REPRESENTATION OF PERMUTATION OPERATORS
IN QUANTUM MECHANICS

by

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Abstract

A simple method is presented for writing the matrix elements of transposition operators for discrete sets of quantum numbers. A proper product of these leads to easy computation of general permutation operators. It is shown how these operators may be constructed with operators defined in angular momentum space. Results agree with Dirac for transposition of two particles of spin $1/2$ and with Kaempffer for spin 1. The calculations are performed to extend the results to spin $3/2$ and 2 along with alternate representations. Special considerations are required for fermion creation and annihilation operators.

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Introduction

Although permutation symmetry is always invoked in the quantum-mechanical description of systems of identical particles, little is found in the literature on the actual construction of permutation operators. A notable exception is an article by Serber (1934), where permutation operators are constructed out of transposition operators for the special case of four objects.

Two other types of symmetries are known. One type, the space time symmetries, can be grasped by introduction of operators whose generators represent constants of the motion. For example, the generator of displacement operators is the momentum of an object, the generator of rotation is the angular momentum and the generator of displacements in time is the energy (see, for example, Merzbacher 1961). The second type of symmetries deals with internal attributes. The physical significance of their generators is not understood at all at present. Most attempts to understand these better use the existence of certain gauge invariances as a starting point (Yang and Mills 1954).

Permutation symmetry is unique in that it is defined only for more than one object. However because of a relation between spin and statistics (Pauli 1940, Luders and Zumino 1958), even single objects reveal through internal attributes

how collections of them will behave under a permutation operator. This is not clearly understood.

A system ψ of n indistinguishable particles labelled $1, 2, 3, \dots, n$ with complete sets of quantum numbers $\tau_1, \tau_2, \dots, \tau_n$ is written

$$(0.1) \quad \psi = |\tau_1, \tau_2, \tau_3, \dots, \tau_n\rangle.$$

"Indistinguishability" of particles means that it is not possible to affix any other label to a particle beyond a complete set τ . Expression (0.1) does not take care of this requirement because there are additional particle labels $1, 2, 3, \dots, n$ beyond the sets τ . Since the state (0.1) is indistinguishable beyond the sets τ , any reordering or "permutation" of the particle labels $1, 2, 3, \dots, n$ would yield a physically indistinguishable state. The operators that perform this symmetry property are called permutation operators.

Since dissection of any permutation into a product of transpositions is always possible, the primary task is indeed the construction of transposition operators. One aims particularly at a construction employing operators defined in the space spanned by the attributes whose particle labels are to be transposed. Dirac (1958) has done this for transposition of particle labels attached

to two angular momenta of amount $j=1/2$. Kaempffer (1964) has extended Dirac's representation to the case of angular momenta of amount $j=1$, and has given representations for transposition of complete sets of attributes in terms of creation and annihilation operators for both fermions and bosons.

In the present work a simple method is presented for writing the matrix elements of transposition operators for a discrete set of quantum numbers. From this representation, general permutation operators are easily computed. It is then shown from another line of reasoning how in angular momentum space these operators may be constructed using operators defined in angular momentum space. Calculations have been performed to extend work by Dirac and Kaempffer mentioned above to include particles of angular momenta $3/2$ and 2 along with alternate representations in each case. Special considerations are required for fermion creation and annihilation operators.

Before one can discuss any possible dynamical significance of permutation operators, one must possess the generators of the operators. This remains an as yet unsolved problem. As a first attempt in this direction, computation of the determinants of the operators was carried out and shown to yield $+1$ or -1 . If the determinant of an operator is $+1$, it should possess a connection with the

identity as a product of infinitesimal operators. The group generated by these infinitesimal operators should then contain the permutations with determinant +1 as a special subgroup. But this group is not known or not recognized.

The following convention about labelling matrix elements will be used. Let $\mu_1, \mu_k, \dots$ be the number of discrete values the quantum numbers $i, k, \dots$ may take. For example, the eigenvalues of J_3 for given eigenvalue $j(j+1)$ of J^2 has $\mu_1 = 2j+1$ values. Now introduce running integers $\alpha_1, \alpha_k, \dots$ to order the possible values of $i, k, \dots$ in a certain sequence. The first value of i corresponds to $\alpha_1=1$ and so on by integer values to $\alpha_1=\mu_1$. All matrices will be labelled $M_{\alpha_1 \alpha_k}$ etc. In particular, a matrix whose rows and columns are characterized by further sets of quantum numbers will be written $D_{\alpha_1 \alpha_k, \alpha_m \alpha_n}$. With this convention the matrix elements of the direct product $C=A \otimes B$ can be written as usual

$$C_{\alpha_1 \alpha_k, \alpha_m \alpha_n} = A_{\alpha_1 \alpha_m} B_{\alpha_k \alpha_n} \quad (\text{Wigner 1959, p. 17}).$$

I. General Permutation Operators for
Discrete Sets of Quantum Numbers

Needed is an operator T which has the following properties:

(a) T is self adjoint and unitary

$$(1.1) \quad T = T^\dagger = T^{-1} \quad \text{so that } T^2 = I.$$

(b) T is defined only in a product space spanned by a direct product of two spaces of equal dimension. It should have the property of interchanging the order of the direct product of any two matrices in the space.

$$(1.2) \quad T A \otimes B T^{-1} = B \otimes A.$$

Thus, if the "first" subspace is associated with the quantum numbers belonging to the "first" particle and the "second" subspace belongs to the "second" particle, the operator T has the effect of transposition of particle labels. This may be expressed formally by denoting any operator belonging to the "first particle"

$$(1.3) \quad F(1) = F \otimes I$$

and any operator belonging to the "second" particle by

$$(1.4) \quad F(2) = I \otimes F$$

remembering the application of (1.2) is

$$(1.5) \quad T E(1) F(2) T^{-1} = F(1) E(2).$$

These requirements are possessed by the following representation

$$(1.6) \quad T_{\alpha_i \alpha_k, \alpha_m \alpha_n} = (\pm) \delta_{\alpha_i \alpha_n} \delta_{\alpha_k \alpha_m}.$$

The proof that (1.6) satisfies (1.1) and (1.2), and unique is straightforward and is relegated to Appendix A 1.

