SADDLEPOINT APPROXIMATIONS TO DISTRIBUTION FUNCTIONS

by

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ABSTRACT

In this thesis we present two approximations to the distribution function of the sum of n independent random variables. They are obtained from generalizations of asymptotic expansions derived by Rubin and Zidek who considered the case of chi random variables. These expansions are obtained from Gurland's inversion formula for the distribution function by using an adaptation of Laplace's method for integrals. By means of numerical results obtained for a variety of common distributions and small values of n these approximations are compared to the classical methods of Edgeworth and Cramér. Finally, the method is used to obtain approximations to the non-central chi-square distribution and to the doubly non-central F-distribution for various cases defined in terms of its parameters.

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INTRODUCTION

It is often necessary to approximate the distribution of a statistic whose exact distribution is unknown or cannot conveniently be calculated. For example, in the evaluation of error probabilities in some types of communications systems or sometimes in the determination of the power function of a likelihood ratio test, the probability $P(\frac{1}{n} \sum_{i=1}^{n} X_i \leq x)$, where the X_i $(i = 1,2,\ldots)$ are independent random variables, is required for some values of x .

A well-known approximation available for this type of problem is due to Edgeworth ([3], pp. 228-229; [7], p. 515). Another is that of Cramér ([7], p. 520; [4]), which was designed to provide an accurate approximation even in the case where x is permitted to depend on n .

When the characteristic function ψ of the statistic is known, inverting the Fourier transform explicitly is often impossible, while a numerical integration routine is too time-consuming for procedures requiring high accuracy, especially when $|\psi(t)|$ is not rapidly decreasing as $|t| \to \infty$.

In this thesis we consider two approximations, suggested by Rubin and Zidek [13], to the distribution function of the sum of n independent random variables. Each of these approximations is called, in [13], a saddlepoint approximation in keeping with the terminology used by Daniels [5], who seems to have been the first to introduce into the literature of probability approximation theory the method upon which the approximations in [13] are based. Whereas in this thesis we are concerned with problems relating to distribution functions, the work of Daniels pertains to density functions.

The efforts of Rubin and Zidek [13] are directed toward the problem of finding an approximation to the distribution function of $(|Z_1| + \dots + |Z_n|)$, where the $\{Z_i\}$ are independent random variables and each Z_i $(i = 1,\dots,n)$ is normally distributed with mean 0 and variance 1. Each of their approximations is roughly equivalent, in terms of required computer time, to those of Edgeworth and Cramér. In [13] it is shown for the problem considered there, on the basis of numerical results, that one of the approximations is superior to either of the two classical methods. Even for the case $n = 10$, where for values of the arguments considered, the older methods yield an accuracy of at most two significant figures, it gives results accurate to five significant figures.

In Chapter I of this thesis all of the approximations mentioned above are presented. Also given is an inversion formula, derived from that of Gurland [9], which forms the basis for the saddlepoint approach.

The contents of Chapter II consist of proofs that the formal expansions derived in [13] are in fact asymptotic expansions. In [13] it is suggested that these results might be obtained by using an adaptation of the argument given by Daniels [5], which is based on the method of steepest descent. Here we give for the case of non-lattice random variables simple direct proofs which use Laplace's method for integrals and special features of the present problem. The results are, in one case, an expansion in powers of $n^{-\frac{1}{2}}$, and, in the other, as in Daniels' case for densities, an expansion in powers of n^{-1}.

The approximations are then compared, in Chapter 3, with those of Edgeworth and Cramér for a variety of special cases on the basis of numerical computations for small values of n. The results are qualitatively the same as those found in [13] for the special case considered there, which is presented for completeness in section (3.4).

The results given in Chapter 3 are obtained with the intention of comparing the various approximations described earlier. Except for the cases considered in sections 3.4 and 3.7, the desired values can be found with reasonable accuracy from existing tables. In Chapter 4, some additional applications of the "saddlepoint method" are considered. These involve the non-central chi-square distribution for large values of its non-centrality parameter, and the doubly non-central F-distribution for selected special cases defined in terms of its four parameters. In the latter case particularly, existing tables are inadequate, and the saddlepoint approximation may be of practical value.

Two appendices are supplied. In the first is given a method of evaluating the normal distribution function for either real or complex values of its argument. The method, which uses continued fractions, is given in [13]. In Appendix

2, we describe a program written in FORTRAN which is suitable for numerically evaluating the better of the two saddlepoint approximations.

CHAPTER I

NOTATION AND PRELIMINARY RESULTS

1.1 Notation.

Let $\{X_1, X_2, \ldots\}$ be a set of independent, identically distributed random variables, each having distribution function F, density function f, mean $\mu = 0$, variance σ^2, and characteristic function φ; that is,

$$\begin{aligned} \varphi(t) &= E[e^{itX}] \qquad\qquad (-\infty<t<\infty) \\ &= \int_{-\infty}^{\infty} e^{itx}\, dF(x) \quad , \end{aligned}$$

where $i = \sqrt{-1}$.

We assume the moment generating function of X_i exists on a non-degenerate interval (a,b) and denote it by M . Let K be the cumulant generating function. Then

$$M(it) = \varphi(t) \quad , \qquad -\infty<t<\infty \quad ,$$

and

$$K(t) = \log M(t) \quad .$$

We take the domains of M and K to be the subset of the complex plane given by $\{z : a < \mathrm{Re}(z) < b\}$.

The distribution and density functions of the standard normal distribution will be denoted by N and n , respectively.

Let F_n denote the probability distribution function of $\sum_{i=1}^{n} X_i/(\sqrt{n}\,\sigma)$, $(n = 1,2,\ldots)$; that is,

$$F_n(x) = F^{n*}(x\,\sigma\,\sqrt{n}) ,$$

where F^{n*} denotes the n-fold convolution of F . More generally, if $\mu \neq 0$, F_n and F will denote the distribution functions of $\sum_{i=1}^{n} (X_i - \mu)/(\sqrt{n}\,\sigma)$ and $(X_i - \mu)$, respectively, while M will denote the moment generating function of $(X_i - \mu)$.

1.2 The Edgeworth Approximation.

THEOREM 1.2.1. *If* $\limsup_{|s|\to\infty} |\varphi(s)| < 1$, *and the* r^{th} *absolute moment of* F *exists, then*

$$F_n(x) = N(x) + n(x) \sum_{k=3}^{r} n^{-\frac{1}{2}k+1} R_k(x) + o(n^{-\frac{1}{2}r+1}) \qquad (n\to\infty) \qquad (1.2.1)$$

uniformly in x , *for some polynomials* $R_1,\ldots,R_r$, *each depending on* $\mu_1,\ldots\mu_r$, *the first* r *moments of* F , *but not on* n , *or otherwise on* F *or* r .

PROOF. See Feller [7], p. 515.

The series in (1.2.1) is known as the Edgeworth expansion of F_n . The construction of the polynomials, R_k $(k = 1,\ldots,r)$, is described in Feller [7], p. 509. With its first few terms given explicitly, this expansion for F_n is

$$
\begin{aligned}
F_n(w_n &= N(w_n) \\
&- n^{-\frac{1}{2}}[\tfrac{\lambda_3}{3!} N^{(3)}(w_n)] \\
&+ n^{-1}[\tfrac{\lambda_4}{4!} N^{(4)}(w_n) + \tfrac{10}{6!} \lambda_3^2 N^{(6)}(w_n)] \\
&- n^{-\frac{3}{2}}[\tfrac{\lambda_5}{5!} N^{(5)}(w_n) + \tfrac{35}{7!} \lambda_3\lambda_4 N^{(7)}(w_n) + \tfrac{280}{9!}\lambda_3^3 N^{(9)}(w_n)] \\
&+ n^{-2}[\tfrac{\lambda_6}{6!} N^{(6)}(w_n) + (35\lambda_4^2 + 56\lambda_3\lambda_5) \frac{N^{(8)}(w_n)}{8!} \\
&+ \tfrac{2100}{10!} \lambda_3^2\lambda_4 N^{(10)}(w_n) + \tfrac{15400}{12!} \lambda_3^4 N^{(12)}(w_n)] - \dots
\end{aligned}
\qquad (1.2.2),
$$

where $w_n = (x - n\mu)/(\sqrt{n}\,\sigma)$, $\lambda_n = \alpha_n/\sigma^n$ (α_n denoting the n^{th} cumulant of F), and $N^{(i)}$ denotes the i^{th} derivative of N .

In practice, it is not advisable to go beyond the second or third term of the series (1.2.2), as a well-known disadvantage of the Edgeworth expansion is that it then tends to give negative values for $F_n(x)$ when x is small or values exceeding 1 when x is large.

1.3 The Cramér Approximation

Cases occur where, in $F_n(x)$, x depends on n , as when computing $P[\overline{X} \leq y] = F_n(\sqrt{n}\, y/\sigma)$ (with $\mu = 0$) . Then (1.2.1) will fail to hold. Since in this case both $F_n(x)$ and $N(x)$ converge to 1 as $n \to \infty$, a more appropriate criterion of the accuracy of the approximation is the relative error. In particular, we would like the relation

$$\frac{1 - F_n(x)}{1 - N(x)} \to 1 \tag{1.3.1}$$

to hold when both x and n tend to infinity. This relation is not true generally, since, for example, in the case of the symmetric binomial distribution, $1 - F_n(x) = 0$ for all $x > \sqrt{n}$. The limit in (1.3.1) does hold if $x \doteq o(n^{1/6})$, this being a consequence of the following more general result. It is due to H. Cramér [4] and was generalized to variable components by Feller ([7], p. 524).

Let the function λ be defined by the equation

$$z^3 \lambda(z) = K(h) - h\, K^{(1)}(h) + \tfrac{1}{2}z^2 \tag{1.3.2}$$

where h is obtained as a power series in z by inverting the series

$$\begin{aligned} \sigma z &= \sum_{r=2}^{\infty} \frac{\alpha_r}{(r-1)!}\, h^{r-1} \\ &= K^{(1)}(h) \end{aligned} \tag{1.3.3}$$

Equation (1.3.2) implies

$$\lambda(z) = \lambda_3/6 + (\lambda_4/24 - \lambda_3^2/8)z$$

$$+ (\lambda_5/120 - \lambda_3\lambda_4/12 + \lambda_3^3/8)z^2 + \ldots \quad (1.3.4).$$

THEOREM 1.3.1 (Cramér). *Suppose there exists a number* h_o *such that* $\int_{-\infty}^{\infty} e^{hx}\, dF(x)$ *exists for all* $h \in (-h_o, h_o)$. *Let* x *be a real number, which may depend on* n , *such that* $x > 1$ *and* $x = o(n^{\frac{1}{2}})$ *as* $n \to \infty$. Then

$$1 - F_n(x) = [1 - N(x)] \exp\left[\frac{x^3}{\sqrt{n}}\, \lambda\left(\frac{x}{\sqrt{n}}\right)\right]\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right] \quad (n \to \infty) \qquad (1.3.5a).$$

For $x < -1$, *the corresponding relation is*

$$F_n(x) = N(x) \exp\left[\frac{x^3}{\sqrt{n}}\, \lambda\left(\frac{x}{\sqrt{n}}\right)\right]\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right] \quad (n \to \infty) \qquad (1.3.5b).$$

PROOF. See Feller [7], p. 517.

When the Cramér approximation is applied, difficulties may be encountered in the inversion of the series (1.3.3), because the series given in (1.3.4) may fail to converge. Examples where this occurs are given in section 3.3. The approximation can still be applied in these cases if it is possible to invert (1.3.3) algebraically.

1.4 The Saddlepoint Method.

We now present two saddlepoint approximations to F_n obtained by Rubin and Zidek [13]. They are based on an inversion formula derived in Lemma 1.4.1 from the Gurland [9] inversion formula. It asserts that

$$1 - F_n\left(\frac{x}{\sqrt{n}\ \sigma}\right) = \tfrac{1}{2} + \lim_{T\to\infty} \lim_{\epsilon\to 0_+} \left\{ \int_{-T}^{-\epsilon} + \int_{\epsilon}^{T} \right\} \frac{e^{-ixt}}{2\pi i t} M^n(it)dt \tag{1.4.1}$$

LEMMA 1.4.1. *Assume that the moment generating function* M *exists in the region* $\{z : a < \mathrm{Re}(z) < b\}$. *Then*

$$\lim_{T\to\infty} \lim_{\epsilon\to 0_+} \left\{ \int_{-T}^{-\epsilon} + \int_{\epsilon}^{T} \right\} \frac{e^{-ixt}}{2\pi i t} M^n(it)dt$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(c+iu)x} M^n(c+iu)\frac{du}{c+iu} - \tfrac{1}{2}\,\mathrm{sign}\,(c) \tag{1.4.2}$$

for every real number $c \neq 0$ *such that* $a < c < b$, *where*

$$\mathrm{sign}(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 0 \end{cases} .$$

PROOF. Suppose $b > c > 0$, and consider the line integral in the complex plane of the function

$f(z) = e^{-zx} M^n(z)/(2\pi iz)$ along the contour $I = I_1 I_2 + \ldots + I_6$, where, for fixed positive constants T and ϵ $(\epsilon < T)$,

$$I_1 = \{z : Re(z) = 0 \ , \ \epsilon < Im(z) \leq T\} \ ,$$

$$I_2 = \{z : Im(z) = T \ , \ 0 < Re(z) \leq c\} \ ,$$

$$I_3 = \{z : Re(z) = c \ , \ -T \leq Im(z) < T\} \ ,$$

$$I_4 = \{z : Im(z) = -T \ , \ 0 \leq Re(z) < c\} \ ,$$

$$I_5 = \{z : Re(z) = 0 \ , \ -T < Im(z) \leq -\epsilon\} \ ,$$

$$I_6 = \{z : z = \epsilon e^{i\theta} \ , \ -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}\} \ .$$

Then $\int_I f(z)dz = 0$.

Now,

$$\left|\int_{I_2} f(z)\, dz\right| \leq \frac{1}{2\pi}\int_0^c \left|\exp\,(-x(y+iT))\, M^n(y+iT)\right| \frac{dy}{|y+iT|}$$

$$\leq \frac{1}{2\pi}\int_0^c e^{-xy} M^n(y) \frac{dy}{|y+iT|} \ ,$$

since

$$|M^n(y+iT)| = |E(\exp[(y+iT)(X_1 + \ldots + X_n)])|$$

$$\leq E(\exp[y(X_1 + \ldots + X_n)])$$

$$= M^n(y) \ .$$

Hence,

$$\left|\int_{I_2} f(z)dz\right| \leq \frac{1}{T}\left[\frac{A}{2\pi}\,\frac{1}{x}\,(1 - e^{-cx})\right] \ ,$$

where $A = \max\limits_{0<y\leq c} M^n(y)$, and therefore

$$\lim_{T\to\infty} \int_{I_2} f(z)\, dz = 0 \quad .$$

Similarly, $\lim\limits_{T\to\infty} \int_{I_4} f(z)dz = 0$.

Consider $\int_{I_6} f(z)dz$. By the residue theorem,

$$\int_{c(\epsilon)} f(z)dz = 1 \quad ,$$

where $c(\epsilon)$ is a circle with centre at the origin, radius ϵ , and the integral is taken in a counterclockwise direction.

We shall now show that

$$\lim_{\epsilon\to 0} \int_{-\pi/2}^{\pi/2} f(\epsilon e^{i\theta})\epsilon i e^{i\theta}\, d\theta = \lim_{\epsilon\to 0} \int_{\pi/2}^{-\pi/2} f(\epsilon e^{i\theta})\epsilon i e^{i\theta}\, d\theta \tag{1.4.3}$$

From this it follows that $\lim\limits_{\epsilon\to 0} \int_{I_6} f(z)dz = \frac{1}{2}$, and the desired result is an immediate consequence.

Now,

$$\Big|\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-x\epsilon(\cos\theta + i\sin\theta)} M^n(\epsilon\cos\theta + \epsilon i \sin\theta)d\theta$$

$$- \frac{1}{2\pi} \int_{\pi/2}^{-\pi/2} e^{-x\epsilon(\cos\theta + i\sin\theta)} M^n(\epsilon\cos\theta + \epsilon i \sin\theta)d\theta\Big|$$

$$= \Big|\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} [e^{-x\epsilon(\cos\theta + i\sin\theta)} M^n(\epsilon\cos\theta + \epsilon i \sin\theta)$$

$$- e^{-x\epsilon(-\cos\theta + i \sin\theta)} M^n(-\epsilon \cos\theta + \epsilon i \sin\theta)]d\theta|$$

$$\leq \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} |e^{-x\epsilon \cos\theta} M^n(\epsilon \cos\theta + \epsilon i \sin\theta) - e^{x\epsilon \cos\theta}$$

$$\times M^n(-\epsilon \cos\theta + \epsilon i \sin\theta)|d\theta \quad .$$

By the Legesgue bounded convergence theorem, the last quantity converges to 0 as $\epsilon \to 0$ and the result, (1.4.3), is established. Combining the results for $\int_{I_2} f(z)dz$, $\int_{I_4} f(z)dz$, and $\int_{I_6} f(z)dz$, we obtain (1.4.2) for $c > 0$. For $a < c < 0$, the proof is similar.

