CITY SIZE DISTRIBUTIONS: FOUNDATIONS OF ANALYSIS

by

GORDON FREDRICK MULLIGAN

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ABSTRACT

While many observers recognize the significance of the city size distribution topic, the resolution of several apparent inconsistencies in the body of literature has not yet been achieved. This may explain why geographers, sociologists, demographers, historians, economists, and planners essentially tend to describe intercity patterns, are biased toward ad hoc interpretations, and are prone to making intuitive statements in their research.

The primary purpose of this thesis is to evolve a more consistent methodological viewpoint within the community size topic. Efforts are made to unite analytical statements resting upon a common premise, to qualify, in this light, the approaches prevalent in empirical research, and to relate theory and empiricism by adopting a flexible explanatory framework. The discussion necessarily involves a critique of existing arguments and certain extensions that we can devise from those arguments. While there is considerable attention directed to presenting empirical methodologies, no original data analysis is included.

Contending that the notions should be bound together within a systems framework, we naturally devote initial emphasis to the features of central place systems

as outlined in the partial equilibrium theory of Christaller (1966) and Losch (1954). We place particular stress upon the Christaller model, the simpler and apparently more realistic of the two approaches.

A major thrust of the paper is an integration of several city size models, all of which display a Christallarian hierarchy. The simplest models are shown to be special cases of a more general formulation given by Dacey (1966). Besides, we illustrate to what degree the characteristic property (that is, the constant proportionality factor) of the most elementary model (Beckmann, 1958) may be considered a limit of empirical generalization.

Using the hierarchial concept, we also provide some rather novel views on the relation between community economic base and the distribution theme. It is felt that this subtopic may be useful in bridging the intra- and interurban scales.

The widely expounded rank-size rule, essentially a consequence of empirical research, is then formally attached to the hierarchial models. At this stage our arguments become increasingly rigorous in order to qualify certain intuitive notions that seem accepted in the literature. The idea of hierarchial sets is crudely developed to complement the uni-hierarchy arguments. The basic conclusion here is that existing city size models hardly explain the rank-size phenomenon but that the two notions cannot be considered totally incompatible.

Empirical research methodologies are stressed as another fundamental subtopic. We suggest certain avenues along which empirical efforts must be strengthened before either (i) rigorous inductive generalizations or (ii) firm theory substantiation become more realizable. Particular attention is given to delimitation of the study area (and, therefore to the scale problem), the comparison of frequency curves, and the value of inferences we can make using rather crude statistical tools. At this stage we introduce other skew distributions that are genetically similar to the rank-size curve. Furthermore, the stochastic models that seemingly account for these distributions are taken to complement the deterministic theory mentioned above. Here we support the central place argument as the only existing source of models that explicate those factors inducing spatial differentiation of economic activities and, as a consequence, urban populations.

Finally, we pursue the idea of growth within the interurban structure. At this time, however, discussion is certainly exploratory and so is limited to developing notions concerning the interrelations of growth variables (population, income, etc.) and hierarchial structure in the broadest sense. Within this analytic framework we can suggest only the most general factors that may be associated with low degrees of primacy (a quality of interurban structure that we view as a deviation from a characteristic skew distribution). This particular subtopic promises to be an exciting research theme in its own right as investigators move from equilibrium to dynamic modelling.

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Chapter 1

INTRODUCTION

Concern over the question of community size distribution is widespread in the geographical literature. In fact, it is a topic that intrigues social scientists in many fields. The theme is given impetus on the empirical side through Jefferson's (1939) study of the primate city and Zipf's (1949) account of rank-size regularities. However, contemporary efforts on the topic often feature a mixture of intuition, weak logic, and rather loose statistical analysis. Among the few accepted generalizations are those that primacy is associated with overurbanization, colonialism, and underdevelopment while rank-size tendencies are associated with the interurban integration of economically advanced regions. Perhaps the most serious recent efforts made to explain city size distributions come from those following Beckmann (1958) who adhere to central place models and from others pursuing Simon(1955) who prefer the stochastic argument. But with little agreement on both theoretical and empirical fronts, and disparate approaches flavoured with inconsistencies and redundancies, it is not surprising that the topic is enveloped by an air of dissatisfaction.

From the standpoint of strengthening harmony among the diverse efforts, there is alone sufficient reason to attempt a rigorous review of existing contributions. Besides, the tenor of present argument in the field of regional development and planning is that a much sounder knowledge of the relationships among urbanization, economic growth, and city size arrangements is decidedly needed. Friedmann (1966), for example, emphasizes that little is understood about the substructures of the space-economy and the influence of spatial activity patterns upon regional growth. Hopefully, then, this study will show practical benefits as well as satisfying personal curiosities.

In this thesis, we analyse the logic of existing theoretical and empirical statements about community size distributions and, when in disagreement, present our counter-arguments. With this in mind we attempt to resolve some of the apparent differences between the deterministic and probabilistic interpretations that support (to some extent) the rank-size principle. Also, attention is devoted to relating seemingly independent geographical concepts (for example, economic base and diffusion) to the discussion of city sizes. The purposes of the thesis are clearly twofold:

(i) To examine and attempt to refine (in explicit fashion) the existing methodology of the general problem area; and

(ii) To offer new ideas within the specific subtopics and to extend notions that bond the general problem area to the growing body of geographical literature and theory.

As the chapters are devised to be somewhat independent, a concise sketch of the entire study seems appropriate at this time.

We are first concerned with presenting a comprehensive review of central place theory as developed by Christaller (1966), Losch (1954), and later students. The review is essential in that it elucidates the drawbacks of the theory and the significant differences between Christallerian and Loschian fundamentals, both of which are needed to realize the domain of existing hierarchial models. Special emphasis is placed on identifying the qualities of hierarchial structure within a set of interrelated communities.

The following chapter is the most rigorous of the thesis. Here, we direct attention to relating the various hierarchial models in explicit fashion. Furthermore, we employ the notions of the central place models to link inter- and intraurban scales via the concept of economic base. The remainder of the chapter is given to introducing the rank-size regularity (the most characteristic concern in the city size discussion) within the central place framework. Some mathematical arguments display the nature of the association between the hierarchial and uni-size class ideas, while qualifying any presently accepted statements that are seen to be invalid. The thrust of this discussion is a demonstration that the rank-size rule and the existing Christaller models are probably, but not necessarily, incompatible and that the chances of coincidence may rise when other independent systems are included as well.

The methodology of empirical research is noticeably weak with regard to the study of interurban structure. Observers consistently fail to give care and thought to the effects of (i) arbitrarily defining study areas, (ii) blending different means of comparing frequency distributions, and (iii) imprecisely evolving statistical analyses. Much of the next chapter is devoted to questions like these in hope that we may become increasingly aware of the value of inferences made from empirical study with improved methodology. On the other hand, the latter portion of this section shows how deterministic and stochastic interpretations of the skewed frequency distributions are not necessarily in opposition.

The fifth chapter completes a circuit with the second, in that it builds upon the ideas of the intervening discussion but also concerns simple micro-economic reasoning. Its primary purpose is to examine within an assumptive framework how growth factors affect intercity structure. Besides, the effects of structure upon growth are suggested within the fundamentals of item diffusion. No attempt is made toward developing a flow chart or feedback model, even of the simplest kind; rather, efforts at this stage are totally directed to displaying impact tendencies alone.

A characteristic feature of the thesis is the adherence to a systems framework for studying the interrelations of population clusters in a spatial setting. It seems absolutely necessary to evoke this framework when trying to integrate the various facets of the literature into a more meaningful whole. Being aware that no real world system illustrates the precise qualities of the central place system, hierarchial notions may, of course, be somewhat relaxed (see Marshall; 1969).

The geographical literature is replete with systems thinking but only recently do we find the concepts formally applied. To be brief, the history of systems thinking is tied up with functional and ecological approaches, the organismic analogy, and the idea of regional synthesis (Harvey, 1969).

Explicit to the definition of a system is that we are concerned not only with a sum of elements whose attributes are directed by causal laws, but by a sum of relations among those units and some environment. The critical point, then, is that a system possesses properties, functions, or purposes that are distinct from its constituent objects, relationships, and attributes (Hall and Fagen, 1956). In our immediate study there is some intent of complementing existing lines of argument with simple fundamentals of general systems theory.

In any case, the systems framework is especially flexible with regard to our level of abstraction and serves as a reminder of the ever-present scale problem. When we talk of regional city systems as opposed to national city systems, the value of a consistent approach should crystallize.

Before closing this introduction, we must comment briefly on the most troublesome aspect of the city size topic; that is, the question of "explanation" per se. On the one hand, we have an ever-improving equilibrium theory dealing with functional allocations in space, but whose domain is typically restricted to activities where input prices vary little over distance. Most empirical studies, however, concern a domain much greater than this and include centers of special site and situation features. In addition, we have an a priori stochastic model that essentially avoids the spatial dimension. It is argued here that despite the fact the so-called entropy approach may describe a larger domain, it fails to satisfy our curiousity to the same degree as the central place approach does. The increased attention we give to the central place scheme, combined with our growing awareness of what the theory lacks, promises to be the best route for suitable explanation in the future.

Chapter 2

THE CENTRAL PLACE SYSTEM

A Review of Central Place Theory

Our discussion of central place theory pursues a synthesis of the fundamental contributions of Christaller, Losch, and more recent advocates of the subject. We plan to effectively defend the notion of a central place system, while developing a strong framework for treating the topic of city size models. Only in this light may the methodology of theory extension be properly understood.

Introductory Remarks

The route to comprehension of the spatial economic systems of Christaller (1966) and Losch (1938, 1954) is through the independent study of single goods or services. While their initial assumptions are not entirely identical, we can nevertheless isolate four general postulates that appear either implicitly or explicitly common:

(i) A homogeneous plain with uniform rural densities;

(ii) A system of f.o.b. pricing;

(iii) Equal demand by all consumers (consuming units) at any real price;

(iv) Free entry of producers into the market.

A clear interpretation of these pricing restrictions is vital to analysis in terms of cost and demand factors. F.o.b. pricing is simply the case where the consumer pays the price for a good at the production site (the f.o.b. price) plus the cost of transportation to his location (the total being the real price). Such a policy seems suitable for firms dealing with (i) large numbers of customers and (ii) goods and services whose distance decay (spatial elasticity of demand) is high.

Losch puts forward his argument in a more rigorous manner, while including settlement geography as only a portion of the general location problem. By presenting his case within the confines of formal economic theory he attaches a strong theoretical tone to the settlement principles of Christaller.

Case of the Single Good

Let's imagine the world as defined by the assumptions of Christaller and Losch. First we consider an individual good or service that is offered at site "0" on the plain. The desires of a consumer residing at the production site are indicated by the usual convex downward-sloping demand curve that intersects both the price and the quantity axes. Since demand "q" is a continuous function of the f.o.b. price, we may consider how demand changes for distinct f.o.b. levels "p_i" in the interval $p_{min} \leq p_i \leq p_{max}$ (where "p_{min}" represents the price at which a consumer at "0" will purchase a maximum quantity of the good and "p_{max}" is that price where the same consumer will purchase a zero quantity).

Let's consider now an identical consumer who resides "x" units distant from the production point. This customer must pay an additional "xt" (where "t" is the transport cost per unit distance) to cover the movement of the commodity to his location. In other words, demand "q" is a continuous function of the real price "$p_i + xt$" in the general case.

With this knowledge we can determine the distance "r_i" to the last customer exerting effective demand for the good or service supplied at "0". This defines a market area of radius "r_i" for any f.o.b. price "p_i":

$$r_i = \frac{p_{max} - p_i}{t} \qquad . . . (2.1)$$

It should be apparent that linear demand is, then, a function of marketing (f.o.b. price "p_i") and transport (cost "t") technologies.

Using our first assumption, we may compute the areal demand facing the firm at "0". By rotating the distance-demand response curve (for given "p_i" and "t") about a vertical axis through "0" we can trace out a demand curve for the typical consumer. Now, when we multiply the volume beneath the demand cone by a constant "D" representing population (consumer) density, the total demanded quantity "Q_i" in the area about "0" is given in integral form:

$$Q_i = D \int_0^{2\pi} \int_0^{r_i} f(p_i + xt)\ x\ dx\ d\theta \qquad \ldots (2.2)$$

If this calculation is repeated for a variety of f.o.b. prices (in the interval $p_{min} \leq p_i \leq pmax$) we can derive different levels of total demand "Q_i" as the cones vary in height and radii. When we plot the values of "p_i" versus those of "Q_i", an aggregate demand curve is constructed for the market area delimited by some radius "r_{max}" (where $p_i = p_{min}$). Since we are in fact dealing with an initial producer and competition is absent, this particular demand curve is termed the free spatial demand curve.

Although in the original literature Losch represents this curve as being concave to the origin it may be shown that, with our initial postulates, the demand curve must be convex (Denike and Parr, 1970). With the aggregate demand curve "D_1" facing our initial producer the next step is to determine the profit maximizing price and output relative to the curve.

Production costs are represented by an average cost curve "AC" that designates the cost of production per unit of output, while a marginal cost curve "MC" shows the increments in total cost as output is extended. Losch illustrates the cost curves as falling at each output

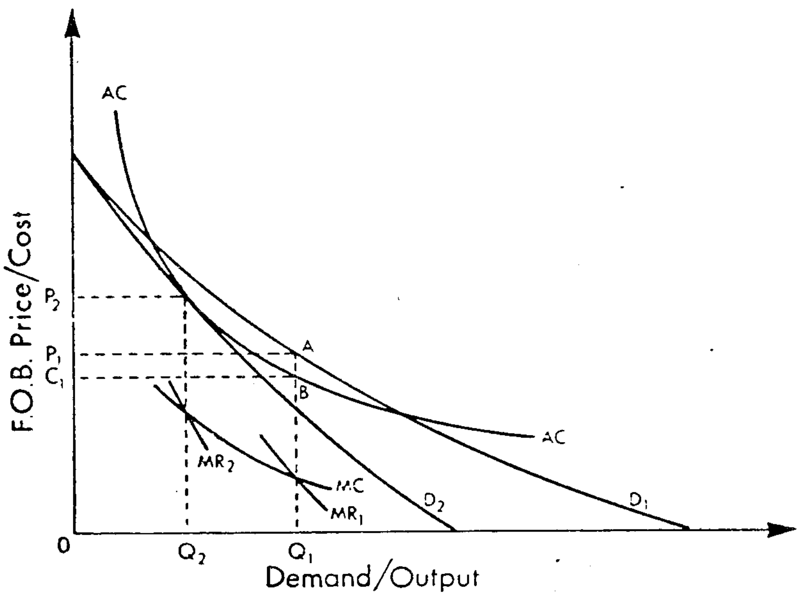

Fig. 1. Price and Output Conditions for the Individual Producer with no Competition and with Free Entry (from Parr and Denike, 1970).

level under monopolistic conditions, but this does not seem to hinder the generality of his argument.

Marginal revenue "MR", on the other hand, refers to the increments of revenue brought into the firm through small production expansions. If we assume that normal profits (including the rate of return that could be earned in other investments) are present in production costs, then the profit maximizing price "p_1" and output "Q_1" are determined by the intersection of the marginal revenue and marginal cost curves.

Losch and Berry (1967) argue that the price charged will be determined by the intersection of the average cost curve and demand curve but this is clearly not a profit maximizing interpretation. This particular price level "p_{min}" (where $p_{min} < p_1$) allows the maximum number of customers to be provided from "0" and may well improve total revenue; unfortunately, these gains are more than offset by climbing operation costs.

The situation changes somewhat when we consider free entry into production activity. The possibility of attaining excess profits encourages new entries into the market while disrupting the initial equilibrium situation. New producers continue to enter until each can only earn normal profits. Now the competitive demand curve "D_2" facing the single initial producer (and all new producers) is shifted to the left of "D_1". This is the case because with unrestricted entry the single original firm loses customers along the edges of its initial market area. The new output equilibrium is indicated by the tangency of the new demand curve with the average cost curve.

We notice, too, that the demand curve shifts even farther to the left if an excessive number of new producers enter the market. Now the demand curve lies below the average production cost at every output level and various sellers are forced out of business. Hence, the point of tangency indicates the minimum or threshold size of the firm.

By observing Fig. 1, we see that with unrestricted entry the equilibrium output "Q_2" is lower and the equilibrium price "p_2" is higher relative to the prior monopolistic conditions. This change is explained by the falling cost curve: as entry into the market continues, production at any one site is limited and prices rise as the opportunities for scale economies are lost.

Two general conditions, then, arise as a consequence of unlimited entry: (i) the loss of excess profits shows that the number of firms is maximized and (ii) each producer seeks a location as distant as possible from his neighbours'. In the ideal case where all suppliers are equally spaced over the homogeneous plain, a uniform triangular arrangement persists. Christaller and Losch argue that this is the most favourable spatial equilibrium pattern and that, as a result, a net of hexagonal market areas is provided.

The monopolistic state defines, in Christaller's terms, the ideal range of the good or service being offered. The new ideal range identified by the higher competitive f.o.b. price cannot be attained, however, since the extent of each seller's market is restricted by his adjacent competitors' market areas. The new boundary that delimits the competitive market area is called the real range of the good.

Obviously this real range is not equal in all directions, (since it defines the extent of identical hexagonal cells) and for this reason we define it as being one-half the distance between adjacent producers of the same good. As Parr and Denike (1970) point out, what Christaller terms the upper limit on the range refers to either the real or the ideal form, depending on whether or not spatial competition exists.

There also exists a minimum limit on the range of a good which Christaller calls the lower limit. Getis and Getis (1966:222) state that this encloses ". . . the number of consumers necessary to provide the mimimum sales volume for the good to be produced and distributed profitably . . . ." For the single original producer this threshold range is equal in all directions. With competition, however, the firm earns only normal profits and only a minimum level of aggregate demand determines the threshold range. Now the lower limit of the range is coincident with the real range and is not equal in all directions.

The fundamental contributions of Christaller and Losch toward a general understanding of the single good pattern are roughly identical. Basically, the former relies upon the concept of threshold range while the latter stipulates that the attainment of normal profits is paramount. Since the demand and cost factors underlying the two concepts are essentially the same, we usually consider Losch's treatment as only a more explicit or sophisticated approach to the same problem that confronted Christaller.

Case of Many Goods

Both Christaller and Losch develop schemes for integrating the features of the various single good nets. We consider Losch's analysis first, since it is the more general of the two, and then go on to summarize Christaller's ideas.

