GENERAL EQUATIONS FOR SHORT-RANGE OPTIMIZATION OF A COMBINED HYDRO-THERMAL ELECTRIC SYSTEM

by

RADEN ARTONO ARISMUNANDAR

B.A.Sc., University of British Columbia, 1958

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

in the Department

of

Electrical Engineering

We accept this thesis as conforming to the standards required from candidates for the degree of Master of Applied Science.

....

....

....

Members of the Department
of Electrical Engineering

THE UNIVERSITY OF BRITISH COLUMBIA

March, 1960

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Electrical Engineering,

The University of British Columbia,
Vancouver 8, Canada.

Date March 8, 1960

ABSTRACT

This thesis offers a review and an analysis of all except the less important advances of the previously developed methods and equations for optimizing the operation of an electric system of m thermal and n hydro plants. In this analysis both short-range (twenty-four hours, seven days) and long-range (one year) periods are involved.

The primary objective of this thesis is to derive, using the Calculus of Variations, general differential equations for short-range optimization of combined hydro-thermal systems. The basic criterion for choosing to solve the short-range instead of the long-term problem lies in the theory of forecasting in general, the theory of forecasting of stream flows in particular, and is based on the aforementioned analysis.

Tests for establishing the fact that the above general equations actually produce the desired minimum cost of operation are given in the form of three other necessary conditions and three sufficient conditions. These conditions are known in this branch of mathematics as the analogue of Legendre's condition, the Weierstrass' analogue of the Jacobi's condition and the Weierstrass' E-function condition for a minimum. A well-known example is worked out using these conditions.

In addition to the above, this thesis also proves that all previously developed methods and equations for short-term optimization are essentially equivalent, and that these formulas are merely simplified forms of the general equations developed in this treatise.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ACKNOWLEDGEMENT

This investigation is sponsored by the National Research Council of Canada.

The author wishes to express his sincere gratitude to Dr. Frank Noakes, thesis supervisor, for guidance, stimulus and advice during the course of this research-work, to Dr. A. D. Moore for his assistance in many aspects of this work, and also to Dr. R. F. Hooley of the Department of Civil Engineering who advised the author on several hydraulic viewpoints of the problem.

The author is grateful to Mr. John H. Drinnan, Planning Engineer of the British Columbia Engineering Company, Vancouver, B. C., and Mr. J. S. Windsor of the Planning Division of the British Columbia Power Commission, Victoria, B. C., with whom the author has had many informative discussions on the practical side of the problem solved in this thesis. The former company, through Mr. L. A. Taylor, Supervisor of the Network Analyzer Section, and the latter company, through Mr. A. W. Lash, Consulting Engineer, have supplied the author with the necessary data for the aforementioned problem.

The author should like to thank Dr. E. Leimanis of the Department of Mathematics for his generous assistance in reriewing the mathematical aspects of a problem which, sometimes, does not seem to lend itself to an exact solution.

Thanks are due to Dr. J. F. Szablya and the author's colleagues in the power group of the Department of Electrical Engineering for much enlightening and stimulating discussion.

The author is indebted to the Colombo Plan Administration in Canada for assistance received in the form of a Scholarship awarded during his stay from 1956 to 1960.

CHAPTER I

INTRODUCTION

For a number of years power system engineers have made several attempts to solve the problem of how to operate an electric system most economically, i.e., to obtain a set of extremals of operation. These extremals can be given in either of the two forms:

(1) as a set of minima, for example, minimization of cost over a predetermined future interval, minimization of losses (due to spillage, due to violations of certain limitations, due to unreliability of service which results in loss of customers), etc., or

(2) as a set of maxima, such as maximization of hydro-energy over a certain period.

The set of extremals discussed in this thesis is either that of minimization of cost or maximization of hydro-energy.

The problem of optimization* differs in complexity with the type of system and with the length of time in which the optimization is considered. The thermal problem differs from the exclusively hydro problem, which is again different from the problem of a combined hydro-thermal electric system. The latter is more difficult to solve than the second which is in turn harder than the first. The problem further increases in complexity if optimization is desired over a long period due to

---

*"Optimization of operation" will often be abbreviated as, simply, "optimization".

the very many uncertainties of the future. Probabilistic methods have been used with the aid of dynamic programming techniques to cope with the above uncertainties, but none of these approaches have come close to the desired minimum results. The short-term problem is therefore relatively simpler than the long-range one, mainly because it involves less vagueness than the latter.

Irrespective of the consideration of long-range or short-range periods the hydro-thermal problem is generally difficult if the number of plants in the system, especially the hydro plants, is large, and if the system is spread over a large geographical area. If the storage elevations vary in their order of magnitude from plant to plant, the system usually cannot be simplified and, therefore, presents extra difficulties. In addition, in various systems many well-established load schedules must be drastically altered to accomodate heavy vessels on the river. The above are some of the problems in a hydro-thermal case. It is evident that the economic problem as a whole seems to be massive and, hence, almost unsolvable. For this reason only part of the whole problem will be solved. In the next paragraph, the purpose and scope of the thesis are introduced.

This thesis discusses the various types of system problems involved for both the long-range and short-term periods, and analyzes all previous methods developed by different authors. The primary objective of this thesis is to derive, using the Calculus of Variations, general differential equations for the short-range problem, realizing that, based on the above analysis, no exact mathematical solution can be obtained for the long-range problem. This treatise also proves that these general equations can be reduced to the several short-term equations developed by different authors previously, when certain simplifying assumptions are applied. At the same time a set of proofs and reasonings, indicating the equivalence of one previously developed equation and another, is given. In addition, for the first time in this field the second, third and fourth necessary conditions, and the first, second and third sufficient conditions for the minimization problem are worked out. For the sake of clarification a simple system of one hydro and one thermal plant is considered imposing the above conditions, along with a well-known example prepared by authorities in the field of economic load-dispatching.

CHAPTER II

REVIEW AND ANALYSIS OF PREVIOUS WORK

2.1 Optimization of a Thermal System

The problem of optimizing the operation of an exclusively thermal system, i.e., a power system of m thermal plants supplying a given load, is relatively simpler than that of optimizing a pure hydro system due to various reasons. It is generally known[1*] that in the latter problem many uncertainties, such as weather conditions, the amount of future inflow to the reservoirs and consequently the amount of water available in storage, are involved. Conversely, in the thermal problem the amount and type of fuel "at hand" and/or "in order" is the amount purchased, and therefore, can be determined more precisely. Furthermore, it is more convenient to assign dollar values to the amount of oil, gas or coal burned to generate certain megawatts of thermal power. It is therefore possible to plot a thermal cost curve, i.e., fuel cost in dollars per hour versus thermal power output in megawatts, from which the incremental thermal cost curve can be derived. On the other hand, it takes a great deal of guess-work to compute the incremental water value of any hydro plant.

The thermal problem may become difficult, however, if the fuel and its price are unknown. In some systems there is a choice of oil and gas carrying a peak escalation charge with

---

* The superscript numerals refer to the list of References on pp. 70 to 79 inclusive.

the added complication that either domestic natural gas or liquid petroleum can be used. In operating the thermal plants the shape of the fuel cost curve may dictate a uniform load for the thermal sources, but it is by no means certain that it will produce the desired optimization, since the possibility of peaking with oil to assist the domestic peak gas load must be included. Furthermore, with different cost and heat content between units within a plant, and between one plant and another, the incremental rate of one thermal plant depends on both the power output of that plant and the anticipated loading of other thermal plants within the system as well.[2] It is also known, that there is that difficulty in determining precisely what the incremental rate is over a small band of output.[3] There is, therefore, a complex interconnection of several problems affecting the optimum thermal schedule.

If the thermal problem is not as complex as the one mentioned in the previous paragraph, then the question of thermal optimization is that of minimizing the total fuel cost $\sum C_j$ over a certain pre-determined and fixed future time interval T, i.e.,*

$$\int_0^T \sum_{j=1}^{m} C_j \; (P_{Tj}) \; dt = \text{minimum}, \qquad \ldots \; (1\text{-}1)$$

where P_{Tj} is the thermal plant output (Mw), when supplying a certain load demand P_D and losses P_L:

$$\sum_{j=1}^{m} P_{Tj} = P_L + P_D. \qquad \ldots \; (1\text{-}2)$$

---

*For Nomenclature see also pp.80-84

The solution of this problem is given by the condition that all plants should be operated at equal incremental rates when transmission losses are neglected,[4-6] and at equal incremental cost of delivered power when transmission losses are taken into account.[7-18] In the former case the condition can be proved when a simple two-plant thermal system is considered. The total fuel cost C_T to be minimized is, then, given by

$$C_T = C_1 + C_2 \qquad \ldots (1\text{-}3)$$

with the subsidiary condition that

$$P_D = P_{T1} + P_{T2},$$

$$\text{or} \quad P_{T2} = P_D - P_{T1}. \qquad \ldots (1\text{-}4)$$

The total differential is given by

$$dP_{T2} = - dP_{T1}, \qquad \ldots (1\text{-}5)$$

since

$$dP_D = 0 \qquad \ldots (1\text{-}6)$$

with a constant load demand at any one time. To minimize C_T its derivative with respect to the two variables must vanish, i.e.,

$$\frac{dC_T}{dP_{T2}} = \frac{dC_T}{dP_{T1}} = 0. \qquad \ldots (1\text{-}7)$$

Using equation (1-5) one obtains the equal incremental rate condition mentioned earlier:

$$\frac{dC_1}{dP_{T1}} = \frac{dC_2}{dP_{T2}} = \text{constant}. \qquad \ldots (1\text{-}8)$$

When transmission losses are included, the problem becomes slightly more complex. However, with the development of the various types of computers, both analogue[19-30] and digital,[31-34] and network analyzers[35] the thermal problem is solvable.

2.2 Optimization of a Hydro System

The problem of determining the most economical method of operating a hydro-electric system of n plants supplying a given load has been attempted for over forty years.[36-44] Several interpretations were given by many authors to the term "optimum" used in this treatise, although very few prove that their results produce the actual desired minimum cost of operation. Some of their findings containing the more important advances of optimization are discussed below.

In 1929, Strowger[45] stated that for best economy the plant should be operated at maximum efficiency and in such a manner that maximum production is realized. Strowger further assumed that this operating procedure is the only way to make the utmost use of the available resource. Schamberger, in 1935, operated his hydro stations on the basis of minimization of loss[46] caused by inefficient loadings of the units and improper loadings of the various connected stations. Four fundamental rules of operation were established, to be followed in order that losses at the stations can be minimized.

An attempt to solve one of the more difficult hydro system problems was made by Burr in 1941, in a Master's thesis at the Massachusetts Institute of Technology.[47] The purpose of his thesis was to develop general principles in determining the loading of "common-flow"* hydro-electric stations, i.e., stations situated on the same stream (river). Realizing the complexity of such a problem, no effort was made to solve any

*Burr's term.[47]

specific case, although Burr did consider a simple illustrative example with two plants on a stream and a number of simplifying assumptions. Ten years later, Johnson of the University of Washington,[48] extended the problem to a three-plant case.

Burr's work was continued for the general case of one-plant-on-one-stream by two of his colleagues at the same institute, who in 1950, wrote a joint Master's thesis[49] on economy loading of hydro systems. In this thesis Chandler and Gabrielle established some mathematical criteria for economy loading and applied them to a simple hydro system in order to obtain general principles and conclusions. The above criteria were also applied to a system problem with actual numerical data. One interesting feature of Chandler's and Gabrielle's work is the inclusion of a number of major factors such as head, flow, storage, plant characteristics (electrical and hydraulic), time delay of flow between plants and transmission losses. A more thorough study on the effect of the last factor on optimum plant loading was considered in a Master's Thesis by Bobo[50] of the University of Pittsburgh. Johannessen[51] of M.I.T. wrote a similar thesis to study the relative changes in costs when transmission losses are included and changes in the predicted stream flow.

Another type of optimization is dealt with in a paper by McIntyre, Blake and Clubb[52] in the form of "an efficient schedule for a daily, weekly, monthly or seasonal basis" to meet an estimated load. A general purpose digital computer (Bendix G-15A) is used for this problem which takes as input data a set of stream flows and maximum capability limits at all plants, specifying the initial storage value at each reservoir. The output for run-of-river plants consists only of peak and average capability and plant discharge, while for storage plants values for "change-in-storage-content", "end storage content" and conversion factors in MW per thousand second feet are typed out in addition. The criterion for this type of optimization is that in the event of an overdraft, overfill, or the violation of a reservoir outlet restriction resulting from an invalid operating instruction, the computer will type out an indicator along with a reservoir identification code and halt. The engineer must then specify some operational procedure which will allow the computer to proceed with a new iteration.

2.3 Optimization of a Hydro-Thermal System

2.3.1 Introduction

The problem of optimizing the operation of an electric power system of m thermal and n hydro plants supplying a given load, is not exactly equivalent to a combination of a pure thermal problem and an exclusively hydro problem.[53-55] In a hydro-thermal problem the object is equivalent to the pure thermal case of minimizing the total cost of operation of the thermal plants only. On the other hand, when these thermal plants are excluded, the problem becomes that of maximizing hydro energy at all plants over a future time interval. This is almost equivalent to Strowger's[45] criteria of "making the utmost use" of the available water resource. All of the above statements mean that in the case of allocating load for the different plants, the optimizing equations can be used both for the restricted thermal and the combined hydro-thermal system. The reverse is not always true, however, since the scheduling equations of a hydro system are not necessarily equal to the scheduling equations of a hydro-thermal system with the thermal equations obliterated.

In general the hydro-thermal problem can be divided into two general groups: (i) short-term or short-range, and (ii) long-term or long-range, both having entirely different characteristics and, hence, require completely different solutions. The first problem is relatively simple as it deals only with a short future time interval (24 hours, one week) and, hence, can be treated with certainty. Conversely, the problem of long-range optimization is much more complex since it deals with many unpredictable variables in a much longer future time interval (one year). For this reason the latter case is more problematical than the first one and, consequently, more difficult to assimilate mathematically.

2.3.2 Previous Long-Term Methods

One of the well-known analytical methods for the long-range problem is that of Cypser[56,57] whose approach is outlined in a doctorate thesis at M. I. T. Cypser dealt with large systems having large storages where plant efficiencies depend on past operations of storage, and where present operations are based on a whole set of predictions of flow and load demand for the whole future long-range period to be optimized. This method is objectionable primarily because, while predicting stream flow for a short period is still possible, long-range predictions, particularly in the west coast, are usually not very accurate. The operating scheme obtained for that year can therefore be grossly misleading. In addition, the fact that the ratio of maximum to extreme low river flow in this part of North America is very high (e.g., Bridge River, British Columbia: 158 to 1) compared to the ratio in the eastern part of this continent (e.g., St. Lawrence River: 2 to 1) almost annihilate any value of forecasting of stream flow in the west coast.[69] Cypser obtained his results by developing a procedure, using the "method of steepest descent", for successively improving a proposed mode of operation such that the effective cost can be continuously reduced.

Contrary to Cypser's approach, Little,[58] in a thesis submitted for the degree of Doctor of Philosophy in Physics at the Massachusetts Institute of Technology, did not assume that future river flow is known in detail a year in advance. For this reason probabilistic methods are used to minimize the expected cost, but not the cost itself. A simple mathematical model comprises one hydroelectric plant and a reservoir, one thermal plant, a given load demand, and a set of stream flows characterized by probability densities. In optimizing the operation the planning period is divided into N smaller intervals (Little used N = 26; one interval is 2 weeks). At the beginning of each interval a decision is made about the use of storage in that interval taking into consideration today's reservoir level and the river flow pattern in the immediately preceding interval. Using a thirty-nine-year record of river flows of the Columbia River for the probability density function and assuming a constant load throughout the year, twenty-six decision functions were obtained. Each is represented by a set of graphs which tell how much stored water should be used for the next two weeks as a function of the volume of the preceding two weeks of flow, with the present volume of water as parameters. The author of this thesis feels that the method described above is deficient in certain ways. While this approach may give a true minimum over-all cost for the whole year, optimization of several intervals are uncertain. It is true that even if it is possible to know what the actual volume of storage is at the beginning of each interval, the volume at any time within the interval is entirely guess-work. This is due to sudden changes of weather conditions which cause drastic changes in rainfall and, hence, riverflow during a preceding short period within the interval itself. If this phenomenon occurred, the whole pattern of planned-storage-use* in that interval should be altered considerably, or else a

---

*Little's term.[58]

certain amount of water must be spilled or flood limitations violated. Consequently, the whole pattern of storage use of the next interval should be altered, which in turn will change all sets of planned-storage-uses of all succeeding intervals. Little's comparison of his method with the well-established rule-curve* operation is also objectionable, primarily because the rule-curve used is based on the driest year of the above thirty-nine-year record. If the flow conditions are not as severe, the rule-curve operation may give a higher percentage of savings than the one per cent calculated by Little. Little's choice of a constant load for a whole year is not practical, and his claim that changing the load from time to time will only add a little complication is not justified. When the system consists of n hydro plants (n larger than 1) the problem is difficult to solve since it will consist of at least n x N decision functions of two variables each.

Little's work was followed by Koopmans'[59] of the Cowles Foundation for Research in Economics at Yale University. Koopmans' paper deals with a simple two-plant system similar to that of his predecessor, and its purpose is to construct a "feasible water storage policy"** which minimizes the thermal cost over a predetermined planning period, while meeting a given load demand. This method

---

*Discussed in a later paragraph.
**Koopmans' term.

is unique in that it offers an additional feature of associating with the above optimal storage policy imputed "efficiency prices" of the power generated and of the water used and in storage, and imputed "efficiency rents" for the use of the plant and the reservoir. However, it has a number of limitations in that certain impractical simplifying assumptions are made: (i) future load and flow conditions are known with certainty (ii) variations in head can be neglected.

One of the most widely used methods for long-range optimization and which will probably produce the closest to the desired minimum cost is called the rule-curve method. This method is extensively covered in a twenty-page transactions paper by Brudenell and Gilbreath of the Tennessee Valley Authority.[60] This paper deals with the subject of "economic integration"* of hydro and thermal plants in delivering the required load to the high voltage transmission system. In supplying this load the basic criterion is that of minimizing the average annual production cost under the most adverse conditions of water. To arrive at the desired economic result several guides in the form of curves are used: (i) the basic rule curve is the diagram which shows the expected or planned reservoir levels or the plot of remaining storage at any given time, taking into account the most critical conditions of stream flow; (ii) the "no-spill rule curve"

---

*TVA's term.

is the curve of the surplus between the firm load and the energy available during the maximum flow period. If the storage content is above this curve,the loss due to spilling should be balanced against the gain in energy from the use of available flow; (iii) the "economy guide-line" is a curve below which the value of incremental storage is greater than the cost of operating the thermal power at any one time. Using this diagram, the engineer would simply shut down or operate the thermal source according to the position of the storage content, whether below or above the line; (iv) the "economy guide curve" is a set of family curves used in the same manner as the previously described curve but over a period of time. Thus for any day and actual storage content the amount of auxiliary power to be used is indicated.

2.3.3 Previous Short-Term Methods and Equations

In the field of short-term optimization several methods and equations have been derived to solve the problem of scheduling the various generating plants to meet a given load at one particular time. These equations are outlined below in three sections according to their similarities.

(i) Ricard*[61]

Ricard derived in 1940, a set of operating schedules for a hydro-thermal system with no losses. His work was

*Ricard is the first person to derive the type of equations developed in this section, hence, the name.

continued by Chandler, Dandeno, Glimn and Kirchmayer[62] (henceforth abbreviated: CDGK) in 1953, who included transmission losses but with constant head. The latter method was improved by Glimn and Kirchmayer[63] (henceforth abbreviated: GK) who included transmission losses and variable-head plants. Kron[63] of the General Electric Company developed equivalent equations.

Their equations are as follows:

Ricard's Equations (September 1940)[61]

thermal: $$dC/dP_T = \lambda, \qquad \text{... (2-9)}$$

hydro: $$\gamma_0 \exp\left[\int_0^t \frac{\partial Q}{\partial h}\frac{dt}{A}\right]\frac{\partial Q}{\partial P_H} = \lambda, \qquad \text{... (2-10)}$$

where λ = constant, Lagrangian multiplier, incremental cost of delivered power (\$/Mw-hr),

γ_0 = conversion constant (\$/ft^3),

Q = hydro plant discharge (cfs),

h = net head (ft),

A = surface area of reservoir (ft^2, acre),

P_H = hydro plant output (Mw).

CDGK's Equations (October 1953)[62]

thermal: $$\frac{dC_j}{dP_{Tj}} + \lambda \frac{\partial P_L}{\partial P_{Tj}} = \lambda, \quad j = n+1,\ldots,n+m \qquad \text{... (2-11)}$$

hydro: $$\gamma_i \frac{dQ_i}{dP_{Hi}} + \lambda \frac{\partial P_L}{\partial P_{Hi}} = \lambda. \quad i = 1, \ldots, n \qquad \text{... (2-12)}$$

GK's Equations (December 1958)[63]

thermal: $$\frac{dC}{dP_T} + \lambda \frac{\partial P_L}{\partial P_T} = \lambda, \qquad \text{... (2-13)}$$

$$\text{hydro:}\quad \gamma_0 \exp\left[\int_0^t \frac{\partial Q}{\partial h}\frac{dt}{A}\right]\frac{\partial Q}{\partial P_H} + \lambda\frac{\partial P_L}{\partial P_H} = \lambda. \qquad \ldots (2\text{-}14)$$

Kron's Equations (December 1958)[63]

$$\text{hydro:}\quad \frac{\lambda}{A}\left[1 - \frac{\partial P_L}{\partial P_H}\right]\frac{\partial P_H}{\partial h} + \frac{d}{dt}\left[\lambda\left(1 - \frac{\partial P_L}{\partial P_H}\right)\frac{\partial P_H}{\partial Q}\right] = 0. \qquad \ldots (2\text{-}15)$$

(ii) M. I. T.

