RANK PRESERVERS ON CERTAIN SYMMETRY CLASSES OF TENSORS

by

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Let $U$ denote a finite dimensional vector space over an algebraically closed field $F$. In this thesis, we are concerned with rank one preservers on the $r$th symmetric product spaces $V^r$ and rank $k$ preservers on the 2nd Grassmann product spaces $\Lambda^2 U$.

The main results are as follows:

(i) Let $T : V^r \to V^r$ be a rank one preserver.

(a) If $\dim U > r + 1$, then $T$ is induced by a non-singular linear transformation on $U$. (This was proved by L.J. Cummings in his Ph.D. Thesis under the assumption that $\dim U > r + 1$ and the characteristic of $F$ is zero or greater than $r$.)

(b) If $2 < \dim U < r + 1$ and the characteristic of $F$ is zero or greater than $r$, then either $T$ is induced by a non-singular linear transformation on $U$ or $T(V^r) = V^r W$ for some two dimensional subspace $W$ of $U$.

(ii) Let $T : V^r \to V^s$ be a rank one preserver where $r < s$. If $\dim U > s + 1$ and the characteristic of $F$ is zero or greater than $\frac{s}{r}$, then $T$ is induced by $s - r$ non-zero vectors of $U$ and a non-singular linear transformation on $U$. 

Supervisor: R. Westwick
(iii) Let $T : \Lambda U \rightarrow \Lambda U$ be a rank $k$ preserver and $\text{char } F \neq 2$. If $T$ is non-singular or $\dim U = 2k$ or $k = 2$, then $T$ is a compound, except when $\dim U = 4$, in which case $T$ may be the composite of a compound and a linear transformation induced by a correlation of the two dimensional subspaces of $U$. 
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INTRODUCTION

Let $U$ be a finite dimensional vector space over an arbitrary field $F$. Let $G$ be a subgroup of the symmetric group $S_m$ and $\chi$ be a character of degree one on $G$. Denote by $U^m_\chi(G)$ the symmetry class of tensors over $U$ associated with $G$ and $\chi$. A non-zero element of $U^m_\chi(G)$ is said to have rank $k$ if it can be expressed as the sum of $k$ but not less than $k$ non-zero decomposable tensors (pure products) in $U^m_\chi(G)$. A linear transformation of $U^m_\chi(G)$ is said to be a rank $k$ preserver if it maps the set of all rank $k$ vectors into itself.

Recently, there have been investigations concerning the structure of rank $k$ preservers on classical symmetry classes of tensors by Beasley [1], Cummings [4], Djoković [6], Marcus and Moyls [12], Moore [16] and Westwick [18;19;20]. The purpose of this thesis is to continue this investigation: mainly on rank one preservers on $r$th symmetric product spaces and partly on rank $k$ preservers on 2nd Grassmann product spaces.

We begin by studying some basic properties of rank $k$ vectors in general symmetry classes of tensors (Chapter I). Most of the theorems are generalizations of well-known results about the classical spaces. For example, we show that (i) the rank of a vector in $U^m_\chi(G)$ is unchanged if we extend $U$; (ii) for each rank $k$ vector in $U^m_\chi(G)$ and each orbit of $G$, we associate a unique subspace of $U$;

(iii) $z_{11} \cdots z_{lm} + \cdots + z_{kl} \cdots z_{km}$ is of rank $k$ if for each orbit $0$ of $G$, the dimension of the subspace spanned by the vectors $z_{jd}$ where $d \in 0$, $j = 1, \ldots, k$ is $k|0|$ where $|0|$ denotes the cardinality
of 0. From our results on rank k vectors, we obtain an application on intersections of symmetry classes of tensors and an application on equalities of two associated transformations (induced transformations).

In Chapter II, we consider rank one preservers on the $r^*$th symmetric product spaces, $\bar{V}_U$, where $U$ is a finite dimensional vector space over an algebraically closed field $F$. We first classify the maximal pure subspaces of $\bar{V}_U$ and study their intersection properties. We are able to determine the structure of an infinite family of certain maximal pure subspaces such that any two of them have a non-zero intersection. With the help of the results on maximal pure subspaces, we prove the following main theorems of this Chapter.

(i) If $\dim U \geq r + 1$, then a rank one preserver on $\bar{V}_U$ is induced by a non-singular transformation on $U$.

(ii) Let $T$ be a rank one preserver on $\bar{V}_U$. Let $2 < \dim U < r + 1$ and the characteristic of $F$ be zero or greater than $r$. Then either $T$ is induced by a non-singular transformation on $U$ or $T(\bar{V}_U) = \bar{W}$ for some two dimensional subspace $W$ of $U$.

(iii) Let $T$ be a rank one preserver from $\bar{V}_U$ to $\bar{V}_U$ where $r < s$. Let $\dim U \geq s + 1$ and the characteristic of $F$ be zero or greater than $s/r$. Then $T$ is induced by $s - r$ vectors $z_1, \ldots, z_{s-r}$ of $U$ and a non-singular transformation $f$ on $U$ in the sense that $T(x_1, \ldots, x_r) = z_1 \cdots \cdot z_{s-r} \cdot f(x_1) \cdots \cdot f(x_r)$, $x_i \in U$, $i = 1, \ldots, r$.

(i) and (ii) partially answer a question raised by Marcus and.
Newman [14, p. 62]. (i) was first proved by Cummings [4] under the assumption that \( \dim U > r + 1 \) and the characteristic of \( F \) is not a prime \( p \leq r \). An example is given to show that there is another type of rank one preserver from \( \mathbb{V}_U \) to \( \mathbb{H}_U \), \( r < s \), if \( F \) is of prime characteristic \( p \) such that \( rp \leq s \).

In Chapter III, we consider rank \( k \) preservers on 2nd Grassmann product spaces, \( \wedge^2 U \), where \( U \) is an \( n \)-dimensional vector space over an algebraically closed field \( F \) with characteristic not equal to two. We show that if \( T \) is a rank \( k \) preserver on \( \wedge^2 U \) and either (i) \( T \) is non-singular; or (ii) \( n = 2k \); or (iii) \( k = 2 \); then \( T \) is a compound of a non-singular transformation on \( U \), except when \( n = 4 \), in which case \( T \) may be the composite of a compound and a linear transformation induced by a correlation of the two dimensional subspaces of \( U \). These results lead to corresponding theorems on rank \( 2k \) preservers on the space of all \( n \)-square skew-symmetric matrices over \( F \). The result on rank two preservers on \( \wedge^2 U \) was also obtained independently by M.J.S. Lim [10].
CHAPTER I

RANK k VECTORS IN SYMMETRY CLASSES

§1. Definitions and Remarks.

Let $F$ be an arbitrary field. Let $G$ be a subgroup of the symmetric group $S_m$ of degree $m$. Let $\chi$ be a character of degree one on $G$; i.e., $\chi : G \to F^*$ is a homomorphism where $F^*$ is the multiplicative group of $F$.

Let $V_1, V_2, \ldots, V_m$ be finite dimensional vector spaces over $F$ such that $V_i = V_{\sigma(i)}$ for $i = 1, 2, \ldots, m$ and all $\sigma \in G$. Let $W$ be the cartesian product of the $V_i$, $W = V_1 \times \cdots \times V_m$.

1.1. Definition. Let $U$ be any vector space over $F$. A multilinear function $f : W \to U$ is said to be symmetric with respect to $G$ and $\chi$ if $f(X_{\sigma(1)}, \ldots, X_{\sigma(m)}) = \chi(\sigma) f(X_1, \ldots, X_m)$ for any $\sigma \in G$ and arbitrary $X_i \in V_i$, $i = 1, 2, \ldots, m$.

1.2. Definition. A pair $(P, \mu)$ consisting of a vector space $P$ over $F$ and a multilinear function $\mu : W \to P$, symmetric with respect to $G$ and $\chi$, is a symmetry class of tensors over $V_1, V_2, \ldots, V_m$ associated with $G$ and $\chi$ if the following universal factorization property is satisfied:

For any vector space $U$ over $F$ and any multilinear function $f : W \to U$, symmetric with respect to $G$ and $\chi$, there exists a unique linear transformation $h : P \to U$ such that $f = h \mu$, i.e., such that the diagram
5.

Given a pair $G$ and $\chi$, a symmetry class of tensors over $V_1, V_2, \cdots, V_m$ associated with $G$ and $\chi$ exists and is unique to within canonical isomorphism (see [13], [17]). We shall denote such a space by $(V_1, \cdots, V_m)^\chi(G)$. If $V_1 = \cdots = V_m = V$, then such a space is usually denoted by $V^m_{\chi}(G)$ (see [13]). The vectors $\mu(X_1, \cdots, X_m)$ are denoted by $X_1 \star \cdots \star X_m$ and are called decomposable elements (pure products).

The notion of a symmetry class of tensors generalizes the classical tensor, Grassmann and symmetric spaces, for an appropriate choice of $G$ and $\chi$, i.e.,

1. If $G = \{e\}$, where $e$ is the identity permutation in $S_m$, $\chi = 1$, then $(V_1, V_2, \cdots, V_m)^\chi(G)$ is the tensor product of vector spaces $V_1, \cdots, V_m \otimes V_1$. In this case, the decomposable element $X_1 \star \cdots \star X_m$ is denoted by $X_1 \otimes \cdots \otimes X_m$.

2. If $G = S_m$ and $\chi = "\text{sign of the permutation}"$ character, then $V^m_{\chi}(G)$ is the Grassmann space $\wedge^m V$. In this case, the decomposable element $X_1 \star \cdots \star X_m$ is denoted by $X_1 \wedge \cdots \wedge X_m$. 
3. If $G = S_m$ and $\chi = 1$, then $V^m(\chi)(G)$ is the symmetric space $W$. The decomposable element $X_1 \ast \cdots \ast X_m$ is denoted by $X_1 \cdots X_m$.

1.3. Remark. Since $u$ is multilinear and symmetric with respect to $G$ and $\chi$, we have:

(i) $X_1 \ast \cdots \ast (\alpha X_i + \beta X_i') \ast \cdots \ast X_m = \alpha X_1 \ast \cdots \ast X_i \ast \cdots \ast X_m + \beta X_1 \ast \cdots \ast X_i' \ast \cdots \ast X_m$

for each $X_i, X_i' \in V_i$, each $\alpha, \beta \in F$ and each $i = 1, 2, \ldots, m$.

(ii) $\chi_{\sigma(1)} \ast \cdots \ast \chi_{\sigma(m)} = \chi(\sigma) X_1 \ast \cdots \ast X_m$, for each $\sigma \in G$, $X_i \in V_i$, $i = 1, \ldots, m$.

1.4. Remark. It can be shown from the universal factorization property that the decomposable elements span $(V_1, \cdots, V_m)_{\chi}(G)$.

1.5. Remark. Let $U_1, \cdots, U_m$ be subspaces of $V_1, \cdots, V_m$ respectively such that $U_i = U_{\sigma(i)}$ for $i = 1, 2, \ldots, m$ and for all $\sigma \in G$.

Let $u_1$ be the restriction of the map $u$ to $U_1 \times \cdots \times U_m$. Then it can be shown that $(\langle \text{range } u_1 \rangle, u_1)$ is a symmetry class of tensors over $U_1, \cdots, U_m$, associated with $G$ and $\chi$ where $\langle \text{range } u_1 \rangle$ denotes the linear closure of the range of $u_1$. Therefore we identify $(U_1, \cdots, U_m)_{\chi}(G)$ with a subspace of $(V_1, \cdots, V_m)_{\chi}(G)$.

Let $T_i : V_i \to V_i$ be linear transformations such that $T_i = T_{\sigma(i)}$ for $i = 1, 2, \ldots, m$ and for all $\sigma \in G$. Define a mapping $\phi : V_1 \times \cdots \times V_m \to (V_1, \cdots, V_m)_{\chi}(G)$ by setting $\phi(X_1, \cdots, X_m) = T_1 X_1 \ast \cdots \ast T_m X_m$. 
It is easily seen that \( \phi \) is multilinear and symmetric with respect to \( G \) and \( \chi \). Hence by the universal factorization property of \((V\chi,\cdots,V\chi)^{(G)}\), there exists a unique linear transformation on \((V\chi,\cdots,V\chi)^{(G)}\), denoted by \( K(T_1,\cdots,T_m) \), such that

\[
K(T_1,\cdots,T_m) x_1 \cdots x_m = T_1 x_1 \cdots T_m x_m.
\]

When \( T_1 = \cdots = T_m = T \), we shall denote \( K(T_1,\cdots,T_m) \) simply by \( K(T) \).

1.6. Definition. The above transformation \( K(T_1,\cdots,T_m):(V\chi,\cdots,V\chi)^{(G)} \rightarrow (V\chi,\cdots,V\chi)^{(G)} \) is called the associated transformation (induced transformation) of \( T_1,\cdots,T_m \).

Our definition of associated transformation generalizes the one in \([11,13]\).

The associated transformations in the classical tensor, Grassmann and symmetric spaces are the \( m^{th} \) tensor product \( T_1 \otimes \cdots \otimes T_m \), the \( m^{th} \) compound \( C_m(T) \) and the \( m^{th} \) induced power of \( T \), \( P_m(T) \), respectively.

1.7. Remark. If \( S_i : V_i \to V_i \) are linear transformations such that \( S_i = S_{\sigma(i)} \) for \( i = 1, \cdots, m \) and \( \sigma \in G \), then we have

\[
K(T_1 S_1,\cdots,T_m S_m) = K(T_1,\cdots,T_m) K(S_1,\cdots,S_m).
\]
1.8. Definition. A non-zero vector in \((V_1, \ldots, V_m)_\chi(G)\) is said to have rank \(k\) if it can be written as a sum of \(k\) but not less than \(k\) non-zero decomposable elements in \((V_1, \ldots, V_m)_\chi(G)\). The set of all rank \(k\) vectors in \((V_1, \ldots, V_m)_\chi(G)\) is denoted by \(R_k((V_1, \ldots, V_m)_\chi(G))\).

1.9. Definition. A subspace \(M \subseteq (V_1, \ldots, V_m)_\chi(G)\) is called a rank \(k\) subspace if every non-zero vector in \(M\) is of rank \(k\). A rank \(k\) subspace \(M\) is said to be maximal if no other rank \(k\) subspaces properly contain \(M\). A rank one subspace is usually called a pure subspace.

1.10. Definition. A linear transformation \(A : (V_1, \ldots, V_m)_\chi(G) \rightarrow (V_1, \ldots, V_m)_\chi(G)\) is called a rank \(k\) preserver if \(A\) maps the set of rank \(k\) vectors into itself.

1.11. Remark. Some knowledge of the structure of rank \(k\) subspaces will be useful in characterizing rank \(k\) preservers. By definition, it is evident that a rank \(k\) preserver \(A\) maps a rank \(k\) subspace \(M\) into a rank \(k\) subspace \(A(M)\). Moreover, \(A|M\) is a monomorphism and hence \(\dim M = \dim A(M)\).

§2. Properties of Rank \(k\) Vectors.

Throughout this section, let \((V_1, \ldots, V_m)_\chi(G)\) denote a symmetry class of tensors over \(V_1, \ldots, V_m\) associated with a subgroup \(G\) of \(S_m\) and a character \(\chi\) on \(G\). We also let \(O_1, \ldots, O_t\) be all the orbits of \(G\).
2.1. Theorem. Let \( x_1 + \cdots + x_k = y_1 + \cdots + y_q \in R_k( (V_1, \cdots , V_m) \chi (G)) \)
where \( x_j = x_{j1} \cdots x_{jm} \), \( y_n = y_{n1} \cdots y_{nm} \) for each \( j = 1, \ldots , k \)
and \( n = 1, \ldots , q \). Then for each orbit \( O_i \), we have

\[
\sum_{j=1}^{k} \langle x_{jd} : d \in O_i \rangle \subseteq \sum_{n=1}^{q} \langle y_{nd} : d \in O_i \rangle
\]

where \( \langle x_{jd} : d \in O_i \rangle \) denotes the subspace spanned by the vectors \( x_{jd} \), \( d \in O_i \).

Proof: Suppose that for some \( j, 1 \leq j \leq k \), \( \langle x_{jd} : d \in O_i \rangle \neq \sum_{n=1}^{q} \langle y_{nd} : d \in O_i \rangle \).

Then for some \( s \in O_i \), \( x_{js} \neq \sum_{n=1}^{q} y_{ns} \).

Consider the associated transformation \( K(T_1, \cdots , T_m) : (V_1, \cdots , V_m) \chi (G) \)
\( \rightarrow (V_1, \cdots , V_m) \chi (G) \) where \( T_{\ell} = T_{\sigma (\ell)} \) for all \( \sigma \in G \), \( \ell = 1, \ldots , m \)
and \( T_1, \ldots , T_m \) are defined as follows:

If \( \ell \in O_i \), \( T_{\ell} : V_{\ell} \rightarrow V_{\ell} \) is a linear transformation such that
\( T_{\ell}(x_{js}) = 0 \) and \( T_{\ell} |_{\sum_{n=1}^{q} \langle y_{nd} : d \in O_i \rangle} \) is the identity mapping.

If \( \ell \notin O_i \), \( T_{\ell} : V_{\ell} \rightarrow V_{\ell} \) is the identity mapping.

We have \( K(T_1, \cdots , T_m)(x_1 + \cdots + x_k) = K(T_1, \cdots , T_m)(y_1 + \cdots + y_q) \).
Since $K(T_1, \ldots, T_m)(y_n) = y_n$ for $n = 1, \ldots, q$ and $K(T_1, \ldots, T_m)(x_j) = T_j x_{j1} \cdots T_m x_{jm} = 0$, it follows that

$$K(T_1, \ldots, T_m)(x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k) = y_1 + \cdots + y_q \in R_k((V_1, \ldots, V_m, \chi)(G)).$$

This is a contradiction since the left hand side is a vector of rank less than $k$ or the zero vector. Therefore $\langle x_d : d \in O_i \rangle \subseteq \sum_{n=1}^{q} \langle y_{nd} : d \in O_i \rangle$ for each $j = 1, \ldots, k$. Hence $\sum_{j=1}^{k} \langle x_d : d \in O_i \rangle \subseteq \sum_{n=1}^{q} \langle y_{nd} : d \in O_i \rangle$.

### 2.2. Corollary

Let $x + y = Z$ where $x = x_1 \cdots x_m$, $y = y_1 \cdots y_m$, and $Z = Z_1 \cdots Z_m$ are non-zero decomposable elements of $(V_1, \ldots, V_m, \chi)(G)$. Then for each orbit $O_i$, we have $\langle Z_d : d \in O_i \rangle \subseteq \langle x_d : d \in O_i \rangle + \langle y_d : d \in O_i \rangle$.

This Corollary generalizes a Theorem of Cummings [4, p. 17].

### 2.3. Theorem

Let $x_1 + \cdots + x_k = y_1 + \cdots + y_k \in R_k((V_1, \ldots, V_m, \chi)(G))$ where $x_j = x_{j1} \cdots x_{jm}$ and $y_j = y_{j1} \cdots y_{jm}$ for each $j = 1, \ldots, k$.

Then for each orbit $O_i$, we have $\sum_{j=1}^{k} \langle x_d : d \in O_i \rangle = \sum_{j=1}^{k} \langle y_d : d \in O_i \rangle$.

**Proof:** This follows immediately from Theorem 2.1.

### 2.4. Corollary

Suppose that $x_1 \cdots x_m = y_1 \cdots y_m \in V^m(G)$ and $x_1 \cdots x_m \neq 0$. Then $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_m \rangle$. 
Corollary 2.4 generalizes a lemma of Marcus and Minc [13].

2.5. Definition. Let \( x = x_1 \cdots x_m \) be a non-zero decomposable element of \( (V_1, \ldots, V_m)_\chi(G) \). For each orbit \( O_i \) of \( G \), define \( < O_i(x) > \) to be the subspace spanned by the vectors \( x_d \) where \( d \in O_i \). By Theorem 2.3, this definition is well-defined.

2.6. Corollary. Let \( x_1 + \cdots + x_k \in R_k((V_1, \ldots, V_m)_\chi(G)) \) where \( x_j \) is a decomposable element for each \( j = 1, \ldots, k \). Let \( x_{k+1} = Z_1 \cdots Z_m \neq 0 \).

If for some orbit \( O_i \), there is a \( s \in O_i \) such that \( Z s \neq 1 \) then \( x_1 + \cdots + x_{k+1} \) has rank greater than or equal to \( k \).

Proof. If \( \sum_{j=1}^{k+1} x_j = 0 \), then \( \sum_{j=1}^{k} x_j = -x_{k+1} \). This implies that \( k = 1 \).

By Theorem 2.3, we have \( Z s \in < O_i(x_1) > \) which contradicts the hypothesis.

If \( \sum_{j=1}^{k+1} x_j = \sum_{j=1}^{n} y_j \in R((V_1, \ldots, V_m)_\chi(G)) \) where \( 1 \leq n < k \), then \( \sum_{j=1}^{k} x_j = \sum_{j=1}^{n} y_j - x_{k+1} \). This implies that \( n = k - 1 \) since \( \sum_{j=1}^{k} x_j \) is of rank \( k \). By Theorem 2.3, we have \( < O_i(y_1) > + \cdots + < O_i(y_{k-1}) > + < O_i(x_{k+1}) > = < O_i(x_1) > + \cdots + < O_i(x_k) > \). Hence \( Z s \in < O_i(x_1) > + \cdots + < O_i(x_k) > \), a contradiction to the hypothesis.

Therefore \( \sum_{j=1}^{k+1} x_j \) is of rank greater than or equal to \( k \).
Corollary 2.6 will be used in Chapter III.

The following corollary of Theorem 2.4 is well-known.

2.7. Corollary. (a) Suppose that \( \sum_{i=1}^{k} x_i \otimes y_i = \sum_{i=1}^{k} u_i \otimes v_i \in R_k(V_1 \otimes V_2) \).

Then

\[
\langle x_1, \ldots, x_k \rangle = \langle u_1, \ldots, u_k \rangle, \quad \langle y_1, \ldots, y_k \rangle = \langle v_1, \ldots, v_k \rangle.
\]

(b) Suppose that \( x_1 \otimes \cdots \otimes x_k = y_1 \otimes \cdots \otimes y_k \neq 0 \) in \( V_1 \otimes \cdots \otimes V_k \).

Then \( \langle x_i \rangle = \langle y_i \rangle, \ i = 1, \ldots, k \).

2.8. Theorem. Let \( U_1, \ldots, U_m \) be subspaces of \( V_1, \ldots, V_m \) respectively such that \( U_i = U_{\sigma(i)} \), \( i = 1, \ldots, m \), for all \( \sigma \in G \). Then

\[
R_k((U_1, \ldots, U_m) (G)) \subseteq R_k((V_1, \ldots, V_m) (G))
\]

Proof: Let \( y = \sum_{j=1}^{k} y_j \in R_k((U_1, \ldots, U_m) (G)) \) where \( y_j \) is a decomposable element for each \( j = 1, \ldots, k \). By definition, if \( \ell \in O_i \), then

\[
\langle o_i(y_j) \rangle \subseteq U_{\ell}
\]

for each \( j = 1, \ldots, k \). Suppose that

\[
y = \sum_{j=1}^{n} Z_j \in R_n((V_1, \ldots, V_m) (G)) \]

where \( Z_j \) is a decomposable element for each \( j = 1, \ldots, n \). Then \( n \leq k \). According to Theorem 2.1, we have

\[
\langle o_i(Z_1) \rangle + \cdots + \langle o_i(Z_n) \rangle \subseteq \langle o_i(y_1) \rangle + \cdots + \langle o_i(y_k) \rangle \subseteq U_{\ell}
\]

if \( \ell \in O_i \).
If \( n < k \), we conclude that the rank of \( y \) is less than \( k \) in \( (U_1, \ldots, U_m)_X(G) \), which is a contradiction. Therefore \( n = k \) and \( y \in \mathbb{R}_K((V_1, \ldots, V_m)_X(G)) \).

