \neq \{(0)\}$.

$$\text{Therefore,}\quad \mathcal{E}' = \overline{\mathcal{J}} = \overline{F} = \overline{FF_g} \,.$$

Q.E.D.

Now if $\mathcal{S}$ is any radical class such that $\mathcal{J} \leq \mathcal{S}$ and $\mathcal{S} \leq F$ or $\mathcal{S} \leq FF_g$ then $\overline{\mathcal{S}} = \mathcal{E}'$. This includes all of the radical classes listed in Chapter I except FF (for example : $\overline{J} = \overline{\mathcal{N}} = \mathcal{E}'$). It also includes the generalized local and elementary classes determined by these radical classes (for example : $\overline{J_g} = \overline{J_{g_1}} = \mathcal{E}'$) and the local and elementary classes determined by these radical classes (for example : $\overline{J^*} = \overline{J'} = \mathcal{E}'$).

Since any ring R can be embedded (as an ideal) in a ring R_1 with identity it is clear that $\overline{\mathcal{E}_{g_1}} = \{(0)\}$. So of course, if $\mathcal{S}$ is any radical class and $\mathcal{S} \geq \mathcal{E}_{g_1}$ then $\overline{\mathcal{S}} = \{(0)\}$.

The radical class FF does not have a local complement; that is, $\overline{FF}$ does not exist. To see this consider the radical classes

$$\mathfrak{I}_n = \bigoplus \{\mathfrak{I}_p(S_n) : p \text{ is a prime}\}$$

where S_n is the set of all positive integers $\leq n$.

Let R be a ring and let $I = \mathfrak{I}_n(R) \cap FF(R)$. Since $I \in \mathfrak{I}_n \leq \mathcal{E}'$, if $I \neq (0)$, I can be homomorphically mapped onto a field K in $\mathcal{E}'$ (see 3.4.3). However, since $I \in \mathfrak{I}_n$, $K \in \mathfrak{I}_n$, so K is a finite field. But then (as in Theorem 46 in Divinsky [7]) $FF(R)$ can be homomorphically mapped onto K . This is a contradiction because K is finite. Therefore $I = (0)$ so $\mathfrak{I}_n \cap FF = \{(0)\}$.

It follows that if $\overline{FF}$ exists then $\mathfrak{I}_n \subseteq \overline{FF}$ for all positive integers n . But then $\bigcup_n \mathfrak{I}_n \subseteq \overline{FF}$ so $\mathcal{E}' = (\bigcup_n \mathfrak{I}_n)' \subseteq (\overline{FF})' = \overline{FF}$. This is impossible since $\mathcal{E}' \cap FF$ contains all infinite algebraic fields of finite characteristic.

Therefore, $\overline{FF}$ does not exist.

We shall now consider the local complements of elementary radicals which are $\leq FC$.

5.4.7 <u>PROPOSITION</u>:

(1) $R \in \overline{FC}_p$ if and only if for all $x \in R$, $a_1x + a_2x^2 + \ldots + a_kx^k = 0$ for some integers $a_1,\ldots,a_k$ such that p divides all a_i for $i > 1$ but p does not divide a_1 .

(2) If $S \neq \emptyset$ then $\overline{\mathfrak{I}_p \mathcal{N}(S)} = \overline{FC}_p$.

(3) $R \in \overline{\mathfrak{I}_p \mathcal{N}(\emptyset)}$ if and only if for all $x \in R$, $a_1 x + \dots + a_k x^k = 0$ for some integers $a_1, \dots, a_k$ such that p does not divide a_1.

(4) If $S \neq \emptyset$ then $R \in \overline{\mathfrak{I}_p(S)}$ if and only if for all $x \in R$, $a_1 x + \dots + a_k x^k = 0$ for some integers $a_1, \dots, a_k$ such that p does not divide a_j for some j but p divides all a_i for which $i \neq j$.

(5) $\overline{\overline{FC}}_p = FC_p$.

(6) $\overline{\overline{\mathfrak{I}_p \mathcal{N}(\emptyset)}} = \mathfrak{I}_p \mathcal{N}(\emptyset)$.

(7) $\overline{\overline{\mathfrak{I}_p(S)}} = \mathfrak{I}_p(Z^+)$ if $S \neq \emptyset$.

Proof:

(1) Suppose R is a ring such that for all $x \in R$, $a_1 x + \dots + a_k x^k = 0$ for some integers $a_1, \dots, a_k$ such that p does not divide a_1 but p divides a_i for all $i > 1$.

