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MAGNETOELASTIC INTERACTIONS IN THE EARTH'S CORE

by

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ABSTRACT

Previous calculations of the interaction of plane elastic waves in a uniform magnetic field in the Earth's liquid core showed negligible damping of such waves. Subsequent extensions of the theory have treated separately the damping of radial oscillations in a uniform field, and the effect of a field gradient on plane waves.

It has been speculated that enhanced attenuation would take place for standing waves in a field gradient. An additional effect might also be expected from a proper treatment of the field geometry, as within the Earth both magnetic and free-oscillation fields can be expanded in spherical harmonics.

In the present thesis a rigorous evaluation of magnetoelastic interactions in a spherical conductor is given, with a view to clarifying these predictions. The results show that within the Earth's core and at seismic frequencies the interaction is indeed weak. Typical values of the Q of the damping due to magnetic effects are at least 10^{13}. Consideration of a wide range of harmonics in the interaction fails to find a significant effect due to field geometry. The role of viscous damping is evaluated using a recent value for the core viscosity and typical viscous Q's were about 10^{16}.

The possibility of gaining useful information from magnetic or viscous damping of the free oscillations is thus remote, but the importance of the results lies in their extension to core oscillations of longer periods. Such oscillations will also be underdamped and their velocity fields may be suitable for the new turbulent dynamo theories of the Earth's main magnetic field.

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SECTION 1

INTRODUCTION

This thesis is principally concerned with the interaction between magnetic fields and elastic waves in a spherical electrically conducting body. Magnetoelasticity is a term often used with reference to the exchange of elastic and magnetic energy in ferromagnetic crystals (Landau and Lifshitz, 1960, p.155). In the present context the term is used, in a macroscopic sense, to describe the effect of a magnetic field on elastic deformations of a continuous medium. The treatment is directed towards evaluating such effects in the core of the Earth.

1.1 Magnetoelastic Interations

The Earth is known to have a surface magnetic field which is predominantly dipole and of internal origin (Hide and Roberts, 1961). This field cannot be a relic of the past and is considered to be generated continuously by some form of induction process in the Earth's core (Roberts, 1967, Ch.3). To maintain the associated currents the core must be electrically conducting.

A large earthquake produces two types of elastic waves within the Earth. One type consists of two travelling body waves P,S, the other is a harmonic series of standing waves. These standing waves are called equivalently normal modes, free oscillations or eigenvibrations, and in recent years their study has considerably refined seismic models of the Earth's interior (Wiggins, 1972). Because a moving electrical conductor in a magnetic field experiences a Lorentz force, it is natural to ask two questions. Are elastic waves, particularly free oscillations, attenuated by the magnetic field in the

Earth's core? What does a measure of that attenuation indicate physically about either the magnetic field or the elastic properties within the core?

The first important attempt to assess the interaction was made by Knopoff (1955). He calculated the effect of a static, uniform magnetic field on the propagation of plane waves in a semi-infinite medium. For values of magnetic field strength and conductivity probable in the Earth's core (Table 1), it was found that negligible attenuation takes place. However propagation of (a) plane waves in (b) a semi-infinite medium in the presence of (c) a uniform magnetic field is not realistic for all interactions within the core. Subsequent developments, including the present investigation, have been aimed at removing these limitations.

The next step was taken by Kraut (1965) who discussed the attenuation of the radial oscillations of a homogeneous conducting fluid sphere, retaining condition (c). Again the interaction was very weak, the longest period of oscillation, taken to be the fundamental radial mode observed for the real Earth, is decreased by less than two parts in 10^8. This represents an effective Q of the oscillation of order 10^{17}.

Nevertheless a further calculation was made by Lilley (1967) who conjectured that there is enhanced attenuation in the presence of a non-uniform field. This is caused by magnetic induction due to translation of an elastic element through a field gradient in addition to the induction due to volume dilatation of the element present in a uniform field. For travelling waves the effective damping was increased by an order of magnitude within half a wavelength of the origin of the coordinate system (Lilley and Smylie, 1968). Although condition (c) had now been relaxed, conditions (a) and (b) were reinstated. Lilley transferred the result to estimate the damping of the long-period eigenvibrations and deduced a Q of about 10^9 might be reached. This is still a factor of 10^6 above indicated Q's for the normal modes and at least 10^5 above current detection levels (Dratler et al., 1971).

The geometry of the Earth's magnetic field in the core can be treated as a combination of spherical harmonic field components (Bullard and Gellman, 1954) in an analogous way to the representation of free oscillations of a spherical elastic body (Alterman et al., 1959). This similarity in geometry between the magnetic and elastic fields suggests the possibility of a resonance interaction between field components which might enhance the damping effect. Further, the magnetic field in some parts of the core must be quite non-uniform, notably near the core-mantle boundary because of the conductivity contrast between the outer core and lower mantle (Rochester and Smylie, 1965).

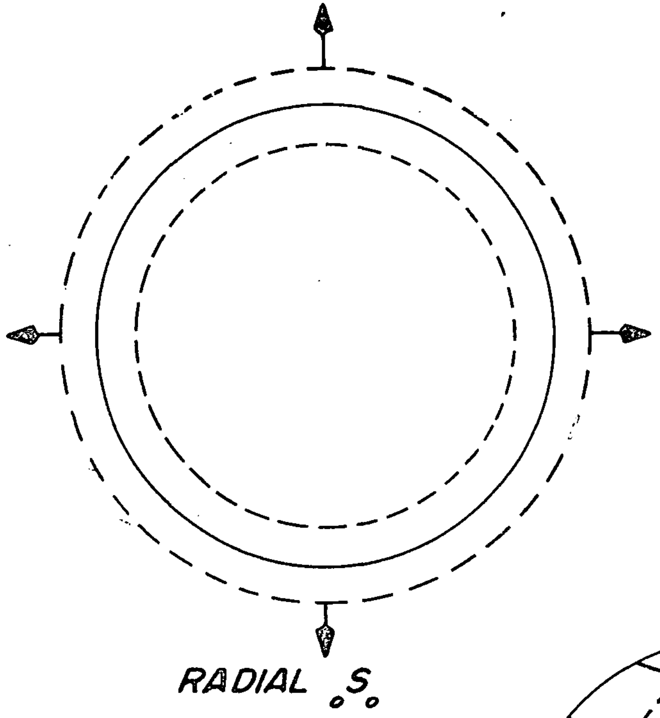
The present thesis utilises the geometry of the core with spherical harmonic expansions of the magnetic and elastic fields and relaxes all three conditions (a), (b), (c) noted above. It is then possible to discuss magnetoelastic interactions in the Earth's core using realistic seismic models and taking advantage of current developments in geomagnetic dynamo mechanisms. The generality of the mathematical approach allows any harmonic displacement field to be substituted in place of the free oscillations e.g. Chandler wobble deformations and earth tide deformations, both of which are forced vibrations of degree two.

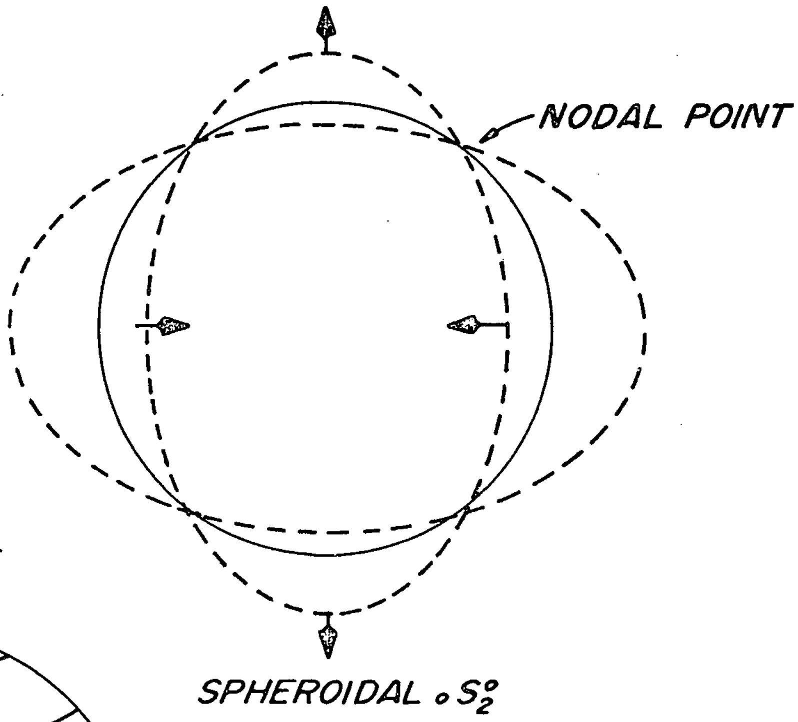
A recent study on the dynamical stability of the fluid core (Higgins and Kennedy, 1971) has raised serious doubts about the existence of large scale convective flow as required by conventional dynamo theory. As a result there is considerable interest in the possibility of turbulent dynamo action, possibly sustained by an oscillatory mechanism. The low viscosity predicted by Gans (1972) therefore gives further stimulus to a detailed study of magnetoelastic interactions in the outer core, as a mechanism of energy dissipation.

1.2 Review of the Free Oscillations

Following a large earthquake, the Earth continues to vibrate in a set of normal modes, three of which are indicated schematically in Fig.1. The early history of the theory of these free oscillations is reviewed by Stoneley (1961). It is convenient to begin here with the work of Lamb (1882) on the free modes of a uniform, incompressible, homogeneous elastic sphere. Lamb's analysis showed that there are two basic modes of vibration which are now referred to as torsional and spheroidal vibrations. The former have no radial displacement and the simplest motion is a twisting about a polar axis. The lowest degree spheroidal mode is a radial contraction and expansion, and the degree two mode oscillates between a prolate and oblate spheriod (Fig.1).

If the material of the Earth is allowed to be compressible and self gravitation is taken into account, the analysis becomes more complicated (Love, 1911, Ch.VII). Gravity not only acts as a restoring force to shorten the eigenperiods but also causes a large initial state of stress throughout the Earth. The method of including this initial stress in the equations of motion was presented originally by Rayleigh (1906). The Earth was by then known to be non-homogeneous





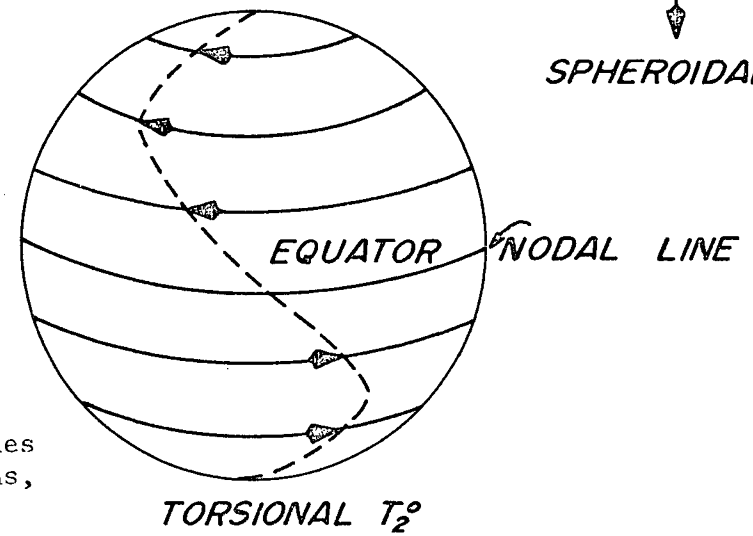


Fig. 1. Fundamental Free Modes of a Sphere. Dashed lines indicate surface motions, greatly exaggerated.

(Oldham, 1906) and Hoskins (1920) extended Love's theory to a radially heterogenous Earth, taking the elastic parameters to be simple algebraic functions of radius.

Takeuchi (1950) used hand calculating machines on a closely related problem, that of determining the response of the Earth to tidal deformation. At the same time Bullen (1950) produced his second whole Earth model, model B. Benioff (1954) then announced that a 57 min. period had been detected on seismic records following the Kamchatka earthquake of 1952. Although this was subsequently questioned as being an eigenvibration (e.g. Bullen, 1963, p.264), several workers began numerical calculations of the eigenperiods using computers (Pekeris and Jarosch, 1958; Jobert, 1957). The most widely quoted results of that time were published by Alterman et al. (1959) using a step-by-step integration of the equations of motion, modified from Love's analysis. The eigenperiods obtained by Alterman et al. were confirmed when seismic records of the Chilean earthquake of 1960 were examined (e.g. Benioff et al., 1961).

In the last decade there has been considerable effort to improve the agreement between theoretical and observed eigenperiods; only a few developments need be cited in this review. The records of Benioff et al. showed very close multiplets of lines where a single frequency was expected theoretically, but there was little hesitation in attributing this effect to the rotation of the Earth (e.g. Pekeris et al., 1961). In effect, the standing nodal patterns on the Earth's surface drift relative to a seismic station. This mode then appears split into 2n+1 components, where n is the degree of the mode. The effect of Earth oblateness on the eigenperiods has been studied by Dahlen (1968), and recently Madariaga (1972) has given an analysis of the effect of large scale lateral heterogeneities on the torsional eigenperiods. These second order departures from spherical symmetry destroy the complete separation of the torsional and spheroidal free oscillations, and result in a certain amount of coupling between modes.

Generally eigenperiods have been obtained by straightforward power spectral analysis, usually performed at each seismic laboratory following an earthquake of sufficient magnitude ($\geqslant 6.5$). Results to 1968 were summarized by Derr (1969). However, by combining a large set of existing records from a single earthquake, Dziewonski and Gilbert (1972) have improved the identification of the eigenperiods. Following the Colombian earthquake of 1970 the records of Dratler et al. (1971) show very clearly the persistence of many overtones of the free oscillations. Such improvements in the quantity and quality of the observed eigenperiods have enabled Earth models to be developed consistent with the mode identifications (e.g. Haddon and Bullen, 1969). The recently developed Earth models are also better constrained, within the quality of the data, by improved fitting techniques (Backus and Gilbert, 1970).

Damping of the free oscillations is generally attributed to anelasticity in the upper mantle (Jackson and Anderson, 1970). Although estimates of Q are somewhat uncertain due to the splitting and coupling of modes mentioned earlier, a typical Q is 300 but for the radial oscillation and most overtones a Q of 10^3 seems indicated (Dratler et al. 1971).

1.3 Review of the Geomagnetic Field

In contrast to the generally excellent confirmation of theory by observations on the free oscillations, there is still considerable uncertainty as to how the Earth's magnetic field is sustained. Hide and Roberts (1961) have given a thorough description of the observations and basic theory, and a review by Weiss (1971) summarises the current position.

To explain the origin of the magnetic field associated with sunspots, Larmor (1919) first suggested a self-generating dynamo mechanism. This was criticised by Cowling (1934) who proved what is now known as Cowling's Theorem, which states that an axisymmetric field cannot be maintained. This ruled out the model proposed by Larmor. Elsasser (1946), in the first of a series of papers on magnetic induction in the Earth's core, discussed the physical conditions required for dynamo action to take place.

Bullard (1949) initiated his own extensive contribution to dynamo theory by proposing a particular model suggested by dynamical motions likely within the core. Initially a dipole field is assumed to exist. The combination of the Earth's rotation and radial convection, driven thermally from the deep core, cause a non-uniform fluid rotation which turns the dipole field around the axis. The field lines then lie in circles of latitude, in opposite sense on either side of the equatorial plane. The field generated is toroidal, with quadrupole symmetry, and has field lines similar to the displacement field T_2^0 of torsional oscillations (Fig.1. and Section 3.2). Because the conversion is an efficient process, driven continuously by rotation and convection, a strong toroidal field can be produced from a weak dipole field.

To sustain the dipole field, the convective flow is imagined to rise in columns and twist the toroidal field into loops through the Coriolis force. The loops are then supposed to coalesce and largely cancel leaving a dipole field as described in detail by Parker (1955). If the whole cycle is efficient then the dipole field can be maintained against ohmic dissipation.

A detailed mathematical formulation of the dynamo process was given by Bullard and Gellman (1954). A general method of expanding both the magnetic and flow velocity fields in spherical harmonics was initiated and the dynamo of Bullard was given exhaustive numerical treatment. Unfortunately a stable solution was not achieved, and there was some indication that energy would be passed to the higher harmonics of the magnetic field instead of returning to the dipole field.

Improved numerical techniques (Gibson and Roberts, 1969) indicated the problem was not trivial, which seemed to confirm a demonstration by Braginskii (1964) that the velocity fields being used were too symmetrical. Noting Braginskii's result, Lilley (1970) introduced a third velocity component into the flow and this seemed to provide a more stable dynamo action. Subsequent calculations by Gubbins (1972) unfortunately showed this dynamo was also unstable.

Meanwhile dynamo mechanisms had been investigated which were proved rigorously to work (e.g. Backus, 1958) but at the expense of rather artificial velocity fields and a new approach was begun by Steenbeck et al. (1966) on the possibility of dynamo action in a turbulent fluid medium. Their work has been reviewed and extended by Moffatt(1970a) who's latest contribution indicates a random driving force can generate dynamo action under certain conditions (Moffatt, 1972).

In a fluid of infinite extent a steady state can be reached in which the magnetic energy density is maintained above the level of kinetic energy density on the assumption of no mean flow. The presence of core boundaries tends to induce a mean flow, so further work is required before a successful mechanism can be claimed.

The large scale flow necessary for the original Bullard dynamo now is challenged on thermodynamic grounds by Higgins and Kennedy (1971). On the basis of new data for the effect of pressure on the melting point of metals and other solids, they argue that the melting point curve lies well above the adiabatic curve. The excess is estimated at 500°C at the core-mantle boundary if the two curves are coincident at the inner-core boundary. The outer core is considered fluid and so its temperature cannot be anywhere less than that given by the melting point curve. Simple dynamical arguments then lead to the conclusion that the outer core is everywhere near the melting point and is quite stable against radial convection.

The effect of this result on dynamo theory has been mentioned briefly by Bullard and Gubbins (1971) and it appears that oscillatory dynamos and those excited by inertial waves (Moffat, 1970b) hold some promise for the future.

1.4 Viscosity

If the outer core of the Earth behaves as a perfect fluid, then anelasticity cannot be a mechanism for attenuating seismic waves. The role of viscosity in the outer core then becomes the only alternative to magnetic dissipation of seismic energy. Although this thesis is mainly concerned with magnetic damping, the effect of viscosity is straightforward to calculate and will perform a minor role in the ensuing discussion.

From the passage of P waves through the outer core, Jeffreys (1970, p.323) has estimated 5 x 10^8 poise as an upper limit for the dynamic viscosity. As Jeffreys points out, this also includes the effect of bulk viscosity. The torsional vibrations of the Earth are confined to the mantle because the fluid core cannot support pure shear; this implies the core-mantle boundary is a free surface for the mantle vibrations. MacDonald and Ness (1961) have given a detailed account of the modification of the eigenperiod of T_2^0 due to viscous and magnetic stresses across this boundary. It was found that there was negligible attenuation of the oscillation due to stiffening of the boundary by these stresses. Similar results were obtained by Sato and Espinoza (1967).

Another approach to the viscosity of the outer core is to compare the relative attenuation of seismic waves which are reflected by the core, ScS and transmitted through the core, SKS. Pairs of rays are chosen to have identical mantle paths. The results again give an upper limit on core viscosity in the region of 10^{10} poise (Suzuki and Sato, 1970).

These seismic estimates of viscosity are markedly higher than those from other sources (Malkus, 1968) probably because they represent upper limits. In an attempt to settle the question Gans (1972) makes use of Andrade's formula to determine the viscosity of iron at the melting point and arrives at the surprisingly low result of about 10^{-1} poise for the dynamic viscosity. The kinematic viscosity is therefore about the same as that of water at 20°C. This result indicates strongly that if the outer core is mainly molten iron, then it behaves as a true liquid.

Some of the quantities discussed in this section are presented in Table 1.

Table 1

Core Parameters

a	Inner core radius	1215	km	Appendix B
b	Outer core radius	3485	km	"
ω	Frequency of typical oscillation	$\sim 10^{-2}$	sec^{-1}	"
v_1	Velocity of typical oscillation	$\sim 5 \times 10^{-4}$	$cm\ sec^{-1}$	"
B_m	Maximum toroidal field strength	480	gauss	Bullard and Gellman(1954)
σ	Electric conductivity	3×10^{5}	$ohm^{-1} m^{-1}$	"
η	Dynamic shear viscosity	0.08	poise	Gans (1972)
μ_0	Permeability	$4\pi \times 10^{-7}$	$henry\ m^{-1}$	-

SECTION 2

BASIC FORMALISM

In this section the equations governing the motion of the medium and the behaviour of the magnetic field are reviewed. It is usual either to combine viscous and elastic forces and form the equation of viscoelasticity (e.g. Bland, 1960), or to add a magnetic field to hydrodynamics and call it magnetohydrodynamics (e.g. Roberts, 1967). Because viscous and magnetic effects are expected to perturb the free oscillations only slightly, for reasons given in Section 1.1, the basic equations are those of elasticity (Love, 1911, Ch.VII). The equations might well be referred to as those of magnetoviscoelasticity.