2.9. Lemma. Let \( f_i : V_1 \to F \) be linear transformations, \( i = 1, \ldots, m \)
where \( V_1 = V_{\sigma(i)} \) for all \( i \) and all \( \sigma \in G \). Let \( f : V_1 \times \cdots \times V_m \to F \)
be defined as follows:

\[
f(w_1, \ldots, w_m) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{m} f_{\sigma(i)}(w_i).
\]

Then \( f \) is multilinear and symmetric with respect to \( G \) and \( \chi \).

Proof: Note that the function \( f \) is well-defined since \( V_1 = V_{\sigma(i)} \),
\( i = 1, \ldots, m, \sigma \in G \). For each \( \alpha, \beta \in F \) and \( w_i, w_i' \in V_i \), we have

\[
f(\alpha w_i + \beta w_i', \ldots) = \sum_{\sigma \in G} \chi(\sigma) \prod_{j=1}^{m} f_{\sigma(j)}(w_j) \cdot \alpha f_{\sigma(i)}(w_i) + \beta f_{\sigma(i)}(w_i')
\]

\[
= \alpha f(w_1, \ldots, w_i, \ldots, w_m) + \beta f(w_1, \ldots, w_i', \ldots, w_m)
\]

Hence \( f \) is multilinear. Also for each \( \tau \in G \),
f(w_1(1), \ldots, w_1(m)) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{m} f_{\sigma(i)}(w_1(i))

= \sum_{\sigma \in G} \chi(\sigma) \prod_{j=1}^{m} f_{\sigma^{-1}(j)}(w_j)

= \sum_{\sigma \in G} \chi(\tau) \chi(\tau^{-1}) \prod_{j=1}^{m} f_{\sigma^{-1}(j)}(w_j)

= \chi(\tau) \sum_{\sigma \in G} \chi(\sigma^{-1}) \prod_{j=1}^{m} f_{\sigma^{-1}(j)}(w_j)

= \chi(\tau) f(w_1, \ldots, w_m).

Hence \( f \) is also symmetric with respect to \( G \) and \( \chi \).

2.10. Lemma. Let \( x = x_1 \ast \cdots \ast x_m \in (V_1, \ldots, V_m)^{\chi}(G) \). Then \( x = 0 \) implies that for some \( i \), \( \dim < O_i(x) > |O_i| \), where \( |O_i| \) denotes the number of elements in \( O_i \).

Proof: Suppose that \( \dim < O_i(x) > |O_i| \) for all \( i = 1, \ldots, t \). For each \( j \), let \( f_j : V_j \to F \) be a linear transformation such that \( f_j(x_j) = 1 \), \( f_j(x_d) = 0 \) for all \( d \) such that \( j \not\parallel d \) and \( j, d \) are in the same orbit of \( G \).

Consider the multilinear function \( f \) defined in Lemma 2.9.

By the universal factorization property of \( (V_1, \ldots, V_m)^{\chi}(G) \), there exists a linear mapping \( h : (V_1, \ldots, V_m)^{\chi}(G) \to F \) such that
h(w_1 * \cdots w_m) = f(w_1, \cdots, w_m).

Since f_{\sigma(j)}(x_j) = 1 if and only if \sigma(j) = j, it follows that

\[ \prod_{j=1}^{m} f_{\sigma(j)}(x_j) = 0 \text{ if } \sigma \neq 1. \]

Hence we have

\[ f(x_1, \cdots, x_m) = \chi(1) \prod_{j=1}^{m} f_j(x_j) = 1. \]

Therefore h(x_1 * \cdots x_m) = f(x_1, \cdots, x_m) = 1 which is a contradiction since x_1 * \cdots x_m = 0. Hence \dim < O_i(x) > < |O_i| \text{ for some } i.

2.11. Theorem. Let x_j = x_{j1} \cdots x_{jm}, j = 1, \cdots, k. Suppose that

for each i = 1, \cdots, t, \dim (\langle O_i(x_1) \rangle + \cdots + \langle O_i(x_k) \rangle) = |O_i| k. Then

\[ \sum_{j=1}^{k} x_j \in R_k((V_1, \cdots, V_m)^\chi(G)). \]

Proof: We proceed by induction with respect to k. By Lemma 2.10, the proposition is true for k = 1. Assume now it is true for k - 1 where k \geq 2. By Corollary 2.6, \[ \sum_{j=1}^{k} x_j \]

has rank equal to k or k - 1. Suppose that

\[ \sum_{j=1}^{k} x_j = \sum_{j=1}^{k-1} y_j \in R_{k-1}((V_1, \cdots, V_m)^\chi(G)) \]

where y_j = y_{j1} * \cdots * y_{jm}, j = 1, \cdots, k - 1. We divide the proof into two parts:
(i) Suppose $G$ is transitive. Let $V = V_1 = \cdots = V_m$.

Then $m \geq 2$. Suppose that for some $1 \leq i \leq k$, \[ \sum_{j \neq i} <0_1(x_j)> \subseteq \sum_{j \neq i} <0_1(y_j)> \, . \]

Then we have \[ \sum_{j \neq i} <0_1(x_j)> = \sum_{j \neq i} <0_1(y_j)> \text{ since } \dim (\sum_{j \neq i} <0_1(x_j)> ) = (k-1)m. \]

Now $x_i = -(x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_k) + y_1 + \cdots + y_{k-1}$. From Theorem 2.1, we have \[ <0_1(x_i)> \leq \sum_{j \neq i} <0_1(x_j)> + \sum_{j \neq i} <0_1(y_j)> = \sum_{j \neq i} <0_1(x_j)> , \]

which contradicts the hypothesis. Therefore for all $i = 1, \cdots, k$, \[ \sum_{j \neq i} <0_1(x_j)> \neq \sum_{j \neq i} <0_1(y_j)> . \]

Hence there are two vectors $x_{nt_1}, x_{dt_2} \neq \sum_{j \neq i} <0_1(y_j)>$ and $n \neq d$.

If $<x_{nt_1}, x_{dt_2}> + (\sum_{j \neq i} <0_1(y_j)>)$ is not a direct sum, then we can choose a vector $x_{st_3}$ such that $x_{st_3} \neq <x_{nt_1}, x_{dt_2}> + \sum_{j \neq i} <0_1(y_j)>$ since \[ \dim (<0_1(x_1)> + \cdots + <0_1(x_k)> ) = mk \text{ and } \dim (<x_{nt_1}, x_{dt_2}> + \sum_{j \neq i} <0_1(y_j)> ) < mk. \]

By the choice of $x_{st_3}$, we have both $<x_{st_3}, x_{nt_1}> + (\sum_{j \neq i} <0_1(y_j)>)$ and $<x_{st_3}, x_{dt_2}> + (\sum_{j \neq i} <0_1(y_j)>)$ are direct. Either $s \neq d$ or $s \neq n$. Therefore in any case, there are two vectors, say, $x_{\ell d_1}, x_{qd_2}$ such that \[ <x_{\ell d_1}, x_{qd_2}> + (\sum_{j \neq i} <0_1(y_j)>)$ is direct and $\ell \neq q$. \]
Consider the associated transformation $K(f) : V^m(G) \rightarrow V^m(G)$

where $f : V \rightarrow V$ is a linear transformation such that

\[ f(x_{d1}) = 0 \quad \text{and} \quad f|_{<x_{qd2}>} + \sum_{j=1}^{k-1} <0_1(y_j)> \]

is the identity mapping.

We then have $K(f)(\sum_{j=1}^{k} x_j) = K(f)(\sum_{j=1}^{k} x_j) = K(f)(\sum_{j=1}^{k} y_j) = \sum_{j=1}^{k-1} y_j$. Since

\[ \sum_{j=1}^{k-1} y_j \]

is of rank $k - 1$, it follows from Theorem 2.3 that

\[ \sum_{j=1}^{k} x_j = \sum_{j=1}^{k-1} f(x_j) + f(x_j) = \sum_{j=1}^{k-1} y_j + y_j \quad \text{and} \quad y_{d2}^j \]

hence $x_{2d}^j \in \sum_{j=1}^{k-1} y_j$, a contradiction. Therefore $\sum_{j=1}^{k} x_j$ is of rank $k$.

(ii) Suppose $G$ has at least two orbits. Consider the associated transformation $K(f_1, \ldots, f_m) : (V_1, \ldots, V_m)^m(G) \rightarrow (V_1, \ldots, V_m)^m(G)$ where

\[ f_n : V_n \rightarrow V_n \]

is a linear mapping such that

\[ f_n|_{<0_1(x_1)>} = 0, f_n|_{<0_1(x_2)> + \ldots + <0_1(x_k)>} = \text{identity if } n \in 0_1 \]

and $f_n : V_n \rightarrow V_n$ is the identity mapping if $n \notin 0_1$.

Let $K(f_1, \ldots, f_m)(\sum_{j=1}^{k} y_j) = \sum_{j=1}^{k} y_j$, where $K(f_1, \ldots, f_m)y_j = y_j$, $j = 1, \ldots, k-1$

Then $K(f_1, \ldots, f_m)(\sum_{j=1}^{k} x_j) = \sum_{j=1}^{k} x_j = \sum_{j=1}^{k-1} y_j$. By the induction hypothesis,

\[ \sum_{j=2}^{k} x_j \]

is of rank $k - 1$, hence by Theorem 2.3, $\sum_{j=2}^{k} <0_2(x_j)> = \sum_{j=1}^{k-1} <0_2(y_j)>$. 

Since \( f = \text{identity if } n \in 0, \) it follows that 
\[
\sum_{j=1}^{k-1} <0_2(y_j)> = \sum_{j=1}^{k-1} <0_2(y'_j)>
\]

Hence 
\[
\sum_{j=2}^{k} <0_2(x_j)> = \sum_{j=1}^{k-1} <0_2(y_j)> .
\]

Now \( x_1 = -(\sum_{j} x_j) + \sum_{j=1} y_j \), by Theorem 2.1, we have

\[
<0_2(x_1)> \subseteq \sum_{j=2}^{k} <0_2(x_j)> + \sum_{j=1}^{k-1} <0_2(y_j)> = \sum_{j=2}^{k} <0_2(x_j)> .
\]

This contradicts the hypothesis. Therefore \( \sum_{j=1}^{k} x_j \) is of rank \( k \).

From Theorem 2.11, we have the following known result.

2.12. Corollary. If \( x_1, \cdots, x_k \) are linearly independent and
\( y_1, \cdots, y_k \) are linearly independent, then \( \sum_{j=1}^{k} x_j \otimes y_j \) is of rank \( k \).

2.13. Remark. Theorem 2.3, Theorem 2.8 and Theorem 2.11 generalize Theorem 3, Theorem 5 and Theorem 6 in [8] respectively.

2.14. Theorem. Let \( x + y = z \) where \( x, y, z \) are non-zero decomposable elements of \( (V_1, \cdots, V_m) (G) \). Then for all \( i \), 
\[
<0_i(x)> = <0_i(y)>,
\]
except possibly for one value \( j \) of \( i \), in which case

\[
\dim <0_j(x)> \leq \dim (<0_j(x)> \cap <0_j(y)>) + 1
\]
and 
\[
\dim <0_j(y)> \leq \dim (<0_j(x)> \cap <0_j(y)>) + 1.
\]
Proof: Suppose that there exist distinct \( s \) and \( q \) such that

\[
<0_s(x)> \n perpendicular \ n perpendicular <0_s(y)> \quad \text{and} \quad <0_q(x)> \n perpendicular \ n perpendicular <0_q(y)>. 
\]

Without loss of generality, we may assume \(<0_s(x)> \n perpendicular \ n perpendicular <0_s(y)>\). Choose \( d \in \mathbb{O}_s \) such that \( x, I <0_s(y)> \) where \( x = x_1 \ast \ldots \ast x_m \).

Let \( T_d : V_d \rightarrow V_d \) be a linear transformation such that \( T_d(x_d) = 0 \) and \( T_d \mid <0_s(y)> \) is the identity mapping. Let \( T_n : V_n \rightarrow V_n \) be the identity mapping if \( n \in \mathbb{O}_s \) and \( T_n = T_d \) if \( n \in \mathbb{O}_s \). Consider the associated transformation \( K(T_1, \ldots, T_m) \) on \( (V_1, \ldots, V_m)_\chi(G) \), we have

\[
K(T_1, \ldots, T_m)(x+y) = K(T_1, \ldots, T_m)z = y \n perpendicular 0.
\]

Since \( T_n \) is the identity mapping if \( n \in \mathbb{O}_s \), by Theorem 2.3, \(<0_i(y)> = <0_i(z)> \) for all \( i, i \n perpendicular s \). In particular, \(<0_q(y)> = <0_q(z)> \n perpendicular <0_q(x)> \). By Corollary 2.2, we have \(<0_q(x)> \subseteq <0_q(y)> \n perpendicular <0_q(z)> = <0_q(y)> \). Therefore \(<0_q(z)> \n perpendicular <0_q(x)> \). Let \( z = z_1 \ast \ldots \ast z_m \). Choose \( r \in \mathbb{O}_q \) such that \( z_r \n perpendicular <0_q(x)> \). Let \( f_r : V_r \rightarrow V_r \) be a linear transformation such that \( f_r(z_r) = 0 \) and \( f_r \mid <0_q(x)> \) is the identity mapping. Let \( f_n : V_n \rightarrow V_n \) be the identity mapping if \( n \in \mathbb{O}_q \) and \( f_n = f_r \) if \( n \in \mathbb{O}_q \). Consider the associated transformation \( K(f_1, \ldots, f_m) \), we have
Therefore $K(f_1, \ldots, f_m)y = -x \not\in 0$. Since $f_n$ is the identity mapping for $n \in 0$, it follows from Theorem 2.3. that $<0_s(x)> = <0_s(y)>$. This yields a contradiction. Therefore there is possibly only one $j$ such that $<0_j(x)> \perp <0_j(y)>$.

Now, assume that such a $j$ exists and

$$\dim <0_j(x)> > 1 + \dim (<0_j(x)> \cap <0_j(y)>).$$

Then it is not hard to see that there are two independent vectors $x_d, x_p$, where $d, p \in 0$, such that $<0_j(y)> + <x_d, x_p>$ is a direct sum. Note that if $<0_j(z)> \subseteq <0_j(y)>$, then $<0_j(x)> \not\subsetneq <0_j(y)> + <0_j(z)>$, a contradiction to Corollary 2.2. Hence there is a $r \in 0$ such that $z \notin <0_j(y)>$.

We have either $<x_d> + (<z_r> + <0_j(y)>)$ is direct or $<x_p> + (<z_r> + <0_j(y)>)$ is direct. We may assume $<x_d> + (<z_r> + <0_j(y)>)$ is direct. Let $g_r : V_r \to V_r$ be a linear transformation such that $g_r(x_d) = 0$, $g_r(z_r) = 0$ and $g_r|<0_j(y)>$ is the identity mapping. Let $g_n : V_n \to V_n$ be the identity mapping if $n \notin 0$ and $g_n = g_r$ if $n \in 0$. Consider the associated transformation $K(g_1, \ldots, g_m)$, we have

$$K(g_1, \ldots, g_m)(x+y) = y = K(g_1, \ldots, g_m)z = 0,$$
a contradiction since \( y \) is assumed to be non-zero. Therefore

\[
\dim <O_j(x)> \leq 1 + \dim (<O_j(x)> \cap <O_j(y)>) , \text{ and similarly }
\]

\[
\dim <O_j(y)> \leq 1 + \dim (<O_j(x)> \cap <O_j(y)>) .
\]

The above theorem contains the known facts in tensor, Grassmann and symmetric spaces as special cases. See Lemma 3.1 [19], Lemma 5 [3] and Theorem 1.14 [4].

2.15. Theorem. Let \( K(T_1, \ldots, T_m) : (V_1, \ldots, V_m, \chi)(G) \to (V_1, \ldots, V_m, \chi)(G) \) be an associated transformation such that \( T_i : V_i \to V_i \) is non-singular for each \( i = 1, \ldots, m \). Then \( K(T_1, \ldots, T_m) \) is a rank \( k \) preserver for all possible \( k \).

Proof: Since \( K(T_1, \ldots, T_m)K(T_1^{-1}, \ldots, T_m^{-1}) = K(I, \ldots, I) \) is the identity mapping on \((V_1, \ldots, V_m, \chi)(G)\), \( K(T_1, \ldots, T_m) \) is non-singular.

Let \( z = z_1 + \cdots + z_k \in R_k((V_1, \ldots, V_m, \chi)(G)) \) where \( z_1, \ldots, z_k \) are decomposable elements. Then \( K(T_1, \ldots, T_m)z \neq 0 \). Let

\[
K(T_1, \ldots, T_m)(\sum_{i=1}^{k} z_i) = \sum_{i=1}^{j} y_i \in R_j((V_1, \ldots, V_m, \chi)(G))
\]

where \( y_1, \ldots, y_j \) are decomposable elements. Clearly \( j \leq k \).
Now, we have $K(T_1^{-1}, \ldots, T_m^{-1})K(T_1, \ldots, T_m)(z) = z$

$= K(T_1^{-1}, \ldots, T_m^{-1})(y_1 + \ldots + y_j)$.

$z$ being of rank $k$ implies that $j \geq k$. Hence $j = k$. Therefore $K(T_1, \ldots, T_m)$ is a rank $k$ preserver.

The problem of rank $k$ preservers is concerned with the converse of Theorem 2.15.

§3. Applications.

Let $U_1, U_2$ be subspaces of $U$ and $V_1, V_2$ be subspaces of $V$.

It is well-known that

(i) $(U_1 \otimes V_1) \cap (U_2 \otimes V_2) = (U_1 \cap U_2) \otimes (V_1 \cap V_2)$ \[7, p. 20\]

(ii) $\bigwedge U_1 \cap \bigwedge U_2 = \bigwedge (U_1 \cap U_2)$.

We now apply Theorem 2.3 and Theorem 2.8 to obtain a result on symmetry classes of tensors generalizing (i) and (ii).

3.1. Theorem. Let $U_i$ and $W_i$ be subspaces of $V_i$ where $U_i = U_{\sigma(i)}$, $W_i = W_{\sigma(i)}$, $V_i = V_{\sigma(i)}$ for $i = 1, \ldots, m$ and for all $\sigma \in G$. Then we have

$$(U_1, \ldots, U_m) \chi(G) \cap (W_1, \ldots, W_m) \chi(G) = (U_1 \cap W_1, \ldots, U_m \cap W_m) \chi(G).$$

Proof: It is obvious that $(U_1 \cap W_1, \ldots, U_m \cap W_m) \chi(G) \subseteq (U_1, \ldots, U_m) \chi(G) \cap (W_1, \ldots, W_m) \chi(G)$ and...
Let \( z \) be a non-zero vector of \( (U_1, \ldots, U_m)_\chi(G) \cap (W_1, \ldots, W_m)_\chi(G) \).

Then \( z \in \mathbb{R}_k((V_1, \ldots, V_m)_\chi(G)) \) for some positive integer \( k \). By Theorem 2.8, \( z \in \mathbb{R}_k((U_1, \ldots, U_m)_\chi(G)) \cap \mathbb{R}_k((W_1, \ldots, W_m)_\chi(G)) \). Therefore

\[
z = x_1 + \cdots + x_k = y_1 + \cdots + y_k \quad \text{for some} \quad x_j \in \mathbb{R}_1((U_1, \ldots, U_m)_\chi(G))
\]

and some \( y_j \in \mathbb{R}_1((W_1, \ldots, W_m)_\chi(G)) \) where \( j = 1, \ldots, k \). By Theorem 2.3, for each orbit \( O_i \) of \( G \), we have

\[
<o_i(x_1)> + \cdots + <o_i(x_k)> = <o_i(y_1)> + \cdots + <o_i(y_k)> \subseteq W_i \cap U_i, \quad q \in O_i.
\]

Therefore \( z \in (U_1 \cap W_1, \ldots, U_m \cap W_m)_\chi(G) \). This completes the proof.

3.2. Corollary. Let \( U_1 \) and \( U_2 \) be two subspaces of \( U \). Then

\[
(VU_1) \cap (VU_2) = V(U_1 \cap U_2).
\]

As an application of Corollary 2.4, we prove the following

3.3. Theorem. Let \( K(T), K(S) : V^m(G) \rightarrow V^m(G) \) be two associated transformations. Suppose (i) \( \rho(T) \), the rank of \( T \), is greater than \( m \) or (ii) \( \chi \equiv 1 \). Then \( K(T) = K(S) \) if and only if \( T = \lambda S \) for some \( \lambda \in F \) with \( \lambda^m = 1 \).

Proof: The sufficiency of theorem is trivial. We proceed to prove the necessity.
Case 1. Let \( v_1 \in V \) be such that \( T(v_1) = z_1 \neq 0 \). Extend \( z_1 \) to a basis \( z_1, \ldots, z_n \) of the range space of \( T \). By hypothesis, \( n > m \).

Let \( T(v_i) = z_i, \ i = 2, \ldots, n \). Since \( \ker(T) = \ker(S) \), we have

\[
K(T) (v_1 \cdots \widehat{v_i} \cdots v_{m+1}) = K(S) (v_1 \cdots \widehat{v_i} \cdots v_{m+1}), \ i = 2, \ldots, m+1.
\]

Hence

\[
z_1 \cdots \widehat{z_i} \cdots z_{m+1} = S_{v_1} \cdots \hat{S}_{v_i} \cdots S_{v_{m+1}}, \ i = 2, \ldots, m+1.
\]

Since \( z_1, \ldots, z_{m+1} \) are linearly independent, \( z_1 \cdots \widehat{z_i} \cdots z_{m+1} \neq 0 \) by Lemma 2.10. Therefore by Corollary 2.4, we obtain

\[
<z_1, \ldots, \widehat{z_i}, \ldots, z_{m+1}> = <S_{v_1}, \ldots, \widehat{S}_{v_i}, \ldots, S_{v_{m+1}}>, \ i = 2, \ldots, m+1.
\]

Therefore

\[
\bigcap_{i=2}^{m+1} <z_1, \ldots, \widehat{z_i}, \ldots, z_{m+1}> = \bigcap_{i=2}^{m+1} <S_{v_1}, \ldots, \widehat{S}_{v_i}, \ldots, S_{v_{m+1}}>	ag{1}
\]

Since \( z_1, \ldots, z_{m+1} \) are linearly independent, the left hand side of (1) is \( <z_1> \). Therefore the right hand side is a one dimensional subspace of \( V \). But \( \bigcap_{i=2}^{m+1} <S_{v_1}, \ldots, \widehat{S}_{v_i}, \ldots, S_{v_{m+1}}> \supseteq <S_{v_1}> \). Hence

\[
<S_{v_1}> = <z_1> = <T_{v_1}> \text{ since } S_{v_1} \text{ is obviously non-zero. By symmetry,}
\]

\[
S(u) \neq 0 \text{ implies that } <S(u)> = <T(u)> \text{ . It follows that } <T(v)> = <S(v)> \text{ for all } v \in V . \text{ This implies that } T = \lambda S \text{ for some } \lambda \in F . \text{ Clearly } \lambda^m = 1 .\]
Case 2. Suppose $\chi \equiv 1$. Let $v \in V$. Then we have

$$K(T) v \ast \ldots \ast v = K(S) v \ast \ldots \ast v = Tv \ast \ldots \ast Tv = Sv \ast \ldots \ast Sv.$$ 

If $Tv = 0$, then we have $Sv = 0$. If $Tv \neq 0$, then $Tv \ast \ldots \ast Tv \neq 0$, and hence by Corollary 2.4, $\langle Tv \rangle = \langle Sv \rangle$. Therefore we have

$$T = \lambda S$$

for some $\lambda \in \mathbb{F}$. Clearly $\lambda^m = 1$.

3.4. Remark. The above theorem (Case 1) is proved by Marcus in [11] under the assumption that the underlying field is the complex numbers.
CHAPTER II

RANK ONE PRESERVERS ON SYMMETRIC SPACES

§1. Maximal Pure Subspaces of Symmetric Spaces.

Throughout this section, let \( U \) denote a finite dimensional vector space over an algebraically closed field \( F \). Let \( \tilde{V}_U \) denote the \( r \)-fold symmetric product space over \( U \) where \( r \geq 2 \).