The personnel of this section have all been members of the Economy Loading Research Group at the Massachusetts Institute of Technology. Their work in this field was initiated by Cypser[56] who developed a set of scheduling equations using the assumption that variations in elevations and plant efficiencies can be neglected. Cypser's equations are not linear and, therefore, not solvable by numerical iterations or by means of an analogue computer of the network analyzer type. Carey's thesis[64] suggested an approach which will linearize Cypser's equations.

The equations developed in this section are:

Cypser's Short-Range Equations (February 1953)[56]

$$\text{thermal:}\quad \frac{\partial C_j}{\partial P_{Tj}} - \mu(t)\frac{\partial P_L}{\partial P_{Tj}} = -\mu(t), \qquad j = n+1,\ldots,n+m \qquad (2\text{-}16)$$

$$\text{hydro:}\quad \frac{\partial C_i}{\partial P_{Hi}} - \lambda_i - \mu(t)\frac{\partial P_L}{\partial P_{Hi}} = -\mu(t), \qquad i = 1,\ldots,n \qquad (2\text{-}17)$$

where λ_i and $\mu(t)$ are Lagrangian multipliers.

Carey's Equations (June 1953)[64,65]

$$\text{thermal:}\quad \frac{\partial C_j}{\partial P_{Tj}} + \beta\frac{\partial P_L}{\partial P_{Tj}} = -\mu(t), \qquad j = n+1,\ldots,n+m \qquad (2\text{-}18)$$

$$\text{hydro:}\quad \beta\frac{\partial P_L}{\partial P_{Hi}} = \lambda_i - \mu(t), \qquad i = 1,\ldots,n \qquad (2\text{-}19)$$

where β = average incremental fuel cost (\$/Mw-hr).

Cypser's Long-Range* Equations (February 1953)[56,57]

hydro: $$\frac{\partial(C_i+C_j)}{\partial S_i} - \frac{d}{dt}\left[\frac{\partial(C_i+C_j)}{\partial(dSi/dt)}\right] = 0, \qquad \ldots (2\text{-}20)$$

where S_i = storage volume at hydro plant i (ft^3).

(iii) Watchorn (April 1955)[66]

Watchorn defined** that maximum economy will be obtained if the following equations are satisfied:

$$\frac{dC}{dP_T}\left[\frac{dP_D/dt}{dQ/dt} - \frac{dP_H}{dQ}\right] = N, \qquad \ldots (2\text{-}21)$$

with $$\frac{dP_H}{dQ} = \frac{\partial P_H}{\partial Q} + \frac{\partial P_H}{\partial h}\left[\frac{(Q-F)dy/dS}{dQ/dt} - \frac{dy_L}{dQ} - \frac{dy_T}{dQ}\right], \qquad \ldots (2\text{-}22)$$

where y = reservoir or pond elevation (ft),

y_L = head loss (ft),

y_T = tailwater elevation (ft),

F = water inflow to reservoir (cfs),

N = incremental water value (\$/hr/1000 cfs).

---

* Included here for comparison with GK's equations.

**Watchorn did not prove his definition.

CHAPTER III

THE SHORT-TERM HYDRO-THERMAL PROBLEM AND ITS GENERAL SOLUTION

3.1 Statement of the Problem

Of the various types of extremals involved (see Chapter I) in solving the short-range optimization problem, this thesis is limited to the derivation of the general solution of the mathematical problem defined in the following paragraphs. The objective is to determine a set of generating schedules for an electric power system of m thermal and n hydro plants such that the total operating cost over a predetermined short-term future interval can be minimized, when it is desired to supply a given load demand. In this problem the following limitations are imposed:

(1) The operating costs involved are only those costs which vary directly with the plants' power output due to the fact that in the process of computations used, only those terms with the derivative of the cost with respect to the power output will appear in the equations. Consequently, capital costs on reservoir and generating stations, or labour costs may be omitted. Maintenance costs are purposely omitted since they are small and relatively indefinite. The dominating cost will therefore be the cost of fuel at the thermal plants.

(2) One set of information is assumed to be known, and that is, the amount of energy available at every hydro plant during the period of optimization. This information can be obtained from the rule-curve,[60] where the energy available is

computed from the knowledge of (i) reservoir elevation at present, called the beginning of the optimizing period, (ii) reservoir elevation at the end of the planning period, and (iii) the total amount of inflow into the reservoir. The last two factors explain the whole criterion for choosing to solve the short-term rather than the long-range problem. While it is practically impossible to determine the expected pattern of flow for a whole year, it is, however, highly probable "to guess with exactness" what the total flow is going to be, for the whole day tomorrow or the whole week next week. The above statement runs parallel with the general theory of forecasting: long-range forecasting (e.g., weather, business conditions, etc.) is more difficult than forecasting over a short period of time.

(3) The period considered could vary from one day or 24 hours to one week or seven days, depending on the reliability of forecasting of water resources of the system. It should be noted, however, that generally a longer optimizing period is desirable for the obvious reason that the longer the period the more economical it is, since less computations are to be performed. On the other hand, the shorter the period the more accurate are the results.

(4) This thesis is limited also to the consideration of fixed periods, i.e., a period with fixed end-points. This means that if the period to be optimized extends from, say, Friday,* January 1, 1960, at 01.00 a.m. till Friday, January 8,

---

*Optimizing periods usually start on a Friday for the simple reason that they will include the week-end's load pattern which is different from that of any week-day.

1960, at 01.00 a.m., then it must start and end exactly at the times indicated. If that is not the case, the problem becomes much more complicated than the present one, since more difficult "variable end-point" cases will enter the picture.

(5) Only one hydro plant will be considered on any one stream. The common-flow problem is more complex due to additional restrictions which must be set for the operation of the hydro plant and for several reasons of interdependency: (i) the amount of water released from any upstream plant will off-set the operation of any downstream plant, and (ii) the operation of a middle plant is governed by both the operation of its upstream and downstream plants.

(6) Various hydro limitations are assumed, given in the forms of project and operating restraints in a subsequent section.

3.2 Mathematical Formulation of the Problem

The problem previously stated will be formulated in the following paragraphs. The object is to minimize the integral I of the fuel cost C_j over a fixed future short-time interval T, i.e.,

$$I = \int_{t_0=0}^{t_e=T} \sum_{j=n+1}^{n+m} C_j \left(P_{Tj}, t\right) dt = \text{minimum}, \qquad \ldots (3\text{-}1)$$

where t_0 and t_e are the two fixed end-points of the interval. The problem admits two sets of restrictions, for energy and for load requirements, i.e.,

$$J_1 = \int_0^T P_{H1} \; (Q_1, h_1, t) \; dt = B_1, \qquad \ldots \; (3\text{-}2.1)$$

$$\vdots$$

$$J_n = \int_0^T P_{Hn} \; (Q_n, h_n, t) \; dt = B_n, \qquad \ldots \; (3\text{-}2.n)$$

or, in general, for hydro plant i

$$J_i = \int_0^T P_{Hi} \; (Q_i, h_i, t) \; dt = B_i, \qquad \ldots \; (3\text{-}2.i)$$

and

$$\emptyset \equiv \sum_{i=1}^{n} P_{Hi} \; (Q_i, h_i, t) + \sum_{j=n+1}^{n+m} P_{Tj} \; (t) - P_L \; (P_{Hi}, P_{Tj}, t) - P_D(t) = 0. \qquad \ldots \; (3\text{-}3)$$

The above restrictions can be changed into a more suitable form (see Appendix A) if the following transformations, using flow variables F_i and storage variables S_i, are performed:

$$P_{Hi}(Q_i, h_i, t) = P_{Hi}(F_i, S_i, \dot{S}_i, t), \qquad i = 1, \ldots n \qquad \ldots \; (3\text{-}4)$$

where $\dot{S}_i = dS_i/dt$ = the rate of change of storage (cfs).*

Since the natural inflows to the reservoirs are "alien" variables, they are uncontrollable and indeterminable.[67] For this reason solutions containing these variables would be meaningless. Assuming that they are known they can be eliminated from the determining equations. The hydro and load restric-

---

*In general, $\dot{q}_u = \frac{dq_u}{dt}$, where q_u is a time variable.

tions can therefore be given in the following forms:

$$\bar{J}_1 = \int_0^T P_{H1}(S_1,\dot{S}_1,t)\, dt = \bar{B}_1, \qquad \text{... (3-5.1)}$$

$$\vdots$$

$$\bar{J}_n = \int_0^T P_{Hn}(S_n,\dot{S}_n,t)dt = \bar{B}_n, \qquad \text{... (3-5.n)}$$

or, in general, for hydro plant i

$$\bar{J}_i = \int_0^T P_{Hi}(S_i,\dot{S}_i,t)dt = \bar{B}_i, \qquad i = 1,\ldots n \qquad \text{... (3-5.i)}$$

and

$$\emptyset \equiv \sum_{i=1}^{n} P_{Hi}(S_i,\dot{S}_i,t) + \sum_{j=n+1}^{n+m} P_{Tj}(t)$$

$$- P_L\left[P_{Hi}(S_i,\dot{S}_i),P_{Tj},t\right] - P_D(t) = 0. \qquad \text{... (3-6)}$$

In addition the following project and operating limitations for the hydro plants must be observed:

(1) Project Limitations, a function of the design and location of the plant, reservoir, channels, turbines, etc., but independent of the operation of the system itself; at hydro plant i, where i = 1, ... , n:*

a. Maximum turbine discharge at maximum gate opening as function of the net head:[56,59]

$$Q_i \leqq Q_{iMT}\,(h_i). \qquad \text{... (3-7)}$$

b. Minimum storage elevation due to location of intake gate:**[56,68]

$$y_i \geqq y_{i\mu G}. \qquad \text{... (3-8)}$$

---

* Subscripts M and μ stand for maximum and minimum respectively.

**Storage at levels below the turbine intake gates is called "dead storage" (Koopmans' term[59]).

c. Minimum storage elevation due to limits of storage basin:[56,68]

$$y_i \geqq y_{i\mu B}. \qquad \ldots (3\text{-}9)$$

d. Maximum of water flow through conduits:[56]

$$Q_i \leqq Q_{iMC}. \qquad \ldots (3\text{-}10)$$

e. Maximum power output at each hydro plant:

$$P_{Hi} \leqq P_{HiM}. \qquad \ldots (3\text{-}11)$$

(2) Operating Limitations, restricting factors in the operation of the system; at hydro plant i, where i = 1,..., n:*

a. Maximum storage elevation due to flood prospects:[56,68]

$$y_i \leqq y_{iMF}. \qquad \ldots (3\text{-}12)$$

b. Minimum plant discharge and spillage for the protection of fish:[56,68]

$$\sigma_i + Q_i \geqq Q_{i\mu f}. \qquad \ldots (3\text{-}13)$$

c. Minimum plant discharge and spillage for navigational purposes:[3,56]

$$\sigma_i + Q_i \geqq Q_{i\mu N}. \qquad \ldots (3\text{-}14)$$

d. Minimum plant discharge and spillage for irrigational purposes:[70]

$$\sigma_i + Q_i \geqq Q_{i\mu I}. \qquad \ldots (3\text{-}15)$$

e. Minimum plant discharge to allow the plant to be operated at minimum load factor during peaking:[68]

$$Q_i \geqq Q_{i\mu P}. \qquad \ldots (3\text{-}16)$$

---

*See corresponding foot-note on page 23.

f. Maximum storage draft at certain lakes for recreational purposes:[68]

$$\dot{S}_i \leqq \dot{S}_{iMR} \cdot \qquad \ldots (3\text{-}17)$$

Most of the operating restrictions apply only during certain periods of the year; some are distinctly seasonal in nature such that the results obtained from the optimizing equations should be checked against these restraints. So long as the optimizing period is shorter than the periods mentioned above, the results mentioned earlier will be usable. On the other hand, all of the project limitations apply at all times of the year.

3.3 General Solutions of the Problem: The First Necessary Condition

3.3.1 Introduction

The problem formulated in the preceding section will be solved using the method of the Calculus of Variations as outlined in several textbooks and papers.[71-75] A summary of this type of calculus, with special reference to this problem, is given in Appendix B. The Calculus of Variations deals with problems of determining extreme values. However, while in the ordinary theory of maxima and minima, the problem is to determine those independent variables x, y, z, ... which will maximize or minimize a given function $f = f(x, y, z, \ldots)$, in the Calculus of Variations definite integrals involving one or more unknowns are considered. The problem in the latter case is to determine these unknown functions such that the definite integral will take maximum or minimum values. The problem of this thesis is more complicated than the one presented above due to the restrictions given by equations (3-5.1) to (3-5.n) and (3-6). The first set of restraints indicates an isoperimetric case, while the latter is a special case of the problem of Lagrange. These two sets of restraints are quite different in nature when solutions for the necessary and sufficient conditions are required, so that the problem can only be solved if certain types of transformations are considered. These transformations will alter the combined problem into either an exclusively isoperimetric one or a pure Lagrange problem. Due to various practical reasons given in Appendix B the transformation into the isoperimetric problem is chosen. This means, that instead of using equation (3-6) itself, the integral of this equation is considered during the planning period T. This in turn implies that, physically, the condition for load requirements is now replaced by the restraints for energy requirements.

The project and operating limitations are given in the form of inequalities and, hence, cannot be included in the variational calculus problem. They will be used, however, in the following sense: If the results produced by the optimizing equations derived below violate any of their restraints, the extreme values (maxima or minima, whichever suitable) should be inserted instead, and the procedure repeated.

The thesis problem, henceforth called the "conditioned" problem,* is exactly equivalent to the following variational problem. Instead of considering the variation of the definite integral given by equation (3-1) with the auxiliary conditions (3-5.i) and the integral of (3-6), consider the modified integral

$$\bar{I} = \int_0^T H dt, \qquad \ldots \text{(3-18)}$$

where

$$H = \sum_{j=n+1}^{n+m} C_j + \sum_{i=1}^{n} \lambda_i P_{Hi} + \lambda_{n+1} \varnothing, \qquad \ldots \text{(3-19)}$$

with no auxiliary conditions. In equation (3-19) λ_i and λ_{n+1}, the Lagrangian multipliers, are to be considered constants relative to the process of variation.

The general solution of the problem of minimizing $\bar{I}$ is given by the Euler's equations:

$$\frac{\partial H}{\partial q_u} - \frac{d}{dt}\frac{\partial H}{\partial \dot{q}_u} = 0, \qquad u = 1,2 \qquad \ldots \text{(3-20)}$$

with q_u as variables and u the type of variable. For $u = 1$ one has $q_1 = P_{Tj}$, and for $u = 2$, $q_2 = S_i$. Thence, there are m variables of the first and n variables of the second type:

$$q_1^1 = P_{T,n+1}; \quad q_1^2 = P_{T,n+2}; \quad \ldots; \quad q_1^k = P_{T,n+k};$$

$$\ldots; \quad q_1^m = P_{T,n+m}; \qquad \ldots \text{(3-21)}$$

---

*Bolza's term.[71] The unconditioned problem is that with no restrictions.

$$q_2^1 = S_1;\; q_2^2 = S_2;\; \ldots;\; q_2^{\ell} = S_{\ell};\; \ldots;\; q_2^n = S_n;$$

$$\dot{q}_2^1 = \dot{S}_1;\; \dot{q}_2^2 = \dot{S}_2;\; \ldots;\; \dot{q}_2^{\ell} = \dot{S}_{\ell};\; \ldots;\; \dot{q}_2^n = \dot{S}_n. \qquad \ldots\ (3\text{-}22)$$

The above Euler's differential equations are the first necessary condition for an extremum, and therefore also for a minimum.

3.3.2 Thermal-Plant Equations

For $q_1^1 = P_{T,n+1}$ one solves, using equations (3-6) and (3-19)

$$\frac{\partial H}{\partial P_{T,n+1}} = \sum_{j=n+1}^{n+m} \frac{\partial C_j}{\partial P_{T,n+1}} + \sum_{i=1}^{n} \lambda_i \frac{\partial P_{Hi}}{\partial P_{T,n+1}} + $$

$$+ \lambda_{n+1}\left(\sum_{i=1}^{n} \frac{\partial P_{Hi}}{\partial P_{T,n+1}} + \sum_{j=n+1}^{n+m} \frac{\partial P_{Tj}}{\partial P_{T,n+1}} - \frac{\partial P_L}{\partial P_{T,n+1}} - \frac{\partial P_D}{\partial P_{T,n+1}}\right)$$

and $\ldots$ (3-23.1)

$$\frac{\partial H}{\partial \dot{P}_{T,n+1}} = \sum_{j=n+1}^{n+m} \frac{\partial C_j}{\partial \dot{P}_{T,n+1}} + \sum_{i=1}^{n} \lambda_i \frac{\partial P_{Hi}}{\partial \dot{P}_{T,n+1}} + $$

$$+ \lambda_{n+1}\left(\sum_{i=1}^{n} \frac{\partial P_{Hi}}{\partial \dot{P}_{T,n+1}} + \sum_{j=n+1}^{n+m} \frac{\partial P_{Tj}}{\partial \dot{P}_{T,n+1}} - \frac{\partial P_L}{\partial \dot{P}_{T,n+1}} - \frac{\partial P_D}{\partial \dot{P}_{T,n+1}}\right). \qquad \ldots\ (3\text{-}24.1)$$

In the above equations, all P_{Hi}'s, P_{Tj}'s and P_D are functions of time. P_L is a function of both P_{Hi} and P_{Tj}, and hence also a function of time.

Substituting equations (3-23.1) and (3-24.1) in the Euler equations (3-20) and combining the summation terms wherever possible, one obtains:

$$\sum_{j=n+1}^{n+m}\left(\frac{\partial C_j}{\partial P_{Tj}} + \lambda_{n+1}\right)\frac{\partial P_{Tj}}{\partial P_{T,n+1}} + \sum_{i=1}^{n}(\lambda_i+\lambda_{n+1})\frac{\partial P_{Hi}}{\partial P_{T,n+1}} +$$

$$- \lambda_{n+1}\left(\frac{\partial P_L}{\partial P_{T,n+1}} + \frac{\partial P_D}{\partial P_{T,n+1}}\right) - \frac{d}{dt}\left[\sum_{j=n+1}^{n+m}\left(\frac{\partial C_j}{\partial P_{Tj}} + \lambda_{n+1}\right) x\right.$$

$$\frac{\partial P_{Tj}}{\partial P_{T,n+1}} \cdot \frac{\partial P_{T,n+1}}{\partial \dot{P}_{T,n+1}} + \sum_{i=1}^{n}(\lambda_i+\lambda_{n+1})\frac{\partial P_{Hi}}{\partial P_{T,n+1}} \cdot \frac{\partial P_{T,n+1}}{\partial \dot{P}_{T,n+1}} +$$

$$\left. - \lambda_{n+1}\left(\frac{\partial P_L}{\partial P_{T,n+1}} + \frac{\partial P_D}{\partial P_{T,n+1}}\right)\frac{\partial P_{T,n+1}}{\partial \dot{P}_{T,n+1}}\right] = 0, \qquad \ldots (3\text{-}25.1)$$

which can be simplified into

$$f^1(t) - \frac{d}{dt}\left[f^1(t) \cdot \frac{\partial P_{T,n+1}}{\partial \dot{P}_{T,n+1}}\right] = 0, \qquad \ldots (3\text{-}26.1)$$

where

$$f^1(t) = \sum_{j=n+1}^{n+m}\left(\frac{\partial C_j}{\partial P_{Tj}} + \lambda_{n+1}\right)\frac{\partial P_{Tj}}{\partial P_{T,n+1}} + \sum_{i=1}^{n}(\lambda_i+\lambda_{n+1})\frac{\partial P_{Hi}}{\partial P_{T,n+1}} +$$

$$- \lambda_{n+1}\left(\frac{\partial P_L}{\partial P_{T,n+1}} + \frac{\partial P_D}{\partial P_{T,n+1}}\right) . \qquad \ldots (3\text{-}27.1)$$

In general, one acquires for the k^{th} thermal plant, where $k = 1, 2, \ldots, m$:

$$f^k(t) - \frac{d}{dt}\left[f^k(t) \cdot \frac{\partial P_{T,n+k}}{\partial \dot{P}_{T,n+k}}\right] = 0, \qquad \ldots (3\text{-}26.k)$$

which can be written as

$$f^k(t) - \left[\frac{df^k(t)}{dt}\left(\frac{\partial P_{T,n+k}}{\partial \dot{P}_{T,n+k}}\right) + f^k(t) \cdot \frac{d}{dt}\left(\frac{\partial P_{T,n+k}}{\partial \dot{P}_{T,n+k}}\right)\right] = 0,$$

$$\ldots (3\text{-}28.k)$$

where

$$f^k(t) = \sum_{j=n+1}^{n+m}\left(\frac{\partial C_j}{\partial P_{Tj}} + \lambda_{n+1}\right)\frac{\partial P_{Tj}}{\partial P_{T,n+k}} +$$

$$+ \sum_{i=1}^{n}(\lambda_i+\lambda_{n+1}) \frac{\partial P_{Hi}}{\partial P_{T,n+k}} - \lambda_{n+1}\left(\frac{\partial P_L}{\partial P_{T,n+k}} + \frac{\partial P_D}{\partial P_{T,n+k}}\right).$$

