STRESSES IN A CIRCULAR DISC CONTAINING A RADIAL CRACK

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Abstract

An analytic solution to a particular boundary value problem in linear elasticity is presented. The problem considered is that of a circular disc, in plane strain or generalized plane stress, which is loaded quite generally at the perimeter by a set of self equilibrating forces. The disc has a radial crack extending completely from the centre to the perimeter.

Three particular load cases are considered: the disc is loaded by uniform radial tension; the disc is pulled by two diametrical concentrated loads with line of action perpendicular to the crack; the disc is pulled by two concentrated forces acting at the mouth of the crack. In each case the stress intensity factor is evaluated and contours of the isochromatics are presented in dimensionless form. The results agree with those derived by other workers but the method presented here is more general, more efficient, and extends the previous results for finite bodies containing cracks.

The three load cases have also been applied to the disc when the tip of the crack has been removed by the presence of a concentric hole. New results are presented for the stress concentration factors for a full range of geometric parameters.

Finally it is shown how the second and third of the load cases may be used for controlled laboratory tests in the study of fracture and fracture toughness of brittle materials.

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Chapter 1

Introduction

In this thesis a method is presented which may be used to determine the stresses in a circular disc with a radial crack with or without a hole at the tip of the crack. The crack extends from the centre of the disc to the edge and is activated by a general set of in-plane self equilibrating surface tractions acting at the perimeter of the disc. The method presented contributes to the small number of analytical solutions available for elastic plates containing cracks. The solution for the second problem, i.e., a disc with a hole at the crack tip, is of much interest in engineering practice.

1.1 Preliminary Remarks

Fracture mechanics is a relative new branch of material science which seeks to quantify the critical combination of stress and crack size for crack extension. It is being increasingly applied to assess the safety of engineering structures containing crack-like defects. Generally, fracture mechanics is applied to estimate the geometry independent resistance to crack extension.

Several circumstances may be cited in order to justify the application of fracture mechanics theory to engineering structures. The most important circumstance is probably that all engineering structures contain cracks or crack-like flaws at some scale of examination.

In recent years, fracture mechanics has received world-wide attention. The concepts and principles of this relatively new discipline, when properly applied, provide a

sound basis for making decisions in engineering design. It offers a systematic approach to failure prevention in aircrafts, ships, rocket casings, pressure vessels, nuclear structures, bridges, pipelines, etc. In view of the rising cost and shortage of raw materials, it becomes increasingly more pertinent to optimize the use of the material with performance. To this end, fracture mechanics can greatly assist in the fabrication of acceptable materials, establishment of proof tests and design codes and even in making managerial decisions. It is becoming a technology that is helpful to those concerned with the developement of heavy industries.

The strength of a structure which has a crack or crack-like flaw can be dangerously over-estimated. The stresses at a crack tip are very high (singularity), and the nature of the singular stress is the critical subject for fracture of brittle materials. The singularity is described by a parameter known as the stress intensity factor K_I . The fracture toughness of a material is specified by a critical value of K_I , denoted by K_{IC} .

The stress intensity factor K_I provides a one parameter characterisation of the stress field ahead of a sharp crack and the energy available to propagate the crack. Failure occurs when it attains the critical value K_{IC} called the fracture toughness.

During the past three decades, the stress intensity factor has been used as a parameter to characterize the loading on cracks in practical fracture tests. Some methods of determining the stress intensity factor are presented here:

1. Effects of geometry and loading on the stress intensity factor

The stress intensity factor K_N ($N = 1, 2, 3$ corresponding to the opening, sliding and tearing modes respectively Fig.(1.1)) can be written

$$K_N = Y \sigma \sqrt{a} \quad (1.1)$$

Figure 1.1: Three modes of the fracture

where σ is a stress, a is a crack length and Y is a geometrical factor which accounts for such things as proximity effects of boundaries or other cracks, orientations of the crack, shape of the crack and the restraints on the structure containing the crack. It is with the determination of this geometrical factor that crack analysis methods are concerned. For example, for an isolated crack in a sheet subjected to a remote uniform stress σ perpendicular to the crack line, the stress intensity factor K_I is given by eqn(1.1) with $Y = \sqrt{\pi}$.

2. Bueckner's principle and superposition

The principle presented by Bueckner is very important in the determination of stress intensity factor. It allows the problem of determining the stress intensity factor of a stress-free crack in a loaded body to be changed to that of the same crack subjected to pressures on its surfaces in the same body; the boundary conditions for the second problem are that all applied traction and displacements are zero. The crack surface

pressures are simply determined as the stresses over the crack site in the uncracked body subjected to the originally applied boundary traction and displacements. This principle of using the distribution of stress over the crack site in the absence of the crack forms the basis of many methods, for example, of the Green's function and the weight function methods and also those based on superposition of polynomial distributions of stress.

Bueckner's principle is illustrated schematically in Fig.1.2. The linearly elastic region inside the boundary O contains a stress-free crack along the line AB . The boundary O_1 subjected to general tractions T and the part (or parts) of the boundary O_2 subjected to general displacements V ; the region inside O is subjected to a body force distribution F . Under this system of loads and displacements the stress intensity factor is given by K_b . In Fig.1.2(c) the body is uncracked with the same tractions T , body forces F and displacements V applied on the boundary; σ_{ij} is the resultant stress distribution along the site of the crack AB . In Fig.1.2(d) the body contains, along AB , a crack with pressure distributions s_i applied to its surfaces; the boundary conditions on the body are $T=V=0$ with $F=0$. Under this pressure distribution the stress intensity factor is given by K_d . Bueckner's principle states that if $s_i = \sigma_{ij}n_j$ then the stress intensity factor for the crack in Fig.1.2(b) and Fig.1.2(d) are identical, i.e.

$$K_d = K_b \quad \text{at both tips}$$

3. Weight functions and Green's functions

The determination of stress intensity factors using either Green's functions or weight functions usually involves the use of Bueckner's principle and always involves evaluating an integral of the form

Elastic solid subjected to arbitrary tractions and displacements. ($K_b = K_d$). (a) Cracked region before loading. (b) Cracked region subjected boundary tractions and displacements. (c) Cracked region subjected to T and V with crack removed. (d) Cracked region with $T = V = 0$ and crack surfaces subjected to pressure distribution $p_{ij} = \sigma_{ij}$.

Figure 1.2: Bueckner's Principle

$$K_N = \int_0^{x'} \sigma_{ij} H(x', x) dx, \quad N = 1, 2, 3 \quad (1.2)$$

where x' is the crack length and x represents a point on the crack. The stress $\sigma_{ij}(x)$ is the pre-cracked stress distribution (or the pressure distribution on the crack surfaces) and $H(x', x)$ is a function which depends on the geometry. This function $H(x', x)$ is independent of the applied tractions, but it will depend on any imposed displacement conditions. Consideration will be limited to the opening mode K_I as this is in practice the more important.

The function $H(x', x)$ may be determined as the stress intensity factor resulting from a unit force on the crack surface. In this case it is proportional to the Green's function $g(x', x)$ for a crack of length x' with a point force located at a coordinate x on the crack surface. The integral in eqn.(1.2) then takes the form, with $N = 1$ and $\sigma_{ij}(x)$ replaced by $\sigma(x)$

$$K_I = \sqrt{\frac{2}{x'\pi}} \int_0^{x_1} \sigma(x) g(x', x) dx \quad (1.3)$$

Alternatively the function $H(x', x)$ may be determined as a weight function $G(x', x)$ defined as

$$G(x', x) = \frac{8\mu}{1 + \kappa} \frac{1}{K_I^*} \frac{\partial v^*}{\partial x'}(x', x)$$

where $\kappa = 3 - 4\gamma$ for plane strain, $\kappa = \frac{3-\gamma}{1+\gamma}$ for plane stress and μ is the shear modulus and γ is Poisson's ratio. K_I^* is a known stress intensity factor solution for the particular geometric configuration being considered and $v^*(x', x)$ is the corresponding displacement of the crack surfaces. Any loading condition may be used to determine K_I^* and $v^*(x', x)$. Equation (1.3) thus takes the following form

$$K_I = \int_0^{x'} \sigma(x) G(x', x) dx$$

4. Simple approximate methods for cracks at stress concentration

It has been shown that, by using certain approximations to the Green's function, expressions for stress intensity factors of cracks at stress concentration can be determined. Consider, for example, a crack of length a_0 at the tip of a notch of root radius ρ . The notch is in a body which is subject to a uniform tensile stress σ acting in a direction perpendicular to the crack line. The precracked stress distribution is $\sigma(x)$, where the distance x is measured along the line of the crack from the notch tip. We have, for the maximum stress method

$$K_I = 1.12\sigma(0)\sqrt{\pi a_0}$$

for the tip stress method

$$K_I = 1.12\sigma(a_0)\sqrt{\pi a_0}$$

and for the mean stress method

$$K_I = 1.12\sigma_{mean}\sqrt{\pi a_0}$$

where

$$\sigma_{mean} = \frac{1}{a_0} \int_0^{a_0} \sigma(x) dx$$

The stress σ_{mean} is the mean stress over the crack site in the absence of the crack.

The maximum stress method has the advantage that only the stress concentration factor of the notch is required. Whereas, the tip stress and the mean stress methods, although more accurate, require that the entire stress distribution over the crack site be known.

5. Superposition of polynomial distributions

If the weight function or the Green's function technique are used to determine a stress intensity factor, it is necessary to evaluate an integral of the type given in eqn (1.3). Several studies have been done to evaluate such integrals, for particular configurations, with $\sigma(x)$ expressed as a polynomial in x . The stress distribution over the crack site in the uncracked body is expressed in the form of

$$\sigma(x) = \sigma \sum_{n=0}^8 A_n \left(\frac{x}{w}\right)^n \quad (1.4)$$

where σ is a suitable reference stress and A_n are known constant coefficients. Both a centre cracked strip of width $2w$ and an edge cracked strip of width w have been studied for such stress distributions by substituting eqn.(1.4) into eqn.(1.3) and replacing $H(x', x)$ by the appropriate Green's function. In both cases the stress intensity factor can be written in the following form

$$K_I = K \left[A_0 + \sum_{n=1}^8 A_n \left(\frac{a}{w}\right)^n \left[B_{0,n} + B_{1,n} \left(\frac{a}{w}\right) + B_{2,n} \left(\frac{a}{w}\right)^2 \right] \right]$$

where K is the stress intensity factor for either the centre crack or edge crack configuration subjected to the uniform stress σ . The coefficients $B_{0,n}$, $B_{1,n}$ and $B_{2,n}$ are given for

the centre cracked strip and edge cracked strip in both configurations, where a is either the half-length of the centre crack or the length of the edge crack.

6. Upper and lower bounds on the stress intensity factors

Under certain conditions, the stress intensity factor K_I for an edge crack of length a in a half-plane will lie between bounds given by

$$[0.29\sigma(a) + 0.83\sigma_{mean}]\sqrt{a\pi} < K_I < 1.12\sigma_{mean}\sqrt{a\pi}$$

where

$$\sigma_{mean} = \frac{1}{a} \int_0^a \sigma(x) dx$$

The conditions are that $\sigma(x)$, the stress over the crack site, in the uncracked body, is a monotonically decreasing function of x , and furthermore, $\sigma(a) \geq 0$. The result expressed above is limited to edge cracks in a half-plane.

7. Compounding technique

Stress intensity factors for many simple configurations are available but they may not be directly applicable to the more complex configurations of typical engineering components. The compounding method is a versatile and quick way of extending these solutions to other more complex configurations for which the stress intensity factors are not known. Often a cracked structural component has several boundaries, such as holes, other cracks or sheet edges all of which will influence the stress intensity factor at the tip of the crack. The compounding technique provides a systematic means by which the contribution of each boundary acting separately can be added to obtain a solution for all the boundaries acting simultaneously.

The choice of a method for determining a stress intensity factor must be assessed in terms of the time available, the required accuracy, the cost, the use (once or many times) and how simple the real structure can be modelled. The stress intensity factor will be affected by other cracks and boundaries, regions of stress concentration, orientation of the crack relative to the applied stress, the degree of constraint on the structure and the position of the applied loading relative to the crack. The method used must take account of these effects and if an approximate method is applied care must be exercised in isolating the dominant effects to be included: it will seldom be possible to consider all when using an approximate method [9, 10].