If $FC_p(R) \neq (0)$ then there is a $x \in FC_p(R)$ such that $x \neq 0$ but $px = 0$. But there are integers $a_1, \dots, a_k$ such that p divides a_i for $i > 1$ but p does not divide a_1 and $a_1 x + \dots + a_k x^k = 0$. Then $a_1 x = 0$ and since p does not divide a_1, $x = 0$. This is a contradiction so $FC_p(R) = (0)$. It follows that every finitely generated subring of R is strongly

FC_p semi-simple so $R \in \overline{FC}_p$.

Let $R \in \overline{FC}_p$ and let $x \in R$. Then $\langle x\rangle$ is strongly FC_p semi-simple so $\langle x\rangle = \langle px\rangle$. Thus there are integers $a_1, \ldots, a_k$ such that p divides $a_1, \ldots, a_k$ and $x = a_1x + \ldots + a_kx^k$. Therefore $(a_1 - 1)x + a_2x^2 + \ldots + a_kx^k = 0$ and clearly p does not divide $a_1 - 1$.

This completes the proof of (1).

(2), (3), (4) The proofs for (2), (3), (4) are in all respects similar to the proof of (1).

(5) We have already seen that $FC_p \leq \overline{\overline{FC}}_p$.

Suppose $R \notin FC_p$. Let $R' = R/FC_p(R)$. Let $0 \neq x \in R'$. If $FC(\langle x\rangle) \neq (0)$ then there is a $y \in \langle x\rangle$ such that $qy = 0$ for some prime $q \neq p$. If $FC(\langle x\rangle) = (0)$ consiser $\mathcal{N}(\langle x\rangle)$. If $\mathcal{N}(\langle x\rangle) \neq (0)$ there is a $y \in \langle x\rangle$ such that $\langle y\rangle \cong C^\infty$ so $\langle y\rangle$ can be homomorphically mapped onto C_q for some prime $q \neq p$. If $\mathcal{N}(\langle x\rangle) = 0$ then as in the proof of 3.3.2 $\langle x\rangle$ can be homomorphically mapped onto a finite field of characteristic $q \neq p$. Thus in any case there is a subring of R which is generated by one element and which can be homomorphically mapped onto a ring of prime characteristic $q \neq p$.

If q is a prime and $q \neq p$ then a ring of characteristic q is in $\overline{FC}_p$ (take $a_1 = q$, $a_i = 0$ for $i > 1$). Therefore, if $R \notin FC_p$ there is an element $x \in R$ such that $\langle x \rangle$ is not strongly $\overline{FC}_p$ semi-simple. Thus $R \notin \overline{\overline{FC}}_p$.

It follows that $\overline{\overline{FC}}_p = FC_p$.

(6),(7) The proofs for (6) and (7) are similar to the proof of (5).

Q.E.D.

In view of the following theorem Proposition 5.4.7 completely determines the local complements of elementary radical classes which are $\leq FC$.

5.4.8 <u>THEOREM</u>:

If $\mathcal{H} \leq FC$, $\mathcal{H}$ is an elementary radical class and $\mathcal{H} = \bigoplus \{\mathcal{H}_p : p \in S\}$ is the representation of $\mathcal{H}$ given in 3.4.14 then $\overline{\mathcal{H}} = \cap\{\overline{\mathcal{H}}_p : p \in S\}$.

<u>Proof</u>:

Let $\mathcal{H} = \bigoplus \{\mathcal{H}_p : p \in S\}$ be an elementary radical.

Since $\mathcal{H} \geq \mathcal{H}_p$ for all $p \in S$, $\overline{\mathcal{H}} \leq \overline{\mathcal{H}}_p$ for all $p \in S$. Therefore $\overline{\mathcal{H}} \leq \cap\{\overline{\mathcal{H}}_p : p \in S\}$.

Suppose $R \in \cap\{\overline{\mathcal{H}}_p : p \in S\}$ and $x \in R$. Then $\langle x \rangle$ is strongly $\mathcal{H}_p$ semi-simple for all $p \in S$ so clearly $\langle x \rangle$ is strongly $\mathcal{H}$ semi-simple. Thus $R \in \overline{\mathcal{H}}$.

Therefore $\overline{\mathcal{H}} = \cap\{\overline{\mathcal{H}}_p : p \in S\}$.

Q.E.D.