... (3-27.k)

3.3.3 Hydro-Plant Equations

As before, for $q_2^1 = S_1$ and $\dot{q}_2^1 = \dot{S}_1$, one solves the Euler's equations for this variable:

$$\sum_{j=n+1}^{n+m} \frac{\partial C_j}{\partial S_1} + \sum_{i=1}^{n} \lambda_i \frac{\partial P_{Hi}}{\partial S_1} + \sum_{i=1}^{n} \lambda_{n+1} \frac{\partial P_{Hi}}{\partial S_1} + \sum_{j=n+1}^{n+m} \lambda_{n+1} \frac{\partial P_{Tj}}{\partial S_1} +$$

$$- \lambda_{n+1}\left(\frac{\partial P_L}{\partial S_1} + \frac{\partial P_D}{\partial S_1}\right) - \frac{d}{dt}\left[\sum_{j=n+1}^{n+m} \frac{\partial C_j}{\partial \dot{S}_1} + \sum_{i=1}^{n} \lambda_i \frac{\partial P_{Hi}}{\partial \dot{S}_1} +\right.$$

$$+ \sum_{i=1}^{n} \lambda_{n+1} \frac{\partial P_{Hi}}{\partial \dot{S}_1} + \sum_{j=n+1}^{n+m} \lambda_{n+1} \frac{\partial P_{Tj}}{\partial \dot{S}_1} - \lambda_{n+1} \frac{\partial P_L}{\partial \dot{S}_1} +$$

$$\left. - \lambda_{n+1} \frac{\partial P_D}{\partial \dot{S}_1}\right] = 0. \qquad \text{... (3-29.1)}$$

Realizing the dependence of the cost on the thermal power output, and the hydro plant output on its storage value, one obtains the following relations:

$$\frac{\partial C_j}{\partial S_1} = \frac{\partial C_j}{\partial P_{Tj}} \cdot \frac{\partial P_{Tj}}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1}, \qquad \text{... (3-30.1)}$$

$$\frac{\partial P_{Hi}}{\partial S_1} = \frac{\partial P_{Hi}}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1}, \qquad \text{... (3-31.1)}$$

$$\frac{\partial P_{Tj}}{\partial S_1} = \frac{\partial P_{Tj}}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1} , \qquad \ldots (3\text{-}32.1)$$

$$\frac{\partial P_L}{\partial S_1} = \frac{\partial P_L}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1} , \qquad \ldots (3\text{-}33.1)$$

$$\frac{\partial P_D}{\partial S_1} = \frac{\partial P_D}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1} , \qquad \ldots (3\text{-}34.1)$$

$$\frac{\partial C_j}{\partial \dot{S}_1} = \frac{\partial C_j}{\partial P_{Tj}} \cdot \frac{\partial P_{Tj}}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1} \cdot \frac{\partial S1}{\partial \dot{S}_1} , \qquad \ldots (3\text{-}35.1)$$

$$\frac{\partial P_{Hi}}{\partial \dot{S}_1} = \frac{\partial P_{Hi}}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1} \cdot \frac{\partial S_1}{\partial \dot{S}_1} , \qquad \ldots (3\text{-}36.1)$$

$$\frac{\partial P_{Tj}}{\partial \dot{S}_1} = \frac{\partial P_{Tj}}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1} \cdot \frac{\partial S_1}{\partial \dot{S}_1} , \qquad \ldots (3\text{-}37.1)$$

$$\frac{\partial P_L}{\partial \dot{S}_1} = \frac{\partial P_L}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1} \cdot \frac{\partial S_1}{\partial \dot{S}_1} , \qquad \ldots (3\text{-}38.1)$$

$$\frac{\partial P_D}{\partial \dot{S}_1} = \frac{\partial P_D}{\partial P_{H1}} \cdot \frac{\partial P_{H1}}{\partial S_1} \cdot \frac{\partial S_1}{\partial \dot{S}1} . \qquad \ldots (3\text{-}39.1)$$

Using equations (3-30.1) to (3-39.1) in equations (3-29.1) one obtains similar to the previous case:

$$g^1(t) - \frac{d}{dt}\left[g^1(t) \cdot \frac{\partial S_1}{\partial \dot{S}_1}\right] = 0, \qquad \ldots (3\text{-}40.1)$$

where

$$g^1(t) = \left[\sum_{j=n+1}^{n+m}\left(\frac{\partial C_j}{\partial P_{Tj}} + \lambda_{n+1}\right)\frac{\partial P_{Tj}}{\partial P_{H1}} + \sum_{i=1}^{n} (\lambda_i+\lambda_{n+1}) \frac{\partial P_{Hi}}{\partial P_{H1}} + \right.$$

$$\left. - \lambda_{n+1}\left(\frac{\partial P_L}{\partial P_{H1}} + \frac{\partial P_D}{\partial P_{H1}}\right)\right]\frac{\partial P_{H1}}{\partial S_1} . \qquad \ldots (3\text{-}41.1)$$

In general, one obtains for the ℓ^{th} hydro plant where $\ell = 1,2,\ldots n$:

$$g^{\ell}(t) - \frac{d}{dt}\left[g^{\ell}(t)\cdot\frac{\partial s_{\ell}}{\partial \dot{s}_{\ell}}\right] = 0, \qquad \ldots (3\text{-}40.\ell)$$

or

$$g^{\ell}(t) - \left[\frac{dg^{\ell}(t)}{dt}\cdot\frac{\partial s_{\ell}}{\partial \dot{s}_{\ell}} + g^{\ell}(t)\cdot\frac{d}{dt}\left(\frac{\partial s_{\ell}}{\partial \dot{s}_{\ell}}\right)\right] = 0, \qquad \ldots (3\text{-}42.\ell)$$

where

$$g^{\ell}(t) = \left[\sum_{j=n+1}^{n+m}\left(\frac{\partial C_j}{\partial P_{Tj}} + \lambda_{n+1}\right)\frac{\partial P_{Tj}}{\partial P_{H\ell}} + \sum_{i=1}^{n}(\lambda_i+\lambda_{n+1})\frac{\partial P_{Hi}}{\partial P_{H\ell}} + - \lambda_{n+1}\left(\frac{\partial P_L}{\partial P_{H\ell}} + \frac{\partial P_D}{\partial P_H}\right)\right]\frac{\partial P_{H\ell}}{\partial s_{\ell}}\ . \qquad \ldots (3\text{-}41.\ell)$$

From equation $(3\text{-}41.\ell)$ one observes the similarity between $f^k(t)$ and $g^{\ell}(t)$ which contains an additional factor $\partial P_{H\ell}/\partial s_{\ell}$.

3.3.4 Substitution of Loss Factors and Cost Functions in the General Equations

It can generally be assumed[4-9] that the fuel cost at any thermal plant is a quadratic function of its power output, i.e., at thermal plant j

$$C_j = \frac{1}{2}\, a_j\, P_{Tj}^2 + b_j\, P_{Tj} + c_j, \qquad \ldots (3\text{-}43.j)$$

where a_j, b_j and c_j are constants. Hence

$$\frac{\partial C_j}{\partial P_{Tj}} = a_j\, P_{Tj} + b_j = \frac{dC_j}{dP_{Tj}}\ . \qquad \ldots (3\text{-}44.j)$$

There are several ways of expressing transmission losses in terms of plant generations for both single[76-90] and interconnected systems,[91-96] but only some of the more important methods and developments are outlined below.

(i) The B-Constant Method, initiated by George[76] in 1943, further developed by Eaton, Ward and Hale,[77] and Kirchmayer and Stagg,[78] is made possible by four basic assumptions.[78] The loss equation developed using these assumptions will be referred to as the simplified loss equation, contrary to the improved and more general formula by Early, Watson and Smith [84] (1955) which ignored the basic assumptions used previously. Early and Watson[85] developed in the same year a new method of determining constants for the General Transmission Loss Equation (GTLE).

(ii) The Voltage Phase-Angle Method developed by Brownlee[81] in 1955 was used by Cahn[83] to determine incremental and total loss formulas. The use of power transfer equations to derive coordination equations expressed as functions of voltage phase-angles resulted in the Miller equations.[89]

(iii) A general method of calculating incremental transmission losses and the GTLE was developed by Watson and Stadlin[88] in 1959.

(iv) A new and revolutionary approach to loss minimization in power systems was developed by Calvert and Sze[86] in 1958, and applied to a simple system by Calvert, Sze and Garnett[87] in 1959.

(v) Another new and fundamentally different method in determining loss formulas from digital load flow studies was developed by George[90] in a recent paper.

Of all the above methods, the most widely used B-Constant approach will be used in this thesis. The simplified loss equation is given by:[77,78]

$$P_L = \sum_{r=1}^{m+n} \sum_{s=1}^{m+n} B_{rs}\, P_r\, P_s \;, \qquad \ldots (3\text{-}45)$$

where B_{rs} is the derived loss formula coefficient (constant at one specific load level), and P_r and P_s are either thermal or hydro plant generations. Differentiating P_L with respect to these generations, one obtains

$$\frac{\partial P_L}{\partial P_{T,n+k}} = 2 \sum_{s=1}^{m+n} B_{n+k,s}\, P_s \;, \qquad \ldots (3\text{-}46.k)$$

$$\frac{\partial P_L}{\partial P_{H\ell}} = 2 \sum_{s=1}^{m+n} B_{\ell s}\, P_s \,. \qquad \ldots (3\text{-}47.\ell)$$

Substitution of equations (3-44.j), (3-46.k) and (3-47.ℓ) in equations (3-27.k) and (3-41.ℓ) yields for the k^{th} thermal plant:

$$f^k(t) = \sum_{j=n+1}^{n=m} (a_j P_{Tj} + b_j + \lambda_{n+1}) \frac{\partial P_{Tj}}{\partial P_{T,n+k}} + \\ + \sum_{i=1}^{n} (\lambda_i + \lambda_{n+1}) \frac{\partial P_{Hi}}{\partial P_{T,n+k}} - \lambda_{n=1} \left(2 \sum_{s=1}^{m+n} B_{n+k,s} P_s + \right. \\ \left. + \frac{\partial P_D}{\partial P_{T,n+k}} \right) , \qquad \ldots (3\text{-}48.k)$$

and for the ℓ^{th} hydro plant:

$$g^{\ell}(t) = \left[\sum_{j=n+1}^{m+n} (a_j P_{Tj} + b_j + \lambda_{n+1}) \frac{\partial P_{Tj}}{\partial P_{H\ell}} + \sum_{i=1}^{n} (\lambda_i + \lambda_{n+1}) \frac{\partial P_{Hi}}{\partial P_{H\ell}} + \right.$$

$$\left. - \lambda_{n+1} \left(2 \sum_{s=1}^{m+n} B_{\ell s} P_s + \frac{\partial P_D}{\partial P_{H\ell}} \right) \right] \frac{\partial P_H}{\partial S_\ell} \quad . \quad \ldots (3\text{-}49.\ell)$$

3.3.5 Two Simplified Cases

Two types of simplified cases will be considered in this section: the case where only one hydro and one thermal plant exists and the case where scheduling of generations is performed at one particular time.

(i) Type A: The Two-Plant Problem

The optimizing equations derived in the preceding sections become much simpler when the load demand is satisfied by one hydro plant ($n = 1$) and one thermal plant ($m = 1$). Hence, using equations (3-26.1) and (3-27.1) one solves for the thermal plant:

$$f^1(t) - \frac{d}{dt} \left[f^1(t) \cdot \frac{\partial P_{T2}}{\partial \dot{P}_{T2}} \right] = 0, \qquad \ldots (3\text{-}50)$$

with $f^1(t)$ taking the special form of

$$f^1(t) = \left(\frac{\partial C_2}{\partial P_{T2}} + \lambda_2 \right) + (\lambda_1 + \lambda_2) \frac{\partial P_{H1}}{\partial P_{T2}} +$$

$$- \lambda_2 \left(\frac{\partial P_L}{\partial P_{T2}} + \frac{\partial P_D}{\partial P_{T2}} \right), \qquad \ldots (3\text{-}51)$$

and for the hydro plant, from equations (3-40.1) and (3-41.1):

$$g^1(t) - \frac{d}{dt} \left[g^1(t) \cdot \frac{\partial S_1}{\partial \dot{S}_1} \right] = 0, \qquad \ldots (3\text{-}52)$$

where

$$g^1(t) = \left[\left(\frac{\partial C_2}{\partial P_{T2}} + \lambda_2\right) \frac{\partial P_{T2}}{\partial P_{H1}} + (\lambda_1+\lambda_2) + \right.$$

$$\left. - \lambda_2\left(\frac{\partial P_L}{\partial P_{H1}} + \frac{\partial P_D}{\partial P_{H1}}\right)\right] \frac{\partial P_{H1}}{\partial S_1} . \qquad \ldots (3\text{-}53)$$

The two equations (3-50) and (3-52) can be combined into one differential equation as in Appendix B:

$$\frac{\partial}{\partial P_{T2}} \left(\frac{\partial H}{\partial \dot{S}_1}\right) - \frac{\partial}{\partial S_1} \left(\frac{\partial H}{\partial \dot{P}_{T2}}\right) + H_1\left(\frac{dP_{T2}}{dt} \cdot \frac{d^2S_1}{dt^2} - \frac{d^2P_{T2}}{dt^2} \cdot \frac{dS_1}{dt}\right) = 0,$$

$$\ldots (3\text{-}54)$$

where

$$H_1 = \frac{1}{(\dot{S}_1)^2} \frac{\partial}{\partial \dot{P}_{T2}} \left(\frac{\partial H}{\partial \dot{P}_{T2}}\right) \qquad \ldots (3\text{-}55.a)$$

$$= \frac{1}{(\dot{P}_{T2})^2} \frac{\partial}{\partial \dot{S}_1} \left(\frac{\partial H}{\partial \dot{S}_1}\right) \qquad \ldots (3\text{-}55.b)$$

$$= \frac{-1}{(\dot{S}_1)(\dot{P}_{T2})} \frac{\partial}{\partial \dot{S}_1} \left(\frac{\partial H}{\partial \dot{P}_{T2}}\right), \qquad \ldots (3\text{-}55.c)$$

and where H can be derived from the general form of equation (3-19):

$$H = C_2 + \lambda_1 P_{H1} + \lambda_2 (P_{H1} + P_{T2} - P_L - P_D). \qquad \ldots (3\text{-}56)$$

It can be seen that the differential equation (3-54) is of the second order. Its general solution contains, therefore, two arbitrary constants of integration α and β, two isoperimetric constants λ_1 and λ_2. Hence

$$P_{T2} = P_{T2} (\alpha, \beta, \lambda_1, \lambda_2, t), \qquad \ldots (3\text{-}57)$$

$$S_1 = S_1\ (\alpha,\ \beta,\ \lambda_1,\ \lambda_2,\ t). \qquad \ldots (3\text{-}58)$$

From equation (3-58), $\dot{S}_1 = \frac{dS_1}{dt}$ can be found which together with the flow $F_1(t)$ will determine

$$P_{H1} = P_{H1}\ (\alpha,\ \beta,\ \lambda_1,\ \lambda_2,\ F_1,\ t). \qquad \ldots (3\text{-}59)$$

From the known B_1 in

$$\int_0^T P_{H1}\ (F_1,\ S_1,\ \dot{S}_1,\ t)\ dt = B_1, \qquad \ldots (3\text{-}60)$$

from the auxiliary condition

$$\emptyset = P_{H1} + P_{T2} - P_L - P_D = 0, \qquad \ldots (3\text{-}61)$$

and the initial conditions

$$\left.\begin{aligned} S_1\ (t = 0) &= S_{10}, \\ P_{T2}\ (t = 0) &= P_{T20}, \end{aligned}\right. \qquad \ldots (3\text{-}62)$$

α, β, λ_1 and λ_2 can be found.

(ii) Type B: The Equivalence of this Method with all Previously Known Methods

In practice, since the curve of load demand does not follow a pattern which is presentable in the form of a simple, continuous and differentiable function of time:

$$P_D = P_D(t), \qquad \ldots (3\text{-}63)$$

the problem can be revised as follows:

Instead of determining what $S_1(t), S_2(t), \ldots, S_\ell(t), \ldots, S_n(t)$ and $P_{T,n+1}(t), \ldots, P_{T,n+k}(t), \ldots, P_{T,n+m}(t)$ are, when $F_1(t), \ldots, F_\ell(t), \ldots, F_n(t)$ are given to meet $P_D(t)$

and $P_L(t)$, one solves the problem of what contribution each plant should make in order to meet the load requirements at any one time t_x where $x = 0,1,\ldots,e$ ($t_0 = 0$; $t_e = T$). In other words, at load level $P_D(t_x)$ what the values of

$$P_{T,n+1}(t_x),\ \ldots,\quad P_{T,n+k}(t_x),\ \ldots,\quad P_{T,n+m}(t_x),\ \ldots \tag{3-64}$$

$$P_{H1}(t_x),\ \ldots\ldots,\quad P_{H\ell}(t_x),\ \ldots\ldots,\quad P_{Hn}(t_x), \tag{3-65}$$

should be, and hence

$$S_1(t_x),\ \ldots\ldots,\quad S_\ell(t_x),\ \ldots\ldots,\quad S_n(t_x), \tag{3-66}$$

combined with a given set of flows

$$F_1(t_x),\ \ldots\ldots,\quad F_\ell(t_x),\ \ldots\ldots,\quad F_n(t_x), \tag{3-67}$$

such that

$$\sum_{i=1}^{n} P_{Hi}(t_x) + \sum_{j=n+1}^{n+m} P_{Tj}(t_x) = P_D(t_x) + P_L(t_x). \tag{3-68}$$

The elements of sets (3-64) to (3-67) above are no longer functions of time, such that expressions as $dP_{H\ell}(t_x)/dt, dP_{T,n+k}(t_x)/dt$ and $dP_D(t_x)/dt$ do not exist. However,

$$\dot{S}_\ell(t_x) = \left.\frac{dS_\ell}{dt}\right]_{t=t_x} \tag{3-69}$$

does exist, since it is one of the three basic variables $P_{T,n+k}$, S_ℓ and $\dot{S}_\ell$ (the t_x's are henceforth omitted for convenience), remembering that each one of them are different independent variables. In addition, basic variables of any one plant are characteristic of that plant only. The scheduling equations can therefore be simplified using the following relations.

For the thermal plants:

$$\frac{\partial C_j}{\partial P_{T,n+k}} = 0 \qquad \text{for } j \neq n+k$$
$$= \frac{\partial C_j}{\partial P_{Tj}} = \frac{\partial C_{n+k}}{\partial P_{T,n+k}}, \text{ for } j = n+k \qquad \ldots (3\text{-}70)$$

$$\frac{\partial P_{Tj}}{\partial P_{T,n+k}} = 0 \qquad \text{for } j \neq n+k$$
$$= 1, \qquad \text{for } j = n+k \qquad \ldots (3\text{-}71)$$

$$\frac{\partial P_{Hi}}{\partial P_{T,n+k}} = 0 \qquad \text{for all } k = 1,\ldots m \qquad \ldots (3\text{-}72)$$

(since P_{Hi} is a function of F_i, S_i and $\dot{S}_i$ only),

$$\frac{\partial P_D}{\partial P_{T,n+k}} = 0, \qquad \text{for all } k = 1,\ldots m \qquad \ldots (3\text{-}73)$$

$$\frac{\partial P_{Tj}}{\partial \dot{P}_{T,n+k}} = 0, \qquad \text{for all } k = 1,\ldots m \qquad \ldots (3\text{-}74)$$

$$\frac{\partial P_{Hi}}{\partial \dot{P}_{T,n+k}} = 0, \qquad \text{for all } k = 1,\ldots m \qquad \ldots (3\text{-}75)$$

$$\frac{\partial P_L}{\partial \dot{P}_{T,n+k}} = 0 \qquad \text{for all } k = 1,\ldots m \qquad \ldots (3\text{-}76)$$

(since P_L is a function of P_{Hi} and P_{Tj} only, but not of their derivatives),

$$\frac{\partial P_D}{\partial \dot{P}_{T,n+k}} = 0. \qquad \text{for all } k = 1,\ldots m \qquad \ldots (3\text{-}77)$$

Substituting the above equations in equations (3-26.k) and (3-27.k) one obtains for the k^{th} thermal plant, or thermal plant j:

$$\frac{\partial C_j}{\partial P_{Tj}} + \lambda_{n+1}\left(1 - \frac{\partial P_L}{\partial P_{Tj}}\right) = 0. \qquad j = n+1,\ldots,n+m \qquad \ldots (3\text{-}78.j)$$

Similarly, for the hydro plants:

$$\frac{\partial P_{Tj}}{\partial P_{H\ell}} = 0, \qquad \text{for all } \ell = 1,\ldots n \qquad \ldots (3\text{-}79)$$

$$\frac{\partial S_i}{\partial S_\ell} = 0 \qquad \text{for } i \neq \ell$$
$$= 1, \qquad \text{for } i = \ell \qquad \ldots (3\text{-}80)$$

$$\frac{\partial P_D}{\partial P_{H\ell}} = 0, \qquad \text{for all } \ell = 1,\ldots n \qquad \ldots (3\text{-}81)$$

$$\frac{\partial P_{Tj}}{\partial P_{H\ell}} = 0, \qquad \text{for all } \ell = 1,\ldots n \qquad \ldots (3\text{-}82)$$

$$\frac{\partial \dot{S}_i}{\partial \dot{S}_\ell} = 0 \qquad \text{for } i \neq \ell$$
$$= 1. \qquad \text{for } i = \ell \qquad \ldots (3\text{-}83)$$

Thus for hydro plant i:

$$(\lambda_i+\lambda_{n+1})\frac{\partial P_{Hi}}{\partial S_i} - \lambda_{n+1}\frac{\partial P_L}{\partial P_{Hi}}\frac{\partial P_{Hi}}{\partial S_i} +$$
$$- \frac{d}{dt}\left[(\lambda_i+\lambda_{n+1})\frac{\partial P_{Hi}}{\partial \dot{S}_i} - \lambda_{n+1}\frac{\partial P_L}{\partial P_{Hi}}\frac{\partial P_{Hi}}{\partial \dot{S}_i}\right] = 0,$$
$$i = 1,\ldots n \qquad \ldots (3\text{-}84.i)$$

which by combining the lost terms can be written in a simpler form:

$$\lambda_i \left(\frac{\partial P_{Hi}}{\partial S_i} - \frac{d}{dt} \frac{\partial P_{Hi}}{\partial \dot{S}_i} \right) + \lambda_{n+1} \left(1 - \frac{\partial P_L}{\partial P_{Hi}} \right) \frac{\partial P_{Hi}}{\partial S_i} +$$

$$- \frac{d}{dt} \left[\lambda_{n+1} \left(1 - \frac{\partial P_L}{\partial P_{Hi}} \right) \frac{\partial P_{Hi}}{\partial \dot{S}_i} \right] = 0. \quad i = 1, \ldots n \qquad (3\text{-}85.i)$$

CHAPTER IV

OTHER NECESSARY AND SUFFICIENT CONDITIONS FOR A MINIMUM OF THE ISOPERIMETRIC PROBLEM

4.1 Introduction

In addition to the first necessary condition of minimizing the conditioned integral $\bar{I}$ given by equation (3-18) of the previous chapter, there are three other necessary and three sufficient conditions to be satisfied. To avoid complexity of symbols and notations involved in the general problem, and to familiarize the reader with the concept and use of such conditions, the simplified problem of type A, Chapter III, will be considered. The extension to the general case of a system of m thermal and n hydro plants is obvious.[71,72] The notations used in the following sections are identical with the ones used in Appendix B, where the two variables x and y are now replaced by P_{T2} and S_1 respectively.

