THE APPLICATION OF LIE DERIVATIVES IN LAGRANGIAN MECHANICS FOR THE DEVELOPMENT OF A GENERAL HOLONOMIC THEORY OF ELECTRIC MACHINES

by

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ABSTRACT

A general approach to the treatment of electrical machine systems is developed. Tensor concepts are adopted; however, metrical ideas are avoided in favour of Hamilton's Principle. Using Lie derivatives and choosing a holonomic reference system, the equations resulting are general, and thus apply to any physical system of machines. These equations are Faraday's Law for the electrical portion and a gradient equation for the mechanical portion.

Transformation characteristics, which are found to be of two independent types, called the v-type and the i-type, are investigated. This leads to tensor character and invariance properties associated with the transformations.

The equations of small oscillation, which are based on the general equations of motion obtained in the thesis, are derived for any physical system.

In the final chapter two examples of application are given; the power selsyn system, and the synchronous machine.

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THE APPLICATION OF LIE DERIVATIVES IN LAGRANGIAN MECHANICS FOR THE DEVELOPMENT OF A GENERAL HOLONOMIC THEORY OF ELECTRIC MACHINES

1. INTRODUCTION

There still remain many important unsolved problems in the theory of electric machines and power systems; the analysis of machine and power system stability, the optimal control criteria, and especially those analyses involving non-linear effects due to saturation in the magnetic circuits or due to cross-product terms of currents or of current and speed.

An extension of static circuit theory to rotating machinery by G. Kron,[7] employing matrix and tensor analysis, provides a means for a general analysis approach. This extension deals principally with the addition of the rotating coil as a generalized inductive circuit element. However, the inclusion of mechanical motion in an electrical circuit introduces an unproportional amount of difficulty, mainly because of the non-linearity resulting.

Kron's adoption of tensors helped because of the generalization and unification achieved thereby. Even though his method of attack was complicated by the introduction of non-holonomiticity, it allowed him to deal with a general class of problems in a set manner. It appeared as though an extension to power system dynamics would readily follow as well as an inclusion of non-linearities caused by saturation and hysteresis. This, however, did not occur.

The main barrier to further development lies in the concept of non-holonomiticity and its implications. According to Kron, Faraday's Law is applicable solely to slip ring machines.[6] For commutator machines an additional term enters because of the independence of the reference frame and coil velocities. This also implies that Lagrange's equations are applicable only for these special cases (slip-ring frames), the more general Boltzmann-Hamel equations being needed for commutator machines. This is a very limiting restriction for it means that neither Lagrangian nor Hamiltonian mechanics can be applied directly to stability or optimization problems involving other than slip ring machines.

In the following chapter, Kron's equations are initially employed to obtain the equations of motion in a slip-ring frame. By defining a transformation from this frame to any general physical frame, it is shown that the equations of motion in the latter remain of the holonomic form. This is followed by a derivation of the general equations of motion using Hamilton's Principle. The generalized coordinates selected are readily seen to be independent proving that any physical machine is a holonomic electro-mechanical system.

In the succeeding chapters, loop equations in machine analysis, Lie derivative concepts of transformation theory, the concept of tensors, and the hunting equations are considered. Examples of application are included in the final chapter.

2. THE BASIC MACHINE EQUATIONS FOR ANY PHYSICAL COORDINATE SYSTEM

2.1 The Transformation of the Machine Equations From a Slip-Ring Coordinate System to Any General Physical Coordinate System

In the general case Kron obtained as the equations of motion for an electrical machine, a variation of the Boltzmann-Hamel equation,[9]

$$e_\gamma = R_{\gamma\alpha}\, i^\alpha + a_{\gamma\alpha} \frac{di^\alpha}{dt} + \Gamma_{\alpha\beta,\gamma}\, i^\alpha i^\beta \qquad (2\text{-}1)$$

where

$R_{\gamma\alpha}$ is the resistance and inertial damping tensor

$a_{\gamma\alpha}$ is the inductance and inertia tensor

$$\Gamma_{\alpha\beta,\gamma} = [\alpha\beta,\gamma] + [-\Omega_{\alpha\beta,\gamma} + \Omega_{\gamma\alpha,\beta} - \Omega_{\beta\gamma,\alpha}]$$

$$\Omega_{\alpha\beta,\gamma} = C^{\alpha'}_{\alpha}\, C^{\beta'}_{\beta}\, a_{\sigma\gamma}\, \partial_{[\beta'} C^{\sigma}_{\alpha']}$$

is the Christofel Symbol in a non-Riemannian space and $C^{\alpha'}_{\alpha}$, the coordinate transformation matrix from a holonomic frame to a general frame of reference.

The last term of (2-1) takes into consideration such effects as the rotation of conductors and coordinate reference frames.

The mechanical equations are included in (2-1).

Consider the case of a coil in which a voltage is induced by the influence of an excitation applied to the "excitation" coil as shown in Fig. 2A. Let m be the induction coil (coil in which a voltage is induced) and n the excitation coil. Also let $p\theta^m$, $p\theta^n$, and $p\theta$, respectively be, the speed of the m-coil commutator axis, the speed of the n-coil commutator axis, and the speed of the rotor, all being independent. It is proposed to determine the voltage in the m-coil by a transformation from the α–β system, which is fixed to the rotor.

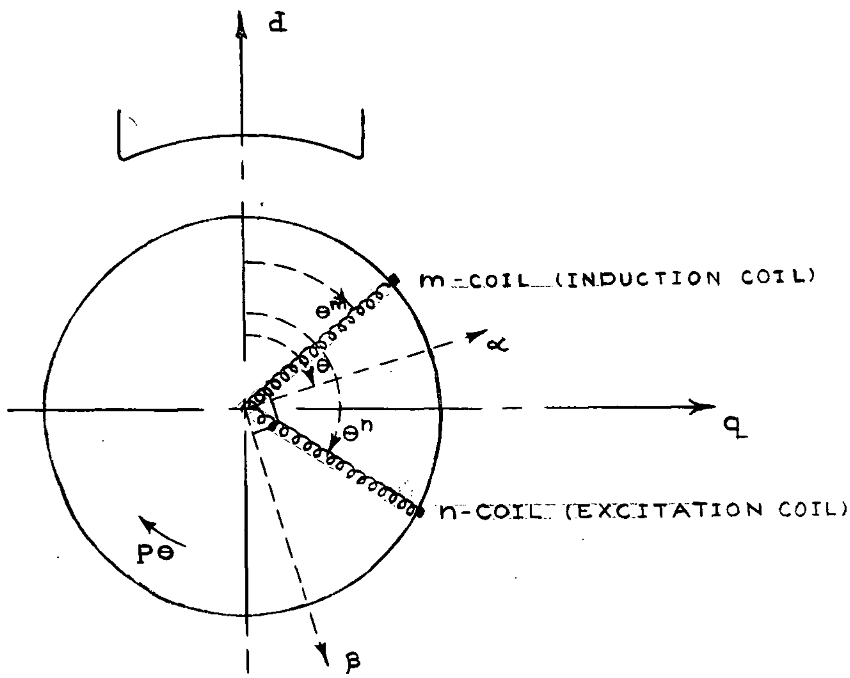


Fig. 2A <u>The Determination of the Voltage in an m-Coil by a Transformation From a Fixed α–β System, Fixed on the Rotor</u>

The instantaneous excitation effect in the m-coil due to the n-coil is independent of the coordinate frame to which the n-coil is referred. It depends only on the angular position and angular speed of the excitation axis (n-coil commutator axis). This implies that the excitation coil can be assumed to be situated in an instantaneous holonomic reference frame.

The equations of motion in the α–β frame, which is holonomic, are

$$e_\gamma = R_{\gamma\alpha} i^\alpha + a_{\gamma\alpha}\frac{di^\alpha}{dt} + [\alpha\beta,\gamma]\, i^\alpha i^\beta \qquad (2\text{-}2)$$

From this the voltage equations in the α and β axes are

$$e_s = R_{st} i^t + L_{st}\frac{di^t}{dt} + [tu,s]\, i^t i^u$$

$$= R_{sn} i^n + L_{sn}\frac{di^n}{dt} + \frac{\partial L_{sn}}{\partial \theta} p\theta\, i^n + \frac{\partial L_{sn}}{\partial \theta^n} p\theta^n i^n,$$

$$(s = \alpha,\beta) \qquad (2\text{-}3)$$

since the voltage in either axis is due to the excitation of the n-coil. Applying Yu's general inductance formula[17] for sinusoidal flux variation, that is

$$L_{ij} = L^d_{ij}\cos\theta^i\cos\theta^j + L^q_{ij}\sin\theta^i\sin\theta^j \qquad (2\text{-}4)$$

gives

$$L_{\alpha n} = L^d_{\alpha n}\cos\theta\cos\theta^n + L^q_{\alpha n}\sin\theta\sin\theta^n$$

$$L_{\beta n} = L^d_{\beta n}\cos(\theta+90^\circ)\cos\theta^n + L^q_{\beta n}\sin(\theta+90^\circ)\sin\theta^n.$$

Assuming $L^d_{\alpha n} = L^d_{\beta n} = L_d$ and $L^q_{\alpha n} = L^q_{\beta n} = L_q$, from (2-4)

$$e_\alpha = R_{\alpha n} i^n + (L_d\cos\theta\cos\theta^n + L_q\sin\theta\sin\theta^n)\frac{di^n}{dt}$$

$$+ (-L_d\sin\theta\cos\theta^n + L_q\cos\theta\sin\theta^n)p\theta\, i^n$$

$$+ (-L_d\cos\theta\sin\theta^n + L_q\sin\theta\cos\theta^n)p\theta^n i^n$$

$$e_\beta = R_{\beta n} i^n + (-L_d\sin\theta\cos\theta^n + L_q\cos\theta\sin\theta^n)\frac{di^n}{dt}$$

$$+ (-L_d\cos\theta\cos\theta^n - L_q\sin\theta\sin\theta^n)p\theta\, i^n$$

$$+ (L_d\sin\theta\sin\theta^n + L_q\cos\theta\cos\theta^n)p\theta^n i^n$$

$$(2\text{-}5)$$

The voltage in the m-coil can be obtained from the transformation

$$e_m = [\cos(\theta^m - \theta),\ \sin(\theta^m - \theta)] \begin{bmatrix} e_\alpha \\ e_\beta \end{bmatrix}$$

Using (2-5), this gives

$$e_m = (R_{\alpha\alpha} + R_{\beta\beta})i^m + (L_d \cos\theta^n \cos\theta^m + L_q \sin\theta^m \sin\theta^n)\frac{di^n}{dt}$$
$$+ (-L_d \sin\theta^m \cos\theta^n + L_q \cos\theta^m \sin\theta^n)p\theta\, i^n$$
$$+ (-L_d \cos\theta^m \sin\theta^n + L_q \sin\theta^m \cos\theta^n)p\theta^n i^n .$$

This is simply

$$e_m = R_{mn}i^n + L_{mn}\frac{di^n}{dt} + \frac{\partial L_{mn}}{\partial\theta^m}i^n p\theta + \frac{\partial L_{mn}}{\partial\theta^n}i^n p\theta^n .$$

If θ^m is considered as the angle of the induction coil, $p\theta^m$ the speed of the space in which this coil is immersed this becomes the holonomic equation of motion.

$$e_m = R_{mn}i^n + \frac{d\Phi_m}{dt}$$

This result suggests the possibility of associating two mechanical variables with each coil, thereby reducing the general machine equations to a holonomic form. The consequence of this would be a simplification of the tedious work that was previously carried out in the form of transformations from primitive machines. Indeed, the basic primitive machines could be discarded or replaced by a more useful model as pointed out by Dr. Yu.

A similar transformation of the torque equation to general coordinates will now be carried out. First, consider the torque equation in the α–β frame of Fig. 2A. From (2-3)

$$[\alpha\beta,\gamma]\, i^{\alpha} i^{\beta} = [\alpha\beta, u]\; i^{\alpha} i^{\beta}$$

where α and β refer to electrical coordinates and u refers to a mechanical coordinate. Signifying this term by $-t_u$, the negative of the electromagnetic torque along the u axis;

$$t_u = \frac{1}{2} \frac{\partial L_{\alpha\beta}}{\partial x^u} i^{\alpha} i^{\beta} \text{ .}$$

In most cases it is desired to find the torque associated with the rotor. Thus $x^u = \theta$ in any reference frame and

$$t = \frac{1}{2} \frac{\partial L_{\alpha\beta}}{\partial \theta} i^{\alpha} i^{\beta} \text{ .}$$

Expanding the two summations in t for Fig. 2A

$$t = \frac{1}{2} \left(\frac{\partial L_{\alpha\alpha}}{\partial \theta} i^{\alpha} i^{\alpha} + 2 \frac{\partial L_{\alpha\beta}}{\partial \theta} i^{\alpha} i^{\beta} + \frac{\partial L_{\beta\beta}}{\partial \theta} i^{\beta} i^{\beta} \right) \tag{2-6}$$

since

$$L_{\alpha\beta} = L_{\beta\alpha} \text{ and } i^{\alpha} i^{\beta} \left(\frac{\partial L_{\alpha\beta}}{\partial \theta} + \frac{\partial L_{\beta\alpha}}{\partial \theta} \right) = 2\, i^{\alpha} i^{\beta} \frac{\partial L_{\alpha\beta}}{\partial \theta} \text{ .}$$

Resolving both i^m and i^n (Fig. 2A) into α–β components gives

$$\begin{bmatrix} i^{\alpha} \\ i^{\beta} \end{bmatrix} = \begin{bmatrix} \cos(\theta^m - \theta) \;, & \cos(\theta^n - \theta) \\ \sin(\theta^m - \theta) \;, & \sin(\theta^n - \theta) \end{bmatrix} \begin{bmatrix} i^m \\ i^n \end{bmatrix} \tag{2-7}$$

The inductances and their derivatives are

$$L_{\alpha\alpha} = L_d \cos^2\theta + L_q \sin^2\theta$$

$$L_{\alpha\beta} = -L_d \cos\theta \sin\theta + L_q \sin\theta \cos\theta$$

$$L_{\beta\beta} = L_d \sin^2\theta + L_q \cos^2\theta$$

$$\frac{\partial L_{\alpha\beta}}{\partial\theta} = -L_d(\cos^2\theta - \sin^2\theta) + L_q(\cos^2\theta - \sin^2\theta)$$

$$\frac{\partial L_{\beta\beta}}{\partial\theta} = 2(L_d \sin\theta \cos\theta - L_q \cos\theta \sin\theta)$$

$$\frac{\partial L_{\alpha\alpha}}{\partial\theta} = 2(-L_d \cos\theta \sin\theta + L_q \sin\theta \cos\theta)$$

The torque t in the α–β frame is

$$\begin{aligned} t = {} & (-L_d \cos\theta \sin\theta + L_q \sin\theta \cos\theta)\, i^\alpha i^\alpha \\ & +(-L_d \cos 2\theta + L_q \cos 2\theta)\, i^\alpha i^\beta \\ & +(L_d \sin 2\theta - L_q \sin 2\theta)\, i^\beta i^\beta \end{aligned}$$

Using the transformation equations (2-7)

$$\begin{aligned} i^\alpha i^\alpha = {} & \cos^2(\theta - \theta^m) i^m i^m + 2\cos(\theta - \theta^m)\cos(\theta^n - \theta) i^m i^n \\ & + \cos^2(\theta - \theta^n)\, i^n i^n \\ i^\alpha i^\beta = {} & \cos(\theta - \theta^m) \sin(\theta^m - \theta) i^m i^m + (\cos(\theta^m - \theta)\sin(\theta^n - \theta) \\ & + \cos(\theta^n - \theta)\sin(\theta^m - \theta))\, i^m i^n + \\ & \qquad + \cos(\theta^n - \theta)\sin(\theta^n - \theta)\, i^n i^n \\ i^\beta i^\beta = {} & \sin^2(\theta^m - \theta) i^m i^m + 2\sin(\theta^m - \theta)\sin(\theta^n - \theta) i^m i^n \\ & + \sin^2(\theta^n - \theta) i^n i^n . \end{aligned}$$

Hence

$$t = (L_d \sin \theta \cos \theta - L_q \sin \theta \cos \theta)(i^\beta i^\beta - i^\alpha i^\alpha)$$
$$+ (\cos^2\theta - \sin^2\theta)(L_d - L_q)\ i^\alpha i^\beta$$

But

$$i^\beta i^\beta - i^\alpha i^\alpha = (\sin^2(\theta^m - \theta) - \cos^2(\theta^m - \theta))i^m i^m$$
$$-2(\cos(2\theta - \theta^m - \theta^n))i^m i^n + (\sin^2(\theta^n - \theta) - \cos^2(\theta^n - \theta))i^n i^n$$

Thus

$$t = (L_d - L_q)(\frac{\sin 2\theta}{2} (i^\beta i^\beta - i^\alpha i^\alpha) - \cos 2\theta(i^\alpha i^\beta))$$
$$= (L_d - L_q) \quad \frac{\sin 2\theta}{2} (-\cos 2(\theta^m - \theta)i^m i^n - 2\cos(2\theta - \theta^m - \theta^n)i^m i^n$$
$$- \cos 2(\theta^n - \theta)i^n i^n) - \frac{\cos 2\theta}{2} (\sin 2(\theta^m - \theta)\ i^m i^m +$$
$$2 \sin (\theta^m + \theta^n - 2\theta)i^m i^n + \sin 2(\theta^n - \theta)\ i^n i^n)$$

Expanding the double angle arguments and collecting terms results in

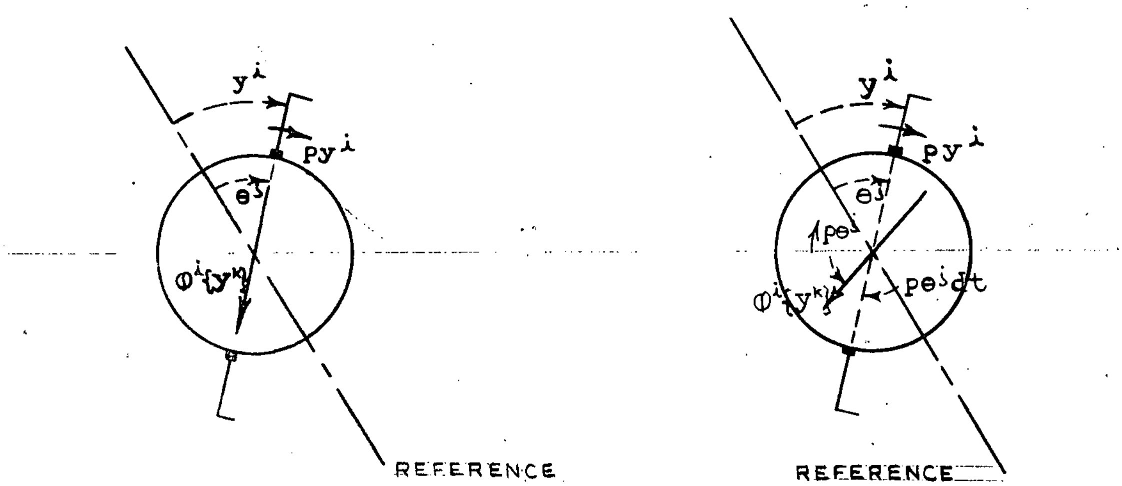
$$t = (L_q - L_q) [\sin\theta^m \cos\theta^m i^m i^n + (\sin\theta^m \cos\theta^n + \cos\theta^m \sin\theta^n)$$
$$\times i^m i^n + \sin\theta^n \cos\theta^n i^n i^n]$$
$$= G_{mn}\ i^m i^n \qquad \text{where } G_{mn} = \frac{\partial L_{mn}}{\partial \theta^m} \tag{2-8}$$

Two important results are observed: 1) The Christofel transformation used by Kron was no more than a mathematical tool for selecting θ^m for differentiation, 2) G_{mn} is simply the coefficient of coil angular speed $p\theta^m$ in the inductive impedance matrix.