1.1. Definition. Let \( x_1, \ldots, x_{r-1} \) be non-zero vectors of \( U \). Then the pure subspace \( \{x_1 \cdot \cdot \cdot x_{r-1} u : u \in U\} \), denoted by \( x_1 \cdot \cdot \cdot x_{r-1} U \), is called a type one pure subspace of \( \tilde{V}_U \).

1.2. Definition. Let \( S \) be a two dimensional subspace of \( U \). Then the pure subspace \( \{s_1 \cdot \cdot \cdot s_r : s_i \in S, i = 1, \ldots, r\} \), denoted by \( S^{(r)} \), is called a type \( r \) pure subspace of \( \tilde{V}_U \).

1.3. Definition. Let \( S \) be a two dimensional subspace of \( U \). Let \( x_1, \ldots, x_{r-k} \) be vectors of \( U \) such that \( x_i \notin S, i = 1, \ldots, r-k \), where \( 1 < k < r \). Then the pure subspace \( \{x_1 \cdot \cdot \cdot x_{r-k} s_{1} \cdot \cdot \cdot s_k : s_i \in S, i = 1, \ldots, k\} \), denoted by \( x_1 \cdot \cdot \cdot x_{r-k} S^{(k)} \), is called a type \( k \) pure subspace of \( \tilde{V}_U \).

The following result is proved by L.J. Cummings in [4]:

1.4. Theorem. (i) If \( \dim U > 2 \), then every type \( i \) pure subspace of \( \tilde{V}_U \)
is a maximal pure subspace where 1 ≤ i ≤ r.

(ii) If dim U > 2 and the characteristic of F is zero or greater than r, then every maximal pure subspace of \( \mathbb{V} \) of type i for some i = 1, \ldots, r.

In this section, we shall show that if the characteristic of F is a prime \( p \leq r \), then there is one more type of maximal pure subspace besides the above mentioned ones.

1.5. Definition. For each \( u, y_1, \ldots, y_{r-k} \in U \), \( k < r \), we denote the pure vector \( u \cdots u \in \mathbb{V} \) by \( u^r \) and the pure vector \( u \cdots u \cdot y_1 \cdots y_{r-k} \in \mathbb{V} \) by \( u^r \cdot y_1 \cdots y_{r-k} \).

1.6. Theorem. Let \( \text{char } F = \text{prime } p \) and \( r > p^t \) for some positive integer t. Let \( x_1, \ldots, x_{r-p^t} \) be \( r-p^t \) non-zero vectors of U. Then the set \( M = \{ x_1 \cdots x_{r-p^t} : u \in U \} \) is a pure subspace of \( \mathbb{V} \). Moreover, \( \dim M = \dim U \).

**Proof:** Let \( k = p^t \). For any \( x, y \in U \), we have

\[
(x+y)^k = x^k + \binom{k}{1} x^{k-1} y + \cdots + \binom{k}{i} x^{k-i} y^i + \cdots + y^k
\]

\[
= x^k + y^k \quad \text{since } P \mid \binom{k}{i} \text{ for all } i = 1, \ldots, k-1 .
\]

Hence \( x_1 \cdots x_{r-k} \cdot x^k + x_1 \cdots x_{r-k} \cdot y^k = x_1 \cdots x_{r-k} \cdot (x+y)^k \).
For any non-zero $a$ in $F$, we have $ax^k = (a^k x)^k$ where $a^k$ is the $k^{th}$ root of $a$. Hence $x_1 \cdots x_{r-k} x^k = x_1 \cdots x_{r-k} (a^k x)^k$.

Therefore $M$ is a pure subspace of $V_U$.

Let $u_1, \ldots, u_n$ be a basis of $U$. Let $a_1, \ldots, a_n \in F$.

Then
\[ \sum_{i=1}^{n} a_i x_1 \cdots x_{r-k} u_i = \sum_{i=1}^{n} x_1 \cdots x_{r-k} (a_i^k u_i)^k \]
\[ = x_1 \cdots x_{r-k} (\sum_{i=1}^{n} a_i^k u_i)^k = 0 \]

implies that $\sum_{i=1}^{n} a_i^k u_i = 0$. Therefore $a_i = 0$, $i = 1, \ldots, n$.

Also for any $y$ in $U$, we have $y = \sum_{i=1}^{n} \beta_i u_i$ for some $\beta_i$ in $F$, $i = 1, \ldots, n$. Therefore

\[ x_1 \cdots x_{r-k} y^k = x_1 \cdots x_{r-k} (\sum_{i=1}^{n} \beta_i u_i)^k \]
\[ = \sum_{i=1}^{n} x_1 \cdots x_{r-k} (\beta_i u_i)^k \]
\[ = \sum_{i=1}^{n} \beta_i^k x_1 \cdots x_{r-k} u_i^k. \]

Hence $x_1 \cdots x_{r-k} u_i^k$, $i = 1, \ldots, n$, is a basis of $M$. 
The pure subspace in Theorem 1.6 will be denoted by 

\[ x_1 \cdots x_{r-p} U^p \]

1.7. Definition. Let \( x = x_1 \cdots x_r \) be a non-zero pure vector of \( V_U \).

We shall use \( U(x) \) to denote \( <x_1, \ldots, x_r> \). The one dimensional subspaces \( <x_i> \), \( i = 1, \ldots, r \) are called the factors of \( x \). A one dimensional subspace \( \langle u \rangle \) is said to be a factor of \( x \) of multiplicity \( k \) if \( x = u^k y_1 \cdots y_{r-k} \) where \( <y_i> \not= \langle u \rangle \), \( i = 1, \ldots, r - k \).

The above definition is well-defined since \( x_1 \cdots x_r = z_1 \cdots z_r \neq 0 \) implies that \( <x_i> = <z_\sigma(i)> \) for some \( \sigma \in S_r \), \( i = 1, \ldots, r \) [4, Corollary 1.9].

Let \( u_1, \ldots, u_n \) be a basis of \( U \). Let \( V_U \) denote the symmetric algebra over \( U \) and \( F[\xi_1, \ldots, \xi_n] \) denote the polynomial algebra in \( n \) determinates \( \xi_1, \ldots, \xi_n \) over \( F \). Then there exists a unique isomorphism \( \phi : V_U \rightarrow F[\xi_1, \ldots, \xi_n] \) such that

\[ \phi u_i = \xi_i \quad (i = 1, \ldots, n) \]

The restriction of \( \phi \) to \( \overline{V_U} \) is an isomorphism of \( \overline{V_U} \) onto the vector space of homogeneous polynomials of degree \( r \) (see [7, pp. 202-203]).
Clearly \( x_1 \cdots x_r \) is a non-zero pure vector of \( \mathbb{V}U \) if and only if \( \phi(x_1 \cdots x_r) \) is a product of \( r \) linear homogeneous polynomials in \( F[\xi_1, \cdots, \xi_n] \). In the Gaussian domain \( F[\xi_1, \cdots, \xi_n] \), linear homogeneous polynomials are irreducible elements.

1.8. Lemma. Let \( x_1, \cdots, x_k \) be \( k \) non-zero vectors of \( U \). Let \( r > k + 1 \) and \( x_1 \cdots x_k \cdot A = z_1 \cdots z_r \neq 0 \) where \( A \in \mathbb{V}U \) and \( z_i \in U \). Then \( \langle x_1 \rangle, \cdots, \langle x_k \rangle \) are \( k \) factors of \( z_1 \cdots z_r \) and \( A \) is a pure vector of \( \mathbb{V}U \).

Proof: Consider the isomorphism \( \phi \) from \( \mathbb{V}U \) onto \( F[\xi_1, \cdots, \xi_n] \). We have

\[
\phi(x_1 \cdots x_k \cdot A) = \phi(z_1 \cdots z_r) \neq 0 .
\]

Therefore \( \phi(x_1) \cdots \phi(x_k) \cdot \phi(A) = \phi(z_1) \cdots \phi(z_r) \neq 0 \). Since \( F[\xi_1, \cdots, \xi_n] \) is a Gaussian domain and since \( \phi(x_1), \cdots, \phi(x_k), \phi(z_1), \cdots, \phi(z_r) \) are linear homogeneous polynomials, it follows that for each \( i = 1, \cdots, k \), \( \langle \phi(x_i) \rangle = \langle \phi(z_j) \rangle \) for some \( j_i \) where \( 1 \leq j_i < r \) and \( j_t \neq j_s \) if \( t \neq s \), and \( \phi(A) \) is a product of \( r - k \) linear homogeneous polynomials.

Hence \( \langle x_1 \rangle, \cdots, \langle x_k \rangle \) are \( k \) factors of \( z_1 \cdots z_r \) and \( A \) is a pure vector of \( \mathbb{V}U \).
The following Lemma is equivalent to Theorem 1.17 and Theorem 1.18 in [4]. The proof given here is rather simple.

1.9. Lemma. Let \( x \) and \( y \) be non-zero pure vectors of \( \mathfrak{F}U \) such that \( \dim (U(x) + U(y)) \geq 3 \) and \( \langle x, y \rangle \) is a pure subspace. Then \( x \) and \( y \) have a common factor.

Proof: If \( x \) and \( y \) are linearly dependent, then clearly \( x \) and \( y \) have a common factor. Hence we assume \( x \) and \( y \) are linearly independent. Let \( x = x_1 \cdots x_r \) and \( y = y_1 \cdots y_r \). Since \( \dim (U(x) + U(y)) \geq 3 \), either \( \dim U(x) \geq 2 \) or \( \dim U(y) \geq 2 \). We may assume that \( \dim U(y) \geq 2 \). Then for some \( i, j, k, x_i, y_j, y_k \) are linearly independent. Without loss of generality, we may assume \( i = 1, j = 1, k = 2 \). Extend \( y_1, y_2, x_1 \) to a basis \( u_1, \ldots, u_n \) of \( U \) where \( y_1 = u_1, y_2 = u_2 \) and \( x_1 = u_3 \).

For each non-zero \( \lambda \in F \), \( x + \lambda y \) is a pure vector by hypothesis. Let \( x + \lambda y = z(\lambda) = z_1(\lambda) \cdots z_r(\lambda) \). Also let \( x_i = \sum_{t=1}^{n} a_{it} u_t \) and \( z_i(\lambda) = \sum_{t=1}^{n} a_{it}(\lambda) u_t \) where \( i = 1, \ldots, r \). Let \( f : U \to U \) be the linear transformation such that \( f(u_1) = 0 \) and \( f(u_i) = u_i, i \geq 2 \).

Consider the induced transformation \( P_r(f) \) on \( \mathfrak{F}U \), we have

\[ P_r(f)(x+\lambda y) = P_r(f)(z(\lambda)) \]
It follows that \( x_1 \cdot f(x_2) \cdots f(x_r) = f(z_1(\lambda)) \cdots f(z_r(\lambda)) \). If 
\( f(x_j) = 0 \) for some \( j \), then \( x \) has a factor \( <y_1> \) and we are done. Hence we may assume \( f(x_j) \neq 0 \) for all \( j = 1, \cdots, r \). Without loss of generality, we may choose vectors \( z_1(\lambda), \cdots, z_r(\lambda) \) such that Corollary 1.9 of [4] implies that \( x_1 = f(z_1(\lambda)) \). Hence we have \( z_1(\lambda) = x_1 + a_{11}(\lambda) u_1 \).

Now, let \( g : U \to U \) be the linear transformation such that 
\( g(u_2) = 0 \) and \( g(u_i) = u_i, i = 1, 3, \cdots, n \). Consider the induced transformation \( P_r(g) \) on \( V U \), we have

\[
P_r(g) (x + \lambda y) = P_r(g) (z(\lambda)) .
\]

It follows that \( g(x_1) \cdots g(x_r) = g(z_1(\lambda)) \cdots g(z_r(\lambda)) \).

If \( g(x_j) = 0 \) for some \( j \), then \( x \) has a factor \( <y_2> \) and we are done. Hence we may assume that \( g(x_j) \neq 0 \) for all \( j = 1, \cdots, r \). Therefore 
\( g(z_1(\lambda)) = \eta g(x_{k\lambda}) \) for some non-zero \( \eta \in F \) and for some \( k_\lambda \) where 
\( 1 \leq k_\lambda \leq r \). It then follows that

\[
z_1(\lambda) = x_1 + a_{11}(\lambda) u_1 = \eta \left( \sum_{t=2}^{n} a_{k_\lambda t} u_t \right) .
\]

Hence \( z_1(\lambda) = \eta (a_{k_\lambda 1} u_1 + a_{k_\lambda 3} x_1) \). This shows that \( z(\lambda) \) has a factor
<a_{k_1} u_1 + a_{k_2} x_1>. Since \{<a_{1}x_1 + a_{1}u_1>, \ldots, <a_{r}x_1 + a_{r}u_1>\}
is a finite family and F is an infinite field, we see that there are
distinct non-zero \lambda_1, \lambda_2 \in F such that z(\lambda_1) and z(\lambda_2) have
a common factor. But <x, y> = <z(\lambda_1), z(\lambda_2)>, we conclude from Lemma
1.8 that x and y have a common factor.

1.10. Theorem. Let \text{dim } U > 2 \text{ and } \text{char } F = \text{prime } p. \text{ Let } x_1, \ldots, x_r
be non-zero vectors of U where r > p^t, t a positive integer. Then
M = x_1 \cdots x_r u^p \text{ is a maximal pure subspace of } VU.

Proof: Let \( y = y_1 \cdots y_r \) be a non-zero pure vector in \( VU \) such that
M \cup \{y\} spans a pure subspace. We shall show that \( y \in M \).

Let \( U_{ij} = \langle x_i, x_j \rangle \) and \( W_{ik} = \langle x_i, y_k \rangle, i, j = 1, \ldots, r - p^t,\)
k = 1, \ldots, r. Let \( V_{it} = \langle y_i, y_t \rangle, i \not\equiv t, i, t = 1, \ldots, r. \)

Since F is algebraically closed, F is an infinite field. Since
\text{dim } U \geq 3, U \nsubseteq (U_{ij}) \cup (W_{ik}) \cup (V_{it}). Choose a vector u \in U such that
u \nsubseteq (U_{ij}) \cup (W_{ik}) \cup (V_{it}). Then we consider the pure vectors
x = x_1 \cdots x_r u^p \text{ and } y.

If for all i and j, \( <x_i> \not\subseteq <y_j> \), then \( y \in M \) and we
are done. Hence we assume now \( <x_{i_0}> \not\subseteq <y_{j_0}> \) for some \( i_0 \) and \( j_0 \).
Since \( u, x_i, y_j \) are linearly independent, it follows from Lemma 1.9 that \( x \) and \( y \) have a common factor. Let \( \langle u_1 \rangle, \cdots, \langle u_k \rangle \) be all the common factors (counting multiplicities) of \( x \) and \( y \). Let

\[
x = u_1 \cdots u_k x', \quad y = u_1 \cdots u_k y'
\]

where \( x' \) and \( y' \) are non-zero pure vectors of \( V^U \). Then \( x' \) and \( y' \) have no common factors. Since \( \langle u \rangle \) is not a factor of \( y \), we see that \( \langle u \rangle \) is a factor of \( x' \) of multiplicity \( p^t \). Since \( \langle x, y \rangle \) is a pure subspace, it follows from Lemma 1.8 that \( \langle x', y' \rangle \) is a pure subspace of \( V^U \). Suppose for some \( d, \langle x_d \rangle \) is a factor of \( x', 1 \leq d \leq r - p^t \).

Let \( \langle y_j \rangle \) be a factor of \( y' \). Then \( u, x_d, y_j \) are linearly independent. This implies that \( \dim (U(x') + U(y')) \geq 3 \) and \( x' \) and \( y' \) have a common factor, a contradiction. Therefore \( x' = \lambda u^p \) for some \( \lambda \) in \( F \) and \( u_1 \cdots u_k = \eta x_1 \cdots x_k \) for some \( \eta \in F \). Since \( \dim (U(x') + U(y')) \leq 2 \), by our choice of \( u \), we see that \( y' = z^p \) for some \( z \) in \( U \). Consequently \( y \in M \). Hence \( M \) is a maximal pure subspace.

1.11. Theorem. Let \( \dim U > 2 \) and \( \text{char } F = \text{prime } p \). Then \( M = \{x^p^t : x \in U \} \) is a maximal pure subspace of \( V^U \) and \( \dim M = \dim U \).

Proof: That \( M \) is a pure subspace and \( \dim M = \dim U \) can be seen from the proof of Theorem 1.6. Suppose \( M \) is properly contained in a
pure subspace \( N \). Let \( x \) be a non-zero vector of \( U \). Then \( x \cdot U^{pt} \) is properly contained in the pure subspace \( \{x \cdot A : A \in N\} \), contradicting the fact that \( x \cdot U^{pt} \) is a maximal pure subspace of \( U^{p+1} \). Therefore \( M \) is a maximal pure subspace.

If \( \text{char } F = p \), we shall denote \( \{x^{pt} : x \in U\} \) by \( U^{pt} \). \( U^{pt} \) and the pure subspace in Theorem 1.6 are called power type pure subspaces of degree \( t \).

The following elementary fact will be needed.

1.12. Lemma. Let \( V \) be a \( k \)-dimensional vector space over an infinite field. Then there is an infinite subset \( W \) of \( V \) such that any \( k \) elements in \( W \) are linearly independent.

1.13. Theorem. Let \( U \) be an \( n \)-dimensional vector space over an algebraically closed field \( F \). Let \( n \geq 3 \) and \( M \) be a maximal pure subspace of \( V U \). Then \( M \) is of type \( i \) for some \( 1 \leq i \leq r \), or \( M \) is of power type of degree \( t \) for some positive integer \( t \).

Proof: Case 1. Suppose that for all non-zero pure vectors \( x \) and \( y \) in \( M \), \( \dim (U(x) + U(y)) \leq 2 \). We have two subcases:

(a) For some non-zero \( x \in M \), \( \dim U(x) = 2 \). Let \( y \) be any non-zero element in \( M \). Since \( \dim (U(x) + U(y)) \leq 2 \), it follows that \( U(x) \supset U(y) \). Let \( S = U(x) \). Then \( M \subseteq S^{(r)} \). Since both \( M \) and \( S^{(r)} \)
are maximal pure subspaces of \( V^U \), we conclude that \( M = S_r \).

(b) For all non-zero \( x \in M \), \( \dim U(x) = 1 \). Then \( M \subseteq \{ u^r : u \in U \} \).

Since \( M \) is a maximal pure subspace, \( \dim M \neq 1 \); for otherwise \( M \) is property contained in a type one pure subspace. Hence there are two independent pure vectors \( y^r \) and \( z^r \) in \( M \). Clearly \( y \) and \( z \) are linearly independent. Note that we have \( y^r + z^r = u^r \) for some non-zero \( u \in U \). In view of Corollary 1.2.2, \( \langle y, z \rangle \supsetneq \langle u \rangle \). Therefore \( u = az + by \) for some \( a, b \in F \). It then follows that

\[
(az + by)^r = a^r z^r + \cdots + (^r_k)(az)^{r-k}b^k y^r = z^r + y^r.
\]

Hence \( a^r = b^r = 1 \) and \( (^r_k)a^{-k}b^k = 0 \), \( k = 1, \ldots, r - 1 \) since \( z^r, z^{r-1}y, \ldots, z^1y, y^r \) are linearly independent vectors of \( V^U \). Thus \( ^r_k = 0 \) for all \( k = 1, \ldots, r - 1 \). This implies that characteristic \( F = \text{prime } p \), \( r = p^t \) for some positive integer \( t \) and hence \( M = U^{p^t} \) in this case.

**Case 2.** There exists a pair of non-zero pure vectors \( x \) and \( y \) of \( M \) such that \( \dim (U(x)+U(y)) \geq 3 \). If \( \dim U(x) = 2 \) and \( \dim U(y) = 1 \), then by Theorem 1.2.14, \( U(x) \supsetneq U(y) \), a contradiction. Similarly it is impossible that \( \dim U(y) = 2 \) and \( \dim U(x) = 1 \).

Let \( z \) be a non-zero vector in \( M \). We shall show that
z has a common factor with x or y. If \( \dim (U(z)+U(x)) \geq 3 \) or \( \dim (U(z)+U(y)) \geq 3 \), then by Lemma 1.9, we are done. Now, assume 
\[ \dim (U(z)+U(y)) < 3 \text{ and } \dim (U(z)+U(x)) < 3. \]
Then \( \dim U(x) = \dim U(y) = 2 \) and \( U(x) \nmid U(y) \). It follows that \( U(x) \supseteq U(z) \) and \( U(y) \supseteq U(z) \); for otherwise \( \dim (U(x)+U(z)) \geq 3 \) or \( \dim (U(y)+U(z)) \geq 3 \). Hence \( \dim U(z) = 1 \) and \( U(x) \cap U(y) = U(z) \). Since x and y have a common factor, say \( <f> \), (Lemma 1.9), we see that \( <f> = U(z) \). Therefore \( U(z) \) is a common factor of \( x, y \) and \( z \).

Assume now \( \dim M = m \). By Lemma 1.12, let \( z_1, \ldots, z_n, \ldots \) be an infinite subset of \( M \) such that any \( m \) \( z_i \) forms a basis of \( M \). By the above argument, for each \( i \), \( z_i \) has a common factor with \( x \) or \( y \).

Since \( x \) and \( y \) have at most \( 2r \) distinct factors, it follows that there are infinitely many \( z_i \) with a common factor, say \( <x_1> \). Hence there is a basis \( z_{i_1}, \ldots, z_{i_m} \) of \( M \) with a common factor \( <x_1> \). By Lemma 1.8, we see that \( <x_1> \) is a factor of any non-zero pure vector in \( M \). Let \( <x_1>, \ldots, <x_j> \) be all the common factors of non-zero elements of \( M \).

If \( j = r-1 \), then clearly \( M = x_1 \cdots x_{r-1} U \). If \( j < r-1 \), let 

\[ M' = \{v_1 \cdots v_{r-j} : x_1 \cdots x_j v_{r-j} \in M \} \]

By Lemma 1.8, we see that \( M' \) is a pure subspace of \( V U \). Since \( M \) is a maximal pure subspace, it follows that \( M' \) is also a maximal pure
subspace. Now \( \dim(U(x') + U(y')) \leq 2 \) for all non-zero \( x', y' \in M' \); for otherwise, by the same argument as above, there is some non-zero \( v \in U \) such that \( \langle v \rangle \) is a common factor for all \( A \in M' \), \( A \neq 0 \). Hence by the same argument as in case 1, \( M' = S_{(r-j)} \) for some two dimensional subspace \( S \) of \( U \) or \( M' = U^{p^s} \) where \( r - j = p^s \) for some positive integer \( s \) and \( \text{char } F = \text{prime } p \). Thus

\[
M = x_1 \cdots x_j \cdot S_{(r-j)} \quad \text{or} \quad M = x_1 \cdots x_{r-p} \cdot U^{p^s}.
\]

In case \( M = x_1 \cdots x_j \cdot S_{(r-j)} \), then we must have \( x_1, \ldots, x_j \notin S \); for if some \( x_i \in S \), then \( M \) is properly contained in the pure subspace

\[
x_1 \cdots \hat{x_i} \cdots x_j \cdot S_{(r-j+1)},
\]

a contradiction to the fact that \( M \) is maximal.

This completes the proof.

§2. Intersections of Maximal Pure Subspaces.

In this section, we study the intersection properties of maximal pure subspaces and determine the form of an infinite family of maximal pure subspaces of type one or power type such that any two members of them have a non-zero intersection. These results will be used in a latter section. Unless otherwise stated, throughout this section, \( U \) is assumed to be a finite dimensional vector space over an algebraically closed field \( F \) such that \( \dim U > 2 \).
2.1. Theorem. Two power type pure subspaces $M = u_1 \cdots u_{r-p} U^t$ and $N = v_1 \cdots v_{r-p} U^t$ (r > p^t) of $\mathbb{R}^U$ are equal if and only if

$$u_1 \cdots u_{r-p} t = \lambda v_1 \cdots v_{r-p} t$$

for some $\lambda \in F$.