Losch's derivations rest upon a modification of our first postulate. He further assumes that the rural population is discontinuously distributed over the isotropic plain and that inhabitants reside in basic settlement units (farmsteads or hamlets) that are arranged on a uniform triangular lattice. Reasoning that the duality of agriculture and industry ideally leads to this punctiform distribution (compromises between proximity to food and industrial production, suitability to most aspects of agricultural production), he stipulates that these basic settlement units lie at the centre of hexagonal farms. The significance of this approach unfolds when he demonatrates that, with this discontinuous rural stratum of population, (i) the possible sizes of the complementary areas for different goods and services and (ii) the number of basic settlement units these areas enclose, likewise grow discontinuously.

To illustrate this condition, a concept that is fundamental to all central place discussions is introduced. Losch formulates a method for determining the number of equivalent basic settlements in any market area. This number equals the sum of the following three: (i) the number of units (or preferably lattice points) interior to the cell, (ii) one-half the number of units on edges of the cell, and (iii) one-third the number of units at vertices of the cell (on a triangular lattice, that is). Using this concept, Losch derives the possible market area sizes in terms of how many basic settlement units are provided. In a similar vein, spatial extent of the market areas is given by "$^{nA}/3$", where "A" is the area of the smallest hexagonal cell and "n" represents the number of equivalent settlements enclosed.

The results of this restrictive approach should be obvious: Losch is arguing that minimum demand for commodities offered at various farmstead locations is usually met by market sizes that offer an unnecessarily large number of basic consuming units. The inflexibility of Losch's derivation means that moderate surplus profits cannot be eliminated by further entry and some producers are certain to benefit. He (1954:120) emphasizes further that ". . . not all possible market areas need occur in reality . . . but conversely, every actual market area must be on the list of possible ones."

From his formulation of market sizes, Losch proceeds to discuss integration of the different market nets. He combines them by (i) ensuring that each good has one common supply center (the metropolis) and (ii) rotating the nets so as to yield a cogwheel pattern of six sectors with few and six sectors with many production sites. He (1954:124) states that:

> . . . with this arrangement the greatest number of locations coincide, the maximum number of purchases can be made locally, the sum of the minimum distances between industrial locations is least, and in consequence not only shipments but also transport lines are reduced to a minimum.

Essentially he is applying rational agglomeration assumptions in order to derive a related set of market nets in hope of defining some reasonable notion of an economic region. The underlying theme of his entire analytic argument is, in fact, that this derived arrangement identifies the most orderly and spatially confined closed system of market areas. As Losch (1938:75) points out: "How many of these self-sufficient systems will come into existence on our plain depends merely upon the commodity which has the largest shipping radius, as long as there are no economic limits to the size of the central city."

While the thrust of Losch's approach involves the concept of regional integration, a very important portion of the discussion concerns the numbers of coincident settlement units at various points on the triangular lattice. Losch does not disclose, however, his interpretation of the size of these aggregate settlement units except through a listing of the functions they provide.

It seems, though, that several interesting properties arise when we superimpose the various market nets in this way:

(i) Some lattice points possess more economic functions than others; hence there is differentiation among farmsteads, towns, etc.;

(ii) Some lattice points possess the same number of functions but these functions may be different; hence there is specialization among centers;

(iii) All lattice points possess at least one function but there are few with many functions; hence, a numerical pyramid in the number of multiple-good supply centers is suggested.

In closing off the Loschian case, we should emphasize that he consistently requires that only one producer of a given good is located in the center where that good is offered. This characteristic limitation is based solely on the rational scheme used to combine the independent market area nets.

Christaller's approach to the multi-good system is less general and we present a summary of his interpretation in a considerably more rigorous manner so as to avoid repetition at a later time.

Generally we might consider a large region in which "y" different goods and services are provided. Designating the first of these as "t_1", we may rank these central goods from "t_1" to "t_y" in ascending order of threshold need; a center offering "t_y", then, requires the greatest amount of consumer purchasing power for supply to persist in the long run. We term such a place an "M" level center and, according to our introductory postulates, it is associated with the largest complementary area on the homogeneous plain.

Of course, only as many "M" level centers emerge in the region as there are threshold markets available to support those firms offering "t_y". Since these firms compete spatially, production sites become arranged so that supply is most efficiently sustained. In other words, by enforcing an implicit assumption that firms offering the good of minimum distance decay organize the spatial pattern of "M" level centers, a triangular lattice develops on the plain. The boundaries between the various "M" level places are determined by the real range of "t_y" and form hexagon shaped market areas about each central place.

If total sales levels are an exact multiple of thresholds for good "t_y", these firms earn only normal profits (since they locate so as to minimize consumer movement). Excess profits may be earned if sales in the region are slightly greater than this exact multiple.

As we noted earlier, the ranges for different goods and services decline with lower threshold requirements; therefore, greater and greater numbers of surplus consumers lie between the threshold market areas of "M" level centers for these same commodities. There may be some good "t_{y-i}" for which the interstitial purchasing power reaches threshold volume itself. In this case, alternate centers evolve to supply "t_{y-i}" (and all other goods and services of lower threshold need) at prices below those at the "M" level places. These "M-1" level centers service the areas between the threshold ranges of those goods supplied exclusively from "M" level centers. Berry and Garrison (1958d) call "t_{y-i}" a hierarchial marginal good.

A similar argument calls for the emergence of "M-2" level centers where some commodity "t_{y-j}" ($j > i$) is the new hierarchial marginal good. These centers service the areas between the threshold ranges of those goods supplied only from higher order centers (i.e. "M" and "M-1" level places). Likewise, goods "t_{y-j}" through to "t_1", are provided at these lower order centers.

A consistent property of the Christaller scheme is that a center of a given order develops equidistant from its neighbouring centers of the next highest order. Getis and Getis (1966:224) add that "In this way, consumer movements are kept to a minimum, and a maximum number of demands are satisfied from a mimimum number of centers." Therefore, just as in the Loschian case, all central places are located on a triangular lattice.[1]

[1]Losch and Christaller are aware of different geometries as well as different market nets on the same lattice.

Certain fundamental characteristics of the spatial pattern we have outlined seem to exist:

(i) All centers but the smallest have other centers dependent upon them for the provision of goods and services; hence, the set of central places displays interdependency;

(ii) Each central place offers all the goods and services that dependent centers supply plus additional ones; hence, the criterion of incremental baskets of goods suggest that these communities show discrete stratification of centrality;

(iii) While the scale of the pattern is changeable, the interstitial placement of orders is a distinctive form;

(iv) There exists a definite numerical pyramid according to the orders of the centers.

Basically Christaller is forwarding a simple geometric argument in which the market area sizes increase in extent by a factor "q". This scheme contrasts with Losch's where there is a considerably smoother progression of possible market area sizes. If we further assume a discontinuous rural population, then in a Christaller $q = 3$ system (where "q" represents the nesting factor for market areas), the possible market area sizes in terms of equivalent basic settlements are 3, 9, 27, 81, etc. Similarly, if we denote the areal extent of the smallest market as "A", then the multiplier "q^zA" ($z = 0, 1, 2, \ldots$)

represents the progression of all possible market sizes.

Differences Between Christaller and Losch

Some of the basic differences between the two approaches have already been mentioned. The critical divergence between the schemes arises out of the different methods employed in combining the market networks of individual goods. Losch considers first the commodity with the smallest market area and then introduces commodities with progressively larger threshold requirements.

In other words, Losch's approach is analytic: it develops in stages from the most general ideas of Chamberlinian economic theory (where the differentiation of the producer's location is but one type of product differentiation). Christaller's case is relatively inductive as he argues from the most particular to the most general.[2] Since Christaller begins with the most "national commodity" while Losch begins with the most "local commodity", von Boventer (1963:171) suggests:

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[2]Central place theory has a lucid deductive structure for the general arguments proceed from a priori premises to statements concerning particular instances (for example, the number of functions coincident at a certain lattice point). Furthermore, Christaller's interpretation really assumes that a community system exists and that a particular community (the "M" level place) is dominant therein. Losch, however, does not rely upon the first of these assumptions in the same sense. As he envisages agglomeration from the most general case while Christaller approaches it from the most particular, we feel the latter has an added grain of inductive reasoning.

In economic-historical terms, Christaller's method of deriving his system may be thought of as describing the population growth in an area which at the beginning is very thinly populated. Losch's system would appear to be a more adequate description of a landscape in which a certain dense ground structure exists, with, in the beginning, entirely self-sufficient small spatial units (if new commodities with ever-increasing internal economies of production are introduced). It is solely this difference in the derivation of the systems which has the effect that Losch's system becomes much more complicated than Christaller's.

As a consequence of these opposite approaches we may identify numerous significant differences between the two schemes:

(i) The deviations from the optimal layout for the individual goods and services are smaller in the Loschian system since a greater number of possible market area sizes exist; the idea that relatively few market area sizes are permissible in the Christaller system provides the opportunity for initial excess profits;

(ii) While the general geometric appearance (triangular lattice) is identical for both, the spatial arrangement of centers is at variance. The Loschian system has one extra degree of freedom left after the metropolis is spatially fixed (hence the city-rich and the city-poor sectors) while Christaller's system is entirely symmetrical;

(iii) Losch does not consider the additional demands of a supplying population in a central place nor does he include the possible effects of multi-purpose trips. Concomitantly, either we must expect significant variation in the sizes of hexagonal cells for the same commodity (Isard, 1956:270-273) or, in order to preserve geometric regularities, we must compose new assumptions to eliminate such change in the spatial demand function (von Boventer, 1963:171-172). Christaller (1966:50-55) appears to implicitly include these features in his scheme. Therefore, the physical extent of the market areas for most commodities in a multi-good system will be smaller in the Christaller case than in the Loschian;

(iv) The level of urban concentration also varies considerably between the two approaches. If we suppose that the smallest market area size is the same in both, then we find fewer central places (and therefore a greater concentration of urban population) in the Christaller formulation. Besides, this level of urban concentration is directly related to the value of the "q" factor in the simpler system;

(v) Losch takes better account of the partial specialization of production in smaller centers; Christaller's assumption that each higher order central place supplies all the commodities (plus some additional ones) is a rather restrictive one. In the real world, smaller communities frequently supply larger places with specialized goods and services;

(vi) Losch's approach is more restrictive in considering entry of competitors at the same place. While the location of more than one producer of the same commodity at the same central place runs contrary to Loschian analysis, Parr and Denike (1971:572) suggest that further entry (at least in the short run) may be taken as another qualification of the inconsistencies related to in (iii) above. On the other hand, Christaller's more inductive derivation does allow for entry of competing producers where excess profits may be gained;

(vii) Christaller's incremental baskets of goods suggest certain natural agglomerative tendencies (inter-industry) among firms, but this condition is not characteristic of the Loschian landscape;

(viii) A clearly identifiable hierarchy is found in the Christaller scheme but not in the Loschian (identifiable, that is, in terms of orders and not individual functions);

(ix) It is much more difficult to make inferences about the population sizes of centers in the Loschian model; Christaller's rigid hierarchy suggests that discrete population levels (that is, different size classes) may be assigned to centers of different order. To make this difference even more apparent, we can relax the assumption of uniform purchasing power in the Christaller model and still develop discrete stratification (Berry and Garrison, 1958d).

Nevertheless it is significant that both observers reach quite similar conclusions while using somewhat different lines of reasoning. In both cases complete systems of networks are derived from an indefinite number of goods and services through partial equilibrium suggestions. In both systems the triangular arrangement of production sites and the hexagonal shaping of market areas for each commodity are found to be optimal.

Perhaps von Boventer (1963:173) best sums up the consequences of the two formulations:

> . . . (i) Losch's system is realistic and capable of an extension in that on a homogeneous plain a specialization of production in different centers, an interregional or inter-urban exchange of industrial goods and a complicated network of markets is derived. (ii) In its final result, as far as the overall system of a hierarchy of cities or central places is concerned, where the individual economic activities are neglected, Christaller's system gives both a better description of reality - at least with regard to Southern Germany in the 'thirties - and has the advantage of being simpler, or more elegant, whereas the Losch system is very difficult to test at all.

Scope and Nature of the Classical Argument

The classical literature attracts considerable comment from geographers for diverse reasons. It is not our purpose here to rigorously qualify the central place approach but to make certain that the reader is fully aware of the more important drawbacks of the theory.

A first group of criticisms deals with the actual assumptions employed by Christaller and Losch. Isard (1956:274), for one, points out that Loschian analysis is limited only ". . . in situations where raw materials are not required (as in service industries) or are ubiquitous and everywhere available at the same costs." In other words, the central place approach is omitting too many production sites (whether market or material oriented) that are strongly influenced by the nature of input prices.

Also, the theory is restricted to those activities not affected by more selective pricing policies, the introduction of which substantially alters the spatial extent of markets for many goods.

However, this only means that central place theory has a somewhat smaller domain of economic operations than some optimistic observers would give it. The theory does possess an analytic framework that allows it to explain certain hypotheses, with attention being devoted to the applicability of its assumptions.

On the other hand, some criticism is devoted to the static or deterministic nature of the theory. As Pred (1967:99) states:

> In order for the fundamental precepts of central place theory to be faithfully reproduced in a real world situation it would be necessary for every tertiary-activity supplier (entrepreneur, firm) to make an optimal location decision (site and situation selection) and every tertiary-activity consumer (service client) to make a totally rational journey-to-consume decision.

This difference seemingly arises from disagreement as to what entails a satisfactory framework for explanation. Classical theory rests strongly upon economic assumptions and Euclidian geometric notions. Efforts continue to be channelled along functional (roles of phenomena within organizations) and morphometric (spatial structure and form) lines (see Harvey, 1969:78-83). Inherent in such central place thinking is that the system of market networks is constantly adjusting itself toward some long run optimal spatial layout.

The behavioral approaches rest upon psychological and sociological postulates so as to avoid a mechanistic view of decision-making. Decisions concerning where to locate and where to buy are variable due to differences in actors' (firms, buyers) information and capability to employ that information. In many respects, production decision-making does involve non-optimal cause and effect behavior, but improvement of the location situation seems to occur with time. In this way, behaviorial matrix approaches may afford good predictions in the short run. On the other hand, consumer decision-making is still confined to descriptive thinking (see Curry, 1962).

Besides, the more temporal or genetic approaches also tend to be descriptive in the sense that transport routes, economic activities, migration, etc. are randomly assigned. However, with stage-by-stage qualifications, simulation studies do afford excellent pictures of reality (Morrill, 1963).

In any case, no other interpretation challenges the classical view in the way it links the governing processes and the resulting spatial structure and form, a synthesis that Harvey (1969:127) suggests is central to geographic theory. Or to take another view, ". . . the synoptic feasibility of a model is enough to justify its use as a basis for empirical research. It is valid to enquire whether a situation which could exist, does exist, even if one has no air-tight logic to account for its emergence from non-existence or chaos." (Marshall, 1969:40). Also, it is certainly to their credit that Christaller (1966:111-112) and Losch (1954:xiii) seem entirely aware of the shortcomings in their central place derivations.

We should view their models as the first attempts to complement the ideas of spatial differentiation (due to economic factors solely) with those of intraregional equilibrium (that is, the simultaneous determination of market locations, production centers, transportation routes, etc.).[3] Basically, they are deriving a system of economic order through a minimum of assumptions in hope of explaining the essentials of spatial differentiation.

Extensions of the Classical Literature

We have outlined central place theory as the partial theory of the location, size, nature, and distribution of activity clusters. In many instances, functional

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[3]Depending upon the scale at which we examine the central place system, equilibrium may be considered intraregional (all activities) or interregional (subsets of activities).

interdependence amongst centers and their market areas has been stressed: therefore, it is not unnatural that the term "system" has become loosely associated with the combination of market nets. In the following discussion we explicitly develop some of the more basic features of a central place system in order to have added rationale for the formulation of city size models. It may be profitable, then, to first expand on the term "system" by seeing how central place theory ties into general systems theory:

> A system is a set of objects (for example, central places), attributes of the objects (population, establishments, business types, traffic generated), interrelations among the objects (midpoint locations for lower level centers, uniform spacing at any given level) and among the attributes (the graphs of log-log relationships) and interdependencies of objects and attributes (the central place hierarchy). (Berry, 1967:76-77).

Aggregate Relations and Elemental Components

Considerable effort is directed toward summarizing the fundamental interdependencies (that is, empirical structural relationships) of central place systems in a closely knit set of equations. (Berry and Barnum, 1962; Berry, Barnum, and Tennant, 1962; Marshall, 1969). The study areas for these empirical investigations are, typically, rural regions so as to fulfill the postulates of central place theory to a reasonable degree. Empirical research is also carried on at the intraurban level in order to facilitate integrating regularities at different scales. (Berry, 1967). To describe these basic relationships we must provide definitions for several variables:

- p: the population of a central place;
- r: the population of the complementary area served by a central place;
- P: the total population served by a central place;
- A: the spatial extent of the complementary areas being serviced;
- Qp: the population density of the entire area serviced by a center, including the outlying area and the center itself;
- Qr: the population density of the outlying trade area;
- y: the number of central functions (separate business types) offered by a central place; hence, the highest level central function performed by the center;
- Dy: the maximum distance that customers travel to a central place; therefore the range of good "y";
- f: reads "some function of";
- log: indicates base 10 logarithms.

To begin with, we may define several equalities as well:

$$P = p+r \qquad \text{. . . (E1)}$$

$$A = f(Dy) \qquad \text{. . . (E2)}$$

$$r = AQr = f(Dy)Qr \qquad \text{. . . (E3)}$$

$$p = AQp = f(Dy)Qp \qquad \text{. . . (E4)}$$

One generally expects that larger central places have more central functions, more establishments, and larger market areas than smaller centers. Loglinear relationships seem to persist between the number of functions performed in central places and (i) the populations in those places or (ii) the total populations served by those places. (Berry, Barnum, and Tennant, 1962:69; Berry, 1968:37-38; Marshall, 1969:163-164). Besides, a linear pattern seems to illustrate the association between the maximum distance consumers are willing to travel to a central place and the number of functions offered there.[4] (Berry, Barnum, and Tennant 1962:100-101; Berry, 1968:28). These and similar arguments may be formalized in structural equations:

$$\log p = a_1 + b_1 y \qquad \ldots (2.3)$$

$$\log P = a_2 + b_2 y \qquad \ldots (2.4)$$

$$Dy = a_3 + b_3 y \qquad \ldots (2.5)$$

$$\text{where } a_2 > a_1 > a_3$$

Various implications may be drawn from these equations:

from (2.3) & (2.4) $$\log p = \frac{a_1 b_2 - a_2 b_1}{b_2} + \frac{b_1}{b_2} \log P \qquad \ldots (2.6a)$$

" (2.3) & (2.5) $$\log p = \frac{a_1 b_3 - a_3 b_1}{b_3} + \frac{b_1}{b_3} Dy \qquad \ldots (2.6b)$$

" (2.3) & (2.4) $$\log P = \frac{a_2 b_1 - a_1 b_2}{b_1} + \frac{b_2}{b_1} \log p \qquad \ldots (2.7a)$$

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[4]This contradicts a statement in Berry and Barnum, 1962 but seems justified by the empirical evidence referred to above.