It is instructive to write T out explicitly for a given case. Take the product of two three-dimensional space, which may be identified with the product space spanned by two subspaces of J_3 each belong to angular momentum $j=1$. Then T has, with the labelling convention stated in the introduction, the form;

$$(1.7) \quad T = (\pm) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

From the properties (1.1) it follows that the eigenvalues of T are ± 1 . Let an eigenstate of T be denoted by

$$(1.8) \quad |\psi\rangle = |\phi\rangle \otimes |\chi\rangle$$

then

$$(1.9) \quad T|\psi\rangle = f|\psi\rangle \quad \text{requires} \quad T^2|\psi\rangle = f^2|\psi\rangle = |\psi\rangle$$

so that

$$(1.10) \quad f^2 = 1 \quad \text{and} \quad f = \pm 1.$$

The proof leading to (1.6) for transposition operators allows an as yet arbitrary sign; this is not a matter of convention however. Consider the eigenstate

$|j,m\rangle$ belonging to $J^2=j(j+1)$ and $J_3=m$ obtained by addition of two angular momenta of amount j_p (say). In particular look at the state $|2j_p, 2j_p\rangle$. It is well known that this state is symmetric and in the usual column-vector representation has element $e^{i\alpha}$ (with arbitrary α usually chosen as $\alpha=0$) in the first position, with all others 0. That is

$$(1.11) \quad |2j_p, 2j_p\rangle = |j_p, j_p\rangle_1 \otimes |j_p, j_p\rangle_2 = \begin{bmatrix} e^{i\beta} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \begin{bmatrix} e^{i\gamma} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} e^{i\alpha} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where 1 and 2 are the particle labels. The element $T_{11,11}$ of T is the only non-zero element in the first column of T and hence is the only element to affect $|2j_p, 2j_p\rangle$ and must be chosen as +1 to leave the state unchanged by transposition of particle labels. That is

$$(1.12) \quad T|2j_p, 2j_p\rangle = (+) \begin{bmatrix} 1 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} e^{i\alpha} \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} e^{i\alpha} \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}.$$

Hence the sign is positive throughout for angular momenta.

The representation (1.6) can be extended to more than two particles by introduction of the identity for unaffected particles. Thus the three possible transposition operators

for three particles are

$$\begin{aligned}
 (T_{12})_{a_i a_k a_l, a_m a_n a_p} &= \delta_{a_i a_n} \delta_{a_k a_m} \delta_{a_l a_p} \\
 (1.13) \quad (T_{13})_{a_i a_k a_l, a_m a_n a_p} &= \delta_{a_i a_p} \delta_{a_k a_n} \delta_{a_l a_m} \\
 (T_{23})_{a_i a_k a_l, a_m a_n a_p} &= \delta_{a_i a_m} \delta_{a_k a_p} \delta_{a_l a_n}.
 \end{aligned}$$

It is important to keep track of the subscripts and the spaces to which they belong. Here "i,m" belong to subspace "i,m", i.e. to the first particle and so on.

By suitable products of transpositions, general permutations are found. For example, for three objects the permutation T_{123} and T_{132} are found as follows (note that the operators are effected from the right)

$$\begin{aligned}
 (1.14) \quad (T_{123})_{a_i a_k a_l, a_m a_n a_p} &= \sum_{a_r a_s a_t} (T_{23})_{a_i a_k a_l, a_r a_s a_t} (T_{13})_{a_r a_s a_t, a_m a_n a_p} \\
 &= \sum_{a_r a_s a_t} \delta_{a_i a_r} \delta_{a_k a_t} \delta_{a_l a_s} \delta_{a_r a_p} \delta_{a_s a_n} \delta_{a_t a_m} \\
 &= \delta_{a_i a_p} \delta_{a_k a_m} \delta_{a_l a_n}
 \end{aligned}$$

$$\begin{aligned}
 (1.15) \quad (T_{132})_{a_i a_k a_l, a_m a_n a_p} &= \sum_{a_r a_s a_t} (T_{23})_{a_i a_k a_l, a_r a_s a_t} (T_{12})_{a_r a_s a_t, a_m a_n a_p} \\
 &= \sum_{a_r a_s a_t} \delta_{a_i a_r} \delta_{a_k a_t} \delta_{a_l a_s} \delta_{a_r a_n} \delta_{a_s a_m} \delta_{a_t a_p} \\
 &= \delta_{a_i a_n} \delta_{a_k a_p} \delta_{a_l a_m}.
 \end{aligned}$$

The remaining permutation of three-particle labels is the identity, giving six in all; I, T_{12} , T_{13} , T_{23} , T_{123} , and T_{132} .