We shall make use of the inversion formula (1.4.2) in the form,

$$1 - F_n(\sqrt{n}\, x/\sigma) = \tfrac{1}{2}(1 - \text{sign}(c)) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-nx(c+iu)+n \log M(c+iu)] \times \frac{du}{c+iu} \qquad (1.4.4)$$

where we shall choose c to be a saddlepoint of the exponent in the integral; that is, c is the solution of the equation

$$\frac{d}{dz} [-nxz + n \log M(z)] = 0 \qquad (1.4.5),$$

or

$$\frac{M^{(1)}(z)}{M(z)} = x \qquad (1.4.6).$$

When $c = 0$, the integral in (1.4.4) will be understood to mean that of the Gurland inversion formula.

Daniels [5] showed that (1.4.6) has a single real root under fairly general conditions.

THEOREM 1.4.1 (Daniels). *Assume that*

$$M(t) = e^{K(t)} = \int_{-\infty}^{\infty} e^{tx}\, dF(x)$$

converges for real t *in* $-c_1 < t < c_2$, *where* $0 < c_1 \leq \infty$, $0 < c_2 \leq \infty$. *Suppose*

$$\begin{aligned} F(x) &= 0 , & x &< a , \\ 0 < F(x) &< 1 , & a &< x < b , \\ F(x) &= 1 , & b &< x , \end{aligned}$$

where, possibly, $a = -\infty$ *or* $b = \infty$ *or both. Then*

(i) a *and* b *are finite if and only if* $K(t)$ *exists for all real* t , *and* (1.4.6) *has no real root whenever*

$$x \notin [a,b] ,$$

(ii) *for every* $x \in (a,b)$, *where* $-\infty < a < b < \infty$, *there exists a unique simple root* c *of* (1.4.6), *and* $K^{(1)}(t)$ *increases continuously from* $x = a$ *to* $x = b$,

(iii) *for every* $x \in (a,b)$, *where* a *and* b *may be infinite, there exists a corresponding* c *in* $(-c_1,c_2)$ *if* $\lim_{t\to+c_2} K^{(1)}(t) = b$ *and* $\lim_{t\to-c_1} K^{(1)}(t) = a$ (*these conditions are satisfied automatically unless* a *or* b *is infinite*).

PROOF. See Daniels [5].

Since $K(t)$ converges for $-c_1 < \mathrm{Re}(t) < c_2$, and $c \in (-c_1, c_2)$, $K(t)$ has a power series expansion about $t = c$ with a non-zero radius of convergence, and hence a uniformly convergent series expansion for all t such that $|t - c| \leq \rho$ for some $\rho > 0$. Then, for values of n in some neighbourhood of the origin we can write

$$-nx(c+iu) + n\,K(c+iu) = -nxc + K(c)n - n\,\frac{u^2}{2}\,K^{(2)}(c) + n\sum_{r=3}^{\infty} \frac{K^{(r)}(c)}{r!}(iu)^r \quad (1.4.7).$$

Let

$$\begin{aligned} \sigma^* &= [K^{(2)}(c)]^{\frac{1}{2}} , \\ b_r &= K^{(r)}(c)\, i^r/(r!\,\sigma^{*r}) \\ a_r &= (-i)^r/(c\sigma^*)^r , \qquad\qquad (1.4.8) \\ K(c,n,x) &= \frac{1}{\sqrt{2\pi}}\, e^{-nxc + n\,K(c)} , \\ I(x,n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-nx(c+iu) + n\,K(c+iu)}\, \frac{du}{c+iu} . \end{aligned}$$

Then, proceeding formally,

$$\begin{aligned} I(x,n) &= \frac{K(c,n,x)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{n\sum_{r=3}^{\infty} \frac{K^{(r)}(c)}{r!}(iu)^r}\, e^{\frac{-n\sigma^{*2}u^2}{2}}\, \frac{du}{c+iu} \\ &= \frac{K(c,n,x)}{c\sigma^*\sqrt{2\pi n}} \int_{-\infty}^{\infty} \left(1+\frac{iy}{c\sigma^*\sqrt{n}}\right)^{-1} e^{n\sum_{r=3}^{\infty} b_r (y/\sqrt{n})^r}\, e^{-y^2/2}\, dy \end{aligned}$$

(1.4.9)

Now,

$$\left(1 + \frac{iy}{c\sigma^*\sqrt{n}}\right)^{-1} = 1 + \sum_{r=1}^{\infty} a_r \; (y/\sqrt{n})^2 \quad , \; |y| < c\sigma^*\sqrt{n} \quad , \tag{1.4.10}$$

and

$$e^{n \sum_{r=3}^{\infty} b_r \; (y/\sqrt{n})^r} = 1 + n^{-\frac{1}{2}} \, b_3 \, y^3 + n^{-1} \, (b_4 y_4 + \tfrac{1}{2} \, b_3^2 y^6)$$
$$+ \, n^{-3/2}(b_5 y^5 + b_3 b_4 y^7 + \tfrac{1}{6} \, b_3^3 y^9)$$
$$+ \, n^{-2}(b_6 y^6 + [\tfrac{1}{2} b_4^2 + b_3 b_5] y^8 + \tfrac{1}{2} b_3^2 b_4 y^{10} + \tfrac{1}{24} b_3^4 y^{12})$$
$$+ \ldots \tag{1.4.11}$$

Hence,

$$\left(1 + \frac{iy}{c\sigma^*\sqrt{n}}\right)^{-1} \exp\left(n \sum_{r=3}^{\infty} b_r \left(\frac{y}{\sqrt{n}}\right)^r\right) = \sum_{m=0}^{\infty} d_m(y) \left(\frac{1}{\sqrt{n}}\right)^m \tag{1.4.12}$$

where

$$d_0(y) = 1$$
$$d_2(y) = a_2 y^2 + (b_4 + a_1 b_3) y^4 + \tfrac{1}{2} \, b_3^2 \, y^6$$
$$d_4(y) = a_4 y^4 + (b_6 + a_1 b_5 + a_2 b_4 + a_3 b_3) y^6 + (\tfrac{1}{2} b_4^2 + b_3 b_5$$
$$+ \, a_1 b_3 b_4 + \tfrac{1}{2} a_2 b_3^2) y^8 + (\tfrac{1}{2} b_4 b_3^2 + \tfrac{1}{6} a_1 b_3^3) y^{10} + \tfrac{1}{24} \, b_3^4 y^{12} \tag{1.4.13}$$

and, in general, $d_{2k-1}(y)$ is an odd polynomial in y, while $d_{2k}(y)$ an even polynomial in y $(k = 1,2,\ldots)$. As odd powers of y vanish upon integrating, the explicit form of $d_{2k-1}(y)$ $(k = 1,2,\ldots)$ is not required below.

It will be shown in the following chapter that

$$I(x,n) \sim \frac{K(c,n,x)}{c\sigma^* \sqrt{n}} \sum_{m=0}^{\infty} d_{2m} \left(\frac{1}{\sqrt{n}}\right)^{2m} \qquad (1.4.14)$$

is an asymptotic expansion. Here,

$$d_{2m} = \int_{-\infty}^{\infty} n(y)\ d_{2m}(y)\ dy \qquad (1.4.15)$$

(recall that $n(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$) , and the series in (1.4.14) is obtained formally by interchanging summation and integration in the expression obtained from (1.4.9) by replacing the first two factors in its integrand by their series expansion.

The first few coefficients in (1.4.14) are

$$\begin{aligned}
d_0 &= 1 \\
d_2 &= a_2 + 3(b_4 + a_1 b_3) + \frac{15}{2} b_3^2 \qquad (1.4.16) \\
d_4 &= 3a_4 + 15(b_6 + a_1 b_5 + a_2 b_4 + a_3 b_3) + 105(\tfrac{1}{2} b_4^2 + b_3 b_4 \\
&\quad + a_1 b_3 b_4 + \tfrac{1}{2} a_2 b_3^2) + 945(\tfrac{1}{2} b_4 b_3^2 + \tfrac{1}{6} a_1 b_3^3) + \frac{10395}{24} b_3^4 \ .
\end{aligned}$$

Explicitly, equations (1.4.16) are, with $K^{(r)}(c) = K^r$ $(r = 1,2,\ldots)$,

$$\begin{aligned}
d_0 &= 1 \\
d_2 &= -(c\sigma^*)^{-2} + \sigma^{*-4}(\tfrac{1}{8}K_4 - \tfrac{1}{2c} K_3) - \tfrac{5}{24} K_3^2 (\sigma^*)^{-6} \qquad (1.4.17) \\
d_4 &= 3(c\sigma^*)^{-4} + \tfrac{5}{8} \sigma^{*-6}(-\tfrac{1}{30} K_6 + \tfrac{1}{5c} K_5 - \tfrac{1}{c^2} K_4 + \tfrac{4}{c^3} K_3) \\
&\quad + \tfrac{35}{24} \sigma^{*-8}(\tfrac{1}{16} K_4^2 + \tfrac{1}{10} K_3 K_5 - \tfrac{1}{2c} K_3 K_4 + \tfrac{1}{c^2} K_3^2) + \tfrac{35}{16} \sigma^{*-10}
\end{aligned}$$

$$\times \left(-\frac{1}{4}K_4K_3^2 + \frac{1}{3c}K_3^3\right) + \frac{385}{1152}\,\sigma^{*-12}K_3^4 \quad .$$

Thus,

$$1 - F_n\left(\frac{\sqrt{n}x}{\sigma}\right) \sim \tfrac{1}{2}(1-\mathrm{sign}(c)) + \frac{e^{-nxc+nK(c)}}{c\sqrt{2\pi nK_2}}\left(1 + \frac{d_2}{n} + \frac{d_4}{n^2}\right) \qquad (1.4.18)\ ,$$

where c is obtained from (1.4.6) and d_2 and d_4 from (1.4.17).

Let us now return to (1.4.4) and by an alternate argument arrive at another saddlepoint approximation to F_n. On letting $\rho = c\sqrt{nK_2}$ and $b_r' = b_r/i^r$ $(r = 1,2,\ldots)$, we have, by regrouping the factors in the integrand of (1.4.4),

$$1 - F_n\left(\frac{\sqrt{n}\;x}{\sigma}\right) = \tfrac{1}{2}(1-\mathrm{sign}(c)) + K(c,n,x)\int_{-\infty}^{\infty}\frac{n(u)}{\rho+iu}\,e^{\sum_{r=3}^{\infty} b_r'(iu)^r/\sqrt{n}^{\,r-2}}\,du \qquad (1.4.19)$$

But $e^{\sum_{r=3}^{\infty} b_r'(iu)^r/\sqrt{n}^{\,r-2}} = \sum_{r=0}^{\infty} g_r(iu)\left(\frac{1}{\sqrt{n}}\right)^r$, where $g_r(y)$ $(r = 0,1,2,\ldots)$ are defined in the obvious manner from equation (1.4.11).

Define $Q_k(\rho)$ $(k = 0,1,2,\ldots)$ by

$$Q_k(\rho) = \int_{-\infty}^{\infty}\frac{n(u)}{\rho+iu}\,(iu)^k\,du \qquad (1.4.20).$$

Then,

$$Q_0(\rho) = \frac{1}{n(\rho)}\left(\tfrac{1}{2}[1 + \mathrm{sign}(c)] - N(\rho)\right) \qquad (1.4.21),$$

$$Q_1(\rho) = 1 - \rho\,Q_0(\rho) \qquad (1.4.22),$$

and $\quad Q_k(\rho) \quad (k = 2,3,\ldots)$ satisfy the recurrence formulae

$$Q_2(\rho) = -\rho \; Q_{2k-1}(\rho) \quad (k = 1,2,\ldots) \qquad (1.4.23),$$

$$Q_{2k-1}(\rho) = (-1)^{k-1}(2k-3)(2k-5)\ldots(3)(1)-\rho Q_{2k-2}(\rho) \,. \qquad (1.4.24).$$

Thus, formally, we obtain, by interchanging summation and integration, a result of the form -

$$1 - F_n\left(\frac{\sqrt{n}x}{\sigma}\right) = \tfrac{1}{2}(1-\text{sign}(c)) + K(c,n,x) \sum_{k=0}^{\infty} h_k(\rho)\left(\frac{1}{\sqrt{n}}\right)^k \qquad (1.4.25),$$

where

$$\begin{aligned}
h_0(\rho) &= Q_0(\rho) \quad , \\
h_1(\rho) &= \tfrac{1}{6}\,\sigma^{*-3}K_3Q_3(\rho) \quad , \\
h_2(\rho) &= \tfrac{1}{24}\,\sigma^{*-4}K_4Q_4(\rho) + \tfrac{1}{72}\,\sigma^{*-6}K_3^2Q_6(\rho) \;, \qquad (1.4.26) \\
h_3(\rho) &= \tfrac{1}{120}\,\sigma^{*-5}K_5Q_5(\rho) + \tfrac{1}{144}\,\sigma^{*-7}K_3K_4Q_7(\rho) \\
&+ \tfrac{1}{1296}\,\sigma^{*-9}K_3^3Q_9(\rho) \quad , \\
h_4(\rho) &= \tfrac{1}{720}\,\sigma^{*-6}K_6Q_6(\rho) + \left(\tfrac{1}{1152}K_4^2 + \tfrac{1}{720}K_3K_5\right)\sigma^{*-8}Q_8(\rho) \\
&+ \tfrac{1}{1728}\,\sigma^{*-10}K_3^2K_4Q_{10}(\rho) + \tfrac{1}{31104}\,\sigma^{*-12}K_3^4Q_{12}(\rho) \quad ,
\end{aligned}$$

and where the $Q_k(\rho)$ $(k = 0,1,\ldots,13)$ are readily obtained using equations (1.4.21), (1.4.22), (1.4.23) and (1.4.24). The approximation obtained from (1.4.25) by deleting all terms involving h_k $(k \geq 5)$ will be referred to as the saddlepoint 2 approximation.

1.5 Remarks

This last approximation will be shown, by numerical means in Chapter 3, to be superior for small samples to both the Edgeworth (1.2.1) and the Cramér (1.3.5) approximation, as well as to the saddlepoint 1 approximation (1.4.18), which fails for $c = 0$.

A family of formal series expansions of $1 - F_n$ can be obtained in the manner of (1.4.25) by letting c take any value in $(-c_1, c_2)$. In particular, $c = 0$ yields the Edgeworth series, which Cramér [3] proved to be asymptotic. Hence, if the solution c of (1.4.6) is 0 , the Edgeworth and saddlepoint 2 series must be identical. However, as c moves away from 0 , the quality of the Edgeworth approximation deteriorates. The explanation for this lies in the fact that c is a saddlepoint, and, as is well-known, through any saddlepoint there is a path of steepest descent (see, for example, [6]). As Daniels [5] has argued, the path of integration $\{z : z = c+iu , -\infty < u < \infty\}$ will closely approximate the path of steepest descent locally in the region near the saddlepoint, from which the only asymptotic contribution to the integral comes.

CHAPTER II

THE SADDLEPOINT APPROXIMATIONS

In this chapter we prove that the expansions given in (1.4.18) and (1.4.25) are asymptotic. To do so we use Laplace's method for integrals and thereby achieve a more straight-forward proof than that provided by the method of steepest descent (see, for example, Watson [16]). Furthermore, this method is readily adaptable to other applications involving parameters different from n which tend to infinity. An example will be given in Chapter 4.

2.1 Asymptotic Expansions

Returning to (1.4.8), for $c \neq 0$,

$$I(x,n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-nx(c+iu) + n K(c+iu)} \frac{du}{c+iu} \qquad (2.1.1).$$

Let $g(u) = (1 + \frac{iu}{c})^{-1}$. Then $g(u)$ is a integrable over the interval $(-R,R)$ for all R and is equal to the convergent power series

$$g(u) = 1 - \frac{iu}{c} + (\frac{iu}{c})^2 - (\frac{iu}{c})^3 + \ldots \qquad (2.1.2)$$

in the interval $[-\delta_1, \delta_1]$, where $\delta_1 < c$.

As we shall show subsequently, the asymptotically dominant contribution to the integral in (2.1.1) comes from that portion of it which is over $[-n^{-1/3}, n^{-1/3}]$. In order that $g(u)$ be adequately represented over this region by the first few terms of its series expansion, $n^{1/3}$ must be large in comparison to c^{-1}. Since in the case of the saddlepoint 1 approximation we assume $g(u)$ is so represented, it tends to give unsatisfactory results when c is near 0 and n is moderate. The saddlepoint 2 approximation leaves g in its original form and tends to give good results even when c is small. However, it is somewhat more complicated than the saddlepoint 1 approximation.

Let

$$h(u) = \sum_{r=2}^{\infty} a_r u^r \quad , \quad |u| < \delta \quad ,$$
$$h_1(u) = -ixu + K(c+iu) - K(c) \quad , \tag{2.1.3}$$

where $a_r = K^{(r)}(c)\, i^r/r!$ $(r = 2,3,\ldots)$ and δ is some positive constant.