By virtue of 5.4.7 and 5.4.8 we can conclude:

Since $\mathcal{E}' = \bigoplus\{\mathcal{J}_p(Z^+) : p \text{ is a prime}\}$

$\overline{\mathcal{E}'} = \cap\{\overline{\mathcal{J}_p(Z^+)} : p \text{ is a prime}\} = \mathcal{N}$.

Since $\mathcal{F}' = +\{\mathcal{J}_p\mathcal{N}(\emptyset) : p \text{ is a prime}\}$

$\overline{\mathcal{F}'} = \cap\{\overline{\mathcal{J}_p\mathcal{N}(\emptyset)} : p \text{ is a prime}\} = \mathcal{E}'$.

Since $\mathcal{L}_R \cap FC = \bigoplus\{\mathcal{J}_p\mathcal{N}(Z^+) : p \text{ is a prime}\}$

$\overline{\mathcal{L}_R \cap FC} = \cap\{\overline{\mathcal{J}_p\mathcal{N}(Z^+)} : p \text{ is a prime}\} = \{(0)\}$.

We have seen that $\overline{\mathcal{E}'} = \mathcal{N}$ and $\overline{\mathcal{N}} = \mathcal{E}'$. In fact, this implies that:

5.4.9 PROPOSITION:

For any local radical class $\mathcal{H}^*$

(1) $\mathcal{H}^* \cap \mathcal{N} = \{(0)\}$ if and only if $\mathcal{H}^* \leq \mathcal{E}'$.

(2) $\mathcal{H}^* \cap \mathcal{E}' = \{(0)\}$ if and only if $\mathcal{H}^* \leq \mathcal{N}$.

Proof:

Let $\mathcal{H}^*$ be a local radical class. Since $\mathcal{H}^*$, $\mathcal{N}$ and $\mathcal{E}'$ are hereditary it follows that if $\mathcal{H}^* \cap \mathcal{N} = \{(0)\}$ then $\mathcal{H}^* \leq \overline{\mathcal{N}} = \mathcal{E}'$ and if $\mathcal{H}^* \cap \mathcal{E}' = \{(0)\}$ then $\mathcal{H}^* \leq \overline{\mathcal{E}'} = \mathcal{N}$.

Clearly if $\aleph^* \cap \mathcal{N} \neq \{(0)\}$ then $\aleph^* \not\leq \mathcal{E}'$ and if $\aleph^* \cap \mathcal{E}' \neq \{(0)\}$ then $\aleph^* \not\leq \mathcal{N}$.

Q.E.D.

5.5 A REPRESENTATION OF $\mathcal{E}'$ AS THE INTERSECTION OF RADICAL CLASSES.

Many of the radical classes which we have discussed can be represented as the intersection of two other radical classes. The following diagram illustrates the situation.

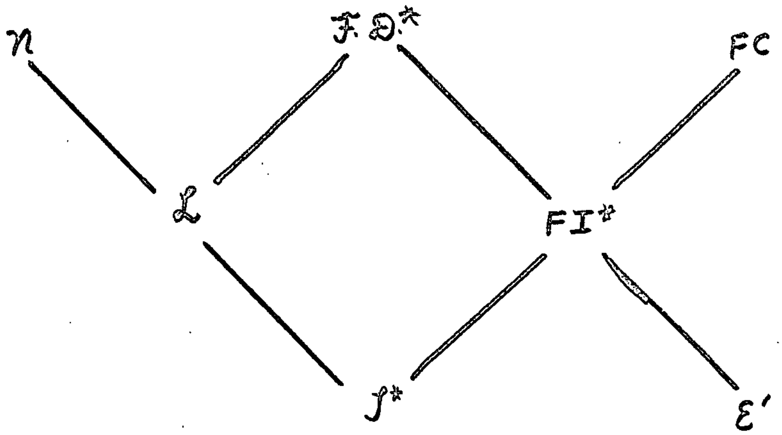


ILLUSTRATION 5

Recall that $\mathcal{L} = \mathcal{N} \cap \mathfrak{F}.\mathfrak{D}.^*$, FI* $= \mathfrak{F}.\mathfrak{D}.^* \cap$ FC and $\mathcal{J}^* = \mathcal{L} \cap$ FI* .

It seems natural to ask if there is a radical class $\aleph \gneq \mathcal{E}'$ such that FI* $\cap \aleph = \mathcal{E}'$. We shall see that no such class exists if we demand that it be a local class. However, we shall prove that $CRH(\mathcal{N}_g) \cap FI^* = \mathcal{E}'$.