" (2.4) & (2.5) $\log P = \dfrac{a_2 b_3 - a_3 b_2}{b_3} + \dfrac{b_2}{b_3} Dy$ . . . (2.7b)

" (2.6a) & (E4) $\log A = \dfrac{a_2 b_1 - a_1 b_2}{b_1} + \dfrac{b_2}{b_1} \log p - \log Qp$ . . . (2.8a)

" (2.6b) & (E4) $\log A = \dfrac{a_2 b_3 - a_3 b_2}{b_3} + \dfrac{b_2}{b_3} Dy - \log Qp$ . . . (2.8b)

" (2.6a) & (E3) $\log r = \dfrac{a_2 b_1 - a_1 b_2}{b_1} + \dfrac{b_2}{b_1} \log p + \log \dfrac{Qr}{Qp}$ . . . (2.9a)

" (2.6b) & (E3) $\log r = \dfrac{a_2 b_3 - a_3 b_2}{b_3} + \dfrac{b_2}{b_3} Dy + \log \dfrac{Qr}{Qp}$ . . . (2.9b)

These statements supplement those that may be formulated for establishments, functional units, etc, in a similar way.

The results of these equations, however, suggest empirical features of central place systems that are simpler than those cited elsewhere (Berry, Barnum and Tennant, 1962; Berry, 1964):

(i) Central place populations are constrained only by the total populations they service. This interpretation strengthens an elementary economic base rationale for central place systems since it avoids gross density as an explicit variable. Besides, the coefficients "b_1" and "b_2" determine how the ratio "P/p" changes as centers take on more and more functions. Empirical evidence (Berry, Barnum, and Tennant, 1962: Figs. 5 & 6) suggests that "b_2" is slightly greater than "b_1" and that, as a consequence, community populations assume a decreasing proportion of total market populations as they grow larger.

(ii) The spatial extent of the complementary area about a central place is constrained by the total population and gross density. This suggests that the area is a function of the number of business types offered by a central place but that this area diminishes as overall densities increase.

(iii) The non-central place population of the complementary area (that is, residents in smaller central places of the trade area or rural inhabitants) depends on the number of business types in the market center and the nature of the density ratio "$^{Qr}/Qp$". These external populations account for an increasing proportion of total population as market areas expand (other things equal).

In short summary, the important aggregate relations of central place systems appear to be exponential: tributary area populations, total market area populations, and the physical extent of these areas are all exponential functions of the population sizes of central places. Also, (2.6b) suggests that the range of the highest level good provided by a central place is exponentially related to that community's population. Moreover, the relationship between the growth rates of the spatial components and that of the associated market center depends upon particular constraints in each case.

Empirical investigation indicates that there must be fewer larger centers with larger trade areas and that these larger centers are more widely spaced than smaller centers. Such properties entirely re-inforce the original analytic statements of central place theory concerning the size, spacing, and functions of urban centers.

Berry and Barnum (1962) add to their derivations a set of empirically based inequalities that identify discontinuities of area and population served at any gross population density. If we recall that major thrust of the Christaller model concerning the existence of discrete orders of central places, then these limits express the maximum size of communities at particular levels of centrality with regard to density constraints.

Hierarchial Structure

Those readers familiar with the central place literature may well be questioning the avoidance of the term "hierarchy" to this point. It is clear that considerable confusion arises over the common use of that term and it remains the author's contention that proper interpretation can only come after a review of the theoretical literature.

The most important notion to remember about "hierarchy" is that it is a spatial term when employed to describe features of a central place system. Therefore, it confines to a city system only spatially related centers among any set of centers. Lukermann (1966) states that we must be explicit about the directions of physical circulation and movement when discussing hierarchial control; in other words, it is not sufficient to only enumerate functions, populations, etc, in a set of cities and extend our knowledge of hierarchial structure.

Therefore, "hierarchy" implies both spatial and functional (order) restrictions. This should be immediately apparent, since the hierarchy bridges the interdependencies of attributes and objects for the entire central place system. Functional restriction is measured by (i) the number of cities having the function, (ii) the size of the population served by the function, and (iii) the area of the population served by the function (Lukermann, 1966).

Spatial restriction, on the other hand, is determined by (i) the interdepedence of centers and (ii) the interstitial placement of orders.

The hierarchy determines the organization of a city system in space. Seen as a consequence of territorial specialization, functional differentiation, and degree of interaction among activity nodes, it emerges only with some maturity in the regional urban structure. Once there, though, it tends to define the limits of individual growth among the urban places.

The Central Place System Reconsidered

The specification of a city system rests upon the delimitation of an initial center for inquiry. Using the hierarchy concept, we can identify those centers of lower order that are commercially linked to the central place.

Recalling our discussion of the Christaller model, we began with the emergence of an "M" level center offering the set $\{t_1, t_2, \ldots, t_y\}$ of functions. On the other hand, the interstially situated "M-1" centers offer the set $\{t_1, t_2, \ldots, t_{y-i}\}$ where function type "M" is the difference between the two sets. In this manner, a city system is developed with "M" hierarchial levels and control is maintained by the property that functions provided by a center at one level are proper subsets of those functions given at higher hierarchial levels.

Besides, a concomitant feature of any central place system is its closure or functional wholeness. This economic integration is determined by the lines of interdependence and the orders in the hierarchy. It is a credit to Christaller and Losch that they offer central place models that combine functional and spatial control, if only in a partial sense.

Chapter 3

CITY SIZE MODELS AND DISTRIBUTIONS

Review of the Hierarchial Models

The hierarchial approach is explicit to the derivation of existing city size models. These models are presently confined to the simpler but more plausible Christaller interpretation; indeed it would be interesting if a model based upon the Loschian landscape were similarly developed. As a result, the city size models evade the postulate of even purchasing power distribution (although we retain it for illustrative ease), but are restricted to cases of discrete functional ordering.

Terms and Notation

Beckmann (1958) provides the initial model of city sizes but the rather debatable properties of this approach coupled with the more recent efforts in the subject by Beckmann and others, requires that we first study a generalized model.

However, before departing on a rather rigorous discussion, the reader should be acquainted with the terminology and notation of the subject. A central place that provides the "m"th bundle (basket) of goods and services is said to possess function type "m" (where "m" represents one of the distinct function subsets between "1" and "M"); also, if that place provides function type "m" but not "m + 1", it is said to have order "m". Since the center provides the "m"th basket for a complementary area, it is said to "m"-dominate the entire population in that surrounding area (including the rural population and the urban population in that center and all smaller centers).

Dacey (1966) refers to the central place system H_{Mq} where "q" indicates the nesting factor (see Chapter 2; that is, the number of places with function type "m-1" that are "m"-dominated by an order "m" place) and "M" denotes the total number of function types offered throughout the system.[1] The following notation is common in the literature:

m: The function provided by a place; hence, the level in the hierarchy as well ($m = 1, 2, \ldots, M$); smallest centers offer only function one;

n: The size class ($n = 1, 2, \ldots, M$); for the single largest center and its associated market area, $n = 1$;

M: the total number of functions provided in the system or the number of levels on the hierarchy; notice $m = M - n + 1$ and

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[1]Implicit to the central place scheme is that these function subsets remain relatively constant in nature; therefore we usually refer to them as simply "functions".

$n = M - m + 1$;

r_m: the population of the complementary area on the "m"th level of the hierarchy; when $m = 1$ the population is entirely rural;

p_m: the population of a center on the "m"th level of the hierarchy;

p_M: the population of the largest center in the system;

P_m: the total population served by a center on the "m"th level of the hierarchy;

k_m: a service or technology multiplier that denotes the proportion of the population in an "m" level complementary area plus an "m*" ($m \leq m^* \leq M$) level central place (servicing the complementary area in the capacity of an "m" level place) that is required to reside in the "m*" level place in order to provide function "m*" to both; a necessary condition exists that $\sum_{m=1}^{M} k_m < 1$;

k: a simple proportionality factor that relates the population of a city to the total population served by that city; a necessary condition exists that $0 < k < 1$;

q: the nesting factor for market areas;

s: the equivalent number of centers of the "m - 1"st level that are dominated by an order "m" place; the geometry of central place systems requires that $s = q - 1$ where "s" and "q" are both constants.

Model I: The General Case

Dacey (1966) first outlines the general city size model that interests us in this discussion. The development of his model is rather sketchy, though, and the reader is greeted by several complicated formulations that are not explicitly derived. Beckmann and McPherson (1970) evolve an identical model in a more elegant fashion.

The derivation of urban populations rests upon three postulates, besides those essential to Christaller's model. The first assumption states that the amount of employment associated with a function depends on the entire population supporting that function. The second assumption states that population in a central place is a linear function of employment (see Dacey, 1966). The combination of these postulates leads to a service multiplier "k_m" characteristic of each function. A third assumption is that "k_m" is identical for all centers offering function "m".

We begin description of the model with those centers providing only the first function to a uniformly dispersed rural population r_1:

$$p_1 = k_1 (p_1 + r_1)$$

$$= \frac{k_1 r_1}{1-k_1} \qquad \ldots (3.1)$$

In this case "k_1" denotes the proportion of the total population demanding function one to the population of the center providing it. Now, consider the case of a larger center that provides both the first and second functions:

$$p_2 = k_1 (p_2 + r_1) + k_2 (p_2 + r_2) \qquad \ldots (3.2)$$

This simply means that the population of a second order center is determined by:

(i) A population group that is related to the supply of function one to the second level center and a first level complementary area (note "p_2" serves "r_1" in the capacity of a "p_1" center);

(ii) A population group that is related to the supply of function two to the second level center and a second level complementary area.

Reasoning in this fashion, we may determine the population resident in a "m"th level center:

$$p_m = \sum_{i=1}^{m} k_i (p_m + r_i) \qquad \ldots (3.3)$$

This premise rooted in Christaller thinking is sufficient for generating a model in which center and complementary area populations are proportional to the basic rural population served by a first order center.

The next step is to determine the nature of the complementary area populations "r_m". Centers of order "m" have "s" satellite cities of order "m-1", each of which is surrounded by a complementary area of population

"r_{m-1}". In other words, the population of the complementary area about a "m" level center consists of "r_{m-1}" in the market area of order "m-1" surrounding the center plus a population of $s(p_{m-1} + r_{m-1})$ in satellite cities and their tributary areas. That is:

$$r_m = sp_{m-1} + (1 + s)\, r_{m-1} \qquad \ldots (3.4)$$

To simplify the substitution method, we employ Beckmann and McPherson's definitions:

$$K_m = \sum_{i=1}^{m} k_m$$

$$D_m = p_m - p_{m-1}$$

so that (3.3) becomes:

$$p_m\,(1-K_m) = \sum_{i=1}^{m} k_i r_i \qquad \ldots (3.5)$$

where:

$$k_m p_m = p_m(1-K_m) - p_{m-1}\,(1 - K_{m-1}) \qquad \ldots (3.6)$$

and since $P_m = p_m + r_m$:

$$D_m = \frac{k_m\, P_m}{1-K_{m-1}} \qquad \ldots (3.7)$$

But from (3.4):

$$P_m = (1 + s)\, P_{m-1} + D_m \qquad \ldots (3.8)$$

or,

$$P_m = \frac{(1 + s)\ (1 - K_{m-1})}{(1 - K_m)}\ P_{m-1} \qquad \ldots (3.9)$$

Through repeated substitutions:

$$P_m = \prod_{i=1}^{m-1} \frac{(1 + s)\ (1 - K_i)\ P_1}{(1 - K_{i+1})}$$

$$= \frac{r_1}{1-k_1} \prod_{i=1}^{m-1} \frac{(1 + s)\ (1 - K_i)}{(1 - K_{i+1})} \qquad \ldots (3.10)$$

and from the definition of D_m:

$$p_m = p_1 + \sum_{i=2}^{m} \frac{k_i}{1-K_{i-1}}\ P_i \qquad \ldots (3.11)$$

Through substitutions in (3.7):

$$p_m = \frac{k_1 r_1}{1-k_1} + \sum_{i=2}^{m} \frac{k_i}{1-K_{i-1}}\ P_i$$

$$= \frac{k_1 r_1}{1-k_1} + \frac{r_1}{1-k_1} \sum_{i=2}^{m} \frac{k_i}{1-K_{i-1}} \prod_{j=1}^{i-1} \frac{(1+s)\ (1-K_j)}{(1-K_{j+1})}$$

$$\ldots (3.12)$$

It should be obvious from (3.12) that a city of order "m" is depicted ". . . as being constructed of layers or segments supplying a nested set of markets, each defined by the bundle of goods and services supplied". (Beckmann and McPherson, 1970:27-28). In the central place framework the population of any community is determined by the nature of decline of the service multipliers (k_i), the geometry (s), and the rural density (r_1).

Model II: The Aggregate Approach

Given the properties of the general city size model we now turn to the discussion of the simple models. The first of these is Beckmann's original hierarchial scheme which employs an assumption that the size of any center is a constant proportion of the population it serves; that is:

$$p_m = k\ P_m \qquad \ldots (3.13)$$

The model is a priori since it rests more upon intuition than development from a theory. Therefore we must be wary of making predictions with this model, at least until we understand better how it relates to central place theory.

Beckmann's initial model, however, displays a glaring inconsistency with central place relationships. On interpreting the geometry of the system, he overstates the total population served by a city on the "m"th level. (Since he seems to equate "s" with "q"). It appears that this error arises from the difference between the total number of settlements in an economic region and the apportioning of those settlements among various hierarchial levels.

In any case, Beckmann (1968) and Parr (1969) rectify the misinterpretation in independent contributions. By adding "p_m" to both sides of (3.4) it should be obvious that:

$$P_m = p_m + sP_{m-1} + r_{m-1} \quad \ldots (3.14)$$

Using (3.13) and (3.14) together it is a simple matter to demonstrate that both city size and total population served increase exponentially with the hierarchial level:

$$p_m = \frac{r_1 k}{1-k}\left(\frac{s}{1-k} + 1\right)^{m-1} \quad \ldots (3.15)$$

$$P_m = \frac{r_1}{1-k}\left(\frac{s}{1-k} + 1\right)^{m-1} \quad . . . (3.16)$$

Parr (1969) illustrates the nature of the error in the early Beckmann model by considering the change in the basic progression component from "$\frac{q}{1-k}$" to "$\frac{s+1}{1-k}$".

At this point in the discussion it may be profitable to compare the attributes of this simple model and the more complex a posteriori model outlined earlier. To begin with, the rationale for the factors "k_m" and "k" rest on quite different central place relationships. The proposal of a distinct "k_m" value for each of the "m" functions seems to be a reasonable derivative of Christallerian theory in that it focuses upon the changing roles of (i) employment-function and (ii) center-tributary area associations as we move through the hierarchy. In other words, while we suppose that the technology used in providing identical functions at different levels remains unchanged, we are introducing systematic changes in city sizes through the unique service mix at each level. On the other hand, the postulate of a constant "k" value is totally arbitrary, though it may indeed have some empirical merit. For instance when we recall (2.6a) we notice that as "b_2" approaches "b_1" in value, a constant relationship between center and total market population is neared (that is, as $k \rightarrow \log^{-1} (a_1 - a_2)$).

In addition to this variance in terms of rationale we note that the two factors cannot be compared by assuming that "k_m" itself is a constant, since the repeated application of the factors affects the models in different ways. For instance, a comparison of (3.1) and (3.13) indicates that "k" would equal "k_1", but this would introduce contradictions in the case when (3.2) and (3.13) are compared with $k = k_1 = k_2$. It should be clear that the two factors have no obvious interrelationship and that, therefore, Dacey (1966:31) is unjustified in criticizing Beckmann's result.

However, Model II can be shown to be only a particular case of the general model by the use of decreasing "k_m" factors. For this to be true, it is only necessary that:

$$k\ (p_m + r_m) = \sum_{i=1}^{m} k_i\ (p_m + r_i) \qquad \ldots\ (3.17)$$

which implies, as we noted above, that $k = k_1$. The determination of remaining "k_i" values is performed one step at a time; for instance:

$$k_2 = \frac{k_1 r_2 - k_1 r_1}{p_2 + r_2} \qquad \ldots\ (3.18)$$

or, in general:

$$k_m = \frac{k_1 r_m - \sum_{i=1}^{m-1} k_i r_i - \sum_{i=2}^{m-1} k_i p_m}{p_m + r_m} \qquad \ldots\ (3.19)$$

Apparently, then, the a priori model and the a posteriori model have a fundamental premise (3.1) in common. The flexibility of the general model, however, comes from a higher level of analysis with the addition of further premises. Rather than using urban centers as study units in the central place system, the a posteriori model is effectively employing functionally determined population subsets (that is: $k_1 p_m$, $k_2 p_m$, . . . , $k_m p_m$) of those centers as elements in a more complex spatial system. Therefore the factor "k" emerges as the aggregate counterpart of the set (k_1, k_2, . . . , k_m) in the simpler system for one particular case. The essential notion is that Models I and II really apply to distinct systems that have the same hierarchy and that are spatially coincident (in that we depict centers as nodes in a geometric network).

Model III - The Geometric Multiplier

Dacey (1966) suggests interpreting "k_m" as an exponential "k^m" to reasonably account for specialization through the service multipliers. Unfortunately he fails to offer any analytic interpretation for his choice. However, a variation of this scheme is an immediate derivative of the Beckmann-McPherson formulation. It is based upon the proposal that market area populations "P_m" increase from level to level by a constant factor. From (3.10) it should be obvious that a sufficient condition for this is that:

$$\frac{1-K_{m-1}}{1-K_m} = \text{constant} = 1 + h \qquad \ldots (3.20)$$

Defining $k_0 = 0$, (3.20) holds for all m   0; however this indicates that:

$$\frac{1}{1-k_1} = 1 + h \qquad \ldots (3.21)$$

or,

$$\frac{k_1}{1-k_1} = h \qquad \ldots (3.22)$$

Besides, the meaning of (3.20) is:

$$k_m = \left(\frac{1}{1+h}\right)^{m-1} k_1 = (1-k_1)^{m-1} k_1 \qquad \ldots (3.23)$$

where the service multiplier decreases in a geometric fashion for the second and higher hierarchial loads. This formulation and the Dacey suggestion are identical for only one value, viz. $k_1 = \frac{1}{2}$. Now, using (3.10):

$$P_m = P_1 \prod_{i=1}^{m-1} (1+s)\ (1+h)$$

$$= P_1\ (1+s)^{m-1}\ (1+h)^{m-1} \qquad \ldots (3.24)$$

and since:

$$D_m = \frac{h}{1+h}\ P_m \qquad \ldots (3.25)$$

it follows that:

$$p_m = r_1\left\{\frac{k_1}{1-k_1} + \left[\frac{h}{(1-k_1)\ (1+h)^2\ (1+s)}\right]\right.$$

$$\left.\left[\frac{(1+h)^{m+1}\ (1+s)^{m+1} - (1+s)^2\ (1+h)^2}{(1+h)\ (1+s) - 1}\right]\right\} \quad \ldots (3.26)$$

which simplifies (see 3.21 or 3.22) to:

$$p_m = r_1\left\{\frac{k_1}{1-k_1} + \left[\frac{k_1}{(1-k_1)^2}\right]\left[\frac{\left(\frac{1}{1-k_1}\right)^{m-1}(1+s)^m - (1+s)}{\left(\frac{1}{1-k_1}\right)(1+s) - 1}\right]\right\} \quad \ldots (3.27)$$

The rationale for this geometric multiplier model is not clear though. Beckmann and McPherson suggest, however, that the growth factor in (3.24) is the same as that in (3.16). Unfortunately, it is easily demonstrated that this interpretation is in error. For instance, assuming that:

$$(1+s)\ (1+h) = \frac{s}{1-k} + 1 \quad \ldots (3.28)$$

means:

$$\frac{1-k}{1-k_1} = 1 - \frac{k}{1+s} \quad \ldots (3.29)$$

But since $k = k_1$ if "P_1" is identical in (3.16) & (3.24) and since $k, k_1 > 0$ we have a contradiction (L.S. > R.S.) in (3.29); in other words, Model II and Model III cannot generate identical market area populations and therefore must be considered distinct.