As we know, the singular stresses, which are the main cause for brittle failure of a structure, have not been determined explicitly before. It is one reason that the stress intensity factor is referred to provide a one parameter characterisation of the stress field just ahead of the crack. Methods used to determine the stress intensity factor as mentioned above, are empirical, inaccurate and with many restrictions. In practice where a particular stress intensity factor is required, it is necessary to obtain an accurate solution. Some simple and more accurate methods are needed when the structure can not be modelled simply. With the above mentioned methods, it is impossible to consider all the effects of the boundary conditions and other restrictions which are responsible for the stress ahead of the crack. Furthermore, more and more recent research and experiments have shown that those parameters, such as K_{IC} , J , COD , etc, do not have the capability to associate the local small crack behavior with the global applied tension or cyclic load. Evidence of this can be clearly seen in past efforts on retardation and crack closure. Current studies have not yet recognized that dynamic fracture is inherently load dependent and can not be characterized by a simple parameter such as K_{IC} , etc. One of the major shortcomings of the conventional approaches is that they failed to separate the fracture energy from other forms of energy dissipation. This is why the parameters, such

as COD ., J ., K_{IC} , etc. are all sensitive to changes in loading and specimen geometry and size. These arguments are not presented in this thesis.

In engineering practice, it is not uncommon to drill a hole at the tip of a crack in order to "release" the stresses and to prevent the crack from propagating. This problem has received considerable attention from mathematicians. Of particular interest is the calculated values of the stress at $\theta = 0$ and $r = a_i$ (a_i is the radius of the hole) to provide some theoretical justification to the empirical experience. This becomes more important for smaller holes which, to some extent, can be treated like a notch (blunt crack).

1.2 Relative Background

The use of linear elasticity in the investigation of the spreading of cracks in materials under stress is now a very well established branch of fracture mechanics. The stress analysis of plane cracks located in elastic material is of interest in the examination of fracture and failure initiation in brittle solids. Considerable efforts have therefore been made in the investigation. The reader is referred to section 1.1, where some techniques are described which are currently available for the determination of stress intensity factors. In order to determine the plane strain fracture toughness of a brittle material it is necessary to carry out a practical fracture test on a specimen of the material which contains a crack. In order to interpret the results of such a test it is necessary to have a theoretical formula for the stress intensity factor at the crack tip for the particular shape of the test specimen and the particular choice of loading used. It is also useful to have a corresponding formula for the opening of the crack at some chosen stations (for example, at the load-line or at the crack mouth). Tada, Paris and Irwin [19] have given some stress intensity factors appropriate to several test specimen configurations. However, since it is often expensive and wasteful of material to fabricate existing test specimens, the American Society for

Testing and Materials Committee on Fracture Testing has introduced the circular disc with a radial edge crack as a new standard specimen. This specimen is easily produced from a cylindrical rod.

One intended application of this thesis is to offer such theoretical formula for the stresses and stress intensity factor at the crack tip for the particular shape of the test specimen and the particular choice of loading used.

In the general field of linear elastic fracture mechanics, there are very few analytical solutions for bodies with finite dimensions. Most solutions involve the infinite plate under uniform tension. However even in the two-dimensional case, very few exact, closed-form solutions are known. This is particularly so in the case of edge cracks, where the only closed-form solutions known appear to be for a straight normal edge crack in a half-plane under various loadings. No non-trivial closed-form solution appears to be known for any case of an edge crack in a plate of finite dimensions. Such a solution would, however, have an importance beyond its intrinsic analytical interest, since it could act as a "model problem", against which the various approximate techniques which are available could be tested for accuracy. There are some particular solutions to the circular plate loaded in a prescribed way. Such problems have been studied by several writers like Tweed (1971), Rooke (1971) [1-7] and Gregory (1977) [8,9]. The general solution is, however, very complicated and consequently only a few numerical results were evaluated for some special cases.

Tweed and Rooke [1-7] have considered a radial crack in a disc which is loaded by a uniform tension. The solution is obtained using integral transforms of the Mellin type. The stress intensity factor is given but detailed stresses are not. This result is valuable in that it gives a closed form solution to an interesting problem but it is of limited practical importance since it is not easy to create a biaxial uniform stress field in tension. They also have considered a number of other related disc problems [2],[3] and

[4] using the Mellin transform. These include the effect of body forces due to rotation, a class of problems which obviously arise in practice for circular plates. Stress intensity factors for a circular disc are given in [14]. In the above papers the integral transform methods reduce to solving, for most cases, singular integral equations. There are three disadvantages of using such methods which make them inconvenient for the standard test. Firstly, the inverse integral transforms have to be used to get the stresses in the disc. However, these integral transforms have not been inverted. Therefore, the stresses in the disc can not be obtained explicitly. Secondly, the singular integral equations are very difficult to invert numerically. Thus, for complex boundary conditions, it would be difficult, or even impossible, to get the numerical solution. Only special solutions for concentrated loading and uniform tension were derived in the papers. Thirdly, it is not easy to apply uniform tension or a concentrated load to a disc in a practical laboratory test.

Other studies [8,9] considered also a circular disc containing a radial edge crack, 'opened' by a constant internal pressure on the crack faces and opposite point forces normal to the crack acting at the crack mouth. The problems are expressed in terms of special stress potential developed by Jeffery using bi-polar coordinates. In this coordinate system, the special case of a disc with a fully radial crack is solved formally using the Wiener-Hopf technique. The corresponding solution for the crack of general length is then obtained from this special case by a transformation. The problem of determining the resulting stress field throughout the disc is solved analytically in a closed-form. However, the method is very complex and it is difficult to get the numerical stresses in the disc. There are two weaknesses in these papers. Firstly, as was mentioned above, it is difficult to set up experiments involving discs having either uniform tension or concentrated loads. Secondly, the methods fail to characterize the stress field internally.

It is clearly desirable to find a theoretical method which may be used to find the

stress intensity factor and the internal stresses corresponding to a simple laboratory test.

A crack originating from a circular hole in a loaded body has been investigated using complex mathematical methods [13] by Morkovin (1944) [11], Buechner (1960), Bowie (1956) [10] and Wigglesworth (1958) [12]. The solutions, however, are very complicated and prohibitive in case of general boundary conditions and subsequently they are very limited in practical applications. The methodology derived during the course of this study is very efficient to solve this kind of problem.

1.3 Objective of the thesis

Referring to the above discussion, the objective of the thesis is to derive an analytical solution to some particular boundary value problems. The solutions can then be used to interpret the results from practical tests on specimens of similar geometry.

Another intended application of this thesis is to offer theoretical formulas for the stresses and stress intensity factor at the crack tip for the particular shape of the test specimen and the particular choice of loading used.

1.4 Outline of the thesis

The mathematical model and the corresponding test pieces are shown in the Fig.(1.3). In the mathematical model (A), the forces act on the outer edge of the disc with an edge crack, α is the angle subtended at the centre by the surface loads. The corresponding test piece is a plexiglas (CR39) disc with a radial crack having two lugs for loading purposes. The isochromatics may be examined by loading the disc and viewing them in polarised light. Fig.1.3(B) models another very interesting experimental test piece. The forces act

on the mouth of the crack. These two tests are very simple and can be done in a very controlled way. In this thesis, model A and B are called "diametrical concentrated force" and "tangential shear forces", respectively.

In chapter 2, an exact closed-form solution for the disc under general boundary conditions is derived. A special stress function is used to get the stresses in the disc with an edge crack. Using Fourier series, the stresses are made to match the boundary conditions. They are obtained directly in terms of the solution of a set of simultaneously linear algebraic equations.

Chapter 3 shows the numerical results for some particular boundary conditions, including a disc under uniform tension; a disc under diametrical concentrated force and a disc with tangential shear forces acting on the mouth of the crack. The closed-form of the stress intensity factors are found for each case. A comparison is made between the results of this work and other related work. Excellent agreement is found. However, the proposed method is more general and efficient in terms of solving complex boundary value problems and getting numerical values for the stresses. This chapter is concluded with an analysis of the numerical error.

In chapter 4, a disc with an edge crack and a hole at the crack tip is solved using both curved beam theory and the proposed method in the thesis. The stress at $r = a_i$ (radius of the hole) and $\theta = 0$ is determined and the results are compared with relative algorithms along with some discussions.

Finally this thesis is concluded with a discussion on the contribution of the thesis along with suggestions for further work.

Figure 1.3: The Mathematical Model and Corresponding Test Piece

Chapter 2

Mathematical Formulation

2.1 General plane strain solution for plate with radial crack

Figure (2.4) shows a circular disc of radius a and thickness t with a radial crack extending outwards from the centre along the full radius $\theta = \pi$. The disc is loaded at its outer edge by surface tractions $T(\theta)$ and $S(\theta)$ where T indicates a normal force and S is a shear force, both having dimension of "force/unit length of circumference". Both T and S must produce a self equilibrating force system. There are no forces acting on the faces of the crack.

A stress function of rather special form is used as follow

$$\begin{aligned}\Phi = & \sum_{m=\frac{3}{2}, \frac{5}{2}, \dots}^{\infty} a_m r^m [m \cos(m-2)\theta - (m-2) \cos m\theta] \\ & + \sum_{n=2, 3, \dots}^{\infty} b_n r^n [\cos(n-2)\theta - \cos n\theta]\end{aligned}\tag{2.5}$$

$$-\pi \leq \theta \leq \pi \quad 0 \leq r \leq a$$

The stresses are then given in their conventional form as follow:

$$\sigma_r = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}$$

Figure 2.4: Description of the Problem

$$\begin{aligned}\sigma_{\theta} &= \frac{\partial^2 \Phi}{\partial r^2} \\ \tau_{r\theta} &= -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right)\end{aligned}\tag{2.6}$$

The stress function is rearranged assuming $a_{-\frac{1}{2}} = a_{\frac{1}{2}} = b_1 = 0$:

$$\begin{aligned}\Phi &= \sum_{m=-\frac{1}{2}}^{\infty} \cos m\theta [-(m-2)r^m a_m + (m+2)r^{m+2} a_{m+2}] \\ &\quad + \sum_{n=1}^{\infty} \cos n\theta [-b_n r^n + b_{n+2} r^{n+2}] + b_2 r^2\end{aligned}\tag{2.7}$$

Using the above stress function, the stresses can be determined as follows:

$$\sigma_r = \sum_{m=-\frac{1}{2}}^{\infty} \cos(m\theta) [m(m-2)(m-1)a_m r^{m-2} - (m+2)(m+1)(m-2)a_{m+2} r^m]$$

$$\begin{aligned}
& + \sum_{n=1}^{\infty} \cos(n\theta) [b_n r^{n-2} n(n-1) - b_{n+2} r^n (n+1)(n-2)] + 2b_2 \\
\sigma_{\theta} = & \sum_{m=-\frac{1}{2}}^{\infty} \cos(m\theta) [-m(m-1)(m-2)a_m r^{m-2} + (m+2)^2(m+1)a_{m+2} r^m] \\
& + \sum_{n=1}^{\infty} \cos(n\theta) [-n(n-1)b_n r^{n-2} + (n+2)(n+1)b_{n+2} r^n] + 2b_2 \quad (2.8) \\
\tau_{r\theta} = & \sum_{m=-\frac{1}{2}}^{\infty} m \sin(m\theta) [-(m-1)(m-2)a_m r^{m-2} + (m+1)(m+2)a_{m+2} r^m] \\
& + \sum_{n=1}^{\infty} n \sin(n\theta) [-(n-1)b_n r^{n-2} + (n+1)b_{n+2} r^n]
\end{aligned}$$

By inspection it is seen

$$\sigma_{\theta} = \tau_{r\theta} = 0 \quad \text{at } \theta = \pm\pi \quad 0 \leq r \leq a$$

Hence equation (2.5) is a stress function corresponding to a disc with a stress free crack at $\theta = \pi$, for all values of a_m and b_n .