2.1 Physical Assumptions

The Earth is to be treated as a spherical, radially heterogeneous, self-gravitating, compressible elastic body. With a quarter of it removed, it appears schematically as in Fig.2. The outer core, discovered first by Oldham (1906), extends just over half way to the surface: the inner core, proposed by Lehmann (1936), occupies about a third of the outer core radius.

The outer core is traditionally assumed to be a liquid with zero rigidity (Jeffreys, 1970, p.285). From the P wave attenuation data already mentioned, Suzuki and Sato (1970) concluded that the outer core behaves more like a viscous liquid than a low-rigidity solid. Evidence on the stiffness of the core mantle boundary from the eigenperiods of torsional oscillations indicates an upper limit of about 10^{10} dynes cm^{-2} (Sato and Espinosa, 1967). This is in agreement with the limit obtained by Takeuchi (1950) from earth tide calculations. By contrast, the effect the

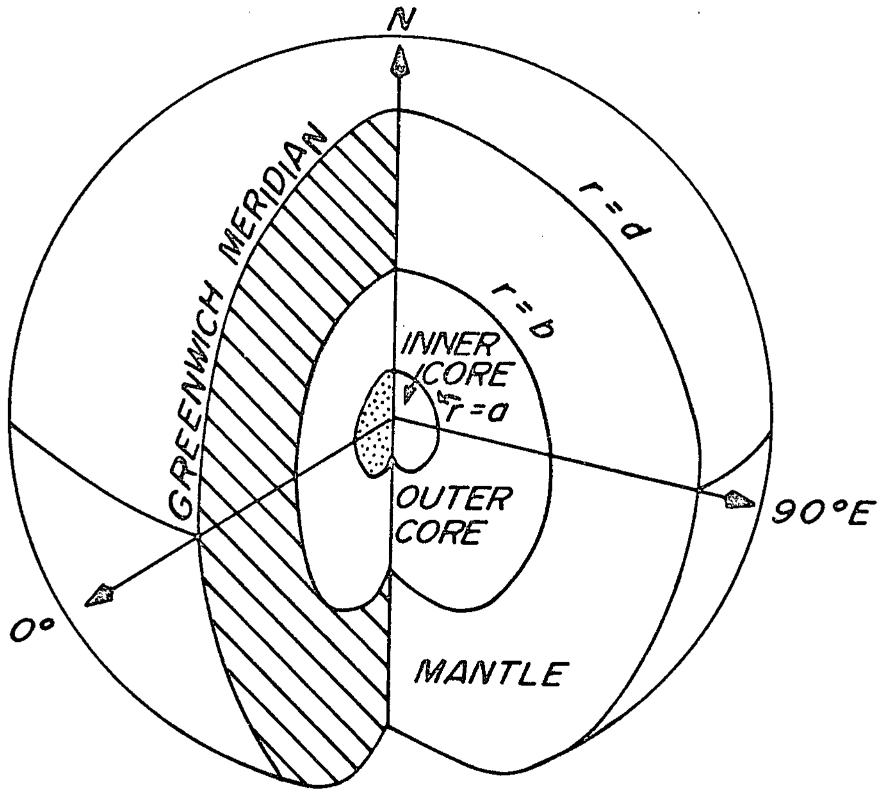


Fig. 2. Earth Model

inertia of the core has on the amplitude of the 19 yr lunar nutation is shown by Jeffreys (1970,p.295) to place an upper limit of 10^9 dynes cm^{-2} on the core rigidity. In comparison, the mantle rigidity averages about 10^{12} dynes cm^{-2}.

The inner core has been generally recognised to be solid (Bullen, p.242) and evidence from the eigenperiods of low degree oscillations definitely favours a normal solid rigidity (Dziewonski and Gilbert, 1972, p.409). There is still some doubt as to the constancy of seismic properties within the inner core. The model used by Dziewonski and Gilbert (1972, Table 6) has both P and S velocities constant, whereas the model used in the present problem (Appendix B) has a small gradient in S velocity, and therefore also in rigidity.

Because there is little reason to do otherwise at this stage, the composition of the inner core is taken to be the same as the outer core (Jeffreys, p.203). The composition of the outer core is of interest because it influences the electrical conductivity which in turn influences ohmic dissipation (Section 4.2). Recent work on the conductivity of liquid iron and metallic alloys by Gardiner and Stacey (1971) and Jain and Evans (1972), extrapolated to core pressures and temperatures, have confirmed Bullard's 1949 estimate (Table 1).

The question of time scales is important. After an earthquake most oscillations die away within a few days of their excitation because of the finite Q (Dratler et al, 1971) whereas dynamo processes associated with variations of the geomagnetic field are by comparison stationary (Hide and Roberts, 1961). This enables the relative rotation of the mantle and core to be ignored and in Fig.2 both parts of the Earth are taken as fixed by the geographic coordinate system (Munk and MacDonald, 1960, p. 11). The rotation of the Earth ensures that the main magnetic field is orientated with the geographic polar axis by dynamo action.

The polar axis of the coordinate system for the free oscillations passes through the epicentre of the earthquake and will be inclined to the geographic polar axis in most cases. However there is no loss of generality in assuming the two axes coincident, because for a given earthquake location a simple transformation of colatitude θ allows for the subtended angle between the two axes. For example, for an earthquake at the equator no transformation is necessary for the ${}_0S_2$ oscillation, and the radial mode has no preferred axis of symmetry (Fig.1).

2.2 Mathematical Notation

Elasticity theory is usually expressed in Cartesian tensor notation, and electrodynamics is nearly always formulated in vector notation. It has been decided to retain the appropriate usage to aid physical understanding, although in some equations both forms will be found. This may look inelegant but is still rigorous for an isotropic medium. As usual, repeated indices imply summation and $\delta_{ij} = 1$ for i = j, otherwise $\delta_{ij} = 0$.

A displacement field is represented throughout by $\underline{u}$ and a velocity field by $\underline{v}$, the three coordinates at a reference point are denoted by the components x_i. The notation for a harmonic component of the magnetic field is T_s^p or S_s^p and for the spheroidal oscillations is ${}_\nu S_n^m$ where s,n are the degree and p,m are the order of the associated Legendre functions and ν is the overtone.

2.3 Elastic Equations

The most general form of the equations of motion for a volume element of density ρ is

$$\rho \frac{Dv_i}{Dt} = \frac{\partial \Pi_{ij}}{\partial x_j} + \rho F_i , \qquad (1)$$

where the substantive derivative refers to a fixed material element (Roberts, 1967, p. 16). The tensor Π_{ij} represents the total stress field and F_i is the body force per unit mass. For the Earth, the body forces may be gravitational or rotational; the gravitational forces arise from either self gravitation or lunar and solar tidal effects. In the present treatment F_i is taken as entirely due to self gravitation and is derived from a potential V. The total stress can be expressed as

$$\Pi_{ij} = X_{ij} + \tau_{ij} + m_{ij}$$

where X_{ij} is the elastic sress, τ_{ij} is the viscous stress and m_{ij} is the magnetic stress.

In a linear isotropic medium m_{ij}, the magnetic part of the Maxwell stress tensor (Stratton, 1941,p.98), is related to the components of magnetic induction by

$$m_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2} B^2 \delta_{ij}\right) ,$$

where μ_0 is the permeability. Because $\underline{B}$ is a vector field, the magnetic stress acting over the surface of a volume is equivalent to a body force within that volume; in the present case the body force per unit volume can be written

$$\frac{\partial m_{ij}}{\partial x_j} = (\underline{J} \times \underline{B})_i ,$$

(Roberts, p.11) where $\underline{J}$ is the current density. The electric part of the Maxwell stress can be neglected by the argument of no free charge (e.g. Elsasser, 1956, p.137).

The viscous stress can be written in several forms, one which separates the different viscosities is

$$\tau_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) + \zeta\frac{\partial v_k}{\partial x_k}\delta_{ij} ,$$

(Landau and Lifshitz, p.214) where η, ζ are the coefficients of dynamic shear and bulk viscosities. The bulk viscosity is often neglected because the rate of shear deformation usually greatly exceeds the rate of compression; it represents resistance to pure expansion (Jeffreys, p.3).

As discussed by Love (1911, p.89) and in a more recent treatment by Smylie and Mansinha (1971), every point within the Earth is under a hydrostatic stress P_{ij} which balances the self gravitation

$$P_{ij} = -p_0\delta_{ij} , \quad \frac{dp_0}{dr} = \rho_0 g_0 .$$

Here ρ_0, g_0 and p_0 are respectively the density, gravity and hydrostatic pressure in the initial state, assumed to vary only with the radius r. Because a volume element carries its initial stress with it during a deformation (Rayleigh, 1906), the element at a reference point x_i had an initial stress

$$P_{ij} - u_k\frac{\partial P_{ij}}{\partial x_k} = -(p_0 - \rho_0 g_0 u_r)\delta_{ij} ,$$

where u_r is the radial displacement. This viewpoint is not immediately obvious and originally caused some confusion in the treatment of self gravitation.

In addition to the hydrostatic stress there is an additional elastic stress T_{ij} related to the deformation u_i by a general form of Hooke's Law (Love, 1944, p.102)

$$T_{ij} = \lambda \Delta \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) ,$$

where λ, μ are Lamé's elastic parameters and Δ is the cubical dilatation, $\Delta = \partial u_k / \partial x_k$. Physically, the meaningful parameters are the rigidity modulus and $k\,(= \lambda + 2\mu/3)$, the bulk modulus. The total elastic stress tensor is

$$X_{ij} = -(p_0 - \rho_0 g_0 u_r)\delta_{ij} + T_{ij} \ .$$

To first order in displacements, the density change of the element is given by

$$\rho - \rho_0 = -\rho_0 \Delta - u_r \frac{d\rho_0}{dr} \quad ,$$

which leads to an additional gravitational potential V_1 satisfying Poisson's equation

$$-\nabla^2 V_1 = 4\pi G \left(\rho_0 \Delta + u_r \frac{d\rho_0}{dr} \right),$$

where G is the gravitational constant. Because F_i is derived from the total gravitational potential,

$$F_i = \frac{\partial V}{\partial x_i} \ , \quad V = V_0 + V_1 \quad \text{and} \quad g_0 = \frac{dV_0}{dr} \ .$$

In taking the equations of motion (1) to represent the vibrations of an elastic medium the displacements can be considered infinitesimal, and to first order

$$\rho \frac{Dv_i}{Dt} \longrightarrow \rho_0 \frac{\partial^2 u_i}{\partial t^2} \ .$$

The elastic equations are then obtained by substituting for Π_{ij} and F_i from the above relations.

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \gamma_i + \frac{\partial T_{ij}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial m_{ij}}{\partial x_j} \ , \qquad (2)$$

where γ_i represents the equivalent gravitational force per unit volume

$$\gamma_i = \rho_0 \frac{\partial V_1}{\partial x_i} - \rho_0 g_{0_i} \Delta + \rho_0 \frac{\partial}{\partial x_i}(g_0 u_r) ,$$

correct to first order in quantities small with the displacements (Hoskins, 1920, pp.7-8). Equations (2) have been given in symbolic form by Smylie and Mansinha (1971, p.332). It is convenient here to first write them

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \gamma_i - \frac{\partial T_{ij}}{\partial x_j} = f_i$$

where

$$f_i = \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial m_{ij}}{\partial x_j}$$

and then in the vector form

$$\left.\begin{aligned} \underline{L}(\underline{u}) &= \underline{f} \\ \underline{f} &= \eta \nabla^2 \underline{v} + (\xi + \tfrac{1}{3}\eta)\nabla(\nabla \cdot \underline{v}) + \underline{J} \times \underline{B} \end{aligned}\right\} \qquad (3)$$

where $\underline{L}$ is a form of vector operator. The terms in $\underline{f}$ are the two viscous volume forces, the second of which is zero if the medium is incompressible, and the magnetic volume force.

2.4 Magnetic Equations

In the core of the Earth the displacement current can be neglected so the field equations are, strictly considered, in pre-Maxwell form,

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} , \quad \nabla \cdot \underline{B} = 0 , \quad \nabla \times \underline{H} = \underline{J} , \quad \underline{B} = \mu_0 \underline{H} , \qquad (4)$$

(Elsasser, 1956). The vectors are $\underline{E}$, the electric field strength, $\underline{B}$, the magnetic induction, $\underline{H}$, the magnetic field strength and $\underline{J}$ the conduction current density. Ohm's law is taken in a form suitable for the relation between current and field in a medium moving with velocity $\underline{v}$,

$$\underline{J} = \sigma(\underline{E} + \underline{v} \times \underline{B}) , \qquad (5)$$

where σ is the electrical conductivity, (Landau and Lifshitz, p.205). By a suitable combination of (4) and (5) the current and electric field can be excluded. The result is known as the induction equation

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) - \nu_m \nabla \times \nabla \times \underline{B} \tag{6}$$

where ν_m is the magnetic diffusivity $(\mu_0 \sigma)^{-1}$.

The induction equation is of central concern in the next section so it is useful to consider the role of the three terms. Remarks similar to the following can be found in most textbooks on the subject of magnetohydrodynamics (e.g. Roberts, 1967, Ch.2).

The terms on the right hand side of (6) can be compared by dimensional analysis

$$\frac{|\nabla \times (\underline{v} \times \underline{B})|}{|\nabla \times (\nabla \times \underline{B})|} = R_m$$

where, if $\mathcal{V}$ and $\mathcal{L}$ are a velocity and a length typical of the system,

$$R_m = \mathcal{V} \mathcal{L} \nu_m^{-1} = \mathcal{V} \mathcal{L} \mu_0 \sigma$$

defines the magnetic Reynolds number. In a medium of high conductivity, R_m is large and there is negligible diffusion of the field; the field is said to be frozen into the medium. Conversely, for small conductivity, R_m is small and the field will diffuse faster than it can be maintained by the flow.

If the medium is stationary, so R_m is zero, (6) is a diffusion equation and in a bounded medium the solution is an infinite series of decay modes (Elsasser, 1946). In a spherical conductor of radius R the mode of longest decay has the form

$$\underline{B}(\underline{r}, t) = \underline{B}(\underline{r}) e^{-t/\tau}, \quad \tau = R^2 \mu_0 \sigma / \pi^2 . \tag{7}$$

If, however, the field is stationary $(\partial \underline{B}/\partial t = 0)$, diffusion and induction are balanced. Since Elsasser (1946) and Bullard (1949) speculated on the possibility of a geodynamo, there has been considerable effort to find a solution $\underline{B}$ which is stationary and which has a steady dipole component. Even if a solution for $\underline{B}$ was found, $\underline{v}$ would have to obey the equations of motion to complete the dynamo mechanism.

The combination of (2) and (6) presents an impressively difficult problem; (6) alone is not simple, as is well known from dynamo theory . Equations (2) have been linearised by Alterman et al (1959) without the viscous and magnetic terms. It is, in principle, possible to treat the additional forces in a manner similar to the Coriolis force (Pekeris et al, 1961) and compute the change in eigenperiod. A simpler approach is to solve the linearised induction equation and determine the energy lost by ohmic dissipation and thus obtain a measure of Q.

2.5 Boundary Conditions

It is appropriate to complete this section with a review of the boundary conditions for the magnetic field and the free oscillations. Because both fields span bounded media, they are modal in character and the boundary conditions are influential.

For the free oscillations Pekeris and Jarosch (1958) give the following conditions;

(a) Regularity of the displacements and stresses at r=0.

(b) Zero surface stress on the deformed surface of the Earth.

(c) The interior and exterior gravitational potentials and their gradients are continuous at the deformed surface of the Earth.

In addition;

(d) Within a liquid or at a liquid-solid interface, the transverse stress is zero.

And, at any discontinuity within the Earth;

(e) Displacements, gravitational potentials and their gradients, and normal stresses are continuous.

These are dynamic boundary conditions and apply to all free oscillations of periods an hour or less. If the frequency decreases to zero (the static limit), certain of the equations of motion become degenerate. To satisfy the surface conditions (b), (c), condition (e) has to be relaxed to allow one extra free constant within the Earth (Smylie and Mansinha, 1971, pp. 342-344). There is still some doubt as to which of the conditions in (e) have to be modified and not all authors agree (e.g. Pekeris and Accad, 1972, p.241). Fortunately the difficulty does not arise for the normal modes.

Condition (a) will be discussed further in Appendix B because it has often been satisfied only approximately (e.g. Alsop,1963,p. 486).

The boundary conditions for the magnetic field at a discontinuity in μ_0 or σ are given by Stratton (pp.34-37) and are derived from (4). They are :

(f) Normal component of $\underline{B}$ continuous, unconditionally.

(g) Transverse component of $\underline{B}$ continuous if there is no surface current.

(h) Transverse component of $\underline{E}$ continuous, unconditionally.

The first condition is derived from $\nabla . \underline{B} = 0$. The second condition is derived from $\nabla \times \underline{H} = \underline{J}$, from which $\nabla . \underline{J} = 0$ and is valid only if the conductivity is finite on both sides of the boundary. Otherwise for an infinite conductivity on one side a surface current density exists.

In the present treatment of the Earth, the mantle can be considered insulating as the time scale of core-mantle coupling, using a finite lower-mantle conductivity, greatly exceeds the time scale of the free oscillations. At the core-mantle boundary the normal components of $\underline{B}$ are continuous, and because there are no sources outside the core, by dynamo hypothesis, the field $\underline{B}$ falls off as r^{-3} in the mantle. No electric currents can flow in the mantle and thus the normal component of $\underline{J}$ must vanish at the boundary.

SECTION 3

THE INDUCTION EQUATION

A solution of the induction equation is now obtained by a linearisation procedure. Sections 3.2 and 3.3 follow the treatment of the dynamo problem by Bullard and Gellman (1954, Sections 4 and 6).

3.1 Linearisation

The induction equation (6) is quite hard to solve unless it is first linearised by a perturbation method. The elastic equations (3) can be included in the scheme, although mainly for completeness because they are not solved directly.

Quantities are expanded in two ways,

$$\underline{B} = \underline{B}_0 + \underline{b} \,, \qquad \underline{J} = \underline{J}_0 + \underline{j} \tag{8a}$$

and

$$\left.\begin{aligned} \underline{u} = \underline{u}_0 + \underline{u}_1 + \underline{u}_2 \,, \quad \underline{f} = \underline{f}_0 + \underline{f}_1 + \underline{f}_2 \\ \underline{v} = \partial \underline{u}/\partial t = \underline{v}_0 + \underline{v}_1 + \underline{v}_2 \end{aligned}\right\} \tag{8b}$$

The following interpretations are implied;

$\underline{B}_0$ Main core magnetic field, sustained by currents $\underline{J}_0$

$\underline{b}$ Perturbation of $\underline{B}_0$ by energy from $\underline{v}_1$, sustained by $\underline{j}$

$\underline{u}_0, \underline{v}_0$ Displacement and velocity associated with fluid flow in the core.

$\underline{u}_1, \underline{v}_1$, Displacement and velocity associated with the elastic free oscillations.

$\underline{u}_2, \underline{v}_2$ Displacement and velocity fields caused by the interaction being perturbations of $\underline{u}_1$, $\underline{v}_1$

$\underline{f}_0$ Body force for fluid flow.

$\underline{f}_1$ Body force for free oscillations, zero by definition.

$\underline{f}_2$ Body force due to the interaction

Because of time scales $\underline{B}_o$, $\underline{J}_o$ and $\underline{f}_o$ are considered static. Perturbation quantities $\underline{b}$, $\underline{j}$, $\underline{u}_2$, $\underline{v}_2$, and $\underline{f}_2$ have the harmonic dependence of the free oscillations, i.e. $\underline{v}_1 = \underline{u}_1 e^{i\omega t}$ etc.

To perform the linearisation it is necessary to assume two conditions

$$|\underline{b}| << |\underline{B}_o| \tag{9a}$$

$$|\underline{b}|/|\underline{B}_o| << |\underline{v}_1|/|\underline{v}_o| \quad , \tag{9b}$$

and define the zero-order induction equation to be

$$-\nabla\times\nabla\times\underline{B}_o = \mu_o\sigma\frac{\partial\underline{B}_o}{\partial t} - \mu_o\sigma\,\nabla\times(\underline{v}_o\times\underline{B}_o) \; . \tag{10}$$

The unperturbed elastic equations are given by equation (3) for the free oscillations

$$\underline{L}(\underline{u}_1) = \underline{f}_1 = 0 \; .$$

Equations (8a) and (8b) are now substituted into (3) and (6), conditions (9a) and (9b) are applied and equation (10) subtracted. To first order in small quantities the resulting equations are

$$-\nabla\times\nabla\times\underline{b} = \mu_o\sigma\frac{\partial\underline{b}}{\partial t} - \mu_o\sigma\,\nabla\times(\underline{v}_1\times\underline{B}_o) \tag{11}$$

and

$$\underline{L}(\underline{u}_2) = \underline{f}_2$$

where

$$\underline{f}_2 = \eta\nabla^2\underline{v}_1 + (\xi+\tfrac{1}{3}\eta)\,\nabla(\nabla.\underline{v}_1) + \frac{1}{\mu_o}\left[(\nabla\times\underline{b})\times\underline{B}_o + (\nabla\times\underline{B}_o)\times\underline{b}\right]$$

assuming

$$|\underline{v}_2| << |\underline{v}_1| \text{, unconditionally.}$$

In a conducting medium a perturbation of the steady field $\underline{B}_o$ is equivalent to disturbing the field lines and producing Alfvén waves (Roberts, Ch.5). In the present situation the elastic forces are controlling the behaviour of the medium and the field $\underline{b}$ is assumed to have the time dependence of $\underline{v}_1$. The hydromagnetic waves are thus in step with the free oscillations and are themselves standing waves in the core. The effect of finite viscosity and electrical conductivity is of course to attenuate the waves. If required, the equations following (11) can be used to compute the perturbation in the free oscillation velocity.