4.2 The Second Necessary Condition

The second necessary condition for a minimum of $\bar{I}$ is given by the analogue of the Legendre's condition:[71,72]

$$H_1 \geqq 0 \qquad \ldots (4\text{-}1)$$

along the extremal C_0, expressed in the form of equations (3-57) and (3-58). H_1 is given by either equation (3-55.a) or (3-55.b) or (3-55.c) which, using the original Euler equations for the thermal plant (or hydro plant), can be written as

$$H_1 = \frac{1}{(\dot{S}_1)^2} \frac{\partial}{\partial \dot{P}_{T2}} \left[f^1(t) \cdot \frac{\partial P_{T2}}{\partial \dot{P}_{T2}} \right] \qquad \ldots (4\text{-}2.a)$$

$$= \frac{1}{(\dot{P}_{T2})^2} \frac{\partial}{\partial \dot{S}_1} \left[g^1(t) \cdot \frac{\partial S_1}{\partial \dot{S}_1} \right] \quad \ldots (4\text{-}2.b)$$

$$= - \frac{1}{(\dot{P}_{T2})(\dot{S}_1)} \frac{\partial}{\partial \dot{S}_1} \left[f^1(t) \cdot \frac{\partial P_{T2}}{\partial \dot{P}_{T2}} \right], \quad \ldots (4\text{-}2.c)$$

where $f^1(t)$ and $g^1(t)$ are given by equations (3-51) and (3-53) respectively.

4.3 The Third Necessary Condition

The third necessary condition is given by Weierstrass's analogue of Jacobi's condition

$$D(t,t_0) = \begin{vmatrix} \omega_1(t_0) & \omega_2(t_0) & \omega_3(t_0) & \omega_4(t_0) \\ \omega_1(t) & \omega_2(t) & \omega_3(t) & \omega_4(t) \\ \int_{t_0}^{t} U\omega_1 dt & \int_{t_0}^{t} U\omega_2 dt & \int_{t_0}^{t} U\omega_3 dt & \int_{t_0}^{t} U\omega_4 dt \\ \int_{t_0}^{t} V\omega_1 dt & \int_{t_0}^{t} V\omega_2 dt & \int_{t_0}^{t} V\omega_3 dt & \int_{t_0}^{t} V\omega_4 dt \end{vmatrix} \neq 0, \quad \ldots (4\text{-}3)$$

where

$$\begin{aligned} \omega_1(t) &= \frac{\partial S_1}{\partial t} \cdot \frac{\partial P_{T2}}{\partial \alpha} - \frac{\partial P_{T2}}{\partial t} \cdot \frac{\partial S_1}{\partial \alpha}, \\ \omega_2(t) &= \frac{\partial S_1}{\partial t} \cdot \frac{\partial P_{T2}}{\partial \beta} - \frac{\partial P_{T2}}{\partial t} \cdot \frac{\partial S_1}{\partial \beta}, \\ \omega_3(t) &= \frac{\partial S_1}{\partial t} \cdot \frac{\partial P_{T2}}{\partial \lambda_1} - \frac{\partial P_{T2}}{\partial t} \cdot \frac{\partial S_1}{\partial \lambda_1}, \\ \omega_4(t) &= \frac{\partial S_1}{\partial t} \cdot \frac{\partial P_{T2}}{\partial \lambda_2} - \frac{\partial P_{T2}}{\partial t} \cdot \frac{\partial S_1}{\partial \lambda_2}, \end{aligned} \quad \ldots (4\text{-}4)$$

and

$$U = \frac{\partial}{\partial P_{T2}}\left(\frac{\partial P_{H1}}{\partial \mathring{S}_1}\right) - \frac{\partial}{\partial \mathring{P}_{T2}}\left(\frac{\partial P_{H1}}{\partial S_1}\right) + G_1\left[\frac{dP_{T2}}{dt} \cdot \frac{d^2S_1}{dt} - \frac{d^2P_{T2}}{dt^2} \cdot \frac{dS_1}{dt}\right], \qquad \ldots (4\text{-}5)$$

$$\text{with } G_1 = \frac{1}{(\mathring{S}_1)^2}\ \frac{\partial}{\partial \mathring{P}_{T2}}\left(\frac{\partial P_{H1}}{\partial \mathring{P}_{T2}}\right) \qquad \ldots (4\text{-}6.a)$$

$$= \frac{1}{(\mathring{P}_{T2})^2}\ \frac{\partial}{\partial \mathring{S}_1}\left(\frac{\partial P_{H1}}{\partial \mathring{S}_1}\right) \qquad \ldots (4\text{-}6.b)$$

$$= -\frac{1}{(\mathring{P}_{T2})(\mathring{S}_1)}\ \frac{\partial}{\partial \mathring{P}_{T2}}\left(\frac{\partial P_{H1}}{\partial \mathring{S}_1}\right). \qquad \ldots (4\text{-}6.c)$$

Similarly:

$$V = \frac{\partial}{\partial \mathring{S}_1}\left(\frac{\partial \phi}{\partial P_{T2}}\right) - \frac{\partial}{\partial \mathring{P}_{T2}}\left(\frac{\partial \phi}{\partial S_1}\right) + \phi_1\left[\frac{dP_{T2}}{dt} \cdot \frac{d^2S_1}{dt^2} - \frac{d^2P_{T2}}{dt^2} \cdot \frac{dS_1}{dt}\right], \qquad \ldots (4\text{-}7)$$

$$\text{with } \phi_1 = \frac{1}{(\mathring{S}_1)^2}\ \frac{\partial}{\partial \mathring{P}_{T2}}\left(\frac{\partial \phi}{\partial \mathring{P}_{T2}}\right) \qquad \ldots (4\text{-}8.a)$$

$$= \frac{1}{(\mathring{P}_{T2})^2}\ \frac{\partial}{\partial \mathring{S}_1}\left(\frac{\partial \phi}{\partial \mathring{S}_1}\right) \qquad \ldots (4\text{-}8.b)$$

$$= -\frac{1}{(\mathring{P}_{T2})(\mathring{S}_1)}\ \frac{\partial}{\partial \mathring{P}_{T2}}\left(\frac{\partial \phi}{\partial \mathring{S}_1}\right), \qquad \ldots (4\text{-}8.c)$$

where ϕ is given by equation (3-61). Computing the derivatives of ϕ with respect to P_{T2}, S_1, $\mathring{P}_{T2}$ and $\mathring{S}_1$, one obtains

$$V = \frac{\partial}{\partial \mathring{S}_1}\left(\frac{\partial P_{H1}}{\partial P_{T2}} + 1 - \frac{\partial P_L}{\partial P_{T2}} - \frac{\partial P_D}{\partial P_{T2}}\right) - \frac{\partial}{\partial \mathring{P}_{T2}}\left[\left(1 + \frac{\partial P_{T2}}{\partial P_{H1}} + \right.\right.$$
$$\left.\left. - \frac{\partial P_L}{\partial P_{H1}} - \frac{\partial P_D}{\partial P_{H1}}\right)\frac{\partial P_{H1}}{\partial S_1}\right] + \phi_1\left[\frac{dP_{T2}}{dt} \cdot \frac{d^2S_1}{dt} - \frac{d^2P_{T2}}{dt^2} \cdot \frac{dS_1}{dt}\right], \qquad \ldots (4\text{-}9)$$

where

$$\phi_1 = \frac{1}{(\mathring{S}_1)^2}\ \frac{\partial}{\partial \mathring{P}_{T2}}\left[\left(\frac{\partial P_{H1}}{\partial P_{T2}} + 1 - \frac{\partial P_L}{\partial P_{T2}} - \frac{\partial P_D}{\partial P_{T2}}\right)\frac{\partial P_{T2}}{\partial \mathring{P}_{T2}}\right] \qquad \ldots (4\text{-}10.a)$$

$$= \frac{1}{(\dot{P}_{T2})^2} \frac{\partial}{\partial \dot{S}_1} \left[\left(1 + \frac{\partial P_{T2}}{\partial P_{H1}} - \frac{\partial P_L}{\partial P_{H1}} - \frac{\partial P_D}{\partial P_{H1}} \right) \frac{\partial P_{H1}}{\partial \dot{S}_1} \right] \qquad \ldots \text{(4-10.b)}$$

$$= - \frac{1}{(\dot{P}_{T2})(\dot{S}_1)} \frac{\partial}{\partial \dot{P}_{T2}} \left[\left(1 + \frac{\partial P_{T2}}{\partial P_{H1}} - \frac{\partial P_L}{\partial P_{H1}} - \frac{\partial P_D}{\partial P_{H1}} \right) \frac{\partial P_{H1}}{\partial \dot{S}_1} \right]. \qquad \text{(4-10.c)}$$

The third necessary condition can also be given as

$$t_e = T \leqq t_0' , \qquad \ldots \text{(4-11)}$$

where t_0' is the conjugate of the point t_0, the beginning of the optimizing period, and t_e the end of this period. The conjugate point t_0' is the root next greater than t_0 of the equation

$$D\ (t,\ t_0) = 0. \qquad \ldots \text{(4-12)}$$

4.4 The Fourth Necessary Condition

The fourth necessary condition for a minimum is given by Weierstrass* as

$$E\ (\ P_{T2},\ S_1,\ \dot{P}_{T2},\ \dot{S}_1,\ \tilde{\dot{P}}_{T2},\ \tilde{\dot{S}}_1;\ \lambda_1,\ \lambda_2\ ) \geqq 0, \qquad \ldots \text{(4-13)}$$

which must be fulfilled along the extremal C_0 for every direction $\tilde{\dot{P}}_{T2}$ and $\tilde{\dot{S}}_1$. The E-function is given by

$$E = H\ (\ P_{T2},\ S_1,\ \tilde{\dot{P}}_{T2},\ \tilde{\dot{S}}_1;\ \lambda_1,\ \lambda_2\ ) - \left(\tilde{\dot{P}}_{T2} \frac{\partial H}{\partial \dot{P}_{T2}} + \tilde{\dot{S}}_1 \frac{\partial H}{\partial \dot{S}_1} \right), \qquad \ldots \text{(4-14)}$$

with

$$H\ (P_{T2}, S_1, \tilde{\dot{P}}_{T2}, \tilde{\dot{S}}_1; \lambda_1, \lambda_2) = C_2\ (P_{T2}, t) + \lambda_1\ P_{H1}\ (S_1, \tilde{\dot{S}}_1, t) + {} + \lambda_2 \left(P_{H1}\ (S_1,\ \tilde{\dot{S}}_1, t) + P_{T2}(t) - P_L(t) - P_D(t) \right), \qquad \text{(4-15)}$$

---

*Bolza[71,72] calls it the Weierstrass' condition.

and

$$\frac{\partial H}{\partial \dot{P}_{T2}} = f^1(t) \cdot \frac{\partial P_{T2}}{\partial \dot{P}_{T2}} , \qquad \ldots (4\text{-}16)$$

$$\frac{\partial H}{\partial \dot{S}_1} = g^1(t) \cdot \frac{\partial S_1}{\partial \dot{S}_1} , \qquad \ldots (4\text{-}17)$$

with $f^1(t)$ and $g^1(t)$ given by equations (3-51) and (3-53) respectively.

4.5 The Three Sufficient Conditions

The three sufficient conditions for a minimum of integral $\bar{I}$ are given by the second, third and fourth necessary conditions with the equality sign omitted, i.e.,

$$H_1 > 0, \qquad \ldots (4\text{-}18)$$

$$t_e < t_0', \qquad \ldots (4\text{-}19)$$

$$E\ (P_{T2}, S_1, \dot{P}_{T2}, \dot{S}_1, \tilde{\dot{P}}_{T2}, \tilde{\dot{S}}_1;\ \lambda_1, \lambda_2) > 0, \qquad \ldots (4\text{-}20)$$

where H_1 and the E-function are given by equations (4-2.a,b,c) and (4-14) respectively, and t_0' defined as previously.

CHAPTER V

EXAMPLE OF A LOSSLESS TWO-PLANT PROBLEM

5.1 Statement of the Problem

As an illustration of how the four necessary and the three sufficient conditions can be applied, the two-plant problem, defined in chapter III as the simplified Type A case, will be considered. In this illustration it is further assumed that the transmission losses can be neglected. The problem is essentially the same as that of Glimn and Kirchmayer (abbreviated: GK)[63] who assumed the following:

(1) The incremental cost of thermal power is constant, or the cost function linear:

$$C_2 = b_2 P_{T2} + c_2 \qquad \ldots (5\text{-}1)$$

$$\partial C_2/\partial P_{T2} = b_2 = dC_2/dP_{T2}. \qquad \ldots (5\text{-}2)$$

(2) The reservoir is a vertical-sided tank, and the tail-water elevation independent of flow, such that (if head loss and spillage are neglected) the net head can be expressed as:

$$h = h_0 + \int_0^t \frac{F-Q}{A}\, dt, \qquad \ldots (5\text{-}3)$$

where F, the inflow to the reservoir is constant.

(3) The load is constant, e.g., numerically equal to the constants α and λ_2:

$$P_D = \alpha + \lambda_2. \qquad \ldots (5\text{-}4)$$

(4) Transmission losses can be neglected:

$$P_L = 0. \qquad \ldots (5\text{-}5)$$

5.2 Solutions of the Problem: The First Necessary Condition

GK's solutions[63] are given in the form of a linear relationship between hydro plant output and time (see Figure 8—GK's paper) and a linear function of head, and therefore storage, with time, i.e., for the storage

$$S_1 = \lambda_1 - \beta t, \qquad \ldots \quad (5\text{-}6)$$

with constants λ_1 and β consistently chosen with the general solution given by equation (3-58). Assuming arbitrarily, that the functional relationship between the hydro plant output and the storage and the change of storage is given by

$$P_{H1} = - S_1 - \dot{S}_1, \qquad \ldots \quad (5\text{-}7)$$

one obtains by using equation (5-6) that

$$P_{H1} = - \lambda_1 + \beta + \beta t, \qquad \ldots \quad (5\text{-}8)$$

which is consistent with both equations (3-59) and Figure 8 of GK's paper.[63]

Due to assumption (4) one acquires for the thermal plant

$$P_{T2} = P_D - P_{H1}, \qquad \ldots \quad (5\text{-}9)$$

which combined with equations (5-4) and (5-8) yields

$$P_{T2} = (\alpha + \lambda_2 + \lambda_1) - \beta\ (t + 1), \qquad \ldots \quad (5\text{-}10)$$

which satisfies equation (3-57).

Since the plant's output and the storage value are all positive physical quantities, the following inequalities hold:

$$\lambda_1 \geqq 0,$$

$$\beta \geqq 0,$$

$$\beta \geqq \lambda_1, \qquad \ldots (5\text{-}11)$$

$$\alpha+\lambda_2+\lambda_1-\beta \geqq 0.$$

The physical meaning of equations (5-8) and (5-10) are that due to the high cost of thermal power, it is desirable to hold the hydro generation low at the beginning of the period and the thermal generation high at the beginning, but low at the end of the period.*

The next step is to prove that equations (5-6) and (5-10) satisfy the Euler's equations. To do this the following derivatives are first computed:

From equations (5-4) and (5-5) one obtains

$$\frac{\partial P_L}{\partial P_{T2}} = \frac{\partial P_L}{\partial P_{H1}} = 0, \qquad \ldots (5\text{-}12)$$

$$\frac{\partial P_D}{\partial P_{T2}} = \frac{\partial P_D}{\partial P_{H1}} = 0, \qquad \ldots (5\text{-}13)$$

from equation (5-9):

$$\frac{\partial P_{H1}}{\partial P_{T2}} = -1, \qquad \frac{\partial P_{T2}}{\partial P_{H1}} = -1, \qquad \ldots (5\text{-}14)$$

from equation (5-10):

$$\mathring{P}_{T2} = \frac{dP_{T2}}{dt} = -\beta, \qquad \ldots (5\text{-}15)$$

$$\frac{\partial P_{T2}}{\partial \mathring{P}_{T2}} = t + 1, \qquad \ldots (5\text{-}16)$$

and using equations (5-6) and (5-7):

$$\mathring{S}_1 = \frac{dS_1}{dt} = -\beta, \qquad \ldots (5\text{-}17)$$

---

*If t is high, P_{T2} is high, and hence C_2 is high.

$$\frac{\partial S_1}{\partial \dot{S}_1} = t, \qquad \ldots (5\text{-}18)$$

$$\frac{\partial P_{H1}}{\partial \dot{S}_1} = -(t+1), \qquad \ldots (5\text{-}19)$$

and

$$\frac{\partial P_{H1}}{\partial S_1} = -\frac{t+1}{t}, \qquad \ldots (5\text{-}20)$$

since

$$P_{H1} = -(\lambda_1 - \beta t) + \frac{\lambda_1 - S_1}{t}$$

$$= -S_1 \cdot \frac{t+1}{t} + \frac{\lambda_1}{t}. \qquad \ldots (5\text{-}21)$$

One may now solve $f^1(t)$ and $g^1(t)$ given by equations (3-51) and (3-53):

$$f^1(t) = (b_2+\lambda_2) - (\lambda_1+\lambda_2) = b_2 - \lambda_1, \qquad \ldots (5\text{-}22)$$

$$g^1(t) = -\left[-(b_2+\lambda_2)+(\lambda_1+\lambda_2)\right]\frac{t+1}{t} = (b_2-\lambda_1)\frac{t+1}{t}. \qquad \ldots (5\text{-}23)$$

Substitution of these in equations (3-50) and (3-52) yields

$$(b_2-\lambda_1) - \frac{d}{dt}\left[(b_2-\lambda_1)\cdot(t+1)\right] =$$

$$(b_2-\lambda_1) - (b_2-\lambda_1)\cdot 1 = 0, \qquad \ldots (5\text{-}24)$$

and

$$(b_2-\lambda_1)\cdot\frac{t+1}{t} - \frac{d}{dt}\left[(b_2-\lambda_1)\cdot\frac{t+1}{t}\cdot t\right] =$$

$$(b_2-\lambda_1)\cdot\frac{t+1}{t} - (b_2-\lambda_1) =$$

$$(b_2-\lambda_1)\frac{1}{t}. \qquad \ldots (5\text{-}25)$$

Equation (5-24) always satisfies Euler's equations while equation (5-25) will be identically zero if and only if, for $t \neq 0$,

$$b_2 = \lambda_1. \qquad \ldots (5\text{-}26)$$

Substitution of this value in equation (5-24) is a trivial solution. As mentioned earlier, the numerical values are chosen arbitrarily such that condition (5-26) can be satisfied if λ_1, the initial storage value, is chosen appropriately.

5.3 Tests for Necessary and Sufficient Conditions

The results of the previous part will now be tested against the three necessary conditions along with the three sufficient conditions.