Daniels [5] showed that if the moment generating function M exists in a region $D = \{z : -c_1 < \mathrm{Re}(z) < c_2\}$, where D is the largest such region, one real root c of $K^{(1)}(z) - x = 0$ exists, and it satisfies $-c_1 < c < c_2$. As $M(z)$ is analytic in D,

$$M(z) = M(c) + M^{(1)}(c)(z-c) + \frac{M^{(2)}(c)}{2!}(z-c)^2 + \ldots \tag{2.1.4}$$

converges in a circle with non-zero radius of convergence R. Therefore, (2.1.4) converges uniformly on $\{z : |z-c| \leq \rho\}$ for some positive $\rho < R$. Hence, $\log M(z) = K(z)$ has a power series expansion about $z = c$ which converges uniformly on $\{z ; |z-c| \leq \delta_2\}$ for some $\delta_2 > 0$. Thus $h_1(u)$ has a power series expansion which is uniformly convergent for $|u| \leq \delta_2$; that is,

$$\begin{aligned} h_1(u) &= h(u) \\ &= a_2 u^2 + \sum_{r=3}^{\infty} a_r u^r \quad (|u| \leq \delta_2) \end{aligned} \tag{2.1.5}$$

LEMMA 2.1.1. *Let* $\delta = \min(\delta_1, \delta_2)$. *If*

$$\int_{-\infty}^{\infty} |\varphi(t)|^j \, dt < \infty \tag{2.1.6(a)}$$

for some real $j \geq 1$, *then*

$$\int_{-\infty}^{-\delta} g(u)\, e^{nh_1(u)} \, du = O(n^{-M}), \tag{2.1.6}$$

and

$$\int_{\delta}^{\infty} g(u)\, e^{nh_1(u)} \, du = O(n^{-M})$$

for each positive integer M.

PROOF. If the $\{X_i\}$, $(i = 1,2,\ldots)$, do not have a lattice distribution, $\left|\dfrac{M(c+iu)}{M(c)}\right| \leq \rho < 1$

since $|u| > \delta > 0$ (Daniels [5]). Condition (2.1.6(a)) implies, after a change of variable, that $\int_{-\infty}^{\infty} |M(c+it)|^j \, dt < \infty$ for some $j \geq 1$, say k. Writing $L_n = \frac{1}{n^{-M}} \int_{\delta}^{\infty} g(u) \, e^{nh_1(u)} \, du$, we obtain

$$|L_n| \leq \frac{1}{n^{-M}} \rho^{n-k} \frac{1}{|M(c)|^k} \int_{-\infty}^{\infty} |M(c+iu)|^k \, du = A \, n^M/\rho^{-n}$$

for some constant $A > 0$. Thus $\lim_{n\to\infty} |L_n| = 0$, since $\rho < 1$.

Similarly, $\int_{-\infty}^{-\delta} g(u) \, e^{nh_1(u)} \, du = O(n^{-M})$.

LEMMA 2.1.2. *There exists a positive constant p such that $\mathrm{Re}\{h(u)\} \leq -pu^2$ for all $|u| \leq \delta_3$ where δ_3 is some positive constant.*

PROOF. Explicitly, when $|u| < \delta$, h is given by

$$h(u) = \sum_{r=1}^{\infty} \frac{K^{(2r)}(c)}{(2r)!} (-1)^r u^{2r} + i \sum_{r=1}^{\infty} \frac{K^{(2r+1)}(c)}{(2r+1)!} (-1)^r u^{2r+1} .$$

Therefore,

$$\frac{\mathrm{Re}\{h(u)\}}{u^2} = -K_2 + K_4 u^2 - K_6 u^4 + \ldots ,$$

where $K_i = K^{(i)}(c)$ $(i = 2,3,\ldots)$. Now, $\mathrm{Re}\{h(u)\}/u^2$ is continuous, and equals $-K_2$ at $u = 0$. Using the terminology and result (ii) of Theorem 1.4.1, $K^{(1)}(t)$ increases continuously from $x = a$ to $x = b$. Hence, $K^{(2)}(c) > 0$.

Select ϵ subject to the requirement $K_2 > \epsilon > 0$. By the definition of continuity, there exists $\delta_3 > 0$ such that if $|u| \leq \delta_3$,

$$\operatorname{Re}\{h(u)\}/u^2 \leq -K_2 + \epsilon < 0 .$$

Thus, $\operatorname{Re}\{h(u)\} \leq -pu^2$ for all $|u| \leq \delta_3$ and for some $p > 0$.

From now on, let $\delta = \min(\delta_1, \delta_2, \delta_3)$.

First consider the saddlepoint 1 approximation. We expand $g(u) \exp\left(n \sum_{r=3}^{\infty} a_r u^r\right) = g(u) \exp\left(nu^3 \sum_{r=3}^{\infty} a_r u^{r-3}\right)$ as a double power series in the two arguments nu^3 and u, convergent for all $|u| \leq \delta$. Denote this power series by

$$P(nu^3, u) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} (nu^3)^i u^j \tag{2.1.7}$$

where the c_{ij} $(i = 0,1,\ldots ; j = 0,1,\ldots)$ are independent of n and u.

In order to approximate P uniformly by its partial sums, we restrict nu^3 to some finite interval, say, $|nu^3| \leq 1$, that is, $|u| \leq n^{-1/3} = \delta_n$.

We can assume $n > \delta^{-3}$, so that $\delta_n < \delta$. Then our region of integration for $I(x,n)$ consists of five intervals: $(-\infty,-\delta)$, $(-\delta,-\delta_n)$, $(-\delta_n,\delta_n)$, (δ_n,δ) and (δ,∞). If δ could be allowed to depend on n, we could take $\delta = \delta_n$ and thereby obtain only three sub-intervals of integration and a simplification of the proof. However, since

$\delta_n \to 0$, $|M(c+iu)/M(c)|$ would not then be uniformly bounded by $\rho < 1$ for $\delta \leq u < \infty$ and all n , as it must if the result of Lemma 2.1.1 is to hold.

<u>LEMMA 2.1.3.</u> <u>Using the same notation as above,</u>

$$\int_{-\delta}^{-\delta_n} g(u)\, e^{nh(u)}\, du + \int_{\delta_n}^{\delta} g(u)\, e^{nh(u)}\, du = O(n^{-M}) \qquad (2.1.8)$$

<u>for each positive integer</u> M .

<u>PROOF.</u> For $u \in (\delta_n, \infty)$,

$$pn(u^2 - \delta_n^2) > pn\,\delta_n\,(u - \delta_n) > p(u - \delta_n) \qquad (n > 1) \,.$$

Then,

$$e^{pn\delta_n^2} \int_{\delta_n}^{\delta} e^{-pnu^2}\, du < e^{pn\delta_n^2} \int_{\delta_n}^{\infty} e^{-pnu^2}\, du$$

$$< \int_{\delta_n}^{\infty} e^{-p(u-\delta_n)}\, du = p^{-1} \,.$$

Hence, $\int_{\delta_n}^{\delta} e^{-pnu^2}\, du = O\left(e^{-pn^{1/3}}\right)$. From this and Lemma 2.1.2,

$$\left|\int_{\delta_n}^{\delta} g(u)\, e^{nh(u)}\, du + \int_{-\delta}^{-\delta_n} g(u)\, e^{nh(u)}\, du\right|$$

$$\leq \int_{\delta_n}^{\delta} |g(u)|\,|e^{nReh(u)}|\,du + \int_{-\delta}^{-\delta_n} |g(u)|\,|e^{nReh(u)}|\,du$$

$$\leq \int_{\delta_n}^{\delta} e^{-pnu^2}\, du + \int_{-\delta}^{-\delta_n} e^{-pnu^2}\, du$$

$$= O(e^{-pn^{1/3}}) \qquad (n > \delta^{-3}) \qquad (2.1.9)$$
$$= O(n^{-M})$$

for any integer $M \geq 0$ and the conclusion follows.

COROLLARY 2.1.3. For each integer $M \geq 0$,

$$\int_{\delta_n}^{\infty} e^{-pnu^2} u^M \, du = O(e^{-\frac{1}{2}pn^{1/3}}) \qquad (2.1.10)$$

PROOF. If $u \geq \delta_n$, we have

$$u^M e^{-\frac{1}{2}pnu^2} \leq u^M e^{-\frac{1}{2}pu^2} \leq K \qquad (n \geq 1)$$

for some constant K independent of n and so $u^M = O(e^{\frac{1}{2}nu}) = O(e^{\frac{1}{2}pnu^2})$. Equation (2.1.10) follows by replacing p with $\frac{1}{2}p$ in

$$\int_{\delta_n}^{\infty} e^{-pnu^2} \, du = O(e^{-pn^{1/3}}) \; .$$

It is now clear why c is chosen to be a saddle-point, that is, a root of equation (1.4.6). If this were not the case, $h(u)$ would include a linear term in u , and P would become $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c'_{ij} (nu)^k u^j$. Restricting nu to a finite interval in order to approximate P uniformly by its partial sum leads to $\delta_n = n^{-1}$ rather than $n^{-1/3}$. The proof of Lemma 2.1.3 will fail since in this case $n\,\delta_n^2 \to 0$ rather than ∞ as $n \to \infty$.

In the remaining interval, $(-\delta_n, \delta_n)$, P is approximated by its partial sums. For any positive integer A we write

$$P_A(nu^3,u) = \sum_{\substack{i\geq 0,\, j\geq 0\\ i+j\leq A}} c_{ij}\,(nu^3)^i\,u^j .$$

<u>LEMMA</u> <u>2.1.4</u>. <u>If</u> $|u| < \delta_n$,

$$P(nu^3,u) - P_A(nu^3,u) = O[(nu^3)^{A+1}] + O[u^{A+1}] \qquad (2.1.11).$$

<u>uniformly with respect to</u> u <u>and</u> n (<u>but not necessarily</u> A).

<u>PROOF</u>. Suppose an arbitrary power series, $\sum_{m\geq 0,n\geq 0} d_{mn} z^m y^n$, converges for $|z| < R$, $|y| < S$. Since the terms of a convergent power series are bounded, $d_{mn} = O(R^{-m}S^{-n})$. Then, if $|z| < R$ and $|y| < S$,

$$\sum_{\substack{m\geq 0,n\geq 0\\ m+n>A}} d_{mn}\, z^m y^n = O\Big(\sum_{\substack{m\geq 0,n\geq 0\\ m+n>A}} |\tfrac{z}{R}|^m |\tfrac{y}{S}|^n\Big)$$

$$= O\Big(\sum_{k=A+1}^{\infty} (|\tfrac{z}{R}| + |\tfrac{y}{S}|)^k\Big)$$

$$= O\big((|\tfrac{z}{R}| + |\tfrac{y}{S}|)^{A+1}\big) .$$

Since, in general, $|a+b|^r \leq 2^{r-1}(|a|^r + |b|^r)$ for all $r > 1$, it follows that $\sum_{\substack{m\geq 0,n\geq 0\\ m+n>A}} d_{mn}\, z^m y^n = O(|z|^{A+1}) + O(|y|^{A+1})$.

Equation (2.1.11) is an immediate consequence.

We now come to our main theorems in which we shall prove, with the aid of the preceding lemmas, that the expansions given in (1.4.18) and (1.4.25) are asymptotic.

THEOREM 2.1.1. *Let* $d_i = (-a_2)^{-i-\frac{1}{2}} \sum_{m=0}^{2i} c_{m,2i-m}(-a_2)^m \times \Gamma(m+i+\frac{1}{2})$, $(i=0,1,2,\ldots)$, *where* $a_2 = -K^{(2)}(c)/2$, *the* $\{c_{mn}\}$ *are defined in equation* (2.1.7), *and* Γ *denotes the gamma function, that is,*

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{n!\,2^{2n}}\,\pi^{\frac{1}{2}} \qquad (2.1.12).$$

Then, if $\int|\varphi|^j < \infty$ *for some* $j \geq 1$,

$$\int_{-\infty}^{\infty} g(u)\, e^{nh_1(u)} \sim \sum_{i=0}^{\infty} d_i\, n^{-\frac{1}{2}-i} \quad (n\to\infty) \qquad (2.1.13)$$

is an asymptotic expansion.

PROOF. From (2.1.10) we obtain, recalling that $a_2 < 0$,

$$\left\{\int_{-\infty}^{-\delta_n} \pm \int_{\delta_n}^{\infty}\right\} P_A(nu^3,u)e^{na_2u^2}\,du = O\left(n^A e^{\frac{1}{2}a_2 n^{1/3}}\right) \quad (n\to\infty) \qquad (2.1.14)$$

for any fixed A .

Hence, combining the above and the results of Lemmas 2.1.1, 2.1.2, 2.1.3, and 2.1.4,

$$|\int_{-\infty}^{\infty} g(u)e^{nh_1(u)}\,du - \int_{-\infty}^{\infty} P_A(nu^3,u)e^{na_2u^2}\,du|$$

$$\leq |\{\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\} g(u)e^{nh_1(u)}\,du| + |\{\int_{-\delta}^{-\delta_n} + \int_{\delta_n}^{\delta}\} g(u)e^{nh_1(u)}\,du|$$

$$+ |\{\int_{-\infty}^{-\delta_n} + \int_{\delta_n}^{\infty}\} P_A(nu^3,u)e^{na_2u^2}\,du|$$

$$+ |\int_{-\delta_n}^{\delta_n}[g(u)e^{nh(u)} - P_A(nu^3,u)e^{na_2u^2}]du|$$

$$= O(n^{-M}) + O\left(e^{\frac{1}{2}a_2n^{1/3}} n^A\right)$$

$$+ O\left(\int_{-\infty}^{\infty} e^{na_2u^2}\left(|nu^3|^{A+1} + |u|^{A+1}\right)du\right) \qquad (n\to\infty)$$

$$(2.1.15).$$

Now let us consider integrals of the type $\int_{-\infty}^{\infty} e^{-tx^2} x^k\,dx$, where $\operatorname{Re}(t) > 0$. For even k, substituting $tx^2 = y$, we obtain

$$\int_{\infty}^{\infty} e^{-tx^2} x^{2m}\,dx = t^{-m-\frac{1}{2}}\,\Gamma(m+\tfrac{1}{2})$$

$$= t^{-m-\frac{1}{2}} \frac{(2m)!}{m!\,2^{2m}}\,\pi^{\frac{1}{2}} \qquad (m=0,1,\ldots)$$

$$(2.1.16).$$

The following estimate is valid for both odd and even k:

$$\int_{-\infty}^{\infty} |e^{-tx^2} x^k|dx = O([\operatorname{Re}(t)]^{-\frac{1}{2}(k+1)}) \qquad (2.1.17)$$

using (2.1.17), we see that the last term in (2.1.15) is

$O(n^{-\frac{1}{2}A-1})$. Therefore, combining (2.1.15), (2.1.16) and (2.1.17),

$$\int_{-\infty}^{\infty} g(u)e^{nh_1(u)}\,du = \sum_{\substack{m\geq 0,\,k\geq 0 \\ m+k\leq A}} c_{mk}\,\varepsilon_{m+k}\,n^{-\frac{1}{2}(m+k+1)}$$

$$\times\ (-a_2)^{-\frac{1}{2}(3m+k+1)}\,\Gamma(\tfrac{1}{2}[3m+k+1]) + O(n^{-\frac{1}{2}A-1}) + O(n^{-M}) \quad (n\to\infty)\ ,$$

where ε_i is 0 or 1 depending on whether i is odd or even, respectively. As A and M are completely arbitrary, we obtain an asymptotic series

$$\int_{-\infty}^{\infty} g(u)e^{nh_1(u)}\,du \sim \sum_{i=0}^{\infty} d_i n^{-\frac{1}{2}-i} \qquad (n\to\infty)$$

where the d_i are the coefficients computed in Chapter 1 (1.4.17) and given in the statement of the theorem.

<u>THEOREM 2.1.2</u>. Let $e^{nu^3\sum_{r=3}^{\infty} a_r u^{r-3}} = \sum_{j=0}^{\infty} \frac{n^j}{j!} \sum_{k=3j}^{K(j)}$ $\times\ c_k(\{a_r\})u^k$, <u>where</u> $a_r = \frac{K^{(r)}(c)i^r}{r!}$ $(r=3,4,\ldots)$, $K(0) = 0$, $K(j) = \infty\ (j > 0)$, <u>and</u> $c_k(\{a_r\})\ (k=0,3,4,\ldots)$ <u>are appropriately defined constants depending only on</u> $\{a_r\}\ (r=3,4,\ldots)$. <u>Then, with</u> $Q_k(z)\ (k=0,1,\ldots)$ <u>defined as before</u> (see equation (1.4.20)),

$$\frac{1}{c\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)e^{nh_1(u)}\,du \sim \sum_{j=0}^{\infty} \sum_{k=3j}^{K(j)} c_k\Big(\Big\{\frac{a_r i^{-r}}{(-2a_2)^{r/2}}\Big\}\Big)$$

$$\times\ n^{j-k/2}\,Q_k(c\sqrt{-2na_2})/j! \qquad (n\to\infty) \qquad (2.1.17)$$

is an asymptotic expansion.