Condition (9a) can be verified in the sequel; condition (9b) requires that the magnitudes of $\underline{v}_o$, $\underline{v}_1$ must be compared.

Bullard and Gellman (1954, p.273) assumed that the transverse component of $\underline{v}_0$ is a measure of the westward drift of the non-dipole surface field. As such it is taken to represent the motion of the outer core past the mantle. An extrapolation of the observed drift of 0.18° per year leads to maximum radial and transverse velocities of 0.014 cm sec^{-1} and 0.04 cm sec^{-1} respectively.

The amplitude of $\underline{v}_1$ in the core requires the variation of amplitude of displacement with depth to be computed for a realistic Earth model, as in Fig.3. In this calculation, the radial surface displacement is usually normalised to be unity, here 1 cm. In a recent study of the source mechanism of the 1964 Alaskan earthquake, Ben-Menahem et al.(1972) have given observed surface amplitudes. When suitably corrected for instrument response and displacement of the epicentre from the station, radial displacements at the surface are of the order of 1 mm for the low-degree spheroidal oscillations. This agrees with Nowroozi's (1965) estimate for ${}_0S_2$ for the same earthquake. For those oscillations with appreciable kinetic energy in the core (Figs. 3,7 and 9) periods range from 3233 sec for ${}_0S_2$ to 244 sec for ${}_4S_0$. Free oscillation velocities in the radial direction are thus in the range 10^{-3} to 10^{-4} cm sec^{-1} with transverse velocities an order of magnitude smaller.

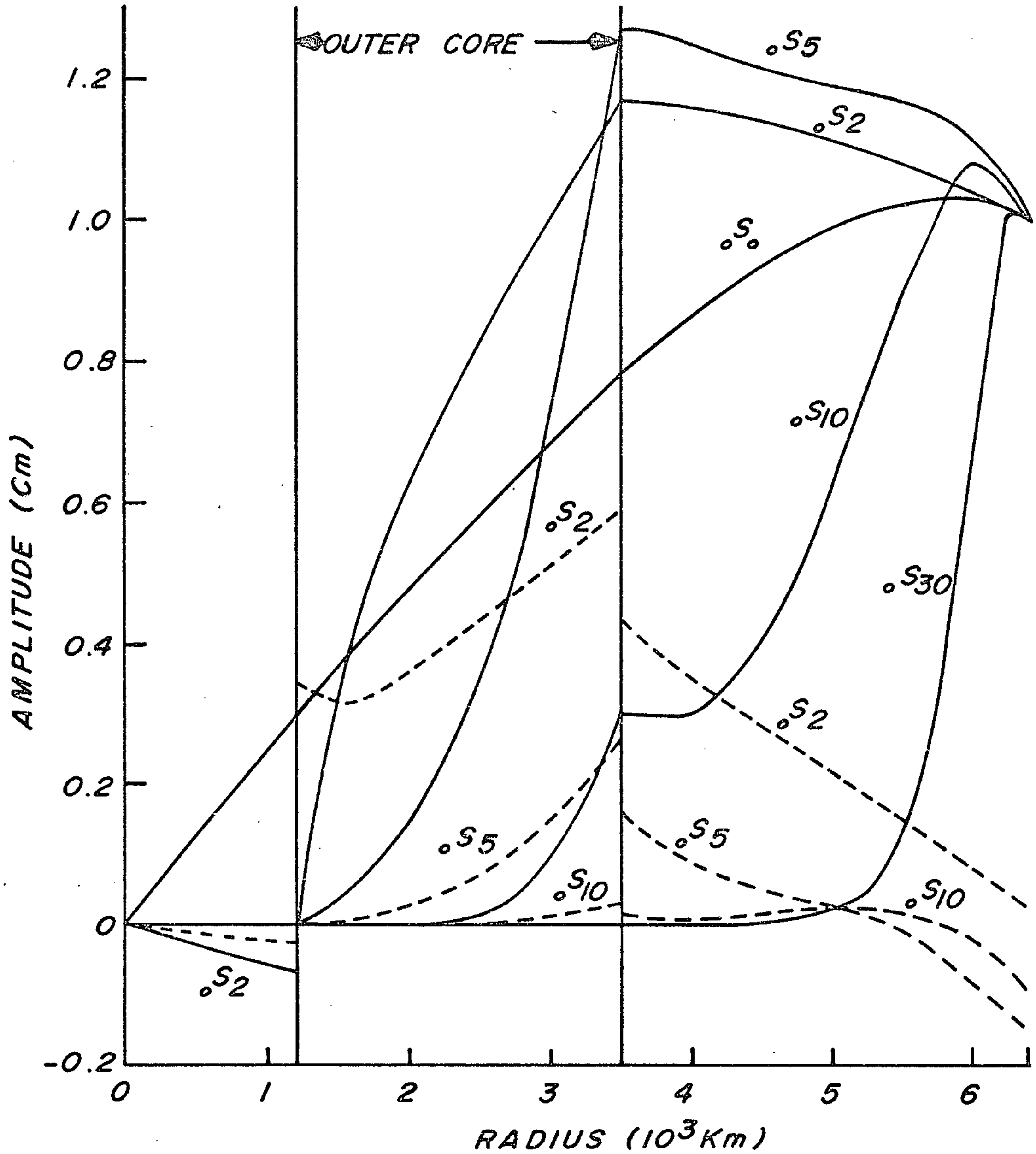


Fig. 3. Amplitudes of Spheroidal Displacements in the Earth. Computed from model JAB1 (Table 10, Appendix B). Full lines are radial displacements, dashed lines are transverse displacements.

Thus

$$10^{-3} < |\underline{v}_1| / |\underline{v}_0| < 1$$

and conditions (9a) and (9b) are satisfied simultaneously if

$$|\underline{b}| << |\underline{B}_0| \times 10^{-3} \tag{12}$$

3.2 Field Expansions

At first sight equation (11) looks similar to the induction equation (10) but there are differences to be noted. In (11) the free oscillation velocity $\underline{v}_1$ replaces the flow velocity $\underline{v}_o$ of (10) and contains an extra part to the field which is lamellar, $\underline{v}_o$ being solenoidal. Second, if (11) is written in the form

$$\nabla \times \nabla \times \underline{b} + \mu_0 \sigma \frac{\partial \underline{b}}{\partial t} = \mu_0 \sigma \nabla \times (\underline{v}_1 \times \underline{B}_0) \tag{13}$$

the left hand side contains the field $\underline{b}$ induced by the interaction term $(\underline{v}_1 \times \underline{B}_0)$ on the right. Because of the perturbation approximation (9a) it is unlikely that dynamo action will take place, particularly as $\underline{B}_0$ is taken to be stationary with respect to $\underline{b}$. Nevertheless because of the spherical symmetry of the problem, field expansions can be used in a similar approach to that of Bullard and Gellman (1954, p.220).

As discussed by Smylie (1965), the magnetic field is solenoidal and can be written as the sum of a poloidal vector $\underline{S}$ and a toroidal vector $\underline{T}$, by a theorem of Backus (1958). Thus

$$\underline{B} = \underline{S} + \underline{T} \quad ; \quad \underline{S} = \nabla \times (\nabla \times \underline{r} S) \quad , \quad \underline{T} = \nabla \times \underline{r} T \tag{14}$$

where S and T are the defining scalars. The degree and order of a particular harmonic are s and p for the main field $\underline{B}_0$, and ℓ and k for the induced field $\underline{b}$. The defining scalars are expanded in spherical harmonics

$$\left.\begin{aligned} S_0 &= \sum_{s=1}^{\infty} \sum_{p=-s}^{s} s_s^p(r,t)\, P_s^p(\cos\theta)\, e^{ip\phi} \\ T_0 &= \sum_{s,p} t_s^p(r,t)\, P_s^p(\cos\theta)\, e^{ip\phi} \end{aligned}\right\} \qquad (15)$$

for $\underline{B}_0$ and similarly for $\underline{b}$, the summation notations in (15) being equivalent. Following Smylie (1965) the associated Legendre functions are defined with the normalisation of Hobson (1955, p.93),

$$\left.\begin{aligned} P_s^p(\mu) &= (-1)^p (1-\mu^2)^{p/2} \frac{d^p}{d\mu^p} P_s^p(\mu) \quad, \quad p \geq 0 \\ P_s^{-p}(\mu) &= (-1)^p \frac{(s-p)!}{(s+p)!} P_s^p(\mu) \quad, \quad -s \leq p \leq s \end{aligned}\right\} \qquad (16)$$

For reference the components of the vector elements are given for $\underline{B}_0$

$$\left.\begin{aligned} (\underline{S}_s^p)_r &= \frac{s(s+1)}{r} s_s^p P_s^p e^{ip\phi} \quad, & (\underline{T}_s^p)_r &= 0 \\ (\underline{S}_s^p)_\theta &= \frac{1}{r}\frac{\partial}{\partial r}(r s_s^p) \frac{dP_s^p}{d\theta} e^{ip\phi} \quad, & (\underline{T}_s^p)_\theta &= \frac{ip}{r\sin\theta} t_s^p P_s^p e^{ip\phi} \\ (\underline{S}_s^p)_\phi &= \frac{ip}{r\sin\theta}\frac{\partial}{\partial r}(r s_s^p) P_s^p e^{ip\phi} \quad, & (\underline{T}_s^p)_\phi &= -t_s^p \frac{dP_s^p}{d\theta} e^{ip\phi}. \end{aligned}\right\} \qquad (17)$$

(Smylie, 1965). Again (16) and (17) are similar for $\underline{b}$.

The components of the free oscillation displacement field $\underline{u}_1$ are well known (e.g. Alterman et al., 1959, pp.84, 86), but it is useful to indicate how they are derived. The Helmholtz separation theorem (Morse and Feshbach, 1953, p.53) allows the displacement field to be written

$$\underline{u}_1 = \nabla L + \nabla \times \underline{A}$$

where L is a scalar and $\underline{A}$ a zero-divergence vector field. If the degree and order of a harmonic of the displacement field are denoted by n and m, the scalar L can be expanded as

$$L = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} l_n^m(r,t)\, P_n^m(\cos\theta)\, e^{im\phi}$$

and the components of $\underline{L} = \nabla L$, a lamellar vector, are

$$(\underline{L}_n^m)_r = \frac{\partial \ell_n^m}{\partial r} P_n^m e^{im\phi}$$

$$(\underline{L}_n^m)_\theta = \frac{1}{r} \ell_n^m \frac{dP_n^m}{d\theta} e^{im\phi}$$

$$(\underline{L}_n^m)_\phi = \frac{im}{r\sin\theta} \ell_n^m P_n^m e^{im\phi}$$

Because $\nabla \times \underline{A}$ is solenoidal, the defining scalars can be expanded as in (15) and the components written as in (17). The lamellar and poloidal fields have the same angular functions and they can be combined into a spheroidal vector field with components

$$\left.\begin{aligned} (\underline{S}_n^m)_r &= u_n^m(r,t) P_n^m e^{im\phi} \\ (\underline{S}_n^m)_\theta &= v_n^m(r,t) \frac{dP_n^m}{d\theta} e^{im\phi} \\ (\underline{S}_n^m)_\phi &= \frac{im}{\sin\theta} v_n^m(r,t) P_n^m e^{im\phi} . \end{aligned}\right\} \qquad (18)$$

The toroidal part $\underline{T}_n^m$ of $\nabla \times \underline{A}$ has the same radial components as the toroidal magnetic field, (17) although the radial function t_s^p is usually written w_n^m. The total displacement field is then the sum of two vector fields, and for a harmonic n, m

$$\underline{u}_{,n}^m = \underline{S}_n^m + \underline{T}_n^m$$

where

$$\underline{S}_n^m = \underline{r}\, u_n^m P_n^m e^{im\phi} + r v_n^m \nabla (P_n^m e^{im\phi})$$

is a spheroidal vector displacement and

$$\underline{T}_n^m = w_n^m \underline{r} \times \nabla (P_n^m e^{im\phi})$$

is a torsional vector displacement.

The expansion of the displacement field in this way has been used by many authors, but a rigorous justification rests on the result of Backus' theorem.

Because the rigidity of the outer core is negligible a purely transverse elastic motion has no restoring force. Therefore only spheroidal oscillations sample the core and the vector components of $\underline{u}_{1n}^{m}$ are given by the components of $\underline{S}_n^{m}$ alone.

Substitution of the fields $\underline{B}_0$ $\underline{b}$ and $\underline{v}_1$ in the form of the expansions (14) - (18) into (13) leads to a rather lengthy manipulation of the angular functions. The procedure closely follows that described by Bullard and Gellman (1954, p.224) and Smylie (1965, p.172). Essentially the angular functions are grouped together in such a way that the poloidal and toroidal parts of (13) are separated. This is accomplished by multiplying (13) first by a poloidal vector S'^{k}_{ℓ} then by a toroidal vector T'^{k}_{ℓ} whose radial functions are chosen to be r^{-1} and 1 respectively. On integration over a spherical surface the orthogonality relations for the associated Legendre functions ensure the separation of the vector equation into poloidal and toroidal parts.

The term $\nabla \times (\underline{v}_1 \times \underline{B}_0)$ also becomes separated into poloidal and toroidal parts each of which contains triple angular integrals K, L. A discussion of these integrals can be found in an appendix by Scott in Gibson and Roberts (1969). They are generally referred to as Gaunt and Elasser integrals and they depend on the six indices of the three fields; by definition

$$K \equiv K^{-k,m,p}_{\ell,n,s} \quad , \quad L \equiv L^{-k,m,p}_{\ell,n,s} \quad ,$$

To ensure that k, m, p are all positive integers it is necessary to have one superscript negative. Definitions of K and L and some of their relevant properties are given in Appendix A.

The equation for the poloidal field is found to be

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rs_\ell^k) - \frac{\ell(\ell+1)}{r^2}s_\ell^k - \mu_0\sigma\frac{\partial s_\ell^k}{\partial t} = i\omega\mu_0\sigma\sum_{n,m}\sum_{s,p}\frac{(-1)^k(2\ell+1)}{4\pi\ell(\ell+1)}\Big\{iu_n^m t_s^p L$$
$$- \frac{1}{2r}u_n^m\frac{\partial}{\partial r}(rs_s^p)\left[n(n+1)-\ell(\ell+1)-s(s+1)\right]K$$
$$-\frac{1}{2r}v_n^m s(s+1)s_s^p\left[n(n+1)-s(s+1)+\ell(\ell+1)\right]K\Big\}, \tag{19}$$

and for the toroidal field

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rt_\ell^k) - \frac{\ell(\ell+1)}{r^2}t_\ell^k - \mu_0\sigma\frac{\partial t_\ell^k}{\partial t} = i\omega\mu_0\sigma\sum_{n,m}\sum_{s,p}\frac{(-1)^k(2\ell+1)}{4\pi\ell(\ell+1)}\Big\{$$
$$-i\frac{s(s+1)}{r}\frac{\partial}{\partial r}(v_n^m s_s^p)L - \frac{i}{r}\frac{\partial}{\partial r}\left(u_n^m\frac{\partial}{\partial r}(rs_s^p)\right)L - \frac{i\ell(\ell+1)}{r^2}v_n^m\frac{\partial}{\partial r}(rs_s^p)L$$
$$+\frac{1}{2r}\frac{\partial}{\partial r}(ru_n^m t_s^p)\left[s(s+1)-n(n+1)+\ell(\ell+1)\right]K$$
$$+\frac{1}{2r}\ell(\ell+1)v_n^m t_s^p\left[n(n+1)-\ell(\ell+1)+s(s+1)\right]K\Big\}, \tag{20}$$

where $\underline{v}_1$ has been written $i\omega\underline{u}_1$. Although these two equations appear complicated, they have a simple physical interpretation which depends on the properties of K, L.

Imagine a particular component of the velocity field $\underline{S}_n^m$ interacting with a component of the main magnetic field, $\underline{S}_s^p$ or $\underline{T}_s^p$. Selection rules, Section 3.3, then determine the values of ℓ, k which give non-zero K,L. The right hand sides of (19) and (20) are then summed over all n, m, s, p to provide the source term for each permissible ℓ, k of the perturbation field. The equation (13) can then be solved for the radial functions s_ℓ^k and t_ℓ^k to determine the harmonic of $\underline{b}$, the induced field.

3.3 Selection Rules

The selection rules are quoted from Scott in Gibson and Roberts (1969, p.588) for k, m, p and ℓ, n, s positive definite integers.

1. For K, L non zero; $-k + m + p = 0$
2. For K non zero;
   (i) $\ell + n + s$ is even.
   (ii) ℓ, n, s can form the sides of a triangle.
3. For L non zero;
   (i) $\ell + n + s$ is odd.
   (ii) ℓ, n, s can form the sides of a triangle.
   (iii) No two superfixes zero, e.g. $m = p = 0$.
   (iv) No two superfix-suffix pairs equal, e.g. $m = p$, $n = s$.

Rules 2(i), 3(i) indicate that either K or L vanishes for any particular selection of ℓ, n, s. Rules 2(ii), 3(ii) ensure that the sums on the right hand sides of (19) and (20) are finite, and this simplifies considerably the evaluation of the perturbation fields. Beyond these observations further illustration of the selection rules is best left to a particular example.

Before this is done however a remark should be made concerning the expansions (15) in comparison with the expansions of Bullard and Gellman. In (15) the spherical harmonies are complex to simplify the algebra. However, the radial functions are not necessarily real, but are only required to satisfy the relation

$$S_s^{-p}(r,t) = (-1)^p \frac{(s+p)!}{(s-p)!} \, S_s^p(r,t)^*$$

(Smylie, 1965), where a * signifies complex conjugation. This ensures that the field $\underline{B}_o$ is real providing a similar condition also holds for the toroidal components. The dynamo expansions of Bullard and Gellman are in terms of sines and cosines and the two forms are of course equivalent. For example, for two real poloidal harmonics $S_n^{m,c}$, $S_n^{m,s}$ of the Bullard-Gellman dynamo, the related complex radial functions are

$$rS_n^m = (-1)^m \frac{1}{2}\left(S_n^{m,c} - i\, S_n^{m,s}\right)$$

$$rS_n^{-m} = \frac{(n+m)!}{(n-m)!}\,\frac{1}{2}\left(S_n^{m,c} + i\, S_n^{m,s}\right)$$

$$rS_n^0 = S_n^{0,c} \quad ,$$

where the superscripts c,s refer to cosine and sine functions. For the dynamo theory the radial functions of the velocity field are real to represent actual flows in the outer core. Because the free oscillations are usually formulated from complex harmonics the magnetic fields for non-zero order are also complex.

It would be impractical to investigate all possible interactions in order to solve (19) and (20) completely. Instead the main magnetic field $\underline{B}_0$ will be assumed to consist of a single component, the axial quadrupole toroidal field T_2^0. Arguments have been forwarded in the Introduction as to why this component is expected to have appreciable strength in the outer core. Restriction to one component considerably simplifies the right hand sides of (19) and (20) as only the sums over n and m have to be considered. An interaction diagram can be constructed similar to that of Gibson and Roberts (1969, p.584) but with a fundamental difference. Instead of specifying a velocity field and examining magnetic field interactions, the main field is specified and the magnetoelastic interactions are evaluated. The diagram for T_2^0 is shown in Fig.4. The presence of an s or t in the diagram indicates a spheroidal oscillation of degree n and order m producing an induced magnetic field of poloidal

SPHEROIDAL OSCILLATION S_n^m

INDUCED FIELD b_l^k	0	-1	0	1	-2	-1	0	1	2	-3	-2	-1	0	1	2	3	-4	-3	-2	-1	0	1	2	3	4
	0	1	1	1	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4
1^{-1}		t				S						t													
1^{0}			t										t												
1^{1}				t				S						t											
2^{-2}					t						S														
2^{-1}		S				t						S													
2^{0}	t						t																		
2^{1}				S				t						S											
2^{2}									t						S										
3^{-3}										t								S							
3^{-2}					S														S						
3^{-1}		t				S						t								S					
3^{0}			t										t												
3^{1}				t				S						t								S			
3^{2}									S														S		
3^{3}																t								S	
4^{-4}																	t								
4^{-3}										S								t							
4^{-2}					t						S								t						
4^{-1}						t						S								t					
4^{0}							t														t				
4^{1}								t						S								t			
4^{2}									t						S								t		
4^{3}																S								t	
4^{4}																									t

Fig. 4. Interactions with the Toroidal Quadrupole Magnetic Field. The induced field is shown s for poloidal, t for toroidal.

or toroidal type with an ℓ, k harmonic. If a square is blank it means either K or L is zero by one or more of the selection rules. Harmonics up to 4,4 only are shown although the diagram continues indefinitely.