Considering the general expression for the shear stress, $\tau_{r\theta}$, in equation (2.8), it is written in terms of $\sin n\theta$ by using Fourier series for each $\sin(m\theta)$ as follows

$$\sin(m\theta) = \frac{2}{\pi} \sin(m\pi) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n \sin(n\theta)}{n^2 - m^2} \quad -\pi < \theta < \pi \quad (2.9)$$

Substituting (2.9) into (2.8) it is seen that the term in $n \sin n\theta$ in the general expression for $\tau_{r\theta}$ is

$$\begin{aligned}
& \sum_{m=-\frac{1}{2}}^{\infty} [-(m-1)(m-2)a_m r^{m-2} + (m+1)(m+2)a_{m+2} r^m] \frac{2m}{\pi} \frac{(-1)^{n+1} \sin(m\pi)}{n^2 - m^2} \\
& -(n-1)b_n r^{n-2} + (n+1)b_{n+2} r^n \quad (2.10)
\end{aligned}$$

For the normal force, σ_r , the fractional trigonometric terms are also eliminated using the following equation

$$\cos(m\theta) = \frac{\sin(m\pi)}{m\pi} + \frac{2}{\pi} \sin(m\pi) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{m \cos n\theta}{n^2 - m^2} \quad (2.11)$$

Substituting (2.11) into (2.8), the terms in $\cos(n\theta)$ in the general expression for σ_r becomes

$$\begin{aligned} & \sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)ma_m r^{m-2} - (m-2)(m+1)(m+2)a_{m+2}r^m] \frac{2}{\pi} \sin(m\pi) \frac{(-1)^{n+1}m}{n^2 - m^2} \\ & + n(n-1)b_n r^{n-2} - (n+1)(n-2)b_{n+2}r^n \end{aligned} \quad (2.12)$$

together with a constant (θ independent) term

$$\sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)ma_m r^{m-2} - (m-2)(m+1)(m+2)a_{m+2}r^m] \frac{\sin(m\pi)}{m\pi} + 2b_2 \quad (2.13)$$

It was mentioned previously that the surface tractions at the boundaries are self equilibrating and denoted by $T(\theta)$ and $S(\theta)$. They may be expanded as Fourier series. Also, if we take $T(\theta)$ is taken as an even function and $S(\theta)$ as an odd function we may write

$$\begin{aligned} T(\theta) &= t\sigma_r|_{r=a} = \frac{t_0}{2} + \sum_{n=1}^{\infty} t_n \cos n\theta \\ S(\theta) &= t\tau_{r\theta}|_{r=a} = \sum_{n=1}^{\infty} s_n \sin n\theta \end{aligned} \quad (2.14)$$

where t_0 , t_n and s_n are assumed to be known. An obvious extension removes the requirement that $T(\theta)$ is even and $S(\theta)$ is odd. The values of σ_r and $\tau_{r\theta}$ at $r = a$ can also be obtained by substituting $r = a$ into (2.10), (2.12) and (2.13). These values must agree with the ones in (2.14). Hence

$$\begin{aligned}
\frac{t_0}{2} &= \sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)ma^{m-2}a_m - (m-2)(m+1)(m+2)a^m a_{m+2}] \frac{\sin(m\pi)}{m\pi} + 2b_2 \\
t_n &= (-1)^{n+1} \sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)ma^{m-2}a_m - (m-2)(m+1)(m+2)a^m a_{m+2}] \frac{2m \sin(m\pi)}{\pi n^2 - m^2} \\
&\quad + n(n-1)b_n a^{n-2} - (n+1)(n-2)b_{n+2} a^n \\
s_n &= (-1)^{n+1} \sum_{m=-\frac{1}{2}}^{\infty} [-(m-1)(m-2)a^{m-2}a_m + (m+1)(m+2)a^m a_{m+2}] \frac{2m \sin(m\pi)}{\pi n^2 - m^2} \\
&\quad - (n-1)b_n + (n+1)b_{n+2}
\end{aligned} \tag{2.15}$$

Where $n = 1, 2, 3, \dots$, $m = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots$ $a_{-\frac{1}{2}} = a_{\frac{1}{2}} = b_1 = 0$

Equations (2.15) are a set of simultaneous linear algebraic equations from which the coefficients a_m and b_n can be found in terms of the given coefficients t_0, t_n and s_n . In the following section these equations are solved for some particular loadings.

2.2 Stresses at the crack tip

The asymptotic form of the stresses near the crack tip are obtained from equation (2.8) by retaining the singular term in $a_{\frac{3}{2}}$. The following are then obtained

$$\begin{aligned}
\sigma_r &= \frac{3}{8}a_{\frac{3}{2}} \left(5\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \right) \frac{P}{at} \sqrt{\frac{a}{r}} \\
\tau_{r\theta} &= \frac{3}{8}a_{\frac{3}{2}} \left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \right) \frac{P}{at} \sqrt{\frac{a}{r}} \\
\sigma_\theta &= \frac{3}{8}a_{\frac{3}{2}} \left(3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \right) \frac{P}{at} \sqrt{\frac{a}{r}}
\end{aligned} \tag{2.16}$$

where P has dimension of force.

For $\theta = 0$ the asymptotic stresses are reduced to

$$\sigma_r = \sigma_\theta = \frac{3}{2}a_{\frac{3}{2}} \frac{P}{at} \sqrt{\frac{a}{r}}$$

$$\tau_{r\theta} = 0 \quad (2.17)$$

The stress intensity factor is then calculated as followed

$$K = \frac{3}{2} a^{\frac{3}{2}} \frac{P}{at} \sqrt{2a\pi} \quad (2.18)$$

As shown above, this method reduces to solving a set of linear algebraic equations. There are three advantages to this method. Firstly, the stresses are accurate not only near the crack tip but also over the whole disc. Secondly, for general boundary tractions acting in the same plane as the disc and for plane strain cases, the method works very well. The third advantage is that this method is not only valuable in theory because it provides a closed-form solution, but also because of its practical application. It is relative easy to get the numerical stresses in the disc and these may be compared with experiments. In the following chapter, it is shown that this method is more convenient as a "model problem" for a test.

Chapter 3

Numerical Results

In this chapter, the method is exemplified with different boundary conditions. Some numerical results and error analysis are presented as well.

3.1 A disc under uniform tensile stress

Let T_0 be the pressure of the normal force shown in Fig.(3.5). For $r = a$,

$$\sigma_r = T_0 \quad \tau_{r\theta} = 0 \quad (3.19)$$

So the coefficients in the Fourier series are as follow:

$$t_0 = 2T_0 \quad t_n = 0 \quad s_n = 0 \quad (3.20)$$

For convenience, replace a_m and b_n by $\hat{a}_m$ and $\hat{b}_n$ given by

$$\begin{aligned} \hat{a}_m &= \frac{a_m a^{m-2}}{T_0 \pi} \\ \hat{b}_n &= \frac{b_n a^{n-2}}{T_0 \pi} \end{aligned} \quad (3.21)$$

Substituting (3.20) and (3.21) into (2.15), and for $r = a$, the following simultaneous equations are obtained.

Figure 3.5: Disc under the Uniform Tension

$$\begin{aligned}
 \sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)m\hat{a}_m - (m-2)(m+1)(m+2)\hat{a}_{m+2}] \frac{\sin m\pi}{m\pi} + 2\hat{b}_2 &= \frac{1}{\pi} \\
 \sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)m\hat{a}_m - (m-2)(m+1)(m+2)\hat{a}_{m+2}] \frac{2}{\pi} \sin m\pi \frac{(-1)^{n+1}m}{n^2 - m^2} \\
 + n(n-1)\hat{b}_n - (n+1)(n-2)\hat{b}_{n+2} &= 0 \\
 \sum_{m=-\frac{1}{2}}^{\infty} [-(m-1)(m-2)\hat{a}_m + (m+1)(m+2)\hat{a}_{m+2}] \frac{2m}{\pi} \frac{(-1)^{n+1} \sin m\pi}{n^2 - m^2} \\
 -(n-1)\hat{b}_n + (n+1)\hat{b}_{n+2} &= 0
 \end{aligned} \tag{3.22}$$

The leading coefficients found from a 45×45 matrix of (3.22) are shown in Table (3.1).

From Table (3.1) it is seen the values of a_m and b_n decrease rapidly with increasing m and n . Thus, only the first few a_m and b_n are important. Table (3.2) shows the stresses on the boundary. The results from the proposed method are in good agreement with the theoretical values.

$\hat{a}_{\frac{3}{2}}$	$\hat{a}_{\frac{5}{2}}$	$\hat{a}_{\frac{7}{2}}$	$\hat{a}_{\frac{9}{2}}$	$\hat{a}_{\frac{11}{2}}$	$\hat{a}_{\frac{13}{2}}$	$\hat{a}_{\frac{15}{2}}$	$\hat{a}_{\frac{17}{2}}$	$\hat{a}_{\frac{19}{2}}$	$\hat{a}_{\frac{21}{2}}$
476.0	-155.9	-2.16	0.510	0.171	0.070	-0.032	0.016	-0.009	0.005
$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$	$\hat{b}_6$	$\hat{b}_7$	$\hat{b}_8$	$\hat{b}_9$	$\hat{b}_{10}$	$\hat{b}_{11}$
150.9	28.3	-3.24	0.0817	-0.27	0.101	-0.04	0.014	-0.003	-0.0001

Table 3.1: First ten coefficients under the uniform tension ($\times 10^{-3}$)

θ	$(\frac{\sigma_r}{T_0})_{cal}$	$(\frac{\sigma_r}{T_0})_{ideal}$	$(\frac{\tau_{r\theta}}{T_0})_{cal}$	$(\frac{\tau_{r\theta}}{T_0})_{ideal}$
0.000	1.00017	1.000	0.00000	0.0000
20.000	0.99996	1.000	-0.00002	0.0000
40.000	1.00013	1.000	0.00005	0.0000
60.000	0.99986	1.000	-0.00007	0.0000
80.000	1.00013	1.000	0.00010	0.0000
100.000	0.99984	1.000	-0.00015	0.0000
120.000	1.00015	1.000	0.00021	0.0000
140.000	0.99977	1.000	-0.00032	0.0000
160.000	1.00043	1.000	0.00058	0.0000
180.000	0.99731	1.000	0.00000	0.0000

Table 3.2: The stresses on the boundary

The asymptotic forms of the stresses at $\theta = 0$ are obtained using the calculated value of $a_{\frac{3}{2}}$ in equation (2.17). We obtain

$$\sigma_r = \sigma_\theta = 2.234T_0\sqrt{\frac{a}{r}} \quad (3.23)$$

which is in good agreement with the results given in [1].

The stress intensity factor for uniform tension is

$$K_I = 3.17T_0\sqrt{\pi a} \quad (3.24)$$

or

$$K_I = 5.6187T_0\sqrt{a} \quad (3.25)$$

where T_0 is the pressure at the boundary.

Contours of the isochromatics (constant maximum shear stress) are shown in Fig.(3.6). The dimensionless stresses $\frac{\sigma_r}{T_0}$, $\frac{\sigma_\theta}{T_0}$ and $\frac{\tau_{r\theta}}{T_0}$ in the disc at $\theta = 0^\circ$, $\theta = 45^\circ$, $\theta = 90^\circ$ and $\theta = 135^\circ$ are shown in Fig.(3.7), Fig.(3.8) and Fig.(3.9).

Selected values of σ_r , σ_θ and $\tau_{r\theta}$ throughout the plate are shown in Table (3.3). Detailed values at 5° interval and increments $dr = 0.02$ are given in [19].

3.2 Disc with diametrical force T

Referring to Fig.(3.10) the disc is loaded by two self equilibrating forces of magnitude F with line of action through the centre of the disc and perpendicular to the crack. Each force T is the resultant of a distributed surface traction $T(\theta)$ distributed over a small sector of angle α (for more detail see Appendix C). The actual distribution of $T(\theta)$ is not

Figure 3.6: Isochromatics under the Uniform Tension

Figure 3.7: Stress σ_r in the Disc under the Uniform Tension

Figure 3.8: Stress σ_θ under the Uniform Tension

Figure 3.9: Stress $\tau_{r\theta}$ under the Uniform Tension

	ξ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$\theta = 0^\circ$	σ_r	7.894	5.397	4.166	3.361	2.765	2.293	1.899	1.562	1.265	1.000
	σ_θ	5.928	3.363	2.066	1.197	0.540	0.006	-0.447	-0.843	-1.196	-1.516
	$\tau_{r\theta}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$\theta = 30^\circ$	σ_r	7.833	5.328	4.105	3.310	2.725	2.261	1.876	1.547	1.258	1.000
	σ_θ	5.580	3.191	1.970	1.147	0.521	0.011	-0.422	-0.800	-1.139	-1.447
	$\tau_{r\theta}$	1.054	0.596	0.399	0.283	0.204	0.145	0.099	0.060	0.028	0.000
$\theta = 60^\circ$	σ_r	7.609	5.113	3.922	3.160	2.604	2.169	1.810	1.504	1.237	1.000
	σ_θ	4.598	2.697	1.691	0.998	0.464	0.025	-0.350	-0.680	-0.976	-1.245
	$\tau_{r\theta}$	1.849	1.074	0.730	0.523	0.380	0.272	0.185	0.114	0.053	0.000
$\theta = 90^\circ$	σ_r	7.111	4.732	3.618	2.918	2.455	2.024	1.706	1.437	1.205	1.000
	σ_θ	3.181	1.953	1.259	0.762	0.369	0.040	-0.245	-0.497	-0.725	-0.934
	$\tau_{r\theta}$	2.164	1.322	0.924	0.675	0.496	0.359	0.247	0.153	0.072	0.000
$\theta = 120^\circ$	σ_r	6.145	4.130	3.184	2.590	2.166	1.840	1.576	1.355	1.165	1.000
	σ_θ	1.656	1.093	0.737	0.463	0.236	0.041	-0.131	-0.285	-0.425	-0.553
	$\tau_{r\theta}$	1.873	1.239	0.907	0.683	0.514	0.378	0.265	0.166	0.079	0.000
$\theta = 150^\circ$	σ_r	4.454	3.198	2.574	2.166	1.864	1.627	1.432	1.268	1.125	1.000
	σ_θ	0.456	0.335	0.243	0.162	0.089	0.021	-0.040	-0.094	-0.140	-0.178
	$\tau_{r\theta}$	1.033	0.773	0.610	0.483	0.377	0.285	0.204	0.131	0.064	0.000
$\theta = 180^\circ$	σ_r	1.824	1.749	1.670	1.589	1.503	1.413	1.318	1.217	1.109	0.997
	σ_θ	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$\tau_{r\theta}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 3.3: Stresses in the disc under uniform tension