(1) The Second Necessary Condition

This condition is given by equation (4-1) together with either of the three equations (4-2.a), (4-2.b) or (4-2c). Using the simplest equation (4-2.a) and making use of the relations (5-15), (5-16), (5-17) and (5-22) one obtains

$$H_1 = \frac{1}{(-\beta)^2} \frac{\partial}{\partial(-\beta)} \left[(b_2 - \lambda_1) \quad (t+1) \right] . \qquad \ldots (5\text{-}27)$$

Equation (5-27) is identically zero due to condition (5-26). Hence, the second necessary condition is satisfied.

(2) The Third Necessary Condition

Using equations (4-4), (5-6) and (5-10) the following functions of time can be calculated:

$$\begin{aligned} \omega_1(t) &= (-\beta)\cdot 1 - (-\beta) \cdot 0 = -\beta, \\ \omega_2(t) &= (-\beta)\cdot -(t+1) - (-\beta)(-t) = \beta, \\ \omega_3(t) &= (-\beta) \cdot 1 - (-\beta)\cdot 1 = 0, \\ \omega_4(t) &= (-\beta) \cdot 1 - (-\beta) \cdot 0 = -\beta. \end{aligned} \qquad \ldots (5\text{-}28)$$

Since $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$ and $\omega_4(t)$ are constant, the first and second rows of the determinant $D(t,t_0)$ are identical and therefore

$$D(t,t_0) = 0, \qquad \ldots (5\text{-}29)$$

and, consequently, the third necessary condition is not satisfied.

(3) <u>The Fourth Necessary Condition</u>

Substituting equations (5-1), (5-2), (5-7), (5-16), (5-18), (5-22) and (5-23) in the Weierstrass's E-function (4-14) one obtains

$$E = b_2P_{T2}+c_2+\lambda_1(-S_1-\tilde{\dot{S}}_1)+\lambda_2\left[(-S_1-\tilde{\dot{S}}_1)+P_{T2}-P_D\right] +$$
$$-\left[\tilde{\dot{P}}_{T2}\ (b_2-\lambda_1)(t+1)\ +\tilde{\dot{S}}_1\ (b_2-\lambda_1)\cdot\frac{t+1}{t}\cdot t\right]. \qquad \ldots (5\text{-}30)$$

In this equation

$$\tilde{\dot{S}}_1 = -\ \beta = \text{constant} = \tilde{\dot{P}}_{T2}, \qquad \ldots (5\text{-}31)$$

and therefore

$$\tilde{\dot{S}}_1 = \dot{S}_1 = \dot{P}_{T2} = \tilde{\dot{P}}_{T2}, \qquad \ldots (5\text{-}32)$$

from which, by substituting equations (5-4), (5-8) and (5-10) in equation (5-30):

$$E = c_2+(b_2-\lambda_1)(\lambda_1+\beta) + b_2(\alpha+\lambda_2) + \beta t(b_2-\lambda_1), \qquad \ldots (5\text{-}33)$$

which, due to condition (5-26) reduces to

$$E = c_2 + b_2\ (\alpha + \lambda_2), \qquad \ldots (5\text{-}34)$$

and hence, since the constants are all positive, the fourth necessary condition is satisfied.

(4) <u>The Three Sufficient Conditions</u>

According to equations (5-27), H_1 and $D(t,t_0)$ respectively always vanish. Therefore, the first and second sufficient conditions, which are equivalent to the second and third necessary conditions without the equality sign, will never be satisfied. However, the third sufficient condition is always satisfied since, due to equation (5-34)

$$E > 0. \qquad \ldots (5\text{-}35)$$

5.4 Conclusions

5.4.1 Discussion on the Solutions

For the specific problem stated in part 5.1 only the first, second, and fourth necessary conditions, and the third sufficient condition, are satisfied if equation (5-26) holds. The solutions of this problem, however, do not satisfy Weierstrass's third necessary condition and the first and second sufficient conditions. These last three conditions could be easily met, if quadratic or higher order solutions were used, in the place of equations (5-6) and (5-10). H_1 would, then, not have to vanish and the determinant $D(t,t_0)$ would not contain identical rows leading to a zero solution. Nevertheless, this example was chosen to enable the reader to compare the method employed in this thesis with methods developed by authorities in this field who obtained linear functions of time for their storage and thermal plant values. The example was made simple enough such that the necessary and sufficient conditions could be easily calculated.

It is to be noted regretfully, that even if the seven conditions are satisfied, the conditioned problem will only lead to a "semi-strong minimum"* due to the fact that, with the first and second sufficient conditions satisfied, no guarantee can be given as to the possibility of constructing Weierstrass' E-function. The problem therefore differs from the unconditioned case where the question of Weierstrass' construction could be answered in the affirmative.

In general, the problem is not finished at the end of part 5.3. If project and operating limitations exist the results would still have to be compared with these limitations. If violations occur, the corresponding extremals should be chosen and the whole procedure repeated. For the same order of magnitude of quantities obtained, i.e., for small violations the tests for necessary and sufficient conditions would probably still be valid, and hence, not to be performed again.

5.4.2 Digital Computer Application

In solving the above two-plant problem, or, in general, for the solution of a combined hydro-thermal problem with m thermal and n hydro plants, high-speed digital computers can be employed most advantageously. These computers can be used in either of the two forms:

(i) directly, by assuming one solution for each plant (in total m+n solutions) and substituting the solutions in the general equations, or,

---

*Bolza's term.[71]

(ii) indirectly, by first solving either of the less complex GK's, Kron's, or Cypser's equations and then substituting them in the general equations derived in this thesis.

There are several criteria which could be used in making the above choice, although the most logical criterion must be that of the speed, in which the correct economic solution can be obtained using the same computer. The direct method would not create difficulties if the operating engineer, based on his experience, knows how to make reliable inital estimates of all solutions. However, the problem becomes insurmountable if many plants are present in the system. In this case, the indirect method would produce the results faster.

After the correct results are obtained they are to be tested against the three other necessary and the three sufficient conditions. The computer program should also contain tests for the various project and operating limitations. In all of the above cases iterative loops should be used for repeating the procedure.

CHAPTER VI

THE GENERAL EQUATIONS COMPARED WITH PREVIOUSLY DEVELOPED FORMULAS

6.1 Introduction

In this chapter the general equations developed in chapter III will be compared with all known formulas summarized in section 2.3.3 of chapter II. It is evident that the methods given in the above section can only be compared with the simplified type B case of the general equations, since only scheduling of generations or load allocations among plants are considered in these methods, instead of the general case using time-functions derived in this thesis. In order that each one of twelve equations in the three short-term groups can be judged, the comparison will be made using the simplified two-plant model. Furthermore, the groupings of equations will be ignored when placing side by side the simplified type B equations (3-78.j) and (3-85.i) with the previously developed formulas (2-9) to (2-22). The comparison will commence with the easiest and finish with the most difficult equation to compare.

6.2 Comparison with Kron's Equation

Kron's problem[63] is not restricted by the first auxiliary condition (3-5.i) used in this treatise. For this reason the Lagrange's multiplier λ_1 will disappear and equation (3-85.i) will reduce to

$$\lambda_2\left(1-\frac{\partial P_L}{\partial P_H}\right)\frac{\partial P_H}{\partial S}-\frac{d}{dt}\left[\lambda_2\left(1-\frac{\partial P_L}{\partial P_H}\right)\frac{\partial P_H}{\partial \dot{S}}\right]=0, \qquad \ldots (6\text{-}1)$$

removing all numbered subscripts of the plant variables. The problem therefore becomes that of comparing Kron's equation (2-15) with equation (6-1).

Kron's variables are q and $\dot{q}$, or q and Q using notations of this text, where

$$q = q(t) = q(0) + \int_0^t Q dt. \qquad \ldots (6\text{-}2)$$

If leakage and evaporation are ignored the inflow to a reservoir equals the outflow and the time rate of change of storage,* i.e.,

$$F(t) = Q(t) + \sigma(t) + \dot{S}(t),$$

which can be written as

$$Q(t) = F(t) - \sigma(t) - \dot{S}(t). \qquad \ldots (6\text{-}3)$$

The storage at any time t can be given as

$$S(t) = S(0) + \int_0^t \dot{S}(t) dt. \qquad \ldots (6\text{-}4)$$

Substituting equation (6-3) in equation (6-2) and substituting equation (6-4) in the resulting equation one obtains

$$q(t) = q(0) + \int_0^t F(t)dt - \int_0^t \sigma(t)dt - S(t) + S(0). \qquad \ldots (6\text{-}5)$$

Let

$$\int_0^t F(t)dt = \Gamma(t) + \Gamma(0), \qquad \ldots (6\text{-}6)$$

$$\int_0^t \sigma(t)dt = \pi(t) + \pi(0), \qquad \ldots (6\text{-}7)$$

---

*See also Appendix A.

then Kron's q(t) will be equivalent to the magnitude of the volume of storage S(t) if

$$q(0) = - \Gamma(t) - \Gamma(0) + \pi(t) + \pi(0) - S(0). \qquad \ldots (6\text{-}8)$$

Since q(0) is not a function of time, the time functions should vanish, thus

$$\Gamma(t) = \pi(t), \qquad \ldots (6\text{-}9)$$

and hence

$$q(0) = - \Gamma(0) + \pi(0) - S(0). \qquad \ldots (6\text{-}10)$$

This last equation stipulates that the volume of storage at the beginning of each planning period depends upon the integrated flow and spillage during all previous planning periods.

Substituting equations (6-6) to (6-10) in equation (6-5) gives

$$q(t) = - S(t), \qquad \ldots (6\text{-}11)$$

from which the equivalence of equations (6-1) and (2-15) can be seen with*

$$\lambda = - \lambda_2, \qquad \ldots (6\text{-}12)$$

and

$$\frac{\partial P_H}{\partial q} = \frac{\partial P_H}{\partial h}\left(- \frac{1}{A}\right) = - \frac{\partial P_H}{\partial S}\,. \qquad \ldots (6\text{-}13)$$

6.3 Comparison with Ricard's Equation

The best way to identify the equivalence between Ricard's equation[61] and the simplified general equations (3-85.i) is by

---

*Quantities or variables of this thesis will henceforth be written on the right-hand side of the equality sign, whenever an equivalence is proven.

comparing the former with Kron's equation (2-15). The link between the two is provided by the equation[63]

$$\gamma = \frac{\lambda}{\partial Q/\partial P_H}, \qquad \ldots (6\text{-}14)$$

where γ as a function of time can be determined from

$$\frac{d\gamma}{\gamma} = \frac{\partial Q}{\partial h}\,\frac{dt}{A}. \qquad \ldots (6\text{-}15)$$

If transmission losses are neglected, Kron's equation reduces to

$$\frac{\lambda}{A}\cdot\frac{\partial P_H}{\partial h} + \frac{d}{dt}\left[\lambda\,\frac{\partial P_H}{\partial Q}\right] = 0, \qquad \ldots (6\text{-}16)$$

with the equivalence in this thesis:

$$\frac{\lambda_2}{A}\cdot\frac{\partial P_H}{\partial h} - \frac{d}{dt}\left[\lambda_2\,\frac{\partial P_H}{\partial \dot{S}}\right] = 0. \qquad \ldots (6\text{-}17)$$

Substituting equation (6-14) in equation (6-15) and by re-arranging one obtains

$$-\frac{\lambda}{A}\,\frac{\partial Q/\partial h}{\partial Q/\partial P_H} + \frac{d}{dt}\left[\frac{\lambda}{\partial Q/\partial P_H}\right] = 0. \qquad \ldots (6\text{-}18)$$

Realizing the dependence of the plant discharge Q on the net head h and the plant output P_H, one may write

$$\Psi = Q - Q\,(h, P_H) = 0,$$

and, hence

$$\frac{\partial \Psi}{\partial Q} = 1 \neq 0, \qquad \ldots (6\text{-}19)$$

from which one derives[63]

$$\frac{\partial Q/\partial h}{\partial Q/\partial P_H} = -\frac{\partial P_H}{\partial h}, \qquad \ldots (6\text{-}20)$$

and

$$\frac{1}{\partial Q/\partial P_H} = \frac{\partial P_H}{\partial Q}. \qquad \ldots (6\text{-}21)$$

Substituting the last two equations in equation (6-18) results in Kron's equation (6-16) and its equivalence (6-17).

6.4 Comparison with CDGK's and GK's Equations

CDGK's thermal plant equations[62] are exactly equivalent with those developed by Glimn and Kirchmayer[63]. Their equations are also the same with the simplified general equation (3-78.j) with the obvious identity

$$\lambda = -\lambda_{n+1}, \qquad \ldots (6\text{-}22)$$

which reduces to identiy (6-12) for a single hydro plant problem ($n = 1$).

With the exception of the loss terms, CDGK's and GK's hydro plants equations are identical with Ricard's equations whose identity with the general equations has been proven. The loss terms can be inserted in the form of penalty factors L_T and L_H which reduce to unity when the losses vanish, i.e.,

$$\left.\begin{aligned} L_T &= \frac{1}{1-\partial P_L/\partial P_T}, \\ L_H &= \frac{1}{1-\partial P_L/\partial P_H}. \end{aligned}\right\} \qquad \ldots (6\text{-}23)$$

Instead of equation (6-14) one therefore has*

$$\gamma = \frac{\lambda}{\partial Q/\partial P_H} \frac{L_T}{L_H},$$

which by substituting the value of λ for the lossless thermal plant system becomes

$$\gamma = \frac{dC/dP_T}{\partial Q/\partial P_H} \frac{L_T}{L_H}. \qquad \ldots (6\text{-}24)$$

---

*λ in this case is lossless.

Integrating equation (6-15) one obtains

$$\gamma = \gamma_0 \exp\left(\int_0^t \frac{\partial Q}{\partial h} \frac{dt}{A}\right), \qquad \ldots (6\text{-}25)$$

from which one derives

$$\frac{dC}{dP_T} = \frac{L_H}{L_T} \frac{\partial Q}{\partial P_H} \gamma_0 \exp\left(\int_0^t \frac{\partial Q}{\partial h} \frac{dt}{A}\right). \qquad \ldots (6\text{-}26)$$

This last equation is exactly equivalent to the two GK's scheduling equations (2-13) and (2-14) combined.

If the variations of the net head can be neglected, h is no longer a variable, thus

$$\frac{\partial Q}{\partial h} = 0, \qquad \ldots (6\text{-}27)$$

and, therefore, from equation (6-25)

$$\gamma = \gamma_0 = \text{constant}. \qquad \ldots (6\text{-}28)$$

This is the problem solved by CDGK given by equation (2-12).

6.5 Comparison with Equations of the M. I. T. Group

6.5.1 Introduction

In the short-range case, Cypser specifies[56] a predetermined amount of water at each hydro plant over a short future time interval. The same specification essentially applies to the problem discussed in this thesis and to the model used by Ricard,[61] GK[63] and CDGK.[62] Cypser, however, distinguishes two cases for his specification: (i) with run-of-river and pondage plants (small storage plants included) the amount of water specified for short term use are the "anticipated availabilities" based on short-range predictions of stream flow; (ii) with large storage plants where appreciable variations in storage elevations and efficiencies occur, the specifications must be the result of a short-term optimization schedule.* For the above reasons, the problem can be solved in two ways: with power outputs as variables and with values of storage as variables. A brief outline of the two cases and their similarity with the general equations of this thesis are given in the following sections. In this connection Carey's linearization[64,65] will be mentioned.

6.5.2 Power Outputs as Variables

The specified amount of water will give a specified amount of energy over a short future time interval and, therefore, an integrated average power from the hydro plant. Hence, for the two-plant problem:

$$\int_0^T (P_H - P_{HA})\, dt = 0. \qquad \ldots (6\text{-}29)$$

The effective cost to be minimized is composed of two parts: the thermal fuel cost and the cost of violating one or more hydro limitations, i.e.,

$$\$ = \int_0^T C_2(P_T)dt + \int_0^T C_1(P_H)dt \;. \qquad \ldots (6\text{-}30)$$

Combining the two equations above together with the

---

*Of the Ricard section (see chapter II), only Kron[63] did not use this specification.

conditions for load requirements, one solves[56]

$$\frac{\partial C_1}{\partial P_H} - \lambda_1 - \mu \frac{\partial P_L}{\partial P_H} = - \mu, \qquad \ldots (6\text{-}31)$$

$$\frac{\partial C_2}{\partial P_{T2}} - \mu \frac{\partial P_L}{\partial P_H} = - \mu. \qquad \ldots (6\text{-}32)$$

Using an assumption similar to the one adopted in this thesis that no violations are allowed, one obtains

$$\frac{\partial C_1}{\partial P_H} = 0 \qquad \ldots (6\text{-}33)$$

which reveals the exact equivalence between Cypser's equations (6-31) and (6-32) and CDGK's equations (2-11) and (2-12), with the following conditions:

$$-\lambda_1 = \gamma \frac{\partial Q}{\partial P_H}, \qquad \ldots (6\text{-}34)$$

and

$$-\mu = \lambda = -\lambda_2 . \qquad \ldots (6\text{-}35)$$

Carey's equations are obtained by using equation (6-33), and by replacing all $-\mu(t)$'s on the left hand side of the equality sign by β which is equal to the weighted average value of the incremental fuel cost.[64,65] Carey's equations are, therefore, necessarily the same as those of Cypser and, hence, equivalent with the general equations.

6.5.3 Plant-Storage Values as Variables

Cypser's general problem involves the use of plant storage as variables and defining error functions such that the storage at any time t equals the optimum storage plus this error function, i.e.,

$$S(t) = S_0(t) + \nu\epsilon(t), \quad \ldots (6\text{-}36)$$

where $\epsilon(t)$ is a fixed undetermined curve constrained by the boundary conditions

$$\epsilon(t=0) = 0, \quad \ldots (6\text{-}37)$$

$$\epsilon(t=T) = 0. \quad \ldots (6\text{-}38)$$

Through the use of the Calculus of Variations it can be proven that the optimum storage curve will be given by Euler's equations

$$\frac{\partial C_2}{\partial S} - \frac{d}{dt}\left(\frac{\partial C_2}{\partial \dot{S}}\right) = 0, \quad \ldots (6\text{-}39)$$

if no hydro violations are allowed. By expansion one obtains*

$$\frac{\partial C_2}{\partial S} = \frac{\partial C}{\partial P_T} \cdot \frac{\partial P_T}{\partial P_H} \cdot \frac{\partial P_H}{\partial S} = \frac{\partial C}{\partial S}, \quad \ldots (6\text{-}40)$$

$$\frac{\partial C_2}{\partial \dot{S}} = \frac{\partial C}{\partial P_T} \cdot \frac{\partial P_T}{\partial P_H} \cdot \frac{\partial P_H}{\partial \dot{S}} = \frac{\partial C}{\partial \dot{S}}, \quad \ldots (6\text{-}41)$$

which, for a lossless system (see equations (3-78.j) and (5-14)), reduce to

$$\frac{\partial C}{\partial S} = \lambda_2 \frac{\partial P_H}{\partial S}, \quad \ldots (6\text{-}42)$$

$$\frac{\partial C}{\partial \dot{S}} = \lambda_2 \frac{\partial P_H}{\partial \dot{S}}. \quad \ldots (6\text{-}43)$$

Substituting equations (6-42) and (6-43) in equation

---

*Only one type of cost is now involved, hence subscript 2 can be omitted.

(6-39) yields

$$\lambda_2 \frac{\partial P_H}{\partial S} - \frac{d}{dt}\left[\lambda_2 \frac{\partial P_H}{\partial \dot{S}}\right] = 0, \qquad \ldots (6\text{-}44)$$

which is the simplified general equation (6-1) with $P_L = 0$.

6.6 Comparison with Watchorn's Equations

Watchorn defines for "maximum economy":[66*]

$$N = \frac{dC}{dP_T} \cdot \frac{dP_T}{dQ} . \qquad \ldots (6\text{-}45)$$

For a lossless transmission system the power outputs are related by

$$P_T = P_D - P_H , \qquad \ldots (6\text{-}46)$$

from which one obtains the differentials

$$dP_T = dP_D - dP_H . \qquad \ldots (6\text{-}47)$$

Taking the derivatives of equation (6-47) with respect to the plant discharge one solves

$$\frac{dP_T}{dQ} = \frac{dP_D/dt}{dQ/dt} - \frac{dP_H}{dQ} . \qquad \ldots (6\text{-}48)$$

Considering a variable-head plant with two variables

$$P_H = P_H\,(Q,h) , \qquad \ldots (6\text{-}49)$$

one derives

$$\frac{dP_H}{dQ} = \frac{\partial P_H}{\partial Q} + \frac{\partial P_H}{\partial h} \cdot \frac{dh}{dQ} . \qquad \ldots (6\text{-}50)$$

But the net head is given by (see Appendix A)

$$h = y - y_T - y_L . \qquad \ldots (6\text{-}51)$$

---

*Watchorn's term.