2.2 The Excitation and Induction Angles of a Coil Winding

A coil is an array of interconnected wires wound in some space, which can be rotating with respect to a chosen inertial frame. A coil winding consists of the coil and the commutator brushes from which effects are observed or introduced. A slip ring machine can be regarded as a special case occurring when the angular speed of the commutator brushes is equal to the angular speed of the rotor.

Consider a rotor with excitations applied directly to its coils (i.e., applied through slip rings). This sets up a flux pattern and in particular a flux $\Phi_i\{y^k\}$ threading the i-th coil winding, the rotor being at standstill. y^i is the angular position of the i-th coil winding commutator axes with respect to a fixed reference. y^i is allowed to rotate over the entire rotor (Fig. 2B). θ^i is the reference angle of the rotor and $p\theta^j$ the rotor speed.



(a) Rotor at Standstill (b) Virtual Rotation

Fig. 2B Incremental Flux Changes

Let a virtual rotation be given to the rotor due to a virtual angular velocity acting for dt seconds. Since the excitations are connected directly to the rotor, the flux pattern rotates by $p\theta^{j}\,dt$ and the flux vector found at y^{i} for the still rotor will now be found at $(y^{i} + \eta^{i}_{j}\,p\theta^{j}dt)$; $\eta^{i}_{j} = 1$. Moreover, the flux measured at the latter point transformed to the coordinate system given by

$$s^{\bar{i}} = \delta^{\bar{i}}_{i}(s^{i} - \eta^{i}_{j}\,p\theta^{j}dt) \qquad (2\text{-}9)$$

where

$$s^{i} = y^{i} + \eta^{i}_{j}\,p\theta^{j}\,dt \qquad (2\text{-}10)$$

will be equal to the flux at y in the still machine since

$$s^{\bar{i}} = \delta^{\bar{i}}_{i}\,y^{i} \qquad (2\text{-}11)$$

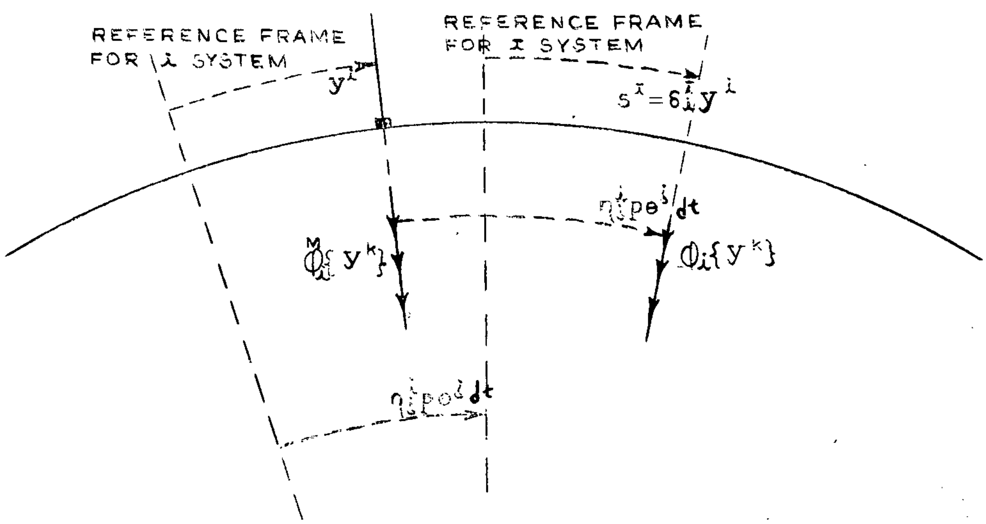


Fig. 2C <u>Virtual Rotation</u>

Denoting $\overset{M}{\Phi}_{i}$ as the flux in the moving machine

$$\overset{M}{\Phi}_{\bar{i}}\{y^{i} + (\eta^{i}_{j}\,p\theta^{j})dt\} = \Phi_{i}\{y^{i}\}\,\delta^{i}_{\bar{i}}. \qquad (2\text{-}12)$$

From (2-9), the coordinate transformation,

$$A_{\jmath}^{\bar{\imath}} = \frac{\partial s^{\bar{\imath}}}{\partial s^{\jmath}} = \delta_{i}^{\bar{\imath}} \left(A_{\jmath}^{i} - \frac{\partial}{\partial s^{\jmath}} (\eta_{\ell}^{i} \; p\theta^{\ell}) dt \right)$$

$$= \delta_{i}^{\bar{\imath}} A_{\jmath}^{i} = \delta_{i}^{\bar{\imath}} \delta_{\jmath}^{i} = \delta_{\jmath}^{\bar{\imath}} \tag{2-13}$$

and thus

$$\overset{M}{\Phi}_{i}\{y^{i} + (\eta_{\jmath}^{i} p\theta^{\jmath}) dt\} = \Phi_{i}\{y^{i}\} \tag{2-14}$$

Taking a linear approximation to the Taylor expansion of the left hand side

$$\overset{M}{\Phi}_{i}\{y^{i}\} + \frac{\partial \overset{M}{\Phi}_{i}}{\partial y^{i}} (\eta_{\jmath}^{i} p\theta^{\jmath}) dt = \Phi_{i}\{y^{i}\} \tag{2-15}$$

The incremental change in the observed flux due to rotor rotation is thus

$$+ \left(\overset{M}{\Phi}_{i}\{y^{i}\} - \Phi_{i}\{y^{i}\} \right) = - \frac{\partial \overset{M}{\Phi}_{i}}{\partial y^{i}} (\eta_{\jmath}^{i} p\theta^{\jmath}) dt \tag{2-16}$$

which is the Lie differential of $\overset{M}{\Phi}_{i}\{y^{k}\}$ over $\eta_{\jmath}^{i} p\theta^{\jmath}$.[11,12] The coordinate system $\bar{\imath}$ is dragged along by the commutator system i under the point transformation

$$s^{i} = y^{i} + (\eta_{\jmath}^{i} p\theta^{\jmath}) dt \tag{2-17}$$

By writing (2-12) as

$$\overset{M}{\Phi}_{i}\{y^{i}\} = \Phi_{i}\{y^{i} - (\eta_{\jmath}^{i} \; p\theta^{\jmath}) dt\}$$

this is also equal to

$$\left(\overset{M}{\Phi}_{i}\{y^{i}\} - \Phi_{i}\{y^{i}\} \right) = - \frac{\partial \Phi_{i}}{\partial y^{i}} (\eta_{\jmath}^{i} p\theta^{\jmath}) dt$$

the negative of the Lie differential of Φ_i over $\eta^i_s p\theta^s$.

The flux will similarly be caused to rotate if the rotor is stationary and the axes of excitation are allowed to rotate. Using standard notation this contribution is

$$\underset{py^s}{\mathfrak{L}}\overset{M}{\Phi}_i\{y^k\} = \lim_{dt \to 0}\left(\frac{\overset{M}{\Phi}_i\{y^k\} - \Phi_i\{y^k\}}{dt}\right)$$

$$= -\frac{\partial \overset{M}{\Phi}_i}{\partial y^s}\, py^s$$

$$= -\frac{\partial \Phi_i}{\partial y^s}\, py^s \tag{2-18}$$

Besides angles the flux is a function of current. As a consequence, the sum of the time and Lie variations must be balanced by current variations. That is

$$\frac{d\overset{M}{\Phi}_i\{y^k, i^l\}}{dt} + \underset{py^s}{\mathfrak{L}}\overset{M}{\Phi}_i\{y^k, i^l\} + \underset{\eta^i_\beta p\theta^\beta}{\mathfrak{L}}\overset{M}{\Phi}_i\{y^k, i^l\} =$$

$$= \frac{\partial \overset{M}{\Phi}_i}{\partial i^l}\frac{di^l}{dt} \quad . \tag{2-19}$$

Thus

$$\frac{d\overset{M}{\Phi}_i}{dt} = \left(\frac{\partial \overset{M}{\Phi}_i}{\partial i^l}\bigg|_{y^k, \theta^k}\right)\frac{di^l}{dt} + \frac{\partial \overset{M}{\Phi}_i}{\partial y^s}\frac{dy^s}{dt} + \frac{\partial \overset{M}{\Phi}_i}{\partial y^i}\eta^i_s\frac{d\theta^s}{dt} \tag{2-20}$$

$$= \frac{\partial \Phi_i}{\partial i^l}\frac{di^l}{dt} + \frac{\partial \Phi_i}{\partial y^s}\frac{dy^s}{dt} + \frac{\partial \Phi_i}{\partial y^i}\eta^i_s\frac{d\theta^s}{dt} \tag{2-21}$$

This can also be written as

$$\frac{d\overset{M}{\Phi}}{dt} = \frac{\partial \Phi_i}{\partial i^l}\frac{di^l}{dt} + \left(\underset{py^i}{\mathfrak{L}}\Phi_i + \underset{\eta^i_s p\theta^s}{\mathfrak{L}}\Phi_i\right) \quad . \tag{2-22}$$

The last term, due to the speed of the space in which the i-th coil is wound, involves differentiation with respect to the i-th commutator axis angle only and is independent of all commutator axes speeds. It has the characteristics of an inductive type of voltage. The commutator axis angle y^i used in this sense will be referred to as the induction angle of the i-th coil winding and will be relabelled x^i in any equations in which it occurs. The contribution $-\underset{\eta^i_i p\theta^i}{\mathfrak{L}}\overset{M}{\Phi}_i$ can then be written

$$+\frac{\partial \Phi_i}{\partial x^i} p x^i, \text{ where } p x^i \triangleq \frac{\delta x^i}{dt} \triangleq \eta^i_j \frac{d\theta^j}{dt} \tag{2-23}$$

The second term on the right hand side of (2-22) involves differentiation over all commutator axes angles and depends on commutator axes angular speeds. It is independent of all coil speeds. "y^i" used in this sense will be referred to as the excitation angle of the i-th coil winding since differentiation with respect to it gives rise to the angular speed of the excitation applied to the i'th coil winding. "y^i" will not be relabelled under these conditions.

Using (2-23), equation (2-22) can be written in a chain rule form

$$\frac{d\overset{M}{\Phi}_i}{dt} = \frac{\partial \Phi_i}{\partial i^j}\frac{di^j}{dt} + \frac{\partial \Phi_i}{\partial y^j}\frac{dy^j}{dt} + \frac{\partial \Phi_i}{\partial x^j}\frac{\delta x^j}{dt} \tag{2-24}$$

We shall now apply the above results to Yu's formula,[17,18] which is of basic importance in machine analysis. In Fig. 2D let the m-coil be wound in a space with angular speed $p\theta^j$. If the n coil is excited the flux in the m-coil is given by

$$\Phi_m = L_{mn} i^n = (L^d_{mn} \cos y^m \cos y^n + L^q_{mn} \sin y^m \sin y^n) i^n \tag{2-25}$$

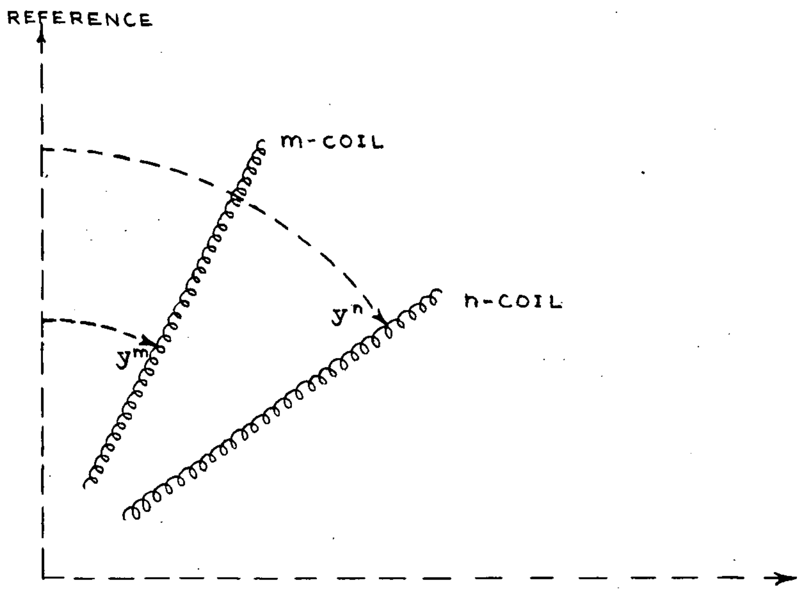


Fig. 2D Two General Machine Coil Windings

where y^m and y^n are the commutator axis angles of the m and n-coils respectively. From (2-21)

$$\frac{d\Phi_m}{dt} = (L^d_{mn} \cos y^m \cos y^n + L^q_{mn} \sin y^m \sin y^n) \frac{di^n}{dt}$$

$$+ (-L^d_{mn} \cos y^m \sin y^n + L^q_{mn} \sin y^m \cos y^n) \frac{dy}{dt} i^n$$

$$+ (-L^d_{mn} \sin y^m \cos y^n + L^q_{mn} \cos y^m \sin y^n)(\eta^m_j \frac{d\theta^j}{dt}) i^n$$

(2-26)

By the previous discussion, y^n is the excitation angle of the n-coil and y^m the induction angle of the m-coil, which is denoted by x^m rather than y^m, (2-25) and (2-26) then become

$$\Phi_m = L_{mn} i^n = (L^d_{mn} \cos x^m \cos y^n + L^q_{mn} \sin x^m \sin y^n) i^n$$

(2-27)

$$\frac{d\Phi_m}{dt} = \frac{\partial \Phi_m}{\partial i^n}\frac{di^n}{dt} + \frac{\partial \Phi_m}{\partial y^n}\frac{dy^n}{dt} + \frac{\partial \Phi_m}{\partial x^m}\left(\eta^m_j \frac{d\theta^j}{dt}\right) \qquad (2\text{-}28)$$

As a special case the n-coil can coincide with the m-coil, y^n becoming y^m which is numerically equal to x^m. However, the operators $\frac{dy^m}{dt}\frac{\partial}{\partial y^m}$ and $\frac{\delta x^m}{dt}\frac{\partial}{\partial x^m}$ are equal only for a slip ring machine.

From (2-23) for a number of coil windings

$$px^m = \frac{\delta x^m}{dt} = \eta^m_j \frac{d\theta^j}{dt} \qquad (2\text{-}29)$$

Here $\frac{\delta x^m}{dt}$ is the speed of the coil comprising the m-th coil winding and $\frac{d\theta^j}{dt}$ that of the j-th space. The quantity connecting these two speeds will be called the induction angle incidence matrix. For the example of Fig. 2E it is given by

$$\begin{bmatrix} px^a \\ px^b \\ px^c \\ px^d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p\theta^s \\ p\theta^r \end{bmatrix}; \quad (\eta) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(\eta)_s = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad (\eta)_r = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

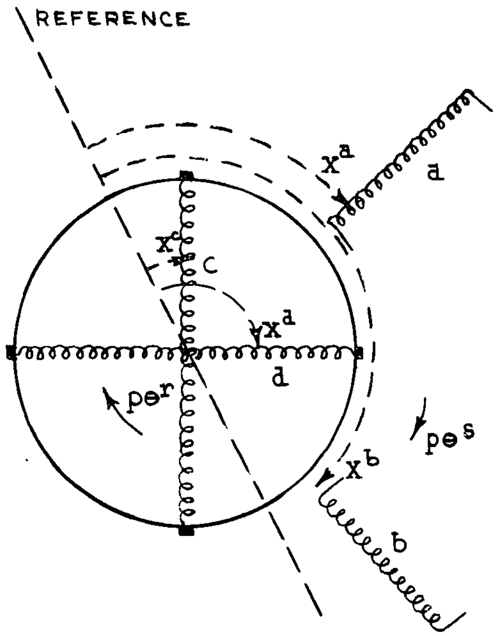


Fig. 2E The Induction Angle

2.3 The Lagrangian Function and The Equations of Motion

The Lagrangian for any electrical machine is given by

$$L = T = (\text{Kinetic Energy Function}) \qquad (2\text{-}30)$$

omitting capacitive effects within the machine. The kinetic energy is the sum of the mechanical energy and the stored magnetic energy.