We shall begin with the following easy lemmas.

5.5.1 <u>LEMMA</u>:

If $R \in FI^*$ and R is $\mathcal{N}_g$ semi-simple then $R \in \mathcal{E}'$.

<u>Proof</u>:

Let R be a $\mathcal{N}_g$ semi-simple ring which is in FI^*. Let $x \in R$. Then $\langle x \rangle$ is finite and $\mathcal{N}(\langle x \rangle) = (0)$ so $\langle x \rangle$ is a commutative Wedderburn ring.

Thus $\langle x \rangle$ is a finite direct sum of fields so $\langle x \rangle = \langle x \rangle^2$. Therefore $R \in \mathcal{E}'$.

Q.E.D.

5.5.2 <u>LEMMA</u>:

$$(CRH(\mathcal{N}_g))^* = \mathcal{E}' = (CRH(\mathcal{N}_g))' .$$

<u>Proof</u>:

If $R \in \mathcal{E}'$ then every subring of R is strongly $\mathcal{N}_g$ semi-simple. Therefore $\mathcal{E}' \leq (CRH(\mathcal{N}_g))^*$.

Suppose $R \in (CRH(\mathcal{N}_g))'$. Let $0 \neq x \in R$. Then $\langle x \rangle / \langle x \rangle^2$ is $\mathcal{N}_g$ semi-simple so $\langle x \rangle = \langle x \rangle^2$. Therefore $R \in \mathcal{E}'$.

Therefore $\mathcal{E}' \leq (CRH(\mathcal{N}_g))^* \leq (CRH(\mathcal{N}_g))' \leq \mathcal{E}'$.

Q.E.D.

Notice that in the proof of 5.5.2 we actually show that if $\langle x\rangle \in CRH(\mathcal{N}_g)$ then $\langle x\rangle \in \mathcal{E}$. Thus $(CRH(\mathcal{N}_g))_{g_1} \leq \mathcal{E}_{g_1}$.

5.5.3 <u>THEOREM</u>:

$CRH(\mathcal{N}_g) \cap FI^* = \mathcal{E}'$ and if $\mathcal{H}$ is a local radical class such that $\mathcal{H} \cap FI^* = \mathcal{E}'$ then $\mathcal{H} \leq CRH(\mathcal{N}_g)$ so $\mathcal{H} = \mathcal{E}'$.

<u>Proof</u>:

As we saw in 5.5.2, $\mathcal{E}' \leq CRH(\mathcal{N}_g)$. Thus $\mathcal{E}' \leq CRH(\mathcal{N}_g) \cap FI^*$.

If $R \in CRH(\mathcal{N}_g) \cap FI^*$ then R is $\mathcal{N}_g$ semi-simple so by Lemma 5.5.1, $R \in \mathcal{E}'$.

Therefore $CRH(\mathcal{N}_g) \cap FI^* = \mathcal{E}'$.

Suppose that $\mathcal{H}$ is a local radical class such that $\mathcal{H} \cap FI^* = \mathcal{E}'$. If $\mathcal{H} \not\leq CRH(\mathcal{N}_g)$ then there is a ring $R \in \mathcal{H}$ such that R is not strongly $\mathcal{N}_g$ semi-simple. Since $\mathcal{H}$ is a local class there is a ring $\langle x\rangle \in \mathcal{H}$ such that $\langle x\rangle$ is nilpotent. Then $\langle x\rangle$ can be homomorphically mapped onto a finite nil ring $\langle x'\rangle$. The ring $\langle x'\rangle \in FI^* \cap \mathcal{H} = \mathcal{E}'$. This is a contradiction so $\mathcal{H} \leq CRH(\mathcal{N}_g)$

Now $\mathcal{H} = \mathcal{H}^* \leq (CRH(\mathcal{N}_g))^* = \mathcal{E}'$ by Lemma 5.5.2. Therefore $\mathcal{H} = \mathcal{E}'$.

Q.E.D.

Of course CRH($\mathcal{N}_g$) $\neq \mathcal{E}'$ since the ring of rational numbers $Q \in$ CRH($\mathcal{N}_g$) . This example also show that CRH($\mathcal{N}_g$) $\not\leq$ FC . On the other hand, $\mathcal{N}_g \cap FC \neq \{(0)\}$ so FC $\not\leq$ CRH($\mathcal{N}_g$) .