Figure 3.10: Disc with Diametrical Force T

important, but in this analysis it is chosen to be in the form of a half sine wave over an angle α where $\alpha^0 = \frac{180^\circ}{K}$ and $K = 4m + 1$, m is integer. The surface value of σ_r is given as:

$$\sigma_r = \frac{T}{at} \frac{K}{2} \sin K\theta \quad \frac{\pi}{2} \left(1 - \frac{1}{K}\right) \leq \theta \leq \frac{\pi}{2} \left(1 + \frac{1}{K}\right)$$

$$\sigma_r = 0 \quad \theta \leq \frac{\pi}{2} \left(1 - \frac{1}{K}\right) \quad \text{or} \quad \theta \geq \frac{\pi}{2} \left(1 + \frac{1}{K}\right)$$

The Fourier series of σ_r is

$$\sigma_r = \frac{T}{at} \left(\sum_{n=1}^{\infty} t_n \cos n\theta + \frac{t_0}{2} \right) \quad \frac{\pi}{2} \left(1 - \frac{1}{K}\right) \leq \theta \leq \frac{\pi}{2} \left(1 + \frac{1}{K}\right)$$

where t_0 and t_n are derived in Appendix D and are

$$t_0 = \frac{2}{\pi}$$

$$\begin{aligned}
t_n &= \frac{K}{\pi} \left[\frac{1}{K+n} \sin \frac{K+n}{2} \pi \sin \frac{K+n}{2K} \pi \right. \\
&\quad \left. + \frac{1}{K-n} \sin \frac{K-n}{2} \pi \sin \frac{K-n}{2K} \pi \right] \\
t_K &= 0
\end{aligned}$$

For convenience, replace a_m and b_n by $\hat{a}_m$ and $\hat{b}_n$ defined as

$$\begin{aligned}
\hat{a}_m &= \frac{a_m a^m t}{T a} \\
\hat{b}_n &= \frac{b_n a^n t}{T a}
\end{aligned}$$

Replacing a_m and b_n by $\hat{a}_m$ and $\hat{b}_n$ in equations (2.15), and setting $r = a$ the following set of equations are derived.

$$\begin{aligned}
&\sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)m\hat{a}_m - (m-2)(m+1)(m+2)\hat{a}_{m+2}] \frac{\sin m\pi}{m\pi} + 2\hat{b}_2 = \frac{1}{\pi} \\
&\sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)m\hat{a}_m - (m-2)(m+1)(m+2)\hat{a}_{m+2}] \frac{2}{\pi} \sin m\pi \frac{(-1)^{n+1}m}{n^2 - m^2} \\
&+ n(n-1)\hat{b}_n - (n+1)(n-2)\hat{b}_{n+2} = \bar{t}_n \\
&\sum_{m=-\frac{1}{2}}^{\infty} [-(m-1)(m-2)\hat{a}_m + (m+1)(m+2)\hat{a}_{m+2}] \frac{2m}{\pi} \frac{(-1)^{n+1} \sin m\pi}{n^2 - m^2} \\
&-(n-1)\hat{b}_n + (n+1)\hat{b}_{n+2} = 0
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
\bar{t}_n &= t_n \quad \text{for } n = K \\
\bar{t}_K &= t_K = 0
\end{aligned}$$

K	$\hat{a}_{\frac{3}{2}}$	$\hat{a}_{\frac{5}{2}}$	$\hat{a}_{\frac{7}{2}}$	$\hat{a}_{\frac{9}{2}}$	$\hat{a}_{\frac{11}{2}}$	$\hat{a}_{\frac{13}{2}}$	$\hat{a}_{\frac{15}{2}}$	$\hat{a}_{\frac{17}{2}}$	$\hat{a}_{\frac{19}{2}}$	$\hat{a}_{\frac{21}{2}}$
	0.672	-0.069	-0.094	-0.063	0.049	0.04	-0.034	-0.029	0.025	0.023
21	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$	$\hat{b}_6$	$\hat{b}_7$	$\hat{b}_8$	$\hat{b}_9$	$\hat{b}_{10}$	$\hat{b}_{11}$
	-0.176	0.051	0.308	0.002	-0.314	0.358	0.309	0.63	-0.304	0.265
	$\hat{a}_{\frac{3}{2}}$	$\hat{a}_{\frac{5}{2}}$	$\hat{a}_{\frac{7}{2}}$	$\hat{a}_{\frac{9}{2}}$	$\hat{a}_{\frac{11}{2}}$	$\hat{a}_{\frac{13}{2}}$	$\hat{a}_{\frac{15}{2}}$	$\hat{a}_{\frac{17}{2}}$	$\hat{a}_{\frac{19}{2}}$	$\hat{a}_{\frac{21}{2}}$
	0.672	-0.069	-0.094	-0.063	0.05	0.041	-0.034	-0.029	-0.026	0.023
29	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$	$\hat{b}_6$	$\hat{b}_7$	$\hat{b}_8$	$\hat{b}_9$	$\hat{b}_{10}$	$\hat{b}_{11}$
	-0.176	0.051	0.309	0.002	-0.317	0.36	-0.314	0.646	-0.311	0.389
	$\hat{a}_{\frac{3}{2}}$	$\hat{a}_{\frac{5}{2}}$	$\hat{a}_{\frac{7}{2}}$	$\hat{a}_{\frac{9}{2}}$	$\hat{a}_{\frac{11}{2}}$	$\hat{a}_{\frac{13}{2}}$	$\hat{a}_{\frac{15}{2}}$	$\hat{a}_{\frac{17}{2}}$	$\hat{a}_{\frac{19}{2}}$	$\hat{a}_{\frac{21}{2}}$
	0.673	-0.069	-0.094	-0.063	0.049	0.041	-0.034	-0.03	0.026	0.023
37	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$	$\hat{b}_6$	$\hat{b}_7$	$\hat{b}_8$	$\hat{b}_9$	$\hat{b}_{10}$	$\hat{b}_{11}$
	-0.176	0.051	0.31	0.002	-0.318	0.345	0.315	0.539	-0.314	-0.462

Table 3.4: The first ten coefficients with diametrical force T

The ten leading coefficients $\hat{a}_m$ and $\hat{b}_n$ are listed in Table (3.4) by inverting a 250×250 matrix. All the $\hat{b}_7, \hat{b}_9$ and $\hat{b}_{11}$ must be multiplied by 10^{-3} , 10^{-4} and 10^{-5} . For $\alpha = 4.9^\circ, 6.2^\circ$ and 8.6° the value of K are 37, 29 and 21, respectively. As expected the coefficients are insensitive to α for small α . Table (3.5) shows the stresses on the boundary when $K = 37$ ($\alpha = 4.9^\circ$). It is seen that the boundary conditions are satisfied well.

The stresses near the crack tip for $K = 37$ ($\alpha = 4.9^\circ$) are obtained by substituting the value of $a_{\frac{3}{2}}$ into equation (2.17).

$$\sigma_r = \sigma_\theta = 1.01 \frac{T}{at} \sqrt{\frac{a}{r}} \quad \text{at } \theta = 0 \quad (3.27)$$

The stress intensity factor for $\alpha = 4.9^\circ$ is

$$K_I = 2.5317 \frac{T}{at} \sqrt{a} \quad (3.28)$$

θ	$(\frac{\sigma_r}{T_0})_{cal}$	$(\frac{\sigma_r}{T_0})_{Fouriour}$	$(\frac{\sigma_r}{T_0})_{ideal}$	$(\frac{\tau_{r\theta}}{T_0})_{cal}$	$(\frac{\tau_{r\theta}}{T_0})_{ideal}$
0.00	0.00735	0.00605	0.0000	0.00000	0.0000
10.00	0.00543	0.00640	0.0000	-0.00013	0.0000
20.00	-0.00854	-0.00644	0.0000	-0.00027	0.0000
30.00	0.00540	0.00651	0.0000	-0.00041	0.0000
40.00	-0.00854	-0.00664	0.0000	-0.00056	0.0000
50.00	0.00551	0.00689	0.0000	-0.00071	0.0000
60.00	-0.00903	-0.00745	0.0000	-0.00088	0.0000
70.00	0.00730	0.00907	0.0000	-0.00107	0.0000
80.00	-0.01915	-0.01798	0.0000	-0.00129	0.0000
85.00	-0.10422	-0.10512	0.0000	0.00140	0.0000
90.00	18.25948	18.26173	18.000	-0.00153	0.0000

Table 3.5: The stresses on the boundary

or

$$K_I = 1.42841 \frac{T}{at} \sqrt{\pi a} \quad (3.29)$$

where T is the total load within the loading area.

Selected stresses are given in Table (3.6) and the isochromatics are shown in Fig.(3.13) with $\alpha = 4.9^\circ$. They are similar to those obtained for $\alpha = 6.5^\circ$, Fig.(3.12) and $\alpha = 8.2^\circ$, Fig.(3.11) except near the regions where the loads are applied. Since the stresses far from the loading area are not sensitive to α , for the same total loads, stresses in the disc under a concentrated tension can be determined assuming the disc is loaded with a tension applied over a very small α . Thus, the model can be used exactly to verify the experiments done on test pieces even with big lugs. The dimensionless stresses $\frac{\sigma_r}{T}$, $\frac{\sigma_\theta}{T}$ and $\frac{\tau_{r\theta}}{T}$ in the disc at $\theta = 0^\circ$, $\theta = 45^\circ$, $\theta = 90^\circ$ and $\theta = 135^\circ$ are shown in Fig.(3.14), Fig.(3.15) and Fig.(3.16). Selected values of $\frac{\sigma_r}{T}$, $\frac{\sigma_\theta}{T}$ and $\frac{\tau_{r\theta}}{T}$ throughout the disc are given in Table (3.6). Detailed stresses for the above three values of α are also given in [19].

Figure 3.11: Isochromatics with Diametric Force T ($\alpha = 8.5^\circ$)

Figure 3.12: Isochromatics with Diametric Force T ($\alpha = 6.2^\circ$)

Figure 3.13: Isochromatics with Diametric Force T ($\alpha = 4.9^\circ$)

Figure 3.14: Stress σ_r with Diametric Force ($\alpha = 4.9^\circ$)

Figure 3.15: Stress σ_θ with Diametric Force ($\alpha = 4.9^\circ$)

Figure 3.16: Stress $\tau_{r\theta}$ with Diametric Force ($\alpha = 4.9^\circ$)

	ξ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$\theta = 0^\circ$	σ_r	2.345	1.371	0.944	0.696	0.528	0.398	0.288	0.188	0.092	-0.009
	σ_θ	2.971	1.853	1.234	0.770	0.384	0.051	-0.239	-0.493	-0.715	-0.917
	$\tau_{r\theta}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$\theta = 30^\circ$	σ_r	2.630	1.592	1.097	0.783	0.564	0.403	0.280	0.179	0.088	0.005
	σ_θ	2.494	1.534	1.045	0.694	0.391	0.106	-0.167	-0.426	-0.663	-0.866
	$\tau_{r\theta}$	1.071	0.798	0.623	0.471	0.334	0.216	0.124	0.060	0.021	0.000
$\theta = 60^\circ$	σ_r	3.198	2.136	1.622	1.262	0.953	0.660	0.395	0.195	0.077	-0.009
	σ_θ	1.356	0.687	0.398	0.254	0.191	0.158	0.088	-0.094	-0.405	-0.774
	$\tau_{r\theta}$	1.489	1.132	0.979	0.883	0.792	0.665	0.481	0.260	0.079	-0.001
$\theta = 90^\circ$	σ_r	3.340	2.371	1.995	1.845	1.843	1.997	2.390	3.310	6.080	18.259
	σ_θ	0.286	-0.093	-0.285	-0.419	-0.526	-0.617	-0.695	-0.752	-0.657	17.665
	$\tau_{r\theta}$	1.066	0.687	0.497	0.370	0.276	0.201	0.139	0.086	0.040	-0.002
$\theta = 120^\circ$	σ_r	2.544	1.683	1.268	0.978	0.726	0.486	0.268	0.113	0.037	-0.008
	σ_θ	-0.159	-0.206	-0.158	-0.068	0.047	0.158	0.208	0.130	-0.092	-0.380
	$\tau_{r\theta}$	0.319	0.058	-0.100	-0.217	-0.288	-0.295	-0.222	-0.098	-0.003	-0.003
$\theta = 150^\circ$	σ_r	1.015	0.531	0.308	0.175	0.095	0.050	0.029	0.019	0.011	0.002
	σ_θ	-0.107	-0.051	0.024	0.080	0.102	0.085	0.039	-0.021	-0.081	-0.125
	$\tau_{r\theta}$	-0.107	-0.134	-0.091	-0.041	0.008	0.046	0.065	0.061	0.038	-0.006
$\theta = 175^\circ$	σ_r	-0.429	-0.450	-0.388	-0.297	-0.204	-0.124	-0.064	-0.026	-0.007	0.005
	σ_θ	-0.004	-0.001	0.002	0.005	0.005	0.004	0.001	-0.002	-0.005	-0.031
	$\tau_{r\theta}$	-0.053	-0.037	-0.013	0.009	0.025	0.032	0.030	0.021	0.008	0.027