Besides, (3.25) indicates that the population differences between centers on adjacent levels are a constant proportion of the total population on the higher level; in fact this necessitates that the populations of urban communities become an increasing proportion of the total market area populations as we ascend the hierarchy. While this is a derivative of the a posteriori general model, we have no reason to expect (3.20) is not a completely arbritrary proposal. Therefore, since this interpretation seems inconsistent with available empirical evidence we consider the simple aggregate model a more valid approach.

Model IV - The Constant Multiplier

A third elementary model is suggested in the literature but is nowhere discussed explicitly. Dacey (1966) introduces the idea of constant service multipliers but we have already demonstrated that this contradicts the assumptions of the general model. However, we may inquire what effect there would be on the size distribution of centers if a constant multiplier were to emerge at the second level. Beckmann and McPherson (1970:33) suggest: ". . . that the large gap between k_1 and k_2 is common, but no clear pattern in the higher service multipliers has appeared . . ."; nevertheless, they do provide data that indicate a constant multiplier for all levels above the first is not unreasonable.

The assumption for this model is that (3.3) may be expressed as:

$$p_m = k_1 (p_m + r_1) + \sum_{i=2}^{m} k'(p_m + r_i) \qquad \ldots (3.30)$$

$k_1 > k'$, or,

$$p_m (1-k_1 - \{m-1\} k') = k_1 r_1 + k' \sum_{i=2}^{m} r_i \qquad \ldots (3.31)$$

which leads to (see 3.4):

$$p_m = \frac{k_1 r_1}{1 - (k_1 + \{m-1\} k')} + \frac{k'\left\{\sum_{j=0}^{m-1} \sum_{i=1}^{m-1} p_i \, s \, (1+s)^j + \frac{r_1(1+s)}{s} \left[(1+s)^{m-1} - 1\right]\right\}}{1 - (k_1 + \{m-1\} k')} \qquad \ldots (3.32)$$

While the proposal for this model is similar to Model III in that a related pattern of service multipliers begins at the second hierarchial level, its results are more like those of Model II. It seems that for certain values in the interval $k_m < k' < k_2$, where $k_2, k_3, \ldots, k_m$ are determined by (3.19), this new city size model approximates the use of a basic progression component. For instance, if we equate "k'" to the mean of $k_2, k_3, \ldots, k_m$ as determined in Model II, then Model IV underestimates their populations at higher levels. Moreover, central places with small to medium populations form a lower proportion of their total market populations than the first level centers do, but the larger centers tend to become a greater part of the total populations they service.

Unfortunately, it is difficult to defend this model with the limited empirical indications we have at this time. Also, if such a "k'" exists, we have little evidence to stipulate that it emerges at the second level. The best we can do is hypothesize that a large gap between "k_1" and "k_2" brings some sort of steady state into being. On the other hand, the empirical evidence we have cited for the support of the aggregate model covers only a number of the larger size classes, and it remains to be empirically substantiated (though it seems intuitively reasonable) that the very largest centers assume a smaller proportion of their total market populations.

All in all, though, it seems improbable that we can discard the constant proportionality model in favor of either of the two remaining elementary models. The fact that it is not completely unsubstantiated by empirical study plus its extreme simplicity suggests that the early Beckmann model (in revised form) is the most practicable of the three. We say practicable because the a posteriori model is not so firmly attached to theory that we can suggest the notion of decline in the "k_m" values and, therefore, we do require some intuitive speculation as to a systematic decline (see 3.13, 3.20, 3.30). In other words, the general model has, at this time, extra unknowns that cannot be deduced from central place theory and therefore it is not workable in generating center populations.

Hierarchial Models and the Economic Base

The economic base concept may be attached to the city size models with little difficulty (see Dacey, 1966). Basic activities provide goods and services for consumers outside the urban community while non-basic production is directed to the residents of the center. With the assumption that all employment is basic or non-basic, we may devise ratios between the two types of employment for each of the hierarchial models. It is intended that the economic base concept should clarify our interpretation of the city size models; besides, we may gain significant evidence toward understanding the changing character of the basic/non-basic ratio (at least within the confines of activities explained by central place theory) as urban centers rise or decline in size.

To begin with, we recall the basic premise (3.3) of the general hierarchial model. The population in center "p_m" that services its complementary area is $\sum_{i=1}^{m} k_i r_i$, while the population fulfilling local need is

$p_m \sum_{i=1}^{m} k_i$. This indicates several properties of the basic/non-basic ratio: $\sum_{i=1}^{m} k_i r_i \Big/ p_m \sum_{i=1}^{m} k_i$:

(i) The ratio is maximized at $i=1$ and minimized at $i=m$;

(ii) The value of the ratio in the interval $1 \leq i \leq m$ depends upon the nature of decline in service multipliers;

(iii) The ratio is a function of the geometry or transport topology of the central place system.

Table 1 indicates the nature of these properties in four central place systems, each with different characteristics but all having seven hierarchial levels. The first and second systems use different multipliers (geometrically declining and constant proportionality factor) but generate data that is topologically comparable to the q=3 Christaller data (see Beckmann and McPherson, 1970) with a relatively constant multiplier "k'" beginning at the second level. The fourth system is a geometrical variant of the second in that it is depicted by the simple aggregate model.

Table 1

Service Multipliers and Basic/Non-Basic Ratios of Four Central Place Systems

	(1) $q = 3$		(2) $q = 3$		(3) $q = 3$		(4) $q = 4$	
	$r_1 = 2000$		$r_1 = 2000$		$r_1 = 2700$		$r_1 = 2000$	
	$k_m = \frac{1}{3}m$		$k = \frac{1}{6}$		empirical k'		$k = \frac{1}{6}$	
m	k_m	ratio	k_m	ratio	k_m	ratio	k_m	ratio
7	.000	1.00	.053	.65	.034	1.30	.054	.56
6	.001	1.01	.060	.81	.030	1.49	.062	.71
5	.004	1.01	.067	1.02	.028	1.64	.072	.91
4	.012	1.02	.076	1.34	.031	1.93	.082	1.18
3	.037	1.08	.085	1.86	.037	2.22	.095	1.69
2	.111	1.25	.098	2.79	.045	2.67	.109	2.63
1	.333	2.00	.167	5.00	.228	3.38	.167	5.00

The nature of the service multipliers is certainly the most significant determinant of the basic/non-basic ratio. The initial constraint is induced by "k_1" in each case but the variability of decline brings out some very interesting patterns.

For instance, when central place functions become extremely specialized (advanced technology, capital intensive perhaps) and rely very little on employment, we might expect a system similar to the first. In this case, the exponentially declining factor levels off the basic/non-basic ratio very quickly. Urban populations in large communities are restricted in the sense that the capture of markets for higher order goods and services brings in little employment; in fact, the community assumes a smaller and smaller proportion of the total market population as both grow larger. Employment becomes increasingly balanced between the basic and non-basic sectors since $\sum_{i=1}^{m} k_i \rightarrow \frac{1}{2}$ as "M" becomes greater.

The second and fourth systems are characterized by gradual functional specialization. Since employment does not taper off rapidly for higher order goods and services, a variety of basic/non-basic ratios is permitted. In both systems, central places form a constant proportion of their total market populations but, as we ascend the hierarchy, both service multipliers and the ratios decline. It seems that the percentage increases in basic activity bring forth even greater percentage increases in non-basic endeavours until absolute increases favor the latter in the smaller size classes. The greatest increments in population increase are apparently determined by:

(i) The capture of the additional markets for the highest order function; and

(ii) The additional demands placed on the first order goods and services by the new members of the basic sector.

The Christaller system introduces another type of multiplier variation, where $k_2, k_3, \ldots, k_m$ are relatively constant. Nevertheless, the basic/non-basic ratio steadily declines as we move up through the hierarchy. This seems further proof that the provision of first-order commodities in response to demands made by additional basic employees is an extremely important determinant of the size of the urban community. We notice, too, that in this empirical example the ratio always exceeds unity (that is, $\sum_{i=1}^{m} k_i < \frac{1}{2}$).

The geometry of the city system also influences the nature of the ratio but in a less spectacular fashion. It appears that with an increase in the number of satellite cities, basic activity gives way to local services in a more rapid fashion as we ascend the hierarchy. Higher multipliers are needed to meet the demands of more smaller centers in the q=4 system; likewise, this introduces the need for further expansion of the non-basic sector (where lower order goods have higher multipliers). It seems that as the individual multipliers converge at high levels of the hierarchy, the resultant basic/non-basic ratios remain significantly different.

The city size models indicate some relevant patterns in the variability of basic/non-basic ratios, at least within the domain of activities that central place theory seems to cover. Besides, we see more clearly how the size distribution of urban communities both constrains and is influenced by the individual urban economies through the central place hierarchy. It seems relevant, then, that we should be more aware of the characteristics of central place size distributions.

Hierarchial Models and the Rank-Size Rule

Geographers direct considerable effort toward describing the frequency distributions of urban centers as based on city size models or empirical evidence. We leave discussion of the latter issue until the next chapter and here we examine the course of arguments concerning the size and frequency distribution of centers in a central place hierarchy.

The rank-size rule continues to be the dominant topic of interest in relation to city size models. It may be represented in the following form:

$$p_M = R^b p_R \qquad \ldots (3.33)$$

where "R" is the rank of the city, 'p_R" is the population of the city of rank "R:, "p_M" is the population of the largest city, and 'b' is a derived constant. If we graph this function on double logarithmic paper, we have a straight line:

$$\log p_R = \log p_M - b \log R \qquad . . . (3.34)$$

As originally (and usually) interpreted, "b" has a value of unity and the population of the "R"th largest center multiplied by its rank "R" equals "p_M".

Hoover (1955) appears to be one of the first to seriously question the relations of Christaller's central place hierarchy and the rank-size principle. He notes ". . . that the Christaller system automatically yields a series of city tributary areas arranged according to the rank-size rule . . . ", but fails to suggest a scheme that links central place populations and the principle (Hoover, 1955:196).

Beckmann (1958) is again the first to explicitly comment on this relationship. While his original model has been shown to be faulty, his clever approach to this new issue merits praise.

We recall that Beckmann's earliest model (in corrected form) employs a constant basic progression component " $\frac{s}{1-k} + 1$". However, if we consider this multiplier as a random variable about that stated constant, then all cities on the same hierarchial level do not necessarily have identical populations. Besides, the component has greater variations as "m" increases: in other words, the city sizes approach a continuous rather than steplike distribution. Beckmann is essentially altering the rigid

Christaller system to random disturbances so that only the midway center of any given hierarchial level is representative of all cities on that level.

Parr (1969) demonstrates that the overall rank "R_n" of a city midway in the "n"th size class can be expressed as:

$$R_n = q^0 + (q^1 - q^0) + (q^2 - q^1) + \ldots + \frac{(q^{n-1} - q^{n-2} + \alpha)}{2} \quad \ldots (3.35)$$

for $n > 1$, $\alpha = 0$ or 1, where "α" is unity if the number of centers in the size class is even and "α" is zero if that number is odd. Since only the second (that is for $n > 1$) size class can possess an odd number of central places, "α" is usually one and "R_n" is written more conveniently as:

$$R_n = \frac{(q^{n-1} + q^{n-2} + 1)}{2} = \frac{(1+s)^{n-1} + (1+s)^{n-2} + 1}{2} \quad \ldots (3.36)$$

Let's first of all consider the aggregate model with the particular b = 1 case of (3.33). Now if Model II can accommodate the rank-size distribution, then the product of the overall rank of a midway city on a particular hierarchial level and the population of that city must equal the population of the largest city in the system. Hence,

$$(R_n) \; \frac{kr_1}{1-k} \left(\frac{s}{1-k} + 1 \right)^{m-1} \stackrel{?}{=} \frac{kr_1}{1-k} \left(\frac{s}{1-k} + 1 \right)^{M-1} \quad \ldots (3.37)$$

or,

$$\frac{(1+s)^{n-1} + (1+s)^{n-2} + 1}{2} \stackrel{?}{=} \left(\frac{s}{1-k} + 1\right)^{n-1} \quad \ldots (3.38)$$

But since $1 + s < \left\{ \left(s/1-k \right) + 1 \right\}$, it is simple to demonstrate that the left side (rank) of (3.38) is always exceeded by the right side (power of the progression component) and no compatibility exists between a central place system based on a constant center/market population ratio and a rank-size distribution with an exponent of one.

Parr also considers the possibility that coincidence of the aggregate model and the rank-size principle may exist for $b \neq 1$; in this case:

$$(R_n)^b \frac{kr_1}{1-k} \left(\frac{s}{1-k} + 1\right)^{m-1} \stackrel{?}{=} (R_{n+1})^b \frac{kr_1}{1-k} \left(\frac{s}{1-k} + 1\right)^{m-2} \quad \ldots (3.39)$$

or,

$$b \stackrel{?}{=} \log\left(\frac{s}{1-k} + 1\right) \Big/ (\log R_{n+1} - \log R_n) \quad \ldots (3.40)$$

By demonstrating that the denominator on the right side of (3.40) is variable, he is able to stipulate that the value of "b" in (3.39) and (3.40) varies with "n" and that, therefore, the initial assumption of a constant "b" is violated. In Parr's (1969:249) words: ". . . it may therefore be concluded that a central place system based on the constant proportionality factor is not compatible with a rank-size distribution even where the value of the constant "b" assumes a value other than unity."[2]

Beckmann and McPherson feel that a sufficient condition for rank-size central place coincidence is that market area populations increase by a constant multiplier from level to level (see 3.24), the assumption they maintain to devise Model III. At this point in the discussion we present an argument that seems to refute this this assertion.

To begin with, we consider the b = 1 case of (3.33). Using (3.1) and (3.27) we see that:

$$\frac{p_M}{p_1} = 1 + \frac{1}{1-k_1}\left\{\frac{\left(\frac{1}{1-k_1}\right)^{M-1}(1+s)^M - (1+s)}{\left(\frac{1}{1-k_1}\right)(1+s) - 1}\right\} \quad . . . (3.41)$$

But employing (3.36), if compatibility occurs then:

$$(1+s)^{M-1} + (1+s)^{M-2} - 1 \stackrel{?}{=} \frac{2}{1-k_1}\left\{\frac{\left(\frac{1}{1-k_1}\right)^{M-1}(1+s)^M - (1+s)}{\left(\frac{1}{1-k_1}\right)(1+s) - 1}\right\} \quad . . . (3.42)$$

[2]Note our substitution of "b" for "q" in Parr's article; this discrepancy is due only to a difference in notation.

or,

$$\left\{\left(\frac{1}{1-k_1}\right)(1+s) - 1\right\} \left\{(1+s)^{M-1} + (1+s)^{M-2} - 1\right\}$$
$$\overset{?}{=} \frac{2}{1-k_1} \left\{\left(\frac{1}{1-k_1}\right)^{M-1} (1+s)^M - (1+s)\right\} \quad \ldots (3.43)$$

where the left side equals:

$$\frac{1}{1-k_1} \left\{(1+s)^M + k_1 (1+s)^{M-1} - (1-k_1)(1+s)^{M-2} - (1+s) + (1-k_1)\right\}$$

Therefore, to state that the left and right sides of (3.43) are identical means:

$$(1+s)^M + k_1 (1+s)^{M-1} - (1-k_1)(1+s)^{M-2} - (1+s) +$$
$$(1-k_1) \overset{?}{=} 2 \left\{\left(\frac{1}{1-k_1}\right)^{M-1} (1+s)^M - (1+s)\right\} \quad \ldots (3.44)$$

or,

$$k_1 (1+s)^{M-1} - (1-k_1)(1+s)^{M-2} + (1+s) + (1-k_1)$$
$$\overset{?}{=} 2 (1+s)^M \left\{\left(\frac{1}{1-k_1}\right)^{M-1} - \tfrac{1}{2}\right\} \quad \ldots (3.45)$$

But for all $s > 0$, $0 < k_1 < 1$, we know:

$$2(1+s)^{M-1} + 2(1+s)^{M-2} + \ldots + 2(1+s)^0 < 2(1+s)^M$$

Hence, by noting that:

$$k_1\ (1+s)^{M-1} - (1-k_1)\ (1+s)^{M-2} + (1+s) + (1-k_1)$$
$$< 2\ (1+s)^{M-1} + \ldots + 2(1+s)^0$$

and that the factor $\left\{ \left(\frac{1}{1-k_1}\right)^{M-1} - \frac{1}{2} \right\}$ exceeds unity for $M \gg 0$, then the left side of (3.45) is exceeded by the right side; the rapidly expanding growth factor disallows the model from being coincident with the rank-size arrangement for b = 1.

Next we consider compatibility of the second elementary model with the rank-size principle when the exponent is not restricted to unity. We should note that Beckmann and McPherson do not allude to this general case but direct their argument to the particular case just refuted.