Notice that CRH($\mathcal{N}_g$) $\cap$ FC $\neq \mathcal{E}'$. For an example consider any field of finite characteristic which is not algebraic.

We may sum up the relationships between these radical classes in the following diagram.

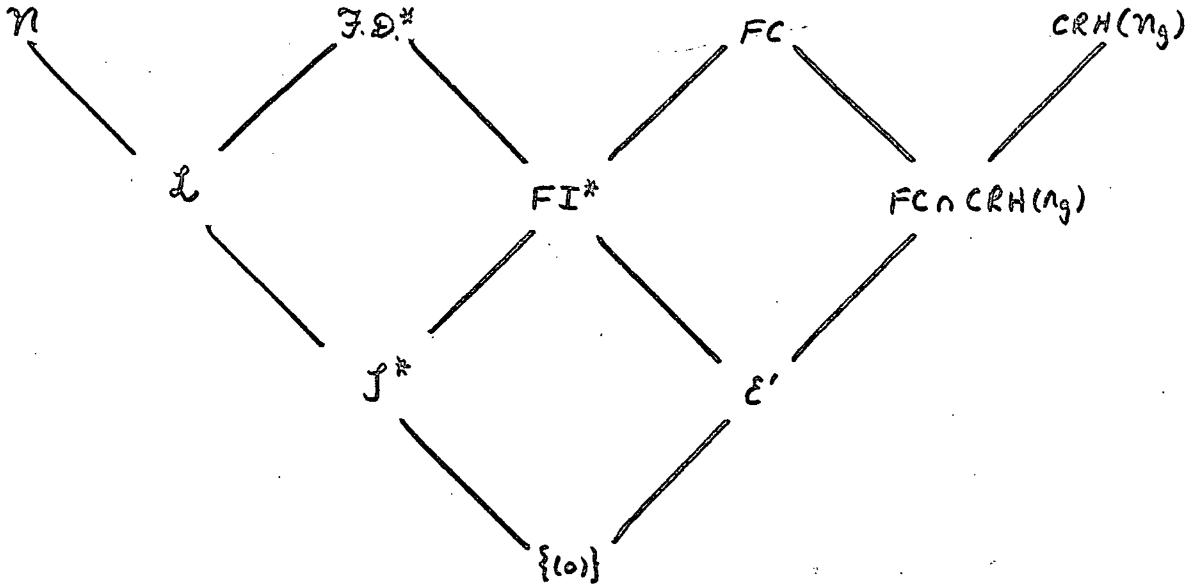


ILLUSTRATION 6

5.6 SEMI-SIMPLE RADICAL CLASSES.

In this section we shall characterize those local classes $\aleph^*$ which are both radical classes and semi-simple classes. We shall see that all radical classes which are also semi-simple classes are in fact local radical classes (indeed, elementary classes) so we shall begin with the more general problem.

5.6.1 LEMMA:

If $\mathfrak{R}$ is a class of rings such that subdirect sums of rings in $\mathfrak{R}$ are in $\mathfrak{R}$ and such that $\mathfrak{R}$ satisfies condition (A) then $\mathfrak{R}$ is strongly hereditary.

Proof:

Let $\mathfrak{R}$ be a class of rings such that subdirect sums of rings in $\mathfrak{R}$ are in $\mathfrak{R}$ and such that $\mathfrak{R}$ satisfies condition (A).

Let $R \in \mathfrak{R}$ and S be a subring of R.

Set $R_i = R$ for all $i \in Z^+$ = the set of positive integers. Now the (discrete) direct sum $\Sigma\{R_i : i \in Z^+\}$ is an ideal of the direct product (complete direct sum) $\prod\{R_i : i \in Z^+\}$. If $s \in S$ let $\hat{s}(i) = s$ for all $i \in Z^+$. Then $S \to \Delta(S) = \{\hat{s} : s \in S\}$ is an embedding of S into $\prod\{R_i : i \in Z^+\}$. $\Delta(S) + \Sigma\{R_i : i \in Z^+\}$ is a subdirect sum of copies of R and hence is in $\mathfrak{R}$, so

$$S \cong \Delta(S) \cong \frac{\Delta(S) + \Sigma\{R_i : i \in Z^+\}}{\Sigma\{R_i : i \in Z^+\}} \in \mathfrak{R} .$$

Q.E.D.