Proof: The sufficiency is clear. We prove the necessity. Choose a vector $z \in U$ such that $z \notin \left< v_1 \right> \cup \cdots \cup \left< v_{r-p} \right>$. Then $M = N$ implies that

$$u_1 \cdots u_{r-p} t z_{r-p}^t = v_1 \cdots v_{r-p} t w_{r-p}^t \neq 0$$

for some $w \in U$. Since $<z> \notin <v_i>$ for all $i = 1, \ldots, r-p^t$, we have $<z> = <w>$. Therefore

$$u_1 \cdots u_{r-p} t = \lambda v_1 \cdots v_{r-p} t$$

for some $\lambda \in F$.

2.2. Theorem. Let $M = x_1 \cdots x_{r-p} U^t$ and $N = y_1 \cdots y_{r-p} U^\ell$ be two distinct power type pure subspaces of $\mathbb{R}^U$ where $t \geq \ell$ and $r \geq p^t$. Then $\dim (M\cap N) = 1$ iff either (i) $t > \ell$ and

$$y_1 \cdots y_{r-p} \ell = \lambda x_1 \cdots x_{r-p} f_{p-t}^\ell$$

for some $f \in U, \lambda \in F$; or

(ii) $r \geq p^t + p^\ell$ and $x_1 \cdots x_{r-p} t = z_1 \cdots z_{r-(p+t^\ell)} a_{p^\ell}$

$$y_1 \cdots y_{r-p} \ell = z_1 \cdots z_{r-(p+t^\ell)} f_{p^\ell}^t$$

for some $a, f, z_1 \in U$. Otherwise $M \cap N = 0$. 
Proof: Suppose that \( MN \neq 0 \). Then there are non-zero vectors \( f \) and \( a \in U \) such that \( x_1 \cdots x_{r-p} f^t = y_1 \cdots y_{r-p} a^p \neq 0 \).

Either \( \langle f \rangle = \langle a \rangle \) or \( \langle f \rangle \neq \langle a \rangle \). If \( \langle f \rangle = \langle a \rangle \), then \( t \neq l \), otherwise by Theorem 2.1, \( M = N \), a contradiction to the hypothesis. Hence \( t > l \) and

\[
y_1 \cdots y_{r-p} = \lambda x_1 \cdots x_{r-p} f^t - p^l \text{ for some } \lambda \in F.
\]

If \( \langle f \rangle \neq \langle a \rangle \), then clearly \( r \geq p + p \) and for some \( z_1, i = 1, \ldots, r - (p + p) \),

\[
x_1 \cdots x_{r-p} = z_1 \cdots z_{r-(p+p)} a^p f^t.
\]

Conversely, if (i) holds, then \( MN = \langle x_1 \cdots x_{r-p} f^t \rangle \) and if (ii) holds, then \( MN = \langle z_1 \cdots z_{r-(p+p)} a^p f^t \rangle \).

2.3. Theorem. Let \( M = x_1 \cdots x_{r-p} u^t \) and \( N = y_1 \cdots y_{r-1} u \) be two maximal pure subspaces of \( VU \) where \( r \geq p \) and \( t \) is a positive integer. Then \( \dim (MN) = 1 \) if and only if either (i) \( y_1 \cdots y_{r-1} = \lambda x_1 \cdots x_{r-1} a^{p-1} \)

for some \( a \in U, \lambda \in F \) or (ii) \( r > p \) and \( y_1 \cdots y_{r-1} = x_1 \cdots x_i \) \( a^t \) for some \( a \in U \) and some \( i \) where \( 1 \leq i \leq r - p \). Otherwise \( MN = 0 \).
Proof: The argument is the same as that given in the proof of Theorem 2.2.

2.4. Theorem. Let \( M = x_1 \cdots x_{r-p} U^t_{r-p} \) be a power type subspace where \( r > p \) and \( N = S \) be a type \( r \) subspace of \( U \) where \( S \) is a two dimensional subspace of \( U \). Then

\[
\dim (M \cap N) = 2 \text{ if and only if } x_1, \ldots, x_{r-p} \in S.
\]

Otherwise \( M \cap N = 0 \).

Proof: If \( M \cap N \neq 0 \), then let \( A \in M \cap N \) and \( A \neq 0 \). We have

\[
A = x_1 \cdots x_{r-p} u^t_{r-p} = s_1 \cdots s_r
\]

for some \( u \in U \), \( s_i \in S \), \( i = 1, \ldots, r \). Clearly this implies that \( x_1, \ldots, x_{r-p} \in S \). Conversely, if \( x_1, \ldots, x_{r-p} \in S \), then

\[
M \cap N = \langle x_1 \cdots x_{r-p} v_1^t, x_1 \cdots x_{r-p} v_2^t \rangle \text{ where } S = \langle v_1, v_2 \rangle.
\]

Hence \( \dim (M \cap N) = 2 \).

2.5. Theorem. Let \( M = U^t_{p^t} \) be a power type pure subspace and let \( N = y_1 \cdots y_{p^t-k} S^{(k)}_{p^t-k} \) be a type \( k \) pure subspace of \( U \) where \( p^t > k > 1 \) and \( S \) is a two dimensional subspace of \( U \). Then (i) \( M \cap N = 0 \) if \( p^t > k \);
and (ii) $M \cap N = \langle s_1^p, s_2^p \rangle$ if $p^t = k$ where $S = \langle s_1, s_2 \rangle$.

**Proof:** (i) Suppose that $p^t > k$ and $M \cap N \neq 0$. Then for some non-zero $u \in U$ and non-zero $v_1, \ldots, v_k \in S$, $u^p = y_1 \cdots y_{p^t-k} v_1 \cdots v_k \neq 0$. This implies $\langle u \rangle = \langle y_1 \rangle = \langle v_1 \rangle$, contradicting the fact that $y_1 \notin S$. Hence $M \cap N = 0$ if $p^t > k$. (ii) is obvious.

2.6. **Definition.** Let $x = x_1 \cdots x_m$ be a non-zero pure vector of $WU$. Suppose that $x = u_1 \cdots u_t \cdot z_{t+1} \cdots z_m$ where $t < m$. Then we denote the pure vector $z_{t+1} \cdots z_m$ in $m_{t+1}^U$ by $x_1 \cdots x_m \mid u_1 \cdots u_t$. The notion $x|d$ products will stand for $x|v_1 \cdots v_d$ for some $v_1, \ldots, v_d \in U$ where $\langle v_1 \rangle, \ldots, \langle v_d \rangle$ are $d$ factors of $x$. This definition is well-defined since $u_1 \cdots u_t \cdot z_{t+1} \cdots z_m = u_1 \cdots u_t \cdot w_{t+1} \cdots w_m \neq 0$ implies that $z_{t+1} \cdots z_m = w_{t+1} \cdots w_m$.

2.7. **Theorem.** Let $M = x_1 \cdots x_{r^p_t}$ be a power type pure subspace of $r^p_t WU$. Let $N = y_1 \cdots y_{r-k} \cdot S(k)$ be a type $k$ pure subspace where $r > k \geq p^t$ and $S$ is a two dimensional subspace of $U$. Let $x = x_1 \cdots x_{r-p^t}$ and $y = y_1 \cdots y_{r-k}$. Then $\dim (M \cap N) \leq 2$. Moreover, we have

(i) $\dim (M \cap N) = 2$ if and only if $x$ and $y$ have exactly $r - k$
common factors while the remaining \( k - p \) factors of \( x \) are contained in \( S \).

(ii) \( \dim (M \cap N) = 1 \) if and only if \( r \geq p + k \) and \( x, y \) have exactly \( r - k - p \) common factors while the remaining \( k \) factors of \( x \) are contained in \( S \) and the remaining \( p \) factors of \( y \) are the same.

Proof: (i) Suppose \( \dim (M \cap N) \geq 2 \). Let \( A, B \) be two independent vectors in \( M \cap N \). Then there are two independent vectors \( u, v \) of \( U \) and \( s_1, \ldots, s_k, s'_1, \ldots, s'_k \in S \) such that

\[
A = x_1 \cdots x_{r-p} u_{p}^t = y_1 \cdots y_{r-k} s_1 \cdots s_k, \\
B = x_1 \cdots x_{r-p} v_{p}^t = y_1 \cdots y_{r-k} s'_1 \cdots s'_k.
\]

If both \( u \) and \( v \in S \), then

\[
x_1 \cdots x_{r-p} = (y_1 \cdots y_{r-k} u_{p}^t) \cdot s_1 \cdots s_k = (y_1 \cdots y_{r-k} v_{p}^t) \cdot s'_1 \cdots s'_k.
\]

Since \( y_1 \notin S \), the above equality implies that

\[
y_1 \cdots y_{r-k} u_{p}^t = \lambda y_1 \cdots y_{r-k} v_{p}^t \quad \text{for some} \quad \lambda \in F.
\]

Clearly this is a contradiction. Therefore either \( u \in S \) or \( v \in S \).
We may assume that \( u \in S \). Since \( <u> \not\subseteq <y_i> \), \( i = 1, \ldots, r - k \), it follows from (1) that \( x \) and \( y \) have exactly \( r - k \) common factors while the remaining \( k - p^t \) factors of \( x \) are contained in \( S \). Conversely if \( x \) and \( y \) have \( r - k \) common factors, say, \( x_1 \cdots x_{r-k} = \lambda y_1 \cdots y_{r-k} \), and the remaining \( k - p^t \) factors \( <x_{r-k+1}>, \ldots, <x_{r-p^t}> \) are contained in \( S \), then by Theorem 2.4 (or Theorem 2.5 (ii) if \( p^t = k \)), we have

\[
\dim (x_{r-k+1} \cdots x_{r-p^t} \cup^t \cap S(k)) = 2.
\]

Therefore \( \dim (M \cap N) = 2 \).

(ii) Suppose \( M \cap N = <A> \) where \( A \neq 0 \). Then we have

\[
A = x_1 \cdots x_{r-k} \cup^t = y_1 \cdots y_{r-k} \cdot s_1 \cdots s_k \quad (2)
\]

where \( s_1, \ldots, s_k \in S \). If \( u \in S \), by the argument in (i), \( \dim (M \cap N) = 2 \) a contradiction. Therefore \( u \notin S \). From (2), we see that \( x \) and \( y \) have exactly \( r - k - p^t \) common factors while the remaining \( k \) factors of \( x \) are contained in \( S \) and the remaining \( p^t \) factors of \( y \) are equal.

The converse of the statement is easily checked.
2.8 Theorem. Let \( M = \underbrace{x_1 \cdots x}_{r-p^t} U^{p^t} \) and \( N = \underbrace{y_1 \cdots y}_{r-k} S^{(k)} \) be two maximal pure subspaces where \( r > p^t > k > 1 \) and \( S \) is a two dimensional subspace of \( U \). Let \( x = \underbrace{x_1 \cdots x}_{r-p^t} \) and \( y = \underbrace{y_1 \cdots y}_{r-k} \).

Then \( \dim (M \cap N) = 1 \) iff \( r > k + p^t \), \( x \) and \( y \) have exactly \( r - k - p^t \) common factors while the remaining \( k \) factors of \( x \) are contained in \( S \) and the remaining \( p^t \) factors of \( y \) are equal. Otherwise \( M \cap N = 0 \).

Proof: Suppose \( M \cap N \neq 0 \). Let \( A \) be a non-zero element of \( M \cap N \). Then

\[ A = \underbrace{x_1 \cdots x}_{r-p^t} U^{p^t} = \underbrace{y_1 \cdots y}_{r-k} S \]

for some \( u \) in \( U \) and \( s_i \) in \( S \). Since \( p^t > k \) and \( y_i \notin S \), it follows that \( u \notin S \). Hence \( y \) has \( p^t \) factors of \( \langle u \rangle \). Hence \( x \) and \( y \) have exactly \( r - k - p^t \) common factors while the remaining \( k \) factors of \( x \) are contained in \( S \) and the remaining \( p^t \) factors of \( y \) are equal.

The converse of the theorem is easily checked.

Combining Theorem 2.1 - Theorem 2.5, Theorems 2.7, 2.8 together with the results in section 2 of Chapter II of [4], we have:

2.9. Theorem. Let \( U \) be a finite dimensional vector space over an algebraically closed field where \( \dim U > 2 \). Then the maximal dimension of
the intersections of any two distinct maximal pure subspaces is 2.

2.10. Remark. Let \( n \) be a positive integer. Let \( A_1, \ldots, A_k \) be a collection of sets such that \( |A_i| = n, i = 1, \ldots, k \) and \( |A_i \cap A_j| = n - 1 \) if \( i \neq j \). If \( k > n + 1 \), then there exists a set \( W \) such that \( W \subseteq \bigcap_{i=1}^{k} A_i \) and \( |W| = n - 1 \).

2.11. Definition. Let \( C_1(\mathring{VU}) \) denote the collection of all type one pure subspaces of \( \mathring{VU} \). For each positive integer \( t \), let \( P_t(\mathring{VU}) \) denote the collection of all power type pure subspaces of degree \( t \) in \( \mathring{VU} \).

2.12. Theorem. Let \( C \subseteq P_t(\mathring{VU}) \) be an infinite family such that \( M_1, M_2 \in C \) implies that \( M_1 \cap M_2 \neq 0 \). Then \( r \geq 2p^t \) and there exist \( r - 2p^t \) non-zero vectors \( y_1, \ldots, y_{r-2p^t} \) with the property that \( M \in C \) implies that

\[
M = y_1 \cdots y_{r-2p^t} a_{M}^{p^t} u_{M}^{p^t}
\]

for some \( a_{M} \in U \). (\( y_1, \ldots, y_{r-2p^t} \) are deleted if \( r = 2p^t \)).

**Proof:** It follows from Theorem 2.2 that \( r \geq 2p^t \). If \( r = 2p^t \), the assertion is clear from Theorem 2.2. Hence we assume \( r > 2p^t \).
For each \( M = x_1 \cdots x_{r-p} U^t \in C \), define a set \( M^* = \{ <x_1^t>, \cdots, <x_{r-p}^t> \} \). This is well-defined because of Theorem 2.1.

Let \( C^* = \{ M^* : M \in C \} \).

We claim that \( C^* \) is infinite. For if \( C^* \) is finite and \( C^* = \{ M_1^*, \ldots, M_j^* \} \) where \( M_i \in C \), \( i = 1, \ldots, j \), then for any \( M = z_1 \cdots z_{r-p} U^t \), we have \( <z_i^t> \in \bigcup_{i=1}^j M_i^* \). But \( \bigcup_{i=1}^j M_i^* \) is a finite set, it follows that \( C \) is finite, a contradiction to the hypothesis.

Let \( N \in C \) such that \( N^* \) has the maximal cardinality \( n \) in \( C^* \). Let \( M \in C \). Since \( M \cap N \neq \emptyset \), by Theorem 2.2,

\[
M = z_1 \cdots z_{r-2p} U^t \quad (1)
\]

\[
N = z_1 \cdots z_{r-2p} a U^t
\]

for some \( z_i \) in \( U \), \( i = 1, \ldots, r-2p^t \) and some \( a, b \) in \( U \).

Since \( |N^*| = n \), we have \( |\{ <z_1^t>, \cdots, <z_{r-2p}^t> \}| \geq n - 1 \).

Observe that \( |M^*| \geq n - 1 \) and

(i) if \( |M^*| = n - 1 \), then \( M^* = \{ <z_1^t>, \cdots, <z_{r-2p}^t> \} \) and hence \( N^* \supseteq M^* \);
(ii) if \( M^* \) contains an element \( <c> \) such that \( <c> \notin N^* \), then we have \( <b> = <c> \) from (1), and hence \( |M^*| = n \).

Since \( C^* \) is infinite, we observe from (ii) that there are infinitely many \( M^* \in C^* \) such that \( |M^*| = n \). Note that if \( M_1, M_2 \in C \) are such that \( |M_1^*| = |M_2^*| = n \) and \( M_1 \neq M_2 \), then \( |M_1^* \cap M_2^*| = n - 1 \).

By Remark 2.10 and (i), we conclude that

\[
|\bigcap_{M \in C} M^*| = n - 1.
\]

Now let \( N_1, N_2 \in C \) such that \( |N_1^*| = |N_2^*| = n \), \( N_1 \neq N_2 \) and

\[
N_1 = y_1 \cdots y_{r-2p} c^p t u^p t
\]
\[
N_2 = y_1 \cdots y_{r-2p} d^p t u^p t
\]

for some \( y_1, c, d \in U \).

Let \( M \in C \). Then \( M^* \nmid N_1^* \) or \( M^* \nmid N_2^* \). We may assume \( M^* \nmid N_1^* \). From Theorem 2.2, we have
for some \( f, g, w_j \) in \( U \). It then follows that

\[
M^* \cap N_1^* = \{<w_1>, \ldots, <w_{r-2p}>\}, \quad |M^* \cap N_1^*| = n - 1.
\]

But we have \( N_1^* \cap N_2^* = \{<y_1>, \ldots, <y_{r-2p}>\} \). Therefore

\[
\{<w_1>, \ldots, <w_{r-2p}>\} = \{<y_1>, \ldots, <y_{r-2p}>\}
\]

since \( M^* \cap N_1^* = N_1^* \cap N_2^* \). It then follows that \( <g> = <c> \). Consequently

\[
w_1 \cdots w_{r-2p} = \lambda y_1 \cdots y_{r-2p}
\]

for some \( \lambda \) in \( F \). Hence \( M = y_1 \cdots y_{r-2p} f^{r-2p} u^{r-2p} \). This completes the proof.

The following result is due to Cummings [4].

2.13. Theorem. Let \( \dim U \geq 2 \). Let \( C = \{M_i : i \in I\} \subseteq C_1(\overline{VU}) \) such that

\[ |C| > r + 2. \]

If for every \( M_1, M_2 \in C \), \( M_1 \cap M_2 \neq \emptyset \), then there exist
non-zero vectors $y_1, \ldots, y_{r-2}$ such that $M_i = y_1 \cdots y_{r-2} a_i U$, $i \in I$, $a_i \in U$.

2.14. Theorem. Let $C$ be an infinite collection of maximal pure subspaces of type one or of power type in $\mathcal{V}U$ such that for every $M_1, M_2$ in $C$, $M_1 \cap M_2 \neq \emptyset$. Then $C \subseteq C_1(\mathcal{V}U)$ or $C \subseteq P_t(\mathcal{V}U)$ for some positive integer $t$, except possibly when $\text{char } F = 2$, in which case, there exist non-zero vectors $x_1, \ldots, x_{r-2s+1}$ in $U$ for some non-negative integer $s$ and a subset $W$ of $U$ such that $C$ has the following form

$$C = \{ M : M = x_1 \cdots x_{r-2s+1} U^{2s} \text{ or } M = x_1 \cdots x_{r-2s+1} w^s U^{2s} \}$$

where $w \in W$.

Proof: Let $D_1 = C \cap C_1(\mathcal{V}U)$ and $Q_t = C \cap P_t(\mathcal{V}U)$. By hypothesis, either $D_1$ is infinite or $Q_t$ is infinite for some positive integer $t$.

'Case 1. Suppose $Q_t$ is infinite for some positive integer $t$.

Then by Theorem 2.12, there exist non-zero vectors $x_1, \ldots, x_{r-2p}$ in $U$ and a subset $W$ of $U$ such that

$$Q_t = \{ x_1 \cdots x_{r-2p} a^t U^t : a \in W \}.$$
Suppose that some type one subspace \( y_1 \cdots y_{r-1} \cdot U \in C \).

Since \( Q_t \) is infinite, we are able to choose \( b \in W \) such that
\[
\langle b \rangle \not\subseteq \{ \langle y_1 \rangle , \cdots , \langle y_{r-1} \rangle \}.
\]
By Theorem 2.3, we have
\[
y_1 \cdots y_{r-1} = \begin{cases} 
  x_1 \cdots x \cdot b^{p^t} \cdot f^{p^{t-1}} \\
  (x_1 \cdots x \cdot b^{p^t} / \text{one product}) \cdot f^{p^t} 
\end{cases}
\]
for some \( f \) in \( U \). In both cases, \( \langle b \rangle \in \{ \langle y_1 \rangle , \cdots , \langle y_{r-1} \rangle \} \), a contradiction. Hence no type one subspace is in \( C \).

Suppose that some power type subspace \( y_1 \cdots y_{r-p} \cdot U^p \in C \)
where \( \ell < t \). Choose \( c \in W \) such that \( \langle c \rangle \not\subseteq \{ \langle y_1 \rangle , \cdots , \langle y_{r-p} \rangle \} \).

By Theorem 2.2, we have
\[
y_1 \cdots y_{r-p} = \begin{cases} 
  x_1 \cdots x \cdot c^{p^t} \cdot f^{p^{t-p\ell}} \\
  (x_1 \cdots x \cdot c^{p^t} / \ell \text{ products}) \cdot f^{p^t} 
\end{cases}
\]
for some \( f \) in \( U \). Since \( p^t > p^\ell \), in both cases, \( \langle c \rangle \in \{ \langle y_1 \rangle , \cdots , \langle y_{r-p} \rangle \} \), a contradiction. Hence \( P_{\ell} (VU) \cap C = \emptyset \) for \( \ell < t \).

Suppose now some \( y_1 \cdots y_{r-p} \cdot U^p \in C \) where \( \ell > t \) and \( r \geq p^\ell \).
Choose $d \in W$ such that $\langle d \rangle \not= \langle y_1 \rangle$ for all $i = 1, \ldots, r - p^t$
and $\langle d \rangle \not= \langle x_j \rangle$ for all $j = 1, \ldots, r - 2p^t$. Then from Theorem 2.2, we obtain

$$\begin{align*}
\sum_{i=1}^{r-p^t} x_i \cdot \phi^{\ell-p^t} & = \begin{cases} 
\sum_{i=1}^{r-p^t} y_1 \cdots y_{r-p^t} \cdot \phi^{\ell-p^t} & (1) \\
(y_1 \cdots y_{r-p^t})^{t \text{ products}} \cdot \phi^{\ell} & (2)
\end{cases}
\end{align*}$$

for some $f \in U$. If (2) holds, then $\langle d \rangle = \langle f \rangle$ and $p^t = p^\ell$ because of our choice of $d$. This yields a contradiction. Hence (1) holds and $\langle d \rangle = \langle f \rangle$, $p^t = p^\ell - p^t$. This implies that $2p^t = p^\ell$ and hence $p$ is divisible by 2. Therefore $p = 2$ and $t + 1 = \ell$. Also if $r > p^\ell$, we have $x_1 \cdots x_{r-2p^t} = \lambda y_1 \cdots y_{r-p^t}$ for some $\lambda$ in $F$. Therefore

$$C = \{ M : M = x_1 \cdots x_{r-2t+1} \cdot U^{t+1} \text{ or } M = x_1 \cdots x_{r-2t+1} \cdot w^t \cdot U^t \text{ where } w \in W \}.$$

Case 2. Suppose that $D_1$ is infinite. By Theorem 2.13, there exist non-zero vectors $x_1, \ldots, x_{r-2}$ in $U$ and $W \subseteq U$ such that

$$D_1 = \{ x_1 \cdots x_{r-2} \cdot a \cdot U : a \in W \}.$$ 

Assume that some power type pure subspace $y_1 \cdots y_{r-p} \cdot U^{p^t} \in C$ where
\[ r \geq p^t \text{ and } t \text{ is a positive integer. Choose } b \in W \text{ such that }<b> \neq <y_i> \text{ for all } i = 1, \ldots, r - p^t \text{ and } <b> \neq <x_j> \text{ for all } j = 1, \ldots, r - 2. \text{ In view of Theorem 2.3, we have}
\]
\[
x_1 \cdots x_{r-2} b = \begin{cases} 
y_1 \cdots y_{r-p^t} f^{p^t-1} \\
y_1 \cdots \widehat{y}_i \cdots y_{r-p^t} f^{p^t}
\end{cases}
\]

for some \( f \) in \( U \). If (4) holds, then \( <b> = <f> \) and \( p^t = 1 \) by our choice of \( b \). This yields a contradiction since \( p^t \neq 1 \). Hence (3) holds. This implies that \( <b> = <f> \) and \( p^t - 1 = 1 \). Hence \( p = 2 \) and \( t = 1 \). Also if \( r > 2 \), we have \( x_1 \cdots x_{r-2} = n y_1 \cdots y_{r-2} \) for some \( n \) in \( F \).