This process may proceed to any number of particles without difficulty. For example the permutation T_{14532} can be written as $T_{132}T_{145}$. From (1.14) and (1.15) it follows that

$$(1.16) \quad (T_{132})_{\alpha_a \alpha_b \alpha_c \alpha_d \alpha_e, \alpha_f \alpha_g \alpha_h \alpha_i \alpha_j} = \delta_{\alpha_a \alpha_g} \delta_{\alpha_b \alpha_h} \delta_{\alpha_c \alpha_f} \delta_{\alpha_d \alpha_i} \delta_{\alpha_e \alpha_j}$$

$$(1.17) \quad (T_{145})_{\alpha_f \alpha_g \alpha_h \alpha_i \alpha_j, \alpha_k \alpha_l \alpha_m \alpha_n \alpha_p} = \delta_{\alpha_f \alpha_p} \delta_{\alpha_g \alpha_l} \delta_{\alpha_h \alpha_m} \delta_{\alpha_i \alpha_k} \delta_{\alpha_j \alpha_n}$$

so that

$$(1.18) \quad (T_{14532})_{\alpha_a \alpha_b \alpha_c \alpha_d \alpha_e, \alpha_k \alpha_l \alpha_m \alpha_n \alpha_p} = \delta_{\alpha_a \alpha_l} \delta_{\alpha_b \alpha_m} \delta_{\alpha_c \alpha_p} \delta_{\alpha_d \alpha_k} \delta_{\alpha_e \alpha_n}$$

As a mnemonic aid, it may be noted the column label, α_l for example, of any given symbol $\delta_{\alpha_a \alpha_l}$, appears always in conjunction with a row label, α_a in this example, which in the labelling of the symbol T have the respective positions required by the permutation of particles with quantum numbers l and a in this case. Thus if the permutation is (21) as in (1.18), α_l is found with α_a among the subscripts of the corresponding δ -symbol.

II. Transposition of Angular Momentum

Necessary properties for transposition operators in angular-momentum space are found by examining the eigenstates for two particles of given angular momentum. Particularly needed are symmetry properties under exchange of particle labels for identical particles. The eigenstate $|j_a, j_b, j, m\rangle$, belonging to $J^2=j(j+1)$ and $J_3=m$ obtained by addition of two angular momenta j_a and j_b , have the well known symmetry property (Condon and Shortley, 1963)

$$(2.1) \quad |j_a, j_b, j, m\rangle = (-1)^{j_a+j_b-j} |j_b, j_a, j, m\rangle.$$

If the particles have the same angular momentum j_p the multiplet $j=2j_p$ is symmetric, the multiplet $j=2j_p-1$ is antisymmetric, and so on alternatingly to $j=0$. Thus the needed property of the transposition operator T is

$$(2.2) \quad T |j, m\rangle = (-1)^{2j_p-j} |j, m\rangle.$$

Now suppose there is an operator F which in the state

$|j, m\rangle$ has the eigenvalue f_j . Further assume the f_j are all different so that there are $2j_p+1$ values of f_j .

Then T can be written as a power series in F ,

$$(2.3) \quad T = X_0 + X_1 F + X_2 F^2 + X_3 F^3 + \dots + X_{2j_p} F^{2j_p}$$

with unknown coefficients $X_0, X_1, X_2, \dots$ to be defined so that the eigenvalues t_j of T must satisfy

$$(2.4) \quad t_j = X_0 + X_1 f_j + X_2 f_j^2 + X_3 f_j^3 + \dots + X_{2j_p} f_j^{2j_p}$$

in accordance with (2.2). This system of $2j_p+1$ equations with $2j_p+1$ unknowns $X_1, X_2, X_3, \dots$ can be put in matrix form,

$$(2.5) \quad \begin{bmatrix} f_0^0 & f_0^1 & f_0^2 & \dots & f_0^{2j_p} \\ f_1^0 & f_1^1 & f_1^2 & \dots & f_1^{2j_p} \\ f_2^0 & f_2^1 & f_2^2 & \dots & f_2^{2j_p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{2j_p}^0 & f_{2j_p}^1 & f_{2j_p}^2 & \dots & f_{2j_p}^{2j_p} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{2j_p} \end{bmatrix} = \begin{bmatrix} (-1)^{2j_p} \\ \vdots \\ \vdots \\ -1 \\ 1 \end{bmatrix}.$$

The coefficient matrix is a Vandermonde's matrix (see, for example, Browne 1958); the inverse of which can be written as (see Appendix A 2)

$$(2.6) \begin{bmatrix} (-1)^{2j_P} {}^0_{2j_P} R & (-1)^{2j_P} {}^1_{2j_P} R & (-1)^{2j_P} {}^2_{2j_P} R & \dots & (-1)^{2j_P} {}^{2j_P}_{2j_P} R \\ \vdots & \vdots & \vdots & & \vdots \\ {}^0_2 R & {}^1_2 R & {}^2_2 R & \dots & {}^{2j_P}_2 R \\ -{}^0_1 R & -{}^1_1 R & -{}^2_1 R & \dots & -{}^{2j_P}_1 R \\ {}^0_0 R & {}^1_0 R & {}^2_0 R & \dots & {}^{2j_P}_0 R \end{bmatrix}$$

where the following notation has been used:

$$s R = \frac{\text{sum of all different products formed with } s \text{ values } f_j}{(f_t - f_0)(f_t - f_1)(f_t - f_2) \dots (f_t - f_{2j_P})}$$

(2.7)

$${}_0^t R = \frac{1}{(f_t - f_0)(f_t - f_1)(f_t - f_2) \dots (f_t - f_{2j_P})}$$

and where the eigenvalue f_t is excluded in the formation of products in the numerator and $f_t - f_t$ is excluded from the denominator.