Assuming no spillage, one has

$$y = y\ (S), \qquad \ldots (6\text{-}52)$$

$$y_T = y_T\ (Q), \qquad \ldots (6\text{-}53)$$

$$y_L = y_L\ (Q), \qquad \ldots (6\text{-}54)$$

from which

$$\frac{dh}{dQ} = \frac{dy}{dS} \cdot \frac{dS}{dQ} - \frac{dy_T}{dQ} - \frac{dy_L}{dQ} . \qquad \ldots (6\text{-}55)$$

Depending on whether or not the inflow exceeds the amount being discharged, one obtains from equation (6-3)

$$\left|\frac{dS}{dQ}\right| = (Q-F)\ \frac{dt}{dQ} = \frac{Q-F}{dQ/dt} = \frac{dS}{dQ} , \qquad \ldots (6\text{-}56)$$

when a positive value of (Q-F) is assumed. Substituting equation (6-56) in equation (6-55) and, then, in equation (6-50) yields

$$\frac{dP_H}{dQ} = \frac{\partial P_H}{\partial Q} + \frac{\partial P_H}{\partial h}\left|\frac{Q-F}{dQ/dt} \cdot \frac{dy}{dS} - \frac{dy_T}{dQ} - \frac{dy_L}{dQ}\right| . \qquad \ldots (6\text{-}57)$$

Using equation (6-57) in equation (6-48) and, then, in equation (6-45) one obtains for the Watchorn's economic equation:

$$N = \frac{dC}{dP_T}\left[\frac{dP_D/dt}{dQ/dt} - \frac{\partial P_H}{\partial Q} - \frac{\partial P_H}{\partial h}\left|\frac{Q-F}{dQ/dt}\ \frac{dy}{dS} - \frac{dy_T}{dQ} - \frac{dy_L}{dQ}\right|\right] . \qquad \ldots (6\text{-}58)$$

To prove the similarity between the Watchorn's equations[66] and those of Glimn and Kirchmayer, and hence of this thesis, the following manipulations are employed. Writing equation (6-45) as

$$N\ \frac{dQ}{dP_T} = \frac{dC}{dP_T} , \qquad \ldots (6\text{-}59)$$

and using equation (6-48) and Ricard's lossless equation (2-9) one obtains

$$N\left(\frac{1}{\frac{dP_D/dt}{dQ/dt} - \frac{dP_H}{dQ}}\right) = \lambda. \qquad \ldots (6\text{-}60)$$

If the variations of load demand with time can be neglected or small compared with the variations of the plant discharge, then

$$\frac{dP_D}{dt} = 0. \qquad \ldots (6\text{-}61)$$

This condition can be made possible if blocks of power are considered during the planning period, or if individual loads are considered such as the type B case of chapter III. Combining equations (6-60) and (6-61) gives

$$-N\frac{dQ}{dP_H} = \lambda. \qquad \ldots (6\text{-}62)$$

With no losses GK's equation can be written as

$$\gamma_0 \exp\left[\int_0^t \frac{\partial Q}{\partial h}\frac{dt}{A}\right]\frac{\partial Q}{\partial P_H} = \lambda, \qquad \ldots (6\text{-}63)$$

which reduces to CDGK's

$$\gamma_0 \frac{dQ}{dP_H} = \lambda, \qquad \ldots (6\text{-}64)$$

if head variations are ignored.

Assuming that Watchorn's claim that N is a constant,[66] then equations (6-62) and (6-64) are necessarily equivalent since γ_0 is also a constant and both quantities share the same unit. The connecting equation between the two methods is given by

$$\gamma_0 = -N, \qquad \ldots (6\text{-}65)$$

from which the sought-for equivalence can be made obvious.

CHAPTER VII

CONCLUDING REMARKS -- FUTURE WORK

In the preceding chapters, all previously known methods and equations for optimizing operation of combined hydro-thermal electric systems, for both short and long periods of time, have been reviewed and analyzed. In conclusion the following comments can be made:

(1) The long-range problem is not amenable to an exact mathematical solution because of the difficulty in stating the anticipated stream flows.

(2) For the short-range problem general differential equations have been derived for optimization of hydro-thermal electric systems employing the Calculus of Variations.

(3) With regard to the previously developed short-range methods it has been shown (i) that these methods are equivalent providing certain conditions are satisfied, and (ii) that the equations derived by the authors of these methods are merely simplified forms of the aforementioned general equations.

(4) In addition to the general equations, several necessary and sufficient conditions for short-range optimization problems have been derived.

(5) The solutions of a standard problem used by well-known authorities in this field have been tested against the various conditions mentioned above.

(6) The use of high-speed digital computers for solving the general equations and testing them for the necessary and

sufficient conditions has been discussed.

In view of the above, it is obvious that much remains to be done. In the following paragraphs, several of the more pertinent fields of work requiring further study are listed.

(i) Digital computer application of the method developed in this thesis, i.e., solutions of general equations, tests of constraints and tests for necessary and sufficient conditions.

(ii) Possibility of digital computer programming of the long-range TVA procedure, after being mathematically formulated. This method appears to offer probable solutions.

(iii) Further study of several long-range methods with regard to their deficiencies, e.g., Cypser, Little and others listed in this thesis.

(iv) Further study of the effect of neglecting transmission losses in the long-range problem.[51]

(v) Inclusion of incremental maintenance cost in the optimizing equations.[97-101]

(vi) Development of a set of definite and reliable forecasting procedures for predicting stream flows.[102-105]

REFERENCES

The following references are numbered according to their order of appearance. They are grouped in several sections in which chronological listings, if possible, are maintained.

A. General Information and Optimization of Thermal Systems

1. Justin, Joel D. and Mervine, William G., Power Supply Economics, John Wiley and Sons, 1934.

2. Klapper, D.B., Discussion on Reference 66.

3. Vencill, G.J., Discussion on Reference 66.

4. Steinberg, M.J. and Smith, T.H., Economic Loading of Steam Power Plants and Electric Systems, McGraw-Hill Book Company, Inc., New York, 1935.

5. Kaufmann, Paul G., "Load Distribution Between Interconnected Power Stations", Journal of the IEE, vol. 90-91, 1943-1944, pt. II (Power), pp. 119-140.

6. Steinberg, M.J. and Smith, T.H.,"Incremental Loading of Generating Stations", Electrical Engineering, vol. 52, October 1933, pp. 674-678.

7. Kirchmayer, L.K., Economic Operation of Power Systems, John Wiley and Sons, Inc., New York, 1958.

8. Kirchmayer, L.K., Economic Control of Interconnected Systems, John Wiley and Sons, New York, 1959.

9. Steinberg, M.J., "How to Calculate Incremental Production Costs", Power, vol. 90, July 1946, pp. 76-78, p. 146.

10. Kirchmayer, L.K. and McDaniel, G.H., "Transmission Losses and Economic Loading of Power Systems", General Electric Review, Schenectady, New York, October 1951.

11. Kirchmayer, L.K. and Stagg, G.W., "Evaluation of Methods of Co-ordinating Incremental Fuel Costs and Incremental Transmission Losses", AIEE Transactions, vol. 71, pt. III, 1952, pp. 513-521.

12. Kron, Gabriel, "Tensorial Analysis of Integrated Transmission Systems - Part III. The 'Primitive' Division", AIEE Transactions, vol. 71, pt. III, October 1952, pp. 814-822.

13. Ward, J.B., "Economy Loading Simplified", AIEE Transactions, vol. 72, pt. III, December 1953, pp. 1306-1311.

14. Harker, D.C., Jacobs, W.E., Ferguson, R.W. and Harder, E.L., "Loss Evaluation - Part I. Losses Associated with Sale Power", AIEE Transactions, vol. 73, pt. III-A, pp. 709-716.

15. Travers, R.H., Harker, D.C., Long, R.W. and Harder, E.L., "Loss Evaluation - Part III. Economic Dispatch Studies of Steam Electric Generating Systems", AIEE Transactions, vol. 73, pt. III-B, October 1954, pp. 1091-1104.

16. LeVesconte, L.B. and Hicks, K.L., "The Common Sense of Incremental Loading", Proceedings of the American Power Conference, Chicago, vol. XIX., 1957, pp. 521-527.

17. Thomas, R.W., Kirchmayer, L.K. and Wilson, J.R., "Economic Considerations in Generation Scheduling for the Southwestern Public Service Company System", AIEE Transactions", vol. 76, pt. III, February 1958, pp. 1545-1552.

18. Lubisich, P.G., "Penalty Factors from Power-System Equations", AIEE Transactions, vol. 77, pt. III, Aug. 1958, pp. 494-501.

B. Computers for Load Dispatching of Thermal Systems

19. Imburgia, C.A., Kirchmayer, L.K. and Stagg, G.W., "A Transmission Loss Penalty Factor Computer", AIEE Transactions, vol. 73, pt. III, pp. 567-571.

20. Early, E.D., Phillips, W.E. and Shreve, W.T., "An Incremental Cost of Power Delivered Computer", AIEE Transactions, vol. 74, pt. III, June 1955, pp. 529-535.

21. Morrill, C.D. and Blake, J.A., "A Computer for Economic Scheduling and Control of Power Systems", AIEE Transactions, vol. 74, pt. III, 1955, pp. 1136-1142.

22. Osterle, W.H. and Harder, E.L., "Loss Evaluation - Part IV. Economic Dispatch Computer: Principles and Application", AIEE Transactions, vol. 75, pt. III, 1956, pp. 387-394.

23. Squires, R.B., Byerly, R.T., Colborn, H.W. and Hamilton, W.R., "Loss Evaluation - Part V. Economic Dispatch Computer: Design", AIEE Transactions, vol. 75, pt. III, 1956, pp. 719-727.

24. Burnett, K.N., Halfhill, D.W. and Shepard, B.R., "A New Automatic Dispatching System for Electric Power Systems", AIEE Transactions, vol. 75, pt. III, 1956, pp. 1049-1056.

25. Glimn, A.F., Kirchmayer, L.K., Peterson, V.R. and Stagg, G.W., "Accuracy Considerations in Economic Dispatching of Power Systems - Part I", AIEE Transactions, vol. 75, pt. III, 1956, pp. 1125-1137.

26. Colborn, H.W., Byerly, R.T. and Squires, R.B.,"Application of Economic Dispatch Computers to Power Systems", Proceedings of the American Power Conference, vol. XVIII, 1956, pp. 515-522.

27. Travers, R.H., "Automatic Economic Dispatching and Load Control - Ohio Edison System", AIEE Transactions, vol. 76, June 1957, pp. 291-297.

28. Farley, A.P., Description - Westinghouse Electronic Dispatch Computer, Pennsylvania Electric Association, Pittsburgh, Pa., October 24-25, 1957.

29. Hamilton, W.R. and Osterle, W.H.,"Operating Experience with West Penn Power Company's Economic Dispatch Computer", AIEE Transactions, vol. 77, pt. III, October 1958, pp. 702-706.

30. Brown, M.J.,"An Automatic Dispatching System", AIEE Transactions", vol. 78, pt. III, October 1959, pp. 957-963.

31. Glimn, A.F., Kirchmayer, L.K., Habermann, R.Jr. and Thomas, R.W., "Automatic Digital Computer Applied to Generation Scheduling", AIEE Transactions, vol. 73, pt. III, Oct. 1954, pp. 1267-1275.

32. Dandeno, P.L., "Use of Digital Computers in Coordination of System Generation", AIEE Conference Paper, CP 56 - 1000.

33. Shipley, R.Bruce and Hochdorf, Martin, "Exact Economic Dispatch - Digital Computer Solution", AIEE Transactions, vol. 75, pt. III, 1956, pp. 1147-1153.

34. Dandeno, P.L., "Applications of Digital Computers in Power System Analysis", Proceedings of the American Power Conference, Chicago, vol. XVIII, 1956, pp. 530-536.

35. George, E.E., Page, H.W. and Ward, J.B., "Coordination of Fuel Cost and Transmission Loss by the Use of the Network Analyzer to Determine Plant Loading Schedules", AIEE Transactions, vol. 68, pt. II, 1949, pp. 1152-1163.

C. Optimization of Hydro Systems

36. Creager, William P. and Justin, Joel D., Hydro-Electric

Handbook, John Wiley and Sons, Inc., New York, 1927, pp. 1075-1126.

37. McCrea, Jr., W.S., "Segregation of Hydro-Electric Power Costs", AIEE Transactions, vol. 52, March 1933, pp. 1-8.

38. Allner, F.A., "Economic Aspects of Water Power", AIEE Transactions, vol. 52, March 1933, pp. 156-166.

39. Frampton, A.H. and Floyd, G.D., "Factors in the Economic Supply of Energy in Hydroelectric Systems", AIEE Transactions, vol. 66, 1947, pp. 1117-1125.

40. Cooper, A.R., "Load Dispatching and the Reasons for It, with Special Reference to the British Grid System", Journal of the IEE, vol. 95, pt. II, 1948, pp. 713-723.

41. Gornshtein, V.M., Most Advantageous Distribution of Loads Among Electric Power Stations Operating in Parallel, State Energy Publishing House, Moscow, U.S.S.R., 1949.

42. Hoard, B.V., "Analysis of Storage Use to Obtain Maximum Incremental Energy from Two Hydro Storage Plants", AIEE Transactions, vol. 73, pt. III-B, 1954, pp. 1041-1048.

43. Monroe, Robert A., "Some Factors in the Economic Generation of Power by the TVA System", Proceedings of the American Power Conference, Chicago, vol. XIX, 1957, pp. 361-368.

44. Bernholtz, B., Shelson, W. and Kesner, O., "A Method of Scheduling Optimum Operation of Ontario Hydro's Sir Adam Beck-Niagara Generating Station", AIEE Transactions, vol. 77, pt. III, December 1958, pp. 981-989.

D. Methods of Optimizing Hydro Systems Being Reviewed and Analyzed

45. Strowger, E.B., "Operating a Hydro-Electric System for Best Economy and Means Employed for Checking Performance of Operation", NELA Bulletin, vol. 16, 1929, pp. 777-780.

46. Schamberger, S.O., "Hydro Station Operation with Minimum Loss", Edison Electric Institute Bulletin, vol. 3, 1935, pp. 19-23.

47. Burr, H.A., "Load Division between Common-Flow Hydroelectric

Stations", S. M. Thesis, Massachusetts Institute of Technology, 1941.

48. Johnson, David L., "Digital Computer Solution to Determine Economical Use of Hydro Storage", AIEE Transactions, vol. 75, pt. III, December 1956, pp. 1153-1156.

49. Chandler, Jr., Webster M. and Gabrielle, Anthony F., "Economy Loading of Hydroelectric Systems", S. M. Thesis, Massachusetts Institute of Technology, September 1950.

50. Bobo, Powell E., "Effect of Transmission Losses on the Optimum Plant Loading Schedule of the West Penn Electric Company", M. S. Thesis, University of Pittsburgh, Pittsburgh, Pa., 1951.

51. Johannessen, Paul R., "Transmission Loss and Stream Flows", S. M. Thesis, Massachusetts Institute of Technology, Cambridge, Mass., June 1953.

52. McIntyre, H.M., Blake, C.W. and Clubb, J.S., "Digital Computer Aids in Power Pool Operation Studies", AIEE Transactions, vol. 77, pt. I, Nov. 1958, pp. 652-657.

E. Optimization of Hydro-Thermal Systems

53. Ruskin, V.W., "What Do Losses Cost in Hydro, Thermal, and Combined Systems", AIEE Transactions, vol. 75, pt. III, 1956, pp. 332-339.

54. Ruskin, V.W., "Thermal Plants for 'Firming Up' Hydro", AIEE Transactions, vol. 76, pt. III, August 1957, pp. 609-612.

55. Dryer, Walter, "Integrating Steam and Hydro Power in Northern and Central California", Proceedings of the American Power Conference, Chicago, vol. XIX, 1957, pp. 352-360.

F. Methods of Long-Range Optimization of Hydro-Thermal Systems Being Reviewed and Analyzed

56. Cypser, Rudolph John, "Optimum Use of Water Storage in Hydro-Thermal Electric Systems", Sc. D. Thesis, Massachusetts Institute of Technology, Cambridge, Mass., February 1953.

57. Cypser, R.J., "Computer Search for Economical Operation of a Hydrothermal Electric System", AIEE Transactions, vol. 73, pt. III-B, 1954, pp. 1260-1267.

58. Little, John D.C., "The Use of Storage Water in a Hydro-Electric System", Journal of the Operations Research Society of America, Baltimore, Md., vol. 3, No. 2, May 1955, pp. 187-197.

59. Koopmans, Tjalling C., "Water Storage Policy in a Simplified Hydroelectric System", Paper No. 115, Cowles Foundation for Research in Economics, Yale University, New Haven, Conn., 1958.

60. Brudenell, Ross N. and Gilbreath, Jack H., "Economic Complimentary Operation of Hydro Storage and Steam Power in the Integrated TVA System", AIEE Transactions, vol. 78, pt. III, June 1959, pp. 136-156.

G. Methods of Short-Range Optimization of Hydro-Thermal Systems Being Reviewed and Analyzed

61. Ricard, J., "The Determination of Optimum Operating Schedule for Interconnected Hydro and Thermal Stations", Revue Générale de l'Electricité, Paris, France, Sept. 1940, p. 167.

62. Chandler, W.G., Dandeno, P.L., Glimn, A.F. and Kirchmayer, L.K., "Short-Range Economic Operation of a Combined Thermal and Hydro-Electric Power System", AIEE Transactions, vol. 72, pt. III, October 1953, pp. 1057-1065.

63. Glimn, A.F. and Kirchmayer, L.K., "Economic Operation of Variable-Head Hydroelectric Plants", AIEE Transactions, vol. 77, pt. III, December 1958, pp. 1070-1078.

64. Carey, John Joseph, "Load Allocation on a Hydrothermal Electric System", S. M. Thesis, Massachusetts Institute of Technology, Cambridge, Mass., June 1953.

65. Carey, John J., "Short-Range Load Allocational Hydro-Thermal Electric System", AIEE Transactions, vol. 73, pt. III-B, 1954, pp. 1105-1111.

66. Watchorn, C.W., "Coördination of Hydro and Steam Generation", AIEE Transactions, vol. 74, pt. III, 1955, pp. 142-150.

H. Miscellaneous

67. Von Neumann, John and Morgenstern, Oskar, Theory of Games and Economic Behaviour, Princeton University Press, Princeton, 1944.

68. Windsor, J.S., Report on Annual Operating Plan for the Vancouver Island System, Planning Division, British Columbia Power Commission, Victoria, B.C., June 2, 1958.

69. Ingledow, T., "Hydro-Electric Power and Hydro-Electric Power Development in the Lower Mainland Coastal Area of British Columbia", (paper read before a Joint Meeting of the Vancouver Section of AIEE, EIC, B. C. Engineering Society, Vancouver Board of Trade, January 8, 1959).

70. Miller, R.H., Discussion on Reference 66.

I. Calculus of Variations

71. Bolza, Oskar, Lectures on the Calculus of Variations, University of Chicago Press, Chicago, 1904.

72. Bolza, Oskar, Vorlesungen ueber Variationsrechnung, Druck und Verlag von B.G. Teubner, Leipzig und Berlin, 1909.

73. Lanczos, Cornelius, The Variational Principles of Mechanics, University of Toronto Press, Toronto, 1949.

74. Smith, D.M., "Jacobi's Condition for the Problem of Lagrange in the Calculus of Variations", Transactions, American Mathematical Society, vol. 17, 1916, pp. 459-475.

75. Bliss, G.A., "The Problem of Lagrange in the Calculus of Variations", American Journal of Mathematics, vol. 52, 1930, pp. 673-744.

J. Transmission Loss Equations and Incremental Transmission Loss Formulas

76. George, E.E., "Intrasystem Transmission Losses", AIEE Transactions, vol. 62, March 1943, pp. 153-158.

77. Ward, J.B., Eaton, J.R. and Hale, H.W., "Total and Incremental Losses in Power Transmission Networks", AIEE Transactions, vol. 69, pt. I, 1950, pp. 626-632.

78. Kirchmayer, L.K. and Stagg, G.W., "Analysis of Total and Incremental Losses in Transmission Systems", AIEE Transactions, vol. 70, pt. II, 1951, pp. 1197-1204.

79. Kron, Gabriel, "Tensorial Analysis of Integrated Transmission Systems - Part I. The Six Basic Reference Frames", AIEE Transactions, vol. 70, pt. II, 1951, pp. 1239-1248.

80. Glimn, A.F., Habermann, R., Kirchmayer, L.K. and Stagg, G. W., "Loss Formulas Made Easy", AIEE Transactions, vol. 72, pt. III, August 1953, pp.730-737.

81. Brownlee, W.R., "Coordination of Incremental Fuel Costs and Incremental Transmission Losses by Functions of Voltage Phase Angles", AIEE Transactions, vol. 73, pt. III, June 1954, pp. 529-541.

82. Harder, E.L., Ferguson, R.W., Jacobs, W.E. and Harker, D.C., "Loss Evaluation - Part II. Current- and Power-Form Loss Formulas", AIEE Transactions, vol. 73, pt. III, 1954, pp. 716-731.

83. Cahn, C.R., "The Determination of Incremental and Total Loss Formulas from Functions of Voltage Phase Angles", AIEE Transactions, vol. 74, pt. III, April 1955, pp. 161-176.

84. Early, E.D., Watson, R.E. and Smith, G.L., "A General Transmission Loss Equation", AIEE Transactions, vol. 74, pt. III, 1955, pp. 510-520.

85. Early, E.D. and Watson, R.E., "A New Method of Determining Constants for the General Transmission Loss Equation", AIEE Transactions, vol. 74, pt. III, 1955, pp. 1417-1423.

86. Calvert, J.F. and Sze, T.W., "A New Approach to Loss Minimization in Electric Power Systems", AIEE Transactions, vol. 76, pt. III, 1957, pp. 1439-1446.

87. Sze, T.W., Garnett, J.R. and Calvert, J.F., "Some Applications of a New Approach to Loss Minimization in Electrical Utility Systems", AIEE Transactions, vol. 77, pt. III, February 1959, pp 1577-1585.