Using a theorem of Amitsur [1] which states that every ring is a homomorphic image of a subdirect sum of total matrix rings of finite order over the ring of all integers, Armendariz in [5] proves that if a hypernilpotent radical class $\mathfrak{R}$ is a semi-simple class, then $\mathfrak{R}$ contains all rings. Recall that a hypernilpotent radical class is a hereditary radical class which contains all nilpotent rings.

5.6.2 THEOREM:

If $\mathfrak{R}$ is a semi-simple radical class and $\mathfrak{R} \not\leq \mathcal{E}'$ then $\mathfrak{R}$ is the class of all rings.

Proof:

Let $\mathfrak{R}$ be a semi-simple radical class. If $\mathfrak{R} \not\leq \mathcal{E}'$ then there is an $R \in \mathfrak{R}$ and an $x \in R$ such that $\langle x\rangle \neq \langle x\rangle^2$. Since $\mathfrak{R}$ is a semi-simple class subdirect sums of rings in $\mathfrak{R}$ are in $\mathfrak{R}$ so by 5.6.1 $\langle x\rangle \in \mathfrak{R}$. Now $\langle x\rangle/\langle x\rangle^2$ is a zero ring on a cyclic group and $\langle x\rangle/\langle x\rangle^2 \in \mathfrak{R}$. Since $\mathfrak{R}$ satisfies (F), $C^\infty \in \mathfrak{R}$. Therefore $\beta \leq \mathfrak{R}$ since β = the lower radical class determined by $\{C^\infty\}$. Therefore $\mathfrak{R}$ is a hypernilpotent radical class so by the preceding remarks $\mathfrak{R}$ is the class of all rings.

Q.E.D.

We can now prove that all semi-simple radical classes must be elementary classes.

5.6.3 LEMMA:

If $\mathfrak{R}$ is a semi-simple radical class then $\mathfrak{R} = \mathfrak{R}'$.

Proof:

Let $\mathfrak{R}$ be a semi-simple radical class. Then by 5.6.1, $\mathfrak{R} \leq \mathfrak{R}'$.

If $\mathfrak{R}$ is the class of all rings then clearly $\mathfrak{R} = \mathfrak{R}'$.

If $\mathfrak{R}$ is not the class of all rings then by 5.6.2, $\mathfrak{R} \leq \mathcal{E}'$ so $\mathfrak{R}' \leq \mathcal{E}'$. Suppose $(0) \neq R \in \mathfrak{R}'$. Then by 3.4.3 R is isomorphic to a subdirect sum of fields $F_\alpha : \alpha \in \Lambda$ where each F_α is an algebraic field of prime characteristic. Let $\beta \in \Lambda$. Then for all $x \in F_\beta$, $\langle x \rangle \in \mathfrak{R}$ since $R \in \mathfrak{R}'$. Therefore the direct product (complete direct sum) $A = \prod\{\langle x \rangle : x \in F_\beta\} \in \mathfrak{R} \leq \mathcal{E}'$. Let $y \in A$ such that $y(x) = x$ for all $x \in F_\beta$. By 3.4.6 $\langle y \rangle$ is finite. Therefore F_β must be finite. Since finite fields are generated by one element each F_α is in $\mathfrak{R}$ and since sub-direct sums of rings in $\mathfrak{R}$ are in $\mathfrak{R}$, $R \in \mathfrak{R}$. Hence $\mathfrak{R}' \leq \mathfrak{R}$.

Therefore $\mathfrak{R} = \mathfrak{R}'$ so $\mathfrak{R}$ is an elementary class.

Q.E.D.

5.6.4 LEMMA:

If $\mathfrak{F}$ is a strongly hereditary finite set of finite fields then a ring R is isomorphic to a subdirect sum of fields in $\mathfrak{F}$ if and only if every finitely generated subring of R is isomorphic to a finite direct sum of fields in $\mathfrak{F}$.

Proof:

Let $\mathfrak{F}$ be a strongly hereditary finite set of finite fields. Then if $F \in \mathfrak{F}$ there is an integer $n(F)$ such that $x^{n(F)} = 1$ for all $x \in F$. Let $N = \prod\{n(F) : F \in \mathfrak{F}\} + 1$. Then if $x \in F \in \mathfrak{F}$, $x^N = x$.