In all subsequent equations the following notation will be used:

a) Greek letters refer to mechanical quantities

$J_{\alpha\alpha}$, the inertia of the α'th space

$D_{\alpha\alpha}$, the damping coefficient of the α'th space

$p\theta^{\alpha}$, the angular speed of the α'th space

b) Roman letters refer to electrical quantities

L_{mn}, the mutual inductance from the n-coil to the m-coil

R_{mm}, the resistance of the m-coil

i^m, the current in the m-coil

c) x and y angles, which arise in both mechanical and electrical equations, will be indexed with i, j, or k unless they take specific indices.

In order to determine the stored magnetic energy the coil currents are raised in succession, the resulting energy contributions are calculated and the individual contributions summed. The order of the currents can be chosen arbitrarily and enumerated as $(i^1, i^2 \ldots i^n)$. The total stored energy is then

$$E_e = \sum_{i=1}^{n} (\Delta E_{ei}) = \int_0^{i^1} \overset{M}{\Phi}_1(\alpha^1)\, d\alpha^1 + \int_0^{i^2} \overset{M}{\Phi}_2(i^1, \alpha^2)\, d\alpha^2 + \cdots + \int_0^{i^n} \overset{M}{\Phi}_n(i^1 \ldots i^{n-1}, \alpha^n)\, d\alpha^n$$

$$= \int_0^{i^\ell} \overset{M}{\Phi}_\ell(i^1 \ldots i^{\ell-1}, \alpha^\ell)\, d\alpha^\ell \qquad (2\text{-}31)$$

using the summation convention and α^ℓ as the dummy variable of integration.

The mechanical energy in the system is

$$E_m = \frac{1}{2} J_{\alpha\alpha}\, p\theta^\alpha\, p\theta^\alpha \qquad (2\text{-}32)$$

and hence the Lagrangian is

$$L = \int_0^{i^\ell} \Phi^{M}_{\ell}(i^1 \ldots i^{\ell-1}, \alpha^\ell)d\alpha^\ell + \frac{1}{2} J_{\alpha\alpha} p\theta^\alpha \, p\theta^\alpha \quad (2\text{-}33)$$

The state of any physically realizable machine is completely described by (q^i, θ^i, y^i): q^i being the total charge passing a reference point on the i'th coil winding lead; θ^i, the angular position of the i'th space with respect to a chosen reference; and y^i, the angular position of the i'th coil winding with respect to a chosen reference. Moreover any "displacement" $(\Delta q^i, \Delta\theta^i, \Delta y^i)$ is a possible "displacement" of the state and hence the description is holonomic and (q^i, θ^i, y^i) is a true "coordinate" system[14]. Consequently, any physically realizable machine is a true electro-mechanical system and Hamilton's Principle applies without subsidiary conditions.

Since $E_e = E_e\{i^1 \ldots i^n, y^1 \ldots y^\ell, x^1 \ldots x^\ell\}$ and $E_m = E_m\{p\theta^\beta\}$, the variation in the action, $A = \int_0^t (L)dt$, due to the independent variations $(\Delta q^i, \Delta\theta^i, \Delta y^i)$ is

$$\Delta A = \int_0^t (\Delta E_e + \Delta E_m)dt$$

$$= \int_0^t \Big(\frac{\partial E_e}{\partial q^n} \Delta q^n + \frac{\partial E_e}{\partial i^n} \Delta i^n + \mathcal{L}_{\eta^i_\alpha \Delta\theta^\alpha} E_e + \mathcal{L}_{\Delta y^i} E_e + \frac{\partial E_m}{\partial \theta^\beta} \Delta\theta^\beta$$

$$+ \frac{\partial E_m}{\partial p\theta^\beta} \Delta(p\theta^\beta)\Big)dt$$

$$= \int_0^t \Big(\frac{\partial E_e}{\partial i^n} \Delta i^n + \mathcal{L}_{\eta^i_\alpha \Delta\theta^\alpha} E + \mathcal{L}_{\Delta y^i} E_e + \frac{\partial E_m}{\partial p\theta^\beta} \Delta \, p\theta^\beta \Big) dt \quad (2\text{-}34)$$

The two Lie variations enter through the flux function (Eqn. (2-31)) and hence from Eqn. (2-16) and (2-18) are equal to

$$\mathcal{L}_{\eta^i_\alpha \Delta\theta^\alpha} E_e = -\frac{\partial E_e}{\partial x^i} \eta^i_\alpha \Delta\theta^\alpha \quad (2\text{-}35)$$

$$\mathcal{L}_{\Delta y^i} E_e = -\frac{\partial E_e}{\partial y^i} \Delta y^i$$

Substituting these into (2-34) and integrating the first and last terms by parts maintaining fixed end points these results:

$$\Delta A = \int_0^t \left(-\frac{d}{dt}\left(\frac{\partial E_e}{\partial i^n}\right)\Delta q^n + \left(-\frac{d}{dt}\frac{\partial E_m}{\partial p\theta^\beta} - \eta^i_\beta \frac{\partial E}{\partial x^i}\right)\Delta\theta^\beta - \frac{\partial E_e}{\partial y^i}\Delta y^i\right)dt \tag{2-36}$$

By Hamilton's Principle this is zero. Since Δq^n, $\Delta\theta^\beta$ and Δy^i are independent the equations of motion including driving forces and dissipation are:

$$e_n = \frac{d}{dt}\frac{\partial E_e}{\partial i^n} + R_{nn}\, i^n \tag{2-37}$$

$$S_\beta = \frac{d}{dt}\frac{\partial E_m}{\partial p\theta^\beta} + D_{\beta\beta}\, p\theta^\beta + \frac{\partial E_e}{\partial x^i}\eta^i_\beta \tag{2-38}$$

$$r_i = \frac{\partial E_e}{\partial y^i} \tag{2-39}$$

The torque r_i arising from the dependence of E_e on the reference axis angle y^i acts on the i'th coil and hence on the space β if η^i_β does not equal zero. t_β the externally applied torque on the β-space is

$$t_\beta = S_\beta + r_i\,\eta^i_\beta \tag{2-40}$$

Since in (2-31) the n-coil can be chosen as last referred to in the summation

$$\frac{\partial E_e}{\partial pq^n} = \frac{\partial}{\partial i^n}\int_0^{i^n} \breve{\Phi}_n\{i^1 \ldots i^{n-1}\alpha^n\}d\alpha^n$$

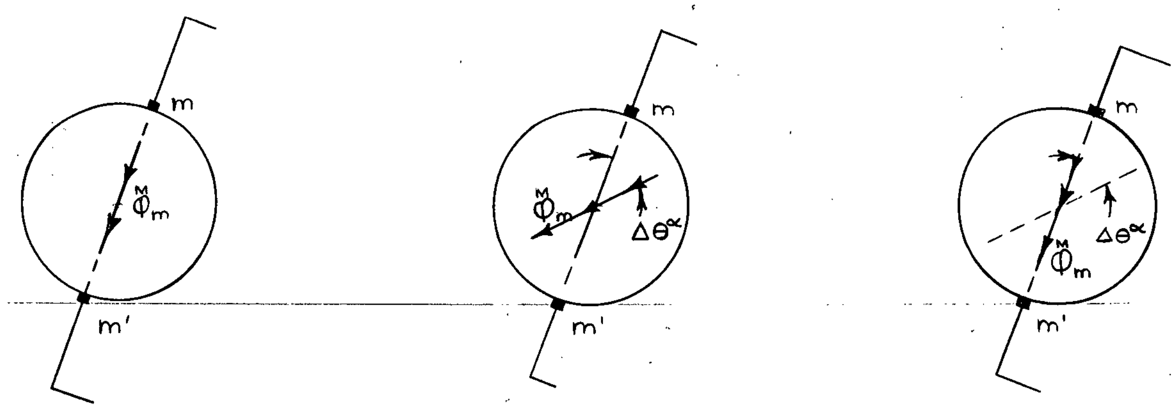
$$= \breve{\Phi}_n\{i^1 \ldots i^n\} \tag{2-41}$$

The three equations (2-34), (2-38) and (2-39) are thus reduced to a system of voltage equations and a system of torque equations.

$$e_n = R_{nn}\, i^n + \frac{d\breve{\Phi}_n}{dt} \tag{2-42}$$

$$t_\beta = D_{\beta\beta}\, p\theta^\beta + J_{\beta\beta}\, p(p\theta^\beta) + \left(\frac{\partial E_e}{\partial y^i} + \frac{\partial E_e}{\partial x^i}\right)\eta^i_\beta \quad (2\text{-}43)$$

The torque equation, (2-43), can be simplified as follows,



a) Before Rotation   b) Rotor Virtual Rotation   c) Excitation Virtual Rotation

Fig. 2F <u>Virtual Rotations</u>

Referring to section 2.2, if the rotor is given a virtual rotation $\Delta\theta$ from the steady state value the Lie variation of flux threading the m coil winding axis is

$$\underset{\eta^i_\alpha \Delta\theta^\alpha}{\mathcal{L}} \overset{M}{\Phi}_m = -\frac{\partial \overset{M}{\Phi}_m}{\partial x^i}\eta^i_\alpha \Delta\theta^\alpha = -\frac{\partial \Phi_m}{\partial x^i}\eta^i_\alpha \Delta\theta^\alpha \qquad (2\text{-}44)$$

the flux having rotated by an additional $\Delta\theta^\alpha$. If on the other hand the rotor excitation axes alone are given a virtual rotation, the i'th axis being rotated by Δy^i, the flux change is

$$\underset{\Delta y^i}{\mathcal{L}} \overset{M}{\Phi}_m = -\frac{\partial \overset{M}{\Phi}_m}{\partial y}\Delta y^i = -\frac{\partial \Phi_m}{\partial y^i}\Delta y^i \qquad (2\text{-}45)$$

By making $\Delta y^{i} = -\eta^{i}_{\alpha}\Delta\theta^{\alpha}$, that is, rotating all rotor excitation coordinates by $-\Delta\theta^{\alpha}$ this change is $+\frac{\partial \Phi_m}{\partial y^{i}}\eta^{i}_{\alpha}\Delta\theta^{\alpha}$. From Fig. 4F(c) the sum of these two flux changes is zero and thus

$$\frac{\partial \overset{M}{\Phi}_m}{\partial x^{i}}\eta^{i}_{\alpha} = \frac{\partial \Phi_m}{\partial x^{i}}\eta^{i}_{\alpha} = \frac{\partial \overset{M}{\Phi}_m}{\partial y^{i}}\eta^{i}_{\alpha} = \frac{\partial \Phi_m}{\partial y^{i}}\eta^{i}_{\alpha} \tag{2-46}$$

Since the derivatives of (2-43) pass under the integrals (2-33) for suitable conditions,

$$\frac{\partial E}{\partial y^{i}}\eta^{i}_{\alpha} = \frac{\partial E}{\partial x^{i}}\eta^{i}_{\alpha} \tag{2-47}$$

The final equations of motion are then

$$\begin{aligned} e_m &= R_{mm} i^m + \frac{d\overset{M}{\Phi}_m}{dt} \\ &= R_{mm} i^m + \frac{\partial \overset{M}{\Phi}_m}{\partial i^n}\frac{di^n}{dt} + \frac{\partial \Phi_m}{\partial x^{i}}\frac{\delta x^{i}}{dt} + \frac{\partial \Phi_m}{\partial y^{i}}\frac{dy^{i}}{dt} \end{aligned} \tag{2-48}$$

$$t_\beta = D_{\beta\beta}\, p\theta^{\beta} + J_{\beta\beta}\, p(p\theta^{\beta}) + 2\frac{\partial E_e}{\partial x^{i}}\eta^{i}_{\beta} \tag{2-49}$$

2.4 Pertinent Equations, Definitions, and Notations

The following pertinent equations, definitions, and notations will be used in subsequent chapters.

a) Voltage equations:

1) $$\begin{aligned} e_m &= R_{mn} i^n + \frac{d\overset{M}{\Phi}_m}{dt} \\ &= R_{mn} i^n + \frac{\partial \Phi_m}{\partial i^n}\frac{di^n}{dt} + \frac{\partial \Phi_m}{\partial x^{i}}\frac{\delta x^{i}}{dt} + \frac{\partial \Phi_m}{\partial y^{s}}\frac{dy^{s}}{dt} \end{aligned}$$

For the linear case $\Phi_m = L_{mn} i^n$ and

$$e_m = R_{mn} i^n + L_{mn}\frac{di^n}{dt} + \frac{\partial L_{mn}}{\partial x^m}\frac{\delta x}{dt}\, i^n + \frac{\partial L_{mn}}{\partial y^n}\frac{dy^n}{dt}\, i^n$$

2) $\frac{\partial L_{mn}}{\partial x^m} = G_{mmn}$ = Induction Coefficient

3) $\frac{\partial L_{mn}}{\partial y^n} = V_{mnn}$ = Excitation Coefficient

4) $\frac{\delta x^m}{dt} = \eta^m_\alpha p\theta^\alpha$; η^m_α = Induction Angle Incident Matrix

5) $\frac{\partial}{\partial x^i} = \overset{x}{\nabla}_i$ = Induction Angle Gradient (i'th Component)

6) $\frac{\partial}{\partial y^i} = \overset{y}{\nabla}_i$ = Excitation Angle Gradient (i'th Component)

7) $\overset{x}{\nabla}_i + \overset{y}{\nabla}_i = \nabla_i$ = Angular Gradient

b) Torque equations:

1) $t_\beta = D_{\beta\beta} p\theta^\beta + J_{\beta\beta} p(p\theta^\beta) + 2(\overset{x}{\nabla} E_e)\cdot(\eta)_\beta$

For the linear case

$$t_\beta = D_{\beta\beta} p\theta^\beta + J_{\beta\beta} p(p\theta^\beta) + (\overset{x}{\nabla} L_{mn})\cdot(\eta)_\beta i^m i^n$$

2) $(\overset{x}{\nabla} L_{mn})\cdot(\eta)_\beta = \frac{\partial L_{mn}}{\partial x^m}\eta^m_\beta = G_{mmn}\eta^m_\beta = T_{mn\beta}$

= Torque Tensor for the β space

3) $\underset{e}{t}_\beta = +2(\overset{x}{\nabla} E_e)(\eta)_\beta$

= Electromagnetic Energy Conversion Torque on the β space

The dot product, as an example, for Fig. 2E is

$$\underset{e}{t}_s = +2\left[\frac{\partial E_e}{\partial x^a}, \frac{\partial E_e}{\partial x^b}, \frac{\partial E_e}{\partial x^c}, \frac{\partial E_e}{\partial x^d}\right]\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

$$\underset{e}{t}_r = +2\left[\frac{\partial E_e}{\partial x^a}, \frac{\partial E_e}{\partial x^b}, \frac{\partial E_e}{\partial x^c}, \frac{\partial E_e}{\partial x^d}\right]\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

or

$$\begin{bmatrix} t_{e\,s} \\ \\ t_{e\,r} \end{bmatrix} = +2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial E_e}{\partial x^a} \\ \frac{\partial E_e}{\partial x^b} \\ \frac{\partial E_e}{\partial x^c} \\ \frac{\partial E_e}{\partial x^d} \end{bmatrix}$$

where $t_{e\,s}$ is the electromagnetic energy conversion torque on the stator and $t_{e\,r}$ is the electromagnetic energy conversion torque on the rotor.