PROOF. The proof is almost identical to the proof of Theorem 2.1.1 and will only be outlined briefly here.

In this case, the factor $e^{nu^3 \sum_{r=3}^{\infty} a_r u^{r-3}}$ in the integrand of (1.4.19) is expanded as a power series in the argument (nu^3). This series is denoted by $P'(nu^3)$. As in the proof of Theorem 2.1.1, nu^3 is restricted to the interval $|nu^3| \leq 1$, that is, $|u| \leq n^{-1/3} = \delta_n$, in order to approximate P' uniformly by its partial sums P'_A. Lemma 2.1.3 can be applied without alteration, while Corollary 2.1.3 is obviously valid with $\frac{e^{-pnu^2}}{c+iu}$ replacing e^{-pnu^2}, since the latter is less in modulus than the former everywhere.

In the same manner as we obtained (2.1.11), we find that, for $|u| < \delta_n$,

$$(P' - P'_A)(nu^3) = O([nu^3]^{A+1})$$

uniformly in u and n. Then

$$\frac{1}{c}\int_{-\infty}^{\infty} g(u)e^{nh_1(u)}\,du = \int_{-\infty}^{\infty} P'_A(nu^3,u)e^{na_2u^2}\frac{du}{c+iu}$$
$$+ O(n^{-M}) + O\left\{\int_{-\infty}^{\infty} e^{na_2u^2}|nu^3|^{A+1}\frac{du}{c+iu}\right\} \qquad (n\to\infty) \tag{2.1.18}$$

Writing $P'_A(nu^3) = \sum_{j=0}^{A} \frac{n^j}{j!}\left(\sum_{r=3}^{\infty} a_r u^r\right)^j$

$$= \sum_{j=0}^{A} \frac{n^j}{j!} \sum_{k=3j}^{K(j)} c_k(\{a_r\})\, u^k$$, equation (2.1.18) becomes

$$\frac{1}{c}\int_{-\infty}^{\infty} g(u)e^{nh_1(u)}du = \sum_{j=0}^{A} \sum_{k=3j}^{K(j)} c_k\left(\left\{\frac{a_r i^{-r}}{(-2a_2)^{r/2}}\right\}\right)$$

$$n^{j-k/2}\, Q_k(c\sqrt{-2na_2})\,\sqrt{2\pi}/j! + O(n^{-M}) + O(n^{-\frac{1}{2}A-1})$$

$$(n\to\infty) \qquad (2.1.19).$$

Again, as M and A are entirely arbitrary, the above, and hence (1.4.25), is an asymptotic series.

This completes the proof of the asymptotic nature of the saddlepoint 1 and 2 approximations. We now discuss briefly a case not included in Theorems 2.1.1 and 2.1.2,

2.2 The Lattice Case.

When the $\{X_i\}$, $(i=1,2,\ldots)$, have a lattice distribution, the preceding argument fails; the tails of the integral (2.1.6) cannot be ignored, since for a distribution having its mass concentrated at points h units apart, the characteristic function is periodic of period $\lambda = 2\pi/h$, with $|\varphi(\lambda)| = 1$ and $|\varphi(s)| < 1$ for $0 < s < \lambda$.

Daniels [5], in his work involving the density function, avoids this complication because he is dealing with densities instead of distributions. In that case, the path of integration stops at $c \pm i\pi$, and no tail regions are present. Using this fact, an approximation to the distribution function could be obtained by numerically integrating the density function.

Feller [7] introduces the concept of a polygonal approximant $F_n^{\#}$ to F_n , where, more generally, $G^{\#}$ is the convolution of G with the uniform distribution on $(-h/2, h/2)$ and h is the span of G . He then shows that the first two terms of (1.2.2) approximate $F_n^{\#}$ with an error of magnitude $o(n^{-\frac{1}{2}})$. This means that at the lattice points of F_n , the error is $o(n^{-\frac{1}{2}})$ when $F_n(x)$ is replaced by $\frac{1}{2}[F_n(x) + F_n(x-)]$. However, for higher order expansions of the type (1.2.2) the additonal assumption that

$$\limsup_{|s|\to\infty} |\varphi(s)| < 1 \tag{2.2.1}$$

is necessary, a condition not met by lattice distributions and a considerable number of other distributions which have their variation concentrated in a set of Lebesgue measure zero. The order of magnitude of the error in approximating F_n by a series of the Edgeworth type depends on the arithmetical nature of the set of possible values of the random variable X_i . Even if all the moments of F are finite, in the case of discrete distributions it is necessary to supplement the expansion (1.2.1) with discontinuous terms.

However, although it is impossible to approximate such discrete distribution functions with continuous functions to an accuracy of within one-half of their maximum jump, local limit theorems for approximating F_n at its points of discontinuity exist (see Gnedenko-Kolmogorov [8]). We reproduce the following proposition of Esseen (see [8], p. 241) which is analogous to the Edgeworth expansion (1.2.1) in the absolutely continuous case.

Suppose the random variables X_i $(i=1,2,\ldots)$ can only take on the values $x_s = a + sh$ $(s = 0, \pm 1, \pm 2, \ldots)$, where h is the maximum span of the distribution F. The random variable

$$Y_n = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{n} (X_k - \mu)$$

can only take on the values

$$y_{ns} = h(s - np)/(\sigma\sqrt{n}) \quad ,$$

where $p = \sum_{i=-\infty}^{\infty} ip_i$ and $p_i = P(X_k = a+ih)$. Let $P_n(s) = P(Y_n = y_{ns})$.

<u>THEOREM 2.2.1.</u> (Esseen). <u>If the identically distributed random variables</u> $X_1,\ldots,X_n$ <u>are independent and have finite absolute moments of the order</u> $k(k \geq 3)$, <u>then</u>

$$P_n(s) = \frac{h}{\sigma\sqrt{n}} [n(y_{ns}) + \sum_{i=1}^{k-2} n^{-i/2} R_i(n(y_{ns}))$$

$$+ O(n^{-(k-1)/2}) \qquad (n\to\infty) \quad .$$

Here,

$$R_1(n(y_{ns})) = -\frac{\lambda_3}{3!} n^{(3)}(y) \quad ,$$

$$R_2(n(y)) = \frac{\lambda_4}{4!} n^{(4)}(y) + \frac{10}{6!} \lambda_3^2 n^{(6)}(y) \quad ,$$

and the $R_i(n(y))$ $(i > 2)$ are obtained in a similar manner from the expansion (1.2.2) by replacing $N^{(i)}(y)$ by $n^{(i)}(y)$.

PROOF. See Gnedenko-Kolmogorov [8], p. 241.

To obtain the values of the function F_n at the points of discontinuity y_{ns} now only requires a summation procedure,

$$F_n(y_{ns}) = F_n(y_{n,s-1} + 0)$$

$$= \sum_{r<s} P_n(r) \quad .$$

CHAPTER 3

COMPUTATIONS

To judge the quality of the saddlepoint approximations in the case of small n, several test cases were considered. Numerical results were obtained in each case for the sake of comparison with the Edgeworth and Cramér approximations. These results were obtained for values of the argument, x, selected to represent the entire admissable range of values and for values of n between 1 and 40 inclusive. For brevity, only a few representative results for each distribution considered are depicted.

It will be noted that whereas the saddlepoint 2 expansion gives uniformly better results than the other three approximations, the Edgeworth series, (1.2.2), is quite good when c is close to 0, as we would expect on the basis of the discussion in section 1.5. However, when x assumes values in the extremities of its range, the saddlepoint method gives substantially better results.

3.1 Remarks on the Tables.

When $c = 0$, the saddlepoint 1 expansion does not exist, and for programming purposes, the Edgeworth approximation is printed in its place.

When $F_n(x)$ is nearly 1, exponents in the calculation of the Cramér formula are excessively large for the computer, and the value $F_n(x) = 1$ is assumed.

Multiple entries under the headings represent successive approximations obtained by adding, at each stage, one more term to the approximations. They are included to facilitate a comparison of the rates of apparent convergence of the various series.

The results are printed in exponential format, and a series of digits, say $0.n_1 \ldots n_k$, followed by "D $\pm$ m" represents the number $0.n_1 \ldots n_k \times 10^{\pm m}$. The letter E occasionally replaces the letter D in this format.

In order to observe the effect of the location of the saddlepoint on the various approximations, the value of c and the accuracy to which it is calculated is given. Hence, "saddlepoint = c_1 + or - δ" means $c \in (c_1-\delta, c_1+\delta)$.

When it is available from existing tables, the correct value of $F_n(x)$ is given for comparison. From these cases it appears that the last entry for the saddlepoint 2 approximation in each case is accurate at least in the digits where it and the next to last entry agree. In the remaining cases, judgement on the quality of the various approximations must be withheld until exact values become available. If the last saddlepoint 2 entry is accurate to the extent just described, as seems likely to be the case, an examination of the tables indicates that this method of approximation gives results of the same comparatively good quality as it did in the earlier cases for which exact values of $F_n(x)$ are known.

3.2 Chi Random Variables.

Let $X_i = |Y_i|$, where Y_i $(i = 1,2,\ldots)$ are independent, standard normal variables. The density function of X_i is given by

$$f(x) = \begin{cases} \sqrt{\frac{2}{\pi}} \; e^{-x^2/2} , & x \geq 0 \\ 0 , & x < 0 , \end{cases}$$

and the moment generating function is

$$M(t) = 2e^{t^2/2} \; N(t) ,$$

where $$N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-y^2/2} \, dy .$$

The cumulants are given by Rubin and Zidek [13] as

$$\mu_1 = \alpha_1 \simeq 0.79788\ 45608\ 03$$

$$\sigma^2 = \alpha_2 = 1-\alpha_1^2 \simeq 0.36338\ 02276\ 32$$

$$\alpha_3 = \alpha_1(\alpha_1^2-\alpha_2) \simeq 0.21801\ 36141\ 45$$

$$\alpha_4 = 2\alpha_1^2(2-3\alpha_1^2) \simeq 0.11477\ 06820\ 54$$

$$\alpha_5 = \alpha_1(3-20\alpha_1^2+24\alpha_1^4) \simeq -\ 0.00443\ 76884\ 6262 .$$

The saddlepoint c is the root of the equation

$$n(z)/N(z) + z = x/n$$

which can be solved numerically using Newton's Method.

In order to compute the correct value of $F_n(x)$, equation (1.4.4) was inverted numerically in [13]. This entailed the evaluation of $M(z)$, and hence, of $N(z)$, for complex values of its argument. Since use of its Taylor expansion results in uncontrollable round-off error, an accurate method using continued fractions was employed. The computation of $Q_o(\rho)$ (see equation (1.4.21)) also requires $N(\rho)$, and for this reason, a detailed account of the method devised in [13] is given in Appendix A.

The derivatives of the cumulant generating function evaluated at c are

$$K_1 = x/n = \overline{x}$$

$$K_2 = c\overline{x} + 1 - \overline{x}^2$$

$$K_3 = c^2\overline{x} + c(1-3\overline{x}^2) - \overline{x}(1-2\overline{x}^2)$$

$$K_4 = c^3\overline{x} + c^2(1-7\overline{x}^2) - c\overline{x}(5-12\overline{x}^2) + \overline{x}^2(4-6\overline{x}^2)$$

$$K_5 = c^4\overline{x}+c^3(1-15\overline{x}^2) - c^2\overline{x}(16-50\overline{x}^2)$$
$$- c(3-35\overline{x}^2+60\overline{x}^4) + \overline{x}(3-20\overline{x}^2 + 24\overline{x}^4)$$

$$K_6 = c^5\overline{x} + c^4(1-31\overline{x}^2) - c^3\overline{x}(42-180\overline{x}^2)$$
$$- c^2(13-191\overline{x}^2+390\overline{x}^4) + c\overline{x}(41-270\overline{x}^2+360\overline{x}^4)$$
$$- \overline{x}^2(28-120\overline{x}^2+120\overline{x}^4) .$$

From now on, $F_n(x)$ will denote $P(\sum_{i=1}^{n} X_i \leq x)$. Then, for the saddlepoint 2 method, the first term approximation to

CHI RANDOM VARIABLES
(=ABS(Y),WHERE Y=NORMAL,MEAN=0,VAR.=1)

N = 10

X = 3.60000
SADDLEPOINT = -0.2119238 E 01 +OR- 0.366E-13
F(X) = 0.410254 E-02

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.1080667090-01	0.4046812430-02	0.4452880830-02	0.3838464480-02
0.4409401340-02		0.3978365120-02	0.4106579370-02
0.4055176800-02		0.4174711670-02	0.4099616090-02
0.4043777420-02			0.4102644590-02
			0.4102950790-02

X = 7.00000
SADDLEPOINT = -0.2939604 E 00 +OR- 0.555E-16
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.3038036570 00	0.3014527160 00	0.6766890570 00	0.3015973490 00
0.3173092810 00		-0.1674674900 01	0.3187874130 00
0.3186011580 00		0.2359028000 02	0.3187134320 00
0.3188148570 00			0.3188532480 00
			0.3188533250 00

X = 10.40000
SADDLEPOINT = 0.5623426 E 00 +OR- 0.619E-14
F(X) = 0.89276430 E 00

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.8979775370 00	0.8883722670 00	0.8457067380 00	0.8880301230 00
0.8922496760 00		0.9589380730 00	0.8925033210 00
0.8929280630 00		0.7223566340 00	0.8927491360 00
0.8927336620 00			0.8927584760 00
			0.8927634590 00

X = 13.90000
SADDLEPOINT = 0.1156851 E 01 +OR- 0.222E-15
F(X) = 0.99729399 E 00

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.9990523730 00	0.9967903210 00	0.9969646250 00	0.9972253000 00
0.9975984990 00		0.9974049340 00	0.9972818850 00
0.9971998510 00		0.9972303460 00	0.9972943640 00
0.9973176900 00			0.9972937810 00
			0.9972939980 00

CHI RANDOM VARIABLES
(=ABS(Y),WHERE Y=NORMAL,MEAN=0,VAR.=1)

N = 40

X = 18.75999
SADDLEPOINT = -0.1307950 E 01 +OR- 0.111E-14
F(X) = 0.6274 E-04

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.2796856250-03	0.632836852D-04	0.659743376D-04	0.612809437D-04
-0.167188214D-04		0.621543542D-04	0.628270854D-04
0.584756959D-04		0.629239305D-04	0.627431954D-04
0.637257643D-04			0.627446602D-04
			0.627443278D-04

X = 24.75999
SADDLEPOINT = -0.5843487 E 00 +OR- 0.694E-15
F(X) = 0.2541083 E-01

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.302715069D-01	0.246549128D-01	0.298779195D-01	0.247259289D-01
0.257362394D-01		0.227366469D-01	0.254272839D-01
0.254511868D-01		0.281840164D-01	0.254101692D-01
0.254154082D-01			0.254113069D-01
			0.254112390D-01

X = 39.00000
SADDLEPOINT = 0.4283156 E 00 +OR- 0.416E-16
F(X) = 0.96412521 E 00

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.968433734D 00	0.963448929D 00	0.955637820D 00	0.963371503D 00
0.963367546D 00		0.970251090D 00	0.964093381D 00
0.964158501D 00		0.956385540D 00	0.964124178D 00
0.964121600D 00			0.964124291D 00
			0.964124454D 00

X = 45.00000
SADDLEPOINT = 0.7239734 E 00 +OR- 0.000E 00
F(X) = 0.99931710 E 00

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.999700453D 00	0.999297909D 00	0.999254728D 00	0.999307342D 00
0.999388199D 00		0.999333277D 00	0.999316180D 00
0.999311194D 00		0.999310014D 00	0.999317116D 00
0.999316909D 00			0.999317100D 00
			0.999317104D 00

$F_n(x)$ is

$$F_n(x) \simeq \tfrac{1}{2}\,(1 + \text{sign}(c)) - e^{-xc+nK(c)}\; Q_o(\rho)/\sqrt{2\pi} \quad ,$$

where $Q_o(\rho)$ is given by (1.4.21).

The results we obtained in this case are listed in Tables I(a), (b). The exact values of $F_n(x)$ given are those computed by Rubin and Zidek [13].

3.3 The Exponential Probability Law.

The probability density function of the exponential distribution is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \; x > 0 \\ 0 & , \; \text{otherwise} \end{cases}$$

where $\lambda > 0$. The moment generating function is

$$M(t) = \lambda/(\lambda - t) \qquad (\mathrm{Re}\{t\} < \lambda) \quad ,$$

from which we obtain the cumulants

$$\alpha_i = K^{(i)}(0) = i!/\lambda^i \qquad (i = 1,2,\ldots) \quad .$$

Solving the saddlepoint equation yields

$$c = \lambda - (x/n)^{-1} \quad ,$$

and hence, $K^{(r)}(c) = r!\ \lambda\ (x/n)^r \qquad (r = 1,2,\ldots)$.