Therefore

\[ C = \{ M : M = x_1 \cdots x_{r-2} \cdot U^2 \text{ or } M = x_1 \cdots x_{r-2} \cdot a \cdot U \text{ where } a \in W \} \]

Hence the proof is complete.

\[ §3. \text{ Rank One Preservers From } U^r \text{ to } U^s, r < s. \]

Let \( U \) and \( W \) be finite dimensional vector spaces over a field \( F \). Let \( f : U \to W \) be a linear transformation. Let \( w_1, \ldots, w_{s-r} \) be \( s - r \) non-zero vectors of \( W \) where \( r < s \). Then the mapping \( \phi : XU^r \to VW^s \) defined by
\( \phi(x_1, \cdots, x_r) = w_1 \cdots w_{s-r} \cdot f(x_1) \cdots f(x_r) \)

is clearly multilinear and symmetric. Hence by the universal factorization property of \( \mathbb{V}_U \), there exists a unique linear transformation \( f^* : \mathbb{V}_U \rightarrow \mathbb{V}_W \) such that

\[ f^*(x_1, \cdots, x_r) = w_1 \cdots w_{s-r} \cdot f(x_1) \cdots f(x_r) \]

for each \( x_1, \cdots, x_r \in U \). \( f^* \) is called the linear transformation induced by \( w_1, \cdots, w_{s-r} \) and the linear transformation \( f : U \rightarrow W \).

It is easy to derive the following two properties:

(i) \( f^* \) is a monomorphism if and only if \( f \) is a monomorphism.

(ii) Let \( g^* \) be induced by \( y_1, \cdots, y_{s-r} \) and \( g : U \rightarrow W \). Let the rank of \( f \) be \( \geq 2 \) and \( F \) be an infinite field. Then \( f^* = g^* \) if and only if \( f = \eta g \) and \( w_1 \cdots w_{s-r} = \lambda y_1 \cdots y_{s-r} \) for some \( \eta, \lambda \in F \) such that \( \lambda \eta^r = 1 \).

3.1. Definition. A linear transformation \( T : \mathbb{V}_U \rightarrow \mathbb{V}_W \) is a rank one preserver if every non-zero pure vector of \( \mathbb{V}_U \) is mapped to a non-zero pure vector of \( \mathbb{V}_W \).

3.2. Definition. A rank one preserver from \( \mathbb{V}_U \) to \( \mathbb{V}_W \) is said to be a type one mapping if every type one subspace of \( \mathbb{V}_U \) is mapped into a type one subspace of \( \mathbb{V}_W \).
Throughout the rest of this section, $U$ is assumed to be a vector space over an algebraically closed field $F$. We shall show that every type one mapping from $V^r_U$ to $V^s_U$, $r < s$, is induced by some $s - r$ vectors of $U$ and a non-singular linear transformation on $U$ provided that $\text{dim } U \geq 2$.

If $\text{dim } U = 1$, it is not hard to see that every rank one preserver from $V^r_U$ to $V^s_U$, $r < s$, is induced by $s - r$ non-zero vectors of $U$ and a scalar multiple of the identity mapping on $U$. Hence, from now on, we assume $\text{dim } U \geq 2$.

3.3. Definition. Let $r > 2$. Two type one pure subspaces $y_1 \cdots y_{r-1}U$ and $z_1 \cdots z_{r-1}U$ of $V^r_U$ are said to be adjacent if $y_1 \cdots y_{r-1}$ and $z_1 \cdots z_{r-1}$ have exactly $r - 2$ common factors counting multiplicities. If $r = 2$, any two distinct type one pure subspaces of $V^2_U$ are said to be adjacent.

It is easy to show that two distinct type one pure subspaces $M$ and $N$ are adjacent if and only if $M \cap N \neq 0$ [4, Theorem 2.18].

3.4. Lemma. Let $T : V^r_U \rightarrow V^s_U$ be a rank one preserver. Then the images of two adjacent type one pure subspaces of $V^r_U$ are distinct.

Proof: Let $M_i = x_1 \cdots x_{r-2}y_iU$, $i = 1, 2$, be two adjacent type one pure subspaces of $V^r_U$. Let $f_i$, $i = 1, 2$, be two linear transformations of $U$
onto $T(M_1)$ defined by $f_1(u) = T(x_1 \cdots x_{r-2} y_1 \cdot u)$ for each $u \in U$.

Since $T$ is a rank one preserver, it follows that $f_1$ is a monomorphism.

Assume that $T(M_1) = T(M_2)$. Then $f_2^{-1} f_1$ is a non-singular linear mapping of $U$. Since $F$ is algebraically closed, $f_2^{-1} f_1$ has a non-zero eigenvector, say $v$. Let $\lambda$ be the corresponding eigenvalue. Then $f_1(v) = \lambda f_2(v)$. Hence

$$T(x_1 \cdots x_{r-2} y_1 \cdot v) = \lambda T(x_1 \cdots x_{r-2} y_2 \cdot v).$$

It follows that $y_1 = \lambda y_2$. Therefore $M_1 = M_2$, contradicting our hypothesis. This proves that $T(M_1) \neq T(M_2)$.

3.5. Remark. The above proof is exactly the same as the case $r = s$ proved in [4, Theorem 3.3]

3.6. Definition. Let $M = x_1 \cdots x_{r-1} U$ be a type one subspace of $V\cdot U$. Then the factors of $x_1 \cdots x_{r-1}$ are defined to be the factors of $M$.

3.7. Lemma. Let $T : \mathbb{V} \mathbb{U} \rightarrow \mathbb{V} \mathbb{U}$, be a type one mapping where $r < s$. Then the images of any two type one subspaces of $\mathbb{V} \mathbb{U}$ have at least $s - r$ common factors counting multiplicities.

Proof: Let $M_1 = x_1 \cdots x_{r-1} U$ and $M_r = y_1 \cdots y_{r-1} U$ be any two type one subspaces. Consider the pure subspaces
57.

\[ M_i = y_1 \cdots y_{i-1} \cdot x_i \cdots x_{r-1} \cdot U, \quad i = 2, \ldots, r - 1. \]

For each \( i = 1, \ldots, r - 1 \), \( T(M_i) \cap T(M_{i+1}) \neq 0 \) since \( M_i \cap M_{i+1} \neq 0 \).

Hence \( T(M_i) \) and \( T(M_{i+1}) \) have at least \( s - 2 \) common factors. Consequently, \( T(M_i) \) and \( T(M_r) \) have at least \( s - r \) common factors.

3.8. Lemma. Let \( T : V^r_U \to V^s_{U} \) be a type one mapping. For any non-zero vectors \( y_1, \ldots, y_{r-2} \) of \( U \), we have

\[ \{T(y_1 \cdots y_{r-2} \cdot u \cdot U) : u \in U, \ u \neq 0\} = \{z_1 \cdots z_{s-2} \cdot w \cdot U : w \in W\}. \]

for some non-zero vectors \( z_1, \ldots, z_{s-2} \in U \) and some \( W \subseteq U \).

Proof: In view of Lemma 3.4, \( C \equiv \{T(y_1 \cdots y_{r-2} \cdot u \cdot U) : u \in U, \ u \neq 0\} \) is an infinite family of type one pure subspaces of \( V^S_U \). Since any two members of \( C \) have a non-zero intersection, it follows from Theorem 2.13 that

\[ C = \{z_1 \cdots z_{s-2} \cdot w \cdot U : w \in W\} \]

for some non-zero vectors \( z_1, \ldots, z_{s-2} \) of \( U \) and some \( W \subseteq U \).

3.9. Lemma. Let \( T : V^r_U \to V^s_{U} \) be a rank one preserver where \( s \geq r > 2 \).

Let \( x_1, \ldots, x_{r-1} \) be \( r-1 \) non-zero vectors in \( U \) such that

\[ \langle x_{r-1} \rangle \nsubseteq \langle x_{r-2} \rangle. \]

If \( \{T(x_1 \cdots x_{r-2} \cdot u \cdot U) : u \in U, \ u \neq 0\} \)

\( \subseteq \{y_1 \cdots y_{s-2} \cdot u \cdot U : u \in U, \ u \neq 0\} \) for some non-zero vectors \( y_1, \ldots, y_{s-2} \)
of $U$, then

$$\{T(x_1 \cdots \hat{x}_{r-2} \cdots x_{r-1} \cdot u \cdot U) : u \in U, u \neq 0\} \not\subseteq \{y_1 \cdots y_{s-2} \cdot u \cdot U : u \in U, u \neq 0\}.$$

Proof: Suppose to the contrary that

$$\{T(x_1 \cdots \hat{x}_{r-2} \cdots x_{r-1} \cdot u \cdot U) : u \in U, u \neq 0\} \subseteq \{y_1 \cdots y_{s-2} \cdot u \cdot U : u \in U, u \neq 0\}. \quad (1)$$

For each non-zero $g \in U$, let

$$M_g = x_1 \cdots x_{r-2} \cdot g \cdot U \quad \text{and} \quad N_g = x_1 \cdots \hat{x}_{r-2} \cdots x_{r-1} \cdot g \cdot U.$$

Since $M_{x_{r-1}} \in \{x_1 \cdots x_{r-2} \cdot u \cdot U : u \in U, u \neq 0\}$, we have

$$T(M_{x_{r-1}}) = y_1 \cdots y_{s-2} \cdot f \cdot U \quad (2)$$

for some non-zero $f \in U$. Now, choose a non-zero vector $y$ such that $<y> \nmid <x_{r-1}'>$, $<y> \nmid <x_{r-2}'>$. Then by assumption

$$T(x_1 \cdots \hat{x}_{r-2} \cdot y \cdot U) = y_1 \cdots y_{s-2} \cdot y' \cdot U \quad (3)$$

for some $y' \in U$ and from $(1)$, $T(x_1 \cdots \hat{x}_{r-2} \cdots x_{r-1} \cdot y \cdot U) = y_1 \cdots y_{s-2} \cdot y'' \cdot U \quad (4)$

for some $y'' \in U$. Since $<x_{r-1}'> \nmid <x_{r-2}'>$, it follows that $M_y$ and $N_y$ are adjacent. Hence by Lemma 3.4, $T(M_y) \nmid T(N_y)$. Therefore $<y'> \nmid <y''>$.
Consider the following equalities:

\[ T(x_1 \cdots x_{r-1}y) = y_1 \cdots y_{s-2} \cdot f \cdot \bar{y} \text{ for some } \bar{y} \text{ (from (2))} \]

\[ = y_1 \cdots y_{s-2} \cdot y' \cdot a \text{ for some } a \text{ (from (3))} \]

\[ = y_1 \cdots y_{s-2} \cdot y'' \cdot b \text{ for some } b \text{ (from (4)).} \]

We have \( <y'> = <b> \), \( <y''> = <a> \) since \( <y'> \neq <y''> \).

It then follows that either \( <f> = <a> = <y''> \) or \( <f> = <b> = <y'> \).

If \( <f> = <y''> \), then \( T(M_{x_{r-1}}) = T(N_y) \) contradicting Lemma 3.4 since \( M_{x_{r-1}} \) and \( N_y \) are adjacent. Similarly, \( <f> = <y'> \) implies that \( T(M_{x_{r-1}}) = T(M_y) \), a contradiction. Hence the Lemma is proved.

3.10. Definition. Let \( V \) be a vector space of dimension \( k \). Then a set of vectors \( z_1, \ldots, z_m \) is said to be in general position if any \( k \) vectors from \( z_1, \ldots, z_m \) are linearly independent.

We shall need the following Theorem in order to prove Theorem 3.14.

3.11. Theorem. Let \( V^m(G) \) be a symmetry class of tensors over \( V \)

associated with a subgroup \( G \) of \( S_m \) and a character \( \chi \) on \( G \). Let \( \dim V = k \). If \( k \geq m \), then \( V^m(G) \) has a basis of tensors of the form \( x_1 \cdots x_m \) in which \( x_1, \ldots, x_m \) are linearly independent. If \( k < m \)
and $F$ is an infinite field, then $V^m(G)_X$ has a basis of tensors of the form $x_1 \otimes \cdots \otimes x_m$ such that $x_1, \ldots, x_m$ are in general position.

**Proof:** Consider the mapping $\phi : \bigotimes_1^m V \rightarrow V^m(G)_X$ defined by

$$
\phi(v_1, \ldots, v_m) = v_1 \otimes \cdots \otimes v_m
$$

for each $v_1, \ldots, v_m$ in $V$. Clearly $\phi$ is multilinear, hence by the universal factorization property of $\bigotimes_1^m V$, there is a unique linear transformation $\bar{\phi} : \bigotimes_1^m V \rightarrow V^m(G)_X$ such that

$$
\bar{\phi}(v_1 \otimes \cdots \otimes v_m) = v_1 \otimes \cdots \otimes v_m
$$

for each $v_1, \ldots, v_m$ in $V$. Since $V^m(G)_X$ is spanned by the set of all decomposable elements, it follows that $\bar{\phi}$ is an onto mapping. Therefore it suffices to prove the theorem for the case of $\bigotimes_1^m V$.

The theorem is trivial in case of $\dim V = 1$. Hence we may assume $k \geq 2$. We proceed by induction on $m$. Suppose first $m = 2$. Let $u$ be any non-zero vector of $V$. Let $v$ be linearly independent of $u$. Then $u \otimes u = u \otimes v - u \otimes (v-u)$. Hence the set of all tensors of the form $x_1 \otimes x_2$ where $x_1$ and $x_2$ are linearly independent, spans $\bigotimes_1^2 V$. 

Assume now the theorem is true for \( m = n \), \( n \geq 2 \).

We have the following two cases:

(i) \( k \geq n + 1 \). Let \( u_1 \otimes \cdots \otimes u_{n+1} \in V^{n+1} \). By the induction hypothesis,

\[
u_1 \otimes \cdots \otimes u_n = \sum_{i=1}^{t} a_i y_{i1} \otimes \cdots \otimes y_{in}
\]

for some \( t \), some \( a_i \) in \( F \) where \( y_{i1}, \ldots, y_{in} \) are linearly independent for each \( i \). If \( y_{i1}, \ldots, y_{in}, u_{n+1} \) are linearly dependent, choose \( z_i \) in \( V \) such that \( y_{i1}, \ldots, y_{in}, z_i \) are linearly independent. Then

\[
y_{i1} \otimes \cdots \otimes y_{in} \otimes u_{n+1} = y_{i1} \otimes \cdots \otimes y_{in} \otimes z_i - y_{i1} \otimes \cdots \otimes y_{in} \otimes (z_i - u_{n+1}) \quad (1)
\]

Note that \( y_{i1}, \ldots, y_{in}, z_i - u_{n+1} \) are linearly independent. Since

\[
u_1 \otimes \cdots \otimes u_{n+1} = \sum_{i=1}^{t} a_i (y_{i1} \otimes \cdots \otimes y_{in} \otimes u_{n+1}),
\]

we conclude from (1) that \( V^{n+1} \) has a basis of the form \( x_1 \otimes \cdots \otimes x_{n+1} \) such that \( x_1, \ldots, x_{n+1} \) are linearly independent.

(ii) \( k < n + 1 \). Let \( u_1 \otimes \cdots \otimes u_{n+1} \in V^{n+1} \). By the induction hypothesis,
for some \( t \) and some \( b_i \) in \( F \) where \( v_{i1}, \ldots, v_{in} \) are in general position for each \( i \). Since \( F \) is infinite, we have for each \( i \),

\[
\forall i \in \{1, \ldots, t\}, \quad v_{i1} \not\in \bigcup_{j \in Q_{k-1,n}} \langle v_{ij1}, \ldots, v_{ijk-1} \rangle
\]

where \( Q_{k-1,n} \) denotes the totality of strictly increasing sequences of \( k-1 \) integers chosen from \( 1, \ldots, n \). Choose \( z_i \in V \) such that

\[
z_i \not\in \bigcup_{j \in Q_{k-1,n}} \langle v_{ij1}, \ldots, v_{ijk-1} \rangle, \quad i = 1, \ldots, t.
\]

Note that the vectors \( z_i - \lambda u_{n+1}, v_{ij1}, \ldots, v_{ijk-1} \) are linearly dependent for at most one value of \( \lambda \) in \( F \). Hence we are able to choose a non-zero \( \lambda_i \) in \( F \) such that \( z_i - \lambda_i u_{n+1}, v_{ij1}, \ldots, v_{ijk-1} \) are linearly independent for all \( j \) in \( Q_{k-1,n} \). We have

\[
v_{i1} \otimes \cdots \otimes v_{in} \otimes u_{n+1} = \lambda_i^{-1} (v_{i1} \otimes \cdots \otimes v_{in} \otimes z_i - v_{i1} \otimes \cdots \otimes v_{in} \otimes (z_i - \lambda_i u_{n+1})).
\]

Hence

\[
u_{i} \otimes \cdots \otimes u_{n+1} = \sum_{i=1}^{t} b_i \lambda_i^{-1} (v_{i1} \otimes \cdots \otimes v_{in} \otimes z_i - v_{i1} \otimes \cdots \otimes v_{in} \otimes (z_i - \lambda_i u_{n+1})).
\]
Since $v_i, \ldots, v_n, z_i$ are in general position and $v_i, \ldots, v_n, z_i - \lambda_i u_{n+1}$ are in general position for each $i$, it follows that $\bigotimes^{n+1} V$ has a basis of tensors of the form $x_1 \otimes \cdots \otimes x_m$ such that $x_1, \ldots, x_m$ are in general position.

3.12. Remark. It is clear that if $V$ is over a finite field, then whenever $m$ is chosen to be greater than the number of vectors in $V$, two of the vectors from $x_1, \ldots, x_m$ of $U$ must be equal and so the above theorem is false.

The following Theorem is due to Cummings [4]:

3.13. Theorem. Let $T : \overline{V} U \rightarrow \overline{V} U$ be a type one mapping. If $\dim U \not\geq 2$, then $T$ is induced by a nonsingular transformation on $U$.

3.14. Theorem. Let $T : \overline{V} U \rightarrow \overline{V} U$, $r < s$, be a type one mapping. If $\dim U > 2$, then $T$ is induced by $s - r$ vectors of $U$ and a nonsingular linear transformation on $U$.

Proof: Case 1: $r > 2$. Let $x_1, \ldots, x_{r-1}$ be $r - 1$ fixed non-zero vectors of $U$ such that $\langle x_i \rangle \not\subseteq \langle x_j \rangle$ if $i \not\subseteq j$. Let

$$T(x_1 \cdots x_{r-1} : U) = z_1 \cdots z_{s-1} : U.$$
From Lemma 3.8, we see that for each \( i = 1, \ldots, r - 1 \),

\[
\{T(x_1 \cdots x_{r-1} u \cdot U) : u \in U, u \perp 0\} = \{y_{i1} \cdots y_{i(s-2)} v \cdot U : v \in U_1 \subseteq U\}
\]

for some vectors \( y_{i1}, \ldots, y_{i(s-2)} \in U \) and for some \( U_1 \subseteq U \).

Since \( z_1 \cdots z_{s-1} \cdot U \in \{y_{i1} \cdots y_{i(s-2)} v \cdot U : v \in U_1 \subseteq U\} \), it follows that

\[
z_1 \cdots z_{s-1} = \lambda_j y_{i1} \cdots y_{i(s-2)} v_i \text{ for some } v_i \in U_1 \text{ and some } \lambda_i \in F \text{ where } i = 1, \ldots, r - 1.
\]

According to Lemma 3.9, we have for each pair of distinct \( k \) and \( j \),

\[
<y_{j1} \cdots y_{j(s-2)}> \perp <y_{k1} \cdots y_{k(s-2)>}.
\]

Therefore we may assume without loss of generality that for each \( i = 1, \ldots, r - 1 \),

\[
y_{i1} \cdots y_{i(s-2)} = \eta_i z_1 \cdots \widehat{z_{i-1}} \cdots z_{s-1}
\]

for some \( \eta_i \in F \) and that any two vectors from \( z_{s-1}, \ldots, z_{s-(r-1)} \) are linearly independent.
Our task is to show that \( T(VU) \subseteq <z_1, \ldots, z_{s-r}, s_1, \ldots, s_r : s_i \in U, i = 1, \ldots, r> \).

Now, let \( u_1, \ldots, u_{r-1} \) be any \( r-1 \) non-zero vectors of \( U \) such that 
\( <u_i> \not\parallel <u_j> \) if \( i \not= j \). Let 
\[
T(u_1, \ldots, u_{r-1}, U) = w_1, \ldots, w_{s-1}, U.
\]

We shall show that \( <z_1>, \ldots, <z_{s-r}> \) are \( s-r \) factors of \( w_1, \ldots, w_{s-1} \).

To do this, we first show that there exists a type one pure subspace \( B \) in \( VU \) with \( r-1 \) distinct factors such that the following hold:

(i) \( T(B) = g_1, \ldots, g_{s-r}, u'_1, \ldots, u'_{r-1}, U \); 

(ii) \( <g_1>, \ldots, <g_{s-r}> \) are \( s-r \) factors of \( w_1, \ldots, w_{s-1} \); 

(iii) \( <u'_i> \not\parallel \{ <z_1>, \ldots, <z_{s-1}> \} \) where \( i = 1, \ldots, r-1 \).

Consider the family \( C = \{ T(u_1, \ldots, u_{r-2}, u, U) : u \in U, u \not= 0 \} \).

Since \( w_1, \ldots, w_{s-1}, U \in C \), it follows from Lemma 3.8 that 
\[
C = \{ w_1, \ldots, \hat{w}_{s-i_1}, \ldots, w_{s-1}, U : u \in V_1 \subseteq U \}
\]

for some \( i_1 \) and some \( V_1 \subseteq U \). We may assume \( i_1 = 1 \). Since \( C \)
is an infinite family, we are able to choose non-zero vectors \( f \in U \) and
f' ∈ V_1 such that

\[ T(u_1 \cdots u_{r-2} \cdot f \cdot U) = w_1 \cdots w_{s-2} \cdot f' \cdot U, \]

\(<f> \notin \{<u_1>, \ldots, <u_{r-2}>\} \text{ and } <f'> \notin \{<z_1>, \ldots, <z_{s-1}>\}. \] Note that the factors of \( u_1 \cdots u_{r-2} \cdot f \cdot U \) are distinct.

Assume now we have shown that for some \( 1 \leq t < r-1 \), there exists a type one pure subspace \( d_1 \cdots d_{r-1} \cdot U \) with \( r-1 \) distinct factors such that

(a) \( T(d_1 \cdots d_{r-1} \cdot U) = h_1 \cdots h_{s-1} \cdot U \);

(b) \( <h_1>, \ldots, <h_{s-(t+1)}> \) are \( s-(t+1) \) factors of \( w_1 \cdots w_{s-1} \);

(c) \( <h_{s-t}>, \ldots, <h_{s-1}> \) \( \notin \{<z_1>, \ldots, <z_{s-1}>\} \).

Consider the families \( C_i \equiv \{T(d_1 \cdots \widehat{d}_{r-1} \cdots d_{r-1} \cdot u \cdot U : u \neq 0, u \in U}\} \) where \( i = 1, \ldots, t+1 \). Since \( h_1 \cdots h_{s-1} \cdot U \in C_i \) for each \( i = 1, \ldots, t+1 \), it follows from Lemma 3.8 that

\[ C_i = \{h_1 \cdots \widehat{h}_{s-k_i} \cdots h_{s-1} \cdot u \cdot U : u \in W_i \subseteq U\} \]
for some \( k_1, 1 < k_1 < s - 1 \) and for some \( W_i \subseteq U \). Since all factors of \( d_1 \cdots d_{r-1} U \) are distinct, by Lemma 3.9, we must have

\[ k_n \neq k_m \quad \text{for} \quad m \neq n \quad \text{where} \quad m, n = 1, \ldots, t + 1. \]

Therefore there is a \( j \) such that \( k_j > t + 1 \). We may assume \( k_j = t + 1 \). Now \( C_j \) is an infinite family implies that there are non-zero vectors \( d \in U \) and \( d' \in W_j \) such that

\[
T(d_1 \cdots \hat{d}_{r-j} \cdots d_{r-1} d U) = h_1 \cdots h_{s-(t+2)} d' h_{s-t} \cdots h_{s-1} U,
\]

\(<d> \notin \{<d_1>, \ldots, <d_{r-1}>\} \quad \text{and} \quad <d'> \notin \{<z_1>, \ldots, <z_{s-1}>\}. \]

Note that the factors of \( d_1 \cdots \widehat{d}_{r-j} \cdots d_{r-1} d U \) are distinct. Therefore our inductive argument shows that there exists a type one pure subspace \( B \) of \( V U \) with \( r - 1 \) distinct factors satisfying conditions (i), (ii) and (iii).