Assume that R has ideals $I_\alpha : \alpha \in \Lambda$ such that $R/I_\alpha \cong F_\alpha \in \mathfrak{F}$ and $\cap\{I_\alpha : \alpha \in \Lambda\} = (0)$. Let R' be a finitely generated subring of R. By 3.4.5 $R' \in \mathcal{E}'$. Then by 3.4.4 $R' \cong A_1 \oplus \dots \oplus A_k$ where the A_i are finite fields. Choose $a_i \in R$ such that $\langle a_i \rangle \cong A_i$. Then $a_i \neq 0$ so $a_i \notin I_{\beta_i}$ for some $\beta_i \in \Lambda$. Now $\langle a_i \rangle \cap I_{\beta_i} \triangleleft \langle a_i \rangle$ so $\langle a_i \rangle \cap I_{\beta_i} = (0)$. Therefore $A_i \cong \langle a_i \rangle \cong (\langle a_i \rangle + I_{\beta_i})/I_{\beta_i}$ is isomorphic to a subring of F_{β_i}. Since $\mathfrak{F}$ is strongly hereditary R' is isomorphic to a finite direct sum of fields in $\mathfrak{F}$.

Conversely, assume that every finitely generated subring of R is isomorphic to a finite direct sum of fields in $\mathfrak{F}$. Then $x^N = x$ for all $x \in R$ so $R \in \mathcal{E}'$. Thus, by

3.4.3 there are ideals $I_\alpha : \alpha \in \Lambda$ of R such $\cap\{I_\alpha : \alpha \in \Lambda\} = (0)$ and R/I_α is a field of finite characteristic for all $\alpha \in \Lambda$. But then R/I_α must be a finite field since $x^N - x = 0 \in I_\alpha$ for all $x \in R$. Therefore, for each $\alpha \in \Lambda$, there is an $x_\alpha \in R$ such that $(\langle x_\alpha\rangle + I_\alpha)/I_\alpha = R/I_\alpha$. But then R/I_α is a homomorphic image of $\langle x_\alpha\rangle$ so R/I_α is isomorphic to a field in $\mathfrak{F}$.

Q.E.D.

5.6.5 THEOREM:

If $\mathfrak{R}$ is a class of rings which is not the class of all rings then the following are equivalent:

(1) $\mathfrak{R}$ is a semi-simple radical class.

(2) There is a finite set of primes T and for each $p \in T$ a finite C.U.D. set of positive integers S_p such that $R \in \mathfrak{R}$ if and only if R is isomorphic to a subdirect sum of fields in $\{F_{p^\alpha} : p \in T \text{ and } \alpha \in S_p\}$.

(3) There is a finite set of primes T and for each $p \in T$ a finite C.U.D. set of positive integers S_p such that $\mathfrak{R} = \bigoplus\{\mathfrak{I}_p(S_p) : p \in T\}$.

Proof:

Let $\mathfrak{R}$ be a class of rings which is not the class of all rings.

Lemma 5.6.4 implies that (2) and (3) are equivalent.

Assume that $\mathfrak{R}$ is a semi-simple radical class. Then by 5.6.2 and 5.6.3 $\mathfrak{R}$ is an elementary radical class and $\mathfrak{R} \leq \mathcal{E}'$. Therefore, by Theorem 3.4.14, $\mathfrak{R} = \bigoplus \{\mathfrak{I}_p(S_p) : p \in T\}$ for some set of primes T and C.U.D. sets of positive integers S_p for each $p \in T$. For each $p \in T$, $1 \in S_p$. Let $R = \prod\{F_p : p \in T\}$. Since $\mathfrak{R}$ is a semi-simple class $R \in \mathfrak{R} \leq \mathcal{E}'$. Because $R \in \mathcal{E}'$, $R \in FC$ so certainly T must be finite. Just as in 5.6.3 we see that each S_p must be finite by considering $\prod\{F_{p^\alpha} : \alpha \in S_p\}$.

Conversely, assume that $\mathfrak{R} = \bigoplus \{\mathfrak{I}_p(S_p) : p \in T\}$ where T is finite and for each $p \in T$, S_p is a finite C.U.D. set of positive integers. Then $\mathfrak{R}$ is a radical class.

Since $\mathfrak{R}$ is an elementary class $\mathfrak{R}$ satisfies condition (E).

Suppose that every non-zero ideal of a ring R can be homomorphically mapped onto a non-zero ring in $\mathfrak{R}$. Then by 5.6.4 every ideal of R can be mapped onto a field in $\{F_{p^\alpha} : p \in T , \alpha \in S_p\}$. From the proof of Theorem 46 in Divinsky [7] we see that this implies that R is isomorphic to a subdirect sum of fields in $\{F_{p^\alpha} : p \in T , \alpha \in S\}$. Then, by 5.6.4 again, $R \in \mathfrak{R}^* = \mathfrak{R}$. Thus $\mathfrak{R}$ satisfies condition (F).