The solution can now be written

$$(2.8) \quad X_k = (-1)^{2j_P - k} \sum_{i=0}^{2j_P} (-1)^{2j_P - i} {}^i_{2j_P - k} R = (-1)^k \sum_{i=0}^{2j_P} (-1)^i {}^i_{2j_P - k} R.$$

Suitable operators F remain to be found. Clearly J^2 with eigenvalues $j(j+1)$ will work. If $\underline{J}$ is written as the sum of the spin vectors $\underline{J}(1)$ and $\underline{J}(2)$ for the first and second particles respectively

$$(2.9) \quad \underline{J} = \underline{J}(1) + \underline{J}(2)$$

it follows that

$$(2.10) \quad J^2 = J^2(1) + J^2(2) + 2 \underline{J}(1) \cdot \underline{J}(2).$$

The eigenvalues g_j of $\underline{J}(1) \cdot \underline{J}(2)$ satisfy then the equation

$$(2.11) \quad j(j+1) = j_P(j_P+1) + j_P(j_P+1) + 2 g_j.$$

Alternatively then, one may use a representation with $\underline{J}(1) \cdot \underline{J}(2)$ for F with the property

$$(2.12) \quad \underline{J}(1) \cdot \underline{J}(2) |j, m\rangle = g_j |j, m\rangle \quad \text{where} \quad g_j = -j_P(j_P+1) + \frac{1}{2} j(j+1).$$

If J^2 is diagonalized such that $V J^2 V^{-1} = J_d^2$, the eigenvalues $j(j+1)$ then appear along the diagonal and J_d^2 can be factored as

$$(2.13) \quad J_d^2 = K_d (K_d + 1)$$

where K_d is a diagonal matrix with eigenvalues j . Then by the operation $V^{-1}J_d^2V$ one has

$$(2.14) \quad J^2 = K(K+1) \quad \text{where} \quad K = V^{-1}K_dV.$$

Thus it is always possible to find a matrix K such that

$$(2.15) \quad K |j, m\rangle = j |j, m\rangle$$

which is another alternative for F .

Results with $\underline{J}(1), \underline{J}(2)$ agree with Dirac (1958) for spin $1/2$ and Kaempfer (1964) for spin 1. These and further results for T are collected in Table I.

An exponential representation of these operators should exist. One would like a matrix A such that T may be written

$$(2.16) \quad T = e^S = e^{i\pi A}$$

where the factor $i\pi$ is introduced for convenience. The eigenvalues a_j of A satisfy

$$(2.17) \quad \begin{aligned} T |j, m\rangle &= e^{i\pi A} |j, m\rangle = e^{i\pi a_j} |j, m\rangle \\ &= (\cos a_j \pi + i \sin a_j \pi) |j, m\rangle. \end{aligned}$$

In accordance with (2.2) one has

$$(2.18) \quad (-1)^{2j_p-j} = \cos a_j \pi + i \sin a_j \pi = (-1)^{a_j}$$

because the eigenvalues of T are real, requiring a_j to be integers. Indeed A can be found by much the same procedure just followed to find T . That is, A may be written as a power series

$$(2.19) \quad A = Y_0 + Y_1 F + Y_2 F^2 + Y_3 F^3 + \dots + Y_{2j_p} F^{2j_p}$$

with the unknown coefficients $Y_0, Y_1, Y_2, \dots$ defined so that the eigenvalues a_j satisfy

$$(2.20) \quad a_j = Y_0 + Y_1 f_j + Y_2 f_j^2 + Y_3 f_j^3 + \dots + Y_{2j_p} f_j^{2j_p}$$

where a_j is a suitable even or odd integer to agree with (2.18). This system of $2j_p+1$ equations with $2j_p+1$ unknowns $Y_0, Y_1, Y_2, \dots$ can be solved in the same manner as the system of equation (2.4). It is seen that A is not unique; the general form would be

$$(2.21) \quad T = e^{i\pi(A+B)} = e^{i\pi A}$$

where B is any matrix with even integer eigenvalues, but

all the representations are equivalent and this is written in (2.21).

Rather than work out these power series, which are not unique for a given spin, one would like a closed form which would always work. Using the operator K of (2.15), T may be written

$$(2.22) \quad T = e^{i\pi (K(1) + K(2) - K)} = e^{i\pi (2j_P - K)}$$

for one has

$$(2.23) \quad \begin{aligned} T |j, m\rangle &= e^{i\pi (2j_P - K)} |j, m\rangle = e^{i\pi (2j_P - j)} |j, m\rangle \\ &= (-1)^{2j_P - j} |j, m\rangle \end{aligned}$$

in accordance with (2.2).

III. Fermion and Boson Operators

The representation T proposed by (1.6) cannot be applied as it stands to spaces spanned by fermion creation and annihilation operators, because fermion states require the sign convention of Jordan and Wigner (1928) which has not been taken care of by (1.6). This sign convention is introduced such that

$$(3.1) \quad \begin{aligned} a^+(\tau_m) | \dots, 0_{\tau_m}, \dots \rangle &= C_{\tau_m} | \dots, 1_{\tau_m}, \dots \rangle \\ a(\tau_m) | \dots, 1_{\tau_m}, \dots \rangle &= C_{\tau_m} | \dots, 0_{\tau_m}, \dots \rangle \end{aligned}$$

where

$$(3.2) \quad C_{\tau_m} = \prod_{m'=1}^{m-1} (1 - 2 N_{\tau_{m'}}) = \begin{cases} +1 & \text{if the No. of occupied states } N < m \text{ is even} \\ -1 & \text{if the No. of occupied states } N < m \text{ is odd} \end{cases}$$

and N_{τ_m} stands for the occupation number of the quantum state τ_m , and $a^+(\tau_m)$ and $a(\tau_m)$ are the well known creation and annihilation operators for these quantum states. The operators then satisfy the anti-commutation relations

$$(3.3) \quad \begin{aligned} \{ a(\tau_m) a(\tau_{m'}) \} &= \{ a^+(\tau_m) a^+(\tau_{m'}) \} = 0 \\ \{ a(\tau_m) a^+(\tau_{m'}) \} &= I(\tau_m) \delta_{mm'}. \end{aligned}$$