Difficulties are encountered in applying the Cramér approximation. For certain choices of the parameter λ and the argument x , series (1.3.4) does not converge. For example, when $\lambda = 1$, (1.3.4) becomes

$$\lambda(z) = \frac{1}{3} - \frac{1}{4}z + \frac{1}{5}z^2 - \frac{1}{6}z^3 + \ldots ,$$

which does not converge for $|z| > 1$. Thus, if $x > 20$ and $n = 10$, $w/\sqrt{n} = (x-n\mu)/(n\sigma) > 1$, and $\lambda(w/\sqrt{n})$ cannot be evaluated using (1.3.4). We overcame this difficulty by inverting directly the equation (1.3.3).

The form of the expression for c indicates that it can be considerably different from 0 for moderate values of x . Hence, the Edgeworth series often yielded inaccurate results. For example, when $x = 4$ and $n = 15$ it is incorrect in the first significant figure.

Results for this case are listed in Tables II(a), (b).

EXPONENTIAL RANDOM VARIABLE
MEAN=1
VARIANCE=1

N = 15

X = 31.00000
SADDLEPOINT = 0.5161290 E 00 +OR- 0.000E 00
F(X) = 0.99948

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.9999819550 00	0.9972110870 00	0.9994176550 00	0.9994470520 00
0.9998733900 00		0.9994914080 00	0.9994757200 00
0.9996344520 00		0.9994697970 00	0.9994762350 00
0.9994320490 00			0.9994763730 00
0.9994370310 00			0.9994763290 00

X = 11.00000
SADDLEPOINT = -0.3636363 E 00 +OR- 0.000E 00
F(X) = 0.14596

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.1508498580 00	0.1339353650 00	0.2011847340 00	0.1339272020 00
0.1495070060 00		0.6175253640-01	0.1454332790 00
0.1466163920 00		0.3823763470 00	0.1458955560 00
0.1460833740 00			0.1459526560 00
0.1459807790 00			0.1459570810 00

X = 5.75000
SADDLEPOINT = -0.1608695 E 01 +OR- 0.000E 00
F(X) = 0.93 E-03

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.8462344170-02	0.9447172850-03	0.9855682510-03	0.8649830210-03
-0.8612170820-03		0.9138603820-03	0.9261830780-03
-0.3184287690-03		0.9347567530-03	0.9279023940-03
0.5964954540-03			0.9283587020-03
0.9014119670-03			0.9284334050-03

X = 4.00000
SADDLEPOINT = -0.2749999 E 01 +OR- 0.000E 00
F(X) = 0.2 E-04

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.2254348950-02	0.2980789600-04	0.2062211930-04	0.1868297240-04
-0.2044016900-02		0.1982582900-04	0.1990022450-04
-0.3932762050-03		0.1995951830-04	0.1992148070-04
0.2349010740-03			0.1993001850-04
0.2127062140-03			0.1993159130-04

EXPONENTIAL RANDOM VARIABLE
MEAN=1
VARIANCE=1

N = 40

X = 15.50000
SADDLEPOINT = -0.1580644 E 01 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.5357784620-04	0.179693256D-06	0.152706336D-06	0.144090112D-06
-0.108795561D-03		0.148444927D-06	0.148869473D-06
0.618016992D-04		0.148919900D-06	0.148838317D-06
0.495412666D-05			0.148847695D-06
-0.697715176D-05			0.148849150D-06

X = 30.00000
SADDLEPOINT = -0.3333333 E 00 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.569233681D-01	0.440145909D-01	0.558901681D-01	0.440102401D-01
0.478872484D-01		0.390066799D-01	0.461927691D-01
0.463812285D-01		0.558087829D-01	0.462471508D-01
0.462576094D-01			0.462527943D-01
0.462517156D-01			0.462532125D-01

X = 45.00000
SADDLEPOINT = 0.1111111 E 00 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.785402325D 00	0.780228869D 00	0.621907080D 00	0.780228927D 00
0.791170960D 00		0.130326203D 01	0.791859834D 00
0.791667702D 00		-0.223710637D 01	0.791602609D 00
0.791612676D 00			0.791618761D 00
0.791618829D 00			0.791618251D 00

X = 55.00000
SADDLEPOINT = 0.2727272 E 00 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.991146968D 00	0.984622937D 00	0.982479782D 00	0.984645487D 00
0.985306908D 00		0.986799003D 00	0.985312753D 00
0.985133038D 00		0.983934896D 00	0.985301129D 00
0.985341275D 00			0.985302996D 00
0.985297550D 00			0.985302810D 00

Table III(a)

NORMAL RANDOM VARIABLES
MEAN=.5
VARIANCE=1

N = 9

X = -6.00000
SADDLEPOINT = -0.1166666 E 01 +OR- 0.000E 00
F(X) = 0.2326291 E-03

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.2326290400D-03	0.2326290400D-03	0.2493379130D-03	0.2326290790D-03
0.2326290400D-03		0.2289837980D-03	0.2326290790D-03
0.2326290400D-03		0.2339684790D-03	0.2326290790D-03
0.2326290400D-03			0.2326290790D-03
0.2326290400D-03			0.2326290790D-03

X = -1.50000
SADDLEPOINT = -0.6666666 E 00 +OR- 0.000E 00
F(X) = 0.227501319 E-01

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.2275015550D-01	0.2275015550D-01	0.2699548330D-01	0.2275013190D-01
0.2275015550D-01		0.2024661240D-01	0.2275013190D-01
0.2275015550D-01		0.2530826560D-01	0.2275013190D-01
0.2275015550D-01			0.2275013190D-01
0.2275015550D-01			0.2275013190D-01

X = 6.00000
SADDLEPOINT = 0.1666666 E 00 +OR- 0.000E 00
F(X) = 0.691462461

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.6914624600D 00	0.6914624600D 00	0.2958693460D 00	0.6914624610D 00
0.6914624600D 00		0.3112391960D 01	0.6914624610D 00
0.6914624600D 00		-0.3068587940D 02	0.6914624610D 00
0.6914624600D 00			0.6914624610D 00
0.6914624600D 00			0.6914624610D 00

X = 9.00000
SADDLEPOINT = 0.4999999 E 00 +OR- 0.000E 00
F(X) = 0.933192056

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.9331926640D 00	0.9331926640D 00	0.9136549360D 00	0.9331920560D 00
0.9331926640D 00		0.9520305200D 00	0.9331920560D 00
0.9331926640D 00		0.9008630750D 00	0.9331920560D 00
0.9331926640D 00			0.9331920560D 00
0.9331926640D 00			0.9331920560D 00

NORMAL RANDOM VARIABLES
MEAN=.5
VARIANCE=1

N = 36

X = 0.00000
SADDLEPOINT = -0.5000000 E 00 +OR- 0.000E 00
F(X) = 0.134989803 E-02

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.134989775D-02	0.134989775D-02	0.147728280D-02	0.134989803D-02
0.134989775D-02		0.131314027D-02	0.134989803D-02
0.134989775D-02		0.136785445D-02	0.134989803D-02
0.134989775D-02			0.134989803D-02
0.134989775D-02			0.134989803D-02

X = 9.00000
SADDLEPOINT = -0.2500000 E 00 +OR- 0.000E 00
F(X) = 0.668072013 E-01

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.668073365D-01	0.668073365D-01	0.863450638D-01	0.668072013D-01
0.668073365D-01		0.479694799D-01	0.668072013D-01
0.668073365D-01		0.991369251D-01	0.668072013D-01
0.668073365D-01			0.668072013D-01
0.668073365D-01			0.668072013D-01

X = 15.00000
SADDLEPOINT = -0.8333331 E-01 +OR- 0.000E 00
F(X) = 0.308537539

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.308537540D 00	0.308537540D 00	0.704130654D 00	0.308537539D 00
0.308537540D 00		-0.211239196D 01	0.308537539D 00
0.308537540D 00		0.316858794D 02	0.308537539D 00
0.308537540D 00			0.308537539D 00
0.308537540D 00			0.308537539D 00

X = 24.00000
SADDLEPOINT = 0.1666666 E 00 +OR- 0.000E 00
F(X) = 0.841344746

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.841344737D 00	0.841344737D 00	0.758029275D 00	0.841344746D 00
0.841344737D 00		0.100000000D 01	0.841344746D 00
0.841344737D 00		0.274087826D 00	0.841344746D 00
0.841344737D 00			0.841344746D 00
0.841344737D 00			0.841344746D 00

3.4 The Normal Probability Law.

This case is considered because extensive and highly accurate tables are available. The results obtained indicate the high accuracy possible with the saddlepoint 2 approximation. In all cases considered, the first term in (1.4.25) gave an answer which is correct to every figure tabulated. However, as the results given in Tables III(a) and (b) indicate, the Edgeworth and Cramer methods generally incorrect in the last two or three figures.

3.5 The Non-Central Chi-Square Probability Law.

The distribution function of the $\{X_i\}$ $(i = 1,2,\ldots)$ is given by

$$F(x|\nu,\lambda) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} F_c(x,\nu + 2j) \quad ,$$

where $\lambda \geq 0$ is termed the non-centrality parameter, ν is the number of degrees of freedom, and

$$F_c(x,k) = [2^{\frac{1}{2}k} \Gamma(\tfrac{1}{2}k)]^{-1} \int_0^x t^{\frac{1}{2}k-1} e^{-\frac{1}{2}t} dt \quad (0 \leq x < \infty)$$

is the central chi-square distribution with n degrees of freedom.

The characteristic function of X_1 is

$$\varphi(t) = \exp[\lambda it/(1-2it)](1-2it)^{-\nu/2} \quad .$$

Using equation (1.4.6), we readily find that c, the saddlepoint, is given by

$$c = \tfrac{1}{2}[1 - \tfrac{1}{2}\frac{\nu}{(x/n)} - (\tfrac{1}{4}\frac{\nu^2}{(x/n)^2} + \frac{\lambda}{x/n})^{\frac{1}{2}}] .$$

Also, the derivatives of the cumulant generating function K evaluated at $t = c$ are given by

$$K^{(j)}(c) = (1-2c)^{-j}\, 2^{j-1}(j-1)!\; [\nu + \lambda j/(1-2c)] .$$

As a first approximation to $F_n(x)$ we can write, using (1.4.25),

$$F_n(x) \simeq \tfrac{1}{2}(1 + \mathrm{sign}(c)) + e^{-xc+nK(c)+\rho^2/2}[\tfrac{1}{2}(1 + \mathrm{sign}(c)) - N(\rho)] ,$$

where $\rho = c\sqrt{nK^{(2)}(c)}$.

Numerical results are tabulated in Tables IV(a), (b) and (c).

Table IV(a)

NON-CENTRAL CHI-SQUARE
1 DEGREE OF FREEDOM
NON CENTRALITY PARAMETER = 2

N = 5

X = 0.50000
SADDLEPOINT = -0.5354101 E 01 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.201524908D-01	0.408479631D-03	0.762591271D-04	0.611544053D-04
-0.462622449D-03		0.744184606D-04	0.727985978D-04
-0.428243247D-02		0.745856488D-04	0.737907413D-04
-0.350359347D-02			0.743107341D-04
-0.236738097D-02			0.744830492D-04

X = 10.00000
SADDLEPOINT = -0.1403881 E 00 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.239750076D 00	0.225509436D 00	0.415353119D 00	0.226020105D 00
0.260255003D 00		-0.180394527D 00	0.258417464D 00
0.260098775D 00		0.269618950D 01	0.259691729D 00
0.260265635D 00			0.260225196D 00
0.260284453D 00			0.260275306D 00

X = 20.00000
SADDLEPOINT = 0.7846480 E-01 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.760249924D 00	0.750808864D 00	0.542413476D 00	0.750672533D 00
0.780754851D 00		0.171323931D 01	0.781703890D 00
0.780911079D 00		-0.657541799D 01	0.780704254D 00
0.781077939D 00			0.781086935D 00
0.781059121D 00			0.781057361D 00

X = 50.00000
SADDLEPOINT = 0.2500000 E 00 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.999999628D 00	0.100000000D 01	0.999736861D 00	0.999747265D 00
0.999993707D 00		0.999769215D 00	0.999761253D 00
0.999955918D 00		0.999760939D 00	0.999763271D 00
0.999838539D 00			0.999763228D 00
0.999682619D 00			0.999763223D 00

NON-CENTRAL CHI-SQUARE
1 DEGREE OF FREEDOM
NON CENTRALITY PARAMETER = 2

N = 15

X = 0.50000
SADDLEPOINT = -0.1544097 E 02 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.139850081D-03	0.293893570D-09	0.827646061D-15	0.748692693D-15
-0.364363892D-03		0.820674756D-15	0.817670141D-15
0.204458697D-03		0.820566522D-15	0.819280304D-15
0.863652947D-04			0.820262712D-15
-0.159474306D-04			0.820522552D-15

X = 35.00000
SADDLEPOINT = -0.8229047 E-01 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.207108148D 00	0.197183833D 00	0.333944115D 00	0.197359615D 00
0.214369460D 00		-0.729580433D-01	0.213253655D 00
0.213783176D 00		0.152355832D 01	0.213614325D 00
0.213724289D 00			0.213702119D 00
0.213711194D 00			0.213707284D 00

X = 50.00000
SADDLEPOINT = 0.3050665 E-01 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.658454280D 00	0.656802920D 00	0.962915067D-01	0.656773159D 00
0.681763554D 00		0.602392513D 01	0.681432263D 00
0.681044790D 00		-0.107226205D 03	0.681076914D 00
0.681172406D 00			0.681164924D 00
0.681161699D 00			0.681162475D 00

X = 80.00000
SADDLEPOINT = 0.1433714 E 00 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.997866638D 00	0.987962604D 00	0.992283246D 00	0.993012895D 00
0.994195089D 00		0.993845643D 00	0.993370262D 00
0.993002421D 00		0.993040480D 00	0.993378925D 00
0.993430984D 00			0.993379875D 00
0.993397009D 00			0.993379870D 00

NON-CENTRAL CHI-SQUARE
1 DEGREE OF FREEDOM
NON CENTRALITY PARAMETER = 2

N = 40

X = 20.00000
SADDLEPOINT = -0.1118033 E 01 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.286651503D-06	0.500525922D-13	0.277554366D-13	0.266054817D-13
-0.137847435D-05		0.274821895D-13	0.275465300D-13
0.251425291D-05		0.274975374D-13	0.274951670D-13
-0.195930593D-05			0.274966303D-13
0.309407186D-06			0.274973883D-13

X = 100.00000
SADDLEPOINT = -0.5825756 E-01 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.158655263D 00	0.150502076D 00	0.229759540D 00	0.150580710D 00
0.158655263D 00		0.322842898D-01	0.157873472D 00
0.158058402D 00		0.568227400D 00	0.157973993D 00
0.157999325D 00			0.157989998D 00
0.157991824D 00			0.157990658D 00

X = 200.00000
SADDLEPOINT = 0.1298437 E 00 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.999683329D 00	0.999586453D 00	0.999731178D 00	0.999745721D 00
0.999874648D 00		0.999757380D 00	0.999752314D 00
0.999778629D 00		0.999750755D 00	0.999752574D 00
0.999746572D 00			0.999752575D 00
0.999751471D 00			0.999752575D 00

X = 500.00000
SADDLEPOINT = 0.2790024 E 00 +OR- 0.000E 00
F(X) =

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.100000000D 01	0.100000000D 01	0.100000000D 01	0.100000000D 01
0.100000000D 01		0.100000000D 01	0.100000000D 01
0.100000000D 01		0.100000000D 01	0.100000000D 01
0.100000000D 01			0.100000000D 01
0.100000000D 01			0.100000000D 01

3.6 The Uniform Probability Law.

The probability density function and moment generating function, respectively, of a random variable distributed uniformly over the interval (a,b) are

$$f(x) = \begin{cases} (b-a)^{-1} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

and

$$M(t) = \frac{e^{bt} - e^{at}}{(b-a)t} .$$

For simplicity, we consider the case $a = -b$.

In general, the cumulants, if they exist, may be expressed in terms of the central moments $\{\mu_i\}$ as

$$\alpha_i = \mu_i \qquad , i = 1,2,3$$

$$\alpha_4 = \mu_4 - 3\mu_2^2$$

$$\alpha_5 = \mu_5 - 10\mu_2(\mu_3 + 4\mu_1^3)$$

$$\alpha_6 = \mu_6 - 15\mu_4\mu_2 + 10\mu_3(\mu_3+6\mu_2\mu_1+2\mu_1^2) + 30\mu_2^3$$

For the uniform probability law, as is easily shown,

$$\mu_i = \begin{cases} 0 & i = 1,3,5,\ldots \\ b^i/(i+1) & i = 2,4,6,\ldots \end{cases} .$$

Thus,

$$\alpha_i = 0 \qquad i = 1,3,5,\ldots$$

$$\alpha_2 = b^2/3$$

$$\alpha_4 = -\frac{2}{15}\, b^4$$

$$\alpha_6 = \frac{5}{18}\, b^6$$

Equation (1.4.6) becomes

$$-\frac{1}{c} + b(e^{cb} + e^{-cb})/(e^{cb} - e^{-cb}) = x/n \quad .$$

To obtain numerical results for this case, the last equation was solved numerically for c with initial iterate $= x/n$, and successive iterates obtained by the Newton technique. Note that since $|K^{(1)}(t)| < b$, a saddlepoint exists only if $|x| < nb$.