Writing the radial function of $\underline{T}_2^0$ by the capital letter T to distinguish it from the toroidal perturbation field, equations (19) and (20) then reduce to

$$\left.\begin{aligned} \frac{d^2}{dr^2}(rs_\ell^k) - \frac{\ell(\ell+1)}{r^2}(rs_\ell^k) - i\omega\mu_0\sigma\, rs_\ell^k &= f(r) \\ \frac{d^2}{dr^2}(rt_\ell^k) - \frac{\ell(\ell+1)}{r^2}(rt_\ell^k) - i\omega\mu_0\sigma\, rt_\ell^k &= g(r) \quad , \end{aligned}\right\} \tag{21}$$

where, for s = 2, p = 0,

$$\left.\begin{aligned} f(r) &= i\omega\mu_0\sigma \sum_{n,m} \frac{(-1)^k(2\ell+1)}{4\pi\ell(\ell+1)} \left\{ i r u_n^m T \right\} L_{\ell,n,2}^{-k,m,0} \\ g(r) &= i\omega\mu_0\sigma \sum_{n,m} \frac{(-1)^k(2\ell+1)}{4\pi\ell(\ell+1)} \left\{ \frac{1}{2}\frac{\partial}{\partial r}(r u_n^m T)\left[6 - n(n+1) + \ell(\ell+1)\right] \right. \\ &\qquad \left. + \frac{1}{2} v_n^m T \ell(\ell+1)\left[n(n+1) - \ell(\ell+1) - 6\right] \right\} K_{\ell,n,2}^{-k,m,0} \end{aligned}\right\} \tag{22}$$

In (21) the time dependence of the perturbation fields has been recognised as $e^{i\omega t}$ and in (22) the Gaunt and Elasser integrals are given their indicial dependence to illustrate the formalism.

3.4 Solution in the Outer Core

Equations (21) now have to be solved in the outer core. At the inner-core boundary r is of the order of 10^6m and for low degree spheroidal modes ω is typically 10^{-2} rad. sec^{-1} The degree ℓ of the induced field is likely to be a low integer, say typically $\ell = 5$, then

$$\frac{\ell(\ell+1)}{r^2} << |i\omega\mu_0\sigma| \text{ , to about } 10^{-8}.$$

The equations are thus well approximated by

$$\left.\begin{aligned}\frac{d^2}{dr^2}(rs_\ell^k) + \alpha^2(rs_\ell^k) &= f(r)\\ \frac{d^2}{dr^2}(rt_\ell^k) + \alpha^2(rt_\ell^k) &= g(r)\end{aligned}\right\} \tag{23}$$

where $\alpha^2 = i\omega\mu_0\sigma$. It is convenient at this point to introduce the electromagnetic skin depth δ (Landau and Lifshitz, p.195) and its relation to α,

$$\delta^2 = \frac{2}{\omega\mu_0\sigma} \quad , \quad \alpha = \frac{1-i}{\delta} \quad .$$

A graph of δ against period T of the field $\underline{b}$, or $\underline{v}$, is shown in Fig.5 for the periods involved in the present discussion.

Independent solutions to the homogeneous form of (23) are $e^{i\alpha r}$, $e^{-i\alpha r}$ for both rs_ℓ^k and rt_ℓ^k. Particular integrals can be found from these solutions and their Wronskian by a well known method (e.g. Morse and Feshbach, p.528). The solutions of (23), valid for $r \geqslant a$ are obtained in the usual way,

$$\left.\begin{aligned} rs_\ell^k &= Ce^{i\alpha r} + De^{-i\alpha r} + \frac{1}{2i\alpha}\Big\{e^{\beta}\int_a^r e^{-\beta'} f(q)\,dq\\ &\qquad - e^{-\beta}\int_a^r e^{\beta'} f(q)\,dq\Big\}\\ rt_\ell^k &= Ee^{i\alpha r} + Fe^{-i\alpha r} + \frac{1}{2i\alpha}\Big\{e^{\beta}\int_a^r e^{-\beta'} g(q)\,dq\\ &\qquad - e^{-\beta}\int_a^r e^{\beta'} g(q)\,dq\Big\}\end{aligned}\right\} \tag{24}$$

where

$$\beta = i\alpha(r-a), \quad \beta' = i\alpha(q-a).$$

The constants in (24) can be found by matching the solutions at the boundaries r = a, r = b to solutions determined for the inner core and the mantle. At the inner-core boundary a choice must be made on the induction probable in the inner core. Using equation (7) the longest

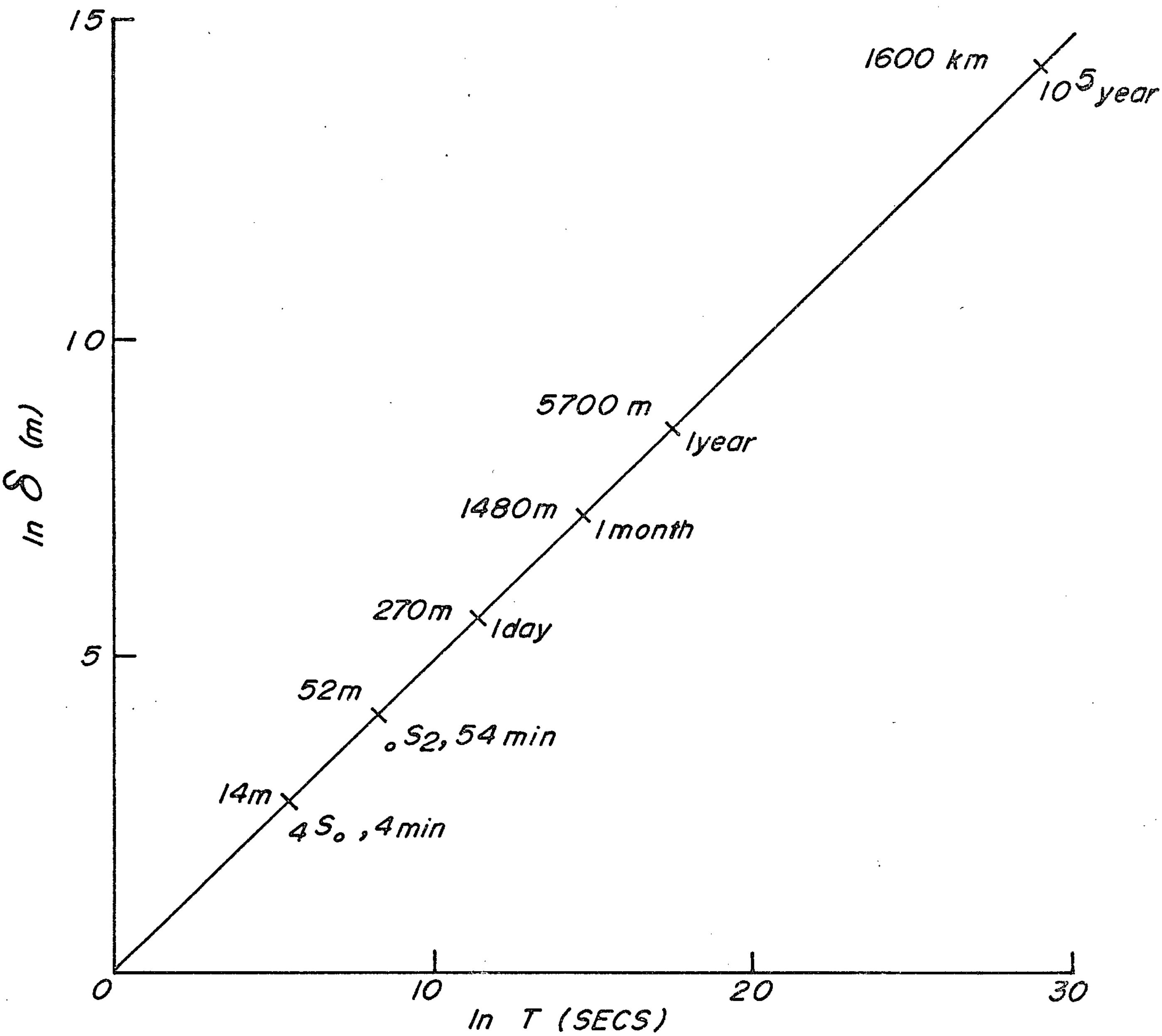


Fig. 5. Electromagnetic Skin Depths for the Core.

decay time for a static field in the outer core is of the order of 15000 yrs and for the inner core is of the order of 2000 yrs. However the main dipole field is at present decreasing, has polarity reversals which occur in less than 10^4 yrs, and can be characterised by variations of the order of 10^5 yrs (Kaula, 1968, p.133). Any harmonic component of these variations with a period of about 10^5 yrs will penetrate through the entire inner core (Fig.5). This assumes the inner core has the same conductivity as the outer core.

There is thus some evidence for a leakage of the field into the inner core, but the nature of the field is uncertain because the temporal variations in the outer core are not well analysed at the present time. Subsequently the inner core field will be ignored and the equations (21) are solved as a homogeneous system with f(r) = g(r) = 0.

Letting $x = \alpha r$, $y = r s_\ell^k$, the first equation of the pair (21) is a Bessel equation,

$$x^2 y'' + [x^2 - \ell(\ell+1)] y = 0,$$

with a solution $y = A x j_\ell(x)$, regular at the origin. A prime denotes radial differentiation and $j_\ell(x)$ is a spherical Bessel function. The solutions are to be evaluated at r = a, so $x \gg 1$, and for x large,

$$x j_\ell(x) = \sin(x - \ell\pi/2) + O(x^{-1}),$$

(Abramowitz and Stegun, 1965, p.364). Near r = a, $e^{ix} \gg e^{-ix}$ and thus the solutions are

$$\left.\begin{aligned} r s_\ell^k &= \tfrac{1}{2} A (-1)^\ell e^{i\alpha r} \\ r t_\ell^k &= \tfrac{1}{2} B (-1)^\ell e^{i\alpha r} \end{aligned}\right\} \qquad (25)$$

The boundary conditions for continuity in normal and transverse fields (Section 2.5) imply, using (17),

Continuity in s_ℓ^k, $\frac{\partial}{\partial r}(r s_\ell^k)$ and t_ℓ^k, $(r=a)$ (26a)

Assuming an insulating mantle, the radial component of $\underline{j} = \frac{1}{\mu_0}(\nabla\times\underline{b})$ must vanish, thus

$$t_\ell^k = 0 \ , \quad (r=b) \tag{26b}$$

The poloidal field within the mantle is then obtained from (21), and it satisfies

$$\frac{d^2}{dr^2}(r s_\ell^k) - \frac{\ell(\ell+1)}{r^2} r s_\ell^k = 0 \ ,$$

with solution

$$s_\ell^k(r) = (d/r)^{\ell+1} s_\ell^k(d) \ ,$$

where $s_\ell^k(d)$ is the field value at the Earth's surface. The poloidal field then satisfies

$$\frac{d}{dr}(r s_\ell^k) + \ell s_\ell^k = 0, \quad (r=b) \ . \tag{26c}$$

There are now five boundary conditions (26a) - (26c) for the six constants in (24) and (25). The sixth condition is supplied by the requirement that the tangential electric field is continuous at r = a. Using (4) and (5)

$$\underline{E} = \frac{1}{\mu_0\sigma}(\nabla\times\underline{B}) - (\underline{v}\times\underline{B}) \ ,$$

and for the perturbation part of $\underline{E}$ this implies continuity in the tangential part of

$$\frac{1}{\mu_0\sigma}(\nabla\times\underline{b}) - (\underline{v}_1\times\underline{B}_0) \ .$$

If $\underline{v}_1$ is continuous, then

Continuity in $\frac{\partial}{\partial r}(r t_\ell^k)$, $(r=a)$ (26d)

supplies the last required condition. A discontinuity in $\underline{v}_1$ at the inner core boundary leads to a boundary toroidal field generated by shear

(Smylie, 1965, p.175). This will not be treated in the present work.

The boundary conditions (26a) - (26d) serve to determine the six constants A - F in (24) and (25). Omitting the algebra, the resulting expressions for the perturbation fields are

$$r s_\ell^k = -\frac{1}{2i\alpha} e^{i\alpha(r-b)} \left[\int_a^b e^{i\alpha(b-r)} f(r)\, dr + \int_a^b e^{-i\alpha(b-r)} f(r)\, dr \right]$$
$$+ \frac{1}{2i\alpha} \left[e^{i\alpha(r-a)} \int_a^r e^{-i\alpha(q-a)} f(q)\, dq - e^{-i\alpha(r-a)} \int_a^r e^{i\alpha(q-a)} f(q)\, dq \right]$$

$$r t_\ell^k = -\frac{1}{2i\alpha} e^{i\alpha(r-b)} \left[\int_a^b e^{i\alpha(b-r)} g(r)\, dr - \int_a^b e^{-i\alpha(b-r)} g(r)\, dr \right]$$
$$+ \frac{1}{2i\alpha} \left[e^{i\alpha(r-a)} \int_a^r e^{-i\alpha(q-a)} g(q)\, dq - e^{-i\alpha(r-a)} \int_a^r e^{i\alpha(q-a)} g(q)\, dq \right].$$

In each of the integrals in these equations, the exponential becomes large at one end of the integration. The asymptotic evaluations can be easily obtained, almost by inspection (Jeffreys and Jeffreys, 1956, p.503),

$$\int_a^b e^{i\alpha(b-r)} f(r)\, dr \sim \frac{1}{i\alpha} f(a) e^{i\alpha(b-a)}$$

$$\int_a^b e^{-i\alpha(b-r)} f(r)\, dr \sim \frac{1}{i\alpha} f(b) \quad .$$

Writing

$$e^{i\alpha(r-a)} = e^{(r-a)/\delta} e^{i(r-a)/\delta}$$

for $r-a > n\delta$ where n is a small integer,

$$\int_a^r e^{i\alpha(q-a)} f(q)\, dq \sim \frac{1}{i\alpha} f(r) e^{i\alpha(r-a)}$$

$$\int_a^r e^{-i\alpha(q-a)} f(q)\, dq \sim \frac{1}{i\alpha} f(a)$$

With these approximations the perturbation fields are

$$\left.\begin{aligned} r s_\ell^k &= \frac{i\delta^2}{4}\left[f(r) + f(b)\,e^{i\alpha(r-b)}\right] \\ r t_\ell^k &= \frac{i\delta^2}{4}\left[g(r) - g(b)\,e^{i\alpha(r-b)}\right] \end{aligned}\right\} \qquad (27)$$

applicable everyshere in the outer core except for the boundary layer near r = a (Region III, Fig.6). The solutions fall naturally into two parts, the first is for Region I (Fig.6) and the exponential is for Region II. Equations (27) indicate that rs_ℓ^k continues on into the mantle across r = b, while rt_ℓ^k drops to zero within the boundary layer.

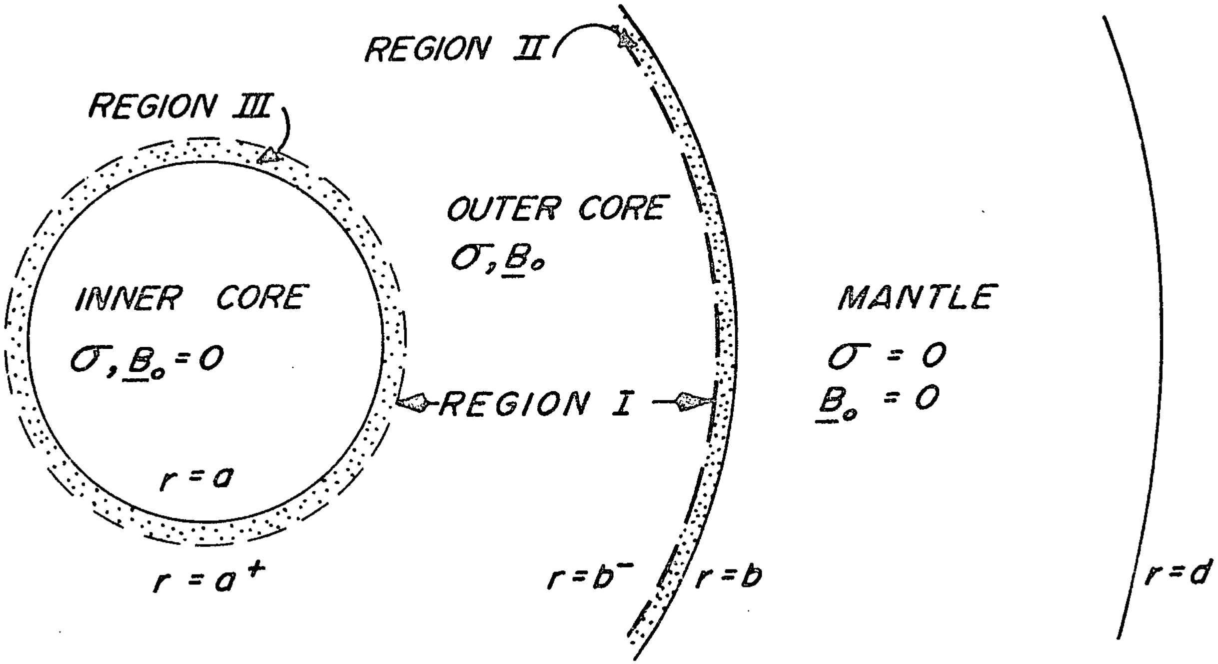


Fig. 6. Induced Field Regions in the Outer Core.

SECTION 4

ENERGY CONSIDERATIONS

The perturbation magnetic fields derived in the last section are generated by the kinetic energy of the free oscillations. To formulate exactly how the transfer of energy takes place, the energy balance is considered. The role of viscosity is treated as an integral part of the discussion.

4.1 Energy Equations

If the Earth were perfectly elastic there would be no damping of the free oscillations. The total energy of each harmonic is then divided equally between kinetic and potential energies, the potential energy being the sum of elastic strain energy and gravitational energy (Kovach and Anderson, 1967, p. 2162). It is useful to remember that over a cycle of an oscillation, velocity dependent forces result in dissipation of energy, whereas amplitude dependent forces do not.

It is clear that one simple model cannot describe the behaviour of the Earth under all conditions within its interior (Jeffreys, pp. 6-13). The mantle behaves as a nearly perfectly elastic medium and the outer core has the properties of a fluid, notably low (or zero) rigidity. To describe the departures from elastic behaviour, the Kelvin-Voigt model (Bland, p. 2) is taken for the outer core because it represents closely the frequency dependence of the attenuation of elastic waves in liquids (Knopoff, 1964, p. 635). The total stress field is taken as simply the sum of the elastic stress, which depends linearly on the strain, and the viscous stress which depends linearly on the strain rate. In this model, when the medium is stressed, there is a delay in the attainment of the strain that would occur in a purely elastic medium. The subsequent loss in elastic energy can be accounted for by the usual expression for dissipation in a viscous medium (Lamb,1932, p. 579).

Consider now, within the core, a small element of material of volume δV, bounded by a surface δS and containing an internal heat energy U per unit mass. With reference to Section 3.2 for the interpretation of quantities, the rate at which body forces do work on the element is

$$v_i \gamma_i + v_i (\underline{J} \times \underline{B})_i \; ,$$

and the rate at which surface stresses do work is

$$\frac{\partial}{\partial x_j} (T_{ij} + \tau_{ij}) v_i \quad ;$$

these forms are obtained by following Love (1944, pp. 93-94). Adding these two expressions, the total rate at which work is done on the element per unit time is

$$\rho_0 v_i \frac{\partial v_i}{\partial t} + T_{ij} \frac{\partial v_i}{\partial x_j} + \tau_{ij} \frac{\partial v_i}{\partial x_j} \tag{28}$$

using the equations of motion (2). The equation of continuity, or mass conservation, requires that

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = 0 \quad ,$$

so neglecting the products of quantities small with the velocity,

$$\rho_0 v_i \frac{\partial v_i}{\partial t} = \frac{\partial}{\partial t} \left(\tfrac{1}{2} \rho v_i^2 \right) . \tag{29}$$

This is the rate of change of kinetic energy density.