3. THE EQUATIONS OF MOTION IN MACHINE SYSTEM ANALYSIS

3.1 Loop Equations in True Coordinate Systems

It has been proven that Faraday's Induction Law is valid for any electrical machine referred to a true coordinate system. Consequently methods of static circuit analysis can be extended to include a system containing electrical machines. For an unsaturated system including capacitive elements, the general loop equations are

$$\begin{aligned} e_m &= R_{mn} i^n + p(L_{mn} i^n) + \frac{1}{C^{mn}} \int i^n dt \\ &= R_{mn} i^n + L_{mn} p i^n + \frac{\partial L_{mn}}{\partial x^m} \eta^m_\beta p\theta^\beta i^n + \frac{\partial L_{mn}}{\partial y^n} p y^n i^n + \\ &\quad + \frac{1}{C^{mn} p} i^n \\ &= z_{mn}(p) i^n \end{aligned} \tag{3-1}$$

where

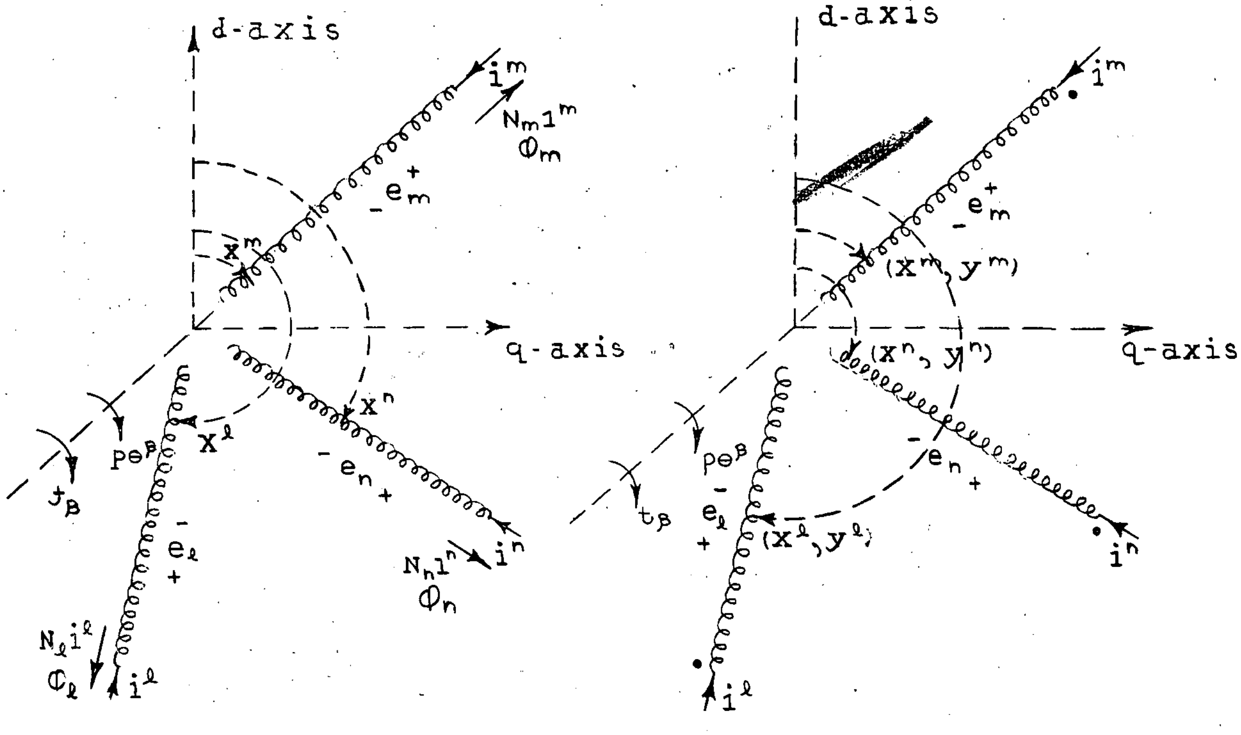
$$z_{mn}(p) = R_{mn} + L_{mn} p + \frac{\partial L_{mn}}{\partial x^m} \eta^m_\beta p\theta^\beta + \frac{\partial L_{mn}}{\partial y^n} p y^n + \frac{1}{C^{mn} p} \tag{3-2}$$

are the impedance matrix elements. Nodal equations could also be used; however, since an electrical machine is an inductive system, and since the driving forces are voltages, loop analysis is more natural.

The torque tensor element $T_{mn\beta} = \frac{\partial L_{mn}}{\partial x^m} \eta^m_\beta$ is the coefficient of $p\theta^\beta$ in $z_{mn}(p)$. Therefore once the loop equations have been determined, the torque equations can be obtained by inspection.

$$t_\beta = D_{\beta\beta} p\theta^\beta + J_{\beta\beta} p(p\theta^\beta) + \frac{\partial L_{mn}}{\partial x^m} \eta^m_\beta i^m i^n \tag{3-3}$$

Using Yu's formula a standard polarity convention can be adopted to determine the signs of all terms in the equations of motion. The coil currents are arbitrarily chosen to flow towards the center of the machine if positive, whereas the M.M.F is referenced in an outward direction, as is the flux. Since the voltage is a driving force the current flows into the coil at its plus reference. From Yu's formula the x and y angles are measured positively in a clockwise direction from the d-axis. Positive torque acts in the direction of increasing θ^{β} since it is a driving force. The positive reference sense for θ^{β} can be chosen arbitrarily, since the signs occurring in the equations are given by η^{n}_{β}. These references are shown in Fig. 3A(a) for three coils.



a) Positive polarity b) Standard Polarity Convention

Fig. 3A Polarity Convention

If the M.M.F. direction in the n-coil is opposite to the chosen reference direction a negative sign appears in front of all L_{mn}. Since in Yu's formula $L^{d}_{mn} = N_m N_n P_d$, this is equivalent to setting N_n negative. Polarity can thus be interpreted as the presence of a positive or negative number of turns in a winding. This will be noted by placing a dot at the plus end of the n-coil if N_n is positive and at the negative end if N_n is negative. This gives the standard polarity convention of Fig. 3B(b).

3.2 Transformation Theory

Many transformations used in machine analysis are hypothetical; that is, they are manipulative inventions which ease the task of dealing with complicated algebra or cumbersome differential equations. Since they are hypothetical, they can be defined in an arbitrary manner. For instance, a current transformation, $i^{n} = C^{n}_{\bar{n}} i^{\bar{n}}$, can involve the x-angles, y-angles, or any other parameters, z^{i} say. The voltage transformation $e_{\bar{n}} = \mathcal{E}^{n}_{\bar{n}} e_{n}$ can be defined independently of the current transformation, involving an entirely different set of parameters. These transformations can also involve complex numbers.

If the transformations are from one true coordinate system to a new true coordinate system, the equations of motion in this new system are derivable directly from Hamilton's Principle. The transformed equations must then be reducible to Lagrange's equations, which restricts the transformations. To obtain these restrictions let the current and voltage transformations from the old (unbarred) to the new (barred) system be respectively

$$i^{\bar{n}} = C^{\bar{n}}_{n}\, i^{n} \tag{3-4}$$

$$e_{\bar{m}} = \epsilon^{m}_{\bar{m}}\, e_{m} \tag{3-5}$$

Dealing with the voltage equations

$$e_{m} = R_{mn}\, i^{n} + \frac{d}{dt}\left(\frac{\partial E}{\partial i^{m}}\right)$$

Under the above transformations these become

$$e_{\bar{m}} = \epsilon^{m}_{\bar{m}}\left(R_{mn} C^{n}_{\bar{n}}\, i^{\bar{n}} + \frac{d}{dt}\left(\frac{\partial E}{\partial i^{\bar{n}}} C^{\bar{n}}_{m}\right)\right)$$

$$= \left(\epsilon^{m}_{\bar{m}} R_{mn} C^{n}_{\bar{n}}\right) i^{\bar{n}} + \epsilon^{m}_{\bar{m}} \frac{d}{dt}\left(C^{\bar{n}}_{m} \frac{\partial E}{\partial i^{\bar{n}}}\right) \tag{3-6}$$

In order for the new system to be a true system (physical system) we must have

$$\epsilon^{m}_{\bar{m}} \frac{d}{dt}\left(C^{\bar{n}}_{m} \frac{\partial E}{\partial i^{\bar{n}}}\right) = \frac{d}{dt}\left(\frac{\partial E}{\partial i^{\bar{m}}}\right) \tag{3-7}$$

where E is the energy function in the new system. This can be satisfied only if $\epsilon^{m}_{\bar{m}}$ or $C^{\bar{n}}_{m}$ or both can be moved through the differentiation and then combined to give a Kronecker delta. The following conditions must be satisfied.

$$1)\quad \epsilon^{m}_{\bar{m}} C^{\bar{n}}_{m} = \delta^{\bar{n}}_{\bar{m}} \tag{3-8}$$

$$2)\quad \frac{d\epsilon^{m}_{\bar{m}}}{dt} = 0 \text{ and/or } \frac{dC^{\bar{n}}_{m}}{dt} = 0 \tag{3-9}$$

The first condition restricts the transformations to those which maintain the power (energy) invariant since

$$e_{\bar{m}} i^{\bar{m}} = e_{m} \epsilon^{m}_{\bar{m}} C^{\bar{m}}_{n}\, i^{n} = \delta^{m}_{n} e_{m} i^{n} = e_{m} i^{m}$$

$$\bar{E} = \int_{0}^{t} e_{\bar{m}}\, i^{\bar{m}}\, dt = \int_{0}^{t} e_{m} i^{m} dt = E$$

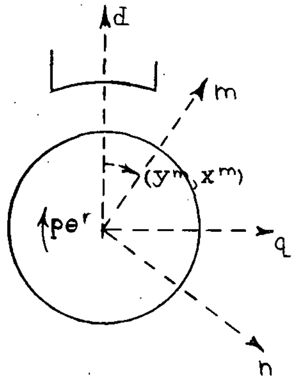
The Kronecker delta obtained in (3-8) is not strictly a constant even though it is a unit matrix since the time derivative, which is Lie, is not zero generally.

The condition $\frac{dC^{\bar{n}}_{m}}{dt} = 0$ allows $C^{\bar{n}}_{m}$ to be taken in front of $\frac{d}{dt}$. It is then combined with $\epsilon^{m}_{\bar{m}}$ to give (3-8). Since no differentiation of (3-8) occurs, the voltage equations remain essentially unchanged and therefore these transformations are of little interest. We are left with $\frac{d\epsilon}{dt} = 0$; $\frac{dC}{dt} \neq 0$ and $\epsilon^{m}_{\bar{m}} C^{\bar{n}}_{m} = \delta^{\bar{n}}_{\bar{m}}$ to be satisfied by transformations from one true system to another true system. These allow $\epsilon^{m}_{\bar{m}}$ to be taken inside $\frac{d}{dt}$ in (3-6) and then combined with $C^{\bar{n}}_{m}$ to give $\delta^{\bar{n}}_{\bar{m}}$. In this case, since $\delta^{\bar{n}}_{\bar{m}}$ is under the differentiation, its Lie derivatives enter, possibly changing the voltage equation quite radically.

Since $\frac{dC^{\bar{n}}_{m}}{dt} \neq 0$, the Lie derivative occurring must be over a non-zero speed. Moreover not all the partial derivatives $\frac{\partial C^{\bar{n}}_{\mathfrak{T}}}{\partial x}$ and $\frac{\partial C^{\bar{n}}_{\mathfrak{T}}}{\partial y}$ can vanish. By (3-8) $\epsilon^{m}_{\bar{m}}$ must be the inverse of C^{n}_{m} with a possible interchange of excitation and induction angles occurring. However, since $\frac{d\epsilon^{m}_{\bar{m}}}{dt} = 0$ and not all $\frac{\partial \epsilon^{m}_{\bar{\mathfrak{T}}}}{\partial x}$ and $\frac{\partial C^{m}_{\bar{\mathfrak{T}}}}{\partial y}$ need be zero, the Lie derivative must be over a zero speed. This implies that the Lie derivative reduces to one over the difference between coil velocities in the new system and the old system since these are the only invariant speeds. Consequently $\epsilon^{m}_{\bar{m}}$ is a function of the difference between new and old induction angles referring to coils in the same space. Since there are no preferred angles and $\frac{dC^{\bar{n}}_{n}}{dt} \neq 0$, $C^{\bar{n}}_{m}$ must be a function of the difference between new and old excitation angles.

Examples of some such transformations are given on the following page.

a) mn–dq Transformation.



$$(y^d, x^d) = (0,0)$$
$$(py^d, px^d) = (0, p\theta^r)$$
$$(py^m, px^m) = (p\theta^r, p\theta^r)$$

All coordinates are referred to the rotor space since the transformation is on the rotor.

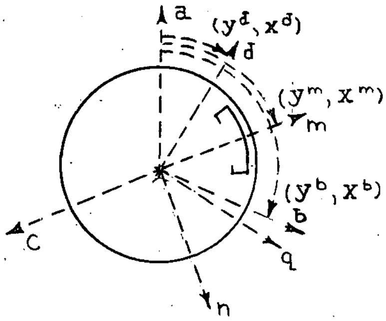
$$\begin{bmatrix} i^m \\ i^n \end{bmatrix} = \begin{bmatrix} \cos(y^m - y^d) & , & \sin(y^m - y^d) \\ -\sin(y^m - y^d) & , & \cos(y^m - y^d) \end{bmatrix} \begin{bmatrix} i^d \\ i^q \end{bmatrix}$$

$$\begin{bmatrix} e_m \\ e_n \end{bmatrix} = \begin{bmatrix} \cos(x^m - x^d) & , & +\sin(x^m - x^d) \\ -\sin(x^m - x^d) & , & \cos(x^m - x^d) \end{bmatrix} \begin{bmatrix} e_d \\ e_q \end{bmatrix}$$

In these

$$\frac{dC^m_{\bar{m}}}{dt} = \mathfrak{T} C^m_{\bar{m}} \neq 0 \quad ; \quad \frac{d\mathcal{E}}{dt} = \mathfrak{T}\mathcal{E}^{\bar{m}}_{m} = 0$$

b) Synchronous Machine Transformations.



$$(py^m, px^m) = (p\theta^r, p\theta^s)$$
$$(py^d, px^d) = (s, p\theta^s)$$
$$(y^a, x^a) = (0,0)$$
$$(py^a, px^a) = (0, p\theta^s)$$

All coordinates are tied to the stator for stator transformations. (Note that $px^m = p\theta^s$). s is the synchronous speed.

$$\begin{bmatrix} i^d \\ i^q \end{bmatrix} = \begin{bmatrix} \cos(y^d - y^m) & , \sin(y^d - y^m) \\ -\sin(y^d - y^m) & , \cos(y^d - y^m) \end{bmatrix} \begin{bmatrix} i^m \\ i^n \end{bmatrix}$$

$$\begin{bmatrix} e_d \\ e_q \end{bmatrix} = \begin{bmatrix} \cos(x^d - x^m) & , \sin(x^d - x^m) \\ -\sin(x^d - x^m) & , \cos(x^d - x^m) \end{bmatrix} \begin{bmatrix} e_m \\ e_n \end{bmatrix}$$

$$\begin{bmatrix} i^m \\ i^n \\ i^o \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(y^a - y^m), & \cos(120 + y^a - y^m), & \cos(240 + y^a - y^m) \\ \sin(y^a - y^m), & \sin(120 + y^a - y^m), & \sin(240 + y^a - y^m) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i^a \\ i^b \\ i^c \end{bmatrix}$$

$$\begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(x^a - x^m) & \sin(x^a - x^m) & \frac{1}{\sqrt{2}} \\ \cos(120 + x^a - x^m) & \sin(120 + x^a - x^m) & \frac{1}{\sqrt{2}} \\ \cos(240 + x^a - x^m) & \sin(240 + x^a - x^m) & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e_m \\ e_n \\ e_o \end{bmatrix}$$

$$\begin{bmatrix} i^d \\ i^q \\ i^o \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(y^a - y^d), & \cos(120 + y^a - y^d), & \cos(240 + y^a - y^d) \\ \sin(y^a - y^d) & \sin(120 + y^a - y^d) & \sin(240 + y^a - y^d) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i^a \\ i^b \\ i^c \end{bmatrix}$$

$$\begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(x^a - x^d) & , \sin(x^a - x^d) & \frac{1}{\sqrt{2}} \\ \cos(120 + x^a - x^d), & \sin(120 + x^a - x^d) & \frac{1}{\sqrt{2}} \\ \cos(240 + x^a - x^d), & \sin(240 + x^a - x^d) & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e_d \\ e_q \\ e_o \end{bmatrix}$$

In all of these the Lie derivative of the voltage transformation is equivalent to zero whereas that of the current transformation is not. That is $\frac{d\mathcal{E}^{m}_{\bar{m}}}{dt} \equiv 0$; $\frac{dC^{m}_{\bar{m}}}{dt} \not\equiv 0$.

c) fb-dq Transformations.

$$\begin{bmatrix} e_f \\ \\ e_b \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -j \\ \\ 1 & j \end{bmatrix} \begin{bmatrix} e_m \\ \\ e_n \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-j(x^m\ x^d)}, & -je^{-j(x^m - x^d)} \\ \\ e^{j(x^m - x^d)}, & je^{j(x^m - x^d)} \end{bmatrix} \begin{bmatrix} e_d \\ \\ e_q \end{bmatrix}$$

Once again

$$\frac{d\mathcal{E}}{dt}{}^{m}_{\bar{m}} = 0 \; ; \; \frac{dC}{dt}{}^{m}_{\bar{m}} \neq 0$$

The torque equation introduces no additional conditions on transformations from one true system to another true system, since for $E = \bar{E}$, the torque equation retains its holonomic form.

$$t_\beta = D_{\beta\beta}\, p\theta \;+ J_{\beta\beta}\, p(p\theta^\beta) \;+ 2\nabla_i E\, \eta^i_\beta \qquad (3\text{-}10)$$

$$t_{\bar\beta} = D_{\bar\beta\bar\beta}\, p\theta^{\bar\beta} + J_{\bar\beta\bar\beta}\, p(p\theta^{\bar\beta}) \;+ 2\nabla_{\bar{i}} \bar{E}\, \eta^{\bar{i}}_{\bar\beta} \qquad (3\text{-}11)$$

where

$$D_{\bar\beta\bar\beta} = \delta^\beta_{\bar\beta}\, \delta^\beta_{\bar\beta}\, D_{\beta\beta} \qquad (3\text{-}12)$$

$$J_{\bar\beta\bar\beta} = \delta^\beta_{\bar\beta}\, \delta^\beta_{\bar\beta}\, J_{\beta\beta} \qquad (3\text{-}13)$$

$$p\theta^{\bar\beta} = \delta^{\bar\beta}_\beta\, p\theta^\beta \qquad (3\text{-}14)$$

$$t_{\bar\beta} = \delta^\beta_{\bar\beta}\, t_\beta \qquad (3\text{-}15)$$

$\bar{E} = E$ in the new system, $\nabla_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}$, $x^{\bar{i}}$ being the i'th induction angle in the new system and $\eta^{\bar{i}}_{\bar\beta} = \eta^{\bar{i}}_{\beta}$, the incidence matrix in the new system.