Let $u = e^{bt} - e^{-bt}$ and $v = e^{bt} + e^{-bt}$. Then the relevant derivatives of the cumulant generating function are

$$K^{(1)}(t) = -t^{-1} + bv/u$$

$$K^{(2)}(t) = t^{-2} - 4b^2/u^2$$

$$K^{(3)}(t) = -2t^{-3} + 8b^3v/u^3$$

$$K^{(4)}(t) = 6t^{-4} + 8b^4(1-3v^2/u^2)/u^2$$

$$K^{(5)}(t) = -24t^{-5} - 32b^5(2-3v^2/u^2)v/u^3$$

$$K^{(6)}(t) = 120t^{-6} - 32b^6(2-15v^2/u^2+15v^4/u^4)/u^2$$

UNIFORM DISTRIBUTION OVER (A,B)
WITH -A=B=2

N = 10

X = -15.00000
SADDLEPOINT = -0.1994526 E 01 +OR- 0.000E 00
F(X) = 0.256647272 E-05

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.1996198530-04	0.4806654010-05	0.2607815870-05	0.2397507550-05
0.1996198530-04		0.2553486560-05	0.2568800810-05
-0.4662959280-05		0.2567846470-05	0.2558957110-05
-0.4662959280-05			0.2563236030-05
0.1949974560-05			0.2564751020-05

X = -5.00000
SADDLEPOINT = -0.3899486 E 00 +OR- 0.125E-15
F(X) = 0.867211958 E-01

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.8545185520-01	0.8396289630-01	0.1142458350 00	0.8493896350-01
0.8545185520-01		0.5623862940-01	0.8726124220-01
0.8665517800-01		0.1473192170 00	0.8672577770-01
0.8665517800-01			0.8673667630-01
0.8671628800-01			0.8672415950-01

X = 2.00000
SADDLEPOINT = 0.1509085 E 00 +OR- 0.486E-15
F(X) = 0.705481321 E 00

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.7080587650 00	0.7081901090 00	0.3714771930 00	0.7078876020 00
0.7080587650 00		0.2450785420 01	0.7047138670 00
0.7055197790 00		-0.1827993540 02	0.7054930290 00
0.7055197790 00			0.7054632890 00
0.7054803980 00			0.7054792630 00

X = 10.00000
SADDLEPOINT = 0.8983779 E 00 +OR- 0.000E 00
F(X) = 0.997530827 E 00

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.9969150510 00	0.9976713580 00	0.9973180520 00	0.9976248840 00
0.9969150510 00		0.9975994960 00	0.9975075100 00
0.9974931720 00		0.9974941300 00	0.9975302620 00
0.9974931720 00			0.9975296770 00
0.9975270390 00			0.9975305920 00

UNIFORM DISTRIBUTION OVER (A,B)
WITH -A=B=2

N = 40

X = -50.00000
SADDLEPOINT = -0.1294131 E 01 +OR- 0.000E 00
F(X) = 0.107542573 E-12

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.378308496D-11	0.242672377D-12	0.108906992D-12	0.105941867D-12
0.378308496D-11		0.107459744D-12	0.107936408D-12
-0.614192450D-11		0.107549638D-12	0.107547449D-12
-0.614192450D-11			0.107548432D-12
0.445536319D-11			0.107546058D-12

X = -10.00000
SADDLEPOINT = -0.1892840 E 00 +OR- 0.347E-15
F(X) = 0.857564838 E-01

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.854518552D-01	0.850771582D-01	0.114127434D 00	0.853269240D-01
0.854518552D-01		0.539766824D-01	0.858966395D-01
0.857526859D-01		0.149766993D 00	0.857568413D-01
0.857526859D-01			0.857572412D-01
0.857565053D-01			0.857565350D-01

X = 5.00000
SADDLEPOINT = 0.9397053 E-01 +OR- 0.847E-15
F(X) = 0.752531582 E 00

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.753218597D 00	0.753286368D 00	0.538637309D 00	0.753168610D 00
0.753218597D 00		0.151999031D 01	0.752321958D 00
0.752534940D 00		-0.475330943D 01	0.752532233D 00
0.752534940D 00			0.752530474D 00
0.752531572D 00			0.752531541D 00

X = 20.00000
SADDLEPOINT = 0.3899486 E 00 +OR- 0.125E-15
F(X) = 0.997061749 E 00

EDGEWORTH	CRAMER	SADDLEPOINT 1	SADDLEPOINT 2
0.996915051D 00	0.997124511D 00	0.996752800D 00	0.997085911D 00
0.996915051D 00		0.997164984D 00	0.997053953D 00
0.997059581D 00		0.997003185D 00	0.997061647D 00
0.997059581D 00			0.997061703D 00
0.997061698D 00			0.997061745D 00

Comparison with the exact values for $F_n(x)$ indicates that the saddlepoint 2 series again yields the most accurate results (see Tables $\underline{V}$ (a) and (b)).

3.7 Remarks

In additional test cases (which for the sake of brevity are omitted) involving random variables from sections 3.2 to 3.6, the results obtained were qualitatively the same as those reported. Although for the reasons cited in section 2.2 these methods of approximation cannot be theoretically justified in the lattice case, discrete random variables distributed according to the Poisson probability law were treated. Predictably, the results were erratic and usually inaccurate, but when the argument x was a point of discontinuity of F_n, the saddlepoint 2 series yielded results which were accurate to two significant figures in almost all cases. Only the Edgeworth expansion, when c was close to 0, yielded results of similar quality.

CHAPTER 4

OTHER APPLICATIONS

4.1 The Non-Central Chi-Square Distribution.

The form of the characteristic function of the non-central chi-square probability law suggests that an alternative approximation to the distribution function of the n-fold convolution of this law may be obtained by expanding the integrand in (1.4.4) in powers of $\lambda^{-\frac{1}{2}}$, where λ is the non-centrality parameter. The objective of this alternative approach would be an approximation which was useful for very large λ and moderate n .

Let $X_1,\ldots,X_n$ be independent, non-central chi-square distributed random variables, each of whose distribution has non-centrality parameter λ . The moment generating function of X_i $(i = 1,2,\ldots)$ is

$$M(t) = \exp[\lambda t/(1-2t)]\,(1-2t)^{-\nu/2}$$

where ν = number of degrees of freedom and λ = non-centrality parameter. Equation (1.4.4) becomes

$$1 - F_n(\lambda x) = \tfrac{1}{2}(1-\text{sign}(c)) + \frac{1}{2\pi}\int_{-\infty}^{\infty}[1-2(c+iu)]^{-\nu n/2}$$

$$\times \exp[-\lambda x(c+iu) + n\lambda(c+iu)/(1-2(c+iu))] \frac{du}{c+iu} \qquad (4.1.1).$$

The saddlepoint c is the root of the equation

$$\frac{d}{dz}[-zx + \frac{nz}{1-2z}] = 0 \quad .$$

Thus, $c = \frac{1}{2}(1 - \sqrt{\frac{n}{x}})$.

If we write $g(z) = nz/(1-2z)$, and proceed formally as in Chapter 1, the integral in equation (4.1.1) becomes

$$\exp[-\lambda xc + \lambda g(c)] \int_{-\infty}^{\infty} [1-2(c+iu)]^{-n\nu/2} \exp[-\lambda(-g^{(2)}(c)u^2/2 + \sum_{r=3}^{\infty} g^{(r)}(c)\ (iu)^r/r!)] \frac{du}{c+iu} \quad .$$

This can be expanded as a series in powers of $\lambda^{-\frac{1}{2}}$, the asymptoticity of which can be demonstrated in a proof very similar to that of Theorem 2.1.2. This expansion, up to the first five terms, is

$$\begin{aligned} F_n(\lambda x) = \tfrac{1}{2}(1 + \mathrm{sign}(c)) - \frac{e^{-2\lambda xc^2}}{\sqrt{2\pi}} &[h_o Q_o(\rho) \\ &+ \lambda^{-\frac{1}{2}}(h_1 Q_1(\rho) + h_o b_3 Q_3(\rho)) + \lambda^{-1}(h_2 Q_2(\rho) \\ &+ (h_1 b_3 + h_o b_4)\ Q_4(\rho) + \tfrac{1}{2} h_o b_3^2 Q_6(\rho)) + \lambda^{-3/2}(h_3 Q_3(\rho) \\ &+ (h_2 b_3 + h_1 b_4 + h_o b_5) Q_5(\rho) + (\tfrac{1}{2} h_1 b_3^2 + h_o b_3 b_4) Q_7(\rho) \\ &+ \tfrac{1}{6} h_o b_3^3 Q_9(\rho)) + \lambda^{-2}(h_4 Q_4(\rho) + (h_3 b_3 + h_2 b_4 + h_1 b_5 \\ &+ h_o b_6) Q_6(\rho) + (\tfrac{1}{2} h_2 b_3^2 + h_1 b_3 b_4 + \tfrac{1}{2} h_o b_4^2 + h_o b_3 b_5) Q_8(\rho) \\ &+ (\tfrac{1}{6} h_1 b_3^3 + \tfrac{1}{2} h_o b_3^2 b_4) Q_{10}(\rho) + \tfrac{1}{24} h_o b_3^4 Q_{12}(\rho))] \quad , \qquad (4.1.2) \end{aligned}$$

where $\rho = c\sqrt{\lambda g^{(2)}(c)}$, $Q_i(\rho)$ $(i = 0,1,\ldots,12)$ are defined by equations (1.4.21), (1.4.22), (1.4.23) and (1.4.24),

$$b_i = \frac{g^{(i)}(c)}{i!\sqrt{g^{(2)}(c)}^{\,i}} = \tfrac{1}{2}\,(nx)^{\frac{1}{2}-\frac{i}{4}} \qquad (i = 3,\ldots,6) \quad , \text{ and}$$

$$h_i = \frac{M_c^{(i)}(c)}{i!\sqrt{g^{(2)}(c)}^{\,i}} \qquad (i = 0,1,2,\ldots) \quad , \quad M_c \text{ denoting the}$$

function $M_c(t) = (1-2t)^{-n\nu/2}$.

For the sake of simplicity, the case $\nu = 1$ was considered. Numerical results given below in Table VI indicate that for moderate values of n and λ , the expansions are nearly equivalent in accuracy and speed of "convergence". As expected, our earlier approximation is superior where n is large. But even in extreme cases, such as when $n = 1$ and $\lambda = 1000$, the improvement achieved by using the new approximation is very slight.

Table VI

A COMPARISON OF APPROXIMATIONS (1.4.25) AND (4.1.2) TO $F_n(\lambda x)$

λ	n	x	EQUATION (1.4.25)	EQUATION (4.1.2)
100	15	15	.423374879	.5000000000
			.428272116	.427895482
			.428277904	.427895482
			.428278337	.428280040
			.428278338	.428280040
100	1	.8	.138175666	.141809322
			.145448561	.145367769
			.145528761	.145535852
			.145545801	.145545588
			.145546418	.145546215
1000	1	.98	.369745696	.373832713
			.375295334	.375299472
			.375308061	.375308827
			.375308876	.375308896
			.375308880	.375308896

4.2 The Doubly Non-Central F-Distribution.

Let χ_1^2 and χ_2^2 be two independent non-central chi-square random variables with degrees of freedom f_1 and f_2 and non-centrality parameters λ_1 and λ_2, respectively. The distribution of $X_F = \dfrac{\chi_1^2/f_1}{\chi_2^2/f_2}$ is called the doubly non-central F-distribution. It occurs in the analysis of variance and is used in engineering problems where it gives the probability of error in certain communications systems. No simple formula for evaluating the probability integral, F, of X_F is available. Tiku [15] developed several series expansions for F which yield satisfactory approximations when

(i) λ_1 is large and λ_2 is small,

(ii) λ_1 is small and λ_2 is large,

(iii) both λ_1 and λ_2 are large.

In this section we obtain an alternative to Tiku's approximation for the case when f_1 and f_2 are large and λ_1 and λ_2 are moderate. It is derived by means of the saddlepoint method and is, in part, intended to demonstrate the versatility of this method.

Gurland [9] shows that if X_1 and X_2 are two independent random variables with characteristic functions φ_1 and φ_2, respectively, then the ratio X_1/X_2 has a distribution function G which satisfies

$$G(x) + G(x-0) = 1 - \frac{1}{\pi i} \lim_{T\to\infty} \lim_{\epsilon\to 0} \left\{\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T}\right\} \varphi_1(t)\varphi_2(-tx)\frac{dt}{t} \tag{4.2.1}$$

Putting $\varphi_j(t) = \exp[\lambda_j it/(1-2it)](1-2it)^{-f_j/2}$ $(j = 1,2)$, we obtain from equation (4.2.1), since F is continuous,

$$2F(f_2x/f_1) = 1 - \frac{1}{\pi i} \lim_{T\to\infty} \lim_{\epsilon\to 0} \left\{\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T}\right\} \exp\left[\frac{\lambda_1 it}{1-2it} - \frac{\lambda_2 xit}{1+2xt}\right.$$

$$\left. - \frac{f_1}{2}\log(1-2it) - \frac{f_2}{2}\log(1+2xit)\right]\frac{dt}{t} \qquad (4.2.2).$$

If we suppose $f_1 \to \infty$ while λ_1, λ_2 and f_2 are fixed, the appropriate saddlepoint equation is

$$\frac{1}{1-2z} = 0 \quad .$$

This equation has no finite solution, c . A similar problem occurs when we try to expand in terms of f_2, λ_1 or λ_2 .

If the saddlepoint method, is to be useful, therefore, we must suppose that $f_1 \to \infty$, $f_2 \to \infty$ and that λ_1 and λ_2 are fixed parameters. Then c is the solution of

$$\frac{d}{dz}\left[\frac{f_1}{2}\log(1-2z) + \frac{f_2}{2}\log(1+2xz)\right] = 0 \quad ,$$

which yields $c = \dfrac{xf_2 - f_1}{2x(f_2+f_1)}$. Now, $-\frac{1}{2x} < c < \frac{1}{2}$; thus, formally at least, we can proceed as in section (1.4) and expand the functions in the integrand of (4.2.2) about c in convergent power series.

Let

$$f(z) = \lambda_1 z/(1-2z) - \lambda_2 xz/(1+2xz) \quad ,$$

$$g(z) = -\tfrac{1}{2}f_1 \log(1-2z) - \tfrac{1}{2} f_2 \log(1+2xz) \quad ,$$

$$\sigma^* = [g^{(2)}(c)]^{\frac{1}{2}} = \frac{\sqrt{2}\, x(f_1+f_2)}{(1+x)} \left(\frac{1}{f_1} + \frac{1}{f_2}\right)^{\frac{1}{2}}$$

Then, equation (4.2.2) can be rewritten

$$F(xf_2/f_1) = \tfrac{1}{2}(1+\mathrm{sign}(c)) - \frac{1}{2\pi} e^{f(c)+g(c)} \int_{-\infty}^{\infty} \exp\left[\sum_{j=1}^{\infty} \frac{f^{(j)}(c)}{j!}(iu)^j\right]$$

$$\times \exp\left[\sum_{r=3}^{\infty} \frac{g^{(r)}(c)}{r!}(iu)^r\right] \exp\left[-\frac{\sigma^{*2}u^2}{2}\right]\frac{du}{c+iu} \qquad (4.2.3)$$

Define constants a_j $(j = 0,1,2,\ldots)$ by

$$\exp\left[\sum_{r=1}^{\infty} \frac{g^{(r)}(c)}{r!} \left(\frac{iy}{\sigma^*}\right)^r\right] = \sum_{j=0}^{\infty} a_j \left(\frac{iy}{\sigma^*}\right)^j$$

and let

$$b_j = g^{(j)}(c)/j! \quad , \quad (j = 3,4,\ldots) \quad , \text{ and } \quad \rho = c\sigma^* \quad .$$

Proceeding as in the derivation of equation (1.4.25), equation (4.2.3) becomes,

$$F(xf_2/f_1) = \tfrac{1}{2}(1+\mathrm{sign}(c)) - \frac{1}{\sqrt{2\pi}} e^{f(c)+g(c)} \sum_{j=0}^{\infty} c_j(f_1,f_2) \qquad (4.2.4).$$

Each $c_j(f_1,f_2)$ $(j = 0,1,\ldots)$ represents a term of the order $(f_i^{-\frac{1}{2}})^j$. The first few terms in the series of equation (4.2.4) are

$$c_o(f_1,f_2) = Q_o(\rho)$$

$$c_1(f_1,f_2) = \frac{a_1}{\sigma^*} Q_1(\rho) + \frac{b_3}{\sigma^{*3}} Q_3(\rho)$$

$$c_2(f_1,f_2) = \frac{a_2}{\sigma^{*2}} Q_2(\rho) + \frac{1}{\sigma^{*4}}(b_4+a_1b_3)Q_4(\rho) + \tfrac{1}{2}\frac{b_3^2}{\sigma^{*6}} Q_6(\rho)$$

$$c_3(f_1,f_2) = \frac{a_3}{\sigma^{*3}} Q_3(\rho) + \frac{1}{\sigma^{*5}}(b_5+a_1b_4+a_2b_3)Q_5(\rho) + \frac{1}{\sigma^{*7}}(b_3b_4 + \tfrac{1}{2}a_1b_3^2)Q_7(\rho) + \frac{1}{6}\frac{b_3^3}{\sigma^{*9}} Q_9(\rho)$$

$$c_4(f_1,f_2) = \frac{a_4}{\sigma^{*4}} Q_4(\rho) + \frac{1}{\sigma^{*6}}(b_6+a_1b_5+a_2b_4+a_3b_3)Q_6(\rho) + \frac{1}{\sigma^{*8}}(\tfrac{1}{2}b_4^2+b_3b_5+a_1b_3b_4+\tfrac{1}{2}a_2b_3^2)Q_8(\rho) + \frac{1}{\sigma^{*10}}(\tfrac{1}{2}b_3^2b_4 + \tfrac{1}{6}a_1b_3^3)Q_{10}(\rho) + \frac{1}{24}\frac{b_3^4}{\sigma^{*12}} Q_{12}(\rho) \quad .$$

Here, $Q_i(\rho)$ $(i = 0,1,\dots)$ is defined as in section 1.4.