Table 3.6: Stresses in the disc with diametrical force T

Figure 3.17: Disc with the Tangential Shear Force S

3.3 Disc with tangential shear forces

The case shown in Fig.(3.17) is now considered in which the crack is opened by two self equilibrating forces, F , applied at the mouth of the crack. S is the resultant of a tangential shear distribution of $S(\theta)$ applied over a small sector α . The distribution of $S(\theta)$ is in the form of a half sine wave, as discussed in section 3.2, which ensures that $\tau_{r\theta} = \tau_{\theta r} = 0$ at $\theta = \pi$, $r = a$. The surface value of $\tau_{r\theta}$ is

$$\begin{aligned} \tau_{r\theta} &= -\frac{S_0}{at} \frac{K}{2} \sin K\theta & \pi(1 - \frac{1}{K}) \leq \theta \leq \pi \\ \tau_{r\theta} &= 0 & \theta \leq \pi(1 - \frac{1}{K}) \end{aligned}$$

The Fourier form of $\tau_{r\theta}$ as well as the coefficients are given below (for detail see Appendix E)

$$\tau_{r\theta} = \frac{S_0}{at} \sum_{n=1}^{\infty} s_n \sin n\theta$$

$$\begin{aligned}
s_n &= \frac{K}{2\pi} \left[-\frac{1}{K+n} \sin(K+n) \frac{K-1}{K} \pi + \frac{1}{K-n} \sin(K-n) \frac{K-1}{K} \right] \\
s_K &= -\frac{1}{2}
\end{aligned} \tag{3.30}$$

To maintain the equilibrium, there must be some normal forces acting on the boundary. Let σ_r act on the same section as $\tau_{r\theta}$ and write σ_r as

$$\sigma_r = A \frac{S}{at} \sin K\theta$$

From the equilibrium equations, the coefficient A is given by Appendix C.

$$A = \frac{1}{2} K \frac{\sin \frac{\pi}{K}}{1 + \cos \frac{\pi}{K}}$$

Hence

$$\sigma_r = \frac{K S_0}{2at} \frac{\sin \frac{\pi}{K}}{1 + \cos \frac{\pi}{K}} \sin K\theta$$

The Fourier form is

$$\begin{aligned}
\sigma_r &= \frac{S_0}{at} \left[\frac{t_0}{2} + \sum_{n=1}^{\infty} t_n \cos n\theta \right] \\
t_n &= \frac{A}{\pi} - \frac{1}{K+n} \left[\cos(K+n)\pi - \cos \frac{(K+n)(K-1)}{K} \pi \right. \\
&\quad \left. - \frac{1}{K-n} \left[\cos(K-n)\pi - \cos \frac{(K-n)(K-1)}{K} \pi \right] \right] \\
t_0 &= \frac{4A}{K\pi} \\
t_K &= 0
\end{aligned} \tag{3.31}$$

The details are given in Appendix D.

a_m and b_n are also replaced by $\hat{a}_m$ and $\hat{b}_n$ as follow:

$$\begin{aligned}\hat{a}_m &= \frac{a^m a_m}{S a} \\ \hat{b}_n &= \frac{a^n b_n}{S a}\end{aligned}\quad (3.32)$$

Substituting (3.30), (3.31) and (3.32) into (2.15) for $r = a$, the final set of equations are

$$\begin{aligned}& \sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)m\hat{a}_m - (m-2)(m+1)(m+2)\hat{a}_{m+2}] \frac{\sin m\pi}{m\pi} + 2\hat{b}_2 = \frac{2A}{K\pi} \\& \sum_{m=-\frac{1}{2}}^{\infty} [(m-2)(m-1)m\hat{a}_m - (m-2)(m+1)(m+2)\hat{a}_{m+2}] \frac{2m}{\pi} (-1)^{n+1} \frac{\sin m\pi}{n^2 - m^2} \\& + n(n-1)\hat{b}_n + (n+1)(n-2)\hat{b}_{n+2} = \bar{t}_n \\& \sum_{m=-\frac{1}{2}}^{\infty} [(m-1)(m-2)\hat{a}_m + (m+1)(m+2)\hat{a}_{m+2}] \frac{2m}{\pi} (-1)^{n+1} \frac{\sin m\pi}{n^2 - m^2} \\& - (n-1)\hat{b}_n + (n+1)\hat{b}_{n+2} = \bar{s}_n\end{aligned}\quad (3.33)$$

$\bar{t}_n$ and $\bar{s}_n$ are as follow:

$$\begin{aligned}\bar{t}_n &= t_n & \bar{s}_n &= s_n & \text{when } n \neq K \\ t_K &= 0 & s_K &= -\frac{1}{2}\end{aligned}$$

The leading coefficients are shown in Table 3.7.

Table (3.8) shows the stresses on the boundary which are in good agreement with the theoretical values.

The stresses at the crack tip are given by eqn.(2.17) and in this case are

$\hat{a}_{\frac{3}{2}}$	$\hat{a}_{\frac{5}{2}}$	$\hat{a}_{\frac{7}{2}}$	$\hat{a}_{\frac{9}{2}}$	$\hat{a}_{\frac{11}{2}}$	$\hat{a}_{\frac{13}{2}}$	$\hat{a}_{\frac{15}{2}}$	$\hat{a}_{\frac{17}{2}}$	$\hat{a}_{\frac{19}{2}}$	$\hat{a}_{\frac{21}{2}}$
2.047	-1.111	0.094	-0.046	0.027	-0.018	0.013	-0.01	0.008	-0.006
$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$	$\hat{b}_6$	$\hat{b}_7$	$\hat{b}_8$	$\hat{b}_9$	$\hat{b}_{10}$	$\hat{b}_{11}$
1.057	-0.052	0.100	-0.082	0.067	-0.056	0.048	-0.042	0.038	-0.034

Table 3.7: The fist ten coefficients with tangential shear force S

θ	$(\frac{\sigma_r}{T_0})_{cal}$	$(\frac{\sigma_r}{T_0})_{Fouriour}$	$(\frac{\sigma_r}{T_0})_{ideal}$	$(\frac{\tau_{r\theta}}{T_0})_{cal}$	$(\frac{\tau_{r\theta}}{T_0})_{Fouriour}$	$(\frac{\tau_{r\theta}}{T_0})_{ideal}$
0.00	0.00009	0.00000	0.0000	0.00000	0.00000	0.0000
20.00	-0.00001	-0.00008	0.0000	-0.00032	-0.00055	0.0000
40.00	0.00002	0.00008	0.0000	0.00050	0.00077	0.0000
60.00	-0.00002	-0.00008	0.0000	-0.00041	-0.00047	0.0000
80.00	0.00004	0.00008	0.0000	0.00002	0.00002	0.0000
100.00	-0.00006	-0.00008	0.0000	0.00058	0.00100	0.0000
120.00	0.00012	0.00008	0.0000	-0.00125	-0.00143	0.0000
140.00	-0.00028	-0.00010	0.0000	0.00166	0.00116	0.0000
160.00	0.00108	0.00018	0.0000	-0.00083	-0.00034	0.0000
176.00	0.42741	0.42257	0.4165	-9.92305	-9.93722	-9.8035
178.00	0.78068	0.75935	0.7554	-17.6972	-17.7269	-17.783

Table 3.8: The stresses on the boundary

$$\sigma_r = \sigma_\theta = 3.07 \frac{S}{at} \sqrt{\frac{a}{r}} \quad (3.34)$$

and the stress intensity factor is

$$K_I = 7.6953 \frac{S}{at} \sqrt{a} \quad (3.35)$$

Where S is the total load at the crack mouth.

Selected stresses are given in Table (3.9) and the isochromatics are shown in Fig.(3.18). The dimensionless stresses $\frac{\sigma_r}{S}$, $\frac{\sigma_\theta}{S}$ and $\frac{\tau_{r\theta}}{S}$ are also given in Fig.(3.19), Fig.(3.20) and Fig.(3.21). Detailed stresses are given in [19].

3.4 Comparison with related results

The comparison with related results such as the ones shown by Tweed [1-7] and Gregory [8,9] are given in table (3.10)

Cases I, II and III are for uniform tension, diametrical concentrated tension and tangential shear force, respectively. $K_0 = P_0 \sqrt{\pi a}$, where K_0 is the stress intensity factor of a Griffith crack of length $2a$ in an infinite elastic sheet which is opened by a constant pressure P_0 , F and S are the total load for Case II and Case III respectively.

The comparison shows good agreement with other related works. However, the method derived in the thesis is more general and powerful.

3.5 An analysis of the error

The errors in the numerical results are introduced three ways. Firstly, when the Fourier series are used to approximate $\sin(m\theta)$ and $\cos(m\theta)$ as series in $\sin(n\theta)$ and

	ξ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$\theta = 0^\circ$	σ_r	11.320	7.449	5.441	4.088	3.069	2.250	1.566	0.977	0.461	0.000
	σ_θ	7.123	3.265	1.257	-0.106	-1.143	-1.985	-2.698	-3.319	-3.871	-4.369
	$\tau_{r\theta}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$\theta = 30^\circ$	σ_r	11.088	7.290	5.333	4.015	3.018	2.216	1.543	0.964	0.454	0.000
	σ_θ	6.848	3.148	1.191	-0.143	-1.160	-1.985	-2.682	-3.287	-3.825	-4.308
	$\tau_{r\theta}$	0.989	0.485	0.303	0.210	0.152	0.110	0.077	0.048	0.023	0.000
$\theta = 60^\circ$	σ_r	10.413	6.806	5.000	3.788	2.865	2.114	1.478	0.926	0.437	0.000
	σ_θ	6.051	2.840	1.036	-0.227	-1.197	-1.983	-2.641	-3.208	-3.707	-4.151
	$\tau_{r\theta}$	1.865	0.895	0.527	0.344	0.238	0.168	0.117	0.074	0.036	0.000
$\theta = 90^\circ$	σ_r	9.392	6.028	4.424	3.383	2.595	1.945	1.380	0.875	0.417	0.000
	σ_θ	4.764	2.429	0.918	-0.243	-1.181	-1.953	-2.594	-3.132	-3.590	-3.985
	$\tau_{r\theta}$	2.524	1.233	0.656	0.354	0.192	0.108	0.065	0.039	0.020	-0.001
$\theta = 120^\circ$	σ_r	8.218	5.191	3.719	2.800	2.159	1.663	1.233	0.821	0.410	0.000
	σ_θ	2.991	1.871	0.945	0.037	-0.868	-1.733	-2.507	-3.156	-3.669	-4.059
	$\tau_{r\theta}$	2.739	1.563	0.848	0.348	0.020	-1.733	-2.507	-3.156	-3.669	-4.059
$\theta = 150^\circ$	σ_r	6.833	4.817	3.638	2.711	1.917	1.269	0.836	0.612	0.412	0.000
	σ_θ	1.016	0.863	0.718	0.504	0.122	-0.565	-1.676	-3.135	-4.559	-5.483
	$\tau_{r\theta}$	1.981	1.542	1.856	0.793	0.316	-0.233	-0.725	-0.912	-0.622	-0.003
$\theta = 175^\circ$	σ_r	4.851	4.557	4.439	4.396	4.398	4.407	4.291	3.494	0.095	-0.002
	σ_θ	0.033	0.036	0.041	0.052	0.074	0.119	0.213	0.337	-1.230	-33.12
	$\tau_{r\theta}$	0.380	0.384	0.403	0.442	0.511	0.628	0.804	0.881	-0.894	-0.124

Table 3.9: Stresses in the disc with tangential shear force S

Figure 3.18: Isochromatics with Tangential Shear Force

Figure 3.19: Stress σ_r , with Tangential Shear Force

Figure 3.20: Stress σ_θ with Tangential Shear Force

Figure 3.21: Stress $\tau_{r\theta}$ with Tangential Shear Force

		$K_{I(Tweed)}$	$K_{I(Gregory)}$	$K_{I(Vaughan)}$
Case I	$\frac{K_I}{K_0}$	3.14	3.17	3.17
Case II	$\frac{K_I}{\frac{F}{a}\sqrt{a}}$	-	-	2.5317
Case III	$\frac{K_I}{\frac{S}{a}\sqrt{a}}$	-	7.6917	7.6953

Table 3.10: Comparison with earlier results

$\cos(n\theta)$. Secondly, when the boundary forces are represented by Fourier series, and finally when the simultaneous algebraic equations are solved numerically, round off error occurs. However, it is obvious that these errors can be deduced by increasing n , the number of terms in the Fourier series. Among these three causes the second one is found to be dominant.