To interpret the second term of (28), the strain energy function W (Love, 1944, p. 94) is introduced and defined by

$$W = \tfrac{1}{2} T_{ij} e_{ij} = \tfrac{1}{2} k \Delta^2 + \mu E_{ij}^2 \tag{30}$$

where e_{ij} is the strain tensor and E_{ij} the strain deviator (Bullen, p. 32). The terms on the right hand side are the compressional and shear elastic energy densities. Denoting a time averaged quantity by an overbar, $\overline{W} = \frac{1}{2} W$. The third term in (28) is the rate of viscous dissipation per unit volume (Landau and Lifshitz, 1960, p.214).

Combining the above relations (2), (28), (29) and (30)

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v_i^2\right) = v_i\gamma_i + v_i(\underline{J}\times\underline{B})_i + \frac{\partial}{\partial x_j}\left(T_{ij}v_i + \tau_{ij}v_i\right) - \frac{\partial W}{\partial t} - \tau_{ij}\frac{\partial v_i}{\partial x_j} \qquad (31)$$

The rate of change of magnetic energy density can be written in a similar form (Roberts, p. 18),

$$\frac{\partial}{\partial t}\left(\frac{1}{2\mu_0}B^2\right) = -\frac{1}{\mu_0}\nabla.(\underline{E}\times\underline{B}) - v_i(\underline{J}\times\underline{B}) - \frac{1}{\sigma}J^2 \qquad (32)$$

where the terms on the right are respectively, the outflow of energy from δV given by the Poynting vector, the rate at which work is done by the magnetic volume force against the deformation and the energy lost by ohmic dissipation. Elastic deformations are assumed adiabatic (Bullen, p. 74), so that heat conduction between the element and its surroundings can be ignored. The energy balance is maintained by the rate of increase of internal energy

$$\frac{\partial}{\partial t}(\rho U) = \frac{\partial W}{\partial t} + \tau_{ij}\frac{\partial v_i}{\partial x_j} + \frac{1}{\sigma}J^2 , \qquad (33)$$

expressing the First Law of Thermodynamics.

Equations (31) - (33) are now added and integrated throughout a volume V to obtain the change in total energy per unit time.

$$\int\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho v_i^2 + \frac{1}{2\mu_0}B^2 + \rho U\right\}dV = \int v_i\gamma_i\,dV$$
$$+\int(T_{ij}+\tau_{ij})v_i n_j\,dS - \int\frac{1}{\mu_0}(\underline{E}\times\underline{B})\cdot\underline{n}\,dS \qquad (34)$$

In (34) Gauss' theorem has been used to get the surface integrals, n_j are components of the normal vector $\underline{n}$ and S is the surface bounding V. The terms on the right hand side of (34) are interpreted as follows. The first term is the gravitational energy passing into V, the second term is the work done by the elastic and viscous stresses over the boundary S and the last term is the electromagnetic energy flowing out of V.

If V is now taken to be the core, S is the core-mantle boundary. It can be shown that, because the mantle is considered an insulator, the Poynting vector vanishes just within the mantle. The mantle also has a high rigidity by comparison with the outer core and this indicates there is negligible transfer of energy across S by the viscous stresses. The energy within the core is thus maintained by the flow of gravitational and elastic energy. For plane wave motion, the elastic energy flow is called the intensity of the wave (Morse and Feshbach, p. 151). Within the core the elastic and gravitational energy is stored as the potential energy of the displacement field. The internal heat energy of the core contains contributions from viscous and ohmic dissipation which are positive definite and therefore result in a net loss of energy per cycle. Finally there is a conservation of the energy transferred between the displacement field and the magnetic field through the action of the Lorentz force.

Writing E_k and E_m for the kinetic energy and the magnetic energy within the volume V,

$$E_k = \int \frac{1}{2} \rho v^2 \, dV \quad , \quad \frac{dE_v}{dt} = \int \tau_{ij} \frac{\partial v_i}{\partial x_j} \, dV \qquad (35)$$

and

$$E_m = \int \frac{1}{2\mu_0} B^2 \, dV \quad , \quad \frac{dE_m}{dt} = \int \frac{1}{\sigma} J^2 \, dV \qquad (36)$$

where dE_v/dt is the rate of change of energy due to viscous dissipation.

In (35) $\underline{v}$ is taken as the free oscillation velocity and the two equations of (36) can be linearised by using (8a) in order to assess the contributions of the main and pertubation magnetic fields. Consider the magnetic energy arising from this linearisation,

$$E_m = \frac{1}{2\mu_0}\int B_0^2\, dV + \frac{1}{\mu_0}\int \underline{B}_0 . \underline{b}\, dV + \frac{1}{2\mu_0}\int \underline{b}^2\, dV$$

The first term on the right hand side gives the magnetic energy stored in the main field and the second term is linear in $\underline{b}$ and so averages to zero over a cycle. The last term is the magnetic energy stored in the perturbation field, and is denoted subsequently by e_m. The ohmic dissipation rate reduces in the same way,

$$\frac{dE_m}{dt} = \frac{1}{\sigma}\int J_0^2\, dV + \frac{2}{\sigma}\int \underline{J}_0 \cdot \underline{j}\, dV + \frac{1}{\sigma}\int j^2\, dV$$

where the first term is the dissipation in the main field and the second term averages to zero over a cycle. For the perturbation field

$$\bar{e}_m = \frac{1}{4\mu_0}\int \underline{b} \cdot \underline{b}^* \, dV \tag{37}$$

and

$$\bar{\dot{e}}_m = \frac{\pi}{\sigma\omega}\int \underline{j} \cdot \underline{j}^* \, dV \tag{38}$$

where a dot signifies time differentiation. Also averaged over a cycle, the kinetic energy and viscous dissipation are respectively,

$$\bar{E}_k = \frac{1}{4}\omega^2 \int \rho\, \underline{u}_1 . \underline{u}_1^* \, dV \tag{39}$$

and

$$\bar{\dot{E}}_v = \frac{\pi}{\omega}\int \tau_{ij} \frac{\partial v_i}{\partial x_j}\, dV \tag{40}$$

Equations (38) - (40) are the basis for assessing the attenuation of the free oscillations.

4.2 Ohmic Dissipation

Consider first the kinetic energy. For spheroidal oscillations the displacement $\underline{u}_1$ is given by the vector components (18) and these are substituted into (39). The angular part of the volume integral can be evaluated using the spheroidal vector orthogonality properties (Smylie and Mansinha, 1971, p. 338). For a particular harmonic the time averaged kinetic energy in the outer core is

$$\bar{E}_k = \frac{\omega^2 \pi}{2n+1} \frac{(n+m)!}{(n-m)!} \int_a^b \rho \left\{ U^2 + n(n+1) V^2 \right\} r^2 \, dr \tag{41}$$

The quantities U and V denote the radial and transverse displacements for a mode where they are taken as

$$U = u_n^m(r), \quad V = v_n^m(r), \quad U^2 = u_n^m u_n^{m*}, \quad V^2 = v_n^m v_n^{m*}.$$

In terms of the magnetic field, equation (38) for the ohmic dissipation per cycle is

$$\bar{e}_m = \frac{\pi}{\mu_0^2 \sigma \omega} \int (\nabla \times \underline{b}) \cdot (\nabla \times \underline{b}^*) \, dV$$

Because the curl of a poloidal vector is a toroidal vector and vice-versa, the angular part of this integral follows from the orthogonality properties of these vectors on a sphere (Smylie, 1965, p. 172).

The energy dissipated per cycle by the perturbation field is

$$\bar{e}_m = \sum_{\ell, k} \frac{4\pi^2 \ell(\ell+1)}{\mu_0^2 \sigma \omega (2\ell+1)} \frac{(\ell+k)!}{(\ell-k)!} \int_a^b \left\{ \bar{S}_\ell^k \bar{S}_\ell^{k*} + \right.$$

$$\left. \frac{\ell(\ell+1)}{r^2} t_\ell^k t_\ell^{k*} + \frac{1}{r^2} \frac{\partial}{\partial r}\left(r t_\ell^k\right) \frac{\partial}{\partial r}\left(r t_\ell^{k*}\right) \right\} r^2 \, dr \tag{42}$$

where

$$\bar{S}_\ell^{\,k} = -\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}(rS_\ell^{\,k}) - \frac{\ell(\ell+1)}{r^2} S_\ell^{\,k}\right)$$

Because only self interactions are involved, it is clear that all perturbation fields dissipate energy.

In a similar way, the magnetic energy averaged over a cycle becomes

$$\bar{e}_m = \sum_{\ell,k} \frac{\pi\ell(\ell+1)}{\mu_0(2\ell+1)} \frac{(\ell+k)!}{(\ell-k)!} \int_a^b \left\{ t_\ell^{\,k} t_\ell^{\,k*} + \frac{\ell(\ell+1)}{r^2} S_\ell^{\,k} S_\ell^{\,k*} \right.$$

$$\left. + \frac{1}{r^2}\frac{\partial}{\partial r}(rS_\ell^{\,k})\frac{\partial}{\partial r}(rS_\ell^{\,k*}) \right\} r^2 dr \tag{43}$$

Expressions (42) and (43) are to be evaluated in the next section for particular interactions.

4.3 Viscous Dissipation

The stress-strain rate relations for viscous deformation can be written

$$\tau_{ij} = \xi \dot{\Delta} \delta_{ij} + 2\eta \dot{E}_{ij}$$

where

$$E_{ij} = e_{ij} - \tfrac{1}{3}\Delta\delta_{ij} .$$

It is easily shown that

$$\tau_{ij}\frac{\partial v_i}{\partial x_j} = \tau_{ij}\dot{e}_{ij} = \xi\dot{\Delta}^2 + 2\eta\dot{E}_{ij}^{\,2} ,$$

which can be compared directly with the strain energy function W, equation (30). Using (40) the viscous dissipation can be expressed as

$$\dot{\bar{E}}_v = \pi\omega \int \{ \xi \Delta^2 + 2\eta E_{ij}^2 \} \, dV ,$$

where the time dependence of $\dot{\Delta}$ and $\dot{E}_{ij}$ is that of $\dot{v}_i$. For comparison the elastic energy stored per cycle is

$$\bar{W} = \frac{1}{4} \int \{ k\Delta^2 + 2\mu E_{ij}^2 \} \, dV ,$$

showing the close equivalence between $\dot{E}_v$ and W.

If the elastic and viscous parameters are constant within a given volume V, the ratio of the rate of viscous dissipation to elastic strain energy can be expressed in terms of these constants and the angular frequency only. To illustrate this property the following table is presented, quantities being averaged over a cycle.

Table 2

Integrals for Viscous Dissipation

and Elastic Energy

Deformation	Viscous Dissipation (joules)	Elastic Strain Energy (joules)	Ratio
Compressional	$\pi\omega\xi \int \Delta^2 dV$	$\frac{1}{4} k \int \Delta^2 dV$	$\frac{4\pi\omega\xi}{k}$
Shear	$2\pi\omega\eta \int E_{ij}^2 \, dV$	$\frac{1}{2}\mu \int E_{ij}^2 \, dV$	$\frac{4\pi\omega\eta}{\mu}$

Explicit expressions for the elastic strain energy are given by Kovach and Anderson (1967 , p. 2157). The final form for the viscous dissipation can be written down by the analogy given above

$$\dot{\bar{E}}_v = \sum_{n,m} \frac{4\pi^2\omega}{2n+1} \frac{(n+m)!}{(n-m)!} \int_a^b \Big\{ \zeta \left[rU' + 2U - n(n+1)V\right]^2 + \eta \Big[-\frac{2}{3}\left(rU' + 2U - n(n+1)V\right)^2 + 2\left(rU'\right)^2 + \left(2U - n(n+1)V\right)^2 + n(n+1)\left(rV' - V + U\right)^2 + n(n+2)(n^2-1)V^2 \Big] \Big\} r^2 dr \tag{44}$$

where a prime denotes radial differentiation. Higgins and Kopal (1968) have derived the dissipation rate for shear viscosity but their final expression contains a typographical error.

4.4 Q

To measure the energy dissipation per cycle of an oscillation, a specific dissipation function Q^{-1} is introduced

$$Q^{-1} = (2\pi E)^{-1} \oint \frac{dE}{dt}\, dt$$

where E is the peak energy and $\frac{dE}{dt}$ the dissipation rate (Knopoff, 1964, p. 626). The particular energy to which E refers depends on the damping mechanism. For instance, in the mantle the damping is due to anelastic effects and so E refers to the strain energy stored per cycle (e.g. Anderson and Archambeau, 1964). Ohmic dissipation can be considered as an imperfection in inertia and thus E must strictly be taken as the kinetic energy. In practice, because energy dissipation is here only expected to be a small effect, it makes little difference which interpretation is placed on E.

One advantage of using Q^{-1} as opposed to Q is that for a layered system with energy E_i per layer, Q^{-1} for all the layers is a linear combination of Q_i^{-1} for each layer,

$$Q^{-1} = \sum_i (E_i/E)\, Q_i^{-1}$$

(Jackson and Anderson, 1970, p. 4). For a two-layered Earth, a core (c) and mantle (m),

$$Q^{-1} = \left(\frac{E_c}{E}\right) Q_c^{-1} + \left(\frac{E_m}{E}\right) Q_m^{-1} \tag{45}$$

where E is the total energy. To observe a Q_c for the core from an observed Q for a mode, suppose a change of 10% can be detected seismically i.e. $\Delta Q/Q \geq 10^{-1}$. Then from the previous equation, this implies

$$(E/Q)_c \geq (E/Q)_m \times 10^{-1}$$

If the mantle has an infinite Q_m or there is no kinetic energy in the mantle, the observed Q will be the Q_c for the core. In this case

$$Q^{-1} = (2\pi E)^{-1} \oint \frac{dE_c}{dt}\, dt \tag{46}$$

so that the kinetic energy integral given by (41) can be extended over the whole Earth, for a Q due to dissipation only in the core. Following Knopoff (1964, p. 626), the Q defined by (46) can be related to a damping factor, for an oscillation with displacement $\underline{u}_n^m$ and angular frequency ω_n by

$$\left.\begin{aligned} \underline{u}_n^m(r,t) &= \underline{u}_n^m(r)\, e^{i\omega_n t}\, e^{-\gamma_n t}, \\ \text{where} \quad \gamma_n &= \omega_n/2Q \end{aligned}\right\} \tag{47}$$

is applicable for the attenuation of a standing wave.

In Table 2, the ratio of viscous dissipation to elastic strain energy is given for medium uniform in the viscous and elastic constants. Regarding viscous dissipation as an imperfection in elasticity, the ratio of energy lost to energy stored defines a Q. For shear the Q is

$$Q^{-1} = \omega \eta / \mu$$

and this agrees with the expression given by Knopoff (1964, p. 635) for acoustic loss in Kelvin-Voigt solid.

SECTION 5

PARTICULAR INTERACTIONS

The discussion in the previous sections can now be applied to calculate the Q of an oscillation due to magnetic and viscous damping. The toroidal quadrupole field component has been chosen as probably the strongest field in the core, and therefore of greatest geophysical interest. Table 9 in Appendix A indicates there are no resonance interactions to be considered and so the choice of a velocity field can be made on dynamical grounds.

5.1. Radial Oscillations

It is well known (Ness et al., 1961) that the fundamental radial oscillation has a high Q, which is explained by the small amount of shear energy, relative to compressional energy, in the mantle (Kovach and Anderson, 1964, p.2162). The radial overtones also have high Q's (Dratler et al., 1971), and it can be seen from Fig.7 that all these oscillations have appreciable energy within the core. It is therefore natural to treat first the radial oscillations; they also have no transverse displacement and this simplifies the mathematical treatment.

From the interaction diagram (Fig 4), the only field induced by the oscillations S_0 is the toroidal quadrupole t_2^0. Defining T(r) as the radial function of the harmonic T_2^0 and U(r) as the radial displacement of the νth overtone of a radial oscillation,

$$g(r) = i\omega\mu_0\sigma\,(rUT)' \quad . \tag{48}$$

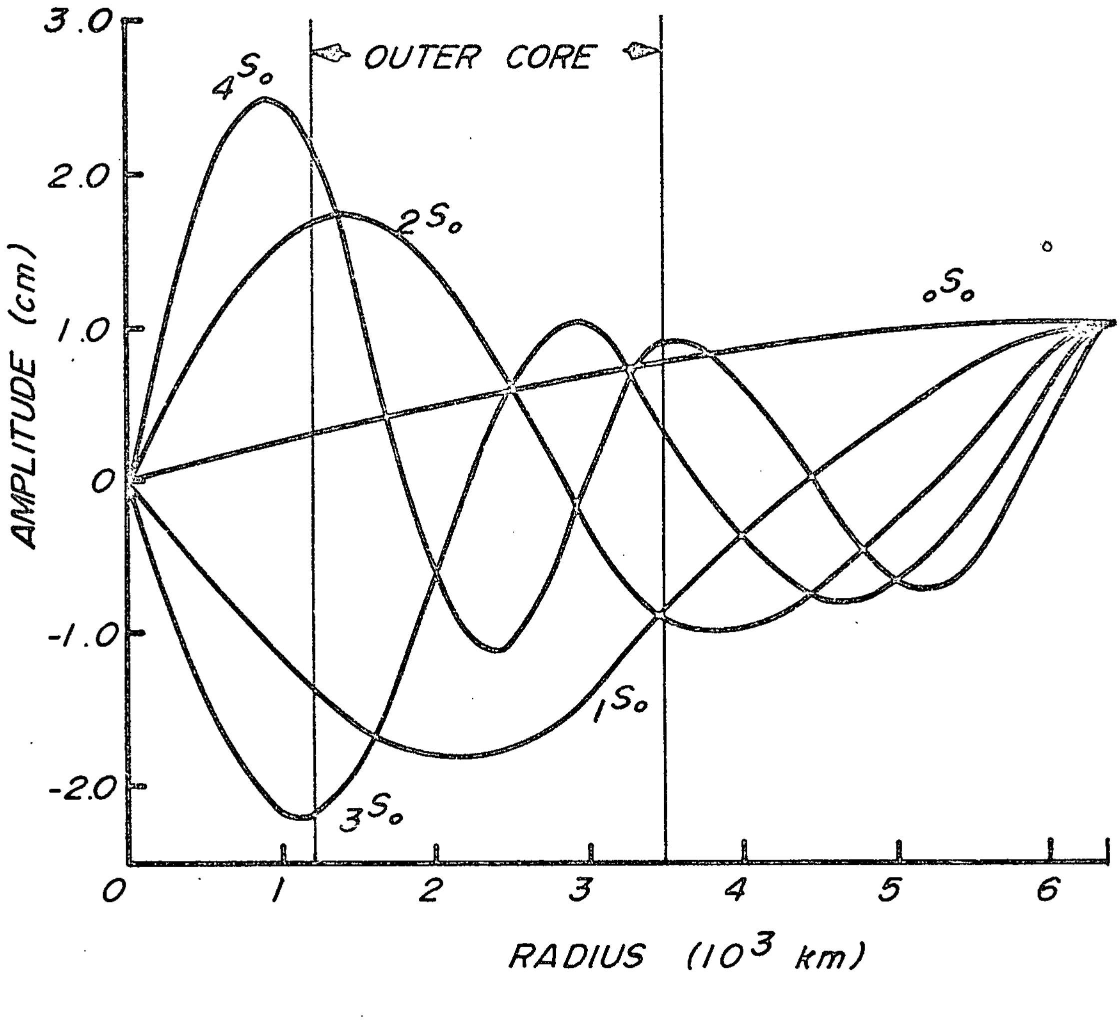


Fig. 7. Amplitudes of the Radial Oscillations in the Earth.

In this and subsequent equations a prime denotes a radial differentiation. The radial function for the induced field becomes

$$r t_2^0 = -\frac{1}{2}(rUT)' + \frac{1}{2}(rUT)'_b\, e^{(1+i)(r-b)/\delta} \tag{49}$$

using (27) and (48). The subscript b indicates the term $(rUT)'$ is to be evaluated at r = b. Similarly

$$r t_2^{0\,*} = -\frac{1}{2}(rUT)' + \frac{1}{2}(rUT)'_b\, e^{(1-i)(r-b)/\delta} ,$$

and so

$$r^2 t_2^0\, t_2^{0\,*} = \frac{1}{4}\Big[(rUT)'^2 - 2(rUT)'(rUT)'_b\, e^{(r-b)/\delta} \cdot \cos\left(\frac{r-b}{\delta}\right) + (rUT)'^2_b\, e^{2(r-b)/\delta}\Big] \tag{50}$$

The relative strengths of the induced and main magnetic fields can be estimated using (49). For a uniform field, T(r) is constant,

$$\frac{|t_2^0|}{T_2^0} = \frac{U}{2r} + \frac{U'}{2} \tag{51}$$

The perturbation expansion is thus invalid near the origin r = 0, and wherever the gradient of the displacement field is of order unity. Neither of these conditions hold in the outer core, the first does not because simply $r \gg a$. The gradient of the radial displacement is generally of order 10^{-8} (Fig.7) and because the radial displacement is required to be continuous across r = a,b, large gradients are unlikely to occur at the boundaries. This limitation on the linearisation is similar to that described by Lilley and Smylie (1968, p.6529).