3.3 Tensors In the Equations of Motion

A tensor index can transform by the voltage transformation matrix $\mathcal{C}$, in which case it will be called a v-index, or it can transform by the current transformation matrix C, in which case it will be called an i-index. Any tensor quantity can have; all i-indices (i-tensor), all v-indicies (v-tensor) or a combination of v and i indices (v-i tensor). If (3-8) is satisfied, so $\mathcal{C}_{\bar{m}}^{m}$ is the inverse of $C_{n}^{\bar{n}}$ except for a possible interchange of induction and excitation angles, all tensor indices transform in the same manner except for a possible interchange of induction and excitation angles. If it is necessary to show explicitly how an index transforms a "v" or an "i" will be placed either above or below it.

Tensor quantities occurring in the equations of motion can be determined from the expansion of (3-6), and from (3-10) through (3-15). They are listed in Table 1 on the following page. Since the excitation and induction angles are not tensor quantities neither are the induction or excitation coefficients, G_{mmn} and V_{mnn}.

It can be seen from the table that in general the torque tensor elements transform as a v-i tensor in the voltage equation whereas they transform as an i-i tensor in the torque equation.

The detailed tensor notation used above is useful for checking tensor character or for investigationg details, however, for computational purposes matrix notation is better. In this form the voltage equation in a true reference frame are

$$(e) = ((R) + p (L)) (i), \; p = \frac{d}{dt} \qquad (3\text{-}16)$$

Tensor Quantities	Transformation Properties
a) Voltage Equation	
$e_{\underset{v}{m}}$	$e_{\bar{m}} = C^{m}_{\bar{m}} e_m$
$i^{\overset{i}{m}}$	$i^{\bar{m}} = C^{\bar{m}}_{m} i^m$
$\frac{\partial E_e}{\partial i^m} = \overset{M}{\Phi}_{\underset{i}{m}}$	$\overset{M}{\Phi}_{\bar{m}} = C^{m}_{\bar{m}} \overset{M}{\Phi}_m$
$\frac{d\overset{M}{\Phi}_m}{dt} = \alpha_{\underset{v}{m}}$	$\alpha_{\bar{m}} = C^{m}_{\bar{m}} \alpha_m$
$L_{\underset{v}{m}\underset{i}{n}}$	$L_{\bar{m}\bar{n}} = C^{m}_{\bar{m}} C^{n}_{\bar{n}} L_{mn}$
$\frac{\partial}{\partial i^n}\left(\frac{\partial \Phi_m}{\partial x^m}\right)\eta^m_\beta = T_{\underset{v}{m}\underset{i}{n}\beta}$	$T_{\underset{v}{\bar{m}}\underset{i}{\bar{n}}\bar{\beta}} = C^{m}_{\bar{m}} C^{n}_{\bar{n}} \delta^{\beta}_{\bar{\beta}} T_{\underset{v}{m}\underset{i}{n}\beta}$
$\frac{\partial}{\partial i^n}\left(\frac{\partial \Phi_m}{\partial y^n}\frac{dy^n}{dt}\right) = \psi_{\underset{v}{m}\underset{i}{n}}$	$\psi_{\bar{m}\bar{n}} = C^{m}_{\bar{m}} C^{n}_{\bar{n}} \psi_{mn}$
b) Torque Equation	
$D_{\beta\beta}$	$D_{\bar{\beta}\bar{\beta}} = \delta^{\beta}_{\bar{\beta}} \delta^{\beta}_{\bar{\beta}} D_{\beta\beta}$
$J_{\beta\beta}$	$J_{\bar{\beta}\bar{\beta}} = \delta^{\beta}_{\bar{\beta}} \delta^{\beta}_{\bar{\beta}} J_{\beta\beta}$
$p\theta^{\beta}$	$p\theta^{\bar{\beta}} = \delta^{\bar{\beta}}_{\beta} p\theta^{\beta}$
t_β	$t_{\bar{\beta}} = \delta^{\beta}_{\bar{\beta}} t_\beta$
$\underset{e}{t}_\beta = 2\,\boxed{\nabla x}_i E \eta^i_\beta$	$\underset{e}{t}_{\bar{\beta}} = \delta^{\beta}_{\bar{\beta}} \underset{e}{t}_\beta$
$\frac{\partial^2}{\partial i^m \partial i^n}\left(\frac{\partial \Phi_m}{\partial x^m}\right)\eta^m_\beta = T_{\underset{i}{m}\underset{i}{n}\beta}$	$T_{\underset{i}{\bar{m}}\underset{i}{\bar{n}}\bar{\beta}} = C^{m}_{\bar{m}} C^{n}_{\bar{n}} \delta^{\beta}_{\beta} T_{\underset{i}{m}\underset{i}{n}\beta}$

Table 1 <u>Tensor Quantities in the Equations of Motion</u>

A matrix multiplication occurs for each β in the calculation of the electromagnetic torque since $T_{mn\beta}$ is a triad. Retaining this index for clarity the torque equations in a true frame are

$$(t_\beta) = (D_{\beta\beta})(p\theta^{\beta}) + (J_{\beta\beta}p)(p\theta^{\beta}) + 2((\nabla E)\circ(\eta)_\beta) \tag{3-17}$$

For the unsaturated case this is

$$(t_\beta) = (D_{\beta\beta})(p\theta^\beta) + (J_{\beta\beta}p)(p\theta^\beta) + 2((i)^t(T)_\beta(i)) \qquad (3\text{-}18)$$

The transformation equations (3-4) and (3-5) are

$$(i) = (C)(\bar{i}) \qquad (3\text{-}19)$$

$$(\bar{e}) = (\epsilon)^t(e) \qquad (3\text{-}20)$$

For these transformations (3-16) and (3-18) become

$$(\bar{e}) = (\epsilon)^t(R)(C)(\bar{i}) + (\epsilon)^t p(L)(C)(\bar{i}) \qquad (3\text{-}21)$$

$$(t_\beta) = (D_{\beta\beta})(p\theta^\beta) + (J_{\beta\beta}p)(p\theta^\beta) + ((\bar{i})^t(C)^t(T)_\beta(C)(i)) \qquad (3\text{-}22)$$

If $p\theta^\beta$ does not equal zero and (3-8) holds so that (C) is replaceable by (ϵ) except for an interchange of induction and excitation angles, the transformed torque tensor $(C)^t(T)_\beta(C)$ is the matrix of coefficients of $p\theta^\beta$ in (3-21) before possible $(\epsilon)^t(L)\,\frac{d(C)}{dt}\,(i)$ contributions are added. In this case then, once the voltage equations have been transformed, the transformed torque equations can be obtained by inspection after a possible interchange of induction and excitation coordinates.

Invariance, a property usually associated with the convection of two tensors of opposite types, is no longer a general result. As an example under (3-4) and (3-5)

$$e_m i^m = e_{\bar{m}} \epsilon^{\bar{m}}_{m} C^{m}_{\bar{n}} i^{\bar{n}} = \epsilon^{\bar{m}}_{m} C^{m}_{\bar{n}} e_{\bar{m}} i^{\bar{n}} \quad . \qquad (3\text{-}23)$$

(3-23) is understandable if it is realized that power and energy can be defined only in a true system for which e_m and i^m are physically measurable. For a transformation from one true system to another (3-8) is satisfied and the invariance property follows.

For an untrue system $e_{\bar{m}} i^{\bar{m}}$ and $\int (e_{\bar{m}} i^{\bar{m}}) dt$ are abstract quantities; they are not the true power and energy. Power in the $(\bar{m})$ system is $C^{\bar{m}}_{m} C^{m}_{\bar{m}} e_{\bar{n}} i^{\bar{n}}$ referenced to the real (m) system. The energy is likewise $\int (C^{\bar{m}}_{m} C^{m}_{\bar{n}} e_{\bar{m}} i^{\bar{n}}) dt$ referenced to the (m) system. In this sense both power and energy are invariants of any transformation.

Another form of invariance occurs in the torque equation. The torque t_{β} is a covariant tensor; however, its transformation matrix is always the Kronecker delta and it is thus actually invariant. Mechanical angles are transformed similarly and hence the entire torque equation is essentially invariant. This applies in particular to the electromagnetic energy conversion torque which is covariant in the mechanical coordinates.

3.4 Extension of the Equations of Motion to a Machine With Rotating Saliency

Saliency in the equations of motion is expressed by a dependence of the flux upon angular displacement, θ^{β}, of the machine spaces. In the equations of motion (2-48) and (2-49) any saliency was assumed fixed with respect to the chosen reference and hence θ^{β} dependencies were irrelevent and could be neglected.

In general then the flux and thus the stored electromagnetic energy are influenced by mechanical rotations as well as the mechanical energy. These added variations are Lie, the generalizations of (2-22) and (2-34) being

$$\frac{d\Phi_i}{dt} = \frac{\partial \Phi_i}{\partial i^{l}} \frac{di^{l}}{dt} + \underset{\Delta y^{j}}{\mathcal{L}} \Phi_i + \underset{pe^{\beta}}{\mathcal{L}} \Phi_i \qquad (3\text{-}24)$$

$$= \frac{\partial \Phi_i}{\partial i^{l}} \frac{di^{l}}{dt} + \frac{\partial \Phi_i}{\partial y^{j}} \frac{dy^{j}}{dt} + \frac{\partial \Phi_i}{\partial \theta^{\beta}} \frac{d\theta^{\beta}}{dt} + \frac{\partial \Phi_i}{\partial x^{j}} \eta^{j}_{\beta} \frac{d\theta^{\beta}}{dt}$$

(3-25)

$$\Delta A = \int_0^t \left(\frac{\partial E_e}{\partial i^n}\ \Delta i^n + \underset{\Delta\Theta^\beta}{\mathcal{I}} E_e + \underset{\Delta y^i}{\mathcal{I}} E_e + \frac{\partial E_m}{\partial p\Theta^\beta}\ \Delta p\Theta^\beta\right) dt$$

$$= \int_0^t \left(-\frac{d}{dt}\frac{\partial E_e}{\partial i^n}\ \Delta q^n - \frac{\partial E_e}{\partial x^i}\eta^i_\beta \Delta\Theta^\beta - \frac{\partial E_e}{\partial y^i}\ \Delta y^i - \frac{\partial E_e}{\partial \Theta^\beta}\ \Delta\Theta^\beta - \frac{d}{dt}\frac{\partial E_m}{\partial p\Theta^\beta}\ \Delta p\Theta^\beta\right) dt \tag{3-26}$$

Proceeding as in 2.3, the equations of motion are easily found to be

$$e_m = R_{mn} i^n + \frac{d\Phi^M_m}{dt} \tag{3-27}$$

$$t_\beta = D_{\beta\beta}\, p\Theta^\beta + J_{\beta\beta}\, p(p\Theta^\beta) + 2\nabla_i E_e\, \eta^i_\beta + \frac{\partial E_e}{\partial \Theta^\beta} \tag{3-28}$$

As a special case to be used later, consider Yu's formula for a rotating d-axis. It can be written in the form

$$L_{mn} = L^d_{mn} \cos X^m \cos Y^n + L^q_{mn} \sin X^m \sin Y^n \tag{3-29}$$

where

$$X^m = x^m - s^m \tag{3-30}$$

$$Y^n = y^n - s^n \tag{3-31}$$

x^m and y^n are the induction and excitation angles of the m and n coils respectively. s^i, which will be called the saliency angle of the i-th coil, is defined by

$$\frac{ds^i}{dt} = \frac{\partial s^i}{\partial \Theta^\beta}\frac{d\Theta^\beta}{dt} = \xi^i_\beta \frac{d\Theta^\beta}{dt} \tag{3-32}$$

"ξ^i_β", the saliency angle incidence matrix, has in this case, elements which are ones if β refers to the space in which the d-axis is located. Otherwise the elements are zeroes. (3-27) and (3-28) are then

$$e_m = R_{mn} i^n + L_{mn} \frac{di^n}{dt} + \frac{\partial L_{mn}}{\partial X^m} i^n pX^m + \frac{\partial L_{mn}}{\partial Y^n} i^n pY^n \tag{3-33}$$

$$t_\beta = D_{\beta\beta}\, p\theta^\beta + J_{\beta\beta}\, p(p\theta^\beta) + 2 \nabla_i E_e \eta^i_\beta - 2 \nabla_i E_e \xi^i_\beta \tag{3-34}$$

$$= D_{\beta\beta}\, p\theta^\beta + J_{\beta\beta}\, p(p\theta^\beta) + 2 \nabla_i E_e \sigma^i_\beta$$

Here

$$pX^m = \frac{\delta}{dt}(x^m - s^m) = (\eta^m_\beta p\theta^\beta - \xi^i_\beta p\theta^\beta) = \sigma^i_\beta p\theta^\beta \tag{3-35}$$

$$pY^n = \frac{d}{dt}(y^n - s^n) = (py^n - \xi^i_\beta p\theta^\beta) \tag{3-36}$$

The form of these equations is identical to (2-48) and (2-49) with σ^i_β as the generalized induction angle incidence matrix. The torque tensor for the β space, as before, is the array of coefficients of $p\theta^\beta$ excluding those from pY^n.

4. THE EQUATIONS OF SMALL OSCILLATIONS

To derive the hunting equations, the general network loop equations and the torque equations in the following form will be used.

$$e_m = R_{mn} i^n + \frac{\partial \Phi_m}{\partial i^n} \frac{di^n}{dt} + \frac{\partial \Phi_m}{\partial x^i} \frac{\delta x^i}{dt} + \frac{\partial \Phi_m}{\partial y^i} \frac{dy^i}{dt} + \frac{1}{C^{mn}} \int_0^t i^n dt \tag{4-1}$$

$$t_\beta = D_{\beta\beta}\, p\theta^\beta + J_{\beta\beta}\, p(p\theta^\beta) + 2(\nabla E)\cdot(\eta)_\beta \tag{4-2}$$

4.1 The Voltage Equations

If small variations in voltage are applied to a system of machines several conditions change slightly, all of which contribute to the "hunting" of the system.

a) The angular positions of the coil spaces and/or of the reference angles can be changed slightly from the steady state values (i.e., the instantaneous non-oscillating values). These variations are analogous to flux variations with θ^β and y^i and hence are Lie variations given by

$$1)\quad \underset{\eta^i_\beta \Delta\theta^\beta}{\mathfrak{L}} e_m = \frac{\partial e_m}{\partial x^m} \eta^i_\beta \Delta\theta^\beta \tag{4-3}$$

$$2)\quad \underset{\Delta y^i}{\mathfrak{L}} e_m = \frac{\partial e_m}{\partial y^i} \Delta y^i \tag{4-4}$$

The applied voltage e_m can only be a function of the reference angle of the coil winding to which it is applied. Therefore $\frac{\partial e_m}{\partial y^i} = \delta^m_i \frac{\partial e}{\partial y^m}$ and $\frac{\partial e_m}{\partial y^m} = \frac{\partial e_m}{\partial x^m}$. The above variation is as a result

$$\frac{\partial e_m}{\partial y^m}\,(\eta^m_\beta\,\Delta\theta^\beta + \Delta y^m) \qquad (4\text{-}5)$$

b) The steady state velocities of the coil spaces and/or of the coil winding commutator axes can change slightly. The resulting variations are obtainable from (4-1)

$$\frac{\partial e_m}{\partial p\theta^\beta}\,\Delta p\theta^\beta + \frac{\partial e}{\partial py^i}\,\Delta\,py^i$$

$$= \frac{\partial \Phi_m}{\partial x^i}\,\eta^i_\beta \Delta p\theta^\beta + \frac{\partial \Phi_m}{\partial y^i}\,\Delta py^i \qquad (4\text{-}6)$$

c) The currents can be oscillating about their steady state values giving a variation

$$\frac{\partial e_m}{\partial i^l}\,\Delta i^l = \left(R_{ml} + \frac{\partial^2 \Phi_m}{\partial i^l \partial i^m}\,p + \frac{\partial^2 \Phi_m}{\partial i^l \partial x^i}\,\frac{\delta x^i}{dt} + \frac{\partial^2 \Phi_m}{\partial i^l \partial y^i}\,\frac{dy^i}{dt}\right)\Delta i^l$$

$$(4\text{-}7)$$

d) The rates of change of currents can be oscillating about their steady state values to give

$$\frac{\partial e_m}{\partial\left(\frac{di^n}{dt}\right)}\,\Delta\,\frac{di^n}{dt} = \frac{\partial \Phi_m}{\partial i^n}\,p\,\Delta\,i^n \qquad (4\text{-}8)$$

e) From the presence of capacitance, the integral of the current can be oscillating

$$\frac{\partial\, e_m}{\partial\left(\int i^n dt\right)}\,\Delta\left(\int_0^t i^n dt\right) = \frac{1}{C^{mn}}\,\Delta\left(\frac{1}{p}\,i^n\right) = \frac{1}{C^{mn}p}\,\Delta i^n \quad (4\text{-}9)$$

In analogy with (2-22) the first approximation of the voltage oscillation equations is

$$\Delta e_m = (e_m - e_{mo}) = \frac{\partial e_m}{\partial p\theta^\beta}\Delta p\theta^\beta + \frac{\partial e_m}{\partial py^i}\Delta py^i + \frac{\partial e_m}{\partial i^l}\Delta i^l$$

$$+ \frac{\partial e_m}{\partial pi^n}\Delta pi^n + \frac{\partial e_m}{\partial \int_0^t i^n dt}\Delta\int_0^t i^n dt$$

$$+ \underset{\eta^i_\beta \Delta\theta^\beta}{\mathcal{L}} e_m + \underset{\Delta y^i}{\mathcal{L}} e_m \qquad (4\text{-}10)$$

Δe_m is the voltage variation in the m'th loop, e_{mo} being the steady state value and e_m the actual value.