From

$$(3.4) \quad I(\gamma_m) - 2N(\gamma_m) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I_3 \text{ (say)}$$

it follows that

$$(3.5) \quad \begin{aligned} a^+(1) &= a^+ \otimes I & a(1) &= a \otimes I \\ a^+(2) &= I_3 \otimes a^+ & a(2) &= I_3 \otimes a \end{aligned}$$

Thus the transposition of particle labels for fermions T_{12} should have the properties

$$(3.6) \quad T_{12} a^+ \otimes I T_{12}^{-1} = I_3 \otimes a^+, \quad T_{12} I_3 \otimes a T_{12}^{-1} = a \otimes I, \text{ etc.}$$

which is not the property assumed in (1.2). The construction of the operator T_{12} in terms of creation and annihilation operators for fermions is relatively easy, because there is only a finite number of bilinear combinations containing the operators $a(1)$, $a^+(1)$, $a(2)$, and $a^+(2)$ from which one can construct the most general unitary operator involving $a(1)$, $a^+(1)$, $a(2)$, and $a^+(2)$ of which T_{12} , if it exists at all, must be a special case. One finds (see Kaempffer, 1964)

$$(3.7) \quad T_{12} = -i e^{i(\frac{\pi}{2})R_{12}} e^{i(\frac{\pi}{2})S_{12}}; \quad T_{12}^{-1} = +i e^{-i(\frac{\pi}{2})S_{12}} e^{-i(\frac{\pi}{2})R_{12}}$$

with

$$(3.8) \quad R_{12} = [a^+(1)a(2) + a^+(2)a(1)]$$

$$(3.9) \quad S_{12} = [1 - a^+(1)a(1) - a^+(2)a(2)]$$

will do the job required by (3.6). Indeed, since

$$(3.10) \quad [S_{12}, a(1)] = a(1) ; [S_{12}, a(2)] = a(2)$$

one has

$$(3.11) \quad e^{i\frac{\pi}{2}S_{12}} a(1) e^{-i\frac{\pi}{2}S_{12}} = a(1) + i\frac{\pi}{2} [S_{12}, a(1)] + (i\frac{\pi}{2})^2 \frac{1}{2!} [S_{12}, [S_{12}, a(1)]] + \dots \\ = a(1) e^{i\frac{\pi}{2}} = i a(1)$$

and similarly

$$(3.12) \quad e^{i\frac{\pi}{2}S_{12}} a(2) e^{-i\frac{\pi}{2}S_{12}} = i a(2)$$

so that, because

$$(3.13) \quad [R_{12}, a(1)] = -a(2) ; [R_{12}, a(2)] = -a(1)$$

one has finally

$$(3.14) \quad T_{12} a(1) T_{12}^{-1} = i e^{i\frac{\pi}{2}R_{12}} a(1) e^{-i\frac{\pi}{2}R_{12}} = i \left\{ a(1) + (i\frac{\pi}{2}) [R_{12}, a(1)] + \dots \right\} \\ = i \left\{ a(1) \cos \frac{\pi}{2} - i a(2) \sin \frac{\pi}{2} \right\} = a(2)$$

and similarly

$$(3.15) \quad T_{12} a(2) T_{12}^{-1} = i \left\{ a(2) \cos \frac{\pi}{2} - i a(1) \sin \frac{\pi}{2} \right\} = a(1).$$

The representation (3.7) for the transposition operator turns out to be equally valid for bosons, because the bilinear nature of the operators R_{12} and S_{12} lead to the commutation relations (3.10) and (3.13) if the fermion operators are replaced everywhere by the corresponding boson operators r_{12} and s_{12} (say). The decisive difference between the two cases is that T_{12} has with fermion operators only the eigenvalue -1 and with boson operators it has only the eigenvalue +1.

For fermion operators S_{12} and R_{12} satisfy the relations

$$(3.16) \quad S_{12}^2 + R_{12}^2 = 1 ; R_{12} S_{12} = 0 ; R_{12}^3 = R_{12} ; S_{12}^3 = S_{12}$$

so that one can write

$$(3.17) \quad e^{i \frac{\pi}{2} S_{12}} = 1 + i S_{12} \sin \frac{\pi}{2} + S_{12}^2 \left[\cos \frac{\pi}{2} - 1 \right] = 1 + i S_{12} - S_{12}^2$$

$$e^{i \frac{\pi}{2} R_{12}} = 1 + i R_{12} - R_{12}^2$$

and thus, for fermions, the operator of transposition can be put in the somewhat more transparent form

$$(3.18) \quad T_{12} = S_{12} + R_{12} = 1 - a^+(1)a(1) + a^+(2)a(2) + a^+(1)a(2) + a^+(2)a(1)$$

The deviation from (1.6) is made more obvious by writing T_{12} explicitly;

$$(3.19) \quad T_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

which differs from (1.6) by the -1 instead of 1 in the fourth row and column. Further one finds

$$(3.20) \quad T_{12}^2 = S_{12}^2 + R_{12}^2 = 1.$$

IV. Determinants of Operators

The determinant of a matrix is the product of its eigenvalues. The eigenvalues of a permutation P are ± 1 ; therefore, the determinant $|P|$ of P is $+1$ or -1 . If the determinant of an operator is $+1$ it enjoys a connection with the identity. That is, the effect of the operator may be reached by successive application of infinitesimal operations. If the determinant is -1 such connection is never possible. One would like to find these infinitesimal operators, which generate some group that possesses the permutations of determinant $+1$ as a special subgroup. The present work points out, by computation of determinants, the operators where such a study might be feasible.