Chapter 4

Application to a More General Case

4.1 Motivations

In this chapter, the method is applied to a more practical and more important problem.

Once a crack initiates in a structure, a hole may be drilled at the crack tip to release the high stresses and to prevent crack propagation. This problem is now explored numerically. One motivation is to provide some theoretical justification to empirical experience.

Fig.(4.22) shows a circular disc of radius a_0 and thickness t with a radial crack extending outwards from the centre along the full radius $\theta = \pi$. There is a hole of radius a_i at the crack tip. The disc is loaded at its outer edge. There are no forces acting on the crack faces and the hole. The problem of determining the stress distribution in this disc is one of great importance. For instance, the solution provides some theoretical justification of the engineering practice of "stress relief", in which circular holes are drilled at the tip of the crack with the objective of reducing the stress singularity at the tip of the crack; the problem considered here would certainly be relevant when the diameter of the hole is small compared with the length of the crack.

Another motivation for considering a disc with a hole is that the method and results shown in Chapter 2 and Chapter 3 are for a mathematically ideal crack. When r tends to zero, the stresses σ_r and σ_θ are infinite, i.e. the stresses are singular at the crack

Figure 4.22: Disc with a hole at the crack tip

tip. The conclusion is only valuable for an ideal crack which has zero radius at the crack tip. However, in laboratory test specimens, cracks are often fabricated or introduced by cutting with a fine saw blade. The radius of the crack tip is then not zero. To this end, the solution is of much interest in engineering practice. When the hole is not too small, the problem can also be treated as a curved beam. For uniform tension, the solution is available from classical elasticity theory.

Approximate solutions of some special cases, such as the stress distribution in an infinite plate with straight radial cracks originating from the boundary of a circular hole and subjected to a prescribed load, have been derived by Morkovin (1944) [11] and Bueckner (1960), but the most complete discussion of the general case would appear to be that due to Bowie (1956) [10]; The analysis of Wigglesworth (1958) [12] is also relevant. The methods used involved mapping the problems to some other plane and then applying complex variable methods. The results give the stress distribution in an infinite plate which contains straight radial cracks originating from the boundary of a circular hole

Figure 4.23: Equivalent problem

and subjected to uniform tension. The references quoted above are important, but the methods become prohibitive in cases of more complicated boundary conditions. Also, they are very difficult to apply to a finite plate problem. This is a major weakness since, as mentioned before, the circular disc with a radial edge crack has been chosen as a new standard specimen for fracture tests. In this chapter a more elegant and general method is applied to this practical problem. Two methods are used in this regard for uniform tension. First the disc is treated as a curved beam using elastical theory; secondly the method developed in Chapter 2 is used. The results are compared and the stress σ_θ at $r = a_i$ and $\sigma = 0$ are obtained for various ratios of $\frac{a_i}{a_0}$. The case of a disc with diametrical concentrated force is also studied, and the results are presented. Finally, some conclusions and recommendations are given.

4.2 The curved beam solution in classical elasticity

When the hole is not too small, the problem is similar to the one shown in Fig.(4.23). The stress, σ_θ , at $\theta = 0$ and $r = a_i$ can be derived by superposition of three different cases shown in Fig.(4.24). Values of T_1 , F_1 and M_1 are calculated from the equilibrium conditions as follow:

$$\begin{aligned} T_1 &= T_0 a_0 t \\ F_1 &= T_0 a_0 t \\ M_1 &= \frac{1}{2} T_0 a_0 t (a_0 + a_i) \end{aligned} \quad (4.36)$$

where T_0 is the pressure on the perimeter, F_1 is the total force and M_1 is the total moment on each end about the middle point of the wall. Theoretical solutions for each case exist.

The first case is the same as the cylinder under the uniform tension at the outer boundary and free on the inner boundary as shown in Fig.(4.24)a. The stress function is thus

$$\Phi = A \ln r + B r^2 \ln r + C r^2 + D \quad (4.37)$$

the stresses are

$$\begin{aligned} \sigma_r &= \frac{1}{r} \frac{\partial \Phi}{\partial r} = \frac{A}{r^2} + B(1 + 2 \ln r) + 2C \\ \sigma_\theta &= \frac{\partial^2 \Phi}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2 \ln r) + 2C \\ \tau_{r\theta} &= 0 \end{aligned} \quad (4.38)$$

With the following boundary conditions

Figure 4.24: Supoposition of the three problems

$$\sigma_r|_{r=a_i} = 0 \quad \sigma_r|_{r=a_0} = T_0 \quad (4.39)$$

Substituting eqn (4.38) into (4.39), the stresses are solved as below

$$\begin{aligned} \sigma_r &= -\frac{a_0^2 a_i^2 T_0}{a_0^2 - a_i^2} \frac{1}{r^2} + \frac{T_0 a_0^2}{a_0^2 - a_i^2} \\ \sigma_\theta &= \frac{a_0^2 a_i^2 T_0}{a_0^2 - a_i^2} \frac{1}{r^2} + \frac{T_0 a_0^2}{a_0^2 - a_i^2} \\ \tau_{r\theta} &= 0 \end{aligned}$$

At $\theta = 0, r = a_i$, the value of σ_θ is

$$\sigma_\theta = \frac{2a_0^2}{a_0^2 - a_i^2} T_0 \quad (4.40)$$

The second case is the pure bending of the curved bar. The stress function is the same as the first case. The boundary conditions are

$$\begin{aligned} \sigma_r &= 0 \quad \text{for } r = a_i \quad \text{and } r = a_0 \\ \int_{a_i}^{a_0} \sigma_\theta dr &= 0 \quad \int_{a_i}^{a_0} \sigma_\theta r dr = -M \\ \tau_{r\theta} &= 0 \quad \text{at the boundaries} \end{aligned}$$

Solving the equations the stresses are

$$\begin{aligned} \sigma_r &= -\frac{4M}{N} \left(\frac{a_i^2 a_0^2}{r^2} \ln \frac{a_0}{a_i} + a_0^2 \ln \frac{r}{a_0} + a_i^2 \ln \frac{a_i}{a_0} \right) \\ \sigma_\theta &= -\frac{4M}{N} \left(-\frac{a_i^2 a_0^2}{r^2} \ln \frac{a_0}{a_i} + a_0^2 \ln \frac{r}{a_0} + a_i^2 \ln \frac{a_i}{r} + a_0^2 - a_i^2 \right) \\ \tau_{r\theta} &= 0 \end{aligned}$$

where

$$\begin{aligned}
 N &= (a_0^2 - a_i^2)^2 - 4a_i^2 a_0^2 \left(\ln \frac{a_0}{a_i}\right)^2 \\
 A &= -\frac{4M}{N} a_i^2 a_0^2 \ln \frac{a_0}{a_i} \\
 B &= -\frac{2M}{N} (a_0^2 - a_i^2) \\
 C &= \frac{M}{N} [a_0^2 - a_i^2 + 2(a_0^2 \ln a_0 - a_i^2 \ln a_i)]
 \end{aligned}$$

The stress σ_θ at $\theta = 0, r = a_i$ is

$$\sigma_\theta = -\frac{4M}{N} \left(-2a_0^2 \ln \frac{a_0}{a_i} + a_0^2 - a_i^2\right) \quad (4.41)$$

The third case is bending of a curved bar by a force at the end.

The stress function is defined as

$$\Phi = \left(Ar^3 + \frac{B}{r} + Cr + Dr \ln r\right) \sin \theta \quad (4.42)$$

and the stresses are as follows

$$\begin{aligned}
 \sigma_r &= \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right) \sin \theta \\
 \sigma_\theta &= \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right) \sin \theta \\
 \tau_{r\theta} &= -\left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right) \cos \theta
 \end{aligned} \quad (4.43)$$

with the following boundary conditions

$$\sigma_r = \tau_{r\theta} = 0 \quad \text{for} \quad r = a_i \quad \text{and} \quad r = a_0$$

$$\int_{a_i}^{a_0} \tau_{r\theta} dr = -T_1 \quad \text{for } \theta = 0 \quad \text{and } \theta = \pi$$

Solving these equations for A, B, C and D

$$A = \frac{P}{2N}$$

$$B = -\frac{P a_i^2 a_0^2}{2N}$$

$$D = -\frac{P}{N}(a_i^2 - a_0^2)$$

$$N = a_i^2 - a_0^2 + (a_i^2 + a_0^2) \ln \frac{a_0}{a_i}$$

The stress σ_θ at $\theta = 0, r = a_i$ is thus

$$\sigma_\theta = -\frac{T_0}{N} 2a_i - \frac{2a_0^2}{a_i} \quad (4.44)$$

The numerical value of σ_θ at $\theta = 0$ and $r = a_i$ for different a_i are shown in Table 4.13.

4.3 Solution using the proposed method

The same disc as shown in Fig.(4.22) is considered where a_0 is outer radius, a_i is the radius of the hole. The disc is loaded at its outer edge by self-equilibrating surface tractions $T(\theta)$ and $S(\theta)$ where T is the normal force and S is a shear force. The stress function is assumed to be the same as section 2.1 except that m and n include negative numbers.

$$\begin{aligned}\Phi &= \sum_{m=m_1}^{\infty} a_m r^m [m \cos(m-2)\theta - (m-2) \cos m\theta] \\ &+ \sum_{n=n_1}^{\infty} b_n r^n [\cos(n-2)\theta - \cos n\theta] \quad -\pi \leq \theta \leq \pi \quad a_i \leq r \leq a_o \quad (4.45)\end{aligned}$$

where $m_1 = -\frac{2k+1}{2}$, k positive integer, n_1 is a negative integer, $b_1 = 0$. For a complete mathematical solution, k and n_1 are both infinite. However, we terminate the series by choosing finite values for k and n_1 . The actual values depend on the boundary conditions prevailing, and are obtained by examining the convergence of the numerical results.

Take $a_{m_1-2} = a_{m_1-1} = 0$ and $b_{n_1-1} = b_{n_1-2} = 0$, Φ may be rewritten as follows

$$\begin{aligned}\Phi &= \sum_{m=m_1-2}^{\infty} \cos(m\theta) [-a_m r^m (m-2) + (m+2) a_{m+2} r^{m+2}] \\ &+ \sum_{n=n_1-2}^{\infty} \cos(n\theta) [-b_n r^n + b_{n+2} r^{n+2}] \quad (4.46)\end{aligned}$$

The stresses are given as follow

$$\begin{aligned}\sigma_r &= \sum_{m=m_1-2}^{\bar{M}} \cos(m\theta) [a_m r^{m-2} (m-2)m(m-1) - a_{m+2} r^m (m+2)(m+1)(m-2)] \\ &+ \sum_{n=n_1-2}^{\bar{M}} \cos(n\theta) [b_n r^{n-2} n(n-1) - b_{n+2} r^n (n+1)(n-2)] \\ \sigma_\theta &= \sum_{m=m_1-2}^{\bar{N}} \cos(m\theta) [-m(m-1)(m-2) a_m r^{(m-2)} + (m+2)^2 (m-1) a_{m+2} r^m] \\ &+ \sum_{n=n_1-2}^{\bar{N}} \cos(n\theta) [-n(n-1) b_n r^{n-2} + (n+2)(n+1) b_{n+2} r^n] \quad (4.47) \\ \tau_{r\theta} &= \sum_{m=m_1-2}^{\bar{M}} m \sin(m\theta) [-(m-1)(m-2) r^{m-2} a_m + (m+1)(m+2) r^m a_{m+2}] \\ &+ \sum_{n=n_1-2}^{\bar{N}} n \sin(n\theta) [-(n-1) b_n r_{n-2} + (n+1) b_{n+2} r^n]\end{aligned}$$

The same Fourier series are used to write $\cos(m\theta)$ and $\sin(m\theta)$ in terms of $\cos(n\theta)$ and $\sin(n\theta)$ as in Chapter 2. Also, the boundary tractions are written in Fourier series. Then after the same derivitions the final set of equation are derived as follow:

$$\begin{aligned}
& \sum_{m=m_1-2}^M [a_m r^{m-2} (m-2)(m-1)m - a_{m+2} r^m (m-2)(m+1)(m+2)] \frac{\sin(m\pi)}{m\pi} + 2b_2 = t_0 \\
& \sum_{m=m_1-2}^M [a_m r^{m-2} (m-2)(m-1)m - a_{m+2} r^m (m-2)(m+1)(m+2)] \frac{2}{\pi} \sin(m\pi) (-1)^{n+1} \frac{m}{n^2 - m^2} \\
& b_n r^{n-2} n(n-1) - b_{n+2} r^n (n+1)(n-2) + n(n+1) b_{-n} r^{-n-2} + (n+2)(-n+1) b_{-n+2} r^{-n} = t_n \\
& \sum_{m=m_1-2}^M [-(m-1)(m-2) r^{m-2} a_m + (m+1)(m+2) r^m a_{m+2}] \frac{2m}{\pi} \frac{(-1)^{n+1} \sin(m\pi)}{n^2 - m^2} \\
& + (n+1) b_{-n} r^{-n-2} - (n-1) b_{-n+2} r^{-n} - (n-1) b_n r^{n-2} + (n+1) b_{n+2} r^n = 0
\end{aligned} \tag{4.48}$$

Where $a_{m_1-2} = a_{m_1-1} = b_{n_1-2} = b_{n_1-1} = b_{n_1-2} = 0$ and

$$t_0 = t_0^0 \quad t_n = t_n^0 \quad s_n = s_n^0 \quad \text{when} \quad r = a_0$$

$$t_0 = t_0^i \quad t_n = t_n^i \quad s_n = s_n^i \quad \text{when} \quad r = a_i$$

t_0^0, t_n^0 and s_n^0 are the coefficients in the Fourier series for the outer boundary, t_0^i, t_n^i and s_n^i are the coefficients of the Fourier series for the inner boundary. The special case of uniform tension on the outer boundary is now considered. Thus

$$t_0^0 = 2T_0 \quad t_n^0 = s_n^0 = 0$$

$$t_0^i = t_n^i = s_n^i = 0$$

The special case of a disc with diametrical concentrated force is also considered and the coefficients are shown in Appendix D.