Numerical results listed in Table VII indicate that good accuracy is achieved when λ_1 and λ_2 are small, say $\lambda_i < 1$. The expansion is not, of course, uniform in λ_1 and λ_2 , and is usally accurate to no more than 2 significant figures for larger λ_i .

When $f_1 = f_2 = f$, the above approximation can be simplified considerably. The saddlepoint c becomes $\frac{1}{4}(1 - \frac{1}{x})$. If we let $g(z) = \lambda_1 z/(1-2z) - \lambda_2 xz/(1+2xz)$ and $h(z) = -\frac{1}{2}\log(1-2z) - \frac{1}{2}\log(1+2xz)$, then

$$F(x) = \tfrac{1}{2}(1+\text{sign}(c)) - \frac{1}{\sqrt{2\pi}} e^{g(c)+fh(c)} \sum_{j=0}^{\infty} c_j\, (f^{-\frac{1}{2}})^j \tag{4.2.5}$$

Table VII

APPROXIMATIONS TO F-RATIO

N1 =NO.OF DEGREES OF FREEDOM IN NUMERATOR
N2 =NO.OF DEGREES OF FREEDOM IN DENOMINATOR
N.C.P.(1)=NON CENTRALITY PARAMETER IN NUMERATOR
N.C.P.(2)=NON CENTRALITY PARAMETER IN DENOMINATOR

N1	N2	N.C.P.(1)	N.C.P.(2)	X	F-RATIO
35	40	0.2500	0.3000	0.8000	0.2494549
					0.2511567
					0.2501807
					0.2500907
					0.2504148
20	15	0.2500	0.3000	0.7500	0.2764190
					0.2779659
					0.2753362
					0.2725183
					0.2745522
45	40	0.7000	0.5000	1.0000	0.5000000
					0.4949650
					0.4949650
					0.4951469
					0.4951469
45	55	1.0000	1.2000	1.0000	0.5000023
					0.5011154
					0.5011153
					0.5020313
					0.5020313
25	20	1.0000	1.2000	0.8000	0.3059350
					0.3182083
					0.3077930
					0.2987578
					0.3041053
40	30	0.1000	0.2000	0.7000	0.1479470
					0.1482007
					0.1478174
					0.1474133
					0.1477020
25	20	0.2000	0.1500	0.8000	0.2981952
					0.2966938
					0.2958719
					0.2954838
					0.2960377
70	65	0.1500	0.1000	1.5000	0.9502987
					0.9501236
					0.9501317
					0.9501472
					0.9501286

The first few coefficients of this expansion are

$$c_0 = Q_0(\rho)$$

$$c_1 = a_1 Q_1(\rho) + b_3 Q_3(\rho)$$

$$c_2 = a_2 Q_2(\rho) + (b_4 + a_1 b_3) Q_4(\rho) + \tfrac{1}{2} b_3^2 Q_6(\rho)$$

$$c_3 = a_3 Q_3(\rho) + (b_5 + a_1 b_4 + a_2 b_3) Q_5(\rho) + (\tfrac{1}{2} a_1 b_3^2 + b_3 b_4)$$
$$\times\ Q_7(\rho) + \tfrac{1}{6} b_3^3 Q_9(\rho)$$

$$c_4 = a_4 Q_4(\rho) + (b_6 + a_1 b_5 + a_2 b_4 + a_3 b_3) Q_6(\rho) + (b_3 b_5 + a_1 b_3 b_4)$$
$$+ \tfrac{1}{2} a_2 b_3^2 + \tfrac{1}{2} b_4^2) Q_8(\rho) + (\tfrac{1}{2} b_3^2 b_4 + \tfrac{1}{6} a_1 b_3^3) Q_{10}(\rho) + \tfrac{1}{24} b_3^4 Q_{12}(\rho) .$$

Here, $\rho = c\sqrt{fh^{(2)}(c)}$, $Q_i(\rho)$ $(i = 0,1,\dots)$ is defined in section 1.4, $b_i = \dfrac{h^{(i)}(c)}{i!\sqrt{h^{(2)}(c)}^{\,i}}$ $(i = 3,4,\dots)$, and a_i $(i = 0,1,\dots)$ is defined by the equation

$$\exp\Big[\sum_{r=1}^{\infty} \frac{g^{(r)}(c)}{r!} \Big(\frac{iy}{\sqrt{h^{(2)}(c)}}\Big)^r\Big] = \sum_{j=0}^{\infty} a_j (iy)^j .$$

Numerical calculations listed in Table VIII again indicate that good accuracy is obtained when λ_1 and λ_2 are small. For small f , approximation 4.2.5 is considerably more accurate than expansion 4.2.4 for unequal f_i of the same order of magnitude.

In the case when $\lambda_1 = \lambda_2 = \lambda$, a more suitable approximation is achieved for larger λ , say $\lambda > 3$, if

Table VIII

APPROXIMATIONS TO F-RATIO

N1 =NO.OF DEGREES OF FREEDOM IN NUMERATOR
N2 =NO.OF DEGREES OF FREEDOM IN DENOMINATOR
N.C.P.(1)=NON CENTRALITY PARAMETER IN NUMERATOR
N.C.P.(2)=NON CENTRALITY PARAMETER IN DENOMINATOR

N1	N2	N.C.P.(1)	N.C.P.(2)	X	F-RATIO
7	7	0.2500	0.0000	0.7875	0.3708766
					0.3601652
					0.3627365
					0.3634248
					0.3634509
35	35	5.0000	1.0000	0.7500	0.1490082
					0.1269457
					0.1199288
					0.1236284
					0.1245618
7	7	0.0000	0.0000	2.0000	0.8154488
					0.8154488
					0.8096126
					0.8096126
					0.8096125
10	10	0.0000	1.0000	2.5000	0.9357405
					0.9384183
					0.9376194
					0.9374207
					0.9373013
10	10	1.0000	1.0000	0.7500	0.3253837
					0.3350003
					0.3288592
					0.3271418
					0.3281402
9	9	0.5000	0.2500	0.6000	0.2176499
					0.2163377
					0.2172729
					0.2173970
					0.2176049
15	15	1.8000	1.2000	1.4000	0.7281970
					0.7048957
					0.7139909
					0.7185232
					0.7170775
40	40	0.2500	6.2500	0.7875	0.3223013
					0.4221069
					0.3909000
					0.3663807
					0.3724224

we expand the integral in powers of $\lambda^{-\frac{1}{2}}$.

The appropriate saddlepoint equation now is

$$\frac{d}{dz}\left[\frac{z}{1-2z} - \frac{xz}{1+2xz}\right] = 0 \quad .$$

This equation has root $c = (-2\sqrt{x} + x + 1)/((2\sqrt{x})(x-1))$
It is seen that $-\frac{1}{2x} < c < \frac{1}{2}$, and thus, the functions in the integrand may be expanded about c .

Again, the approximation is not uniform in the remaining parameters, f_1 and f_2 , but for moderate values of f_i , say $f_i < 15$, fairly accurate results (3 or more significant figures) seem to be obtained. These and others are tabulated in Table IX.

4.3 Remarks.

The results of this chapter suggest that the saddlepoint method can be effectively applied to an integral of the form $\int_{-i\infty}^{i\infty} g(z)\, e^{\lambda h(z)} dz$, provided that the equation $h^{(1)}(z) = 0$ has a finite, real solution c . If it does, and the integral can be put in the form $\int_{-\infty}^{\infty} g_1(u)\, e^{\lambda h_1(u)} du$, where g_1 and h_1 satisfy the conditions of Theorem 2.1.1, this method will yield an asymptotic expansion in powers of $\lambda^{-\frac{1}{2}}$. If, as in the example considered in section 4.2, the problem has other parameters in addition to λ , the expansion need not be uniform with respect to them.

APPENDIX

A1. Computing $N(z)$.

In this appendix is presented a method of evaluating $N(z)$ for complex values of z which was devised by Rubin and Zidek [13]. It uses continued fractional expansions for N and thereby avoids the uncontrollable round-off errors which accrue in using the Taylor's expansion. Their method involves the complex form of Shenton's [14] continued fraction for small values of $|z|$ and Laplace's continued fraction (see Kendall and Stuart [11], p. 138) otherwise.

Since $N(z) = 1 - N(-z)$, we can without loss of generality assume $\text{Re}(z) \geq 0$. Writing

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} = \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+} \cdots \tag{A1.1}$$,

we obtain, using the Shenton fraction,

$$N(z) = \tfrac{1}{2} + n(z)\left(\frac{z}{1-}\ \frac{z^2}{3+}\ \frac{2z^2}{5-}\ \frac{3z^2}{7+} \cdots\right), \quad \text{Re}(z) > 0 \tag{A1.2}$$

and, using Laplace's fraction,

$$N(z) = 1 - n(z)\left(\frac{1}{z+}\ \frac{1}{z+}\ \frac{2}{z+}\ \frac{3}{z+}\ \cdots\right)\ , \quad Re(z) > 0 \qquad (A1.3).$$

Rewriting equation (A1.2), with $t = z^{-2}$,

$$N(z) = \tfrac{1}{2} + n(z)\left(\frac{z}{1-}\ \frac{1/3}{t+}\ \frac{2/(3\times5)}{1-}\ \frac{3/(5\times7)}{t+}\ \cdots\right)\ , \quad Re(z) > 0 \qquad (A1.4).$$

We shall call

$$\frac{z}{1-}\ \frac{1/3}{t+}\ \cdots\ \frac{(2n-1)/[(4n-3)(4n-1)]}{t} \quad (n = 1,2,\ldots) \qquad (A1.5)$$

the $2n^{th}$ approximant to the continued fraction in (A1.4). The remainder satisfies

$$\frac{2n/[(4n-1)(4n+1)]}{1-}\ \frac{(2n+1)/[(4n+1)(4n+3)]}{t+}\ \cdots$$

$$\sim \frac{1/8n}{1-}\ \frac{1/8n}{t+}\ \frac{1/8n}{1-}\ \cdots \quad (n \to \infty) \qquad (A1.6).$$

The continued fraction on the right side of (A1.6) represents the function $u(t)$ which satisfies the equation

$$u = a_n[1 - a_n(t + u)^{-1}]^{-1} \qquad (A1.7),$$

that is,

$$u(t) = (a_n - t/2) + [(t/2)^2 + a_n^2]^{\frac{1}{2}} \qquad \text{(A1.8)},$$

where $a_n = 1/8n$.

Let

$$R_n(t) = \text{Re}\{(t/2)^2 + a_n^2\}$$

$$I_n(t) = \text{Im}\{(t/2)^2 + a_n^2\} \qquad \text{(A1.9)}$$

$$R_n'(t) = [\tfrac{1}{2}(R_n(t) + \{R_n^2(t) + I_n^2(t)\}^{\frac{1}{2}})]^{\frac{1}{2}} \quad (n=1,2,\ldots).$$

Then (see Ahlfors [2], p.3)

$$[R_n^2(t) + I_n^2(t)]^{\frac{1}{2}} = \begin{cases} \pm\, [R_n'(t) + \frac{i}{2} I_n(t)/R_n'(t)] & (R_n'(t) \neq 0) \\ 0 & \text{otherwise} \end{cases} \qquad \text{(A1.10)}.$$

The square root in (A1.10) has branch points at $\pm\, 2a_n i$, and the function obtained by choosing either sign is a branch of the square root. Rather than fix the sign, we take

$$[R_n^2(t) + I_n^2(t)]^{\frac{1}{2}} = \begin{cases} \text{sign}[\text{Re}(t)][R_n'(t) + \frac{i}{2} I_n(t)/R_n'(t)] \ , & \text{Re}(t) \neq 0, R_n'(t) \neq 0 \\ R_n'(t) + \frac{i}{2} I_n(t)/R_n'(t) \ , & \text{Re}(t) = 0, R_n'(t) \neq 0 \\ 0 & R_n'(t) = 0 \end{cases} \qquad \text{(A1.11)}$$

to obtain a continuous approximation to the continued fraction.

Using (A1.8) we obtain, as an asymptotic approximation to the continued fraction of (A1.4),

$$\frac{z}{1-}\ \frac{1}{3t+}\ \frac{2}{5-} \cdots \frac{(2n-1)}{(4n-1)(t/2+a_n+\sqrt{(t/2)^2+a_n^2})}\ ,\ \mathrm{Re}(z) > 0$$

$$(n = 1,2,\ldots) \qquad (A1.12).$$

This also gives satisfactory results when $\mathrm{Re}(z) = 0$.

Similarly, we obtain an approximation to the continued fraction of (A1.3),

$$\frac{1}{z+}\ \frac{1}{z+}\ \frac{2}{z+}\ \cdots \frac{2(n-1)}{z+(z^2+4n)^{\frac{1}{2}}}\ ,\ \mathrm{Re}(z) > 0\ ,\qquad (n = 2,3,\ldots)$$

$$(A1.13).$$

One additional modification is introduced to further improve these approximations. If

$$H_n(z) = \int_0^\infty \frac{t^{n-1}}{(n-1)!} \exp[-\tfrac{1}{2}(t+z)^2]dt\ ,\ \mathrm{Re}(z) > 0\ ,\ (n = 1,2,\ldots)\ ,$$

$$(A1.14)$$

and

$$H_o(z) = e^{-z^2/2}\ ,$$

then

$$H_n(z) = (n+1)\ H_{n+2}(z) + z\ H_{n+1}(z)\ ,\ (n = 0,1,\ldots)$$

$$(A1.15).$$

Letting $Q_n = H_{n-1}/H_n$ yields

$$Q_n(z) = z + n/Q_{n+1}(z) \quad , (n = 1,2,\ldots) \quad . \qquad (A1.16).$$

Hence,

$$Q_1(z) = n(z)/(1-N(z)) = z + 1/Q_2(z)$$

$$= z + \frac{1}{z+}\ \frac{2}{z+} \cdots \frac{(n-1)}{Q_n(z)} \quad , (n = 2,3,\ldots) \qquad (A1.17).$$

We now replace $4n$ by a_n' in (A1.13), where the $\{a_n'\}$, (n = 2,2,...) , are chosen so that the approximation $\frac{1}{2}(z+\sqrt{z^2+a_n'})$ to $Q_n(z)$ is exact at $z = 0$, that is,

$$a_n' = 8\,\Gamma^2(\tfrac{n+1}{2})/\Gamma^2(\tfrac{n}{2}) \quad , (n = 2,3,\ldots) \quad , \qquad (A1.18)$$

where Γ denotes the Gamma function.

Let $R_n(z)$ and $I_n(z)$ denote $\mathrm{Re}(z^2+a_n')$ and $\mathrm{Im}(z^2+a_n')$, respectively. Then, if

$$R_{k,n}(z) = [\tfrac{1}{2}(R_n(z)(-1)^{k-1} + (R_n^2(t) + I_n^2(t))^{\frac{1}{2}})]^{\frac{1}{2}} \quad ,$$

$$(k = 1,2; n = 2,3,\ldots) \quad , \qquad (A1.19)$$

we take, as we did in the derivation of (A1.11),

$$[R_n(z) + iI_n(z)]^{\frac{1}{2}} = \begin{cases} \mathrm{sign}[I_n(z)][I_n(z)/2R_{2,n}(z) + i\,R_{2,n}(z)] \\ \qquad R_n(z) < 0,\ I_n(z) \neq 0, R_{2,n}(z) \neq 0 \quad , \\ I_n(z)/2R_{2,n}(z) + i\,R_{2,n}(z) \quad , \\ \qquad R_n(z) < 0, I_n(z) = 0,\ R_{2,n}(z) \neq 0 \quad , \\ R_{1,n}(z) + i\,I_n(z)/2R_{1,n}(z) \quad , \\ \qquad R_n(z) > 0,\ R_{1,n}(z) \neq 0 \quad , \\ 0 \qquad , R_{2,n}(z) \text{ or } R_{1,n}(z) = 0 \quad . \end{cases}$$

(A1.20).