Some properties of determinants will be needed. First, the determinant of a product of two matrices A and B is the product of the determinants of A and B .

$$(4.1) \quad |A B| = |A| |B|.$$

Further the determinant of a direct product has value:

$$(4.2) \quad |C \otimes E| = |C|^{d_E} |E|^{d_C}$$

where d_C and d_E are the dimensions of matrices C and E respectively. This is shown by diagonalizing $C \otimes D$; so that the eigenvalues of C appear d_E times and those of E appear d_C times along the diagonal.

The determinants of transposition operators for a given set of particle labels are either all $+1$ or all -1 . It follows from property (4.1) that the determinant of an even permutation is always $+1$, and that the determinant of an odd permutation is the same as that of the transposition operators. (An even permutation is one formed by an even number of transpositions, and an odd permutation is one formed by an odd number of transpositions).

Angular-momentum operators will now be used as an illustration. The number of antisymmetric states determines the number of eigenvalues -1 of T . The product space of two particles of angular momenta of amount j_p , with dimension $d=2j_p+1$ each, has dimension d^2 . If j is

$\begin{pmatrix} \text{half-odd integer} \\ \text{integer} \end{pmatrix}$, it follows from (2.1) that the

$\begin{pmatrix} \text{singlet} \\ \text{triplet} \end{pmatrix}$ is antisymmetric, and each consecutive anti-

symmetric multiplet where j is $\begin{pmatrix} \text{even} \\ \text{odd} \end{pmatrix}$ has four more

values of m . There are in all $\begin{pmatrix} \frac{d}{2} \\ \frac{d-1}{2} \end{pmatrix}$ antisymmetric

multiplets. It follows from the formula for an arithmetic sum that in either case there are $\frac{d(d-1)}{2}$ antisymmetric eigenstates. Thus the determinant of T may be written

$$(4.3) \quad |T| = (-1)^{\frac{d(d-1)}{2}} = \begin{cases} (-1)^{\frac{d}{2}} & \text{for even } d, \text{ and half-odd} \\ & \text{integer spin} \\ (-1)^{\frac{d-1}{2}} = (-1)^{\frac{d}{2}} & \text{for odd } d, \text{ and} \\ & \text{integer spin.} \end{cases}$$

As an illustration of these results the determinants of the six possible permutations of three objects will be worked out. From (1.13), (4.2) and (4.3) it follows that

$$(4.4) \quad \left. \begin{aligned} |T_{12}| &= |T \otimes I| = |T|^d |I|^{d^2} \\ |T_{23}| &= |I \otimes T| = |I|^{d^2} |T|^d \\ |T_{13}| &= |T_{12} T_{23} T_{12}| = |T_{12}| |T_{23}| |T_{12}| \end{aligned} \right\} = \begin{cases} +1 & \text{for even } d, \text{ and half-odd} \\ & \text{integer spin} \\ (-1)^{\frac{d-1}{2}} & \text{for odd } d, \text{ and} \\ & \text{integer spin} \end{cases}$$

and thus

$$(4.5) \quad \left. \begin{aligned} |I| &= |T_{23} T_{23}| \\ |T_{123}| &= |T_{23} T_{13}| \\ |T_{132}| &= |T_{23} T_{12}| \end{aligned} \right\} = +1.$$

From (4.4) it follows that the determinant of transposition operators for more than two particles is always +1 in case of half-odd integer spin, but formula (4.3) remains valid in case of integer spin.

The representation (1.6) is quite general except for sign. The discussion thus far has been with the plus sign as demanded for angular momentum. Relations corresponding to (4.3), (4.4) and (4.5) for the minus sign in (1.6) can be found by replacing T by $-T$ in the above derivation. Affixing a $(-)$ to T is in effect multiplication by a diagonal matrix with all elements equal -1 ; from (4.1) it follows that $|-T| = (-1)^{d^2} |T| = (-1)^d |T|$. One finds the general results

$$(4.6) \quad |T_{ab}| = \begin{cases} \text{even } d \\ (+ \text{ or } - \text{ in (1.6)}) & \begin{cases} (-1)^{\frac{d}{2}} \text{ for two particles} \\ +1 \text{ for more than two particles} \end{cases} \\ \text{odd } d \\ (\text{any number of particles}) & \begin{cases} (-1)^{\frac{d-1}{2}} \text{ for plus sign on (1.6)} \\ (-1)^{\frac{d+1}{2}} \text{ for minus sign on (1.6)} \end{cases} \end{cases}$$

remembering " d " is the dimension of the subspace of a single particle ($d=2j_p+1$ for spin), and the plus sign must be chosen in the representation (1.6) for spin space. Results are collected in Table II.

From (3.9) it follows that the determinant of fermion transposition operators are +1. From (4.3) it follows that for two-particle states, the transposition operators for particles of spin $1/2$, $5/2$, $9/2$, ... have $|T| = -1$. Such particles cannot be identified as fermions unless they have further attributes so that the total transposition operator has determinant +1.