Now, we shall show that \( <g_1 \cdots g_{s-r}> = <z_1 \cdots z_{s-r}> \). For each \( i = 1, \ldots, r - 1 \), choose a vector \( f_i \in U_i \) such that

\[ <f_i> \notin \{<g_1>, \ldots, <g_{s-r}> \ldots, <u_i>, \ldots, <u_{r-1}>\}. \]

From Lemma 3.7, it follows that \( g_1 \cdots g_{s-r} u_1 \cdots u_{r-1} \) and \( z_1 \cdots \widehat{z}_{s-1} \cdots z_{s-1} f_i \) have at least \( s - r \) common factors for each \( i = 1, \ldots, r - 1 \). Since \( <u_1>, \ldots, <u_{r-1}> \notin \{<z_1>, \ldots, <z_{s-1}>\} \).
and $\langle f_1 \rangle \not\subseteq \{ \langle g_1 \rangle, \ldots, \langle g_{s-r} \rangle, \langle u'_1 \rangle, \ldots, \langle u'_{r-1} \rangle \}$, it follows that $\langle g_1 \rangle, \ldots, \langle g_{s-r} \rangle$ are $s-r$ factors of $z_1 \cdots \hat{z}_{s-i} \cdots \hat{z}_{s-1}$ for each $i = 1, \ldots, r-1$.

Consider the isomorphism $\phi$ from the symmetric algebra $V_U$ onto the polynomial algebra $F[\xi_1, \ldots, \xi_n]$ described in §1. The polynomial $\phi(g_1 \cdots g_{s-r})$ is a factor of $\phi(z_1 \cdots \hat{z}_{s-i} \cdots \hat{z}_{s-1})$ in $F[\xi_1, \ldots, \xi_n]$ for each $i = 1, \ldots, r-1$. Since $\langle \phi(z_{s-k}) \rangle \not\subseteq \langle \phi(z_{s-m}) \rangle$ for $k \neq m$ where $k, m = 1, \ldots, r-1$, it follows that $\phi(z_{s-k})$ and $\phi(z_{s-m})$ are relatively prime in the Gaussian domain $F[\xi_1, \ldots, \xi_n]$, and therefore $\phi(z_1 \cdots \hat{z}_{s-r})$ is a greatest common divisor of $\phi(z_1 \cdots \hat{z}_{s-i} \cdots \hat{z}_{s-1})$, $i = 1, \ldots, r-1$. This implies that $\phi(g_1 \cdots g_{s-r})$ is a factor of $\phi(z_1 \cdots \hat{z}_{s-r})$. Hence $\langle \phi(g_1 \cdots g_{s-r}) \rangle = \langle \phi(z_1 \cdots \hat{z}_{s-r}) \rangle$ since they are both products of $s-r$ linear homogeneous polynomials in $F[\xi_1, \ldots, \xi_n]$. Hence $\langle g_1 \cdots g_{s-r} \rangle = \langle z_1 \cdots \hat{z}_{s-r} \rangle$. This shows that $\langle z_1 \rangle, \ldots, \langle z_{s-r} \rangle$ are $s-r$ factors of $w_1 \cdots w_{s-1}$. Consequently

$$T(u_1 \cdots u_{r-1} \cdot U) = z_1 \cdots z_{s-r} \cdot D \cdot U$$

for some non-zero pure vector $D$ in $r^{-1}V U$. Since $\dim U > 2$, Theorem 3.11 implies that $r^{-1}V U$ has a basis of pure vectors of the form $u_1 \cdots u_{r-1}$ where $\langle u_i \rangle \not\subseteq \langle u_j \rangle$ if $i \neq j$. Therefore we have
Define a mapping \( T^* : VU \rightarrow VU \) as follows:

\[
T^*(C) = C' \quad \text{if} \quad T(C) = z_1 \cdots z_{s-r} C' \quad \text{for} \quad C' \in VU.
\]

\( T^* \) is well-defined since \( z_1 \cdots z_{s-r} C' = z_1 \cdots z_{s-r} C'' \) will imply \( C' = C'' \). Since \( T \) is a rank one preserver, it follows from Lemma 1.8 that \( T^* \) is also a rank one preserver. Moreover, \( T \) is a type one mapping implies that \( T^* \) is a type one mapping. By Theorem 3.13, \( T^* \) is induced by a non-singular linear transformation \( f \) on \( U \). Consequently,

\[
T(d_1 \cdots d_r) = z_1 \cdots z_{s-r} f(d_1) \cdots f(d_r)
\]

for each \( d_i \) in \( U \).

**Case 2:** \( r = 2 \). By Lemma 3.8, we have

\[
\{T(u \cdot U) : u \in U, u \neq 0\} \subseteq \{a_1 \cdots a_{s-2} \cdot u \cdot U : u \in U, u \neq 0\}
\]

for some \( a_i \) in \( U \), \( i = 1, \cdots, s - 2 \). Hence

\[
T(\hat{V}U) \subseteq \langle a_1 \cdots a_{s-2} \cdot s_1 \cdot s_2 : s_1, s_2 \in U \rangle.
\]
By the same argument as in case 1, we see that $T$ is induced by $a_1, \ldots, a_{s-2}$ of $U$ and a non-singular transformation on $U$.

Hence the proof is complete.

3.15. Remark. Theorem 3.14 is no longer valid if $\mathbf{S}^rU$ is replaced by $\mathbf{S}^rW$ with $\dim W > \dim U$. For example, if $\dim W > \dim \mathbf{S}^rU$ and $z_1, \ldots, z_{s-r}$ are non-zero vectors of $W$, then there is a monomorphism $T : \mathbf{S}^rU \to \mathbf{S}^rW$ such that

$$T(\mathbf{S}^rU) = z_1 \cdot \ldots \cdot z_{s-1} \cdot W_o$$

where $W_o \subseteq W$ and $\dim W_o = \dim \mathbf{S}^rU$. Clearly $T$ is a type one mapping which is not induced by $s - r$ vectors of $W$ and a linear transformation from $U$ to $W$.

§4. Rank One Preservers On $\mathbf{S}^rU$.

Let $U$ be a finite dimensional vector space over an algebraically closed field $F$. In this section we show that: if $\dim U > r + 1$, then every rank one preserver on $\mathbf{S}^rU$ is induced by a non-singular linear transformation on $U$; if $2 < \dim U < r + 1$ and the characteristic of $F$ is greater than $r$ or equal to zero, then every rank one preserver on $\mathbf{S}^rU$ is either induced by a non-singular linear transformation on $U$, or the image of $\mathbf{S}^rU$ under the rank one preserver is a type $r$ pure subspace.
We first establish the following Lemmas.

4.1. Lemma. Let $T : VU \to VU$ be a rank one preserver where $r \leq s$. Assume that $\dim U > 2$ and $\text{char } F = 2$. Let $x_1, \ldots, x_{r-2}$ be non-zero vectors of $U$. Then it is impossible that

$$\{T(x_1 \cdots x_{r-2} y : y \in U, y \neq 0) : y \in U, y \neq 0\} = \{z_1 \cdots z_{s-2k+1} \cdot u^{2k+1} : z_{s-2k+1} \cdot u^{2k} \cdot u^2 : w \in W\}$$

for some non-zero vectors $z_1, \ldots, z_{s-2k+1}$ of $U$ and some $W \subseteq U$ where $k$ is a non-negative integer.

Proof: Suppose to the contrary that

$$\{T(M_y) : y \in U, y \neq 0\} = \{z_1 \cdots z_{s-2k+1} \cdot u^{2k+1} \} \cup \{z_1 \cdots z_{s-2k+1} \cdot w^k \cdot u^2 : w \in W\}$$

where we denote $x_1 \cdots x_{r-2} \cdot y \cdot U$ by $M_y$.

Assume that $T(M_{u_1}) = z_1 \cdots z_{s-2k+1} \cdot u^{2k+1}$. Let $u_2 \in U$ be independent of $u_1$. Let $v_1 = u_1 - u_2$ and $v_2 = u_2$. By Lemma 3.4, $T(M_{u_1}) \not\subset T(M_{v_1})$ and $T(M_{u_1}) \not\subset T(M_{v_2})$. Therefore

$$T(M_{v_1}) = z_1 \cdots z_{s-2k+1} \cdot v_1^{2k} \cdot u^2$$

(1)

$$T(M_{v_2}) = z_1 \cdots z_{s-2k+1} \cdot v_2^{2k} \cdot u^2$$

(2)
for some $v_1, v_2$ in $W$. Let $z$ and $y$ be two independent vectors of $U$ such that $z, y \notin \langle v_1', v_2' \rangle$. Since $T(M_1) = z_{s-2k+1} U^{2k+1}$, it follows that

$$T(x_1 \cdots x_{r-2} u_1 f) = z_1 \cdots z_{s-2k+1} v_{2k} v_1'$$

$$T(x_1 \cdots x_{r-2} u_1 g) = z_1 \cdots z_{s-2k+1} v_{2k} v_2'$$

for some $f, g$ in $U$. Clearly $<z> \notin <y>$ implies that $<f> \notin <g>$. We have either $<f> \notin <u_1>$ or $<g> \notin <u_1>$. We may assume that $<f> \notin <u_1>$.

Let $A = x_1 \cdots x_{r-2} u_1 f$. In view of (1) and (2), we have

$$B = x_1 \cdots x_{r-2} v_1 f + z_1 \cdots z_{s-2k+1} v_{2k} v_1'$$

$$C = x_1 \cdots x_{r-2} v_2 f + z_1 \cdots z_{s-2k+1} v_{2k} v_2'$$

for some $f_1, f_2$ in $U$. On the other hand, since $<f> \notin <u_1>$, by Lemma 3.4, we have

$$T(M_f) = z_1 \cdots z_{s-2k+1} f_{2k} U^{2k}$$

for some $f'$ in $W$. Since $T(A), T(B), T(C) \in z_1 \cdots z_{s-2k+1} f_{2k} U^{2k}$, it follows that $<f'> = <z> = <f_1> = <f_2>$ by our choice of $z$. 

Now, let \( f_1 = \alpha z \) and \( f_2 = \beta z \) where \( \alpha, \beta \) are non-zero elements of \( F \). We obtain

\[
T(B+C) = z_1 \cdots z_{s-2+k+1} \cdot f_1^k \cdot v_1^k + z_1 \cdots z_{s-2+k+1} \cdot f_2^k \cdot v_2^k
\]

\[
= z_1 \cdots z_{s-2+k+1} \cdot (\alpha z)^k \cdot v_1^k + z_1 \cdots z_{s-2+k+1} \cdot (\beta z)^k \cdot v_2^k
\]

\[
= z_1 \cdots z_{s-2+k+1} \cdot z^k \cdot (\alpha v'_1)^k + z_1 \cdots z_{s-2+k+1} \cdot z^k \cdot (\beta v'_2)^k
\]

\[
= z_1 \cdots z_{s-2+k+1} \cdot z^k \cdot (\alpha v'_1 + \beta v'_2)^k \quad \text{since char } F = 2
\]

However, since \( u_1 = v_1 + v_2 \), we also have

\[
T(B+C) = T(A) = z_1 \cdots z_{s-2+k+1} \cdot z^{k+1}
\]

This implies that \( \alpha v'_1 + \beta v'_2 \) and \( z \) are linearly dependent, contradicting our choice of \( z \). This completes the proof.

4.2. Lemma. Let \( T : V^r U \to V^s U \) be a rank one preserver where \( s \geq r > 2 \).

Suppose \( \dim U > 2 \) and \( \text{char } F = \text{prime } p \). Let \( x_1, \ldots, x_{r-1} \) be non-zero vectors of \( U \) such that \( \langle x_{r-1} \rangle \nless \langle x_{r-2} \rangle \). If

\[
\{ T(x_1 \cdots x_{r-2} \cdot y \cdot U) : y \in U, y \neq 0 \} \subseteq \{ z_1 \cdots z_{s-2p}^t \cdot u^p \cdot u^p : u \in U, u \neq 0 \}
\]
for some non-zero \(z_1, \ldots, z \) of \(U\) where \(t\) is a positive integer, then

\[
\{T(x_1 \cdot \cdot \cdot x_{r-2} \cdot x_{r-1} \cdot y \cdot U) : y \in U, y \neq 0\} \subseteq \{z_1 \cdot \cdot \cdot z_{s-2p}u^t \cdot u^t : u \in U, u \neq 0\}
\]

**Proof:** The argument is similar to that used in the proof of Lemma 3.9.

**4.3. Theorem.** Let \(T : VU \to VU\) be a rank one preserver. Let \(\dim U > s + 1\). Then (i) if \(r = s\), \(T\) is induced by a non-singular transformation on \(U\); (ii) if \(r < s\) and the characteristic \(F\) is either zero or equal to a prime \(p > \frac{s}{r}\), \(T\) is induced by \(s - r\) vectors of \(U\) and a non-singular transformation on \(U\).

**Proof:** We shall show that under these conditions, \(T\) preserves type one subspaces and then the theorem will follow from Theorem 3.13 and Theorem 3.14.

Let \(M\) be a type one subspace of \(VU\). Since \(T\) is a rank one preserver, \(T(M)\) is a pure subspace of \(VU\). Moreover, \(\dim M = \dim T(M) = \dim U > s + 1\). Let \(T(M) \subseteq N\) where \(N\) is a maximal pure subspace of \(VU\). If \(N\) is of type \(k\) where \(1 < k \leq s\), then \(\dim N = k + 1 \leq s + 1\), contradicting the fact that \(\dim T(M) > s + 1\). Hence \(N\) is of type one or power type pure subspace. Moreover, \(T(M) = N\).

Suppose that \(T(x_1 \cdot \cdot \cdot x_{r-1} \cdot U) \in P_t^s(VU)\) for some non-zero
\[ x_1, \ldots, x_{r-1} \text{ in } U \text{ and some positive integer } t. \text{ Then } \text{char } F = \text{ a prime } p. \]

If \( r = 2 \), then \( u_1 \cdot U \cap u_2 \cdot U \neq 0 \) for all \( u_1, u_2 \neq 0 \). Hence

\[ T(u_1 \cdot U) \cap T(u_2 \cdot U) \neq 0 \text{ for all } u_1, u_2 \neq 0. \text{ Also } T(u_1 \cdot U) \neq T(u_2 \cdot U) \]

for all \( u_1 \neq u_2 \) by Lemma 3.4. In view of Theorem 2.14 and Lemma 4.1, we have

\[ \{ T(u \cdot U) : u \in U, u \neq 0 \} \subseteq P_t(\mathcal{V}U). \]

Hence by Theorem 2.12, \( s \geq 2p^t \), a contradiction to the hypothesis on \( s \). Now consider \( r > 2 \). Let \( y_1, \ldots, y_{r-1} \) be non-zero vectors of \( U \) such that \( \langle y_i \rangle \neq \langle y_j \rangle \) for \( i \neq j \). Consider the following type one pure subspaces of \( \mathcal{V}U \):

\[ M_1 = x_1 \cdots x_{r-1} \cdot U, M_2 = x_1 \cdots x_{r-2} \cdot y_1 \cdot U, \ldots, M_r = y_1 \cdots y_{r-1} \cdot U; \]

that is, \( M_i = x_1 \cdots x_{r-1} \cdot y_1 \cdots y_{i-1} \cdot U, i = 1, \ldots, r \). Since

\[ T(x_1 \cdots x_{r-2} \cdot u_1 \cdot U) \cap T(x_1 \cdots x_{r-2} \cdot u_2 \cdot U) \neq 0 \]

for all non-zero \( u_1, u_2 \) in \( U \), by Theorem 2.14 and Lemma 4.1,

\[ \{ T(x_1 \cdots x_{r-2} \cdot u \cdot U) : u \in U, u \neq 0 \} \subseteq P_t(\mathcal{V}U). \]

Therefore, \( T(M_2) \in P_t(\mathcal{V}U) \). Similarly \( T(M_2) \in P_t(\mathcal{V}U) \) implies that
Let $T(M_r) = z_1 \cdots z_{s-p} t^{t \cdot u^p s-p-t}$. In view of Theorem 2.12, Theorem 2.14 and Lemma 4.1, we have for each $i = 1, \ldots, r - 1$,

$$
\{T(y_1 \cdots \hat{y}_i \cdots y_{r-1} \cdot u \cdot U) : u \in U, u \neq 0\} \subseteq \{w_{i1} \cdots w_{i(s-2p^t)} \cdot u^p \cdot u^p t^{t \cdot u^p s-p-t} : u \in U, u \neq 0\}
$$

for some non-zero $w_{i1}, \ldots, w_{i(s-2p^t)}$ of $U$. Since

$$
T(M_r) \in \{T(y_1 \cdots \hat{y}_i \cdots y_{r-1} \cdot u \cdot U) : u \in U, u \neq 0\},
$$

it follows that

$$
z_1 \cdots z_{s-p-t} \cdot u^p t^{t \cdot u^p s-p-t} = w_{i1} \cdots w_{i(s-2p^t)} \cdot u^p t^{t \cdot u^p s-p-t}
$$

for some non-zero $u_i$ in $U$ where $i = 1, \ldots, r - 1$. Hence $z_1, \ldots, z_{s-p-t}$ has at least $p^t$ factors of $<u_i>$ for each $i$ by Theorem 2.1.

Since $<y_k> \nmid <y_j>$ for $k \nmid j$, according to Lemma 4.2,

$$
<w_{j1} \cdots w_{j(s-2p^t)}> \nmid <w_{k1} \cdots w_{k(s-2p^t)}>
$$

for $k \nmid j$. This shows that $<u_j> \nmid <u_k>$ for $j \nmid k$. Consequently
$z_1 \cdots z_{s-p^t}$ has at least $(r-1)p^t$ factors. This implies that
s - p^t \geq (r-1)p^t and thus $s \geq r p^t$, a contradiction to the hypothesis
on $s$. Therefore no type one pure subspace of $\mathbb{V}_U$ is mapped onto
a power type subspace of $\mathbb{V}_U$. This proves that $T$ is a type one mapping.
Our result thus follows from Theorem 3.13 and Theorem 3.14.

That Theorem 4.3 is not true if $s \geq r p$ where $\text{char } F = p$
is shown by the following:

4.4. Example. Let $\text{char } F = p$ and $s \geq r p^t$ for some positive integer
$t$. Let $k = p^t$. Let $u_1, \ldots, u_n$ be a basis of $U$ and $w_1, \ldots, w_n$
be another basis of $U$. Let $f : U \to U$ be a mapping defined by

$$f(\sum_{i=1}^{n} \lambda_i u_i) = \sum_{i=1}^{n} \lambda_i^k w_i$$

for any $\lambda_i$ in $F$ where $\lambda_i^k$ is the $k^{th}$ root of $\lambda_i$.

Then it is easily checked that $f$ has the following property:

$$f(\lambda v + \eta w) = \lambda^k f(v) + \eta^k f(w)$$

for any $\lambda, \eta$ in $F$, $v, w$ in $U$ and $f$ is one to one.

Define a mapping $\phi : XU \to \mathbb{V}_U$ by

$$\phi(v_1, \ldots, v_r) = z_1 \cdots z_{s-rk} \cdot f(v_1)^k \cdots f(v_r)^k$$

where $z_1, \ldots, z_{s-rk}$ are fixed non-zero vectors of $U$. We have
\[
\phi(v_1, \ldots, v_r) = z_1 \cdots z_{s-rk} f(v_1)^k \cdots f(v_r)^k = z_1 \cdots z_{s-rk} f(v_1)^k \cdots f(v_r)^k \\
\]

\[
= z_1 \cdots z_{s-rk} f(v_1)^k \cdots (\lambda f(v_1) + \eta f(v_1)) f(v_2)^k \cdots f(v_r)^k \\
= z_1 \cdots z_{s-rk} f(v_1)^k \cdots (\lambda f(v_1)) f(v_2)^k \cdots f(v_r)^k \\
+ z_1 \cdots z_{s-rk} f(v_1)^k \cdots (\eta f(v_1)) f(v_2)^k \cdots f(v_r)^k \\
= \lambda z_1 \cdots z_{s-rk} f(v_1)^k \cdots f(v_r)^k + \eta z_1 \cdots z_{s-rk} f(v_1)^k \cdots f(v_r)^k \\
= \lambda \phi(v_1, \ldots, v_r) + \eta \phi(v_1, \ldots, v_r).
\]

Hence \( \phi \) is multilinear. Clearly \( \phi \) is symmetric. Hence there exists a unique linear transformation \( \bar{\phi} : \mathbb{V}^r \to \mathbb{V}^s \) such that

\[
\bar{\phi}(v_1 \cdots v_r) = z_1 \cdots z_{s-rk} f(v_1)^k \cdots f(v_r)^k
\]

for each \( v_1, \ldots, v_r \) in \( U \). Clearly \( \bar{\phi} \) is a rank one preserver from \( \mathbb{V}^r \) to \( \mathbb{V}^s \) and is not induced by \( s-r \) vectors of \( U \) and a linear transformation on \( U \).

4.5. Theorem. Let \( T : \mathbb{V}^r \to \mathbb{V}^s \) be a rank one preserver. Let \( \dim U = s + 1 \).
We have (i) if \( r = s \), then \( T \) is induced by a non-singular transformation on \( U \);

(ii) if \( r < s \) and the characteristic of \( F \) is either zero or equal to a prime \( p > \frac{s}{r} \), then \( T \) is induced by \( s-r \) vectors of \( U \) and a non-singular linear transformation on \( U \).
Proof: Since $\dim U = s + 1$, every pure subspace of type one or of power type or of type $s$ in $\mathbb{V}^U$ has dimension $s + 1$. Also every type one pure subspace of $\mathbb{V}^U$ has dimension $s + 1$. If a type one subspace of $\mathbb{V}^U$ were mapped into a type $k$ pure subspace of $\mathbb{V}^U$, $1 < k < s$, then the image would have dimension $k + 1 < s + 1$. This would imply that some non-zero pure vector is mapped into 0, which is not the case. Hence a type one pure subspace of $\mathbb{V}^U$ is mapped only onto a type one pure subspace or a power type pure subspace or a type $s$ subspace of $\mathbb{V}^U$.

Suppose that some type one subspace $x_1 \cdots x_{r-2} y U$ is mapped onto a type $s$ subspace $V(s)$ of $\mathbb{V}^U$, where $V$ is a two dimensional subspace of $U$. We shall show that this leads to a contradiction.

Let $C = \{T(x_1 \cdots x_{r-2} u U) : u \in U, u \neq 0\}$. By Lemma 3.4, $C$ is an infinite family. We shall show that $V(s)$ is the only type $s$ subspace in $C$. Suppose that $C \supseteq \{V(s), V^*_s\}$ where $V^*_s$ is another type $s$ subspace of $\mathbb{V}^U$. Then $V \cap V^*_s$ is 1-dimensional since $V(s) \cap V^*_s \neq 0$.

Choose a non-zero $z$ in $U$ such that

$$T(x_1 \cdots x_{r-2} y z) = v_1 \cdots v_s$$

where $\dim <v_1, \ldots, v_s> = 2$, $<y> \neq <z>$ and $V \cap V^*_s \neq <v_i>$ for all $i = 1, \ldots, s$.

Note that $T(x_1 \cdots x_{r-2} z U) \cap V(s) \neq 0$ and $T(x_1 \cdots x_{r-2} z U) \cap V^*_s \neq 0$. 


If \( T(x_1 \ldots x_{r-2} \cdot z \cdot U) = z_1 \ldots z_{s-1} \cdot U \) for some \( z_1 \) in \( U \), then 
\( z_1, \ldots, z_{s-1} \in V \cap V^* \); hence \( \langle z_1 \rangle = \cdots = \langle z_{s-1} \rangle = V \cap V^* \). But 
\( v_1 \ldots v_s \in z_1 \ldots z_{s-1} \cdot U \), thus for some \( i \), \( \langle v_i \rangle = V \cap V^* \), a contradiction. 