Summarizing these results we obtain

$$N(z) \sim \tfrac{1}{2} + n(z)\left(\frac{z}{1-}\ \frac{1}{3t+}\ \frac{2}{5-}\ \cdots (2n-1)/[(4n-1)(t/2 + 1/8n + \sqrt{(t/2)^2 + 1/64n^2})]\right) \quad , \ \mathrm{Re}(z) \geq 0 \quad , \qquad (A1.21)$$

and

$$N(z) \sim 1 - n(z)\left(\frac{1}{z+}\ \frac{1}{z+}\ \frac{2}{z+}\ \cdots 2(n-1)/[z+\sqrt{3^2+8\Gamma^2(\tfrac{n+1}{2})/\Gamma^2(\tfrac{n}{2})}]\right) \quad , \quad \mathrm{Re}(z) > 0 \quad , \qquad (A1.22)$$

where the square roots are calculated according to (A1.11) and (A1.20).

The approximations were computed in [13] on a grid for z comprised of 231 points spread over the region $D = \{z: -5 \leq \mathrm{Re}(z) \leq 5, 0 \leq \mathrm{Im}(z) \leq 5\}$, and Table X was completed on the basis of the results. It lists suitable approximations for different subregions of D. For simplicity the approximations in (A1.21) and (A1.22) are denoted by S_r and C_r $(r = 1,2,\ldots)$, respectively, where r is a value of n sufficiently large to give an accuracy of at least 10 significant figures over that subregion.

TABLE X

APPROXIMATIONS TO $N(z)$, $z \in \underset{\smile}{D}$ WHICH ARE ACCURATE TO AT LEAST TEN SIGNIFICANT PLACES

REGIONS $R(z)$	REGIONS $I(z)$	APPROXIMATION
(3.5, 5]	[0, 5]	C_{10}
(-4.25, -3.75]	(4.25, 5]	C_{10}
(-4.75, -4.25]	(3.25, 5]	C_{10}
[-5, -4.75]	(2.25, 5]	C_{10}
(2.25, 5]	(0, 5]	C_{20}
(-2.25, -1.75]	(3.75, 5]	C_{20}
(-2.75, -2.25]	(2.25, 5]	C_{20}
(-3.75, -2.75]	[0, 5]	C_{20}
(-4.25, -3.75]	[0, 4.25]	C_{20}
(-4.75, -4.25]	[0, 3.25]	C_{20}
[-5 -4.75]	[0, 2.25]	C_{20}
(-2.75, -2.25]	[0, 2.25]	C_{30}
(.75, 1.75]	[0, .25]	S_5
(-.25, .75]	[0, 1.75]	S_5
(-1.25, -,25]	[0, .75]	S_5
(-1.75, -1.25]	[0, .25]	S_5
(.75, 1.75]	(.25, 3.25]	S_{10}
(-.25, .75]	(1.75, 3.25]	S_{10}
(-1.25, -.25]	(.75, 2.75]	S_{10}
(-1.75, -1.25]	(.25, 2.25]	S_{10}
(-2.25, -1.75]	[0, 1.25]	S_{10}
(1.75, 2.25]	[0, 2.75]	S_{10}
(1.75, 2.25]	(2.75, 5]	S_{15}
(.75, 1.75]	(3.25, 5]	S_{15}
(-.25, .75]	(3.25, 5]	S_{15}
(-1.25, -.25]	(2.75, 5]	S_{15}
(-1.75, -1.25]	(2.25, 5]	S_{15}
(-2.25, -1.75]	(1.25, 3.75]	S_{15}

A2 Computer Program for Evaluating the Saddlepoint 2 Approximation.

To evaluate the saddlepoint 2 approximation (1.4.25), a computer program was written in FORTRAN and run on the IBM 360/67 computer using the Waterloo University compiler (WATFOR). The entire program was written in double precision to keep round-off error to a minimum.

Included in this appendix is a listing of a sample run to calculate the approximation (1.4.25) in the case of non-central random variables. Several of the subroutines, such as SUBROUTINE UU, FUNCTION CUMUL, FUNCTION K, FUNCTION KP and FUNCTION KPP, which calculate the constants K_j (see section 1.4), the cumulants, the cumulant generating function and its first and second derivatives, respectively, have to be rewritten for different distributions of the variables X_i. In addition, the few lines in the main program (lines 47 and 48) which find the saddlepoint are altered with different cases.

A sample set of data cards will contain the following information:

i) Cards 1,2 and 3 - title or comments.

ii) Card 4 - the constant PARA, which may be any parameter that the user wishes to vary during the problem. If PARA $\geq$ 1000, the program terminates.

iii) Card 5 - an indicator showing whether the cumulants are read in (for example, in the case of chi random variables) or whether they are generated by the function CUMUL; the card will read 2 or

1, respectively.

(iv) Card 6 - the constant NCUM, which specifies the number of cumulants to be read in or generated.

(v) Cards 6 + 1, 6 + 2,... - (optional) if Card 5 reads 2, the cummulants will have to be read in as data.

(vi) Card 7 - the number n ; if $n \geq 1000$, the program returns to (ii).

(vii) Card 8 - the number x ; if $x \geq 1000$, the program returns to (vi); otherwise, additional values of x are read in.

```
$COMPILE
      DIMENSION ZID(5),Q(13),G(5),U(6)
      COMMON CUM(10),STDEV,PARA,NCUM
      DOUBLE PRECISION CUM,MU1,STDEV,ENN,PI1,PI2,XBAR,U,PARA,X,RT,STAR
      DOUBLE PRECISION Q,G,W,ABSRHO,SPERR,C,DEXP,DSQRT,KAY,K,KP,KPP,FI
      DOUBLE PRECISION CUMUL,CEE,ESS,ARGUM,ENU,BR,RHO,A2,RUTN,ZID,CO,C1
      DOUBLE PRECISION TEMP
      EQUIVALENCE(U(1),XBAR)
      INTEGER OPT
      PI1=.3989422804014327
      PI2=2.D0*PI1
      TOL=5.E-14
C
C     READ IN TITLE
C
      READ(5,500)
      READ(5,501)
      READ(5,502)
  500 FORMAT(70H
     1                                                                      )
  501 FORMAT(70H
     1                                                                      )
  502 FORMAT(70H
     1                                                                      )
C
C     READ IN VARIABLES
C
   22 READ(5,400)PARA
      IF(PARA.GT.999.)GO TO 20
      READ(5,100)OPT
  100 FORMAT(I3)
C
C IF OPT=(1,2),CUMULANTS ARE (GENERATED,READ IN)
C
      READ(5,100)NCUM
C
C     MAXIMUM NCUM = 7
C
      IF(OPT.EQ.1)GO TO 1
      READ(5,300)(CUM(I),I=1,NCUM)
  300 FORMAT(3D26.16)
      GO TO 222
    1 DO 3 I=1,NCUM
    3 CUM(I)=CUMUL(I)
  222 READ(5,102)N
```

```
      WRITE(6,499)
  499 FORMAT(1H1)
      WRITE(6,500)
      WRITE(6,501)
      WRITE(6,502)
      WRITE(6,900)N
  900 FORMAT(//34X,3HN =,I3//)
  102 FORMAT(I5)
C
C     MAXIMUM N = 9999
C
      IF(N.GT.9999)GO TO 22
   21 READ(5,400)X
C
C     MAXIMUM X = 999.
C
      IF(X.GT.999.)GO TO 222
  400 FORMAT(F10.5)
C
C  START OF INITIAL CALCULATIONS
C
      MU1=CUM(1)
      STDEV=DSQRT(CUM(2))
      ENN=N
      W=(X-ENN*MU1)/(DSQRT(ENN)*STDEV)
      XBAR=X/ENN
C
C     FIND SADDLEPOINT
C
      RT=.25D0/XBAR**2+PARA/XBAR
      C=.5D0*(1.-.5D0/XBAR-DSQRT(RT))
      SPERR=0.
   73 DO 12 I=2,6
   12 U(I)=0.
      CALL UU(U,C)
      STAR=DSQRT(U(2))
      ARGUM=-C*X+ENN*K(C)
      KAY=DEXP(ARGUM)     *PI1
      ENU=ENN*U(2)
C
C  PROCEED WITH SECOND SADDLEPOINT APPROXIMATION
C
   23 RHO=C*DSQRT(ENU)
      A2=RHO**2*.5D0
      RO=RHO
      ABSRHO=ABS(R O)
      IF(ABSRHO-2.25)38,38,37
   37 Q(1)=CEE(30,ABSRHO)*SIGNUM(C)
      GO TO 39
   38 Q(1)=(DEXP(A2)/PI2-ESS(20,ABSRHO))*SIGNUM(C)
   39 FACT=1.
      DO 10 I=1,11,2
      FACT=-1.*FLOAT(I-2)*FACT
      Q(I+1)=FACT-RHO*Q(I)
   10 Q(I+2)=-RHO*Q(I+1)
      G(1)=Q(1)
      G(2)=U(3)*Q(4)/(STAR**3*6.D0)
      G(3)=U(4)*Q(5)/(U(2)**2*24.D0)+U(3)**2*Q(7)/(U(2)**3*72.D0)
      G(4)=U(5)*Q(6)/(STAR**5*120.D0)+U(3)*U(4)*Q(8)/(STAR**7*144.D0)+
     1U(3)**3*Q(10)/(STAR**9*1296.D0)
```

```
            G(5)=U(6)*Q(7)/(U(2)**3*720.D0)+(U(4)**2/1152.D0+U(3)*U(5)/720.D0)
           1*Q(9)/U(2)**4+U(3)**2*U(4)*Q(11)/(U(2)**5*1728.D0)+U(3)**4*Q(13)/(
           2U(2)**6*31104.D0)
            RUTN=1./DSQRT(ENN)
            ZID(1)=.5D0*(1.+SIGNUM(C))-KAY*G(1)
            DO 11 I=2,5
   11       ZID(I)=ZID(I-1)-KAY*G(I)*RUTN**(I-1)
C
C     PREPARE OUTPUT
C
            XD=X
            CS=C
            SPE=SPERR
            WRITE(6,1000)XD,CS,SPE
 1000 FORMAT(10X,1HX,11X,1H=,F10.5/10X,13HSADDLEPOINT =,E16.8,2X,4H+OR-,
           1E10.3/10X,4HF(X),8X,1H=        /)
            WRITE(6,1100)(ZID(I),I=1,5)
 1100 FORMAT(4X,13HSADDLEPOINT 2/5(1X,D16.9/) )
            GO TO 21
   20       CONTINUE
            STOP
            END

            FUNCTION SIGNUM(T)
            DOUBLE PRECISION T
            IF(T)1,2,3
          1 SIGNUM=-1.
            GO TO 4
          2 SIGNUM=0.
            GO TO 4
          3 SIGNUM=1.
          4 RETURN
            END

            DOUBLE PRECISION FUNCTION FI(T)
C
C           NORMAL DISTRIBUTION FUNCTION
C
            DOUBLE PRECISION DEXP,DSQRT,ESS,CEE,A,FACTOR,FF,PI1,T,TT
            PI1=.7978845608028654
            A=-.5D0*T**2
            FACTOR=.5D0*DEXP(A)*PI1
            TT=T
            TSP=TT
            A=T
            IF(TT)6,7,8
          7 FF=.5D0
            GO TO 15
          6 A=-T
          8 IF(ABS(TSP)-1.75)1,1,2
          1 FF=.5D0+FACTOR*ESS(8,A)
            GO TO 5
          2 IF(ABS(TSP)-2.25)3,3,4
          3 FF=.5D0+FACTOR*ESS(13,A)
            GO TO 5
          4 FF=1.-FACTOR*CEE(25,A)
          5 IF(TT)9,15,15
          9 FI=1.-FF
```

```
      GO TO 16
   15 FI=FF
   16 RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION ESS(N,Z)
C
C     SHENTON CONTINUED FRACTION
C
      DOUBLE PRECISION Z,DENOM,BR,RUT,T,EN,DEXP,DSQRT
      ESS=0.
      IF(Z.EQ.0.)RETURN
      EN=N
      T=1./Z**2
      RUT=.25D0*T**2+1./(64.D0*EN**2)
      MULT=4*N-1
      NUM=2*N-1
      SIGN=-1.
      DENOM=FLOAT(MULT)*(.5D0*T+.125/EN+DSQRT(RUT))
      LIM=NUM
      DO 1 I=1,LIM
      MULT=MULT-2
      DENOM=FLOAT(MULT)*((SIGN+1.)*T-SIGN+1.)*.5D0+SIGN*FLOAT(NUM)/DENOM
      NUM=NUM-1
    1 SIGN=-SIGN
      ESS=Z/DENOM
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION CEE(N,Z)
C
C     LAPLACE CONTINUED FRACTION
      DOUBLE PRECISION DEXP,DSQRT,Z,DENOM,RUT,A1,A2,GAMM
      A1=(FLOAT(N)+1.)/2.D0
      A2=FLOAT(N)/2.D0
      RUT=Z**2+8.*(GAMM(A1)/GAMM(A2))**2
      DENOM=Z+DSQRT(RUT)
      LIM=2*N-2
      DO 1 I=1,LIM
      NUM=2*N-1-I
    1 DENOM=Z+FLOAT(NUM)/DENOM
      CEE=1./DENOM
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION GAMM(X)
C
C     GAMMA FUNCTION
C
      DOUBLE PRECISION X,XX,FACT
      IF(X.LE.1.)GO TO 10
      N=X
      XX=N
      IF(XX.NE.X)GO TO 2
      FACT=1.
      N1=N-1
      DO 1 I=1,N1
```

```
    1     FACT=FACT*FLOAT(I)
          GAMM=FACT
          GO TO 11
        2 LIM=2*N-1
          FACT=1.
          DO 3 J=1,LIM,2
        3 FACT=FACT*FLOAT(J)
          GAMM=FACT*1.772453850905516D0/2.D0**N
          GO TO 11
       10 WRITE(6,100)
  100     FORMAT(5X,20H ERROR IN GAMMA FCN. )
   11     RETURN
          END

          SUBROUTINE UU(                     U,C)
C
C         DERIVATIVES OF C.G.F. CALCULATED AT C
C
          DIMENSION U(6)
          COMMON CUM(10),STDEV,PARA,NCUM
          DOUBLE PRECISION CUM,STDEV,U,RT,PARA,CAL,DSQRT,C
          RT=.25D0/U(1)**2+PARA/U(1)
          CAL=.5D0/U(1)+DSQRT(RT)
          U(2)=2./CAL**2*(1.+2.*PARA/CAL)
          U(3)=8.D0/CAL**3*(1.+3.D0*PARA/CAL)
          U(4)=48.D0/CAL**4*(1.+4.D0*PARA/CAL)
          U(5)=384.D0/CAL**5*(1.+5.D0*PARA/CAL)
          U(6)=3840.D0/CAL**6*(1.+6.*PARA/CAL)
          RETURN
          END

          DOUBLE PRECISION FUNCTION CUMUL(J)
C
C         CALCULATION OF CUMULANTS
C
          COMMON CUM(10),STDEV,PARA,NCUM
          DOUBLE PRECISION CUM,STDEV,PARA,UMUL
          UMUL=1.+   PARA
          JM1=J-1
          FACT=1.
          IF(J-1)1,1,2
        2 DO 3 K=1,JM1
        3 FACT=FACT*2.*FLOAT(K)
          UMUL=FACT*(1.+   PARA*FLOAT(J))
        1 CUMUL=UMUL
          RETURN
          END

          DOUBLE PRECISION FUNCTION K   (S)
C
C         CUMULANT GENERATING FUNCTION (C.G.F.)
C
          COMMON CUM(10),STDEV,PARA,NCUM
          DOUBLE PRECISION CUM,STDEV,S,PARA,ARG,DLOG
          IF(S-.5)1,2,2
        2 WRITE(6,100)
      100 FORMAT(5X,13H K UNDEFINED     )
```

```
      K=0.
      GO TO 5
    1 ARG=1.-2.*S
      K=-.5D0*DLOG(ARG)+PARA*S/ARG
    5 RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION KP (S)
C
C     FIRST DERIVATIVE OF C.G.F.
C
      COMMON CUM(10),STDEV,PARA,NCUM
      DOUBLE PRECISION CUM,STDEV,S,PARA,ARG
      IF(S-.5)1,2,2
    2 WRITE(6,100)
  100 FORMAT(5X,13H K UNDEFINED    )
      K=0.
      GO TO 5
    1 ARG=1.-2.*S
      KP=1./ARG+PARA/ARG**2
    5 RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION KPP(S)
C
C     SECOND DERIVATIVE OF C.G.F.
C
      COMMON CUM(10),STDEV,PARA,NCUM
      DOUBLE PRECISION CUM,STDEV,S,PARA,ARG
      IF(S-.5)1,2,2
    2 WRITE(6,100)
  100 FORMAT(5X,13H K UNDEFINED    )
      K=0.
      GO TO 5
    1 ARG=1.-2.*S
      KPP=2./ARG**2+PARA*4./ARG**3
    5 RETURN
      END
```

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