Appendix A 1: Proof that (1.6) satisfies (1.1) and (1.2)

By interchanging in

$$(1.6) \quad T_{\alpha_i \alpha_k, \alpha_m \alpha_n} = \delta_{\alpha_i \alpha_n} \delta_{\alpha_k \alpha_m}$$

the row and column labels one has

$$(A 1.1) \quad \tilde{T}_{\alpha_i \alpha_k, \alpha_m \alpha_n} = \delta_{\alpha_m \alpha_k} \delta_{\alpha_n \alpha_i} = \delta_{\alpha_i \alpha_n} \delta_{\alpha_k \alpha_m} = T_{\alpha_i \alpha_k, \alpha_m \alpha_n} = \tilde{T}_{\alpha_i \alpha_k, \alpha_m \alpha_n}^* \\ \text{i.e., } T = T^\dagger$$

and also

$$(A 1.2) \quad (T^2)_{\alpha_i \alpha_k, \alpha_m \alpha_n} = \sum_{\alpha_p \alpha_r} \delta_{\alpha_i \alpha_r} \delta_{\alpha_k \alpha_p} \delta_{\alpha_p \alpha_n} \delta_{\alpha_r \alpha_m} = \delta_{\alpha_i \alpha_m} \delta_{\alpha_k \alpha_n}; \\ \text{i.e., } T = T^{-1}.$$

Thus relations (1.1) are verified. Further it follows that

$$(A 1.3) \quad (I \otimes F)_{\alpha_i \alpha_k, \alpha_m \alpha_n} = \delta_{\alpha_i \alpha_m} F_{\alpha_k \alpha_n}$$

so that

$$(A 1.4) \quad (T(I \otimes F))_{\alpha_i \alpha_k, \alpha_m \alpha_n} = \sum_{\alpha_p \alpha_r} \delta_{\alpha_i \alpha_r} \delta_{\alpha_k \alpha_p} \delta_{\alpha_p \alpha_m} F_{\alpha_r \alpha_n} = \delta_{\alpha_k \alpha_m} F_{\alpha_i \alpha_n}$$

and also

$$(A 1.5) \quad (T(I \otimes F)T)_{\alpha_i \alpha_k, \alpha_m \alpha_n} = \sum_{\alpha_p \alpha_r} \delta_{\alpha_k \alpha_p} F_{\alpha_i \alpha_r} \delta_{\alpha_p \alpha_n} \delta_{\alpha_r \alpha_m} = F_{\alpha_i \alpha_m} \delta_{\alpha_k \alpha_n} \\ = (F \otimes I)_{\alpha_i \alpha_k, \alpha_m \alpha_n}$$

or,

$$(A 1.6) \quad TF(2)T^{-1} = F(1).$$

Then

$$(A\ 1.7) \quad TTF(2)TT = TF(1)T$$

so that

$$(A\ 1.8) \quad TF(1)T^{-1} = F(2)$$

For another operator E it follows that

$$(A\ 1.9) \quad TE(1)T^{-1} = E(2)$$

so from (A 1.6)

$$(A\ 1.10) \quad TE(1)T^{-1}TF(2)T^{-1} = E(2)F(1)$$

which gives (1.2) or (1.5),

$$(A\ 1.11) \quad TE(1)F(2)T^{-1} = F(1)E(2)$$

as required.

Assume there is another operator S with the properties (1.1) and (1.2); that is,

$$(A\ 1.12) \quad S = S^{\dagger} = S^{-1}, \quad SE(1)F(2)S^{-1} = F(1)E(2).$$

It follows that

$$(A\ 1.13) \quad TSE(1)F(2)ST = TF(1)E(2)T = E(1)F(2)$$

or

$$(A\ 1.14) \quad TSE(1)F(2) = E(1)F(2)TS.$$

That is TS commutes with all $E(1)F(2)$ and hence must be a multiple of the identity (Wigner 1959, p. 75). Hence

$$(A\ 1.15) \quad TS = nI, \quad n \text{ a constant}$$

and thus

$$(A\ 1.16) \quad T = nS, \quad T^2 = n^2 S^2 = I = n^2 I, \quad \text{and } n = \pm 1.$$

The uniqueness except for sign is thus proved.

Appendix A 2: Inversion of Vandermonde's Matrix

Vandermonde's matrix V has the form

$$(A\ 2.1) \quad V = \begin{bmatrix} a_0^0 & a_0^1 & a_0^2 & \dots & a_0^n \\ a_1^0 & a_1^1 & a_1^2 & \dots & a_1^n \\ a_2^0 & a_2^1 & a_2^2 & \dots & a_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n^0 & a_n^1 & a_n^2 & \dots & a_n^n \end{bmatrix}$$

where a_i^h is the number a_i to the power h .

The notation ${}_s^t R$ is given by

$$(A\ 2.2) \quad {}_s^t R = \frac{\text{sum of all different products with } s \text{ of the values } a_i}{(a_t - a_0)(a_t - a_1)(a_t - a_2) \dots (a_t - a_n)}$$

$$(A\ 2.3) \quad {}_0^t R = \frac{1}{(a_t - a_0)(a_t - a_1)(a_t - a_2) \dots (a_t - a_n)}$$

and where the element a_t is excluded in formation of the products in the numerator and $a_t - a_t$ is excluded from the denominator. With four numbers a_0, a_1, a_2, a_3 as examples one has

$$(A\ 2.4) \quad {}^1_0 R = \frac{1}{(a_1 - a_0)(a_1 - a_2)(a_1 - a_3)}, \quad {}^3_2 R = \frac{a_0 a_1 + a_0 a_2 + a_1 a_2}{(a_3 - a_0)(a_3 - a_1)(a_3 - a_2)}, \text{ etc.}$$