$\hat{a}_{-\frac{9}{2}}$	$\hat{a}_{-\frac{7}{2}}$	$\hat{a}_{-\frac{5}{2}}$	$\hat{a}_{-\frac{3}{2}}$	$\hat{a}_{-\frac{1}{2}}$	$\hat{a}_{\frac{1}{2}}$	$\hat{a}_{\frac{3}{2}}$	$\hat{a}_{\frac{5}{2}}$	$\hat{a}_{\frac{7}{2}}$	$\hat{a}_{\frac{9}{2}}$
0.000	0.000	-0.0013	0.150	18.011	-409.105	2232.3	-807.32	-36.30	3.1306
$\hat{b}_{-4}$	$\hat{b}_{-3}$	$\hat{b}_{-2}$	$\hat{b}_{-1}$	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$
-0.0003	0.0055	1.7992	36.885	0.0000	672.003	191.91	-2.8605	0.8460	-0.1114

Table 4.11: The important coefficients under the uniform tension (scaled by 10^{-3})

4.4 Numerical results

Disc under uniform tension

a_m and b_n are replaced by $\hat{a}_m$ and $\hat{b}_n$ in the same way as before:

$$a_m = \frac{\hat{a}_m T_0}{a_0^{m-2} t}$$

$$b_n = \frac{\hat{b}_n T_0}{a_0^{n-2} t}$$

A 66×66 matrix was used to solve the equations. Some important coefficients are shown in Table (4.11). Table (4.12) shows how well the boundary conditions are satisfied.

Fig.(4.25) shows $\frac{\sigma_\theta}{T_0}$ at $r = a_i, \theta = 0$, for the range of $\frac{a_i}{a_0}$ from 10^{-5} to 0.95. Fig.(4.26) is the stress σ_θ for $\theta = 0$. Table (4.13) compares the stress at $r = a_i$ and $\theta = 0$ obtained from curved beam theory with those obtained using the method derived in the thesis. Good agreement can be seen.

Disc with diametrical concentrated force

a_m and b_n are replaced by $\hat{a}_m$ and $\hat{b}_n$ in the following form:

$$a_m = \frac{\hat{a}_m T}{a_0^{m-1} t}$$

$\frac{r}{a_0}$	θ	$(\frac{\sigma_r}{T_0})_{cal}$	$(\frac{\sigma_r}{T_0})_{ideal}$	$(\frac{\tau_{r\theta}}{T_0})_{cal}$	$(\frac{\tau_{r\theta}}{T_0})_{ideal}$
1.0	0.00	0.99974	1.000	0.00000	0.000
	30.00	1.00016	1.000	-0.00019	0.000
	60.00	1.00009	1.000	0.00028	0.000
	90.00	0.99963	1.000	-0.00017	0.000
	120.00	1.00054	1.000	-0.00024	0.000
	150.00	0.99975	1.000	0.00118	0.000
	170.00	1.00155	1.000	0.00243	0.000
0.1	0.00	0.00155	0.000	0.00000	0.000
	30.00	-0.00103	0.000	-0.00033	0.000
	60.00	0.00006	0.000	0.00070	0.000
	90.00	0.00030	0.000	-0.00111	0.000
	120.00	0.00002	0.000	0.00143	0.000
	150.00	0.00066	0.000	-0.00009	0.000
	170.00	-0.00415	0.000	0.00812	0.000

Table 4.12: Stresses on the boundaries

$\frac{a_i}{a_0}$	0.05	0.10	0.20	0.30	0.40	0.5
$\sigma_\theta(c.b)$	33.4830	27.2005	26.2219	29.9934	37.7409	51.6981
$\sigma_\theta(v.w)$	32.8785	26.6839	25.6424	29.4010	37.1403	51.1109
$\frac{a_i}{a_0}$	0.55	0.60	0.70	0.80	0.90	0.95
$\sigma_\theta(c.b)$	62.6871	78.1968	136.1550	302.5295	1202.32	4802.36
$\sigma_\theta(v.w)$	62.1084	77.6275	135.6084	302.0283	1202.14	4802.36

Table 4.13: The stress σ_θ at $\theta = 0$ and $r = a_i$ obtained using different methods

Figure 4.25: Stress σ_θ at $\theta = 0$, $r = a_i$

Figure 4.26: Stress σ_θ at $\theta = 0$

$\hat{a}_{-\frac{9}{2}}$	$\hat{a}_{-\frac{7}{2}}$	$\hat{a}_{-\frac{5}{2}}$	$\hat{a}_{-\frac{3}{2}}$	$\hat{a}_{-\frac{1}{2}}$	$\hat{a}_{\frac{1}{2}}$	$\hat{a}_{\frac{3}{2}}$	$\hat{a}_{\frac{5}{2}}$	$\hat{a}_{\frac{7}{2}}$	$\hat{a}_{\frac{9}{2}}$
0.000	0.000	0.0145	1.344	73.112	-806.642	-123.40	-102.05	-62.733	49.012
$\hat{b}_{-4}$	$\hat{b}_{-3}$	$\hat{b}_{-2}$	$\hat{b}_{-1}$	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$
0.0000	0.0025	0.1356	1.747	0.0000	-149.178	72.059	312.722	1.850	-310.31

Table 4.14: The important coefficients with diametrical force (scaled by 10^{-3})

$$b_n = \frac{\hat{b}_n T}{a_0^{n-1} t}$$

A 600×600 matrix was used to solve the equations. Some important coefficients are shown in Table (4.14). Table (4.15) shows how well the boundary conditions are satisfied.

Fig. (4.27) shows $\frac{\sigma_\theta}{\sigma_0}$ at $\theta = 0$ and $r = a_i$, for the range of $\frac{a_i}{a_0}$ from 0.025 to 0.95.

4.5 Discussion

From Fig.(4.25), Fig.(4.26) and Fig.(4.27), it is seen that for small holes, $(\frac{a_i}{a_0}) \leq 0.16$, stress σ_θ at $\theta = 0, r = a_i$ increases as the $\frac{a_i}{a_0}$ decreases. It means the hole should not be too small when it is drilled to release the stress singularity. Also, when $\frac{a_i}{a_0} \geq 0.16$, σ_θ at $\theta = 0$ and $r = a_i$ increases as $\frac{a_i}{a_0}$ rises. It indicates that the property of the stresses are governed primarily by the local stress field at the hole. Hence, to prevent the propagation of the crack, the ratio of $\frac{a_i}{a_0}$ should be between 0.13–0.26. The minimum stress occurs when $\frac{a_i}{a_0}$ is 0.16 for both cases of uniform tension and diametrical force. When $\frac{a_i}{a_0}$ is greater than 0.26, the stress concentration will also be high. The same conclusion can be made from Fig.(4.27), the case of diametrical concentrated force.

Table 4.13 shows good agreement between the data using the curved beam theory and the proposed method for uniform tension. Consequently this problem can be treated

$\frac{r}{a_0}$	θ	$(\frac{\sigma_r}{T_0})_{cal}$	$(\frac{\sigma_r}{T_0})_{ideal}$	$(\frac{\tau_{r\theta}}{T_0})_{cal}$	$(\frac{\tau_{r\theta}}{T_0})_{ideal}$
1.0	0.00	0.00001	0.00000	0.00000	0.000
	30.00	0.00332	0.00082	-0.00262	0.000
	60.00	0.00013	0.00039	0.00414	0.000
	80.00	-0.02128	-0.01906	-0.00043	0.000
	90.00	10.57990	10.58377	0.00360	0.000
	100.00	-0.01387	-0.01906	0.00032	0.000
	120.00	-0.00776	0.00039	0.00009	0.000
	150.00	-0.01329	0.00082	-0.00840	0.000
	170.00	0.02961	-0.00070	-0.01771	0.000
	180.00	0.00001	0.00000	0.00000	0.000
0.1	0.00	0.00001	0.000	0.00000	0.000
	30.00	0.00000	0.000	0.00000	0.000
	60.00	0.00000	0.000	0.00001	0.000
	90.00	0.00001	0.000	-0.00001	0.000
	120.00	0.00002	0.000	-0.00001	0.000
	150.00	0.00002	0.000	-0.00004	0.000
	170.00	-0.00014	0.000	0.00001	0.000

Table 4.15: Stresses on the boundaries

Figure 4.25: Stress $\frac{\sigma_\theta}{\sigma_0}$ at $\theta = 0, r = a_i$

as a curved beam when the hole is not too small.

However, the method derived in the thesis is more general since it can deal with any kind of boundary conditions with good accuracy. Also the cases of greatest practical use are when the loads are concentrated, as shown in Figure 3.10 and 3.17. Curved beam theory cannot handle these cases.

Chapter 5

Concluding Remarks

A fairly simple method has been derived which gives the stresses in a disc with a radial crack for any distribution of surface forces. Except for the transform method of Tweed [1-7] and Gregory [8,9], this work is original.

The two cases presented for diametric concentrated forces, particularly the tangential shear forces acting at the mouth of the crack, provides us with a new experimental test for studying fracture mechanics in a very controlled way. Many problems which could not be solved well before, can now be solved easily with the method shown in the thesis.

Closed-form solutions for elastic bodies containing a crack are sparse, particularly for bodies of finite dimension. The solutions obtained here for circular discs add to the few available solutions. It is valuable in not only providing additional theory but also because it is of practical importance, being a representation of a "model problem". The two solutions involving concentrated loads are particularly useful for examining fracture strength since specimens may easily be machined and the load can be accurately and simply controlled. In this regard the solutions provide a useful test for comparative studies. The results are also particularly adapted to photoelastic methods in which isochromatics similar to those given graphically may be obtained.

It has been noted that the method will certainly give accurate values for the stress intensity factor and the stresses in the disc. It was also shown that the effect of relatively small differences in the loading area is negligible so that these formula are useful in the

calibration of a fracture test specimen.

The results for the case in which a hole has been drilled at the crack tip are new. The graphs of stress cocentration factors versus $\frac{a_i}{a_o}$ are of practical importance and significance. It is shown that there is an optimum ratio of $\frac{a_i}{a_o}$ for stress relief in a disc but that as $a_i \rightarrow 0$, the stress concentration factor becomes unbounded.