The proof in this case is not as elegant as the one just outlined since several of its statements involve taking limits when $M \gg 0$. We recall (3.41), which gives in simpler form:

$$\frac{p_M}{p_1} = \frac{\left(\frac{1}{1-k_1}\right)^{M-1} (1+s)^M - (1-k_1)}{s + k_1} \quad \ldots (3.46)$$

Therefore:

$$\frac{p_2}{p_1} = \frac{\left(\frac{1}{1-k_1}\right)(1+s)^2 - (1-k_1)}{s + k_1} \quad \ldots (3.47)$$

Now if the rank-size rule holds for the model and $b \neq 1$, then:

$$p_1 R_M^{\,b} = p_2 R_{M-1}^{\,b} = \ldots = p_M$$

Here we have two cases (among "M-1") of immediate interest:

$$① \quad R_M^{\,b} \overset{?}{=} \frac{p_M}{p_1} \qquad \ldots (3.48)$$

$$② \quad \left(\frac{RM}{R_{M-1}}\right)^b \overset{?}{=} \frac{p_2}{p_1} \qquad \ldots (3.49)$$

meaning:

$$b\ ① = \frac{\log (p_M/p_1)}{\log R_M} \qquad \ldots (3.50)$$

$$b\ ② = \frac{\log (p_2/p_1)}{\log (R_M/R_{m-1})} \qquad \ldots (3.51)$$

Considering b ①, we can define b' ① $>$ b ① where:

$$b'\ ① = \frac{\log\left\{\dfrac{\left(\dfrac{1}{1-k_1}\right)^{M-1} (1+s)^M}{s}\right\}}{\log R_M} \qquad \ldots (3.52)$$

$$= \frac{(M-1) \log \left(\dfrac{1}{1-k_1}\right)}{\log R_M} + \frac{M \log \left(\dfrac{1+s}{s}\right)}{\log R_M} \qquad \ldots (3.53)$$

Now the first of these terms is less than $\dfrac{M-1 \log \left(\dfrac{1}{1-k_1}\right)}{M-2 \log (1+s)}$

and the second term is less than $\dfrac{M \log \left\{ \dfrac{(1+s)}{s} \right\}}{M-2 \log (1+s)}$

(by substituting (3.36) into the denominator of (3.53)).

As M >> 0, these terms converge toward $\dfrac{\log\left(\dfrac{1}{1-k_1}\right)}{\log (1+s)}$

and $\dfrac{\log\left(\dfrac{1+s}{s}\right)}{\log (1+s)}$ respectively.

In other words, b′ ① itself converges at the sum:

$$\frac{\log\left(\dfrac{1}{1-k_1}\right) + \log\left(\dfrac{1+s}{s}\right)}{\log (1+s)}$$

or,

$$b' \; ① = \frac{\log \left\{ \dfrac{\left(\dfrac{1}{1-k_1}\right)(1+s)}{s} \right\}}{\log (1+s)} \quad \text{for } M >> 0 \qquad \ldots (3.54)$$

On the other hand, we may rewrite (3.51) as:

$$b \; ② = \frac{\log \left\{ \dfrac{\left(\dfrac{1}{1-k_1}\right)(1+s)^2 - (1-k_1)}{s+k_1} \right\}}{\log (1+s)} \quad \text{for } M >> 0 \qquad \ldots (3.55)$$

However, when we compare (3.54) and (3.55) we find that the numerator in the former is always exceeded by that in the latter. Therefore, for $s > 0$, $0 < k_1 < 1$, $M >> 0$, we have b ① < b′ ① < b ② and the rank-size rule is not valid for an exponent "b" unequal to unity.

The disassociation of the geometric multiplier model from the rank-size approach should come as no surprise in light of Parr's earlier analysis. In that case, market populations grow at a constant rate but city sizes grow too quickly for city rank declines when these centers expand at the same rate. In this later case, market populations grow at a constant rate but city growth exceeds that rate; hence, we can expect again that city size will outweigh the rank value and that a constant product of rank and size cannot be realized.

The third simple model, as typified by a constant service multiplier that emerges at the second level, is more difficult to relate to rank-size thinking. As we noted before, no explicit rationale determines the nature of "k " and we cannot develop an argument similar in form to that for Models II and III. Nevertheless, we intuitively expect that sizes expand too rapidly for rank declines, since this model overestimates the aggregate model at high levels (that is when "k'" depends on the sum of k_2, k_3, . . . ,k_m in Model II). Therefore we do not consider Model IV and the rank-size principle as being compatible concepts.

Dacey is unsuccessful in defining a sequence of service multipliers that permits the general model to conform to a rank-size distribution, but from the tone of his article he may well be restricting his search to a set of functionally related "k_i's". Nevertheless, if we can identify any set of multipliers that gives compatibility, then we cannot accept that populations in a central place system are at variance with the rank-size rule.

Beginning with the lowest levels of the hierarchy, a sufficient condition for "p_1" and "p_2" to be rank-size related is:

$$\frac{R_M}{R_{M-1}} \; p_1 = p_2 \qquad \ldots (3.56)$$

Now by employing (3.56) to define central place populations, one can stipulate a service multiplier "k_2" by introducing (3.17). In other words we can construct a more general statement than (3.19) in which service multipliers (one at a time) are designated so as to generate a rank-size distribution among urban communities:

$$k_m = \frac{p_m - \sum_{i=1}^{m-1} k_i \, (p_m + r_i)}{p_m + r_m} \qquad \ldots (3.57)$$

While this approach is inductive and totally lacking in theoretical rationale, it does establish some association between the deductive features of central place systems and the more empirically founded (see Chapter 4) rank-size principle. On the other hand, there is no suggestion as yet that the rank-size rule may be interpreted as a law ststement within the framework of central place theory.

In table 2 we present in a rather comprehensive fashion the various properties of four central place hierarchies, each generated from $M = 7$, $k_1 = 0.333$, $s = 2$, $r_1 = 2000$:

(i) Model I formulated to conform to constant rank-size products;

(ii) Model II;

(iii) Model III;

(iv) Model IV with "k'" estimated from the service multipliers (k_2, k_3, . . ., k_m.) defined in (i).

The table is useful for qualifying any of the statements we have made to this point in the discussion.

Table 2

Fundamental Properties of Midway Cities in Related Central Place Systems via Diverse Modelling Approaches

Model I: The General Case - Rank-Size Pattern							
m	rank	p_m	r_m	p_m	k_m	p_m/p_m	Rank x Size
7	1	486,500	4,223,940	4,710,440	.021	.103	486,500
6	2.5	194,700	1,278,180	1,472,880	.029	.132	486,500
5	6.5	75,000	376,060	451,060	.044	.166	486,500
4	18.5	26,300	107,820	134,120	.060	.196	486,500
3	54.5	8,940	29,980	38,920	.084	.229	486,500
2	162.5	2,990	8,000	10,990	.120	.273	486,500
1	486.5	1,000	2,000	3,000	.333	.333	486,500

Table 2 (Continued)

Model II: The Aggregate Approach

7	1	4,096,000	8,192,000	12,288,000	.040	.333	4,096,000
6	2.5	1,024,000	2,048,000	3,072,000	.053	.333	2,560,000
5	6.5	256,000	512,000	768,000	.070	.333	1,664,000
4	18.5	64,000	128,000	192,000	.094	.333	1,184,000
3	54.5	16,000	32,000	48,000	.125	.333	872,000
2	162.5	4,000	8,000	12,000	.167	.333	650,000
1	486.5	1,000	2,000	3,000	.333	.333	486,500

Model III: The Geometric Multiplier

7	1	10,450,000	14,143,000	24,593,000	.029	.428	10,450,000
6	2.5	2,330,000	3,161,000	5,491,000	.043	.428	5,800,000
5	6.5	532,000	699,000	1,231,000	.067	.428	3,460,000
4	18.5	115,500	156,000	271,500	.099	.427	2,140,000
3	54.5	25,500	35,000	60,500	.148	.422	1,390,000
2	162.5	5,500	8,000	13,500	.222	.406	892,000
1	486.5	1,000	2,000	3,000	.333	.333	486,500

Model IV: The Constant Multiplier - "k′" est. from Model I

7	1	1,025,000	3,727,300	4,752,300	.060	.222	1,025,000
6	2.5	251,000	1,075,100	1,326,100	.060	.190	628,000
5	6.5	64,000	315,700	379,700	.060	.169	416,000
4	18.5	17,300	93,700	111,000	.060	.156	320,000
3	54.5	5,150	27,800	32,950	.060	.155	281,000
2	162.5	1,900	8,000	9,890	.060	.191	309,000
1	486.5	1,000	2,000	3,000	.333	.333	486,500

Hierarchial Sets and the Rank-Size Rule

We have already indicated that the aggregate model appears to be the most suitable approach in light of (i) existing theory, (ii) existing empirical evidence, and (iii) elegance. However, table 2 illustrates that apparently similar declines of the service multipliers is not a sufficient reason to expect similarity in the nature of rank-size products.

Parr suggests that the size distribution of centers on the endpoints of each hierarchial level gives descriptive support to coincidence of the basic progression component model and the rank-size rule. On the other hand, he rightfully notes that compatibility of the two notions cannot be inferred as such. This raises the question of whether or not the aggregate model, through any reasonable modification, can be aligned to rank-size thinking?

Surely, though, internal modification defeats the purpose of a model whose strength lies in its simplicity. For instance, variation of the "k" factor from level to level adds more unknowns to the argument, while changes in the number of satellite cities erases the concept of a general progression multiplier. Since we may interpret the distribution of urban sizes through one rather intricate a posteriori model, it seems unreasonable to manipulate an elementary model having its own distinct advantages.

However, an extension of our single system framework allows a new association between Model II and the rank-size principle to arise. If we consider a set of independent central place systems, then we may consider the overall pattern of city size formed by the various independent hierarchies. We may pursue the approach that we used to compute the "k_i's" for the general model so as to relate to the rank-size principle (see (3.56) and (3.57)). In this case we generate a hypothetical "M" level system of central places and demonstrate that, with the addition of various smaller systems (that is, ones with M-1, M-2, . . ., 2, or 1 levels), an overall rank-size arrangement may evolve.

Clearly, the idea is to determine ranks when a constant growth factor " $\frac{s}{1-k} + 1$ " is supposed for an entire set of cities. We use the same conceptual method of the initial Beckmann contribution where actual populations vary about this modal value and add, too, that an increased number of centers in each aggregate size class (except the first) means that a smoother decline in city sizes is now more likely.

The single largest center of the "M" level system gives us the central place of rank one for the entire set. Our next step is construct a hypothetical size class of population p_{M-1} so that the midpoint of the group has rank " $\frac{s}{1-k} = 1$ ". Recalling (3.36), we know that there are "s" centers of that second size class in the complete

"M" level system; hence, we add "x_1" more centers so that:

$$\frac{1 + s + (x_1 + 1)}{2} = \frac{s}{1-k} + 1 \qquad \ldots (3.58)$$

Obviously we are building up "x_1" independent central place systems of "M-1" levels each to supplement the first system. Likewise, we continue to the third class where we already have "s(s+1)" centers in the complete system plus "sx_1" centers in the smaller systems. Now we must add "x_2" new centers of "M-2" level systems by solving:

$$1 + s + x_1 + \frac{s(s+1) + s(x_1) + (x_2+1)}{2} = \left(\frac{s}{1-k} + 1\right)^2 \qquad \ldots (3.59)$$

In this manner we can develop additional smaller central place systems for varying size classes "n" so that:

$$x_{n-1} = 2\left(\frac{s}{1-k} + 1\right)^{n-1} - (s+2)\sum_{i=0}^{n-2} x_i\,(s+1)^{n-2-i} - 1 \qquad \ldots (3.60)$$

To clarify this argument, "x_i's" are determined for the initial central place system in the following table:

Table 3

Constant Rank-Size Products Given by Independent Hierarchial Sets via Model II

Size Class	Population	Rank	h_0[a]	h_1	h_2	h_3	h_4	h_5	h_6	Total
1	4,096,000	1	1	-	-	-	-	-	-	1
2	1,024,000	4	2	3(x_1)	-	-	-	-	-	5
3	256,000	16	6	6	7(x_2)	-	-	-	-	19
4	64,000	64	18	18	14	27(x_3)	-	-	-	77
5	16,000	256	54	54	42	54	103(x_4)	-	-	307
6	4,000	1,024	162	162	126	162	206	411(x_5)	-	1,229
7	1,000	4,096	486	486	378	486	618	822	1,639(x_6)	4,915

[a]The "h_i" columns indicate the total number of places in each size class of independent hierarchial sets; "i" refers to the number of levels missing from "M".

We should emphasize, however, that the various systems must be assumed to be integrated in a size distribution sense alone, since we are assuming that spatial integration of the systems is non-existent. It is difficult to justify this supposition in a real world case, but the idea of a very large territory that diplays regional economic closure approximates the idea. We will summarize the empirical size distribution literature in the next chapter and this should shed some light on the nature of large territories and closure in the real world. On the other hand, we must consider this argument as an extremely hypothetical one that only demonstrates how the aggregate model can conform to the rank-size rule and not why it does.

Chapter 4

EMPIRICAL ANALYSIS AND INTERPRETATION

In this chapter we have three general and interrelated objectives:

(i) To critically examine the particular techniques employed in empirical city size studies;

(ii) To review existing stochastic interpretations so that they supplement one another;

(iii) To stipulate whether or not these interpretations are satisfactory explanations of city size patterns.

Discussion is arranged to highlight the improvement of methodological concern in the subject, while emphasizing the explicit role of theory in explaining the population distribution amongst urban places.

Background

> Pick any large area. It will likely contain many small cities, a lesser number of medium-size cities, and but few large cities. This pattern of city sizes has been observed to be quite regular from one area to another. That is, when the frequency of occurrence of city sizes in any area is compared with the frequency of occurrence of sizes in another area, the two frequencies are very much alike . . . Such empirical regularities of city size have been noted many times and have long posed a challenge to those who would explain or interpret them.
>
> (Berry and Garrison, 1958a:83)

In a few words we are briskly introduced to the more relevant features of empirical city size discussions. The literature is characterized by numerous disparate contributions and, unfortunately, few reviews attempt to integrate the various concepts and schemes into a meaningful whole. In the interests of avoiding repetition in terms (as well as in hypotheses for that matter) we adopt a consistent model framework for explanation: that being, of course, the concept of city system.

Berry (1964), for one, particularly advocates the grounding or urban theory in a general systems approach. The flexibility of systems inquiry and its stressing of interactions and interdependencies suggest that the spatial system is a most adequate conceptual device to bridge theoretical and empirical contributions. However, where the notion is applied with regard to city systems, it is often done implicitly and the student wonders why attention is devoted to the idea at all. With explicit use of the city system idea however, empirical generalizations may become increasingly substantive since it affords a consistent base to rationally order sense-perception data (Harvey, 1969:33). Besides, central place theory, as we have seen, is naturally couched in this framework and the system notion seems advantageous (if not essential) for stating deduced propositions so that they may be empirically tested. In other words, perceiving city sets as systems may be defended as a most satisfactory methodological device in that it serves to (i) initiate and (ii) substantiate geographic theory.

Upon considering such a diffuse topic as city systems, where debate covers a number of issues, it may be more productive to isolate several points for discussion rather than attempt to unite the frequently independent ideas in a chronological sense. The statement used to introduce this section appears to highlight these issues:

(i) What constitutes "any large area"?

(ii) How do we descrive a "pattern of city sizes"?

(iii) How do we demonstrate that "two frequencies are very much alike"?

(iv) In what manner do we "explain or interpret" these empirical regularities?

It is hoped that through an examination of these basic questions we can define which methodological qualities are wanting in existing investigations.

The Study Area

The initial matter is by far the most neglected although it should be considered critical to any interpretation of city size distributions. Without devoting some attention to the nature of the study area a consistent point of view is forgone and comparability of different investigations becomes impossible. In its loosest interpretation, a "large area" is an intuitive general classification in that we are abstracting a subset of urban centers from the universal set of all centers (whatever our definition of "urban" may be). But such an arbitrary classification process seems hardly acceptable toward offering consistent selection measures throughout space and time.

The earliest studies consider entire nations as appropriate study regions. Auerbach (1913), Lotka (1924, 1941), and Zipf (1949) give original impetus to the rank-size thesis as a description of the size of all cities in a country above some designated population threshold. In fact, there is a definite theme of national integration in many of the rank-size arguments (Zipf, 1949; J. Q. Stewart, 1947; Berry, 1961).

Jefferson (1939:231), on the other hand, evokes the principle of the primate city where "A country's leading city is always disproportionately large and exceptionally expressive of national capacity and feeling." He considers only the trio of largest centers throughout a sample of national units in order to index size relations and, therefore, the domain of his statement is severely restricted. While the largest center appears to be much greater than the second and third centers, it is never clear how it is considered disproportionately greater.

This difference of opinion is only resolved when we stipulate what is considered an appropriate sample space in each national unit. For example, when we generated populations in the first two size classes of the hierarchial models (see Chapter 3), the largest center was always much greater than the following two, but only to a degree that was determined by the parameters of the entire system. A review of Jefferson's (1939:228) data indicates there is sufficient reason to doubt his law statement on the small samples alone; to also infer a property that is supposedly characteristic of the entire national system from such a small sample is yet another matter and must be treated with additional skepticism.

However, a more essential problem must be settled before we can even consider the comparison of the overall city size distributions in these national territories: we must state unequivocally whether the argument concerns city sets or city systems. The latter term, of course, calls for additional functional relationships among the urban centers and likewise suggests that there exist some criteria of self-sufficiency or closure within a city set. The importance of this dichotomy is that inferences derived from systems may be carried into sets but not vice versa. Or, to take an example, Zipf (1949) cannot really state that the exponent "b" in a rank-size relationship indicates whether or not a national system of urban communities is integrated when he selects the elements of that system in a priori fashion.

It seems that our choice must rest solely upon the immediate purpose of our argument. If we wish to suggest a rather general empirical relationship that appears to persist (i) internationally at one point in time or

(iii) nationally for several time periods, then the simple grouping of cities (above some minimum threshold level) seems a reasonable approach. On the other hand, if we wish to formulate (i) comparisons between subnational and national units or (ii) amongst the subnational units themselves, or if we plan to (iii) offer some economic rationale for this regularity, then the systems concept seems absolutely necessary. Adherence to this systems framework is simply a measure for ensuring consistency or accuracy in our inferences since we have no reason to expect that the size distributions are independent of how study areas are delimited. Since it is a most common thesis that urban populations are determined by functional differentiation and the degree of interaction among economic activities distributed in space, it is sensible that these urban elements be delimited by the very factors that determine their magnitude.

Spatial interaction (socio-economic flows) between city pairs seems adequately described by several interactance hypotheses (gravity models and graph theory applications, in particular), but deviations result from physical, cultural, and political barriers. Hence, when we postulate that a set of cities in a national territory is distinct from sets in adjacent territories (and therefore corresponds to the ". . . general intuitive notion as regards classification, namely, that the classes be as distinct from one another as possible and internally as homogeneous as possible" (Harvey, 1969:339)), we are really asserting that international boundaries are barriers of paramount influence. Fortunately, empirical evidence tends to firmly support this supposition. Mackay (1958) and Nystuen and Dacey (1961), for example, find that the international boundary between Canada and the United States considerably reduces traffic between city pairs. However, it is hardly clear at this time how interaction tapers off with (i) varying distance decay qualities of the flow commodities or (ii) the population potential of the general area. Nevertheless there are reasonable signs that city sets and city systems are highly coincident at the national level. Studying the changing features of national city size distributions through time seems to follow in the same vein. While annexation or cession of areas (and urban centers) or development forced upon resource frontiers (Friedmann, 1966) differentially shock stable patterns of national growth, the failure of these new elements to enter the national space-economy can only persist in the short run.