The energy in the perturbation magnetic field, averaged over a cycle, is obtained from (43)

$$\bar{e}_m = \frac{6\pi}{5\mu_0}\int_a^b t_2^0\, t_2^{0*}\, r^2\, dr \quad , \tag{52}$$

and can be computed directly using (50). Contributions from the boundary layer, (Region II, Fig.6) can be obtained analytically. The exponential terms in (50) decrease rapidly to zero within a few skin depths of r = b. Providing the term $(rUT)'$ does not grow exponentially within the boundary layer, and this is arranged by the choice of T, the term can be considered constant in the boundary layer. The contributions to (52) from the boundary layer are then

$$\frac{6\pi}{5\mu_0}\int_{z_0}^{0}\frac{1}{4}\left\{-2(rUT)_b'^2\, e^{z}\cos z + (rUT)_b'^2\, e^{2z}\right\}dz$$

where

$$z = \frac{(r-b)}{\delta} \qquad \text{and} \qquad z_0 = \frac{(b^- - b)}{\delta}$$

On integration this contribution becomes

$$\frac{3\pi}{10\mu_0}\left[-(rUT)_b'^2\,\frac{\delta}{2}\right] , \qquad e^{z_0} \gg 1 ,$$

and the magnetic energy is now

$$\bar{e}_m = \frac{3\pi}{10\mu_0}\left\{-(rUT)_b'^2\,\frac{\delta}{2} + \int_a^b (rUT)'^2\right\} \tag{53}$$

For the main field the magnetic energy density is $B_0^2/2\mu_0$ which gives, for T_2^0 alone,

$$E_m = \frac{12\pi}{5\mu_0}\int_a^b (T)^2 r^2\, dr \tag{54}$$

The total energy dissipated per cycle is obtained from (42),

$$\bar{\dot{e}}_m = \frac{24\pi^2}{5\mu_0^2\sigma\omega}\int_a^b \left\{ \frac{6}{r^2} t_2^0 t_2^{0*} + \frac{1}{r^2}(rt_2^0)'(rt_2^{0*})' \right\} r^2 dr ,$$

where (50) is appropriate for the first term in the integrand. The second term can be readily reduced by substituting with (49) and the rate of ohmic dissipation then follows,

$$\bar{\dot{e}}_m = \frac{6\pi^2}{5\mu_0^2\sigma\omega}\left\{ \int_a^b \left[\frac{6}{r^2}(rUT)'^2 + (rUT)''^2 \right] dr \right.$$

$$+ 2(rUT)'_b \int_0^{z_0} \left[\frac{6\delta}{r^2}(rUT)'\cos z + (rUT)''(\cos z$$

$$\left. - \sin z)\right] e^z dz + (rUT)'^2_b \int_{z_0}^{0} \left(\frac{2}{\delta} + \frac{6\delta}{r^2} \right) e^{2z} dz \right\}$$

The approximations

$$\frac{6\delta}{r^2} << \frac{2}{\delta} , \qquad \frac{6\delta}{r^2}(rUT)'_b << (rUT)''_b$$

are found to be valid near r = b and the rate of ohmic dissipation per cycle reduces to

$$\bar{\dot{e}}_m = \frac{6\pi^2}{5\mu_0^2\sigma\omega}\left\{ -2(rUT)''_b(rUT)'_b + \frac{1}{\delta}(rUT)'^2_b \right.$$

$$\left. + \int_a^b \left[\frac{6}{r^2}(rUT)'^2 + (rUT)''^2 \right] dr \right\} \qquad (55)$$

The kinetic energy per cycle is simply

$$\bar{E}_k = \pi\omega^2 \int \rho U^2 r^2 dr \qquad (56)$$

Following the discussion in Section 4.4, the limits to the last integral can be taken as 0 and d if the dissipation within the core is considered as the only energy loss in the Earth.

The effective Q associated with the magnetic damping is thus, from (46),

$$Q^{-1} = (4\pi \bar{E}_k)^{-1} \bar{e}_m \quad , \tag{57}$$

where the peak kinetic energy is used in accordance with the definition of Knopoff (1964, p.626).

Equations (53) to (57) are now used to compute the various quantities for the interaction. It remains to specify the radial functions U, T within the regions of the integrations.

Consider first the radial displacement U. In Fig.7 the amplitude of the radial displacement is plotted versus radius for the purely radial mode and the first four overtones. A recent publication on the modes of the Alaskan earthquake (Dziewonski and Gilbert, 1972) does not list observed eigenperiods above ${}_4S_0$ for the radial oscillations (their Table 7). The amplitudes of U in Fig.7 were established using a recent Earth model supplied by Jordan and Anderson (1972). Some details of the computations for the displacement field are to be found in Appendix B. As mentioned in the Introduction, the agreement between theoretical and observed eigenperiods gives considerable confidence in the broad properties of the Earth models currently in use. For the present purpose the choice of model is not critical because amplitudes are not very sentitive to minor changes in Earth structure.

Choice of a radial function for the toroidal magnetic field is open to some speculation. Only approximate indications of the strength of T_2^0 have been obtained by dynamo theory. Following Bullard and Gellman (1954, p.275) the maximum field strength B_m is taken as 480 gauss. To cover the simple types of field, three elementary fuctions are used in the computations. The first field, called Type A, assumes a constant value throughout the outer core. The two functions called Types B and C have uniform gradients, and the fourth field, Type D, is a sinusoidal function with a variable number of oscillations in the radial direction (given by the index n). These functions are shown in Fig.8 together with their radial forms and first and second derivatives.

In Table 3 the magnetic energy within the core is shown for the various fields, computed from equation (54), together with the field and its first two derivatives at the core-mantle boundary.

Table 3

Main Magnetic Field Parameters

Field Type	Energy E_m (joules)	Field at Core-Mantle Boundary (m.k.s. units)		
		T $\times 10^{-4}$	T' $\times 10^{-8}$	T'' $\times 10^{-19}$
A	1.17×10^{18}	480	0.0	0.0
B	5.72×10^{17}	480	2.1	0.0
C	2.19×10^{17}	0.0	-2.1	0.0
D (n = 1)	5.61×10^{17}	0.0	-6.6	-1.4
(n = 5)	5.86×10^{17}	0.0	-33.2	15.5
(n =10)	5.86×10^{17}	0.0	66.4	1629.3

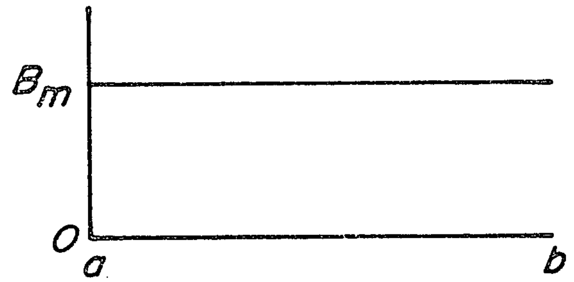
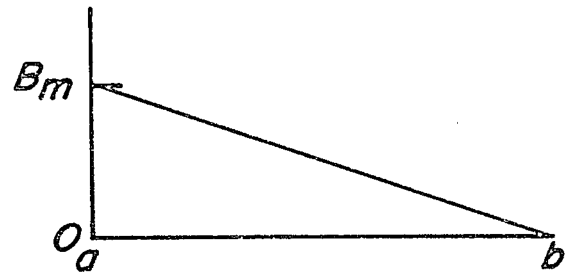
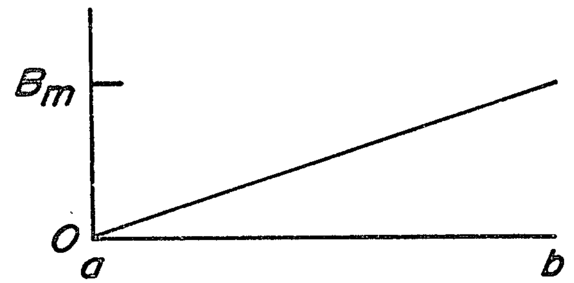
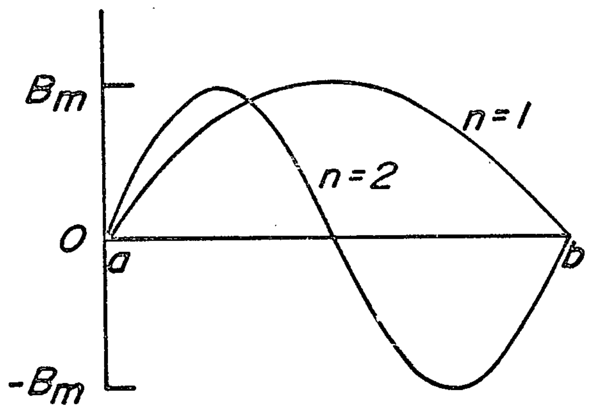
FIELD TYPE	AMPLITUDE IN OUTER CORE	RADIAL FUNCTION
A		$T(r) = B_m$ $T' = 0$ $T'' = 0$
B		$T(r) = B_m(r-a)(b-a)$ $T' = B_m/(b-a)$ $T'' = 0$
C		$T(r) = B_m(b-r)/(b-a)$ $T' = -B_m/(b-a)$ $T'' = 0$
D		$T(r) = B_m \sin\psi$ $T' = \frac{n\pi}{b-a} B_m \cos\psi$ $T'' = -\left(\frac{n\pi}{b-a}\right)^2 B_m \sin\psi$ $\psi = n\pi(r-a)/(b-a)$

Fig. 8. Radial Functions for the Toroidal Quadrupole Field.

5.2 Results for the Radial Oscillations.

The results of the computations of Q values are presented in Tables 4, 5 and 6. It is immediately clear that the Q's for the interactions are so high that the oscillations are virtually unattenuated. Observations of these Q's can probably be safely dismissed under all conditions. Further evaluation and discussion of the results is interesting from a physical, rather than a practical, point of view.

A comparison of Tables 4 and 5 for the fundamental radial oscillation confirms the gradient effect (Lilley and Smylie, 1968). From Table 3, field Type A has about twice the energy of Type B and when this difference is allowed for, the energy dissipated in the Type B field is an order of magnitude larger than that dissipated in the Type A field. For the overtones however the gradient effect is not evident, e.g. ${}_3S_0$ has a relatively large amount of dissipation for both field types. In Fig.7 it can be seen that ${}_3S_0$ has the steepest gradient at r = b in Region II. The damping is thus seen to be dependent on the combinations of gradients in both T and U, as expected from the terms appearing in equation (55).

It is also clear that most of the ohmic dissipation takes place in Region II where there is a large gradient in the induced field due to the requirement of no induction in the mantle. In Table 5, Region I dissipation has been excluded because it is negligible.

In Table 6, where the field Type D is sinusoidal, again most of the dissipation occurs in Region II, and it is proportional to n^2 .

Table 4

Radial Oscillations in a Uniform Field

Oscillation	Period (secs)	Kinetic Energy per cycle (joules)	Skin Depth (m)	Induced Field Energy per cycle (joules)	Energy Dissipated per cycle (joules)		Effective Q
					Region I	Region II	
$_0S_0$	1227.65	2.9×10^{15}	32.2	4.4×10^{5}	0.0085	52.0	7×10^{14}
$_1S_0$	614.15	1.6×10^{16}	22.8	1.9×10^{6}	0.035	430.0	5×10^{14}
$_2S_0$	398.56	2.3×10^{16}	18.3	4.7×10^{6}	0.030	200.0	1×10^{15}
$_3S_0$	305.62	3.6×10^{16}	16.1	6.8×10^{6}	0.088	1000.0	4×10^{14}
$_4S_0$	243.64	5.5×10^{16}	14.3	1.2×10^{7}	0.11	64.0	1×10^{16}

Table 5

Radial Oscillations in a Field with a Uniform Gradient

Oscillation	Induced Field Energy per cycle (joules)	Energy Dissipated per cycle (joules) Region II		Effective Q	
		Type B	Type C	Type B	Type C
$_0S_0$	1.0×10^5	200.0	49.0	1.8×10^{14}	7.6×10^{14}
$_1S_0$	8.9×10^5	210.0	39.0	9.6×10^{14}	5.2×10^{15}
$_2S_0$	9.4×10^5	420.0	39.0	6.8×10^{14}	7.4×10^{15}
$_3S_0$	2.1×10^6	930.0	3.0	4.8×10^{14}	1.4×10^{17}
$_4S_0$	3.4×10^6	180.0	30.0	3.8×10^{15}	2.2×10^{16}

Table 6

Radial Oscillations in a Sinusoidal Field

Oscillation	n	Energy per cycle inInduced Field(joules)	Energy Dissipated per cycle(joules)		Effective Q
			Region I	Region II	
$_0S_0$	1	$9.3x10^5$	0.03	$4.8x10^2$	$2.6x10^{13}$
	5	$2.2x10^7$	7.3	$1.2x10^4$	$3.0x10^{12}$
	10	$8.7x10^7$	110.0	$4.8x10^4$	$2.6x10^{11}$
$_1S_0$	1	$6.6x10^6$	0.11	$3.9x10^2$	$5.2x10^{14}$
	5	$1.3x10^8$	21.0	$9.7x10^3$	$2.0x10^{13}$
	10	$5.1x10^8$	320.0	$3.9x10^4$	$5.2x10^{12}$
$_2S_0$	1	$3.9x10^6$	0.09	$3.8x10^2$	$7.4x10^{14}$
	5	$4.6x10^7$	6.1	$9.6x10^3$	$3.0x10^{13}$
	10	$1.8x10^8$	79.0	$3.8x10^4$	$7.4x10^{12}$
$_3S_0$	1	$5.5x10^6$	0.11	32.0	$1.4x10^{16}$
	5	$5.0x10^7$	5.4	$8.0x10^2$	$5.6x10^{14}$
	10	$1.9x10^8$	66.0	$3.2x10^3$	$1.4x10^{14}$
$_4S_0$	1	$6.5x10^6$	0.15	$3.0x10^2$	$2.4x10^{15}$
	5	$4.0x10^7$	4.4	$7.5x10^3$	$9.2x10^{13}$
	10	$1.4x10^8$	43.0	$3.0x10^4$	$2.4x10^{13}$

This can easily be verified by substituting the radial function

$$T(r) = B_m \sin\left\{ n\pi \frac{(r-a)}{(b-a)} \right\}$$

into (55). Thus Q is proportional to n^{-2}, a clear indication of the gradient effect. It should be noticed however that for a sinusoidal field, the energy dissipated in Region I increases faster than n^2 and for n greater than about 10^4, Region I becomes the dominant region for dissipation. This would be a rough field indeed, with a peak-to-peak amplitude of nearly 10^3 gauss, and a wavelength about 200m.

If a comparison is to be made between the fields here and the field functions resulting from dynamo theory, the choice of a sinusoidal field with n = 1 is probably the closest (Bullard and Gellman, 1954, p.246). A typical Q for the radial modes can therefore be taken as 10^{15} for this type of field.

5.3 Non-Radial Oscillations

The discussion of Section 5.1 can be easily extended to the case of non-radial oscillations. With the notation used in Appendix A, equations (22) can be written.

$$f(r) = i\omega\mu_0\sigma \tfrac{1}{2}\left\{ rUTa_1 \right\}$$

$$g(r) = i\omega\mu_0\sigma \tfrac{1}{2}\left\{ (rUT)'b_1 + VTb_2 \right\}$$

where, for the non-radial oscillations, the transverse displacement is added to the term on the right hand side of (48). The previous theory can then be extended by substituting

$$\tfrac{1}{2}\left\{ (rUT)'b_1 + VTb_2 \right\}$$

in place of

$$(rUT)'$$

in the expressions (53) and (55). According to Table 9 (Appendix A), a_1 is zero for all the fundamental oscillations and so only toroidal fields will be induced.

Several oscillations are chosen as representative motions in the outer core. Fig.9 shows a plot of the amplitudes with depth for the radial displacements of several overtones. The fundamental oscillations ${}_0S_2$, ${}_0S_5$, and ${}_0S_{10}$ have been shown in Fig.3. The overtones of the spheroidal oscillations S_2 are seen to have appreciable energy within the inner core, while the amplitudes of the overtones ${}_2S_2$ and ${}_1S_{10}$ are large at the core-mantle boundary. Such oscillations, with most of their energy at the core-mantle interface, are referred to as Stoneley waves (e.g. Alsop, 1963, pp.498-499).

In Table 7 results are presented for the ohmic dissipation of these modes with the energy dissipated given mainly for Region II as before. The energy dissipated is shown for each of the induced fields produced by the interactions in Fig.4 and Table 9 (Appendix A). To obtain the Q for the mode these energies are then summed to find the total dissipation.

5.4 Viscous Damping

Expression (44) gives the energy dissipated as viscous heating for an oscillation in the outer core. It is straightforward to program

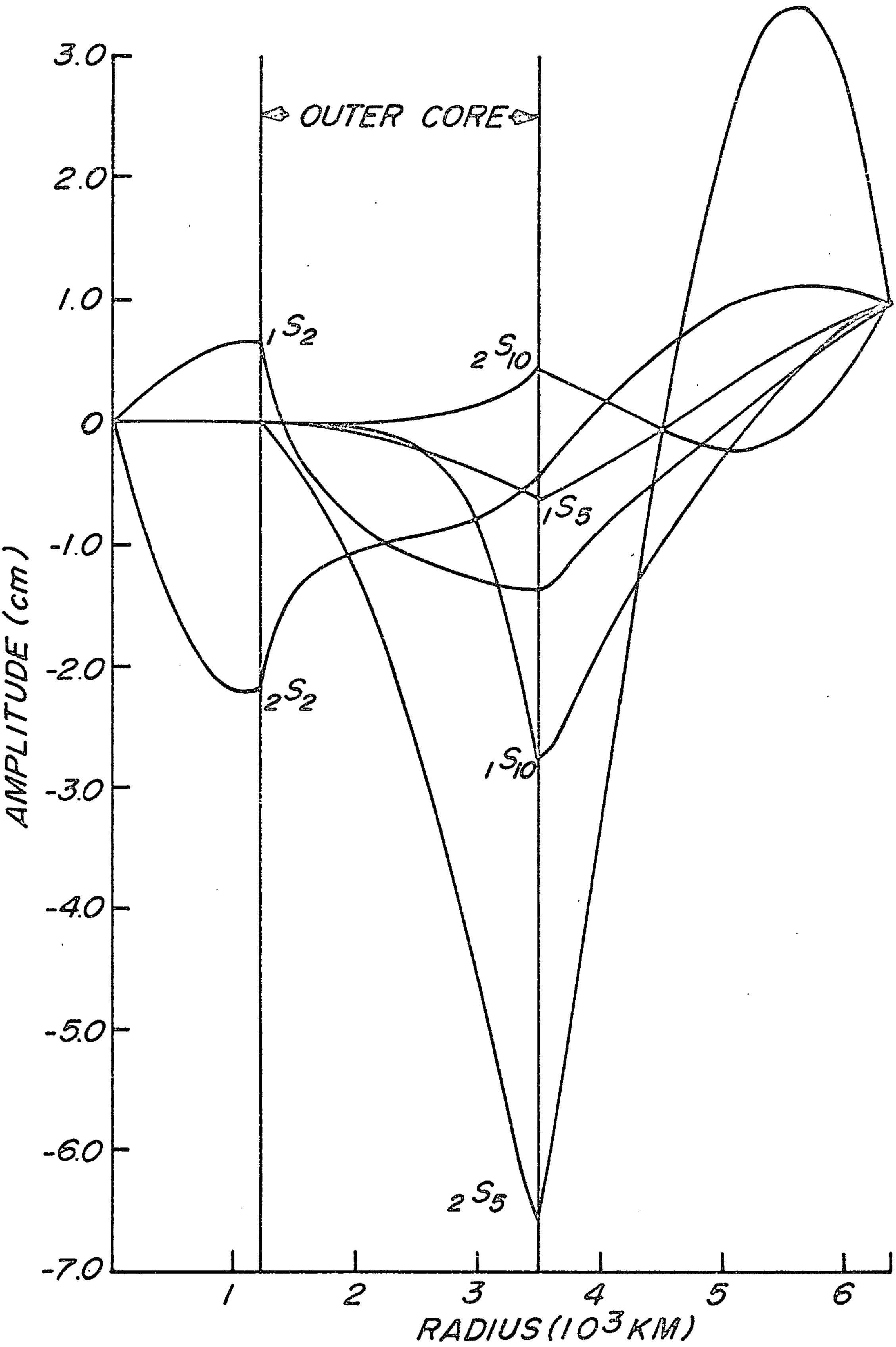


Fig. 9. Amplitudes of Spheroidal Overtones in the Earth.