88. Watson, R.E. and Stadlin, W.O., "The Calculation of Incremental Transmission Losses and the General Transmission Loss Equation", AIEE Transactions, vol. 78, pt. III, April 1959, pp. 12-18.

89. Miller, A.R., Koen, Jr., H.R. and Deliyannides, J.S., "The Use of Power Transfer Equations to Derive Economic Coordination Relationships Expressed as Functions of Voltage Phase Angles", AIEE Transactions, vol. 78, pt. III, October 1959, pp. 747-753.

90. George, E.E., "A New Method of Making Loss Formulas Directly from Digital Power Flow Studies", Transactions Paper 59-1081, AIEE Fall General Meeting, October 11-16, 1959.

91. Glimn, A.F., Kirchmayer, L.K. and Stagg, G.W., "Analysis of Losses in Interconnected Systems", AIEE Transactions, vol. 71, pt. III, October 1952, pp. 796-808.

92. Hale, H.W., "Power Losses in Interconnected Transmission Networks", AIEE Transactions, vol. 71, pt. III, December 1952, pp. 993-998.

93. Kron, Gabriel, "Tensorial Analysis of Integrated Transmission Systems - Part IV. The Interconnection of Transmission Systems", AIEE Transactions, vol. 72, pt. III, August 1953, pp. 827-839.

94. Glimn, A.F., Kirchmayer, L.K. and Stagg, G.W., "Analysis of Losses in Loop-Interconnected Systems", AIEE Transactions, vol. 72, pt. III, October 1953, pp. 944-953.

95. Glimn, A.F., Kirchmayer, L.K. and Skiles, J.J., "Improved Method of Interconnecting Transmission Loss Formulas", AIEE Transactions, vol. 77, pt. III, October 1958, pp. 755-760.

96. Kerr, R.H. and Kirchmayer, L.K., "Theory of Economic Operation of Interconnected Areas", AIEE Transactions, vol. 78, pt. III, August 1959, pp. 647-653.

K. Maintenance Cost and Incremental Maintenance Cost

97. Robertson, A.S. and Standing, R.O., "The Maintenance of Hydroelectric Generating Units", AIEE Transactions, vol. 66, 1947, pp. 776-782.

98. Steinberg, M.J., "Incremental Maintenance Costs of Steam-Electric Generating Stations", AIEE Transactions, vol. 76, February 1958, pp. 1251-1254.

99. Light, F.H.,"Application of Digital Computer Technique for Development of the Incremental Maintenance Cost", AIEE Transactions, vol. 78, pt. III, February 1959, pp. 1562-1568.

100. Zelenka, D.B. and Travers, R.H.,"Fundamental Concepts of Incremental Maintenance Costs as Used by Ohio Edison Company", AIEE Transactions, vol. 78, pt. III, June 1959, pp. 163-165.

101. Nichols, C. (chairman), AIEE Committee Report: "Report on Present-Day Practices of Handling Incremental Maintenance Costs as They Apply to Economic Dispatch of Power", AIEE Transactions, vol. 78, pt. III, June 1959, pp. 279-282.

L. Forecasting Procedures Using Probability Theory

102. Foster, H.Alden, "Theoretical Frequency Curves and Their Application to Engineering Problems", Trans. Am. Soc. C. E., vol. LXXXVII, 1924, pp. 142-173.

103. Loane, E.S. and Watchorn, C.W., "Probability Methods Applied to Generating Capacity Problems of a Combined Hydro and Steam System", AIEE Transactions, vol. 66, 1947, pp. 1645-1654.

104. McIntyre, Henderson N. and Sachs, Milton S., "Forecasting Procedures Advance Effective Water Routings on the U.S. Columbia River Hydro-electric System", AIEE Transactions, vol. 77, pt. III, February 1959, pp. 1588-1593.

105. Kartvelishvili, N.A., "Probability Theory and Operating Modes for Power Systems", Izvestia Akad. Nauk Otdelenie Tekh. Nauk (Energetika i Automatika') No.1; 3-10, Jan-Feb 1959 (excerpts in 'Power Express', vol. 1, No. 0, July 1959, International Physical Index, Inc., New York).

NOMENCLATURE

The following notations are given in an alphabetical order. Notations using subscripts i or j indicate generalities; these subscripts may be removed for convenience if simplified two-plant problems are considered. Units actually used in practice are the ones indicated between brackets or multiples of ten of them.

Latin Alphabets

A_i	=	surface area of reservoir at hydro plant i (acre)
a_j, b_j, c_j	=	constants of the cost function at thermal plant j
B_i	=	constant of J_i (Mw-hr)
$\bar{B}_i$	=	constant of $\bar{J}_i$ (Mw-hr)
B_{rs}	=	derived loss formula coefficient $(1/(\text{Mw})^2)$
C_i	=	cost of violating hydro-limitations at hydro plant i (\$/hr)
C_j	=	fuel cost at thermal plant j (\$/hr)
$D(t,t_0)$	=	determinant used in the third necessary condition and the second sufficient condition
E	=	Weierstrass' function
F_i	=	water inflow to reservoir or pond at hydro plant i (cfs)
G_1, U, V	=	time functions to compute $D(t,t_0)$
H	=	integrand of $\bar{I}$ (\$/hr)
H_1	=	second partial derivative used in the second necessary condition and first sufficient condition
h_i	=	net head at hydro plant i (ft)
h_{Gi}	=	gross head at hydro plant i (ft)
I	=	integral to be minimized with no auxiliary condition (\$)
$\bar{I}$	=	integral to be minimized with auxiliary conditions (\$)
J_i	=	isoperimetric condition with h_i and Q_i as variables

$\bar{J}_i$ = isoperimetric condition with S_i and $\mathring{S}_i$ as variables

L_{Hi} = penalty factor at hydro plant i (dimensionless)

L_{Tj} = penalty factor at thermal plant j (dimensionless)

m = number of thermal plants

N = incremental water value used by Watchorn (\$/hr/1000cfs)

n = number of hydro plants

P_D = load, demanded or delivered power (Mw)

P_{Hi} = power output of hydro plant i (Mw)

P_L = power transmission losses (Mw)

P_{Tj} = power output of thermal plant j (Mw)

$\dot{P}_{Tj}$ = dP_{Tj}/dt = time rate of change of P_{Tj} (Mw/hr)

Q_i = discharge at hydro plant i (cfs)

q = Kron's storage variable (ft^3)

q_u^k = k^{th} variable of the u^{th} type

$\mathring{q}_u^k$ = dq_u^k/dt = time rate of change of q_u^k

S_i = storage volume at hydro plant i (acre-ft)

$\dot{S}_i$ = dS_i/dt = time rate of change of S_i (cfs)

T = length of the optimizing period (hrs, days)

t = time variable (hrs)

t_0 = beginning of optimizing period

t_0' = conjugate of point t_0

t_e = end of optimizing period

y_i = reservoir or pond elevation at hydro plant i (ft)

y_{Li} = head loss due to friction, etc. at hydro plant i (ft)

y_{Ti} = tailwater elevation at hydro plant i (ft)

Greek Alphabets

α, β = integration constants for the two-plant problem

β = weighted average incremental fuel cost used by the MIT Group (\$/Mw-hr)

γ = conversion factor used for the variable-head case

γ_i = conversion constant at hydro plant i for the constant-head case (\$/million ft^3)

γ_0 = conversion constant for the variable-head case (\$/million ft^3)

$\Gamma(t)$, $\Gamma(0)$ = time functions of the integrated flow (ft^3)

$\epsilon(t)$ = error function of S_i

λ = Lagrangian multiplier, used by the Ricard Group as incremental cost of delivered power (\$/Mw-hr)

λ_i = Lagrangian multiplier, used in this thesis as incremental cost at hydro plant i (\$/Mw-hr)

$-\lambda_i$ = Lagrangian multiplier, used by the MIT Group as part of the incremental cost at hydro plant i (\$/Mw-hr)

$-\lambda_{n+1}$ = Lagrangian multiplier, used in this thesis as incremental cost of delivered power (\$/Mw-hr)

$-\mu(t)$ = Lagrangian multiplier, used by the MIT Group as incremental cost of delivered power (\$/Mw-hr)

ν = parameter of error function $\epsilon(t)$

$\pi(t)$, $\pi(0)$ = time functions of the integrated spillage (ft^3)

σ_i = spillage at hydro plant i (cfs)

$\emptyset$ = auxiliary conditions for load requirements (Mw)

Ψ_i = function of net head h_i, discharge Q_i, and hydro plant output P_{Hi}

$\omega_1(t), \ldots \omega_4(t)$; $\emptyset_1$ = time functions to compute $D(t, t_0)$

Derivatives

dC_j/dP_{Tj} = incremental fuel cost at thermal plant j (\$/Mw-hr)

$\partial C_i / \partial P_{Hi}$ = incremental cost at hydro plant i, used by the MIT Group (\$/Mw-hr)

dP_H / dQ = incremental hydro equivalent, used by Watchorn (Mw/1000cfs)

$\partial P_L / \partial P_{Hi}$ = incremental transmission loss of hydro plant i (dimensionless)

$\partial P_L / \partial P_{Tj}$ = incremental transmission loss of thermal plant j (dimensionless)

dQ_i / dP_{Hi} = incremental water rate at hydro plant i, used by the Ricard Group (ft^3/Mw-hr)

Subscripts and Superscripts

The following subscripts and/or superscripts are used in connection with:

i, ℓ - hydro plants

j,k,n - thermal plants

r,s - generations, either thermal or hydro

x - time variables

u - control variables

The following subscripts stand for:

B - storage basin

C - conduits, channels

D - demand, delivered

e - end of time interval

F - flood

f - fish

G - intake gates or gross head

H - hydro plant

I - irrigation

L	- losses
M	- maximum
μ	- minimum
N	- navigation
O	- zero time
P	- peaking
R	- recreation
T	- thermal plant, total or turbine

APPENDIX A

THE PROOF OF EQUIVALENCE OF $P_{Hi}(Q_i,h_i,t)$ AND $P_{Hi}(F_i,S_i,\mathring{S}_i,t)$

It is generally known that for any hydro plant i, considering one plant on one stream only, there is a definite relationship between the hydro plant output P_{Hi}, the net head h_i, and the plant discharge Q_i. This relationship can be given either as[56,68]

$$P_{Hi} = P_{Hi}(Q_i,h_i,t), \qquad \ldots \text{(A-1)}$$

or, as[63]

$$Q_i = Q_i(h_i,P_{Hi},t). \qquad \ldots \text{(A-2)}$$

Both equations are the same, and therefore only the equivalence of equation (A-1) with

$$P_{Hi} = P_{Hi}(F_i,S_i,\mathring{S}_i,t) \qquad \ldots \text{(A-3)}$$

will be proven. The reason for using equation (A-3) is purely mathematical: (1) to conform with the method used the presence of the factor $\mathring{S}_i = dS_i/dt$, the time rate of change of the storage value S_i, is essential; (2) the flow factor F_i is an "alien variable" which cannot be controlled and indeterminable and, hence, can be omitted from the optimizing equations, i.e., it is wasteful to determine what the flow should be if it is known that the result obtained is going to be useless. This last assumption will greatly simplify a rather complex problem.

The net head is the difference between the reservoir elevation y_i and the tailwater elevation y_{Ti} and the head loss y_{Li}, i.e.,

$$h_i(t) = y_i(t) - y_{Ti}(t) - y_{Li}(t), \qquad \ldots \text{(A-4)}$$

which can be written as

$$h_i(t) = h_{Gi}(t) - y_{Li}(t), \qquad \ldots \text{ (A-5)}$$

where

$$h_{Gi}(t) = y_i(t) - y_{Ti}(t) \qquad \ldots \text{ (A-6)}$$

$$= \text{the gross head at hydro plant i.}$$

To investigate the dependence of the net head and the plant discharge on the flow and the storage factors, the following relations will be revealed:

(1) There is a definite relationship between the reservoir elevation and its storage volume:[56,68]

$$y_i = y_i(S_i, t). \qquad \ldots \text{ (A-7)}$$

(2) The tailwater elevation depends on the discharge of water through the plant and the amount being spilled through the spillways:[56,68]*

$$y_{Ti} = y_{Ti}(Q_i, \sigma i, t). \qquad \ldots \text{ (A-8)}$$

(3) The head loss is composed of skin friction and eddy losses, the latter caused by sudden changes in the direction of flow or by sudden changes in velocity. This head loss is a function of the discharge through the plant:[36]

$$y_{Li} = y_{Li}(Q_i, t). \qquad \ldots \text{ (A-9)}$$

(4) Generally, the spillage depends only on one variable, the elevation of the reservoir:[68]

$$\sigma_i = \sigma_i(y_i, t), \qquad \ldots \text{ (A-10)}$$

---

*A very exceptional case occurs when water discharges, or is being spilled, into a very large river or other very large drainage areas. Then, the tailwater elevation is independent of any of these variables.

which combined with equation (A-7) gives

$$\sigma_i = \sigma_i(S_i, t). \qquad \ldots \text{(A-11)}$$

In the case of a "free" spill* at plants with no spill-gates (e.g., at run-of-river plants), the spillage is a continuous function of elevation, see Fig. 1. In the case of a "controlled"

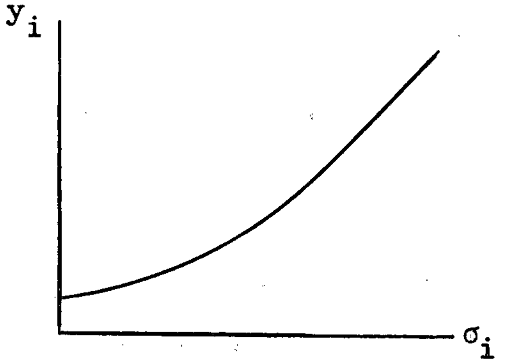

Fig. 1. "Free" spill curve

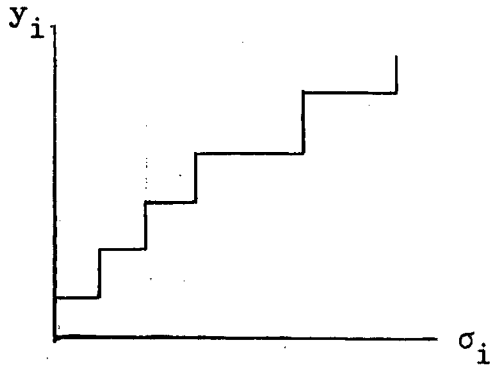

Fig. 2. "Controlled" spill curve

spill,* spillage is adjusted manually: when the reservoir reaches a certain height, a certain amount of water must be spilled through the spillways; the spill curve is therefore a step-function of elevation, see Fig. 2.

(5) The relation between the discharge and all other variables are given by the continuity equation at any hydro plant, i.e., ignoring leakage and evaporation, the inflow must equal the outflow plus the time rate of change of storage,** or,

$$F_i(t) = Q_i(t) + \sigma_i(t) + \dot{S}_i(t), \qquad \ldots \text{(A-12)}$$

which can be written as

$$Q_i(t) = F_i(t) - \sigma_i(t) - \dot{S}_i(t), \qquad \ldots \text{(A-13)}$$

---

* Windsor's term.[68]

**This value can be either positive or negative.

whose functional relationship is given by

$$Q_i = Q_i(F_i, \sigma_i, \dot{S}_i, t). \qquad \text{... (A-14)}$$

Substituting equation (A-11) in the last equation yields

$$Q_i = Q_i(F_i, S_i, \dot{S}_1, t). \qquad \text{... (A-15)}$$

(6) Substituting equations (A-11) and (A-15) in equation (A-8) gives

$$y_{Ti} = y_{Ti}(F_i, S_i, \dot{S}_i, t), \qquad \text{... (A-16)}$$

which combined with equations (A-4), (A-7), (A-9) and (A-15) gives the functional character of the net head

$$h_i = h_i(F_i, S_i, \dot{S}_i, t). \qquad \text{... (A-17)}$$

To prove the sameness of equation (A-1) and equation (A-3) one substitutes equations (A-15) and (A-17) in equation (A-1).

APPENDIX B

AN OUTLINE OF THE VARIATIONAL CALCULUS PROBLEM WITH AUXILIARY CONDITIONS

1. The Unconditioned Problem and the First Necessary Condition for an Extremum

The variational problem in which no auxiliary conditions are involved is called the unconditioned problem.[71] It is the case of finding the extreme value (maximum or minimum) of a definite integral*

$$I = \int_{t_0}^{t_e} F(q_1,\ldots,q_p,\dot{q}_1,\ldots,\dot{q}_p,t)dt, \qquad \ldots \text{(B-1)}$$

with the boundary conditions $q_u(t_0)$ and $q_u(t_e)$ given $(u=1,\ldots,p)$; thus, their variations at the two end-points must vanish:

$$\delta q_u(t)\Big]_{t=t_0} = 0, \qquad \delta q_u(t)\Big]_{t=t_e} = 0. \qquad \ldots \text{(B-2)}$$

The variables $q_1,\ldots,q_p$ are unknown functions of t, to be determined such that the integral I has an extreme value, hence[71-73]

$$\delta I = 0 \qquad \ldots \text{(B-3)}$$

for independent variations of q_u, subject only to the boundary conditions (B-2).

Consider at this moment one variable (p=1), then equation (B-1) reduces to

$$I_1 = \int_{t_0}^{t_e} F_1(q_1,\dot{q}_1,t)dt. \qquad \ldots \text{(B-4)}$$

---

*$\dot{q}_u$ means dq_u/dt.

Suppose that the function $q_1 = f(t)$ is an extremal of I_1, then

$$\delta I_1 = 0. \qquad \ldots \text{(B-5)}$$

In order to obtain the first necessary condition for an extremum, consider the modified function

$$\bar{f}(t) = f(t) + \varepsilon \nu(t), \qquad \ldots \text{(B-6)}$$

where $\nu(t)$ is some arbitrary, continuous and differentiable new function.[71-73] One must now prove that the change of the integral due to the change in the function becomes zero. Using the small variable parameter ε one can modify the function $f(t)$ by arbitrarily small amounts. Comparing the values of the original function $f(t)$ with the modified function $\bar{f}(t)$ at a certain definite point t by forming the difference between the two functions, one obtains

$$\delta q_1 = \bar{f}(t) - f(t) = \varepsilon \nu(t). \qquad \ldots \text{(B-7)}$$

This difference is called the "variation" of $f(t)$,[73] see Fig. 3 below.

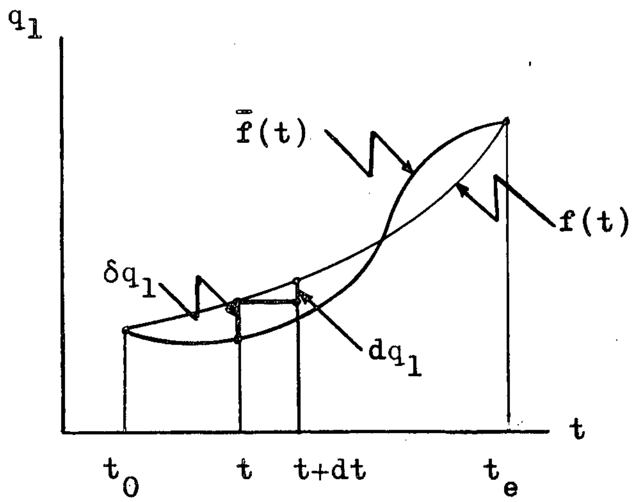

Fig. 3
Variation of $f(t)$

Using equation (B-7), the variation of the integrand F_1 of the integral I_1, caused by the variation of q_1, can now be computed:[71-73]

$$\delta F_1(q_1,\dot{q}_1,t) = F_1(q_1+\epsilon\nu, \dot{q}_1+\epsilon\dot{\nu},t) - F_1(q_1,\dot{q}_1,t)$$

$$= \epsilon\left[\frac{\partial F_1}{\partial q_1}\nu + \frac{\partial F}{\partial \dot{q}_1}\dot{\nu}\right], \qquad \ldots \text{(B-8)}$$

when the higher order terms of the Taylor series are neglected since ϵ approaches zero.