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Appendix A

First mode linking between shear and normal stress

In processing the algebraic equations, it was observed that when $n = 1$, the boundary condition on $\tau_{r\theta}$ and that on σ_r yield identical equations. The reason for this unexpected result is now examined and explained.

$$\tau_{r\theta} = f_1(\theta), \quad \sigma_r = f_2(\theta), \quad r = a \quad -\pi \leq \theta \leq \pi$$

Since $T(\theta)$ and $S(\theta)$ keep self equilibrium

$$\begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum M_0 &= 0 \end{aligned} \tag{A.49}$$

where F_x and F_y are the resultant forces in x and y directions and M_0 is the moment about O as shown in Fig.(A.28). Thus,

$$\begin{aligned} \int_{-\pi}^{\pi} (\sigma_r \cos\theta - \tau_{r\theta} \sin\theta) a d\theta &= 0 \quad (\sum F_x = 0) \\ \int_{-\pi}^{\pi} (\sigma_r \sin\theta + \tau_{r\theta} \cos\theta) a d\theta &= 0 \quad (\sum F_y = 0) \\ \int_{-\pi}^{\pi} (\tau_{r\theta} a) a d\theta &= 0 \quad (\sum M_0 = 0) \end{aligned} \tag{A.50}$$

Expanding the integral

Figure A.28: Disc under equilibrating force system

$$\begin{aligned} \int_{-\pi}^{\pi} \sigma_r \sin \theta d\theta + \int_{-\pi}^{\pi} \tau_{r\theta} \cos \theta d\theta &= 0 \\ \int_{-\pi}^{\pi} \tau_{r\theta} d\theta &= 0 \end{aligned} \quad (\text{A.51})$$

σ_r and $\tau_{r\theta}$ are written in Fourier series as follow,

$$\sigma_r = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos n\theta + \gamma_n \sin n\theta)$$

$$\tau_{r\theta} = \bar{\alpha}_0 + \sum_{n=1}^{\infty} (\bar{\alpha}_n \cos n\theta + \bar{\gamma}_n \sin n\theta)$$

$$\alpha_1 = \int_{-\pi}^{\pi} \sigma_r \cos \theta d\theta$$

$$\bar{\alpha}_1 = \int_{-\pi}^{\pi} \tau_{r\theta} \sin \theta d\theta$$

and from (A.51) the following is concluded

$$\alpha_1 = \bar{\alpha}_1$$

Appendix B

Normal force requirement for crack loaded by tangential shears

In Chapter 3, it is mentioned that to maintain equilibrium, there must be some normal forces acting on the boundary in addition to the active shear forces. These normal forces are now evaluated. A disc in which the crack is opened by two self equilibrating forces, F_τ , applied at the mouth of the crack as shown in Fig.(B.29) is considered. The surface forces are expressed in terms of the surface stresses by

$$F_\tau = \int_{\pi-\alpha}^{\pi} at\tau_{r\theta}d\theta$$

$$F_\sigma = \int_{\pi-\alpha}^{\pi} at\sigma_r d\theta$$

There is to be no resultant force in the x direction, hence

$$\int_{\pi-\alpha}^{\pi} at\tau_{r\theta}\sin\theta d\theta = \int_{\pi-\alpha}^{\pi} at\sigma_r\cos\theta d\theta \quad (B.52)$$

$\tau_{r\theta}$ is defined in Chapter 3 as follow:

$$\tau_{r\theta} = -\frac{K}{2} \frac{P_0}{at} \sin K\theta$$

$$F_\tau \sin\theta = \int_{\pi-\alpha}^{\pi} \tau_{r\theta} \sin\theta d\theta$$

$$= -\frac{K}{2} \frac{P_0}{at} \int_{\pi-\frac{\pi}{K}}^{\pi} \sin K\theta \sin\theta d\theta$$

Figure B.29: Disc with tangential shear forces

$$\begin{aligned}
 &= \frac{K}{4} \frac{P_0}{at} \int_{\pi - \frac{\pi}{K}}^{\pi} \cos(K+1)\theta - \cos(K-1)\theta d\theta \\
 &= \frac{K}{4} \frac{P_0}{at} \frac{1}{K+1} \sin(K+1)\theta \Big|_{\pi - \frac{\pi}{K}}^{\pi} - \frac{1}{K-1} \sin(K-1)\theta \Big|_{\pi - \frac{\pi}{K}}^{\pi} \\
 &= \frac{K}{4} \frac{P_0}{at} \frac{1}{K+1} [-\sin \frac{\pi}{K}] - \frac{1}{K-1} \sin \frac{\pi}{K} \\
 &= -\frac{K}{4} \frac{P_0}{at} \left(\frac{1}{K+1} \sin \frac{\pi}{K} + \frac{1}{K-1} \sin \frac{\pi}{K} \right)
 \end{aligned}$$

Assuming $\sigma_r = A \frac{P_0}{at} \sin K\theta$

$$\begin{aligned}
 F_\sigma \cos \theta &= \frac{1}{2} A \frac{P_0}{at} \int_{\pi - \frac{\pi}{K}}^{\pi} \sin(K+1)\theta + \sin(K-1)\theta d\theta \\
 &= -\frac{1}{2} A \frac{P_0}{at} \frac{1}{K+1} \pi \sin(K+1)\theta \Big|_{\pi - \frac{\pi}{K}}^{\pi} - \frac{1}{K-1} \cos(K-1)\theta \Big|_{\pi - \frac{\pi}{K}}^{\pi}
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}A - \frac{1}{K+1} \left[1 - \cos(K-1)\pi + \frac{K-1}{K}\pi \right] - \frac{1}{K-1} \left[1 - \cos(K-1)\pi - \frac{K-1}{K}\pi \right] \\
&\quad - \frac{1}{2}A - \frac{1}{K+1} \left[1 + \cos\frac{\pi}{K} \right] - \frac{1}{K-1} \left[1 + \cos\frac{\pi}{K} \right] \\
&= A \left(1 + \cos\frac{\pi}{K} \right) \frac{K}{(K-1)(K+1)}
\end{aligned}$$

From equation (B.52), A is given as

$$A = \frac{1}{2}K \frac{\sin\frac{\pi}{K}}{1 + \cos\frac{\pi}{K}}$$

$$\sigma_r = \frac{K}{2} \frac{P_0}{at} \sin K\theta \frac{\sin\frac{\pi}{K}}{1 + \cos\frac{\pi}{K}}$$

Appendix C

It is discribed in Chapter 3 that T is the resultant of the distribution of $T(\theta)$ in the case of diametric concentrated force.

$$T(\theta) = \sigma_r t.$$

σ_r is as follow

$$\sigma_r = \frac{K}{2} \frac{T}{at} \sin K\theta$$

In Chapter 3 T is assuming as follow

$$T = \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \sigma_r a t d\theta$$

However, the real resultant force, F , is supposed to be

$$F = \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \sigma_r \sin\theta a t d\theta$$

Since α is small, the error between T and F is very low as

$$\begin{aligned} I &= T - \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \frac{K}{2} \frac{T}{at} \sin K\theta \sin\theta a t d\theta \\ &= T - \frac{K}{2} T \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \sin K\theta \sin\theta d\theta \\ &= T + \frac{K}{2} T \left[\frac{1}{K+1} \sin(K+1)\theta \Big|_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} - \frac{1}{K-1} \sin(K-1)\theta \Big|_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \right] \end{aligned}$$

$$\begin{aligned}
&= T + \frac{K}{2} T \left[\frac{1}{K+1} \cos \frac{(K+1)\pi}{2} \sin \frac{(K+1)\pi}{2K} - \frac{1}{K-1} \cos \frac{(K-1)\pi}{2} \sin \frac{(K-1)\pi}{2K} \right] \\
&= T \left[1 + \frac{K}{2} \left[-\frac{1}{K+1} \cos \frac{\pi}{2K} - \frac{1}{K-1} \cos \frac{\pi}{2K} \right] \right] \\
&= T \left[1 - \frac{K^2}{(K+1)(K-1)} \cos \frac{\pi}{2K} \right]
\end{aligned}$$

Suppose E as the error

$$E = \frac{I}{T} \times 100 \%$$

$$E < 0.017 \% \quad \text{When } K = 37$$

$$E < 0.053 \% \quad \text{When } K = 21$$

Appendix D

In this part, the coefficients in Fourier series of the normal stress defined in Chapter 3 are derived. The normal force σ_r is as

$$\sigma_r = \frac{T}{at} \frac{K}{2} \sin K\theta$$

The Fourier series is as follow:

$$\sigma_r = \frac{T}{at} \left(\sum_{n=1}^{\infty} t_n \cos n\theta + \frac{t_0}{2} \right)$$

$$t_n = \frac{2}{\pi} \frac{at}{T} \int_0^{\pi} \sigma_r \cos n\theta d\theta$$

$$t_0 = \frac{2}{\pi} \frac{at}{T} \int_0^{\pi} \sigma_r d\theta$$

$$t_0 = \frac{2}{\pi} \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \frac{K}{2} \sin K\theta d\theta$$

$$= \frac{1}{\pi} \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \sin K\theta dK\theta$$

$$= \frac{1}{\pi} (-\cos K\theta) \Big|_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}}$$

$$= -\frac{1}{\pi} \left[\cos\left(\frac{K}{2}\pi + \frac{\pi}{2}\right) - \cos\left(\frac{K}{2}\pi - \frac{\pi}{2}\right) \right]$$

$$= \frac{2}{\pi} \sin \frac{K}{2} \pi = \frac{2}{\pi}$$

For $n \neq K$

$$\begin{aligned}
 t_n &= \frac{2}{\pi} \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \frac{K}{2} \sin K \theta \cos n \theta d\theta \\
 &= \frac{K}{\pi} \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \frac{1}{2} [\sin(K+n)\theta + \sin(K-n)\theta] d\theta \\
 &= \frac{K}{2\pi} \left[\frac{1}{K+n} \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \sin(K+n)\theta d(K+n)\theta + \frac{1}{K-n} \int_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \sin(K-n)\theta d(K-n)\theta \right] \\
 &= \frac{K}{2\pi} \left[\frac{1}{K+n} (-\cos(K+n)\theta) \Big|_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} + \frac{1}{K-n} (-\cos(K-n)\theta) \Big|_{\frac{\pi}{2} - \frac{\pi}{2K}}^{\frac{\pi}{2} + \frac{\pi}{2K}} \right] \\
 &= \frac{K}{2\pi} \left(-\frac{1}{K+n} \left[\cos(K+n) \frac{K+1}{2K} \pi - \cos(K+n) \frac{K-1}{2K} \pi \right] \right. \\
 &\quad \left. - \frac{1}{K-n} \left[\cos(K-n) \frac{K+1}{2K} \pi - \cos(K-n) \frac{K-1}{2K} \pi \right] \right) \\
 &= \frac{K}{\pi} \left[\frac{1}{K+n} \sin \frac{K+n}{2} \pi \sin \frac{K+n}{2K} \pi + \frac{1}{K-n} \sin \frac{K-n}{2} \pi \sin \frac{K-n}{2K} \pi \right]
 \end{aligned}$$

$$t_K = 0$$

Also, as shown in section 3.3, the σ_r can be written as

$$\sigma_r = \frac{S_0}{at} \frac{K}{2} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} \sin K \theta$$

The coefficients of Fourier series are thus

$$\begin{aligned}
 t_0 &= \frac{2}{\pi} \int_{\frac{K-1}{K}\pi}^{\pi} \frac{K}{2} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} \sin K\theta d\theta \\
 &= \frac{K}{\pi} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} \int_{\frac{K-1}{K}\pi}^{\pi} \sin K\theta d\theta \\
 &= \frac{1}{\pi} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} (-\cos K\theta) \Big|_{\frac{K-1}{K}\pi}^{\pi} \\
 &= -\frac{1}{\pi} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} (\cos K\pi - \cos(K-1)\pi) \\
 &= \frac{2}{\pi} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} \\
 t_n &= \frac{2}{\pi} \int_0^{\pi} \frac{K}{2} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} \sin K\theta \cos n\theta d\theta \\
 &= \frac{K}{\pi} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} \int_{\frac{K-1}{K}\pi}^{\pi} \frac{1}{2} [\sin(K+n)\theta + \sin(K-n)\theta] d\theta \\
 &= \frac{K}{2\pi} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} \left[-\frac{1}{K+n} \cos(K+n)\theta \Big|_{\frac{K-1}{K}\pi}^{\pi} - \frac{1}{K-n} \cos(K-n)\theta \Big|_{\frac{K-1}{K}\pi}^{\pi} \right] \\
 &= \frac{K}{2\pi} \frac{\sin \frac{\pi}{K}}{1 + \frac{\pi}{K}} \left(-\frac{1}{K+n} [\cos(K+n)\pi - \cos(K+n)\frac{K-1}{K}\pi] - \frac{1}{K-n} [\cos(K-n)\pi - \cos(K-n)\frac{K-1}{K}\pi] \right) \\
 t_K &= 0
 \end{aligned}$$

Appendix E

In this part, the coefficients in Fourier series of the tangential shear force acting on the mouth of the crack, as shown in section 3.3, are derived. Stress $\tau_{r\theta}$ is defined as

$$\tau_{r\theta} = -\frac{K}{2} \frac{S_0}{at} \sin K\theta$$

or in term of Fourier series

$$\tau_{r\theta} = \frac{S}{at} \sum_{n=1}^{\infty} s_n \sin n\theta$$

for $n \neq K$

$$\begin{aligned} s_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} -\frac{K}{2} \sin K\theta \sin n\theta d\theta \\ &= -\frac{K}{\pi} \int_{\frac{K-1}{K}\pi}^{\pi} -\frac{1}{2} [\cos(K+n)\theta - \cos(K-n)\theta] d\theta \\ &= \frac{K}{2\pi} \left[\frac{1}{K+n} \sin(K+n)\theta \Big|_{\frac{K-1}{K}\pi}^{\pi} - \frac{1}{K-n} \sin(K-n)\theta \Big|_{\frac{K-1}{K}\pi}^{\pi} \right] \\ &= \frac{K}{2\pi} \left[-\frac{1}{K+n} \sin \frac{(K+n)(K-1)}{K} \pi + \frac{1}{K-n} \sin \frac{(K-n)(K-1)}{K} \pi \right] \\ s_K &= -\frac{1}{2} \end{aligned}$$