The inverse V^{-1} has the elements b_{ij} where

$$(A\ 2.5) \quad b_{ij} = (-1)^{n-i} {}^j_{n-i} R$$

That is

$$(A\ 2.6) \quad V^{-1} = \begin{bmatrix} (-1)^n {}^0_n R & (-1)^n {}^1_n R & (-1)^n {}^2_n R & \cdots & (-1)^n {}^n_n R \\ \vdots & \vdots & \vdots & & \vdots \\ {}^0_2 R & {}^1_2 R & {}^2_2 R & \cdots & {}^n_2 R \\ -{}^0_1 R & -{}^1_1 R & -{}^2_1 R & \cdots & -{}^n_1 R \\ {}^0_0 R & {}^1_0 R & {}^2_0 R & \cdots & {}^n_0 R \end{bmatrix}.$$

This is realized by computing the product VV^{-1} . An element c_{ij} of VV^{-1} has the form

$$(A\ 2.7) \quad c_{ij} = a_i^n {}^j_0 R - a_i^{n-1} {}^j_1 R + a_i^{n-2} {}^j_2 R - \cdots + (-1)^{n-1} a_i {}^j_{n-1} R + (-1)^n {}^j_n R.$$

The denominator is common to all ${}_s^j R$. If $i \neq j$ then a_1 is used in formation of the numerators of ${}_s^j R$ and the product $a_1^{n-s} {}_s^j R$ can be written in the form

$$(A\ 2.8) \quad a_i^{n-s} \underset{s}{j} R = a_i^{n-s+1} \underset{s-1}{j,i} R + a_i^{n-s} \underset{s}{j,i} R$$

where $\underset{s}{j,i} R$ means exclusion of both i and j information of the numerator but still only $a_j - a_j$ is left out of the denominator. Now it is seen that $c_{ij} = 0$ when $i \neq j$. If $i = j$, one should have $c_{ij} = 1$. The denominator of c_{jj} has the factors $a_j - a_s$ with $a_j \neq a_s$, so it is sufficient to show that the numerator factors the same way. Substitution of root $a_i = a_s \neq a_j$ then should yield zero as has already been shown above. Thus $c_{ij} = \delta_{ij}$ and the proposed inversion is verified.

Table I. Transposition Operators in Angular Momentum Space

j_P	F	$f_j = f_0, f_1, f_2, \dots, f_{2j_P}$	T
$\frac{1}{2}$	$\underline{J}(1) \cdot \underline{J}(2)$ S^2 K	$-\frac{3}{4}, \frac{1}{4}$ 0, 2 0, 1	$2 \underline{J}(1) \cdot \underline{J}(2) + \frac{1}{2}$ $S^2 - 1$ $2K - 1$
1	$\underline{J}(1) \cdot \underline{J}(2)$ J^2 K	-2, -1, 1 0, 2, 6 0, 1, 2	$(\underline{J}(1) \cdot \underline{J}(2))^2 + (\underline{J}(1) \cdot \underline{J}(2)) - 1$ $\frac{1}{4} (J^2)^2 - \frac{3}{2} (J^2) + 1$ $2K^2 - 4K + 1$
$\frac{3}{2}$	$\underline{J}(1) \cdot \underline{J}(2)$ J^2 K	$-\frac{15}{4}, -\frac{11}{4}, -\frac{3}{4}, \frac{9}{4}$ 0, 2, 6, 12 0, 1, 2, 3	$\frac{2}{9} (\underline{J}(1) \cdot \underline{J}(2))^3 + \frac{11}{18} (\underline{J}(1) \cdot \underline{J}(2))^2 - \frac{9}{8} (\underline{J}(1) \cdot \underline{J}(2)) - \frac{67}{32}$ $\frac{1}{36} (J^2)^3 - \frac{17}{36} (J^2)^2 + \frac{11}{6} (J^2) - 1$ $\frac{4}{3} K^3 - 6K^2 + \frac{20}{3} K - 1$
2	$\underline{J}(1) \cdot \underline{J}(2)$ J^2 K	-6, -5, -3, 0, 4 0, 2, 6, 12, 20 0, 1, 2, 3, 4	$\frac{1}{36} (\underline{J}(1) \cdot \underline{J}(2))^4 + \frac{1}{6} (\underline{J}(1) \cdot \underline{J}(2))^3 - \frac{13}{36} (\underline{J}(1) \cdot \underline{J}(2))^2 - \frac{5}{2} (\underline{J}(1) \cdot \underline{J}(2)) - 1$ $\frac{1}{576} (J^2)^4 - \frac{1}{16} (J^2)^3 + \frac{95}{144} (J^2)^2 - \frac{25}{12} (J^2) + 1$ $\frac{2}{3} K^4 - \frac{16}{3} K^3 + \frac{40}{3} K^2 - \frac{32}{3} K + 1$

Table II. Determinants of Operators

(a) All even permutations have determinant +1.

(b) Determinant of odd permutations are

number of particles		two		many	
sign of eq. (1.6)		(+)	(-)	(+)	(-)
dimension	d	$d=2j_p+1$		$d=2j_p+1$	
	2	-1	-1	+1	+1
	3	-1	+1	-1	+1
	4	+1	+1	+1	+1
	5	+1	-1	+1	-1
	6	-1	-1	+1	+1
	7	-1	+1	-1	+1
	8	+1	+1	+1	+1
	9	+1	-1	+1	-1
	10	-1	-1	+1	+1

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