Substituting (4-2) through (4-9) into (4-10),

$$\Delta e_m = \Big(R_{mn} + \frac{\partial \Phi_m}{\partial i^n}p + \frac{\partial^2 \Phi_m}{\partial i^n \partial i^l}\frac{di^l}{dt} + \frac{\partial}{\partial i^n}\Big(\frac{\partial \Phi_m}{\partial x^m}\Big)\frac{\delta x^m}{dt} + \frac{\partial}{\partial i^n}\Big(\frac{\partial \Phi_m}{\partial y^i}\Big)\frac{dy^i}{dt}$$

$$+ \frac{1}{c^{mn}p}\Big)\Delta i^n + \frac{\partial \Phi_m}{\partial x^m}\Delta px^m + \frac{\partial \Phi_m}{\partial y^n}\Delta py^n + \frac{\partial e_m}{\partial y^m}(\eta^m_\beta \Delta\theta^\beta + \Delta y^m)$$

$$(4\text{-}11)$$

There are three distinct contributions in the above equation.

a) $\Big(R_{mn} + \frac{\partial \Phi_m}{\partial i^n}p + \frac{\partial^2 \Phi_m}{\partial i^n \partial i^l}\frac{di^l}{dt} + \frac{\partial}{\partial i^n}\Big(\frac{\partial \Phi_m}{\partial x^m}\Big)\frac{\delta x^m}{dt} +$

$$\frac{\partial}{\partial i^n}\Big(\frac{\partial \Phi_m}{\partial y^i}\Big)\frac{dy^i}{dt} + \frac{1}{c^{mn}p}\Big)\Delta i^n$$

$\triangleq \mathfrak{z}_{mn}(p)\Delta i^n$: a variation arising from the effect of the oscillation impedance ($\mathfrak{z}_{mn}$) acting on oscillating currents. Here $\mathfrak{z}_{mn}(p) \triangleq z_{mn}(p) + \frac{\partial^2 \Phi_m}{\partial i^n \partial i^l}\frac{di^l}{dt}$. The last term vanishes for unsaturated systems.

b) $\Big(\frac{\partial \Phi_m}{\partial y^n}p + \frac{\partial e_m}{\partial y^m}\delta^m_n\Big)\Delta y^n$: voltages arising from oscillating reference axes.

c) $\Big(\frac{\partial e_m}{\partial y^m} + \frac{\partial \Phi_m}{\partial x^m}p\Big)\eta^m_\beta \Delta p\theta^\beta$: voltages arising due to oscillating coil spaces.

For the unsaturated case $\Phi_m = L_{mn} i^n$, (4-10) becomes

$$\Delta e_m = z_{mn}(p)\,\Delta i^n + V_{mnn} i^n \Delta p y^n + \frac{\partial e_m}{\partial y^m} \Delta y^m$$

$$+ G_{mmn} i^n \eta^m_\beta \Delta p \theta^\beta + \frac{\partial e_m}{\partial y^m} \eta^m_\beta \Delta \theta^\beta \qquad (4\text{-}12)$$

where

$$z_{mn}(p) = (R_{mn} + L_{mn} p + G_{mmn} \eta^m_\beta p \theta^\beta + V_{mnn} p y^n + \frac{1}{c^{mn} p}$$

As an example consider the induction motor of Fig. 4A with fixed commutator axes. $py^i = 0$ and $\frac{\partial e_m}{\partial y^i} = 0$ since the applied voltages are a function of time only. The general equation reduces to

$$\Delta e_m = (R_{mn} + L_{mn} p + G_{mmn} \eta^m_\beta p\theta^\beta)\Delta i^n + G_{mmn} i^n \eta^m_\beta (\Delta p \theta^\beta) \qquad (4\text{-}13)$$

4.2 The Mechanical Equations of Oscillation

For the voltage oscillation equations, a variation of the loop equations was taken; equations containing relationships between different machines. Torque variations, in contrast, must deal with individual machine spaces and in this sense the equations are separated. This separation arises computationally when one calculates the torque tensor for the component spaces ($\frac{\partial L_{mn}}{\partial x^m} \eta^m_\beta$ for the β'th space). Since for each β, a matrix results, the torque tensor is a triadic. Operations (transpose, inverse, etc.) on the torque tensor apply for each β individually.

Proceeding as in the derivation of the voltage oscillation equations contributions to torque oscillations arise from

a) Variations due to oscillating mechanical angles, Θ^{β}, mechanical angular speeds, $p\Theta^{\beta}$, and mechanical angular accelerations, $p\,(p\Theta^{\beta})$. From (4-2) the sum of these contributions is

$$\frac{\partial t_{\beta}}{\partial \Theta^{\beta}}\Delta\Theta^{\beta} + \frac{\partial t_{\beta}}{\partial p(p\Theta^{\beta})}\Delta(p(p\Theta^{\beta})) + \frac{\partial t_{\beta}}{\partial p\Theta^{\beta}}\Delta(p\Theta^{\beta}) = \frac{\partial t_{\beta}}{\partial \Theta^{\beta}}\Delta\Theta^{\beta} + D_{\beta\beta}\Delta p\Theta^{\beta}$$

$$+ J_{\beta\beta}\Delta p(p\Theta^{\beta}) \qquad (4\text{-}14)$$

b) A variation due to oscillating currents given by

$$\frac{\partial t_{\beta}}{\partial i^{\ell}}\Delta i^{\ell} = \frac{\partial t_{e\beta}}{\partial i^{\ell}}\Delta i^{\ell} = 2\left(\nabla\!\!\!\!\nabla \frac{\partial E}{\partial i^{\ell}}\right)\circ(\eta)_{\beta}\Delta i^{\ell}$$

$$= 2(\nabla\!\!\!\!\nabla\, \Phi_{\ell})\circ(\eta)_{\beta}\Delta i^{\ell} \qquad (4\text{-}15)$$

c) A variation due to oscillating reference axes

$$\left(\frac{\partial t_{\beta}}{\partial y^{i}}\bigg|_{\Theta^{\alpha},\ i^{s}}\right)\Delta y^{i} = \left(\frac{\partial t_{\beta}}{\partial y^{i}}\bigg|_{x^{i}} + \frac{\partial t_{\beta}}{\partial x^{s}}\bigg|_{y^{i}}\frac{\partial x^{s}}{\partial y^{i}}\right)\bigg|_{\Theta^{\alpha},\ i^{s}}$$

$$= \left(\frac{\partial}{\partial y^{i}} + \delta^{s}_{i}\frac{\partial}{\partial x^{s}}\right)t_{e\beta}\Delta y^{i}$$

$$= 2\ \nabla_{i}((\nabla\!\!\!\!\nabla E)\circ(\eta)_{\beta})\Delta y^{i} \qquad (4\text{-}16)$$

To a first approximation the sum of these variations is equal to the applied torque variation $\Delta t_{\beta} = t_{\beta} - t_{\beta o}$. The mechanical oscillation equations are thus

$$\Delta t_{\beta} = \frac{\partial t}{\partial \Theta^{\beta}}\Delta\Theta^{\beta} + (D_{\beta\beta} + J_{\beta\beta}p)\Delta p\Theta^{\beta} + 2\nabla_{i}((\nabla\!\!\!\!\nabla E)\circ(\eta)_{\beta})\Delta y^{i}$$

$$+ 2(\nabla\!\!\!\!\nabla\,\Phi_{s})\circ(\eta)_{\beta}\ \Delta i^{s} \qquad (4\text{-}17)$$

$$= \frac{\partial t_{\beta}}{\partial \Theta^{\beta}}\Delta\Theta^{\beta} + (D_{\beta\beta} + J_{\beta\beta}\,p)\Delta p\Theta^{\beta}$$

$$+ \nabla_{s}(\nabla_{i}E\ \ \Delta y^{i} + \Phi_{i}\Delta i^{i})\,\eta^{s}_{\beta} \qquad (4\text{-}18)$$

For unsaturated systems $\Phi_j = L_{ji} i^i$ and $E = \frac{1}{2} L_{ji} i^j i^i$.

This gives

$$2 \nabla_i((\nabla E)\circ(\eta)_\beta)\Delta y^i = \nabla_i \underset{e}{t}_\beta \Delta y^i$$

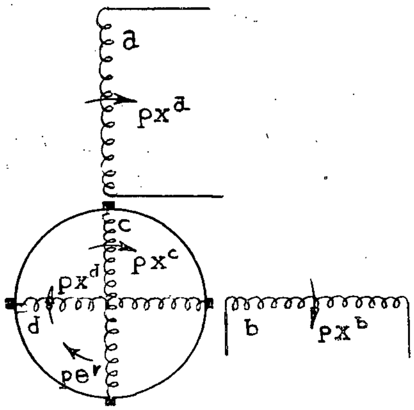
$$= \left(\frac{\partial}{\partial x^j} \Delta y^j + \frac{\partial}{\partial y^n} \Delta y^n\right)\frac{\partial L_{jn}}{\partial x^j} \eta^j_\beta i^j i^n \qquad (4\text{-}19)$$

$$\nabla_i \Phi_j \eta^i_\beta \Delta i^j = \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right) L_{jl} i^l \eta^i_\beta \Delta i^j$$

$$= \left(\frac{\partial L_{jl}}{\partial x^j} i^l \eta^j_\beta \Delta i^j + \frac{\partial L_{lj}}{\partial x^l} i^l \eta^l_\beta \Delta i^j\right)$$

(4-20)

In matrix notation retaining the mechanical index these contributions are $(\nabla \underset{e}{t}_\beta)(\Delta y)$ and $((T)_\beta(i\ ) + (T)^t_\beta(i))^t(\Delta i)$ where t is the transpose operator. As an example consider Fig. 4A, the two phase induction motor. (t) the electromagnetic



$$\eta^i_\beta = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cc} S & r \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

Fig. 4A <u>Two Phase Induction Motor</u>

energy conversion torque vector is $\begin{bmatrix} \underset{e}{t}_s \\ \underset{e}{t}_r \end{bmatrix}$ where $\underset{e}{t}_s$ is that acting on the stator and $\underset{e}{t}_r$, that acting on the rotor.

$$(\nabla \underset{e}{t}_{\beta})(\Delta y) = \begin{bmatrix} \nabla \underset{e}{t}_s \\ \\ \nabla \underset{e}{t}_r \end{bmatrix} \begin{bmatrix} \Delta y^a \\ \Delta y^b \\ \Delta y^c \\ \Delta y^d \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a} \underset{e}{t}_s \,, & \frac{\partial}{\partial x^b} + \frac{\partial}{\partial y^b} \underset{e}{t}_s, & 0 & , \; 0 \\ \\ 0 & , \; 0 & , \; \frac{\partial}{\partial x^c} + \frac{\partial}{\partial y^c} \underset{e}{t}_r, & \frac{\partial}{\partial x^d} + \frac{\partial}{\partial x^d} \underset{e}{t}_r \end{bmatrix} \begin{bmatrix} \Delta y^a \\ \Delta y^b \\ \Delta y^c \\ \Delta y^d \end{bmatrix}$$

The zeroes are present since $\underset{e}{t}_s$ is independent of c or d angles ($\eta^i_\beta = 0$ for $\beta = s$ and $i = c, d$) and similarly for $\underset{e}{t}_r$. For the second contribution, denote $(T)_s$ as the torque tensor for the stator; $(T)_r$, the torque tensor the rotor; and (i) as the column vector

$$\begin{bmatrix} i^a \\ i^b \\ i^c \\ i^d \end{bmatrix}$$

$((T)_\beta(i) + (T)^t_\beta(i))^t(\Delta i)$ is then equal to

$$\left[\begin{bmatrix} (T)_s \\ \\ (T)_r \end{bmatrix} (i) + \begin{bmatrix} (T)^t_s \\ \\ (T)^t_r \end{bmatrix} (i) \right]^t (\Delta i)$$

The torque oscillation equation in matrix form for the unsaturated case is

$$(\Delta t_\beta) = \left(\frac{\partial t_\beta}{\partial \theta^\beta} \Delta\theta^\beta\right) + ((T + T^t)(i)^t)(\Delta i) + (D + Jp)(\Delta p\theta)$$
$$+ (\nabla \underset{e}{t}_\beta)(\Delta y) \qquad (4\text{-}21)$$

4.3 The Combined Equations of Motion

The system equations

$$e_m = (R_{mn} + pL_{mn})i^n$$
$$t_\beta = (D_{\beta\beta} + J_{\beta\beta}p)p\theta^\beta + \nabla_i E_e \eta^i_\beta$$
$$\Delta e_m = z_{mn}(p)\Delta i^n + \left(\frac{\partial \Phi_m}{\partial x^m}p + \frac{\partial e_m}{\partial x^m}\right)\eta^m_\beta \Delta\theta^\beta + \left(\frac{\partial \Phi_m}{\partial y^n} + \frac{\partial e_m}{\partial y^m}\delta^m_n\right)\Delta y^n$$
$$\Delta t_\beta = Z_{\beta\alpha}(p)\Delta p\theta^\alpha + \frac{\partial t_\beta}{\partial \theta^\beta}\Delta\theta^\beta + \nabla_i \Phi_s \eta^i_\beta \Delta i^s + \nabla^2_{is} E_e \eta^i_\beta \Delta y^i$$

(4-22)

where $Z_{\beta\alpha} = (D_{\beta\alpha} + J_{\beta\alpha}p)$ and $\nabla^2_{is} = \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)\left(\frac{\partial}{\partial x^s} + \frac{\partial}{\partial y^s}\right)$ form a set of coupled matrix differential equations which determine the steady state and hunting state of a machine system once the initial conditions are known.

The oscillation equations can be conveniently combined to give a single matrix equation of oscillation

$$\begin{bmatrix} (\Delta e) \\ \\ (\Delta t) \end{bmatrix} = \begin{bmatrix} (z_{mn}(p)), & \left(\frac{\partial \Phi_m}{\partial x^m} + \frac{\partial e_m}{\partial y^m}\frac{1}{p}\right)\eta^m_\beta \\ \\ \nabla_i \Phi_n \eta^i_\beta, & Z_{\beta\alpha}(p) + \frac{\partial t_\beta}{\partial \theta^\beta}\frac{1}{p} \end{bmatrix} \begin{bmatrix} (\Delta i^n) \\ \\ (\Delta p\theta^\beta) \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\partial e_m}{\partial y^m} + \frac{\partial \Phi_m}{\partial y^n}p \\ \\ \nabla^2_{ns} E \eta^s_\beta \end{bmatrix} (\Delta y^n) \qquad (4\text{-}23)$$

When saturation is absent this simplifies to

$$\begin{bmatrix} (\Delta e) \\ \\ (\Delta t) \end{bmatrix} = \begin{bmatrix} (z_{mn}(p)) & , \left(\frac{\partial L_{mn}}{\partial x^m} i^n \eta^m_\beta\right) + \left(\frac{\partial e}{\partial y^m} \eta^m_\beta\right) \frac{1}{p} \\ \\ ((T+T^t)(i))^t, & (D + Jp) + \frac{\partial t_\beta}{\partial \theta^\beta} \frac{1}{p} \end{bmatrix} \begin{bmatrix} (\Delta i^n) \\ \\ (\Delta p\theta^n) \end{bmatrix}$$

$$+ \begin{bmatrix} \left(\frac{\partial e_m}{\partial y^m}\right) + \left(\frac{\partial L_{mn}}{\partial y^n} i^n\right) p \\ \\ \nabla_{i} t^{\beta}_{e} \end{bmatrix} (\Delta y^n) \tag{4-24}$$

In the general hunting equations (Δy) and (Δpy) refer to commutator axes and hence are constrained externally.

5. APPLICATIONS TO THE DERIVATION OF MACHINE SYSTEM EQUATIONS

5.1 The Power Selsyn System (6)

Two induction motors interconnected as shown in Fig. 5A form a selsyn system. The transmitter is driven externally. The reciever runs at the same speed with an angle of lag δ.