Hence \( T(x_1 \ldots x_{r-2} \cdot z \cdot U) \) cannot be a type one subspace. Similarly, if 
\( \text{char } F = p, s > p^t \), then 
\[ T(x_1 \ldots x_{r-2} \cdot z \cdot U) \nsubseteq P_{t}(V U) \],

where \( t \) is a positive integer. Also if \( \text{char } F = p, s = p^k \) for some 
positive integer \( k \), \( T(x_1 \ldots x_{r-2} \cdot z \cdot U) \nsubseteq U^k \) since \( \dim \langle v_1, \ldots, v_s \rangle = 2 \) 
and \( v_1 \ldots v_s \in T(x_1 \ldots x_{r-2} \cdot z \cdot U) \). Hence 
\[ T(x_1 \ldots x_{r-2} \cdot z \cdot U) = W(s) \]

for some type \( s \) subspace \( W(s) \). Since 
\[ x_1 \ldots x_{r-2} \cdot y \cdot z \in x_1 \ldots x_{r-2} \cdot z \cdot U \cap x_1 \ldots x_{r-2} \cdot y \cdot U \],

it follows that \( v_1 \ldots v_s \in V(s) \cap W(s) \). This implies that 
\( \langle v_1, \ldots, v_s \rangle = V = W \), a contradiction to Lemma 3.4 since \( x_1 \ldots x_{r-2} \cdot y \cdot U \) 
and \( x_1 \ldots x_{r-2} \cdot z \cdot U \) are adjacent pure subspaces. Therefore \( V(s) \) is the 
only type \( s \) subspace in \( C \).
Let $C'$ be the collection of all type one subspaces in $C$.

We shall show that $C'$ is finite. Suppose that $C'$ is infinite. Since $M_1 \cap M_2 \neq 0$ for every $M_1, M_2$ in $C'$, it follows from Theorem 2.13 that

$$C' = \{z_1, \ldots, z_{s-2}, u \cdot U : u \in W \subseteq U\}$$

for some non-zero vectors $z_1, \ldots, z_{s-2}$ in $U$ and some $W \subseteq U$. According to Theorem 2.14, $C = C' \cup \{V(s)\}$ or $C = C' \cup \{V(s)\} \cup \{z_1, \ldots, z_{s-2}, u^2\}$.

In the latter case, char $F = 2$.

If $A = z_1, \ldots, z_{s-2}, u^2 \in C$, we let $f \in U$ such that $T(x_1, \ldots, x_{r-2}, f \cdot U) = A$. Let $v$ be a fixed non-zero vector of $U$ such that $<v> \neq <f>$ and $<v> \neq <y>$. Then by Lemma 3.4,

$$T(x_1, \ldots, x_{r-2}, v \cdot U) = z_1, \ldots, z_{s-2}, v' \cdot U$$

for some non-zero vector $v'$ of $W$. Now, for any non-zero $x$ in $U$ such that $<x> \neq <y>$, $<x> \neq <v>$ and $<x> \neq <f>$, let

$$T(x_1, \ldots, x_{r-2} \cdot x \cdot U) = z_1, \ldots, z_{s-2}, x' \cdot U.$$  \hspace{1cm} (2)

Since $z_1, \ldots, z_{s-2}, x', U \cap V(s) \neq 0$, we have $x' \in V$. From (2), we have

$$T(x_1, \ldots, x_{r-2} \cdot x \cdot v) = z_1, \ldots, z_{s-2}, x' \cdot v \cdot x.$$
for some $v_x \in U$. Since $z_1 \cdots z_{s-2} \cdot x' \cdot v_x \in z_1 \cdots z_{s-2} \cdot v' \cdot U$ (from (1)) and $<x'> \nmid <v'>$ (Lemma 3.4), it follows that $<v_x> = <v'>$. Hence

$$T(x_1 \cdots x_{r-2} \cdot v \cdot U) \subseteq z_1 \cdots z_{s-2} \cdot v' \cdot V \cup <T(x_1 \cdots x_{r-2} \cdot v \cdot y) \cup <T(x_1 \cdots x_{r-2} \cdot f \cdot v)> \cup <T(x_1 \cdots x_{r-2} \cdot v \cdot v)>.$$ 

This is impossible since $\dim T(x_1 \cdots x_{r-2} \cdot v \cdot U) = \dim U > 2$.

If $A \nmid C$, by the same argument as above, we have

$$T(x_1 \cdots x_{r-2} \cdot v \cdot U) \subseteq z_1 \cdots z_{s-2} \cdot v' \cdot V \cup <T(x_1 \cdots x_{r-2} \cdot v \cdot y) \cup <T(x_1 \cdots x_{r-2} \cdot v \cdot v)>,$$

which is also impossible.

Hence $C'$ is finite. Since $C$ is infinite, it follows that $\text{char } F = \text{a prime } p$ and for some positive integer $t$, the collection $\mathcal{D}_t$ of all power type subspaces of degree $t$ in $C$ is infinite. In view of Theorem 2.12, we have $s > 2p^t$ and

$$\mathcal{D}_t = \{y_1 \cdots y_{s-2p^t} \cdot w^{p^t} \cdot v \cdot w^* : w \in W^*\}$$

for some $y_i$ in $U$ and some $W^* \subseteq U$. By Theorem 2.14, we see that there is no other type one subspace or power type subspace in $C$ except
possibly \( y_1 \cdots y_{s-2} t^{t+1} \) in which case \( \text{char } F = p = 2 \).

Since \( T(x_1 \cdots x_r \cdot y \cdot U) = V_{(s)} \), we may choose a non-zero vector \( u \) in \( U \) such that \( \langle u \rangle \nmid \langle y \rangle \) and

\[
T(x_1 \cdots x_{r-2} \cdot y \cdot u) = v_1 \cdots v_s
\]

where \( \langle v_i \rangle \nmid \langle v_j \rangle \) if \( i \nmid j \). By Lemma 3.4, we see that

\[
T(x_1 \cdots x_{r-2} \cdot u \cdot U) \in D_t \text{ or } T(x_1 \cdots x_{r-2} \cdot u \cdot U) = y_1 \cdots y_{s-2} t^{t+1} \cdot u^{2 t+1}.
\]

Hence \( v_1 \cdots v_s \) has a factor of multiplicity at least \( p^t \), a contradiction.

Therefore no type one pure subspace of \( rU \) is mapped onto a type \( s \) subspace of \( sU \). This means that every type one subspace of \( rU \) is mapped onto a type one subspace or a power type subspace of \( sU \). By the argument used in the proof of Theorem 4.3, we conclude that \( T \) is a type one mapping. Our result thus follows from Theorem 3.13 and Theorem 3.14.

4.6. Lemma. Let \( W \) be a finite dimensional vector space over a field \( F \) of characteristic 0 or of characteristic greater than a positive integer \( t \). Then every non-zero pure vector of the form \( x^{t-j} y^j \), \( 1 \leq j \leq t \), in \( VW \) is in the span of \( M = \{ u^t : u \in W \} \).
Proof: Let \( \lambda_1, \ldots, \lambda_{t+1} \) be \( t+1 \) distinct elements of \( F \). Then

\[
(x+\lambda_i y)^t = x^t + \cdots + \binom{t}{k} x^{t-k} \cdot (\lambda_i y)^k + \cdots + \lambda_i^t y^t \in M \quad \text{for each} \quad i = 1, \ldots, t+1.
\]

Consider the following system of non-homogeneous equations in \( t+1 \) variables \( \xi_1, \ldots, \xi_{t+1} \):

\[
\begin{align*}
\sum_{i=1}^{t+1} \lambda_i^m \xi_i &= 0 \quad \text{if} \quad m \neq j, 1 \leq m \leq t, \\
\sum_{i=1}^{t+1} \lambda_i^j \xi_i &= 1, \\
\sum_{i=1}^{t+1} \xi_i &= 0.
\end{align*}
\]

(1)

Since the determinant of the coefficients of (1) is

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_{t+1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^t & \lambda_2^t & \cdots & \lambda_{t+1}^t \\
\end{vmatrix}
\]

it follows that (1) has a solution, say \( \xi_i = d_i, i = 1, \ldots, t+1 \).

Hence

\[
\sum_{i=1}^{t+1} d_i (x+\lambda_i y)^t = \binom{t}{j} x^{t-j} y^j \quad \text{is in the span of } M.
\]

Since \( \binom{t}{j} \neq 0 \), we conclude that \( x^{t-j} y^j \) is in the span of \( M \).

4.7. Theorem. Let \( W \) be a finite dimensional vector space over a field \( F \) of characteristic 0 or of characteristic greater than \( r \). Then \( rW \) is spanned by \( M = \{ u^r : u \in W \} \).
Proof: We proceed by induction and assume that any pure vector having less than \( k + 1 \) distinct factors is in \([M]\), the span of \( M \), where \( 2 \leq k < r \). Let \( A \in \mathbb{W} \) have exactly \( k + 1 \) distinct factors. It is clear that we may write \( A = z_1 \cdots z_t \cdot z_{t+1} \cdots z_r \) in such a way that \( z_1 \cdots z_t \) has exactly 2 distinct factors and \( z_{t+1} \cdots z_r \) has exactly \( k - 1 \) distinct factors. By Lemma 4.6, we have

\[
z_1 \cdots z_t = \sum_{i=1}^{t+1} d_i v_i^t
\]

for some \( d_i \) in \( F \) and some \( v_i \) in \( W \). Hence

\[
A = (\sum_{i=1}^{t+1} d_i v_i^t) \cdot z_{t+1} \cdots z_r = \sum_{i=1}^{t+1} d_i (v_i^t \cdot z_{t+1} \cdots z_r).
\]

By the induction hypothesis, \( v_i^t \cdot z_{t+1} \cdots z_r \in [M] \) since \( v_i^t \cdot z_{t+1} \cdots z_r \) has at most \( k \) distinct factors. Therefore \( A \in [M] \). This proves that \( \mathbb{W} \) has \( M \) as a spanning set.

Theorem 4.7 is well-known (see [2], p. 131). We have included a proof for the sake of completeness.

From now on, we assume that \( U \) is an \( n \)-dimensional vector space over an algebraically closed field \( F \) with \( \text{char } F = 0 \) or \( \text{char } F > r \). We also assume that \( 3 \leq n < r + 1 \). Let \( T : \mathbb{U} \to \mathbb{U} \) be a
rank one preserver. Since every type \( k \) subspace of \( \mathbb{V}^r \) has dimension \( k + 1 < r + 1 \) where \( 1 < k < r \), and every type one subspace of \( \mathbb{V}^r \) has dimension \( n < r + 1 \), we see that every type \( r \) subspace of \( \mathbb{V}^r \) is mapped onto a type \( r \) subspace of \( \mathbb{V}^r \) under \( T \).

4.8. Lemma. If there are two distinct type \( r \) pure subspaces \( M \) and \( N \) of \( \mathbb{V}^r \) such that \( M \cap N \neq 0 \) and \( T(M) = T(N) \), then \( T(\mathbb{V}^r) = T(M) \).

Proof: Let \( M = S_1^{(r)} \), \( N = S_2^{(r)} \), and \( T(M) = T(N) = S^{(r)} \) where \( S, S_1, S_2 \) are two dimensional subspaces of \( U \). By hypothesis,

\[
M \cap N = S_1^{(r)} \cap S_2^{(r)} = (S_1 \cap S_2)^{(r)} \neq 0.
\]

Hence \( S_1 \cap S_2 \neq 0 \). Let \( S_1 = \langle y_1, y_2 \rangle \), \( S_2 = \langle y_1, y_3 \rangle \). Consider \( S_3 = \langle y_2, y_3 \rangle \). Then \( S_3^{(r)} \cap S_2^{(r)} = \langle y_3^{(r)} \rangle \), \( S_3^{(r)} \cap S_1^{(r)} = \langle y_2^{(r)} \rangle \).

Hence \( T(S_3^{(r)}) \cap S^{(r)} \supseteq \langle T(y_3^{(r)}), T(y_2^{(r)}) \rangle \). Since \( T \) is a rank one preserver and \( \langle y_2^{(r)}, y_3^{(r)} \rangle \) is a two dimensional pure subspace, it follows that \( \langle T(y_2^{(r)}), T(y_3^{(r)}) \rangle \) is two dimensional. Hence \( T(S_3^{(r)}) = S^{(r)} \) because any two distinct type \( r \) subspaces of \( \mathbb{V}^r \) have at most one dimension in common.

Let \( z = \alpha y_1 + \beta y_2 + \gamma y_3 \) where \( \alpha, \beta, \gamma \) are all non-zero scalars. Consider \( S_4 = \langle y_1, z \rangle = \langle y_1, \beta y_2 + \gamma y_3 \rangle \). Since
\[ S_4(r) \cap S_3(r) \supseteq \langle (\beta y_2 + \gamma y_3)^T \rangle, \quad S_4(r) \cap S_1(r) \supseteq \langle y_1^r \rangle, \]

we have \( T(S_4(r)) \cap S_3(r) \supseteq \langle T(y_1^r), T((\beta y_2 + \gamma y_3)^r) \rangle \) which is two dimensional.

Hence \( T(S_4(r)) = S_3(r) \). Consequently, \( T(V\langle y_1, y_2, y_3 \rangle) = S_3(r) \) since \( V\langle y_1, y_2, y_3 \rangle \) is spanned by all pure products \( y^r \) where \( y \in \langle y_1, y_2, y_3 \rangle \) (Theorem 4.7).

Now, let \( w \in U \) such that \( w \notin \langle y_1, y_2, y_3 \rangle \). Let \( W = \langle y_1, w \rangle \).

Consider the type one subspace \( y_1 \cdots y_1 U \). Let \( P = y_1 \cdots y_1 U \). Since \( \dim (P \cap V\langle y_1, y_2, y_3 \rangle) = 3 \), we have \( \dim (T(P) \cap S_3(r)) \geq 3 \). Since the maximal dimension of the intersections of two distinct maximal pure subspaces is 2, we conclude that \( T(P) \subseteq S_3(r) \). We observe that \( T(W(r)) \cap S_3(r) \supseteq \langle T(y_1^r), T(y_1 \cdots y_1 w) \rangle \). Since \( \langle y_1^r, y_1^{r-1} w \rangle \) is a two-dimensional pure subspace, \( \langle T(y_1^r), T(y_1 \cdots y_1 w) \rangle \) is also two-dimensional. Hence \( T(W(r)) = S_3(r) \). By Theorem 4.7, we conclude that \( T(VU) = S_3(r) \). This completes the proof.

4.9. Lemma. Suppose that for any two distinct type \( r \) subspaces \( M, N \) such that \( M \cap N \neq 0 \), we have \( T(M) \neq T(N) \). Then \( T \) is induced by a non-singular transformation on \( U \).

Proof: Let \( y \) be any non-zero vector of \( U \). Let \( y, y_1, y_2 \) be linearly independent vectors. Let \( S_1 = \langle y, y_1 \rangle, S_2 = \langle y, y_2 \rangle \). Let
\[ T(S_1) = S'_1, \quad T(S_2) = S'_2 \]
where \( S'_1, S'_2 \) are two dimensional subspaces of \( U \). Since \( S_1 \not\perp S_2, S_1 \cap S_2 \not\perp 0 \), by hypothesis, \( S'_1 \not\perp S'_2 \).

Therefore \( S'_1 \cap S'_2 = T(S_1 \cap S_2) = \langle y'^r \rangle \) for some \( y' \in U \).

Hence \( T(y^r) = \lambda y'^r \) for some \( \lambda \) in \( F \).

We claim that \( T(y \cdots y \cdot U) = y' \cdots y' \cdot U \). Since \( T(y \cdots y \cdot U) \) is a pure subspace, it is contained in a maximal pure subspace. If \( T(y \cdots y \cdot U) \) is contained in a type \( k \) pure subspace \( g_1 \cdots g_{r-k} W(k) \) where \( 2 \leq k < r \), then \( y'^r \in g_1 \cdots g_{r-k} W(k) \) and hence \( \langle g_1 \rangle = \langle y' \rangle \), \( y' \in W \).

This implies that \( g_1 \in W \), a contradiction. If \( T(y \cdots y \cdot U) \) is contained in a type \( r \) subspace \( W(r) \), then

\[ \dim (S_1 \cap y \cdots y \cdot U) = 2 \implies \dim (T(S_1) \cap W(r)) \geq 2; \]
and \( \dim (S_2 \cap y \cdots y \cdot U) = 2 \implies \dim (T(S_2) \cap W(r)) \geq 2 \).

Since \( T(S_1) \) and \( T(S_2) \) are both type \( r \) subspaces, it follows that

\[ T(S_1) = W(r) = T(S_2), \quad \text{a contradiction to our hypothesis.} \]

Hence \( T(y \cdots y \cdot U) \) is a type one pure subspace of \( V \cdot U \) since there are no power type pure subspaces under the assumption that \( \text{char } F = 0 \) or \( \text{char } F > r \).

Since \( y'^r \in T(y \cdots y \cdot U) \), it follows that \( T(y \cdots y \cdot U) = y' \cdots y' \cdot U \).
By Theorem 4.7, let $x_1^{r-1}, \ldots, x_t^{r-1}$ be a basis of $V^r U$.

Note that $3 \leq \dim U < r + 1$ implies that $r \geq 3$. Clearly, if $i \neq j$, then $x_i$ and $x_j$ are linearly independent. Consider any type one subspace $z_1^{r-1} \cdot U$. Let $z_1^{r-1} \cdot z_{r-1} = \sum_{i=1}^{t} \lambda_i x_i^{r-1}$ where $\lambda_i \in F$ and $i = 1, \ldots, t$.

We shall show that $T(z_1^{r-1} \cdot U)$ is a type one pure subspace. Suppose to the contrary that (i) $T(z_1^{r-1} \cdot U) \subseteq S_r$ or (ii) $T(z_1^{r-1} \cdot U) \subseteq v_1^{r-k} \cdot S_k$, $2 < k < r$, for some two dimensional subspace $S$ of $U$ and some $v_1^{r-k} \in U - S$.

Let $T(x_1^{r-1} \cdot U) = x_1^r \cdot x_1^{r-1} U$, $i = 1, \ldots, t$. Note that we have $T(x_i^r) = \eta_i x_i^r$ for some $\eta_i \in F$ where $i = 1, \ldots, t$. For $j \neq i$, $<x_i^r, x_j^r>$ is a two dimensional pure subspace of $V^r U$ implies that $T(x_i^r) = x_i^r \cdot x_i^r >$ is a two dimensional pure subspace of $V^r U$.

Therefore $x_i^r$ and $x_j^r$ are linearly independent if $i \neq j$.

Consider case (ii). We choose a vector $v$ of $U$ such that $v \notin <v_1^{r-1} \cup \ldots \cup v_{r-k}^{r-1} \cup S \cup (\bigcup_{i \neq j} <x_i^r, x_j^r>)$. This is possible since $<v_1^r, S,\ldots>$, $i \neq j$ and $<x_1^r, x_1^r>$ are all proper subspaces of $U$. Let $T(x_1^{r-1} \cdot U) = x_1^{r-1} \cdot v$.

For each $i > 2$, let $T(x_i^{r-1} \cdot u) = x_i^{r-1} \cdot u_i$.

We shall show that $<u_i^r > = <v>$ for $i = 2, \ldots, t$. Since $<x_1^{r-1} \cdot u, x_i^{r-1} \cdot u>$ is a pure subspace for $i \geq 2$, $<x_1^r \cdot v, x_i^r \cdot u_i>$ is also
a pure subspace. By our choice of \( v \), \( \langle x_1', v, x_j' \rangle \) is three dimensional, hence from Lemma 1.9, \( x_1' \cdot v \) and \( x_j' \cdot u \) have a common factor, say, \( \langle \omega_i \rangle \). Note that we must have \( \langle \omega_i \rangle = \langle w_i \rangle \), \( i > 2 \). Either \( \langle \omega_i \rangle = \langle x_1' \rangle \) or \( \langle \omega_i \rangle = \langle v \rangle \). If \( \langle \omega_i \rangle = \langle x_1' \rangle \), then \( \langle x_1' \cdot v, x_j' \cdot u \rangle \) is a pure subspace implies that \( \langle x_1^{r-2} \cdot v, x_1^{r-1} \rangle \) is a pure subspace (Lemma 1.8).

Consequently, by Lemma 1.9, \( x_1^{r-2} \cdot v \) and \( x_1^{r-1} \) have a common factor, a contradiction. Therefore \( \langle \omega_i \rangle = \langle v \rangle \), \( i > 2 \).

Now, we have \( u_i = a_i \cdot v \) for some \( a_i \in F \), \( i > 2 \) and

\[
T(z_1 \cdots z_{r-1} \cdot u) = T(\sum_{i=1}^{t} \lambda_i x_i^{r-1} \cdot u) = \lambda_1 x_1^{r-1} \cdot v + \sum_{i=2}^{r-1} \lambda_i x_i^{r-1} \cdot a_i \cdot v
\]

\[
= (\lambda_1 x_1^{r-1} + \sum_{i=2}^{r-1} \lambda_i a_i x_i') \cdot v
\]

Let \( T(z_1 \cdots z_{r-1} \cdot u) = w_1 \cdots w_r \). Then

\[
(\sum_{i=2}^{r-1} \lambda_i a_i x_i') \cdot v = w_1 \cdots w_r \# 0.
\]

In view of Lemma 1.8, \( \langle w_j \rangle = \langle v \rangle \) for some \( j \), \( 1 \leq j \leq r \). Since \( w_1 \cdots w_r \in v_1 \cdots v_{r-k} \cdot S(k) \), we have \( \langle v \rangle = \langle v_i \rangle \) for some \( i \) or \( v \perp S \).

This contradicts our choice of \( v \). Hence

\[
T(z_1 \cdots z_{r-1} \cdot u) \notin v_1 \cdots v_{r-k} \cdot S(k).
\]
Similarly $T(z_1 \cdots z_{r-1} U) \subseteq S(r)$. Hence $T(z_1 \cdots z_{r-1} U)$ is a type 

one pure subspace of $V^U$. In view of Theorem 3.13, $T$ is induced by a non-singular linear transformation on $U$.

Combining the above two Lemmas, we have the following main result:

4.10. Theorem. Let $T : V^U \to V^U$ be a rank one preserver where $U$ is 

a finite dimensional vector space over an algebraically closed field 

$F$ with characteristic equal to $0$ or greater than $r$. If $3 \leq \dim U < r + 1$, 

then either $T$ is induced by a non-singular linear transformation on $U$ 

or $T(V^U)$ is a type $r$ subspace. In particular, if $T$ is non-singular, 

then $T$ is induced by a non-singular transformation on $U$.

We have so far not been able to determine whether there does 

in fact exist a rank one preserver on $V^U$ such that its image is a type 

$r$ subspace when $3 \leq \dim U < r + 1$. 


CHAPTER III

RANK K PRESERVERS ON GRASSMANN SPACES

Let $U$ be an $n$-dimensional vector space over an algebraically closed field $F$ of characteristic not equal to two. Let $\Lambda^r U$ be the $r^{th}$ Grassmann product space of $U$. If $T : \Lambda^r U \to \Lambda^r U$ is a rank one preserver, then the structure of $T$ is known: $T$ is an $r^{th}$ compound of a non-singular linear transformation of $U$, except possibly when $n = 2r$, in which case $T$ may be the composite of a compound and a linear transformation induced by a correlation of the $r$-dimensional subspaces of $U$ [18].

In this chapter, we shall study the structure of rank $k$ preservers on $\Lambda^2 U$ for a fixed positive integer $k$. We show that a rank $k$ preserver $T$ on $\Lambda^2 U$ is also a rank one preserver on $\Lambda^2 U$, provided that $T$ is non-singular or $n = 2k$ or $k = 2$.

Denote by $H_n$ the set of all $n$-square skew-symmetric matrices over $F$. A linear transformation on $H_n$ is called a rank $2k$ preserver if every rank $2k$ matrix in $H_n$ is mapped into another rank $2k$ matrix.