Therefore $\mathfrak{R}$ is a semi-simple radical class.

Q.E.D.

The relationships between the local radical classes which we have discussed can be illustrated in the following diagram.

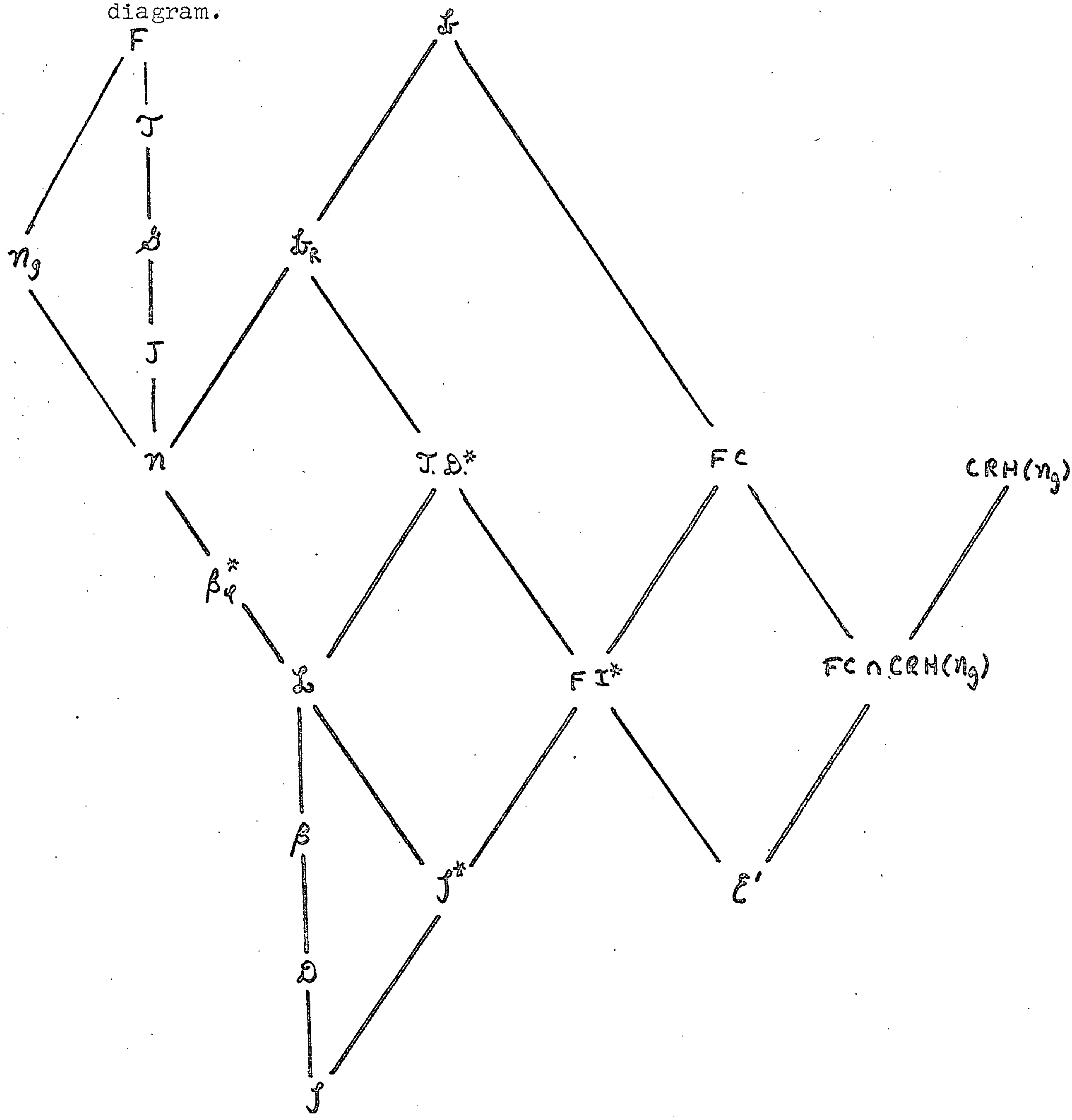


ILLUSTRATION 7

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