However, retaining the systems expression in but an implicit role would still prevent an apparent misconception concerning international comparison. Consider the nature of urbanization in today's typical underdeveloped (low per capita income, technological and regional dualism) nations just prior to the introduction of investment and technical innovation, Their urban structures at this time of initial awareness (or adoption) of the merits of economic competition may be generally represented by a number of comparatively small and independent agricultural communities. On the other hand, in some of these nations there persists (where production surpluses allow) the lineaments of a rather elaborate social and/or political hierarchy that accounts for size differences among the large centers.

Now the evolution of a progressive space-economy from this subsistence base is accompanied by the emergence of a city system defined by principles of economic comparative advantage. However, in the earliest stages of development, a nation's economic space is somewhat directed by the existing lines of administrative and social organization (for example, larger urban centers offer more concentrated markets, transportation routes tend to focus on the larger places, etc.). In fact, the original socio-political hierarchy may be considered as the most prevalent of several initial conditions - demographic, physical (terrain and amount of arable land), and cultural variables - which constrain the initial configuration of economic development. The suggestion is offered that the economic hierarchy is more a condition of convergence that dominates spatial organization only in the long run (when factors of production tend to be mobile and activities whose input prices vary relatively little in space become increasingly demanded) and then makes possible the thorough integration of social, political, and economic spaces as a unit (Friedmann, 1961).

The important point here, though, is that nations with comparable low indices of economic development may be represented by considerable variety in their city size distributions. Also, it becomes dangerous to assume that international cross-time data and national time-series data are simply interchangeable (Lasuen, Lorca, and Oria, 1967). This implies that Berry (1961:585) is not really justified in making the time-series statement ". . . that different city size distributions are in no way related to the relative economic development of countries. Rank size is not the culmination of a process in which national unity is expressed in a system of cities." when his observations are of a cross-time international nature.

In the more recent contributions, there is increasing stress on investigating city size distributions in sub-national or regional territories, especially in the role of a crude planning device (Dziewonski, 1964; Lasuen, Lorca, and Oria, 1967; Vapnarsky, 1969). However the same studies suggesting that national administrative units necessarily delimit city systems tend to refute the value of subnational political units as parallel cases. Mackay's (1958) study of interaction indicates that internal administrative boundaries are considerably more permeable than international boundaries. Nystuen and Dacey (1961) illustrate that centers in the political region may dominate peripheral communities in an adjacent region. Clearly, then, the existence of a regional city system that displays functional structuring is rather independent of state or provincial boundaries, except where national policy advocates high closure in these units. We should stress that there exists great variety in the nature of these regional economies (as delimited by political boundaries); so much, in fact, that one wonders if we are drawing inferences from or providing interpretations for comparable sets of cities.

C. T. Stewart (1958) is perhaps the earliest observer to propose a more adaptable definition of the study area. He suggests that self-sufficiency is the criterion that may lead to generalization. Vapnarsky (1969) extends this view by framing the complementary terms of rank-size and primacy within the condition of closure. The tenor of his argument is that unless subnational administrative units display properties of nodal regions, then they are ". . . totally unrelated to either closure or interdependence in an ecological sense. . ." and it is difficult to defend hypotheses and explanation based on such arbitrary areas (Vapnarsky, 1969:589).

Similarly, the urban system seems to be the only means of carrying generalizations from the regional level to the national level without being totally confused by the scale problem (Harvey, 1969:352-353, 452-454). It seems, though, that we should be more confident in making interregional comparisons than in carrying generalizations through different resolution levels.

We see within the mainstream of the city size topic a certain change in the purpose of investigation and the concomitant increase in attention devoted to refining methodology. While opinions may still differ on how to explain city size patterns it is clear that an interpretation resting upon economic rationale requires some comparability in defining systems. Unfortunately, this comparability is only given in terms like closure and interdependence that are hardly objectively stated as yet.

We note, too, that there is a noticeable tendency to avoid formulating these properties along the lines of central place theory. In other words, city systems are commonly delimited a priori through an areal (hinterland of the largest center) point of view, when the circulation that bonds the system in a whole is clearly linear. With an approach more in line with theory, logical division of the centers is called for, so that the lines of dominance may be studied (Marshall, 1966). Unfortunately, the Loschian model (that seems more satisfactory for the secondary sector) is not so flexible as the Christaller model in this regard, and the idea is still confined to more agricultural regions where the tertiary sector is paramount. On the other hand, we expect that the coincidence of a city system and the related nodal region of the dominant center breaks down in only the periphery where smaller centers may be evenly attracted to adjacent systems.

In summary, "any large area" is adequate as a comparable base in only a qualified sense. Comparing the city size distributions of Korea and Washington State (Berry and Garrison, 1958a:83), for example, is an interesting descriptive undertaking but it clouds the case for rational economic explanation. Only through the use of notions like closure and interdependency, which are firmly based in our only theory of the size, spacing, and functions of urban centers, can we hope to give a consistent tone to this explanation.

City Size Patterns: Skew Distributions and Related Concepts

A second general issue concerns the description of the city size arrangement. While all observers agree that the frequency distribution of cities by size appears to be highly skewed in the shape of a reverse-J, opinion is divided over the nature of the size classes of urban centers and their role in determining these frequency distributions. To add to the confusion, critics point to a number of probability distributions (each showing a family resemblance through positive skewness) that adequately describe the pattern of city sizes in many areas. The first of these problems is taken up at a later time in the discussion while the latter is of immediate concern here.

The Rank-Size and Pareto Distributions

A large portion of the literature is devoted to the applicability of the rank-size principle. For convenience, we recall the relevant equations stated earlier:

$$p_M = R^b p_R \qquad \ldots (3.33)$$

$$\log p_R = \log p_M - b \log R \qquad \ldots (3.34)$$

In empirical approaches, however, we have no justification for expecting a precise least squares fit to (3.34) and it is frequently proposed that a constant "B" (where B $\approx$ p_M) affords an improved statistical description of the relationship. This approximation is defensible in light of (i) the notion that populations may be accepted as being the same when they differ only by chance (Thomas, 1961) and (ii) the data liabilities regarding the definition of the individual urban centers (that is, are corporate or metropolitan entities most appropriate?), but must be determined in a totally objective manner.

By plotting ranks versus sizes on double logarithmic paper, straight line tendencies may be observed where the sample follows (3.34). Goodness of fit may be determined through the linear regression model. The use of "B", then, is feasible if (i) we are confident in the overall covariation of the two variables with "p_M" as the largest center and (ii) "B" itself lies within some limiting error band determined by the sample (Thomas, 1967).

Hence, in general, the rank-size equations may be reinterpreted as:

$$B = R^b p_R \qquad \ldots (4.1)$$

or,

$$\log p_R = \log B - b \log R \qquad \ldots (4.2)$$

where $B \approx p_M$

Singer (1936) and Allen (1954) argue the case for a similar Pareto curve representation, where regularity in the city size pattern is characterized by:

$$A = R p_R{}^a \qquad \ldots (4.3)$$

or,

$$\log R = \log A - a \log p_R \qquad \ldots (4.4)$$

The Pareto form is less useful in the interests of comparing theoretical central place systems but the authors do suggest (if ambiguously) a practical interpretation for the exponent "a". Since the slope in (4.4) designates the relative number of small, medium, and large centers, it becomes a satisfactory "index of metropolization" (Singer, 1936:254). When "a" is small and the least squares line is flat, the greater becomes the proportion of large cities in a given number of cities. In other words, "a" is a useful index for historical description and international comparison since it illustrates how total urban populations are distributed among different sized centers.

On the other hand, "b" may be given a related interpretation, since (4.1) and (4.3) are coincident with $a = \frac{1}{b}$ and $A = B^{\frac{1}{b}}$. Also, when $a = b = 1$, the simplest formulation of the rank-size (Auerbach, 1913) and Pareto relationships requires that "A" and "B" are identical constants.

However, the complementary approaches bring out a unique and rather fundamental feature of the rank-size relationship. Since it is solely concerned with the manner in which all centers in an urban system are individually related to the largest center therein, the possibility of employing grouped data is obviated. Yet these empirical investigations that suggest data follow a Pareto (or some other member of the reverse-J family) distribution are all characterized by the use of grouped data with an open class interval[1] for the largest population values. In this case, "R" is really interpreted as the number of centers with population greater than or equal to the stipulated class boundary.

Due to the aforementioned data liabilities, in form and nature, and to facilitate analysis we frequently consider the discrete sample space of city sizes as being continuous or countably infinite. In fact (4.1) and (4.3) may be thought of as discrete analogues of specific

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[1]Class interval is not mistaken for size class which is a device attached to central place theory.

probability density functions. Nevertheless, it is helpful to be aware of what we are postulating when we employ a continuous approach (since analysis deals with intervals[2] and not points).

To begin with, the discrete approaches may have considerable discrepancies regarding their information in the upper tails (that is, large population values). To be more specific, the population of the largest center may deviate noticeably from the estimate of "A" given in (4.4). Unfortunately, it is in this realm of the size distribution that we are frequently most interested.

It seems that if we wish to make inferences from the discrete Pareto (using grouped data) to the rank size relationship, then we must make two basic assumptions:

(i) That the logarithms of city size are rather evenly distributed in each class interval; this supposition is more important in the size intervals of the upper tail where the sample data become increasingly sparse;

(ii) That $p_M \approxeq A$.

The second assumption is especially relevant because single city primacy may be inadvertedly clouded by use of the open class interval.

With these postulates, we may suggest how the index of metropolization is related to the usual index

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[2]Class intervals also refer to the range a variable assumes over a function.

of urbanization: the latter being, of course, the proportion of total population in an area that is classified as urban. Populations for the system of cities may be calculated through (4.1) and (4.3) when we are confident of a good empirical fit. For instance, if $a > 1$, $b < 1$, and "p_{max}" is the population of the threshold (smallest) place ("R" is maximized), (4.3) implies that total urban population is:

$$\int_{p_{max}}^{A} x \, d\left(\frac{-A}{x^a}\right) = aA \int_{p_{max}}^{A} x^{-a} dx$$

$$= \frac{aA}{1-a} \left\{ \frac{1}{A^{a-1}} - \frac{1}{p_{max}^{\;a-1}} \right\} \qquad \ldots (4.5)$$

Which converges (as $A >> 0$) at:

$$\lim_{A \to \infty} \int_{p_{max}}^{A} = \frac{aA}{a-1} \, p_{max}^{\;1-a} \qquad \ldots (4.6)$$

or,

$$\lim_{B^{\frac{1}{b}} \to \infty} \int_{p_{max}}^{B^{\frac{1}{b}}} = \frac{B^{\frac{1}{b}}}{1-b} \, p_{max}^{\;\frac{b-1}{b}} \qquad \ldots (4.7)$$

which are corrected forms of the Beckmann (1958:247) derivation.

On the other hand if $a < 1$, $b > 1$, (4.1) suggests that total urban population is:

$$\int_1^M By^{-b}\ dy = B\int_1^M y^{-b}\ dy$$

$$= \frac{B}{1-b}\left\{\frac{1}{M^{b-1}} - 1\right\} \qquad \ldots (4.8)$$

which converges ($M >> 0$) at:

$$\lim_{M \to \infty} \int_1^M = \frac{B}{b-1} \qquad \ldots (4.9)$$

$$= \frac{aA^{\frac{1}{a}}}{1-a} \qquad \ldots (4.10)$$

Unfortunately, a similar harmonic series derivation is not possible for a = b = 1, but total urban population in this case is always less than A ($\log_e R + 1$). J. Q. Stewart (1947) provides convenient approximations to finite summations for all three varities. It is interesting to note that when convergence does occur it may be attributed to either size (4.6) or rank (4.9) depending on the nature of the index of metropolization.

In any event, when a rank-size or Pareto relationship holds for a system of cities we can formulate an index of urbanization for the nodal region through (i) the index of metropolization "a", (ii) the constant "A", (iii) the size of the threshold center ($a > 1$), and (iv) total population. It is suggested that this frequency parameter (metropolization) complements an aggregate urbanization index that may be blurred by varying international conceptions of rural-urban distinction.

Steady-state Distributions

Some very intriguing efforts treat city-size patterns as equilibrium states of an underlying stochastic process. The approach therefore, is totally a priori: we are developing some real world interpretation to an abstract calculus in the hope that the structure of a related theory may be implied. For this reason it is not surprising that diverse phenomena (word frequencies in prose samples, income distributions, etc.) are given identical probability interpretations.

The arguement may be illustrated by considering a system of cities at time "t_1" divided into class intervals of equal proportionate width (that is, the logarithms of class boundaries are evenly spaced). Now, as forces (for instance: migration, investment, technology, entrepreneurship) operate on the individual communities through time (consider t_2, t_3, ..., t_n), the initial class intervals likely assume varying proportions of the total population in the system.

The redistribution of centers among the class intervals may be described by transition probabilities in a regular matrix (Adelman, 1958). If we assume:

(i) That the distribution of percentage changes over a time interval is the same in each class interval;

(ii) That these changes remain invariant over all time intervals;

then any initial distribution of city sizes approaches a unique equilibrium state as $t_n >> t_1$. When we depict the first postulate as a normal distribution (that is, the frequency distributions of percentage changes in population size of small, medium, and large communities all approach normal distributions with the same parameters), then we are essentially assuming the law of proportionate effect:

> A variate subject to a process of change is said to obey the law of proportionate effect if the change in the variate at any step of the process is a random proportion of the previous value of the variate.
> (Aitchison and Brown, 1957:22).

The implication here, of course, is that population sizes are lognormally distributed in the steady state, when the average number of centers entering each class interval per time period equals the average number departing.

The second assumption allows a crude estimation of the total population in the system to be given. For instance, if we think of all the centers having nearly identical propulations at time "t_1", then the spreading out of individual populations through the stochastic matrix accounts for a growth in total population (Adelman, 1958, assumes that the mean value in each class interval is the same at "t_n" as in "t_1"). We should realize, however, that this estimate is based on rather inflexible postulates since (i) further entry into the lowest class and (ii) extension beyond the highest class are disallowed.

Simon (1955) avoids the first restriction by allowing a steady introduction of new communities over the threshold size. The thrust of his argument is to provide an interpretation for the probability density function:

$$f(x) = \begin{cases} 0 & \text{for } x \leq T \\ C\beta(x, \quad +1) & \text{for } x \geq T \end{cases} \quad \ldots (4.11)$$

where $\beta(x,\rho+1)$ is the Beta function of "x" (a random variable comparable to "p_R" in (4.3); that is, urban size) and "$\rho+1$" (where "ρ" ≥ 1 is a constant that determines the weighting of the new centers but is really employed analagously to "a"), "f(x)" is the number of cities of size "x" and "C" is a normalizing constant. "T" may be interpreted as the lowest value of "x" for which "f(x)" exceeds zero.

However, where $x \gg 0$, Simon demonstrates that:

$$f(x) = C\Gamma(\rho+1)x^{-(\rho+1)} \quad \ldots (4.12)$$

where "Γ" represents the gamma function. (4.12) is clearly the density function of the Pareto law. Therefore, by making the conceptual leap to the continuous case once again (where the probability that the variable "x" assumes a given value "x_i" is zero but the probability that it assumes a value between "x_i" and "x_j" may be determined), we may integrate "f(x)" between "1" and infinity to establish the distribution function "F(x)" where:

$$F(x) = \begin{cases} 0 & \text{for } x \leq T \\ 1-T^{\rho}x^{-\rho} & \text{for } x \geq T \end{cases} \quad \ldots (4.13)$$

This indicates the probability that the continuous variable "x" assumes a value in a specified class interval, a condition that we may employ to determine the number "N(x)" of centers having populations greater than or equal to "x" ($x \geq T$):

$$N(x) = \frac{XT^{\rho}}{x^{\rho}} \quad . . . (4.14)$$

where "X" is the total number of communities in the system. This derivation is certainly analagous to the Pareto relationship given in (4.3). Besides, this interpretation demonstrates precisely how the simple Berry-Garrison (1958a:88-89) application may be considered a valid discrete representation of Simon's stochastic model (despite the apparent conceptual flaw in their effort: see footnote 37 of page 88 of Berry and Garrison article).

The lognormal distribution truncated at point "T" (Aitchison and Brown, 1957:87-99) was given illustration in matrix form above and Adelman (1958:894) provides a link to the Yule interpretation in a Markov process via ". . . a resevoir of potential entrants." While it is difficult to specifically compare the probability densities of the two distributions through their parameters, it appears that the Yule approach gives a superior fit near the point of truncation. In fact, by generating a perfect hypothetical rank-size distribution it is simple to demonstrate by graphical method that the best fitting lognormal tail overpredicts the number of communities in the first class interval.

Without laboring this point, we should conclude that the differences between the two distributions may be neglected if our analysis is relatively imprecise. For instance, in Berry's (1961) study of city size distributions many of the supposed lognormal distributions are concave to the size axis when plotting is properly scrutinized. While this suggests that the Yule framework achieves an improved fit, certainty is obscured by factors like (i) data reliability, (ii) the effects of unequal (in a proportionate sense) size intervals on small sample spaces, (iii) the open class interval for the very largest communities, and (iv) the nature of the distribution parameters. (It seems that the lognormal provides a superior fit for the hypothetical rank-size distribution when the exponent "b" in (4.1) is less than unity).

Comparison of Distributions

To a certain extent, this topic has been covered in various parts of the previous discussion. However, there do appear to exist additional misconceptions in the literature concerning the similarity of frequency distributions.

To begin with, it seems we must qualify a superfluous dichotomy of the primacy and rank-size terms that is widely held. The distinction, though, seems to arise from Jefferson's (1939) appeal for a uniqueness thesis. More recent social scientists emphasize regularities instead, in hope of realizing general law statements of human behaviour. In any case, adherence to the idea of a unique primate city precludes rational explanation of its economic attributes. The persistence of this division seems related to the notion that primate cities are". . . associated with overurbanization and superimposed colonial economies in underdeveloped countries or with political-administrative controls in indigenous subsistence and peasant societies. . ." and rank-size relations depict complex city systems in more advanced nations (Berry, 1961:574).

Surely, no generality is lost by treating primacy as but a deviation from the overall pattern of city sizes in a system. It may be accounted for in two general ways:

(i) By the emergence of one great "capital" as in Jefferson's extreme conception; and,

(ii) By a deficiency or void of intermediate size centers so that a group of large centers are distinguished from another group of small centers.