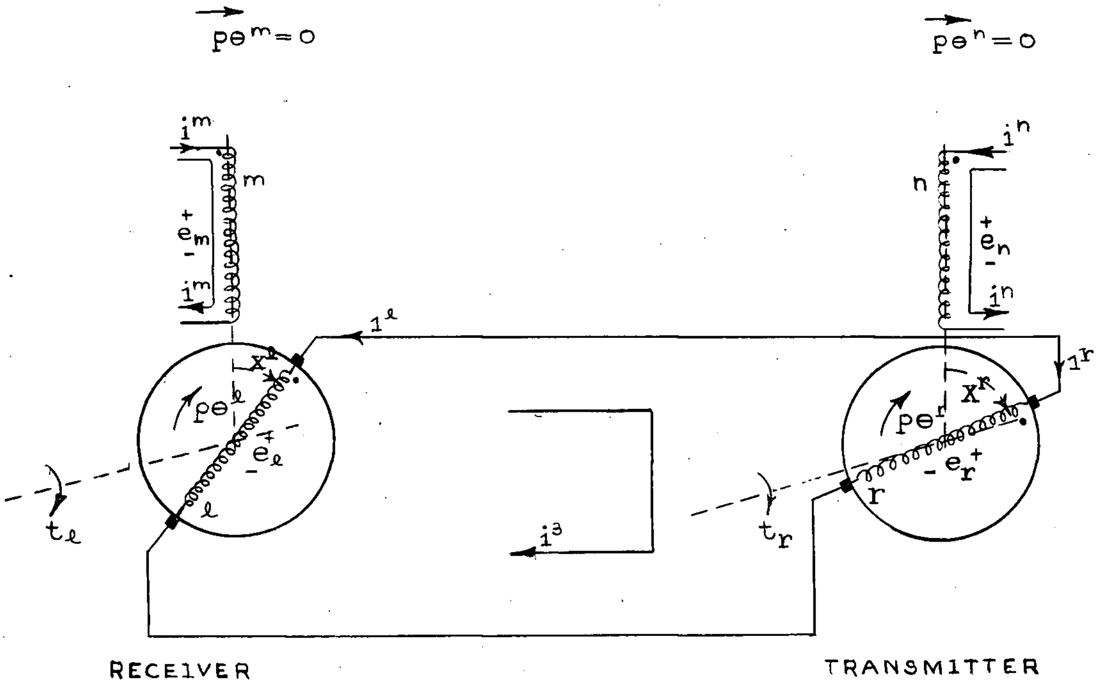


Fig. 5A Selsyn System

The loop equations for a single phase connection are

$$\begin{bmatrix} e_m \\ e_n \\ 0 \end{bmatrix} = \begin{bmatrix} R_{mm} + pL_{mm}, & 0 & -pL_{m\ell} \\ 0 & R_{nn} + pL_{nn}, & pL_{nr} \\ -pL_{m\ell} \quad , & pL_{rn} & R_{33} + pL_{33} \end{bmatrix} \begin{bmatrix} i^m \\ i^n \\ i^3 \end{bmatrix} \tag{5-1}$$

The inductances are obtained from Yu's formula (2-27)

$$p\mathcal{L}_{m\ell} = L_{m\ell}(\cos y^{\ell} p - \sin y^{\ell} p y^{\ell}) \qquad (5\text{-}2)$$

$$p\mathcal{L}_{\ell m} = L_{m\ell}(\cos x^{\ell} p - \sin x^{\ell} p x^{\ell}) \qquad (5\text{-}3)$$

$$p\mathcal{L}_{nr} = L_{nr}(\cos y^{r} p - \sin y^{r} p y^{r}) \qquad (5\text{-}4)$$

$$p\mathcal{L}_{rn} = L_{rn}(\cos x^{r} p - \sin x^{r} p x^{r}) \qquad (5\text{-}5)$$

$$p\mathcal{L}_{ii} = pL_{ii} = L_{ii}p \quad (i = m, l, n, r) \qquad (5\text{-}6)$$
$$p\mathcal{L}_{33} = p(L_{\ell\ell} + L_{rr})$$

The incidence matrix is given by

$$\begin{bmatrix} px^{m} = 0 \\ px^{\ell} \\ px^{r} \\ px^{n} = 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p\theta^{m} = 0 \\ p\theta^{\ell} \\ p\theta^{r} \\ p\theta^{n} = 0 \end{bmatrix} \qquad (5\text{-}7)$$

Substituting (5-6) and (5-7) into (5-1),

$$\begin{bmatrix} e_m \\ e_n \\ 0 \end{bmatrix} = \begin{bmatrix} R_{mm} + pL_{mm}, & 0, & L_{m\ell}(\cos y^{\ell} p - \sin y^{\ell} p y^{\ell}) \\ 0, & R_{nn} + pL_{nn}, & -L_{nr}(\cos y^{r} p - \sin y^{r} p y^{r}) \\ -L_{m\ell}(\cos x^{\ell} p - \sin x^{\ell} p\theta^{\ell}), & L_{nr}(\cos x^{r} p - \sin x^{r} p\theta^{r}), & R_{33} + pL_{33} \end{bmatrix} \begin{bmatrix} i^{m} \\ i^{n} \\ i^{3} \end{bmatrix}$$
$$(5\text{-}8)$$

The torque tensor for the two rotor spaces is

$$T = \begin{matrix} (T)_{\ell} \\ (T)_{r} \end{matrix} = \left[\begin{array}{ccc} 0, & 0, & 0 \\ 0, & 0, & 0 \\ L_{m\ell}\sin x^{\ell}, & 0, & 0 \\ \hline 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & -L_{nr}\sin x^{r}, & 0 \end{array}\right] \qquad (5\text{-}9)$$

The corresponding torque equations for the system are

$$\begin{aligned} t_\ell &= (J_{\ell\ell}\, p + D_{\ell\ell})\; p\theta^\ell \;+ L_{m\ell} \sin x^\ell i^m i^3 \\ t_r &= (J_{rr}\, p + D_{rr})\; p\theta^r \;- L_{nr} \sin x^r i^n i^3 \end{aligned} \qquad (5\text{-}10)$$

Assuming that all angles are measured in terms of electrical angles, and that the machine operation is balanced, an analogous set of equations for a system operation on a greater number of phases can be obtained as follows:

For a two phase arrangement, the currents in the second phase lag those of the first by an angle of $(\pi/2)$. In matrix form

$$\begin{bmatrix} i^m(1) \\ i^m(2) \\ i^n(1) \\ i^n(2) \\ i^3(1) \\ i^3(2) \end{bmatrix} = \begin{bmatrix} 1 & & \\ -j & & \\ & 1 & \\ & -j & \\ & & 1 \\ & & -j \end{bmatrix} \begin{bmatrix} i^m(1) \\ i^n(1) \\ i^3(1) \end{bmatrix} \qquad (5\text{-}11)$$

where the bracketed numbers refer to the phase number. Similarly for a polyphase machine, adding the factor $\frac{1}{\sqrt{p}}$ one obtains

$$\begin{bmatrix} i^1(o) \\ i^2(o) \\ \cdot \\ \cdot \\ i^n(o) \\ i^1(1) \\ \cdot \\ \cdot \\ i^n(1) \\ \cdot \\ i^1(p-1) \\ \cdot \\ \cdot \\ i^n(\dots) \end{bmatrix} = \frac{1}{\sqrt{p}} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \cdot & \\ & & & \cdot \\ & & & 1 \\ e^{-j\frac{2\pi}{p}} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & e^{-j\frac{2\pi}{p}} \\ \cdot & & & \\ e^{-j2\pi\frac{p-1}{p}} & & & \\ \cdot & & & \\ & \cdot & & \\ & & & e^{-j2\pi\frac{p-1}{p}} \end{bmatrix} \begin{bmatrix} i^1(o) \\ i^2(o) \\ \cdot \\ \cdot \\ i^n(o) \end{bmatrix} \qquad (5\text{-}12)$$

$$i^{n}(\phi) = \frac{1}{\sqrt{p}} \; e^{-j\frac{2\pi\phi}{p}} \; i^{n} \tag{5-13}$$

where

ϕ is the phase number, the lowest being zero
p is the number of phases present.
n refers to the n'th coil.

Note that a two phase system is actually a semi-quarter phase system (p=4).

Taking a tensor approach, this matrix equation can be thought of as a transformation of the form

$$i^{n(\alpha)} = C^{n(\alpha)}_{\bar{n}(\bar{\alpha})} \, i^{\bar{n}(\bar{\alpha})} \tag{5-14}$$

in which the unbracketed indices refer to the coil number and the bracketed indices (dead) to the phase number. Here $\bar{\alpha} = 0$: and

$$C^{n(\alpha)}_{\bar{n}(\bar{\alpha})} = \frac{\delta^{m}_{\bar{m}}}{\sqrt{p}} \; e^{-j\frac{2\pi\alpha}{p}} \; \delta_{(\bar{\alpha})(0)} \tag{5-15}$$

Since the transformation is assumed to be from one true multi-phase representation to a true single phase representation

$$e_{\bar{n}(\bar{\alpha})} = C^{n(\alpha)}_{\bar{n}(\bar{\alpha})} \, e_{n(\alpha)} \tag{5-16}$$

The transformed impedance tensor is then

$$z_{\bar{m}\bar{n}(\bar{\alpha})(\bar{\beta})} = C^{n(\alpha)}_{\bar{n}(\bar{\alpha})} \, z_{mn(\alpha)(\beta)} \, C^{n(\beta)}_{\bar{n}(\bar{\beta})} \tag{5-17}$$

$$z_{\bar{m}\bar{n}(\bar{\alpha})(\bar{\beta})} = \frac{\delta_{(\bar{\alpha})(0)}\,\delta^{m}_{\bar{m}}}{\sqrt{p}} \; e^{+j\frac{2\pi\alpha}{p}} \; z_{m(\alpha)n(\beta)} \, \frac{\delta^{n}_{\bar{n}}}{\sqrt{p}} e^{-j\frac{2\pi\beta}{p}} \; \delta_{(\bar{\beta})(0)} \tag{5-18}$$

Dropping the (o) indices

$$z_{\bar{m}\bar{n}} = \frac{z_{\bar{m}(\alpha)\bar{n}(\beta)}}{p} \; e^{j\frac{2\pi}{p}(\alpha-\beta)} \tag{5-18}$$

Or in terms of inductances

$$L_{\bar{m}\bar{n}} = \frac{L_{\bar{m}(\alpha)\bar{n}(\beta)}}{p} \; e^{j\frac{2\pi}{p}(\alpha-\beta)} \tag{5-19}$$

In matrix form this is

$$(\bar{z}) = \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array}\begin{bmatrix} 1 & & & e^{-j\frac{\pi}{p}} & \cdots & e^{-j\frac{\pi}{p}(p-1)} \\ & 1 & & & & \\ & & \ddots & \ddots & & \ddots \\ & & 1 & e & & e^{-j\frac{\pi}{p}(p-1)} \end{bmatrix}_{0,1,2,\ldots,p-1} (Z)^{0,1,2,\ldots,p-1} \begin{array}{c} 1 \ldots n \\ \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & & 1 \\ e^{-j\frac{2\pi}{p}} & & & \\ & \ddots & & \\ & & & e^{-j\frac{2\pi}{p}} \\ \vdots & & & \\ e^{-j2\pi\frac{p-1}{p}} & & & \\ & \ddots & & \\ & & & e^{-j2\pi\frac{p-1}{p}} \end{bmatrix} \end{array} \qquad (5\text{-}20)$$

These results will now be used to reduce the equations of a two phase system to an equivalent single phase system. The impedance tensor of a single two-phase machine is obtained from Yu's formula. "s" denotes stator coils, v, rotor coils and the bracketed index, phase number (fig. 5c)

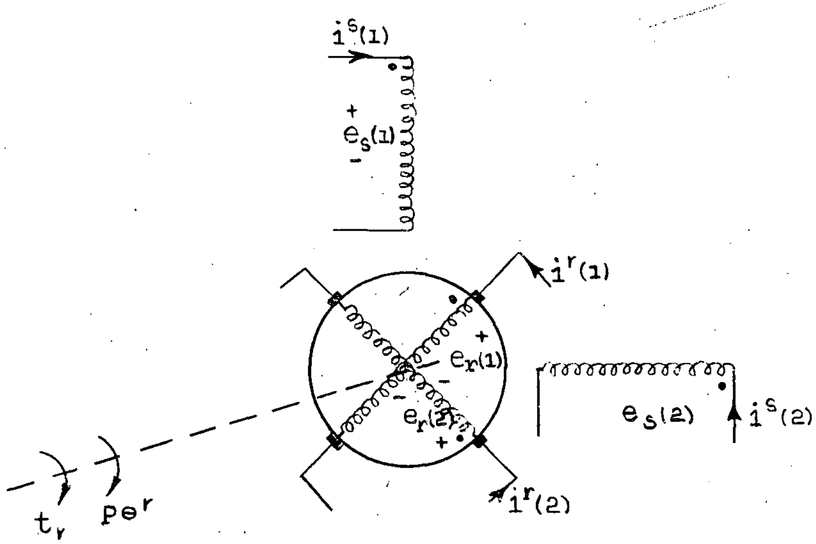


Fig. 5B Two-Phase Selsyn Unit

$$
(z_{mn}) = \begin{array}{c} s(1) \\ s(2) \\ r(1) \\ r(2) \end{array}
\begin{bmatrix}
R_{ss}+pL_{ss}, & 0, & L_{sr}p(\cos y^r), & -L_{sr}p(\sin y^r) \\
0, & R_{ss}+pL_{ss}, & L_{sr}p(\sin y^r), & L_{sr}p(\cos y^r) \\
L_{sr}p(\cos x^r), & L_{sr}p(\sin x^r), & R_{rr}+pL_{rr}, & 0 \\
-L_{sr}p(\sin x^r), & L_{sr}p(\cos x^r), & 0, & R_{rr}+pL_{rr}
\end{bmatrix}
\tag{5-21}
$$

The balanced single phase equivalent is thus

$$
\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & j & 0 & 0 \\ 0 & 0 & 1 & j \end{bmatrix} \circ (z_{mn}) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ j & 0 \\ 0 & 1 \\ 0 & j \end{bmatrix} =
\begin{array}{c|cc}
 & s(1) & r(1) \\
\hline
s(1) & R_{ss}+pL_{ss}, & L_{sr}pe^{jy^r} \\
r(1) & L_{sr}pe^{-jx^r}, & R_{rr}+pL_{rr}
\end{array}
\tag{5-22}
$$

Having obtained the equivalent impedance matrix of a single machine, loop equations for the system can once again be obtained, These are given by (5-1) with the mutual impedance elements obtained in (5-22). Using the notation of Fig. (5A)

$$
\begin{bmatrix} e_m \\ e_n \\ o \end{bmatrix} =
\begin{bmatrix}
R_{mm}+pL_{mm}, & 0, & -L_{m\ell}pe^{jy^\ell} \\
0, & R_{nn}+pL_{nn}, & L_{nr}pe^{jy^r} \\
-L_{m\ell}pe^{-jx^\ell}, & L_{nr}pe^{-jx^r}, & R_{33}+pL_{33}
\end{bmatrix}
\begin{bmatrix} i^m \\ i^n \\ i^3 \end{bmatrix}
\tag{5-23}
$$

These equations are best handled by rotating the reference axis of the 1-coil into the n-axis by

$$
\begin{bmatrix} i^{\bar{m}} \\ i^{\bar{n}} \\ i^{\bar{3}} \end{bmatrix} =
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{j(y^\ell - y^d)} \end{bmatrix}
\begin{bmatrix} i^m \\ i^n \\ i^3 \end{bmatrix}
\tag{5-24}
$$

$$
\begin{bmatrix} e_{\bar{m}} \\ e_{\bar{n}} \\ e_{\bar{3}}=o \end{bmatrix} =
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-j(x^\ell - x^d)} \end{bmatrix}
\begin{bmatrix} e_m \\ e_n \\ e_3=o \end{bmatrix}
$$

This gives, using the incidence matrix of (5-7),

$$
\begin{bmatrix} e_{\bar{m}} \\ e_{\bar{n}} \\ 0 \end{bmatrix} =
\begin{bmatrix}
R_{mm}+pL_{mm}, & 0, & -L_{m\ell}p \\
0, & R_{nn}+pL_{nn}, & L_{nr}e^{+j(y^r - y^\ell)}(p+j(py^r - py^\ell)) \\
-L_{m\ell}(p-jp\theta^\ell), & L_{nr}e^{j(x^\ell - x^r)}(p-jp\theta^r), & R_{33}+L_{33}(p-jpy^\ell)
\end{bmatrix}
\begin{bmatrix} i^m \\ i^n \\ i^3 \end{bmatrix}
\tag{5-25}
$$

where $x^{\ell}-x^{r} = y^{\ell}-y^{r} = \delta$. These equations are in a true system since $\frac{d\epsilon}{dt} = 0$ for both (5-24) and (5-16).

The possibility of commutator rotation is included in these equations through py^{r} and px^{r}.

As an example assume a steady state condition: $py^{r} = py^{\ell} = 0$; $p\theta^{r} = p\theta^{\ell}$ and $p=jw$. Letting $w-p\theta = w(1-v) = w(s)$ and $wL_{i\xi} = X_{i\xi}$ the reactance, one obtains

$$\begin{bmatrix} E_m \\ E_n \\ 0 \end{bmatrix} = \begin{bmatrix} R_{mm}+jX_{mm}, & 0, & -jX_{m\ell} \\ 0, & R + jX_{nn}, & jX_{nr}e^{-j\delta} \\ -jX_{m\ell}, & jX_{nr}e^{j\delta}, & \frac{R_{33}}{s} + \frac{jX_{33}}{s} \end{bmatrix} \begin{bmatrix} I^{\bar{m}} \\ I^{\bar{n}} \\ I^{\bar{3}} \end{bmatrix} \qquad (5\text{-}26)$$

$$(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ jL_{m\ell} & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -jL_{nr}e^{j\delta} & 0 \end{bmatrix} \qquad (5\text{-}27)$$

and the torque equations are

$$\begin{aligned} t_r &= D_{rr}p\theta^{r} + J_{rr}p\,(p\theta^{r}) + jL_{m\ell}i^{\bar{m}}i^{\bar{3}} \\ t_\ell &= D_{\ell\ell}p\theta^{\ell} + J_{\ell\ell}p\,(p\theta^{\ell}) - jL_{nr}e^{j\delta}\,i^{\bar{n}}i^{\bar{3}} \end{aligned} \qquad (5\text{-}28)$$

The stability of the selsyn system can be studied using the equations of small oscillation, which apply directly since the machine is in a true reference system. Since $\frac{\partial e_m}{\partial y^m} = 0$,

$\frac{\partial t_\beta}{\partial \theta^\beta} = 0_{(\beta = r, l)}$, the oscillations equations are

$$\begin{bmatrix} (\Delta e) \\ (\Delta t) \end{bmatrix} = \begin{bmatrix} (z_{mn}(p) & , \frac{\partial L_{mn}}{\partial x^m} i^n \eta^m_\beta \\ +(T + T^t)\ (i))^t, & (D + Jp) \end{bmatrix} \begin{matrix} (\Delta i^n) \\ (\Delta p\theta^n) \end{matrix} + \begin{bmatrix} \frac{\partial L_{mn}}{\partial y^n} i^n p \\ (-\nabla_{i_e} t_\beta) \end{bmatrix} (\Delta y) \qquad (5\text{-}29)$$