Substituting equation (B-8) in equations (B-4) and (B-5) yields

$$\delta\int_{t_0}^{t_e} F_1 dt = \int_{t_0}^{t_e} \delta F_1 dt = \epsilon\int_{t_0}^{t_e}\left[\frac{\partial F_1}{\partial q_1}\nu + \frac{\partial F_1}{\partial \dot{q}_1}\dot{\nu}\right]dt = 0, \qquad \ldots \text{(B-9)}$$

which through integration by parts becomes

$$\delta\int_{t_0}^{t_e} F_1 dt = \epsilon\int_{t_0}^{t_e} \frac{\partial F_1}{\partial q_1}\nu\, dt + \epsilon\left[\frac{\partial F_1}{\partial \dot{q}_1}\nu\right]_{t_0}^{t_e} - \epsilon\int_{t_0}^{t_e} \frac{d}{dt}\left(\frac{\partial F_1}{\partial \dot{q}_1}\right)\nu dt = 0. \qquad \ldots \text{(B-10)}$$

Since $\nu(t)$ vanishes at the two end-points due to condition (B-2), the second term of equation (B-10) drops out. Thus, this equation reduces to

$$\frac{\delta I_1}{\epsilon} = \frac{1}{\epsilon}\int_{t_0}^{t_e} \delta F_1 dt = \int_{t_0}^{t_e}\left[\frac{\partial F_1}{\partial q_1} - \frac{d}{dt}\frac{\partial F_1}{\partial \dot{q}_1}\right]\nu\, dt, \qquad \ldots \text{(B-11)}$$

which can be written as

$$\int_{t_0}^{t_e} E_1(t)\nu(t)dt = 0. \qquad \ldots \text{(B-12)}$$

According to the Fundamental Lemma of the Calculus of Variations,[71] if $E_1(t)$ is continuous in (t_0, t_e), and if $\nu(t)$ vanishes at t_0 and t_e and admits a continuous derivative in (t_0, t_e), then

$$E_1(t) = 0 \qquad \text{... (B-13)}$$

in (t_0, t_e). This leads to

$$E_1(t) = \frac{\partial F_1}{\partial q_1} - \frac{d}{dt}\frac{\partial F_1}{\partial \dot{q}_1} = 0, \qquad \text{... (B-14)}$$

a differential equation discovered by Euler (1744) and will be referred to as Euler's (differential) equation.*

The same result may be obtained for the general case of p (p>1) variables by selecting one definite q_u leaving the other variables unchanged, and repeating the above process. In this case, one obtains a system of simultaneous differential equations

$$E_u(t) = \frac{\partial F}{\partial q_u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{q}_u} = 0. \quad u = 1,\ldots,p \qquad \text{... (B-15)}$$

These equations are the first necessary condition for an extremum (i.e. minimum or maximum) for the unconditioned problem. Since the problem of the thesis is that of minimization, only the first necessary condition for a minimum will be considered in the following sections.

2. The Conditioned Problem and the First Necessary Condition for a Minimum

The process of finding, in general, the extreme value, and in this treatise, the minimum value of the integral I given by equation (B-1), will now be combined with two auxiliary

---

*Lanczos[73] calls this equation Euler-Lagrange's equation, Kneser and Hilbert call it Lagrange's equation, but Lagrange himself attributes it to Euler.[71]

conditions: (i) the conditions of the problem of Lagrange and (ii) the isoperimetric conditions. The variational problem of this type is called the conditioned problem.[71]

2.1 The Problem of Lagrange

The problem of section 1 is now modified such that the variables $q_1,\ldots,q_p$ and $\dot{q}_1,\ldots,\dot{q}_p$ are no longer independent, but restricted by the conditions

$$\emptyset_\alpha(q_1,\ldots,q_p,\dot{q}_1,\ldots,\dot{q}_p,t)=0. \quad \alpha=1,\ldots,a(a<p) \qquad \ldots \text{(B-16)}$$

If the curves of the family

$$q_u = q_u(t,e_1,\ldots,e_p) \quad u=1,\ldots,p \qquad \ldots \text{(B-17)}$$

pass through the points t_0 and t_e, satisfy equation (B-16), and contain, for the parameter value $e_u = 0$, the minimizing curve C_0:

$$q_u = q_u(t,0,\ldots,0), \quad u=1,\ldots,p \qquad \ldots \text{(B-18)}$$

then the function

$$I(e) = \int_{t_0}^{t_e} F\left[t, q_u(e_1,\ldots,e_p),\ \dot{q}_u(e_1,\ldots,e_p)\right] dt \qquad \ldots \text{(B-19)}$$

must have a minimum for all $e_u = 0$.[74] The arc C_0 must be an extremal, and according to the Euler-Lagrange Multiplier Rule,[72] there exists a set of function $\lambda_\alpha(t)$ where $\alpha=1,\ldots,a(a<p)$, such that if

$$K = F + \sum_{\alpha=1}^{a} \lambda_\alpha \emptyset_\alpha, \qquad \ldots \text{(B-20)}$$

then the derivative $I'(0)$ is expressible in the form[74]

$$I'(0) = \int_{t_0}^{t_e} \left[\sum_{u=1}^{p} \left(\Gamma_u \frac{\partial K}{\partial q_u} + \dot{\Gamma}_u \frac{\partial K}{\partial \dot{q}_u} \right) \right] dt, \qquad \ldots \text{(B-21)}$$

where the functions $\Gamma_u(t)$ are the variations of the family (B-17), defined by the relations

$$\left.\frac{\partial q_u(t,e_1,\ldots,e_p)}{\partial e_u}\right]_{e_u\to 0} = \Gamma_u(t). \qquad \ldots \text{(B-22)}$$

If the arc C_0 minimize the integral I, it is necessary that the first variation I'(0) given by equation (B-21) vanishes, from which the Euler-Lagrange Rule is obtained:[72,74,75]

If C_0 is a minimizing arc there exists a set of multipliers $\lambda_\alpha(t)$ such that at every point of C_0, the equations

$$\frac{\partial K}{\partial q_u} - \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_u} = 0 \qquad u=1,\ldots,p \qquad \ldots \text{(B-23)}$$

are satisfied, with K given by equation (B-20). Equations (B-23) are the first necessary conditions for a minimum of the Lagrange problem, and are analogous to equations (B-15) for the unconditioned problem of section 1.

2.2 The Isoperimetric Problem

Auxiliary conditions appearing in the form of definite integrals J_i which must have prescribed constant values B_i

$$\begin{aligned} J_1 &= \int_{t_0}^{t_e} G_1(q_1,\ldots,q_p,\dot{q}_1,\ldots,\dot{q}_p,t)\,dt = B_1, \\ &\vdots \\ J_n &= \int_{t_0}^{t_e} G_n(q_1,\ldots,q_p,\dot{q}_1,\ldots,\dot{q}_p,t)\,dt = B_n, \end{aligned} \qquad \ldots \text{(B-24)}$$

are called "isoperimetric" conditions.*

---

*This term is derived from the first historically recorded extremum problem of finding the maximum area bounded by a perimeter of a given length.[73]

Repeating the process of varying the integral as in the unconditioned problem, one obtains

$$\delta J_1 = \int_{t_0}^{t_e}\left(\frac{\partial G_1}{\partial q_1} - \frac{d}{dt}\frac{\partial G_1}{\partial \dot{q}_1}\right)\delta q_1 dt + \dots + \int_{t_0}^{t_e}\left(\frac{\partial G_1}{\partial q_p} - \frac{d}{dt}\frac{\partial G_1}{\partial \dot{q}_p}\right)\delta q_p dt = 0,$$
$$\vdots \qquad \dots \text{(B-25)}$$
$$\delta J_n = \int_{t_0}^{t_e}\left(\frac{\partial G_n}{\partial q_1} - \frac{d}{dt}\frac{\partial G_n}{\partial \dot{q}_1}\right)\delta q_1 dt + \dots + \int_{t_0}^{t_e}\left(\frac{\partial G_n}{\partial q_p} - \frac{d}{dt}\frac{\partial G_n}{\partial \dot{q}_p}\right)\delta q_p dt = 0.$$

Again, multiplying equations (B-25) by some undetermined constant λ_i and adding the result to δI, one obtains

$$\begin{aligned}\delta I' = &\int_{t_0}^{t_e}\left[\left(\frac{\partial F}{\partial q_1} - \frac{d}{dt}\frac{\partial F}{\partial \dot{q}_1}\right) + \lambda_1\left(\frac{\partial G_1}{\partial q_1} - \frac{d}{dt}\frac{\partial G_1}{\partial \dot{q}_1}\right) + \dots + \lambda_n\left(\frac{\partial G_n}{\partial q_n} + \right.\right.\\ &\left.\left. - \frac{d}{dt}\frac{\partial G_n}{\partial \dot{q}_1}\right)\right]\delta q_1 dt + \dots \quad + \dots\\ &+ \int_{t_0}^{t_e}\left[\left(\frac{\partial F}{\partial q_p} - \frac{d}{dt}\frac{\partial F}{\partial \dot{q}_p}\right) + \lambda_1\left(\frac{\partial G_1}{\partial q_p} - \frac{d}{dt}\frac{\partial G_1}{\partial \dot{q}_p}\right) + \dots + \lambda_n\left(\frac{\partial G_n}{\partial q_p} + \right.\right.\\ &\left.\left. - \frac{d}{dt}\frac{\partial G_n}{\partial \dot{q}_p}\right)\right]\delta q_p dt = 0,\end{aligned} \qquad \dots \text{(B-26)}$$

which can be written as

$$\begin{aligned}\delta I' = &\int_{t_0}^{t_e}\left[\frac{\partial}{\partial q_1}(F + \lambda_1 G_1 + \dots + \lambda_n G_n) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_1}(F + \lambda_1 G_1 + \dots + \lambda_n G_n)\right]\delta q_1 dt +\\ &+ \dots \quad + \dots\\ &+ \int_{t_0}^{t_e}\left[\frac{\partial}{\partial q_p}(F + \lambda_1 G_1 + \dots + \lambda_n G_n) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_p}(F + \lambda_1 G_1 + \dots + \lambda_n G_n)\right]\delta q_p dt = 0.\end{aligned}$$
$$\dots \text{(B-27)}$$

According to the Fundamental Lemma of section 1, the coefficients of δq_u must vanish, thus

$$\frac{\partial}{\partial q_u}(F + \lambda_1 G_1 + \dots + \lambda_n G_n) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_u}(F + \lambda_1 G_1 + \dots + \lambda_n G_n) = 0. \qquad u = 1, \dots, p$$
$$\dots \text{(B-28)}$$

The last two equations show that the isoperimetric problem can be transformed into a free variational problem with no auxiliary conditions, by changing the original F into a new function[72,73]

$$L = F + \sum_{i=1}^{n} \lambda_i \, G_i, \qquad \text{... (B-29)}$$

where λ_i is an undetermined constant.

2.3 A Combination of the Problem of Lagrange and the Isoperimetric Problem

All, except the first, necessary and sufficient conditions for the Lagrange problem are quite different in nature from the necessary and sufficient conditions for the isoperimetric problem. For this reason, the combined problem can only be solved if, either the isoperimetric problem is transformed into a Lagrange problem, or vice versa.

The first type of transformation is performed as follows:[75] Rewriting, for convenience, the isoperimetric conditions (B-25) as

$$J_i = \int_{t_0}^{t_e} G_i(t,q,\dot{q})dt = B_i, \qquad \text{... (B-30)}$$

where the set $(t,q_1, \ldots ,q_p,\dot{q}_1, \ldots ,\dot{q}_p)$ is now represented by $(t,q,\dot{q})$, one introduces new variables

$$Z_i(t) = \int_{t_0}^{t_e} G_i(t,q,\dot{q})\, dt. \qquad \text{... (B-31)}$$

The problem is now to find two sets of extremals

$$q = q(t) \text{ and} \qquad \text{... (B-32)}$$

$$Z_i = Z_i(t), \quad i = 1, \ldots ,n \qquad \text{... (B-33)}$$

satisfying the conditions

$$G_i(t,q,\dot{q}) - \frac{dZ_i}{dt} = 0,$$

$$q(t_0) = q_0, \; q(t_e) = q_e, \qquad \ldots \text{(B-34)}$$

$$Z_i(t_0) = 0, \; Z_i(t_e) = B_i,$$

one which minimizes I.

The function L' analogous to that given by equation (B-29), now has the form

$$L' = F + \sum_{i=1}^{n} \lambda_i \left(G_i - \frac{dZ_i}{dt} \right), \qquad \ldots \text{(B-35)}$$

and the differential equations determining the extremals are

$$\frac{\partial L'}{\partial q} - \frac{d}{dt} \frac{\partial L'}{\partial \dot{q}} = 0. \qquad \ldots \text{(B-36)}$$

The n equations

$$\frac{\partial L'}{\partial Z_i} - \frac{d}{dt} \frac{\partial L'}{\partial (dZ_i/dt)} = \frac{d\lambda_i}{dt} = 0 \qquad i=1,\ldots,n$$

$$\ldots \text{(B-37)}$$

show that in this case the multipliers λ_i are all constants.

A second type of transformation is also possible provided that each of the functions $\emptyset_\alpha$ given by equation (B-16) rewritten as

$$\emptyset_\alpha = \emptyset_\alpha(t,q,\dot{q}) = 0 \qquad \alpha=1,\ldots,a(a<p) \qquad \ldots \text{(B-38)}$$

can be integrated. Integrating equation (B-38) yields

$$\int_{t_0}^{t_e} \emptyset_\alpha(t,q,\dot{q})\, dt = 0, \; \alpha=1,\ldots,a(a<p) \qquad \ldots \text{(B-39)}$$

which is a simplified form of the isoperimetric condition.

Several criteria must be observed in making the choice of the first or the second type of transformation mentioned previously: (i) integrability of the functions $\emptyset_{\alpha}$, (ii) the number of isoperimetric conditions n compared to the number of Lagrange conditions a, and (iii) ease of application with regard to the various necessary and sufficient conditions. Since in the thesis problem n is generally larger than a, where a = 1, the second transformation from the Lagrange problem to the isoperimetric problem will be used. Furthermore, the functions for the special problem discussed here are definitely integrable, as they contain simple summation terms and functions which are identical to the isoperimetric conditions given by equations (B-24) or (B-30). Another advantage of the isoperimetric problem is the fact that it contains constant Lagrangian multipliers instead of the time-variable multipliers as in the case of the problem of Lagrange.

3. Other Necessary and Sufficient Conditions for a Minimum of the Isoperimetric Problem

3.1 Introduction

In line with the thesis problem conditions for minimization of integral I with isoperimetric conditions will be considered. For the sake of simplicity, the discussion will be limited to the two-variable case (p=2). The problem can now be reformulated as follows:

Minimize the integral*

$$I = \int_{t_0}^{t_e} F(x, y, x', y', t)\, dt \qquad \text{... (B-40)}$$

---

*The primes immediately following the variables x and y are derivatives with respect to t of those variables.

with isoperimetric conditions

$$J = \int_{t_0}^{t_e} G(x,y,x',y',t)\ dt = B \qquad \ldots \text{(B-41)}$$

and

$$\int_{t_0}^{t_e} \emptyset(x,y,x',y',t)\ dt = 0, \qquad \ldots \text{(B-42)}$$

replacing q_1, q_2, $\dot{q}_1$ and $\dot{q}_2$ by x, y, x' and y' respectively.

The first necessary conditions for a minimum of I defined by equation (B-40) with isoperimetric conditions (B-41) and (B-42) are given by equations

$$\begin{aligned} \frac{\partial H}{\partial x} - \frac{d}{dt}\frac{\partial H}{\partial x'} &= 0, \\ \frac{\partial H}{\partial y} - \frac{d}{dt}\frac{\partial H}{\partial y'} &= 0, \end{aligned} \qquad \ldots \text{(B-43)}$$

which are equivalent to the one differential equation*

$$\frac{\partial}{\partial y'}\ \frac{\partial H}{\partial x} - \frac{\partial}{\partial x'}\ \frac{\partial H}{\partial y} + H_1(x'y'' - x''y') = 0, \qquad \ldots \text{(B-44)}$$

where

$$H = F + \lambda_1 G + \lambda_2 \emptyset, \qquad \ldots \text{(B-45)}$$

and

$$\begin{aligned} H_1 &= \frac{1}{(y')^2}\frac{\partial}{\partial x'}\ \frac{\partial H}{\partial x'} \\ &= \frac{-1}{x'y'}\frac{\partial}{\partial x'}\ \frac{\partial H}{\partial y'} \\ &= \frac{1}{(x')^2}\frac{\partial}{\partial y'}\frac{\partial H}{\partial y'}. \end{aligned} \qquad \ldots \text{(B-46)}$$

The general solution of the differential equation (B-44) is

---

*Analogous to Bolza's[71,72] "pure" isoperimetric problem.

given by*

$$C_0: \quad \begin{aligned} x &= x(t,\alpha,\beta,\lambda_1,\lambda_2), \\ y &= y(t,\alpha,\beta,\lambda_1,\lambda_2), \end{aligned} \qquad \text{... (B-47)}$$

where α and β are the two constants of integration.** Equations (B-47) are the set of extremals for the thesis problem.

3.2 Necessary Conditions

(1) The second necessary condition for a minimum is that*

$$H_1 \geqq 0 \qquad \text{... (B-48)}$$

along the extremal C_0, defined by equation (B-47), which satisfies the boundary and auxiliary conditions, and where H_1 is given by equation (B-46). This is the analogue of Legendre's condition for the unconditioned problem.

(2) The third necessary condition for a minimum is given by Weierstrass's analogue of Jacobi's condition*

$$D(t,t_0) = \begin{vmatrix} \omega_1(t_0) & \omega_2(t_0) & \omega_3(t_0) & \omega_4(t_o) \\ \omega_1(t) & \omega_2(t) & \omega_3(t) & \omega_4(t) \\ \int_{t_0}^{t} U\omega_1 dt & \int_{t_0}^{t} U\omega_2 dt & \int_{t_0}^{t} U\omega_3 dt & \int_{t_0}^{t} U\omega_4 dt \\ \int_{t_0}^{t} V\omega_1 dt & \int_{t_0}^{t} V\omega_2 dt & \int_{t_0}^{t} V\omega_3 dt & \int_{t_0}^{t} V\omega_4 dt \end{vmatrix} \neq 0 \qquad \text{(B-49)}$$

for

$$t_0 < t < t_e,$$

* Analogous to Bolza's[71,72] "pure" isoperimetric problem.

**The general solutions of Euler equations with p variables are dependent on 2p-2 integration constants.

or

$$t_0' \geqq t_e \qquad \text{... (B-50)}$$

where t_0', called the conjugate point of t_0 , is the root next greater than t_0 of the equation

$$D(t,t_0) = 0. \qquad \text{... (B-51)}$$

The t-functions are given by the following relations:

$$\begin{aligned}
\omega_1(t) &= \frac{\partial y}{\partial t}\frac{\partial x}{\partial \alpha} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial \alpha} , \\
\omega_2(t) &= \frac{\partial y}{\partial t}\frac{\partial x}{\partial \beta} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial \beta} , \\
\omega_3(t) &= \frac{\partial y}{\partial t}\frac{\partial x}{\partial \lambda_1} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial \lambda_1} , \\
\omega_4(t) &= \frac{\partial y}{\partial t}\frac{\partial x}{\partial \lambda_2} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial \lambda_2} ,
\end{aligned} \qquad \text{... (B-52)}$$

$$U(t) = \frac{\partial}{\partial y'}\frac{\partial G}{\partial x} - \frac{\partial}{\partial x'}\frac{\partial G}{\partial y} + G_1\,(x'y'' - x''y') , \qquad \text{... (B-53)}$$

where

$$\begin{aligned}
G_1 &= \frac{1}{(y')^2}\frac{\partial}{\partial x'}\frac{\partial G}{\partial x'} \\
&= -\frac{1}{x'y'}\frac{\partial}{\partial x'}\frac{\partial G}{\partial y'} \\
&= \frac{1}{(x')^2}\frac{\partial}{\partial y'}\frac{\partial G}{\partial y'} ,
\end{aligned} \qquad \text{... (B-54)}$$

and

$$V(t) = \frac{\partial}{\partial y'}\frac{\partial \emptyset}{\partial x} - \frac{\partial}{\partial x'}\frac{\partial \emptyset}{\partial y} + \emptyset_1\,(x'y'' - x''y') , \qquad \text{... (B-55)}$$

where

$$\begin{aligned}
\emptyset_1 &= \frac{1}{(y')^2}\frac{\partial}{\partial x'}\frac{\partial \emptyset}{\partial x'} \\
&= -\frac{1}{x'y'}\frac{\partial}{\partial x'}\frac{\partial \emptyset}{\partial y'} \\
&= \frac{1}{(x')^2}\frac{\partial}{\partial y'}\frac{\partial \emptyset}{\partial y'} .
\end{aligned} \qquad \text{... (B-56)}$$

(3) The fourth necessary conditions for a minimum is given by Weierstrass' inequality*

$$E\ (x,\ y,\ x',\ y',\ \widetilde{x}',\ \widetilde{y}';\ \lambda_1,\ \lambda_2) \geqq 0\ , \qquad \ldots\ (B\text{-}57)$$

which must be fulfilled along the extremal C_0, defined by equation (B-47), for every direction $\widetilde{x}'$ and $\widetilde{y}'$, where*

$$E\ (x,\ y,\ x',\ y',\ \widetilde{x}',\ \widetilde{y}';\ \lambda_1,\ \lambda_2) =$$
$$H\ (x,\ y,\ \widetilde{x}',\ \widetilde{y}';\ \lambda_1,\ \lambda_2) - \Big[\widetilde{x}' \frac{\partial}{\partial x'} H(x,\ y,\ x',\ y';\ \lambda_1,\ \lambda_2) +$$
$$+ \widetilde{y}' \frac{\partial}{\partial y'} H\ (x,\ y,\ x',\ y';\ \lambda_1,\ \lambda_2)\Big]. \qquad \ldots\ (B\text{-}58)$$

3.3 Sufficient Conditions

The extremal C_0, defined by equation (B-47), furnishes a "semi-strong" minimum for the integral I given by equation (B-39) with auxiliary conditions (B-40) and (B-41), if the conditions

$$H_1 > 0\ , \qquad \ldots\ (B\text{-}59)$$

$$t_e < t_0'\ , \qquad \ldots\ (B\text{-}60)$$

$$E > 0 \qquad \ldots\ (B\text{-}61)$$

are fulfilled.[71,72] The above sufficient conditions are, therefore, exactly equivalent to the last three necessary conditions except for the equality sign.

---

*Analogous to Bolza's[71,72] "pure" isoperimetric problem.