Table 7

Non-Radial Oscillations in a Sinusoidal Field (n=1)

Oscillation	Period (secs)	Kinetic Energy per cycle (joules)	Energy Dissipated per cycle for each Induced Field (joules)			Effective Q
			t_2^o	t_4^o		
$_0S_2$	3232.35	1.9×10^{14}	36.9	119.0		1.5×10^{13}
$_1S_2$	1469.34	5.7×10^{15}	33.7	109.0		5.0×10^{14}
$_2S_2$	904.32	2.7×10^{15}	1.56	5.06		5.1×10^{15}
			t_3^o	t_5^o		
$_0S_5$	1190.64	5.2×10^{14}	28.7	0.81		2.2×10^{14}
$_1S_5$	729.40	6.6×10^{14}	5.53	0.16		1.5×10^{15}
$_2S_5$	660.06	6.1×10^{16}	584	16.5		1.2×10^{14}
			t_8^o	t_{10}^o	t_{12}^o	
$_0S_{10}$	580.20	3.8×10^{14}	0.28	2.5×10^{-3}	0.24	9.6×10^{15}
$_1S_{10}$	466.09	2.4×10^{15}	21.2	0.18	17.7	7.7×10^{14}
$_2S_{10}$	416.03	9.2×10^{14}	0.45	3.9×10^{-4}	0.37	1.4×10^{16}

this expression and insert the estimates for the viscous coefficients assumed for the core. Gans' (1972) estimate for η is used (Table 1) but there is no available estimate for ζ so it arbitrarily taken equal to η. To calculate the Q for the dissipation only shear energy is used. The evaluations are given separately for the contributions from the bulk and shear viscosities and are shown in Table 8, for those oscillations considered for magnetic dissipation.

As expected, those oscillations with large amplitudes at the core-mantle boundary have more shear dissipation than the other modes. This does not necessarily lead to a lower Q for these modes because Q is amplitude independent (the energy of the mode also depends on amplitude). It can be seen that the shear dissipation for the fundamental radial mode ${}_0S_0$ is small compared to the other oscillations. This accords with the case of elastic strain energy in the mantle (Kovach and Anderson, 1967), as it should since the basic integral is the same (Table 2). Aside from the radial oscillation a typical Q for shear viscous dissipation can be taken as 10^{15}, of the same order of magnitude as the magnetic Q.

Table 8

Viscous Damping

Oscillation ${}_{\nu}S_n$ Degree n	Overtone ν	Energy Dissipated per cycle (joules) Shear	Bulk	Effective Q for Shear Dissipation
0	0	4.0×10^{-3}	0.95	9.2×10^{18}
	1	9.6	7.2	2.1×10^{16}
	2	22.9	16.1	1.2×10^{15}
	3	48.7	25.5	9.2×10^{14}
	4	82.3	55.7	8.4×10^{14}
2	0	0.41	2.8×10^{-2}	5.8×10^{15}
	1	1.78	4.9×10^{-2}	4.0×10^{16}
	2	2.64	0.85	
5	0	1.53	2.5×10^{-3}	4.3×10^{15}
	1	0.76	2.0×10^{-2}	1.1×10^{16}
	2	100.64	4.8	7.6×10^{15}
10	0	0.24	2.8×10^{-3}	2.0×10^{16}
	1	134.55	3.0	2.3×10^{14}
	2	0.70	1.4×10^{-2}	1.7×10^{16}

SECTION 6

SUMMARY AND CONCLUSIONS

The outcome of the discussion in the previous Section is that the free oscillations of the Earth are negligibly attenuated by either the magnetic field or the fluid viscosity within the core. It remains now to give a brief summary of the results in comparison with former investigations, to indicate the direct geophysical consequences, and to offer some speculation on possible implications.

6.1 Comparison with Previous Results

As reviewed in the Introduction, Knopoff (1955) calculated that magnetoelastic interactions were unimportant within the Earth's fluid core. The work of Kraut (1965) and the refinements of Lilley and Smylie (1968) did not significantly alter this conclusion. The suggestion of Lilley (1967) that the gradient effect might lead to enhanced damping of standing elastic waves has now been followed up with the refined analysis presented in this thesis.

The Q's obtained by Kraut and Lilley and Smylie were of the order of 10^{17}. The Q's determined from the present study are variable, depending, as expected, on the oscillation chosen and the radial function of the toroidal magnetic field. For a field suggested by the Bullard-Gellman dynamo model, that is a half-sinusoid within the outer core, the lowest Q obtained was 10^{13} for the fundamental spheroidal oscillation ${}_0S_2$. At its strongest the interaction is thus a few orders of magnitude larger than previously obtained. The Q can be arbitrarily lowered by making the radial function rougher, but the justification is minimal, and there does not appear to be a further mechanism for lowering the Q values obtained here.

The reason for the difference between the Q's here and those obtained before lies in the rigorous treatment of the geometry of the fields and their gradients. However there can be no hope of measuring such high Q's seismically because of the domination of the mantle dissipation, evident by the observed Q's of 10^2 to 10^3. The speculation of Lilley is seen to be unsubstantiated, indicating, as he himself recognised, the caution necessary in interpreting order-of-magnitude estimates.

Due to the lowering of the viscosity of the outer core to the value suggested by Gans (1972), the role of viscous dissipation is also insignificant. For the ${}_0S_2$ oscillation the Q due to shear viscosity is now of the order of 10^{16}, even higher than the magnetic Q. This does not allow for the effect of the second coefficient, or bulk, viscosity.

6.2 Geophysical Implications

Several comments can be made concerning the direct implications of the theory. It will be assumed in the following that a 'normal' free oscillation refers to the mode ${}_0S_2$ which has a period of 3 x 10^3 secs. and a Q of 10^{13}, obtained from ohmic dissipation in the boundary layer (Region II) of a magnetic field of Type D with n = 1 (Fig. 8). Using equation (47), the damping time of such an oscillation in the core can be written

$$\tau = \gamma^{-1} = \frac{2Q}{\omega} = Q \times \text{Period}/\pi \qquad (58)$$

and is equal to about 10^8 years.

The dependence of the kinetic energy and ohmic dissipation on the pertinent parameters in the core can be obtained by considering equations (55) to (57). Denoting dependency by the symbol $\sim$, from (55), and for Region I

$$\bar{\dot{e}}_m \sim \frac{U^2 B^2}{\mu_0^2 \sigma \omega r_c} , \qquad (59)$$

where U and B are typical values of oscillation amplitude and magnetic field strength, and r_c is a representative core radius. Similarly from (56)

$$\bar{E}_K \sim \omega^2 \rho U^2 r_c^3 , \qquad (60)$$

thus using (57), (59) and (60)

$$Q_I \sim \frac{\mu_0^2 \sigma \omega^3 \rho r_c^4}{B^2} . \qquad (61)$$

For a Q in the boundary layer

$$\bar{\dot{e}}_m \sim \frac{U^2 B^2}{\mu_0^2 \sigma \omega} \cdot \frac{1}{\delta} ,$$

and so

$$Q_{II} \sim \frac{\mu_0^{3/2} \sigma^{1/2} \omega^{5/2} \rho r_c^3}{B^2} . \qquad (62)$$

These expressions for Q, (61) and (62), show a strong dependency on angular frequency such that, as the frequency is lowered, the dissipation in the bulk of the core becomes more important. The skin depth also increases. Considering the boundary layer dissipation, the relation of Q with period is

$$\frac{Q_1}{Q_2} = \left[\frac{Period_2}{Period_1} \right]^{5/2} \qquad (63)$$

so that for a displacement field ${}_0S_2$ at the period of the Chandler Wobble for example, the Q is about 10^3 . This is not directly applicable to the actual Chandler Wobble because at such a long period the displacement field is not expected to have the character of a free oscillation.

The results so far have been discussed for their negative aspect, that is the unlikely use of the damping in determining the structure of the magnetic field within the core. However interesting implications can be suggested in connection with the behaviour of the interaction at periods longer than the free modes and their overtones .

It has long been known that free oscillations can exist at longer periods than the fundamental for each degree of the harmonic expansion of the displacement field (Alterman et al., 1959). The presence of these oscillations depends on the density distribution within the core and Alterman et al. found the oscillations for only one of their trial Earth models, Bullen B. These oscillations have most of their energy in the core, and so by the previous reasoning might be expected to be undamped by magnetoelastic or viscous interactions. With damping times for two oscillations given by the ratio

$$\frac{\tau_1}{\tau_2} = \left[\frac{\text{Period}_2}{\text{Period}_1} \right]^{3/2} \qquad (64)$$

obtained from (58) and (63), the damping time of a core oscillation might be about 10^7 years if there is no energy dissipation in the mantle. Is is difficult, as pointed out by Alterman et al., to understand how these core modes can be excited by a source in the mantle because of the very fact which makes them interesting here, namely their being confined to the core.

At periods corresponding to the bodily tides of the Earth, that is at periods at multiples of 12 hours, the Earth responds in a second degree forced oscillation. The theory of the oscillations of a real Earth model at such frequencies is not as straightforward as for the normal free modes because the Earth's diurnal rotation might be expected to be important. Also, in the static limit (when the frequency is allowed to be zero), the behaviour of the liquid core becomes quite different from the dynamical case. As Smylie and Mansinha (1971) have shown, the system of equations (equation (B1), Appendix B) degenerates to a second order system with the motion determined entirely by gravitational forces. That is, the core responds passively to the gravitational perturbations and the elastic stresses are no longer important.

Pekeris and Accad (1972) have discussed the behaviour of the liquid core at tidal frequencies, but there is some doubt as to the correctness of their asymptotic theory in the manner of letting the angular frequency go to zero (Smylie, personal communication). The nature of the oscillations in the core at these frequencies is at the present stage unclear.

The damping of these oscillations can however be inferred in a cautious manner from (63) and (64) without knowing precisely the displacements in the core, at least for an order of magnitude estimate. At periods of 24 hours the magnetic Q is found to be of the order of 10^9 and the damping time is about 10^5 years. These oscillations will then also be considerably underdamped and may be expected to persist as long as the longest temporal variations of the geomagnetic field.

Such a coincidence of time scales is in all probability fortuitous, and could be dismissed entirely but for the following speculation.

It has already been mentioned that the arguments of Higgins and Kennedy (1971) have raised serious doubts as to the existence of the large scale convection of the core required for the conventional dynamo theory. The recent emphasis on turbulent induction processes (Moffatt, 1972) and the development of models with a cellular flow in the core (Gubbins, 1972) has indicated that alternative mechanisms are possible. A suitable energy source has yet to be established.

As Moffatt indicated, the requirements on the velocity field are that it should have no mean flow and that the motion lack reflexional symmetry. The time scale of the fluctuations also has to be long compared to the diurnal rotation, at periods of a month or more (Moffatt, 1972, p. 398). If the core does respond to oscillations of such long period, and this will depend on the adiabaticity, or degree of stability (Pekeris and Accad, 1972), the associated velocity fields may well satisfy the conditions noted above.

The results of the present work then ensure that the damping of such oscillations is only appreciable over geologic times, which may be an important contribution to future studies of the behaviour of the core of the Earth.

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APPENDIX A

THE GAUNT AND ELSASSER INTEGRALS

In section 3.2 two integrals K L are introduced which are the product of three angular functions. They are defined by

$$K^{-k,m,p}_{l,n,s} = \int_0^{2\pi}\int_0^{\pi} P_l^{-k} P_n^m P_s^p e^{i(-k+m+p)\phi} \sin\theta \, d\theta \, d\phi$$

$$L^{-k,m,p}_{l,n,s} = \int_0^{2\pi}\int_0^{\pi} P_l^{-k} \left\{ p \frac{dP_n^m}{d\theta} P_s^p - m P_n^m \frac{dP_s^p}{d\theta} \right\} e^{i(-k+m+p)\phi} \, d\theta \, d\phi$$

following Bullard and Gellman (1954, p.224). All the indices in these expressions are positive definite and the associated Legendre functions have been defined according to Hobson's normalisation. The selection rules, Section 3.3, follow from the behaviour of these integrals for various combinations of the indices. From the azimuthal parts of the integrals,

$$\int_0^{2\pi} e^{i(-k+m+p)\phi} \, d\phi = 0,$$

unless $-k+m+p = 0$, when this integral has the value 2π : this is the first selection rule. The other rules follow from a more detailed examination of the properties of the associated Legendre functions (e.g., Infeld and Hull, 1951, pp.52-54).

As noted in Section 3.3, the properties of K L are used to establish the fact that only a finite number of interactions contribute to a particular induced field. Because this is an important constraint on the allowed interactions producing a self-inducing magnetic field, it is not surprising to find K L discussed at some length in dynamo theory (e.g., Bullard and Gellman, 1954). An appendix by Scott in Gibson and Roberts (1969) contains a discussion of the properties of K L and a listing of the values of two closely related integrals G^+, E^+ for various indices. These are defined by

$$G^+(m,n,p,s,\ell) = \int_{-1}^{1} P_\ell^k P_n^m P_s^p \, d\mu$$

$$E^+(m,n,p,s,\ell) = \int_{-1}^{1} P_\ell^k \left\{ p P_s^p \frac{dP_n^m}{d\mu} - m P_n^m \frac{dP_s^p}{d\mu} \right\} d\mu$$

where the associated Legendre functions are used in the Ferrer form as used by Scott. The relations between K, L (Hobson's form) and G^+, E^+ (Ferrer form) are found to be

$$\left.\begin{matrix} K_{\ell,n,s}^{-k,m,p} \\ \\ L_{\ell,n,s}^{-k,m,p} \end{matrix}\right\} = 2\pi (-1)^k \frac{(\ell-k)!}{(\ell+k)!} \left\{\begin{matrix} G^+(m,n,p,s,\ell) \\ \\ E^+(m,n,p,s,\ell) \end{matrix}\right.$$

for $-k = m+p \geq 0$, and

$$\left.\begin{matrix} K_{\ell,n,s}^{-k,-m,p} \\ \\ L_{\ell,n,s}^{-k,-m,p} \end{matrix}\right\} = 2\pi (-1)^k \frac{(\ell-k)!}{(\ell+k)!} \left\{\begin{matrix} G^+(p,s,m-p,n,\ell) \\ \\ -E^+(p,s,m-p,n,\ell) \end{matrix}\right.$$

for $-k = m-p \geq 0$.

Equations (22), defining the source functions for the induced fields, can be written

$$f(r) = i\omega\mu_0\sigma \cdot \tfrac{1}{2} \left\{ i r U T a_1 \right\}$$

$$g(r) = i\omega\mu_0\sigma \cdot \tfrac{1}{2} \left\{ (rUT)' b_1 + V T b_2 \right\}$$

where U,T are the radial functions for a displacement of harmonic n,m and for a main magnetic field T_2^0 respectively. The constants a_1, b_1

and b_2 are related to G^+, E^+ by

$$a_1 = c\, E^+$$
$$b_1 = c \cdot \tfrac{1}{2}\left[6 - n(n+1) + \ell(\ell+1)\right] G^+$$
$$b_2 = c\,\ell(\ell+1) \cdot \tfrac{1}{2}\left[n(n+1) - \ell(\ell+1) + 6\right] G^+ ,$$

where

$$c = \frac{(2\ell+1)}{\ell(\ell+1)} \, \frac{(\ell-k)!}{(\ell+k)!} .$$

In order to evaluate the magnetoelastic interactions, the values of a_1, b_1 and b_2 are determined for the various indices of G^+, E^+ This involves knowing G^+ and E^+ and these integrals were programmed using the formulae given by Scott (Gibson and Roberts, 1969). Table 9 shows the values obtained for G^+, E^+, a_1, b_1 and b_2 for indices up to (4,4) for (n,m); only combinations which satisfy the selection rules are included. The constants a_1, b_1 and b_2 for negative m are found from

$$G^+(p,s,m-p,n,\ell) = G^+(m,n,p,s,\ell)$$
$$E^+(p,s,m-p,n,\ell) = -E^+(m,n,p,s,\ell) .$$

It can be seen from the table that, although G^+,E^+ can become large, the constants are generally all of order unity. This indicates there are no resonance interactions which might be expected to produce a large perturbation magnetic field. A search for all indices up to (10,10) for (n,m) and (s,p) also did not reveal a resonance value. It must be concluded that the geometry of the fields does not produce any particularly interesting effects.

Table 9

Gaunt and Elsasser Integrals

m	n	p	s	l	G^+	E^+	a_1	b_1	b_2
0	0	0	2	2	0.40			2.00	0.00
0	1	0	2	1	0.27			1.20	2.40
0	1	0	2	3	0.17			0.80	-2.40
0	2	0	2	2	0.11			0.29	1.71
0	2	0	2	4	0.11			0.51	-4.11
0	3	0	2	1	0.17			-0.51	4.11
0	3	0	2	3	0.08			0.13	1.60
0	3	0	2	5	0.09			0.38	-5.71
0	4	0	2	2	0.11			-0.38	5.71
0	4	0	2	4	0.06			0.08	1.56
0	4	0	2	6	0.07			0.30	-7.27
1	1	0	2	1	-0.27			-0.60	-1.20
1	1	0	2	2		-2.40	-0.33		
1	1	0	2	3	0.69			0.27	-0.80
1	2	0	2	1		-2.40	-1.80		
1	2	0	2	2	0.34			0.14	0.86
1	2	0	2	3		-4.11	-0.20		
1	2	0	2	4	1.14			0.26	-2.06
1	3	0	2	1	0.69			-1.03	8.23
1	3	0	2	2		-4.11	-0.57		
1	3	0	2	3	0.69			0.10	1.20
1	3	0	2	4		-5.71	-0.13		
1	3	0	2	5	1.56			0.23	-3.43
1	4	0	2	2	1.14			-0.63	9.52
1	4	0	2	3		-5.71	-0.28		
1	4	0	2	4	0.98			0.07	1.32
1	4	0	2	5		-7.27	-0.09		
1	4	0	2	6	1.96			0.20	-4.85
2	2	0	2	2	-2.74			-0.29	-1.71
2	2	0	2	3		-41.4	-0.20		
2	2	0	2	4	6.86			0.09	-0.69
2	3	0	2	2		-41.14	-1.43		
2	3	0	2	4		-137.14	-0.17		
2	3	0	2	5	21.82			0.11	-1.71
2	4	0	2	2	6.86			-0.95	14.29
2	4	0	2	3		-137.14	0.67		
2	4	0	2	4	8.31			0.03	0.62
2	4	0	2	5		-305.45	-0.13		
2	4	0	2	6	46.99			0.12	-2.91
3	3	0	2	3	-68.57			-0.17	-2.00
3	3	0	2	4		-1440.00	-0.13		
3	3	0	2	5	174.55			0.04	0.57
3	4	0	2	3		-1440.00	-1.17		
3	4	0	2	4	-101.82			-0.03	-0.55
3	4	0	2	5		-7330.91	-0.13		
3	4	0	2	6	845.87			0.06	-1.45
4	4	0	2	4	-3258.18			-0.11	-2.18
4	4	0	2	5		-87970.	0.09		
4	4	0	2	6	8458.74			0.02	-0.48

APPENDIX B

COMPUTING FREE OSCILLATIONS

Numerical integration of the equations of motion of the free oscillations of a model Earth generally follows the treatment of Alterman et al. (1959). The basic equations, formulated by Love (1911, Ch.VII), are first written in spherical polar coordinates and then transformed into linear first-order differential equations. With routine computing facilities these equations can be integrated simultaneously using a step-by-step procedure, such as a Runge-Kutta algorithm. The method is well-known and details of the computations have been presented several times (e.g., Bolt and Dorman, 1961; Alsop, 1963).

The purpose of this Appendix is to present details of the Earth model used in the computations of Section 5, and to examine the condition of regularity at the origin for the integration of the equations.

B.1 Starting Conditions

Referring to Section 2.3 the equations of elasticity (2) can be written in the form of six coupled linear equations (Alterman et al., 1959)

$$\left.\begin{aligned}
y_1' &= -\frac{2\lambda\beta}{r}y_1 + \beta y_2 + \frac{n(n+1)\lambda\beta}{r}y_3 \\
y_2' &= \left[-\omega^2\rho_0 - \frac{4\rho_0 g_0}{r} + \frac{2\delta}{r^2}\right]y_1 - \frac{4\mu\beta}{r}y_2 + n(n+1)\left[\frac{\rho_0 g_0}{r} - \frac{\delta}{r^2}\right]y_3 \\
&\qquad + \frac{n(n+1)}{r}y_4 - \rho_0 y_6 \\
y_3' &= -\frac{1}{r}y_1 + \frac{1}{r}y_3 + \frac{1}{\mu}y_4 \\
y_4' &= \left[\frac{\rho_0 g_0}{r} - \frac{\delta}{r^2}\right]y_1 - \frac{\lambda\beta}{r}y_2 + \left[-\omega^2\rho_0 + \frac{\epsilon}{r^2}\right]y_3 - \frac{3}{r}y_4 - \frac{\rho_0}{r}y_5 \\
y_5' &= 3\gamma y_1 + y_6 \\
y_6' &= -\frac{3\gamma n(n+1)}{r}y_3 + \frac{n(n+1)}{r^2}y_5 - \frac{2}{r}y_6
\end{aligned}\right\} \quad \text{(B1)}$$

where ω is the angular frequency, and

$$\beta = (\lambda + 2\mu)^{-1}, \quad \delta = 2\mu(3\lambda + 2\mu)\beta,$$

$$\epsilon = 4\mu n(n+1)(\lambda+\mu)\beta - 2\mu, \quad \gamma = \tfrac{4}{3}\pi G\rho_0.$$

The six variables $y_1 \ldots y_6$ have the following interpretations;

$y_1 = U$	radial displacement
$y_2 = \lambda X + 2\mu U'$	change in normal stress
$y_3 = V$	transverse displacement
$y_4 = \frac{\mu}{r}[rV' - V + U]$	transverse shear stress
$y_5 = P$	decrease in gravitational potential
$y_6 = P' - 4\pi G\rho_0 U$	change in gravitational flux density.