Let $u_1, \ldots, u_n$ be a basis of $U$. Let $\phi : \Lambda U \to H_n$ be a mapping defined by

$$\phi(u_i \wedge u_j) = E_{ij} - E_{ji}, \quad 1 \leq i < j \leq n,$$

and extend linearly to all $\Lambda^2 U$. $E_{ij}$ denotes the $n$-square matrix with 1 in position $i$, $j$ and 0 elsewhere. It is shown in [15], that $\phi$ is an isomorphism of $\Lambda^2 U$ onto $H_n$ such that for each positive integer $k$, the set, $R_k(\Lambda U)$, of all rank $k$ vectors in $\Lambda^2 U$ is mapped under $\phi$ onto the set of all rank $2k$ matrices in $H_n$. Moreover, $T$ is a rank $k$ preserver on $\Lambda^2 U$ if and only if $\phi T \phi^{-1}$ is a rank $2k$ preserver on $H_n$. 
1. **Definition.** For each \( z \in R_k(\Lambda U) \), we write \( R(z) = k \).

2. **Theorem.** Let \( x_1, y_1, \ldots, x_k, y_k \) be 2k vectors of \( U \). Then
\[
\sum_{i=1}^{k} x_i \Lambda y_i \in R_k(\Lambda U) \text{ if and only if } x_1, y_1, \ldots, x_k, y_k \text{ are linearly independent (see [8], Theorem 7).}
\]

3. **Lemma.** Let \( T : \Lambda^2 U \rightarrow \Lambda^2 U \) be a rank \( k \) preserver for some positive integer \( k \). If \( z \in R_\ell(\Lambda U) \), \( \ell < k \), then \( R(T(z)) \leq k \).

**Proof:** Clearly we are able to choose a vector \( w \) in \( \Lambda U \) such that \( z + \lambda w \in R_k(\Lambda U) \) for all non-zero \( \lambda \) in \( F \). Suppose that \( R(T(z)) = m \). Then \( \phi(T(z)) \) is of rank \( 2m \). Since \( T \) is a rank \( k \) preserver,
\[
T(z+w) = T(z) + \lambda T(w) \text{ is of rank } k \text{ for all } \lambda \neq 0.
\]
It follows that \( \phi(T(z) + \lambda T(w)) = \phi(T(z)) + \lambda \phi(T(w)) \) is a rank \( 2k \) skew-symmetric matrix for each non-zero \( \lambda \) in \( F \). On the other hand, since \( F \) is an infinite field, we can choose a \( \lambda_o \) in \( F \), \( \lambda_o \neq 0 \), such that \( \phi(T(z)) + \lambda_o \phi(T(w)) \) is of rank greater than or equal to \( 2m \). Hence \( 2m \leq 2k \). This proves the Lemma.

4. **Lemma.** If \( T : \Lambda^1 U \rightarrow \Lambda^1 U \) is a rank one preserver, then \( T \) is also a rank \( k \) preserver for all possible \( k \).

**Proof:** From [18], we know that \( T^2 \) is a compound of a non-singular linear transformation of \( U \). In view of Theorem I.2.15, \( T^2 \) is a rank \( k \) preserver. Now, let \( x = \sum_{i=1}^{k} x_i \in R_k(\Lambda U) \) where \( x_i \) is of rank one for each \( i \). Let
\[ T(x) = \sum_{i=1}^{t} y_i \in R_t(\Lambda U) \] where \( y_i \in R_t(\Lambda U), 1 \leq i \leq t \). Since
\[ \sum_{i=1}^{k} y_i = \sum_{i=1}^{t} T(x_i), \] we have \( t \leq k \). Now \( T^2(x) = T(\sum_{i=1}^{t} y_i) = \sum_{i=1}^{t} T(y_i) \)
and \( T(y_i) \in R_t(\Lambda U), i = 1, \ldots, t \). Since \( T^2 \) is a rank \( k \) preserver,
\( t \geq k \). Consequently \( t = k \). Therefore \( T \) is a rank \( k \) preserver.

5. Theorem. Let \( T : \Lambda U \to \Lambda U \) be a rank \( k \) preserver. If \( T \) is
non-singular, then \( T \) is a rank one preserver.

Proof: Case 1. \( \dim U \geq 2k + 2 \). By the non-singularity of \( T \), we
shall show that if there is an \( A \in R_t(\Lambda U) \) such that \( R(T(A)) = \ell \leq k \),
then there is a \( B \in \Lambda U \) such that \( R(B) \leq t + 1 \) and \( R(T(B)) \geq \ell + 1 \).

Indeed, suppose
\[ T(A) = u_1 \wedge u_2 + \cdots + u_{2 \ell - 1} \wedge u_{2 \ell} \]
for some independent vectors \( u_1, \ldots, u_{2 \ell} \) of \( U \) where \( 1 \leq \ell \leq k \).

Extend \( u_1, \ldots, u_{2 \ell} \) to a basis \( u_1, \ldots, u_{2 \ell}, \ldots, u_n \) of \( U \).

Since \( T \) is onto, there exists a vector \( z \in \Lambda U \) such that
\[ T(z) = u_{2 \ell + 1} \wedge u_{2 \ell + 2} \]. Let \( z = z_1 + \cdots + z_s \in R_s(\Lambda U) \) where
\( z_m \in R_t(\Lambda U), m = 1, \ldots, s \). Let \( T(z_m) = \sum_{i<j}^{s} \alpha_{mij} u_i \wedge u_j \) where
\( \alpha_{mij} \in F \). Then
\[ T(z) = \sum_{m=1}^{s} T(z_m) = \sum_{m=1}^{s} (\sum_{i<j}^{s} \alpha_{mij} u_i \wedge u_j) = u_{2 \ell + 1} \wedge u_{2 \ell + 2} \].
This implies that for some \( m_0 \), we have \( a_{m_0,2\ell+1,2\ell+2} \neq 0 \).

Hence there is a \( w \) in \( R_1(\Lambda U) \) such that

\[
T(w) = \sum_{i<j} a_{ij} u_i u_j + a_{2\ell+1,2\ell+2} u_{2\ell+1,2\ell+2} + 0.
\]

Hence the \( 2\ell+1,2\ell+2 \) entry of \( \phi(T(w)) \) is \( a_{2\ell+1,2\ell+2} \). For each \( \lambda \) in \( F \), the minor of order \( 2\ell + 2 \) in the upper left corner of the matrix \( \phi(T(A+\lambda w)) \) is of the form

\[
a_{2\ell+1,2\ell+2}^2 \lambda^2 + b_1 \lambda^3 + b_2 \lambda^4 + \cdots
\]

Since \( F \) is infinite and \( a_{2\ell+1,2\ell+2} \neq 0 \), (1) is non-zero for some non-zero \( \lambda_0 \) in \( F \). Hence \( \phi(T(A+\lambda_0 w)) \) has rank \( \geq 2\ell + 2 \).

Therefore \( R(T(A+\lambda_0 w)) \geq \ell + 1 \). However \( R(A+\lambda_0 w) \leq R(A) + 1 \).

Now, assume that \( T \) is not a rank one preserver. Then some rank one element \( A_1 \) is mapped to a rank \( j \) element, \( j > 1 \), since \( T \) is non-singular. By Lemma 3, \( j \leq k \). In view of the above argument, there is an \( A_2 \in \Lambda U \) such that \( R(A_2) \leq 2 \) and \( R(T(A_2)) \geq j + 1 > 2 \). If \( j + 1 \leq k \), continue the process as above, we see that eventually some element of \( \Lambda U \) of rank less than or equal to \( k \) is mapped to an element of \( \Lambda U \) of rank greater than \( k \). This contradicts Lemma 3. Hence \( T \) is a rank one preserver.
Case 2. \( \dim U \leq 2k + 1 \). We first note that the maximal rank of all elements in \( \mathcal{A}U \) is \( k \) (Theorem 2). Assume that \( T^{-1} \) is not a rank one preserver. Then some rank one element is mapped under \( T^{-1} \) to a rank \( \ell \) element, \( \ell > 1 \). If \( \ell \not\equiv k \), by the non-singularity of \( T^{-1} \), there exists a \( B \in \mathcal{A}U \) such that \( R(B) \leq 2 \) and \( R(T^{-1}(B)) \geq \ell + 1 \). If \( \ell + 1 \not\equiv k \), continue the process, we see that eventually there is a \( z \in \mathcal{A}U \) such that \( T^{-1}(z) = w \in R_k^{2}(\mathcal{A}U) \) for some \( w \in \mathcal{A}U \) and \( R(z) < k \).

Hence \( T(w) = z \), contradicting the fact that \( T \) is a rank \( k \) preserver. Therefore \( T^{-1} \) is a rank one preserver. By Lemma 4, \( T^{-1} \) preserves all ranks. Hence \( T \) is a rank one preserver.

6. Remark. The proof of Theorem 6 is analogous to that of Theorem 3.1 [1] and Theorem 1 [6].

7. Remark. It is well-known that the maximal dimension of a rank \( n \) subspace (a subspace with every non-zero element of rank \( n \)) in \( M_{n,n}(F) \), the vector space of all \( n \)-square matrices over an algebraically closed field \( F \), is one. Hence the maximal dimension of a rank \( k \) subspace in \( \mathcal{A}U \) is one if \( \dim U = 2k \), since \( \phi: \mathcal{A}U \rightarrow H_n \) is an onto isomorphism such that \( \phi(R_k^{2}(\mathcal{A}U)) \) is the set of all rank \( 2k \) matrices in \( H_n \).

8. Theorem. Let \( T: \mathcal{A}U \rightarrow \mathcal{A}U \) be a rank \( k \) preserver. If \( \dim U = 2k \), then \( T \) is a rank one preserver.

Proof: In view of Theorem 5, it suffices to show that \( T \) is non-singular.
Assume that $T$ is singular. Then some $z \in R_{2k}(\wedge U)$ is such that 

$$T(z) = 0, \ z \neq 0.$$ 
Clearly $t \neq k$, since $T$ is a rank $k$ preserver.

Let $w \in \wedge U$ such that $R(z+w) = k$ and $R(w) = k-t$. Hence 

$$T(z+w) = T(z) + T(w) = T(w) \in R_{2k}(\wedge U).$$

Assume that $w = u_1 \wedge u_2 + \cdots + u_{2j-1} \wedge u_{2j}$ where $j = k-t$.

Extend $u_1, \cdots, u_{2j}$ to a basis $u_1, \cdots, u_{2j}, \cdots, u_{2k}$ of $U$.

Let $w^* = u_2 \wedge u_3 + \cdots + u_{2k} \wedge u_1$. Then for $a, b$ in $F$,

$$aw + bw^* = u_2 \wedge (bu_3 - au_1) + \cdots + u_{2j} \wedge (bu_{2j+1} - au_{2j-1}) + bu_{2j+2} \wedge u_{2j+3} + \cdots + bu_{2k} \wedge u_1.$$ 

Clearly if $b \neq 0$, $R(aw+bw^*) = k$ and hence $T(aw+bw^*) \in R_{2k}(\wedge U)$.

If $b = 0$ and $a \neq 0$, then $T(aw+bw^*) = T(aw) \in R_{2k}(\wedge U)$.

Therefore if $a$ and $b$ are not both zero, then $T(aw+bw^*) \in R_{2k}(\wedge U)$.

It follows that $<T(w), T(w^*)>$ is a rank $k$ subspace of $2\wedge U$ with dimension equal to two. This contradicts the fact that the maximal dimension of a rank $k$ subspace of $2\wedge U$ is one. Therefore $T$ is non-singular and the proof is complete.

9. Lemma. If \[ \dim \langle x_1, y_1, \cdots, x_{k+1}, y_{k+1} \rangle = 2k+1 \] and $x_i \wedge y_i \neq 0$ for all $i = 1, \cdots, k+1, k \geq 1$, then

$$\sum_{i=1}^{k+1} x_i \wedge y_i \in R_{2k}(\wedge U).$$
Proof: It is clear that \( \sum_{i=1}^{k+1} x_i \wedge y_i \) has rank less than \( k+1 \) because \( x_1, y_1, \ldots, x_{k+1}, y_{k+1} \) are linearly dependent. Also there is one \( j \) such that \( \dim \langle x_1, y_1, \ldots, \hat{x}_j, \hat{y}_j, \ldots, x_{k+1}, y_{k+1} \rangle = 2k \). Hence \( \sum_{i+j} x_i \wedge y_i \in R_k(\wedge U) \). Since \( x_j \) or \( y_j \notin \langle x_1, y_1, \ldots, \hat{x}_j, \hat{y}_j, \ldots, x_{k+1}, y_{k+1} \rangle \), by Corollary I.2.6, \( \sum_{i=1}^{k+1} x_i \wedge y_i \) is of rank at least \( k \). This proves that \( \sum_{i=1}^{k+1} x_i \wedge y_i \) is of rank \( k \).

10. Example. The following is a 3-dimensional rank \( k \) subspace of \( \wedge U \) when \( \dim U = 2k + 1, k \geq 2 \).

Let \( u_1, \ldots, u_{2k+1} \) be a basis of \( U \). Let

\[
\begin{align*}
f_1 &= u_1 \wedge u_{k+1} + u_2 \wedge u_{k+2} + \cdots + u_k \wedge u_{2k} \\
f_2 &= u_1 \wedge u_{k+2} + u_2 \wedge u_{k+3} + \cdots + u_k \wedge u_{2k+1} \\
f_3 &= u_{k+1} \wedge u_{k+2} + u_1 \wedge u_{k+3} + \cdots + u_{k-1} \wedge u_{2k+1}.
\end{align*}
\]

Clearly \( f_1, f_2, f_3 \in R_k(\wedge U) \). Let \( a_1, a_2, a_3 \) be non-zero elements in \( F \). We have

\[
\sum_{i=1}^3 a_i f_i = a_3 u_{k+1} \wedge u_{k+2} + u_{k+1} \wedge (a_1 u_{k+1} + a_2 u_{k+2} + a_3 u_{k+3}) \\
+ \cdots + u_{k-1} \wedge (a_1 u_{2k-1} + a_2 u_{2k} + a_3 u_{2k+1}) + u_k \wedge (a_1 u_{2k} + a_2 u_{2k+1}).
\]
Since \( \dim <u_{k+1} u_{k+2} u_1, \sum_{i=1}^{3} a_i u_{k+i}, \cdots, u_{k-1}, \sum_{i=1}^{3} a_i u_{2k-2+i}, u_k, a_1 u_{2k} + a_2 u_{2k+1} > = 2k+1 \), by Lemma 9, \( \sum_{i=1}^{3} a_i f_i \) is of rank \( k \).

Similarly, using Lemma 9, we check that both \( a_1 f_1 + a_3 f_3 \), \( a_2 f_2 + a_3 f_3 \) are of rank \( k \). By Theorem 2, \( a_1 f_1 + a_2 f_2 \) is of rank \( k \).

Hence \( f_1, f_2, f_3 \) are linearly independent and \( <f_1, f_2, f_3> \) is a rank \( k \) subspace of \( \Lambda^2 U \).

It is known that every rank \( m \) subspace of \( M_{m+1,m+1}(F) \) is at most 4-dimensional (see [21]). We feel that every rank \( 2k \) subspace in \( H_{2k+1} \) is most likely of dimension \( \leq 3 \).

**11. Theorem.** Let \( T : \Lambda^2 U \to \Lambda^2 U \) be a rank \( k \) preserver. Suppose \( \dim U = 2k+1 \) and the maximal dimension of all rank \( k \) subspaces of \( \Lambda^2 U \) is \( 3 \). Then \( T \) is a rank one preserver.

**Proof:** Assume that \( T \) is singular. Then \( k \geq 2 \) and for some \( z \in R_t(\Lambda^2 U) \), \( T(z) = 0 \) where \( t < k \). Since \( T(R_k(\Lambda^2 U)) \subseteq R_k(\Lambda^2 U) \), \( t \perp k \).

Choose a vector \( f_0 \) in \( R_{k-t}(\Lambda^2 U) \) such that \( z + f_0 \) is of rank \( k \).

Denote \( f_0 \) by \( u_1 \land u_k + u_2 \land u_{k+1} + \cdots + u_j \land u_{j+k-1} \) where \( j = k - t \).

Then \( T(z + f_0) = T(z) + T(f_0) = T(f_0) \in R_k(\Lambda^2 U) \).
Extend \( u_1, u_k, \ldots, u_j, u_{j+k-1} \) to a basis \( u_1, \ldots, u_{2k+1} \) of \( U \). Consider

\[
\begin{align*}
\mathbf{f}_1 &= u_1 \land u_{k+1} + u_2 \land u_{k+2} + \cdots + u_k \land u_{2k} \\
\mathbf{f}_2 &= u_1 \land u_{k+2} + u_2 \land u_{k+3} + \cdots + u_k \land u_{2k+1} \\
\mathbf{f}_3 &= u_{k+1} \land u_{k+2} + u_1 \land u_{k+3} + \cdots + u_{k-1} \land u_{2k+1} 
\end{align*}
\]

Let \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) be four non-zero scalars in \( F \). Then

\[
\sum_{i=0}^{3} \alpha_i \mathbf{f}_i = \alpha_3 u_{k+1} \land u_{k+2} + u_1 \land (\alpha_0 u_{k+1} + \alpha_1 u_{k+2} + \alpha_2 u_{k+3} + \alpha_3 u_{k+4}) + \cdots \\
+ u_j \land (\alpha_0 u_{k+j-1} + \alpha_1 u_{k+j} + \alpha_2 u_{k+j+1} + \alpha_3 u_{k+j+2}) + \cdots + u_k \land (\alpha_1 u_{2k} + \alpha_2 u_{2k+1}) .
\]

Since

\[
\dim <u_{k+1}, u_{k+2}, u_1, \sum_{i=0}^{3} \alpha_i u_{k+i}, \ldots, u_j, \sum_{i=0}^{3} \alpha_i u_{k+j-i}, u_{j+1}, \sum_{i=1}^{3} \alpha_i u_{k+j+i}, \ldots, u_{k+1}, \alpha_1 u_{2k} + \alpha_2 u_{2k+1} > = 2k + 1 ,
\]

by Lemma 9, \( \sum_{i=0}^{3} \alpha_i \mathbf{f}_i \) is of rank \( k \). Similarly, using Lemma 9, we check that \( \alpha_0 \mathbf{f}_0 + \alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2 + \alpha_3 \mathbf{f}_3 \) are of rank \( k \).

Also we have \( \alpha_0 \mathbf{f}_0 + \alpha_1 \mathbf{f}_1 \in R_k(\wedge U) \) and \( \alpha_0 \mathbf{f}_0 + \alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2 \in R_k(\wedge U) \).
Since \( T \) is a rank \( k \) preserver and \( T(f_0) \in R^2_k(\Lambda U) \), from the above argument, we see that

\[
\sum_{i=0}^{3} c_i T(f_i) = \sum_{i=0}^{3} T(c_i f_i) \in R^2_k(\Lambda U)
\]

if one of \( c_i \neq 0 \). Hence \( \sum_{i=0}^{3} d_i T(f_i) = 0 \) implies that all \( d_i = 0 \).

Therefore \( T(f_0), \ldots, T(f_3) \) generate a 4-dimensional rank \( k \) subspace of \( \Lambda U \) which is a contradiction to our hypothesis. Hence \( T \) is non-singular. This proves that \( T \) is a rank one preserver by Theorem 5.

12. **Corollary.** Let \( T \) be a rank 2 preserver on \( \Lambda^2 U \). If \( \dim U = 5 \), then \( T \) is a rank one preserver.

**Proof:** This follows from Theorem 11 and the fact that every rank 2 subspace of \( \Lambda^2 U \) has dimension less than or equal to 3 if \( \dim U = 5 \) [9].

13. **Theorem.** Let \( T : \Lambda^2 U \rightarrow \Lambda^2 U \) be a rank 2 preserver. Then \( T \) is a rank one preserver.

**Proof:** Because of Theorem 8 and Corollary 12, it suffices to consider the case when \( \dim U > 6 \). It has been shown in [9], that every rank 2 subspace of \( \Lambda^2 U \) has dimension \( \leq n - 3 \) if \( \dim U = n \geq 6 \).
Assume that $T$ is not a rank one preserver. By Lemma 3, some rank one vector is mapped to the zero vector or a rank 2 vector. If some rank one vector is mapped to the zero vector, it is clear that some rank one vector is mapped to a rank 2 vector since $T$ is a rank 2 preserver. Hence $T(u_1 \wedge v_1) \in R_2(\Lambda U)$ for some independent vectors $u_1, v_1$.

Extend $u_1, v_1$ to a basis $u_1, u_2, v_1, \ldots, v_{n-2}$ of $U$.

Consider the following vectors:

$$f_1 = u_1 \wedge v_1$$
$$f_2 = u_1 \wedge v_2 + u_2 \wedge v_1$$

$$f_{n-2} = u_1 \wedge v_{n-2} + u_2 \wedge v_{n-3}.$$ 

Let $a_1, \ldots, a_{n-2}$ be $n-2$ elements in $F$. Then

$$\sum_{i=1}^{n-2} a_i f_i = u_1 \wedge (\sum_{i=1}^{n-2} a_i v_i) + u_2 \wedge (\sum_{i=1}^{n-3} a_{i+1} v_i).$$

$$\sum_{i=1}^{n-2} a_i f_i \in R_1(\Lambda U) \text{ implies that } \sum_{i=1}^{n-2} a_i v_i \text{ and } \sum_{i=1}^{n-3} a_{i+1} v_i \text{ are linearly dependent and thus } a_{n-2} = \ldots = a_2 = 0, a_1 \neq 0. \text{ Hence } \sum_{i=1}^{n-2} a_i f_i \in R_2(\Lambda U)$$

when some $a_i \neq 0$ for $2 \leq i \leq n-2$. Since $T$ is a rank 2 preserver
and \( T(u_1 \wedge v_1) \in R_2(\Lambda U) \), it follows that \( R(T(\sum_{i=1}^{n-2} a_i f_i)) = 2 \) whenever some \( a_i \neq 0 \), \( 1 \leq i \leq n - 2 \). This shows that \( T(f_1), \ldots, T(f_{n-2}) \) are linearly independent and \( \langle T(f_1), \ldots, T(f_{n-2}) \rangle \) is a rank 2 subspace of \( \Lambda U \) of dimension equal to \( n - 2 \). This contradicts the fact that the maximal dimension of a rank 2 subspace is \( n - 3 \). Hence \( T \) is a rank one preserver.

14. **Remark.** Theorem 13 is also obtained independently by M.J.S. Lim [10].

Combining Theorem 5, Theorem 8 and Theorem 13, we have

15. **Theorem.** Let \( T : \Lambda U \to \Lambda U \) be a rank \( k \) preserver. If \( T \) is non-singular or \( \dim U = 2k \) or \( k = 2 \), then \( T \) is a compound of a non-singular transformation of \( U \), except when \( n = 4 \), in which case \( T \) may be the composite of a compound and a linear transformation induced by a correlation of the two dimensional subspaces of \( U \).

Using the results in [15], Theorem 15 can be stated in matrix language as follows:

16. **Theorem.** Let \( S : H \to H \) be a rank \( 2k \) preserver. If \( S \) is non-singular or \( n = 2k \) or \( k = 2 \), then there exists a non-singular \( n \)-square matrix \( P \) such that \( S(A) = PAP' \) for any \( A \) in \( H_n \), except when \( n = 4 \), in which case \( S \) may possibly be of the form:
104.

\[ S(A) = P \begin{bmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{bmatrix} P' \]

where \( A = [a_{ij}] \), \( a_{ij} = -a_{ji} \).

17. Corollary. Let \( S : H_n \rightarrow H_n \) be a linear transformation such that \( \det A = \det S(A) \) for all \( A \) in \( H_n \) where \( n \) is even. Then \( S \) has the form as described in Theorem 16 where \( \det P = \pm 1 \).

Proof: By hypothesis, \( S \) is a rank \( n \) preserver on \( H_n \). Therefore this Corollary follows from Theorem 16 and the fact that

\[
\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix} = \det
\begin{vmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{vmatrix}
\]
BIBLIOGRAPHY