The point is that even if we prefer a strictly inductive or empirical approach to the question of city sizes, then the idea of primacy cannot be detached from the whole distribution of community sizes. This implies, of course, that any rigorous treatment of the problem must employ double-logarithmic plotting (in either size versus rank or size versus cumulative percentage form) with attention devoted to correlations and residuals of the linear regression model.

Besides, inferences based upon limited sample spaces (C. T. Stewart, 1958; Berry, 1961; Mehta, 1964; Linsky, 1965; Rosing, 1966) must be treated with considerable reserve. Unfortunately, Berry appears to be alone in recognizing the importance of this issue.

Investigators also seem hesitant about accepting the significance of the exponent "b" in the rank-size formulation. Since the exponent varies with the type of data collected (Allen, 1954) and it is given some interpretation through the hierarchial and Simon models, it would appear that the acceptance a priori of a "b" value of unity (Vapnarsky, 1969) is rather questionable.

Another most meaningful point should be reiterated at this time. The use of grouped data facilitates analysis to a great degree but tends to obscure the more traditional form of primacy (that is, the "capital" city) through an open class interval for the largest population sizes. While we should agree with Lasuen, Lorca, and Oria (1967) that Berry's lognormal technique weights the effect of the large number of small centers in the system, we cannot agree that it adequately describes the deviation of the very greatest community. (Notice that the plottings for the sample of Spanish cities are done incorrectly in both articles). All in all, it may be best to retain a consistent approach of displaying the frequency distributions throughout any single study.

In conclusion we note that investigation overlooks variation (intercepts and slopes) among systems that are all supposedly lognormally characteristic. While Aitchison and Brown (1957:94) emphasize the difficulties of estimating parameters from grouped data, we may forfeit this operation and still gain insight by studying the properties of the graphs alone. In fact, this is a major issue in the next chapter of the discussion.

Interpretation and Explanation

Acceptable explanation of these empirical regularities is a rather debatable topic but the general argument seems to revolve about the role assigned to theory in explaining reality. The objectives of this section are (i) to evaluate the nature of explanation found in stochastic reasoning, (ii) to complement this line of thinking with certain notions from general systems theory, and (iii) to hopefully clarify the relation between these probabilistic uni-size arguments and the properties of deterministic hierarchial models.

Stochastic Approaches

Berry and Garrison (1958a:90) summarize the case of stochastic argument in the closing pages of their often cited article about rank-size relationships: "For one thing, a probabilistic explanation in some sense refers to the presence of an infinite number of causes and the ability to predict in these terms is not enough;

we wish explanations viable in explicit ways within a broad theoretical context." Mapping the properties of the real world into the law of proportionate effect is an interesting exercise, but the theory that is portrayed by the a priori model is not clear at all.

However, an interpretation of the statistical statement does provide some helpful insights. The feelings of Simon (1955:437) are that the Yule distribution

> . . . would hold if the growth of population were due solely to the net excess of births over deaths, and if this net growth were proportional to present population. This assumption is certainly satisfied at least roughly. Moreover, it need not hold for each city, but only for the aggregate of cities in each population band. Finally, the equation would still be satisfied if there were net migration to or from cities of particular regions provided the net addition or loss of population of individual cities within any region were proportional to city size."

Ward (1962) extends this thesis in a typical Predian (information and ability to act) discussion of aggregate market expansion opportunities that are realized in the long run. The Paretian distribution is attained when the relative frequency of occuring opportunities is randomly proportionate to the size of urban markets. He also provides several qualifications of Simon's model:

(i) The stochastic basis requires that cities in a particular class interval that are becoming proportionately larger must be matched by a similar number becoming proportionately smaller;

(ii) Migration from rural areas or abroad is not accomodated (more than likely this is directed toward the largest centers);

(iii) The usefulness of the approach is hindered by the nature of the data: city sizes in metropolitan or conurbation form have characteristically lower exponents than in corporate form - the applicability of the Yule distribution, though, is restricted to those values greater than or equal to unity.

Unfortunately, there appears to be at best very sketchy support for the type of growth postulated in the stochastic matrix: Madden's (1956) study of the stability of urban growth in the United States indicates that the law of proportionate effect may well be approximated in the real world when we consider the growth of all centers (no class intervals) through equal time periods. Obviously, however, the investigation refutes qualification (i) above: this points to a fundamental distinction between the frequency distributions of percentage changes in size of communities and, say, firms (Simon and Bonini, 1958) in an industry. There is clearly less tendency for urban centers to take proportionate losses in population (especially the large centers) than for firms to take similar losses in employees, value added, etc. Simon's model seems to indicate a better description of the data than of the processes involved.

In any case, the essence of the problem from a scientific viewpoint is that the probabilistic scheme avoids introducing those factors that supposedly cause the stochastic mechanism to operate. The most critical of these is certainly the notion of distance or separation, for only through study of this variable can we understand why and how urban communities are functionally related. Or, to phrase the point differently, the stochastic argument is a totally aggregated one in the sense that it cannot allocate weightings for different orders of opportunities. Without knowledge of how individual elements are related it becomes impossible to make predictions about their attributes at a later time.

However, such discussions that simply weigh the differential merits of positivist (or rationalist) and more conventional (or symbolic) viewpoints toward theory-reality interrelations (Lukermann, 1961) tend to obscure an unnoticed distinction between the statistical and spatial economic arguments. To illustrate, the law of proportionate effect typifies a model dealing with individual population members (recall Simon's 1955 interpretation that the probability the "k+1"st person is found in a center of size "x" is proportional to "x f*(x)", the total population of communities of this size), while the Christaller or Loschian models deal with population groups (the Christaller model is depicted by population subsets associated with baskets of goods). In terms of systems, the Yule distribution portrays random behaviour at a high resolution level while central place theory provides prediction at a lower level (Burton, 1963 discusses this general point with regard to quantification). In modeling terms, the former approach is descriptive, micro, and probabilistic while the latter is analytical, macro, and deterministic. While this scale dichotomy does relax philosophical debate it cannot really refute the superior value of the economic argument as a methodological device.

General Systems Theory

General systems thinking essentially suggests a methodology or viewpoint focussing on properties common to all types of systems. It is an approach that favors studying the totality of relations among elements and emphasizes qualities like wholeness and organization (Rapoport, 1966).

Hall and Fagen (1956) stipulate two macroscopic properties of systems that appear relevant to the present discussion. If we recall our simple illustrative model of how an expanding space-economy induces an urban structure, then this is called progressive systematization, since we witness:

(i) Strengthening of pre-existing interdependencies;

(ii) Development of relations among members previously unrelated;

(iii) Addition of parts and relations to the existing system.

Besides, centralization or dominance by a leading member (the primate center) is a common trait that may accompany an increase in the sum of relations. Beckmann (1958) suggests that coincidence of allometry and the Pareto distribution (see corrected form in (4.6) above) may mean some optimal association exists between population in the leading part and population throughout the entire city system. In our rather functional view of the urban system, we stress its behaviour (flows, responses, etc.) both internally and through transactions with its environment (Harvey, 1969:456). We simply consider that environment as a higher order system (a socio-economy of individual consumers) from which the city system is conveniently, yet necessarily, abstracted. From our strictly economic interpretation, we may determine (at least qualitatively) the spatial extent of the environment via consideration of the proportion of demands exercised through local or foreign markets. Needless to say, as progressive systematization occurs in the urban system, the local or domestic portion of the environment becomes relatively more important.

Systems are classified into closed[3] or open types according to the nature of energy exchange (information, commodities, innovations, capital, etc.) with this

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[3]Note that closure in this interpretation is significantly different from its meaning earlier in the discussion; there, it referred to spatial isolation which accounts for only a part of the total environment.

environment. Obviously the hierarchial structure of a city system denotes an open condition, since it is derived from a competitive process directly related to the wants and needs of individual consumers. Berry (1967:76-78) presents some interesting ideas concerning this notion. Most important, though, is the contention that: ". . . living systems can be defined as hierarchially organized open systems, maintaining themselves, or developing toward a steady state." (von Bertalanffy, 1962:7). Spatial competition by urban communities is a good example of a process that tends to maintain an equilibrium state (homeostasis) despite environmental disturbances.

However, such an argument is restricted to only cultures that value such competition and to those types of activities that relate to a domestic environment (that is, where input prices vary relatively little in space). Urban growth attributed mostly to demands placed in a more distant part of the environment (foreign markets) is directed by linkages that are rather independent of those between elements in the system. When a large proportion of the total market is not in the city system itself, centralization or primacy (at the national and regional levels) is a natural occurrence. In other words, negative feedbacks and the urban hierarchy evolve together but where such a hierarchy does not exist (as in the most primitive space-economies), positive feedbacks and concomitant cumulative-causation expansion are possible in particular subsystems

(Maruyama, 1963). Only as the local environment increasingly controls the total market, and planning favors regional convergence, may these positive feedbacks be checked.

Information is the term we employ to describe the organization of a system. Its thermodynamic counterpart, entropy, is said to increase when a system becomes more randomized (information and negative entropy are analagous). Entropy is maximized in an urban system when community sizes differ only by chance.

To clarify this argument we consider the statistical definition of entropy (Klir and Valach, 1967:61):

> If out of "n" events, each can occur with the probabilities 0_1, 0_2. . . . 0_n, where $\sum_{i=1}^{n} 0_i = 1$ (i.e. some of the events do take place), then the formulation $H = -\sum_{i=1}^{n} 0_i \log_a 0_i$ is called entropy.

It should be obvious that when all "0_i" are identical, "H" is maximized.

Now we may interpret (in descriptive terms at least) the distribution of the total urban population "P" amongst "n" centers in entropy terms. By considering "0_i" as the ratio of a community's population "p_i" to the total "P" we may illustrate that:

(i) Minimum entropy occurs when all "P" is resident of one community and $H = 0$;

(ii) Maximum entropy occurs when $p_i = P/n$ and $H = \log_a n$.

However, in a city system maximum organization is attained through the spatial hierarchy and the trivial case where H = 0 is dismissed. For instance, if we know the population of a center and its level in the hierarchy, we can compute the populations of all the centers in the system. In the real world, hierarchies are never perfect nor complete but it is hypothesized by Berry (1967:71) that the rank-size rule is the steady state that balances hierarchial organization with randomization due to chance local variabilities.

Obviously in terms of population figures alone, a more organized state may be attained then through hierarchial constraint but theory precludes such a condition. This demonstrates precisely why we need a guiding theory in explanation, for a state of perfect primacy represents maximum organization in a population system but central place theory and its related models tell us that this is irrelevant in an economic system. Only through theory are we certain of eliminating absurdities.

Therefore it seems more natural that we should speak of a condition of desirable entropy, but not of minimum or maximum entropy when we speak of city hierarchies and the rank-size rule. However, it may be useful to provide an expression of entropy for a discrete rank-size distribution of total population "P"[4]:

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[4]Our derivation contradicts a formulation devised by von Foerster (1966), restated by Curry (1963) and included in reviews by Berry (1964) and Ollson (1966). While

$$H = \frac{p_1}{1^b P} \log \frac{1^b P}{p_1} + \ldots + \frac{p_1}{n^b P} \log \frac{n^b P}{p_1}$$

$$= \log \frac{P}{p_1} \left\{ \frac{p_1}{1^b P} + \frac{p_1}{2^b P} + \ldots + \frac{p_1}{n^b P} \right\} +$$

$$\frac{p_1}{1^b P} \log 1^b + \frac{p_1}{2^b P} \log 2^b + \ldots + \frac{p_1}{n^b P} \log n^b$$

$$= \log \frac{P}{p_1} + \frac{b p_1}{P} \sum_{i=1}^{n} \frac{\log i}{i^b}$$

Fulfillment of this steady state through time requires that as "P" and "n" grow, "H" remains relatively stable. This, in fact, hypothesizes that the principles of equifinality are met and population values are independent of central conditions (recall that this is a natural result of a stochastic interpretation).

The meaning of the entropy approach may be enhanced by considering the difference between independent settlements in a backward economy and interdependent communities in a progressive economy. The initial case represents a

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our interpretations of entropy are rather different, the initial effort suffers in several respects: (i) the combinatorial redefinition of entropy means that the sum of the logarithms in the second term of their expression must be minimized and not maximized, and (ii) correct use of Lagrangian multipliers leads to this minimization. It is not clear at all how entropy is maximized through the rank-size rule. Note, too, that in our formulation above "p_n" is the same as "p_{max}" employed earlier in this chapter and each is not to be confused with the notation used in the previous chapters; "p_1", of course, is now the largest center in the system.

weakly linked and relatively closed system that possesses maximum entropy: attributes of centers remain uncorrelated with their size. Integration of the space economy, on the other hand, is persistently bringing information into the system through the framework of economic competition and specialization. Vapnarsky (1969) presents an interesting argument along these lines, while focussing on the environment of the system, that depicts four general stages in this integration process.

Unfortunately, it is difficult to objectively specify the effects of added population on the entropy value. If all centers grow at about the same percentage rate then entropy remains constant in the long run. However, the vigorous growth of small centers relative to large centers may indicate an increase in entropy and, vice versa, the concentration of population in larger centers implies a decrease in entropy.

The Aggregate Model Reconsidered

In different parts of the thesis we have emphasized that unique size classes of urban centers cannot be justified on a population basis alone but only through understanding how economic indivisibilities and population become intertwined. It is not unexpected that random variation of individual community populations may express a size continuum within an urban system despite strong hierarchial qualities for the entire set. To hopefully illustrate how growth may be considered to induce such a continuum, let's recall

Beckmann's simple model.

The characteristic feature of this deterministic model is the use of a basic progression component "$\frac{s}{1-k} + 1$" that relates populations on adjacent hierarchial levels. Within this rigid framework, spatial variation of population sizes is severely restricted. As a result, we mentioned in the previous chapter that it is helpful to view the component as a random variable about the mean "$\frac{s}{1-k} + 1$".

In a growth context, let's begin by considering a vector "$\underset{\sim}{p_1}$" that represents urban populations at time "t_1" in a city system. If relative growth in the first time interval is "$\frac{s}{1-k}$", then populations at time "t_2" are expressed as:

$$\underset{\sim}{p_2} = \underset{\sim}{p_1} \quad \frac{s}{1-k} + 1 \qquad \ldots (4.16)$$

In this manner, we can show:

$$\underset{\sim}{p_m} = \underset{\sim}{p_1} \left(\frac{s}{1-k} + 1\right)^{m-1} \qquad \ldots (4.17)$$

which clearly resembles (3.15). Now, assuming that:

(i) Values in "$\underset{\sim}{p_1}$" differ only by chance;

(ii) "$\underset{\sim}{p_m}$" is lognormally distributed, a property that is suggested by Beckmann (1958) and substantiated to a reasonable degree via plotting;

(iii) The time periods are reasonably comparable; then the random variable "$\frac{s}{1-k} + 1$" may also be lognormally distributed (Aitchison and Brown, 1957:12). Unfortunately, this discussion suffers in two respects. It is difficult to interpret independent growth in the successive time intervals, a condition that necessitates lognormal distribution of the component. Besides, the idea conflicts with those notions of cross-time analysis that essentially assume "$\frac{s}{1-k} + 1$" is normally distributed. Nevertheless, the approach provides a more suitable framework for describing urban growth than can be attained via equilibrium adjustments alone. Clearly, this is a theoretical topic that deserves increased attention.

While it is certain that central place theory has a somewhat limited domain in the real world, at this time we have no other analytic statements to direct explanation of the nature of urban systems. However, keen awareness of the fundamentals of that theory gives valuable insight to a rigorous methodology for empirical research.

In the following chapter we take a more selective viewpoint and, after attaching various growth factors to the central place framework, attempt to explain or refute certain inductive generalizations concerning primacy and rank-size relationships.

Chapter 5

CHANGING PATTERNS OF INTERURBAN STRUCTURE

This chapter is largely devoted toward sketching the interrelations of various growth factors and interurban structure. Beckmann's simple hierarchial model provides the framework for a type of comparative-statics analysis, where optimal equilibrium conditions are assumed to persist both before and after an impact is introduced (Nourse, 1968:273). Besides, the concomitant effect of structure upon growth is set within a more descriptive discussion of the diffusion process.

A graphic interpretation of the simple model is outlined near the chapter's end. With this idea in mind, there are some attempts given toward aligning or qualifying inductive generalizations with the central place principles. (Burton, 1963 emphasizes relating hypotheses to a developing body of theory).

The methods used in the argument, however, require considerable refinement before we can establish strong statements relating distribution patterns and independent factors.

Growth in a Theoretical Context

The properties of the aggregate model lend insight to the nature of change in urban structure as fashioned by population and economic growth in a region. The particular drawbacks of this theoretical argument involve the assumption that development may be characterized in a deterministic model that precludes disequilibrium. Our procedure is to study each factor in isolation (that is, holding all other factors constant) and then attempt to conceptually integrate their diverse effects.

Population

To begin with, let's consider a city system that evolves on an isotropic plain so that population growth is spatially directed by the existing structure. More precisely, we suppose that (i) centers attract population increments proportional to their initial sizes and (ii) spatial extension (or contraction) of the entire system proceeds symmetrically about the dominant center. Of course, our analysis is characterized by viewing functional differentiation through bundles of commodities since we assume Christallerian agglomeration.

Now we can analyse the effects of population growth by referring back to Figure 1 in the second chapter. Curve "D_2" represents the aggregate demand facing a firm that is earning normal profits in the competitive single good case. Under multiple good conditions, approximate tangency of the average cost and demand curves may only characterize those firms providing hierarchial marginal goods (except, of course, where low order functions are given in relatively high level centers that possess rather substantial intraurban markets). In any case, for the remaining submarginal goods in the same basket, threshold requirements are less and excess profits are likely greater for the related firms.

A uniform increase in population shifts the demand curve "D_2" to the right (for illustrative purposes, say to "D_1") and allows excess profits to be attained for any particular commodity that is initially hierarchial marginal. Depending upon the amount of population increase, this surplus of purchasing power may or may not be sufficiently large enough to induce entry of a competing firm(s) offering this same commodity at each center on this level. However, certain goods that were formerly supramarginal and only produced at a higher level probably find a sufficient threshold base in these lower level places while other submarginal commodities surely stimulate additional intraurban competition.[1] Considering all hierarchial levels together, we observe that bundles of goods become redefined according to the emergence of new hierarchial marginal goods due to increases of interstitial purchasing power.

[1] The same good may be considered supramarginal or submarginal depending upon which endpoint of the basket we relate to.