The displacements U, V are identical with the radial functions u_n^m, v_n^m of equation (18), as the equations (B1) refer to a spheroidal displacement of a particular degree and order. The cubical dilatation Δ and the perturbation in the gravitational potential V_1 are each written in spherical harmonic form

$$\Delta = X(r)\, P_n^m e^{im\phi}, \quad X(r) = U' + \frac{2U}{r} - \frac{n(n+1)V}{r},$$

$$V_1 = P(r)\, P_n^m e^{im\phi},$$

thus defining the radial functions X,P.

Two properties possessed by (B1) can readily be identified: the first is that no derivatives of the elastic moduli occur, the second is that many of the coefficients on the right hand side are singular at the origin, r=0. The elastic moduli are not known accurately within the Earth and it is to avoid errors involved in taking their derivatives that the form (B1) is preferred.

The singularity at the origin is avoided by several devices. The simplest technique is to begin the integration away from the origin and to establish starting values of the variables y_i by repeated integration. Three of the variables y_1, y_3, and y_5 are zero at the origin and for a solid Earth it would require only three integrations to obtain the values of y_2, y_4 and y_6 at the starting depth. In practice an error is introduced by this method for those oscillations with displacements near the origin (Fig.3). A depth r_0 is sought, generally by trial and error (Bolt and Dorman, 1961, p.2963), below which y_1, y_3 and y_5 can be considered zero to the order of accuracy of the integration method. Such a starting set can be represented by

$$y_i(r_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{for } i = 2,4,6\ ,$$

and $y_i(r_0) = 0$ for i= 1,3,5 . Clearly r_0 cannot be chosen as zero, for the derivatives of the y's are then singular.

An alternative approach is to choose an r_0 below which the Earth may be considered to be homogeneous (e.g. Wiggins, 1968). Within the sphere $r = r_0$, the equations (B1) are solved for a homogeneous medium and the solutions , obtained analytically, are Bessel functions (Love, 1911, Ch.VII). The integration is then begun at $r = r_0$ with the values for y_i determined by the solution for $r < r_0$. These solutions are discussed by Takeuchi and Saito (1971) and are here reproduced in the same notation as in (B1). One independent solution is given here by

$$\left.\begin{aligned} y_1 &= n r^{n-1} \\ y_2 &= 2\mu n(n-1) r^{n-2} \\ y_3 &= r^{n-1} \\ y_4 &= 2\mu(n-1) r^{n-2} \\ y_5 &= [n\gamma - \omega^2] r^n \\ y_6 &= n[(n-3)\gamma - \omega^2] r^{n-1} \end{aligned}\right\} \quad \text{(B2)}$$

noting that Takeuchi and Saito define y_6 differently from above.

Two other independent solutions are given by

$$\left.\begin{aligned} y_1 &= -\frac{r^{n+1}}{2n+3}\left\{\tfrac{1}{2} n h \psi_n(x) + f\phi_{n+1}(x)\right\} \\ y_2 &= -(\lambda+2\mu) r^n f \phi_n(x) + \frac{\mu r^n}{2n+3}\left\{-n(n+1) h \psi_n(x) \right. \\ &\qquad \left. + 2[2f + n(n+1)]\phi_n(x)\right\} \\ y_3 &= -\frac{r^{n+1}}{2n+3}\left\{\tfrac{1}{2} h \psi_n(x) - \phi_{n+1}(x)\right\} \\ y_4 &= \mu r^n\left\{\phi_n(x) - \frac{1}{2n+3}[(n-1) h \psi_n(x) + 2(f+1)\phi_{n+1}(x)\right\} \\ y_5 &= r^{n+2}\left\{\frac{a^2 f - n(n+1) b^2}{r^2} - \frac{3\gamma f}{2(2n+3)}\psi_n(x)\right\} \\ y_6 &= \frac{n y_5}{r} + \frac{3n\gamma h}{2(2n+3)}\psi_n(x) \end{aligned}\right\} \quad \text{(B3)}$$

where
$$\phi_n(x) = \frac{(2n+1)!!}{x^n} j_n(x)$$
$$\psi_n(x) = 2(2n+3)[1-\phi_n(x)]/x^2 \quad , x = kr$$

and
$$a^2 = (\lambda+2\mu)/\rho_0 \quad , \quad h = -\omega^2/\gamma - (n+1)$$
$$b^2 = \mu/\rho_0 \quad , \quad k^2 = \frac{1}{a^2}\left[\omega^2 + 4\gamma - n(n+1)\gamma^2/\omega^2\right]$$

as given by Takeuchi and Saito.

In the present work the singularity is treated by a power-series expansion about the origin for each of the six variables. The series are given by
$$y_i = r^\alpha \sum_{\nu=0}^{\infty} A_{i\nu}\, r^\nu \tag{B4}$$
for i = 1, 6 where α, ν are integer variables and $A_{i\nu}$ are constant coefficients to be determined. Gravity has the expansion
$$g_0(r) = \gamma r + \gamma_1 r^2 + \dots \quad , \quad \gamma = \frac{4}{3}\pi G \rho_0(0) \; ,$$

where γ is the value of gravity within an arbitrarily small uniform sphere of density $\rho_0(0)$ near r = 0.

Because of the complexity of the coefficients in (B1), the derivations of the equations resulting from a substitution of (B4) are quite tedious. When lowest powers of r are compared for each equation, three independent values for α are allowed; $\alpha = n$, $\alpha = n-1$ and $\alpha = n-2$. For each α there are six solutions to the simultaneous equations and all eighteen equations can be written by the form
$$y_i^{(\alpha)} = A_{i,k}^{(\alpha)} r^{k+\alpha} + A_{i,k+2}^{(\alpha)} r^{k+\alpha-2} + \dots$$

The index k determines the r-dependence of the first term in the series

$$\begin{array}{llll} k=0 & \text{for } i=2,4 \ (\alpha=n \text{ or } n-2), & i=6 & (\alpha=n-1), \\ k=1 & i=1,3,6 & i=2,4,5 & \\ k=2 & i=5 & i=1,3 & \end{array}$$

It is found that for the case $\alpha=n-1$ only one constant is required for the solution; for $\alpha=n, n-2$ two constants each are required giving a total of five independent constants. However the general solution near the origin must be taken as the linear combination

$$y_i = c_\alpha A^{(\alpha)}_{ik} \quad ,$$

and then the constants are reduced to three. The solution with these constants is

$$\left.\begin{aligned} y_1 &= A r^{n-1} + A' r^{n+1} + \dots \\ y_2 &= B r^{n-2} + B' r^{n} + \dots \\ y_3 &= C r^{n-1} + C' r^{n+1} + \dots \\ y_4 &= D r^{n-2} + D' r^{n} + \dots \\ y_5 &= E r^{n} + E' r^{n+2} + \dots \\ y_6 &= F r^{n-1} + F' r^{n+} + \dots \end{aligned}\right\} \qquad (B5)$$

where two constants are given by

$$\left.\begin{aligned} B &= 2\mu(n-1)A \quad , \quad C = 1/n\, A \\ D &= 2\mu(n-1)A/n \quad , \quad E = 3\gamma A/n + F/n . \end{aligned}\right\} \qquad (B6)$$

The third constant is contained in the system

$$\left.\begin{aligned} C' &= -\frac{c_2}{c_1} D' + \frac{\rho_0}{c_1}\left[F + (\omega^2 + (3-n)\gamma)A\right] \\ \beta B' &= b_1 C' + b_2 D' \\ A' &= -nC' + D'/\mu \\ E' &= 3\gamma(n+3)A'/a_1 - 3\gamma n(n+1)C'/a_1 \\ F' &= (n+2)E' - 3\gamma A' \end{aligned}\right\} \qquad (B7)$$

where
$$a_1 = 2(2n+3)$$
$$b_1 = -n[(n+1)+(n+3)\lambda\beta]; \quad b_2 = \frac{1}{\mu}(n+1+2\lambda\beta)$$
$$c_1 = 2\lambda n^2(n+2) + 2\mu n(n^2+2n-1); \quad c_2 = -n(n+5) - n(n+3)\frac{\lambda}{\mu} .$$

The set (B7) can be expressed in terms of only one constant by allowing the relation

$$F = -(\omega^2 + (3-n)\gamma)A \tag{B8}$$

and with A = n, the constants in (B6) are identical with those in (B2).

To show the equivalence between (B3) and (B5) with (B7), the set (B3) is approximated by letting $r \to 0$, $(x \to 0)$ whence

$$\left.\begin{aligned}
y_1 &= -\left(\frac{nh}{2} + f\right)\frac{1}{2n+3} r^{n+1} + \dots \\
y_2 &= \left\{-(\lambda+2\mu)(2n+3)f - \mu(n-1)nh + 2\mu[2f + n(n+1)]\right\}\frac{r^n}{2n+3} + \dots \\
y_3 &= \left(1-\frac{h}{2}\right)\frac{1}{2n+3} r^{n+1} + \dots \\
y_4 &= -\mu(nh+h+1)\frac{1}{2n+3} r^n + \dots \\
y_5 &= \frac{1}{\rho_0}[(\lambda+2\mu)h + (n+1)(\lambda+\mu)]r^n - \frac{3\gamma f}{2(2n+3)} r^{n+2} + \dots \\
y_6 &= \frac{n}{\rho_0}[(\lambda+2\mu)h + (n+1)(\lambda+\mu)]r^{n-1} - \frac{3\gamma n(n+1)}{2(2n+3)} r^{n+1} + \dots
\end{aligned}\right\} \text{(B9)}$$

It is then found that (B5) with (B6) to (B8) is identical to (B2) and (B9), as they should be in the limit $r \to 0$.

The system (B5) has y_i and y_i' zero for $n \geq 3$ and is thus unsuitable for a Runge-Kutta integration procedure at the origin. Following the method of Smylie and Mansinha (1971, p. 344) a change of variable is now made

$$z_i = y_i\, r^{-(n-2)} \qquad , \; i = 1, 6 \tag{B10}$$

giving the expansions near the origin as

$$\left.\begin{aligned}
z_1 &= Ar + A'r^3 + \cdots & z_1 &= 0 \;,\; z_1' = A \\
z_2 &= B + B'r^2 + \cdots & z_2 &= B \;,\; z_2' = 0 \\
z_3 &= Cr + C'r^3 + \cdots & z_3 &= 0 \;,\; z_3' = C \\
z_4 &= D + D'r^2 + \cdots & z_4 &= D \;,\; z_4' = 0 \\
z_5 &= Er^2 + E'r^4 + \cdots & z_5 &= 0 \;,\; z_5' = 0 \\
z_6 &= Fr + F'r^3 + \cdots & z_6 &= 0 \;,\; z_6' = F \;,
\end{aligned}\right\} \tag{B11}$$

valid at $r = 0$. The system to be integrated is then

$$z_i' = C_{ij}\, z_j - (n-2)\frac{z_i}{r} \qquad , \quad i = 1, 6$$

where C_{ij} is the matrix of coefficients for the right hand side of (B1). Two integrations throughout the Earth are required to obtain the constants A, F in (B11); the most convenient choice is (1,0) and (0,1) respectively. The system (B12) can be integrated up to some radius r when the reverse transformation of (B10) restores the system to (B1).

To isolate the third constant, a transformation of the form

$$z_i = y_i\, r^{-(n-1)} \qquad , \quad i = 1, 6$$

is required. Then, with A–F zero,

$$\begin{aligned}
z_1 &= A'r^2 + \cdots & z_1 &= 0 \;,\; z_1' = 0 \\
z_2 &= B'r + \cdots & z_2 &= 0 \;,\; z_2' = B' \\
z_3 &= C'r^2 + \cdots & z_3 &= 0 \;,\; z_3' = 0 \\
z_4 &= D'r + \cdots & z_4 &= 0 \;,\; z_4' = D' \\
z_5 &= E'r^3 + \cdots & z_5 &= 0 \;,\; z_5' = 0 \\
z_6 &= F'r^2 + \cdots & z_6 &= 0 \;,\; z_6' = 0 \;,
\end{aligned}$$

valid at $r = 0$.

The equations of motion are in this case

$$z_i' = C_{ij} z_j - (n-1) z_i / r \quad , i = 1,6$$

and one integration is sufficient to find the constant.

This method is then a rigorous way of starting the Runge-Kutta integration for the system (B1). In practise it has been found appropriate for low-degree spheroidal oscillations, but is inefficient for other oscillations because of error accumulation in the integration.

The radial oscillations are simple to integrate because only three equations are involved

$$y_1' = -\frac{2\lambda\beta}{r} y_1 + \beta y_2$$

$$y_2' = \left[-\omega^2 \rho_0 + \frac{4\rho_0 g_0}{r} - \frac{2\delta}{r^2}\right] y_1 - \frac{4\mu\beta}{r} y_2$$

$$y_5' = 3\gamma y_1 \quad .$$

The starting solutions are

$$y_1' = Ar + A' r^3 + ..$$

$$y_2' = B + B' r^2 + ..$$

$$y_5' = E + E' r^2 + .. \quad ,$$

with $B = (3\lambda + 2\mu) A$.

For the variables y_1, y_2 and y_5 the starting set is (0,1,0) at r = 0, and at the eigenfrequency, where y_2 changes sign, this set becomes

$$(0, \ 1/y_1(d) \ , \ -y_5(d)/y_1(d) \,) ,$$

if d denotes surface values.

B.2 Normalisation

To stabilise the system (B1) and prevent overflow in the computations for large n, the coefficients and the variables can be arranged to be of order unity by a simple scaling operation (Wiggins, personal communication). The appropriate changes are

$$y_1 \times 10^{-9},\ y_2 \times 10^{-12},\ y_3 \times 10^{-9},\ y_4 \times 10^{-12},\ y_5 \times 10^{-12},\ y_6 \times 10^{-3},$$

for the variables, and

$$\lambda \times 10^{-12},\ \mu \times 10^{-12},\ r \times 10^{-9},\ g_0 \times 10^{-3},\ G \times 10^{6},\ \omega \times 10^{3},$$

for the Earth parameters, the original values being in c.g.s. units.

B.3 The Numerical Earth Model

An Earth model was supplied by Jordan and Anderson (1972). A free oscillation routine was written to obtain the amplitudes of the oscillations in the core rather than rely on published amplitudes or a packaged program. The Jordan-Anderson model (denoted here by JAB1) was used for three reasons. It

(a) has a solid inner core,

(b) was supplied with a complete list of properties and eigenperiods useful for checking the integrations, and

(c) has a good least-squares fit to observed travel-time data and observed eigenperiods.

The properties of the model are shown in Table 10. Linear interpolation was used to interpolate the model as the results were found similar to those obtained by cubic spline interpolation. A fourth-order Runge-Kutta routine was used, supplied by the Computing Centre at the University of British Columbia and incorporating an automatic error control on the step size, after Christiansen (1970).

Table 10

Parameters for Earth Model JAB1

Radius km	Depth km	Density gm cm^{-3}	Gravity cm sec^{-2}	Bulk Modulus $x10^{12}$ dyne cm	Rigidity $x10^{12}$ dyne cm
0	6371	12.58	0.0	12.694	1.540
100	6271	12.57	52.0	12.689	1.541
200	6171	12.56	78.0	12.674	1.539
300	6071	12.53	110.0	12.648	1.535
400	5971	12.52	144.0	12.643	1.532
500	5871	12.51	178.0	12.642	1.531
600	5771	12.51	212.0	12.639	1.528
700	5671	12.50	247.0	12.633	1.523
800	5571	12.50	281.0	12.632	1.517
900	5471	12.49	316.0	12.623	1.510
1000	5371	12.46	350.0	12.610	1.499
1100	5271	12.39	385.0	12.581	1.485
1215	5156	12.28	423.0	12.513	1.467
1215	5156	12.11	423.0	12.460	0.0
1300	5071	12.08	450.0	12.444	0.0
1400	4971	12.04	482.0	12.411	0.0
1500	4871	11.99	514.0	12.334	0.0
1600	4771	11.93	546.0	12.219	0.0
1700	4671	11.87	578.0	12.042	0.0
1800	4571	11.80	609.0	11.805	0.0
1900	4471	11.72	640.0	11.561	0.0
2000	4371	11.64	671.0	11.307	0.0
2100	4271	11.56	701.0	11.048	0.0
2200	4171	11.47	731.0	10.785	0.0
2300	4071	11.39	760.0	10.542	0.0
2400	3971	11.30	790.0	10.309	0.0
2500	3871	11.21	818.0	10.032	0.0
2600	3771	11.11	846.0	9.718	0.0
2700	3671	11.00	874.0	9.388	0.0
2800	3571	10.88	901.0	9.023	0.0
2900	3471	10.76	928.0	8.628	0.0
3000	3371	10.62	954.0	8.209	0.0
3100	3271	10.48	979.0	7.797	0.0
3200	3171	10.33	1003.0	7.400	0.0
3300	3071	10.19	1026.0	7.026	0.0
3400	2971	10.04	1049.0	6.676	0.0
3485	2886	9.90	1068.0	6.373	0.0
3485	2886	5.58	1068.0	4.523	2.948
3510	2861	5.56	1064.0	4.510	2.934
3550	2821	5.54	1059.0	4.498	2.916
3625	2746	5.50	1049.0	4.471	2.871
3700	2671	5.46	1041.0	4.412	2.822
3775	2596	5.42	1034.0	4.309	2.775
3850	2521	5.38	1027.0	4.194	2.730
3925	2446	5.34	1021.0	4.079	2.686
4000	2371	5.30	1016.0	3.966	2.642

Table 10 (continued)

4075	2296	5.26	1011.0	3.863	2.601
4150	2071	5.22	1008.0	3.765	2.555
4225	2146	5.19	1001.0	3.583	2.465
4375	1996	5.11	999.0	3.487	2.422
4450	1921	5.07	997.0	3.391	2.380
4525	1846	5.04	996.0	3.309	2.336
4600	1771	5.00	994.0	3.226	2.293
4675	1696	4.96	994.0	3.136	2.253
4750	1621	4.92	993.0	3.058	2.214
4825	1546	4.89	993.0	2.977	2.174
4900	1471	4.85	993.0	2.888	2.136
4975	1396	4.81	993.0	2.819	2.089
5050	1321	4.77	993.0	2.740	2.044
5125	1246	4.72	994.0	2.651	1.998
5200	1171	4.68	994.0	2.549	1.956
5275	1096	4.64	995.0	2.435	1.920
5350	1021	4.59	996.0	2.346	1.871
5425	946	4.55	997.0	2.253	1.824
5500	871	4.50	998.0	2.154	1.778
5550	821	4.47	999.0	2.087	1.746
5600	771	4.44	1000.0	2.020	1.712
5650	721	4.41	1000.0	1.957	1.677
5700	671	4.38	1001.0	1.901	1.642
5700	671	4.05	1001.0	1.911	1.098
5725	646	4.02	1001.0	1.846	1.092
5750	621	4.00	1000.0	1.777	1.088
5775	596	3.97	1000.0	1.706	1.086
5800	571	3.95	1000.0	1.632	1.084
5825	546	3.92	999.0	1.558	1.084
5850	521	3.90	999.0	1.482	1.084
5875	496	3.87	999.0	1.408	1.084
5900	471	3.85	998.0	1.338	1.083
5925	446	3.82	998.0	1.272	1.079
5951	420	3.80	997.0	1.210	1.074
5951	420	3.58	997.0	1.186	0.780
5975	396	3.57	997.0	1.151	0.775
6000	371	3.54	996.0	1.118	0.768
6050	321	3.49	994.0	1.076	0.739
6100	271	3.44	992.0	1.067	0.695
6150	221	3.39	990.0	1.066	0.648
6175	196	3.37	989.0	1.051	0.633
6200	171	3.35	988.0	1.009	0.633
6225	146	3.34	987.0	0.962	0.637
6250	121	3.33	986.0	0.889	0.656
6271	100	3.32	986.0	0.810	0.681
6271	100	3.32	986.0	0.810	0.681
6290	81	3.32	985.0	0.730	0.708
6310	61	3.32	984.0	0.643	0.738
6330	41	3.31	984.0	0.568	0.762
6350	21	3.30	983.0	0.524	0.769
6350	21	2.79	983.0	0.427	0.322
6360	11	2.79	982.0	0.427	0.322
6371	0	2.79	981.0	0.427	0.322