The most noticeable effect, then, of population growth is a tendency toward a reduction of functional concentration in the city system. If new functions are not added to the original "M" baskets, density increases may lead to the spreading of these bundles over more than "M" levels. More significant, perhaps, is the result of this decentralization:

(i) Similar numbers of functions may shift downward in each hierarchial level, but firm multiplication is relatively more rapid in the lower levels (see Parr and Denike, 1970);

(ii) Identical bundles may become characteristic of the two smallest size classes.

Besides, the simple model does qualify our understanding of how urban structures are transformed, at least in the more advanced regions. By stressing that changes in the size and frequency distribution of centers at higher levels depend upon the entry thresholds for places at all lower levels, it eliminates the more intuitive views such as: "If the population should only double, there would arise twice as many cities in each rank order,. . . " (Nourse, 1968:210).

Recalling that the correct sequence for satellite cities is $s(s + 1)^{n-2}$ (size classes greater than the first), we can reinterpret "$P_M(t_1)$", the total population served in the central place system at time "t_1" prior to population growth, in urban and rural compenents:

$$P_M(t_1) = p_M(t_1) + s \sum_{n=2}^{M} (s+1)^{n-2} p_{M-n+1}(t_1) + (s+1)^{M-1} r_1 \qquad \ldots (5.1)$$

Now if other factors are constant, a relatively small population increase leads to the beginning of a "M+1" level hierarchy. Hence "$P_{M+1}(t_2)$" denotes the population in the same system after this increase ($P_{M+1}(t_2) = v\, P_M(t_1)$, where "$2P_M(t_1)$" simply means the original population is doubled) and the number "f" of centers at each level of the transformed system is derived from:

$$f(p_{m+1}, t_2) = \frac{v\,(1-k)}{(s+1-k)}\, f(p_m, t_1) \qquad \ldots (5.2)$$

and

$$f(p_1, t_2) = (s+1)\, f(p_2, t_2) \qquad \ldots (5.3)$$

where "k" is the proportionality factor and not all "f's" are of integer form.

In any case, population increases designate concomitant rural density increases so that more central places emerge and centers of the same size move closer together.

Parr and Denike (1970) mention that this characteristic decentralization (import substitution) is espeically prevalent among the higher levels of the urban hierarchy in the United States and is due partly to increased regional demand for certain specialized (professional, financial, etc.) services. The authors stress, on the other hand, the well-known case of decline in rural populations and how this causes threshold ranges to eventually become unattainable from the smallest centers. Unfortunately, our model is not flexible enough to simultaneously account for those functional transfers that converge at the intermediate size classes.

Income

Since we are now holding total population constant, an income increase amounts to a per captia income increase that is the same for all consumers. Since we also assume resource use is not handled more efficiently via improved technology, this income increase may arise from an absolute increase in the amount of productive resources used per head of population. (Lampard, 1968, emphasizes differences between growth and development).

Therefore, just as in the previous case, there is increased purchasing power in each areal unit and the aggregate demand curve facing the various firms shifts to the right in a fashion characteristic of the particular commodity and the level on which it is being offered.

Nourse (1968:212-215) argues correctly that the income increase allows each individual to purchase more of all goods and that fewer people are needed to comprise a threshold size market for any particular central function. For purposes of analysis, he stipulates that the income increase does not affect the supplying population needed for each bundle of first order goods; as a consequence, the factor "k" relating community and market populations increases at the first level. In keeping with the properties of the simple model he extends the progression component " $\frac{s}{1-k} + 1$ " as well and higher level places are relatively greater than before. In general terms, the number of central places of the first level after the income increase becomes:

$$f(p_1,t_2) = \left\{ \frac{1-k(t_1)}{k(t_1)} \times \frac{k(t_2)}{1-k(t_2)} \right\} f(p_1,t_1) \qquad \ldots (5.4)$$

where $p_1(t_2) = p_1(t_1)$. Retaining the supposition of a constant number of satellites, he proceeds to exhaust "$P_M(t_1)$" via application of the formula:

$$p_m(t_2) = \left\{ \frac{s+1-k(t_2)}{1-k(t_2)} \times \frac{1-k(t_1)}{s+1-k(t_1)} \right\}^{m-1} p_m(t_1) \qquad \ldots (5.5)$$

until he defines all the urban populations in a system "P_{M*}" ($M* < M$).

Nourse's essential thesis is that a per capita increase of income extends the number of centers in the system so that they become closer together. At the same time the hierarchy contracts to accommodate the expanded low level urban population totals.

Unfortunately, it seems that this analysis suffers in several respects. First of all, the lines of interdependencies among the different sized places are severed by considering such a truncated hierarchy. The system cannot be considered in equilibrium since (i) no satisfactory market exists for high order commodities and (ii) the stability of rural populations is neglected. Furthermore, his employment of a "k" increase determined by reduced external markets is a rather debatable feature even within the confines of our crude model. A more thorough examination of the case is clearly required.

Let's concentrate entirely upon the demand side. By using an approach that is rather more appealing than Nourse's, we can avoid some of the conceptual flaws that mar his argument. Recalling the fundamental assumptions (see Chapter 3) that link urban and market populations we observe that supply characteristics become at best an implicit factor in the modeling scheme.

Table 2 of the third chapter illustrates the case where a market threshold of 3000 is needed to uphold the first bundle of goods and services. This figure may be considered halved, for example, if per capita income is doubled throughout the region. By keeping our reasoning more coincident with those notions of central place theory that advocate maximum spatial competition (see the purchasing power argument of Berry and Garrison: 1958d) it seems more plausible that this new purchasing threshold is met by populations drawn evenly out of existing centers and complementary areas. In other words, two centers with "p_1" equal to 500 and "r_1" equal to 1000 replace the single center. It is also apparent that this argument avoids

Nourse's limitation to small income increments that do not induce rapid growth of the "k" factor. Now, the general effect of the increase is described by:

$$f(p_m, t_2) = v\ f(p_m, t_1) \qquad \ldots (5.6)$$

where "v" denotes the proportion between per capita income after and before the increment.

The result of this argument should be clear. An income increase spread evenly over all consumers expands the number of places in a city system and lowers the populations of places on levels comparable to the initial case. In fact, sufficient increments may induce two or more identical and adjacent subsystems to replace the original system.

Obviously, though, our argument is somewhat weaker than Nourse's on attacking the issue of supply. Nevertheless we may think of the process just outlined as being constrained by some lower limit of the supplying population at the first (and every other) level although the definition of this bound lies outside our a priori structure.

In both of these cases, too, we assume away an added behaviouristic implication. Certainly as incomes rise, customers increasingly turn toward income-elastic commodities (high style furniture, fashion clothing, specialized medical services, etc.) rather than items like agricultural staples and home fuels. Now it is exceedingly difficult to state just how this new dimension affects the initial system through an income increase but we intuitively expect that (i) the transfer of near-hierarchial marginal goods to lower levels and (ii) the mixture of firm multiplication in existing centers with the entry of bundles in new centers are suitably pronounced. The tendency to agglomerate seems to go hand in hand with income-elasticity and may well serve to sustain the inertia (hierarchial levels and number of communities on each level) of the original system.

All in all, it is impossible to present an accurate picture of progressive systematization in a simple set of equations. While it is true that our views and Nourse's are at odds on certain relevant points, it is significant to note that either approach suggests a constancy or contraction in both hierarchial levels and functional centralization may be realized via per capita income increases.

Innovations

Innovations of knowledge and technique may be generalized as functions of interaction probability or information exchange in open systems (Berry and Horton, 1970). Economic development may then be viewed from the perspective of such innovations occurring in the largest centers and spreading through time to other communities in the system. Lampard's (1968:106) cybernetics framework emphasizes stability of interurban structure since ". . . the transformation of human settlement patterns (the evolving system of cities, for example) involves the emergence and generalization of novelty within the population system. . .", although growth may appear deviation-amplifying in the various smaller subsystems.

In our present discussion, four general cases appear to be of special interest:

(i) Innovations in transportation technology, including both new means and route improvements;

(ii) Innovations that strengthen rural (farming) productivity;

(iii) Innovations in marketing technology that permit the entry of scale economies into certain existing activities;

(iv) Innovations that bring entirely new activities into the different sized urban communities.

Within our comparative-statics framework we may give more specific attention to cases (i) and (ii) as they rest on the demand side like the factors just analysed above. Nevertheless, we can allude to structural transformations for the remaining cases; besides, here we emphasize how structure channels economic development as well.

In the first instance, a transportation innovation clearly affects only demand conditions within the f.o.b. supposition. Generally, we may consider such an improvement as being similar to an increase in per capita income, although its benefits only accrue to the complementary area populations on each hierarchial level (since we assume the supplying population to be located at the production site). Intuitively, we expect this impact to change the urban structure roughly along the lines we hypothesized for an income increase. On the other hand, this statement must be qualified according to variables like (i) the initial value of the proportionality factor "k", (ii) the very magnitude of this benefit given to external consumers, and (iii) changing behaviour (for example, multipurpose trips) due to new means as opposed to simple decreases in spatial friction. By itself, improved customer mobility points to increased functional decentralization in a city system.

The improvement of rural productivity through innovations of agricultural methods, mechanization, etc, changes the urban structure if concomitant rural to urban migration is assumed. By viewing this redistribution process in a vein similar to that outlined earlier for population growth, we expect a slight increase in the "k" factor and the basic progression component to accompany the emergence of a new hierarchial level. This follows because as rural densities diminish, first level places become smaller and tend to lose functions (except the lowest order convenience goods) to second level places. Such an upward transfer occurs throughout the entire hierarchy and essentially suggests increasing functional centralization with a new dominant center entering the system. While the number of communities increases, their populations on levels comparable to the original hierarchy tend to decrease.

Scale economies, allowing large plants to produce at lower marginal and average costs, place more emphasis on the supply side of the argument. Parr and Denike (1970) give lucid illustration of how such scale changes permit functions to be transfered from lower to higher levels of the hierarchy. Considered in isolation, the effects of scale extensions are more noticeable among the lower hierarchial levels where (i) intraurban thresholds are not too substantial and (ii) convenience retailing, as characterized by minimal captial outlays, dominates the basket items. The transfer of functions suggests a contraction of the urban hierarchy as many of the inefficient firms in smaller communities are priced out of their markets. As the possibilities of realizing scale economies for individual goods in the same basket are not identical, improved marketing technology may well evoke increased functional concentration in the city system.

Innovations that bring totally new commodities into the regional market cannot be related to urban structure in the same explicit fashion as the previous factors. Obviously, though, a high incidence of low order innovations relative to high order types may somewhat strengthen functional decentralization.

On the other hand, the innovation idea provides a convenient means for discussing the other side of the coin: that being, of course, how interurban structuring channels the course of economic growth. Let's consider how items that are neither resource-oriented nor regional-specific spread from an innovation origin. It appears that (i) the stronger the distance decay, the closer will diffusion follow the constraints of distance while (ii) the weaker the distance decay, the closer will diffusion follow the size distribution of urban communities (Pederson, 1970).

In any case, without certain hierarchial aspects, an economic region may be typified by curtailed diffusion of both low and high order goods. The integrative role of peripheral centers is emphasized as a means to offset this concentrated and frequently weak economic environment. Observers increasingly stress the focussing of regional policy upon the location and functional characteristics of growth centers (among others, Friedmann, 1966, Lithwick and Paquet, 1968). Though investments are usually associated with the provision of high order goods, suggestions are forwarded that regional convergence may be sought via low order goods as well. Policy implications here include (i) the increase of diffusion sources, (ii) the improvement of accessibility to the primary innovation center, and (iii) the speeding up of the urban growth process in low density areas (Pederson, 1970). The important point being signified is that the extent of urban amenities tends to determine the very economic health (growth, stability, etc.) of the entire region.

Despite considerable variety in how investment is spatially allocated, the principles of comparative advantage evoke hierarchial symptoms at some later stage. Whether peripheral growth naturally follows a frontier or diminishing returns set in at the largest centers or growth is thoughtfully redirected, the regional periphery is eventually taken up by urban subsystems. Along the path toward hypothetical regional convergense, ". . . the diffusion of innovations down the system of city-sizes is the means by which growth and change are transmitted throughout the economy and integrated national development is achieved and maintained." (Bèrry and Horton, 1970:67). Differential urban growth may itself be considered some function of the process of innovation diffusion (Pred, 1966).

In various parts of this thesis we have stressed the manner in which a well defined hierarchy constrains the growth of different sized communities. Our series of equilibrium adjustments are really taken to illustrate a city system after complete saturation of the particular diffusion process. Population members and innovations are the critical items signified. Rationale for equilibrium tendencies depends upon the maximization of intraregional flows of mobile factors of production. In this way employment (and, therefore, population) increments tend to be proportional to urban size while economic activities filter down from innovating areas (Thompson, 1968:52-59 outlines this phenomenon). Unfortunately, not even a smoothly operating market process can promise fulfillment of total adjustment in the long run.

A Brief Synthesis

It would be hazardous, indeed, to infer anything but tendencies from the foregoing diverse impact arguments. Several general possibilities seem to exist:

(i) Regions in which population growth completely outstrips technological advance and income extension may well be characterized by a multi-level hierarchy and high rural densities; besides, the "k" factor will be relatively low;

(ii) Regions in which population growth is retarded but other growth variables continue may show stability or contraction of the hierarchial structure over time; the "k" factor likely increases, especially with significant rural-to-urban migration;

(iii) Regions displaying more balanced growth of population and economic factors probably preserve the gross features of the initial hierarchy to a great extent.

On the other hand, it becomes essential to qualify these statements by considering that:

(i) A finite set of community sites would possibly alter the various impacts in a rather typical fashion: for instance, population growth may be simply attracted to existing places so that "k" increases;

(ii) Innovations may have clear thresholds when proceeding independently but may act quite differently with the simultaneous change of other factors. Parr and Denike (1970) illustrate the case where a scale change in marketing is brought about by an improvement in transportation (or an increase of population or per capita income for that matter), a condition that may well lead to shrinking of the urban hierarchy.

Of course, diverse forces that we cannot contain in our theoretical argument are persistently altering real world interurban structures as well. We should be pleased, therefore, if we approximately account for the directions of real world changes alone. In conclusion, then, the confidence we place in explaining these adjustments depends largely upon our feelings toward the merits of central place theory.

Graphing the Aggregate Model

The simplest illustration of the size distribution of places via the aggregate model is given by plotting values for a hypothetical system on logarithmic-normal probability paper. Comparing different systems by this method makes several graphic properties obvious:

(i) A roughly straight line that seemingly indicates a truncated lognormal distribution is, at closer inspection, slightly convex to the size axis for few hierarchial levels and more concave to that axis for many levels;

(ii) The slope of the apparently straight line depends upon the number of hierarchial levels when the "k" factor remains constant; on the other hand, a reduction of "k" means a steeper line, ceteris paribus;

(iii) A change in geometry affects the slope too; as "s" increases the line becomes flatter.

(iv) The arbitrary point of truncation above minimum sized places affects different systems in various ways; a multi-leveled hierarchy may give a flat line if "k" is small (perhaps large rural densities) since many centers lie in the first grouped interval.

These notions are somewhat useful when related to Berry's (1961) empirical study. They suggest, for instance, why there is considerable variety in these city size distributions though many are nearly lognormal. Investigation along this path seems to be a logical step toward strengthening cross-cultural comparisons of city systems.

Furthermore, the a priori statements suggest what may be the most relevant factors in promoting cross-time similarity of international data. Total population of the urban system, independent of the manner (birth rates, rural-to-urban migration, etc.) in which it is devised, seems to be the one critical variable that expands the urban hierarchy. In cases where some hierarchial aspects are presented at a point in time (either for economic, social, or administrative reasons), these aspects are likely solidified by the population growth and redistribution within a maturing space-economy. Since no real world hierarchy approximates perfection, the emergence of new levels simply makes the lognormal distribution more plausible. To the extent that population totals and national areas show some positive correlation, it is not surprising that these two variables are pointed to in most empirical studies as being conducive to low degrees of primacy (Berry, 1961; Mehta, 1964; Linsky, 1965). On the other hand, we should be somewhat hesitant, then, of believing that income per capita alone always varies positively with the degree of rank-size shown by individual national systems that unfold over time (Lasuen, Lorca, and Oria, 1967, assume this to be the case).

Clearly, added theoretical arguments and more precise inductive approaches are needed before sufficient confidence may be placed in the role of disparate growth variables. This promises to be an important and controversial topic in future interurban research.

Chapter 6

CONCLUDING REMARKS

The problem area directing the discussion in this thesis concerns city size distribution. Considerable attention is given to various subtopics on both theoretical and empirical fronts within an explicit systems framework. The conscious support of this framework throughout the entire discussion is perhaps the most salient feature of the thesis. The essential purposes of this thesis are to improve the prevalent methodologies now in use and to anchor the city size topic more firmly into the growing body of geographical literature and theory.

The findings of the different arguments are rather diverse. Generally, though, the tone is that logical analysis should replace intuition as geographic endeavours proceed scientifically. Unfortunately, in this problem area there is an atmosphere of data malleability and wishful thinking that suggests intuitive tendencies are common. Furthermore, disregard of research methodology (including points such as the definition of study areas and techniques of statistical analysis) severely constrains the value of inferences that many observers put forward.

If efforts are taken to continue scientific research on the city size topic then they must be funnelled along two related paths:

(i) Toward improved interpretative schemes on the theoretical side; an argument of this thesis is that central place theory offers a strong base for such contributions; and

(ii) Toward carefully structured empirical studies on different scales that, in conjunction with theoretical extensions, will suggest those factors that primarily determine the form of the frequency distribution of city sizes in a particular region.

Besides, our review has revealed that several less general subtopics deserve increased attention as well:

(i) The scheme of hierarchial sets, which deals with one complete and many partial hierarchies of an independent nature, suggests a more flexible viewpoint (at least where resolution levels are relatively low) toward compatibility of central place thinking with the characteristic uni-size class distributions of empirical research;

(ii) Concern over the Loschian model should enhance this same compatibility;

(iii) Extensions of the economic base concept via central place theory may well provide valuable feedback at both the intra- and interurban levels;

(iv) Added efforts are needed in the attempt to give central place arguments a reasonable temporal dimension (that is, when a hierarchy is already assumed to exist);

(v) Emphasis on general systems concepts should complement the deterministic and stochastic arguments; perhaps, indeed, the entropy idea can assist in describing non-equilibrium features within a dynamic framework (for example, it may have promise as a device to describe inter-urban structuring prior to hierarchal maturity);

(vi) A crude analytical base has been set for investigating the interrelations of growth and interurban structure; model building within this subtopic may have interesting theoretical and practical implications;

(vii) Emphasis placed upon the peculiar aspects (slopes, etc.) of individual distributions may prove fruitful in the search for cross-cultural regularities.

The research possibilities about the topic of city size distributions are virtually unlimited. It is hoped that this thesis assists others proceeding along these lines.

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