In this set of equations $z_{mn}(p)$ is given by (5-25). Directly from these

$$\frac{\partial L_{mn}}{\partial x^m} \; i^n \eta^m_\beta = \begin{array}{c} \\ \bar{m} \\ \bar{n} \\ \bar{3} \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ jL_{m\ell} i^{\bar{m}}, & -jL_{rr} e^{j\delta} \end{bmatrix} \end{array} \qquad (5\text{-}30)$$

$$\frac{\partial L_{mn}}{\partial y^n} \; i^n \Delta p y^n = \begin{array}{c} \bar{m} \\ \bar{n} \\ \bar{3} \end{array} \begin{bmatrix} 0 \\ jL_{nr} e^{-j\delta} \Delta(py^r - py^\ell) \\ -jL_{rr} \Delta p y^\ell \end{bmatrix} \qquad (5\text{-}31)$$

$$\begin{aligned} ((T + T^t)(i))^t &= (i)^t (T + T^t) \\ &= \begin{bmatrix} (i^{\bar{m}}, i^{\bar{n}}, i^{\bar{3}}) \begin{bmatrix} 0, & 0, & jL_{m\ell} \\ 0, & 0, & 0 \\ jL_{m\ell}, & 0, & 0 \end{bmatrix} \\ (i^{\bar{m}}, i^{\bar{n}}, i^{\bar{3}}) \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & -jL_{rr} e^{j\delta} \\ 0, & -jL_{nr} e^{j}, & 0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} +jL_{m\ell} i^{\bar{3}}, & 0, & +jL_{m\ell} i^{\bar{m}} \\ 0, & -jL_{nr} e^{j\delta} i^{\bar{3}}, & -jL_{nr} e^{j\delta} i^{\bar{n}} \end{bmatrix} \end{aligned} \qquad (5\text{-}32)$$

$$\begin{aligned} \nabla_i t_{e\beta} \Delta y^i &= \nabla_i \begin{bmatrix} jL_{m\ell} i^{\bar{m}} i^{\bar{3}} \\ -jL_{nr} e^{j\delta} i^{\bar{n}} i^{\bar{3}} \end{bmatrix} \Delta y^i \\ &= \begin{bmatrix} 0 \\ -jL_{nr} i^n i^3 e^{j\delta} (j\Delta y^\ell - j\Delta y^r) \end{bmatrix} \end{aligned} \qquad (5\text{-}33)$$

The hunting equations for the single phase equivalent of the two-phase selsyn system are thus

$$\begin{bmatrix} \Delta e_{\bar{m}} \\ \Delta e_{\bar{n}} \\ 0 \\ \Delta t_\ell \\ \Delta t_r \end{bmatrix} = \begin{bmatrix} R_{mm} + pL_{mm}, & 0, & -L_{m\ell} p, & 0, & 0 \\ 0, & R_{nn} + pL_{nn}, & L_{nr} e^{-j\delta} (p + jp(y^r - y^\ell)), & 0, & 0 \\ -L_{m\ell}(p - jp\theta^\ell), & L_{nr} e^{j\delta} (p - jp\theta^r), & R_{33} + L_{33}(p - jpy^\ell), & jL_{m\ell} i^{\bar{m}}, & -jL_{nr} e^{j\delta} \\ +jL_{n\ell} i^{\bar{3}}, & 0, & +jL_{m\ell} i^{\bar{m}}, & D_{\ell\ell} + J_{\ell\ell} p, & 0 \\ 0, & -jL_{nr} e^{j\delta} i^{\bar{3}}, & -jL_{nr} e^{j\delta} i^{\bar{n}}, & 0, & D_{rr} + J_{rr} p \end{bmatrix} \begin{bmatrix} \Delta i^{\bar{m}} \\ \Delta i^{\bar{n}} \\ \Delta i^{\bar{3}} \\ \Delta p\theta^\ell \\ \Delta p\theta^r \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ jL_{nr} e^{-j\delta} \Delta(py^r - py^\ell) \\ -jL_{rr} \Delta p y^\ell \\ 0 \\ -jL_{nr} i^n i^3 e^{j\delta} (j\Delta y^\ell - j\Delta y^r) \end{bmatrix} \qquad (5\text{-}34)$$

The brush axis angles and their speeds as well as their variations are constrained externally and are thus known in all equations.

This example illustrates a general procedure for setting up the equations of motion for any system, transforming them, and deriving the hunting equations. To solve these equations and study their general behavior is complicated by the non-linearity involved.

5.2 Synchronous Machines (1-6,10,13,15,16)

In the conventional analysis of synchronous machines the d-q axes are connected to the rotating salient poles, which are viewed as stationary. This is convenient for handling single machine problems; however, when multi-machine problems are attempted, complicated interconnection matrices arise.

This can be avoided by considering a common d-axis for the complete system and applying the equations of motion with rotating saliency to individual machines. Moreover, each machine can be viewed with its stator stationary, rather than from an arrangement in which the stator rotates.

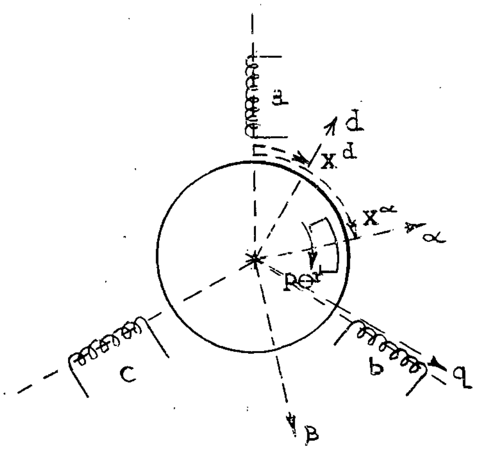


Fig. 5C Synchronous Machine

The new reference scheme is illustrated in Fig. 5C. The $\alpha-\beta$ system is tied to the rotar. The d-q system is rotating at the synchronous speed. For a system of interconnected machines, the d-q systems are parallely displaced, each rotating at the synchronous speed. Consequently interconnection can be made directly along the d-q axis and loop analysis used to obtain the system equations.

The general impedance matrix for a single machine will now be derived using Yu's general formula (3-29) and (3-33). The conventional theory will be considered as a special case obtained when the (d-q) axes are constrained to move with the rotor. The standard polarity of Fig. 5D will be used.

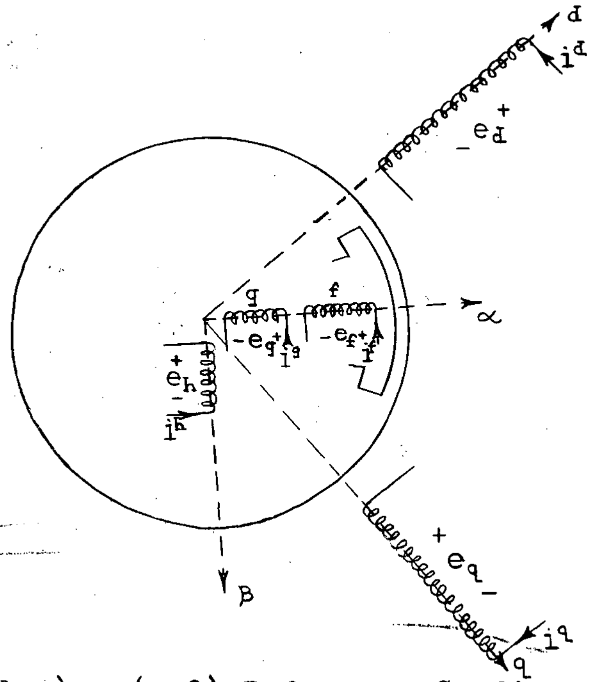


Fig. 5D (d-q) - (α-β) Reference Configuration

S in this case is x, using c for cosine and s for sine, the impedance elements are therefore:

$$pL_{mn} = (L^d_{mn} cX^m cY^n + L^q_{mn} sX^m sY^n)p$$

$$+ \ (-L^{d}_{mn} sX^{m} cY^{n} + L^{q}_{mn} cX^{m} sY^{n}) \frac{\delta X^{m}}{dt}$$

$$+ \ (-L^{d}_{mn} cX^{m} sY^{n} + L^{q}_{mn} sX^{m} cY^{n}) \frac{dY^{n}}{dt} \qquad (5\text{-}35)$$

From Fig. 5D

$$\begin{aligned} p\text{Ł}_{mn} = & \ (L^{d}_{mn} cX^{m} cY^{n} + L^{q}_{mn} sX^{m} sY^{n}) \ p \\ & + (-L^{d}_{mn} sX^{m} cY^{n} + L^{q}_{mn} cX^{m} sY^{n}) \ px^{\alpha} \\ & + (-L^{d}_{mn} cX^{m} sY^{n} + L^{q}_{mn} sX^{m} cY^{n}) \ (py^{n} - py^{\alpha}) \qquad (5\text{-}36) \end{aligned}$$

$$(m, n = d, q)$$

$$\begin{aligned} p\text{Ł} = & \ (L^{d}_{mn} cX^{m} cY^{n} + L^{q}_{mn} sX^{m} sY^{n}) \ p \\ & - (-L^{d}_{mn} sX^{m} cY^{n} + L^{q}_{mn} cX^{m} sY^{n}) \ px^{\alpha} \qquad (5\text{-}37) \end{aligned}$$

$$(m=d, q; \ n=f, g, h)$$

$$\begin{aligned} p\text{Ł}_{mn} = & \ (L^{d}_{mn} cX^{m} cY^{m} + L^{q}_{mn} sX^{m} sY^{n}) \ p \\ & + (-L^{d}_{mn} cX^{m} sY^{n} + L^{q}_{mn} sX^{m} cY^{n}) \ (py^{n} - py^{\alpha}) \qquad (5\text{-}38) \end{aligned}$$

$$(m=f, g, h; \ n=q, d)$$

$$p\text{Ł}_{mn} = L^{d}_{mn} \, p \qquad (m, n, = f, g)$$

$$p\text{Ł}_{mn} = L^{q}_{mn} \, p \qquad (m, n = h) \qquad (5\text{-}39)$$

$$p\text{Ł}_{qh} = pL_{hf} = pL_{hq} = pL_{fh} = 0 \qquad (5\text{-}40)$$

In all these elements $py^{\alpha} = px^{\alpha} = p\theta^{r}$

These could also be obtained by using the synchronous machine transformations of 2:2 b). However, the procedure is more tedious.

The zero sequence equation is omitted in the present analysis since it is uncoupled from the remaining set of equations.

From section 3:5 the torque tensor elements are the coefficients of $p\theta^{\beta}$ excluding contributions from pY^{n}. For the rotor space they are

$$-(-L^{d}_{mn} sX^{m} cY^{n} + L^{q}_{mn} cX^{m} sY^{n}) \qquad (m=d, q, i, n=d, q, f, g, h)$$

The above procedure can be extended to mult-machine systems using loop-analysis.

The conventional synchronous machine equations are obtained by restricting (x^d, y^d) such that; $X^d = 0$, $pX^d = -px^\alpha = -p\theta^r$, $pY^d = 0$. Physically the d-q axis is tied to the $\alpha-\beta$ axis in the stator space. Assuming $N_d = N_q$ and using $L^d_{mn} = L^d_{nm}$; $L^q_{mn} = L^q_{nm}$, the impedance matrix in standard notation is

$$\left[\begin{array}{ccc|cc} R_f+pL_f, & M_{fg}p, & 0 & M_{fd}p, & 0 \\ M_{fg}p, & R_g+L_gp, & 0 & M_{gd}p, & 0 \\ \hline 0, & 0, & R_h+pL_h & 0, & M_{hq}p \\ M_{fd}p, & M_{gd}p, & M_{hq}p\theta & R_d+L_dp, & M_qp\theta^r \\ -M_{fd}p\theta, & -M_{gd}p\theta, & M_{hq}p & -M_dp\theta^r, & R_q+L_qp \end{array}\right] \qquad (5\text{-}41)$$

In the literature, for a generator (alternator), the polarity references of i^m, $p\theta^r$, and t_r are opposite to those shown in Fig. 5D Adopting this reference scheme and partitioning, the equations for an alternator are

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \left[\begin{array}{c|c} Z_1 & Z_2 \\ \hline Z_3 & Z_4 \end{array}\right] \begin{bmatrix} i^1 \\ i^2 \end{bmatrix}; \quad (e_1) = \begin{bmatrix} e_f \\ 0 \\ 0 \end{bmatrix}; \quad i^1 = \begin{bmatrix} i^f \\ i^g \\ i^h \end{bmatrix}$$

$$(e_2) = \begin{bmatrix} e_d \\ e_q \end{bmatrix}; \quad i^2 = \begin{bmatrix} i^d \\ i^q \end{bmatrix}$$

$$(z_1) = \begin{bmatrix} -R_f-pL_f, & -M_{fg}p, & 0 \\ -M_{fg}p, & -R_g-L_gp, & 0 \\ 0, & 0, & -R_h-pL_h \end{bmatrix}; \quad (z_2) = \begin{bmatrix} -M_{fd}p, & 0 \\ -M_{gd}p, & 0 \\ 0, & -M_{hq}p \end{bmatrix}$$

$$(z_3) = \begin{bmatrix} -M_{fd}p, & -M_{gd}p, & M_{hq}p\theta \\ -M_{fd}p\theta, & -M_{gd}p\theta, & -M_{hq}p \end{bmatrix}; \quad (z_4) = \begin{bmatrix} -R_d-L_dp, & M_qp\theta^r \\ -M_dp\theta^r, & -R_q-L_qp \end{bmatrix} \qquad (5\text{-}41)$$

(e_2) is then the d and q axis generated voltage

The first matrix equation in the partitioned set can be used to eliminate i^1 from the second set giving

$$e_2 = (z_3)(z_1)^{-1}(e_1) - (z_3)(z_1)^{-1}(z_2)(i^2) + (z_4)\,i^2 \qquad (5\text{-}42)$$

Written in full this is

$$\begin{bmatrix} e_d \\ e_q \end{bmatrix} = \begin{bmatrix} G(p)pE \\ G(p)p\theta^r E \end{bmatrix} + \begin{bmatrix} -r_d - L_d(p)p \;, & L_q(p)p\theta^r \\ -L_d(p)p\theta^r \;, & -r_d - L_d(p)p \end{bmatrix} \begin{bmatrix} i^d \\ i^q \end{bmatrix} \qquad (5\text{-}43)$$

assuming $r_d = r_q$

$$L_d(p) = M_d - \frac{p^2(L_g(M_{fd})^2 - 2M_{gd}M_{fg}M_{fd} + L_f(M_{gd})^2 + p(R_f(M_{gd})^2 + R_g(M_{fd})^2)}{p^2(L_f L_g - (M_{fg})^2) + p(R_g L_f + R_f L_g) + R_g R_f}$$

$$L_q(p) = M_q - \frac{(M_{hq})^2 p}{R_h + L_h p} \qquad (5\text{-}44)$$

$$G(p) = \frac{p(L_g M_{fd} - M_{fg} M_d + R_g M_{fd}}{p^2(L_f L_g - (M_{fg})^2) + p(R_g L_f + R_f L_g) + R_g R_f}$$

$$E = e_f$$

Writing these equations in the form

$$\begin{bmatrix} e_d - G(p)pE \\ e_q - G(p)p\theta^r E \end{bmatrix} = \begin{bmatrix} -r - L_d(p)p \;, & L_q(p)\,p\theta^r \\ -L_d(p)p\theta^r \;, & -r - L_q(p)\,p \end{bmatrix} \begin{bmatrix} i^d \\ i^q \end{bmatrix} \qquad (5\text{-}45)$$

it is seen that for the armature axes, the impedance tensor has exactly the same form as previously, (z_4), the only difference being that all open circuit quantities are replaced by short circuit quantities. In per unit notation, adding the zero sequence equation $(i^o = \frac{1}{\sqrt{3}}(i^a + i^b + i^c)$, where i^a, i^b, i^c are the phase quantities) which remains unchanged throughout, Parks equations are obtained.[6]

$$e_d = G(p)pE - z_d(p)\, i + x_q(p)\, i^q$$

$$e_q = G(p)p\theta^r E - x_d(p)p\theta^r i^d - z_q(p) i^q \qquad (5\text{-}46)$$

$$e_o = -z_o i^o$$

The torque equation (3-34) i.e.

$$t_r = D_{rr}p\theta^r + J_{rr}p(p\theta^r) + \frac{\partial L_{mn}}{\partial X^m}\sigma^m_r i^m i^n$$

The electromagnetic energy conversion torque $\underset{e}{t}_r$ is the torque to be overcome by the prime mover.

$$t_{e}{}_{r} = \frac{\partial L_{mn}}{\partial X^m}\sigma^m_r i^m i^n = (i)^t (T)_r (i)$$

$$= (i^d, i^q) \begin{bmatrix} 0, & 0, & M_{hq} & 0 & M_q \\ -M_{fd}, & -M_{gd}, & 0 & -M_d & 0 \end{bmatrix} \begin{bmatrix} i^f \\ i^g \\ i^h \\ i^d \\ i^q \end{bmatrix}$$

$$= (i^q)(-M_{fd}\, i^f - M_{gd} i^g - M_d i^d) + i^d (M_{hq} i^h + M_q i^q) \qquad (5\text{-}47)$$

Using once again the first partition of (5-41) to eliminate (i')

$$t_{e}{}_{r} = i^d i^q (x_q(p) - x_d(p)) + i^q G(p)\, e_f \qquad (5\text{-}48)$$

in Park's per-unit system.

A different form of the equations (5-46) and (5-48) can be obtained as follows. For the unsaturated case, once the impedance matrix has been obtained its coefficients no longer display any angular dependence. We can write for the alternator

$$(e) = -(R)(i) - \frac{d}{dt}\ (L)(i) + (T)_r (i) p\theta^r \qquad (5\text{-}49)$$

where

$$(L) = \begin{bmatrix} +L_f & M_{fg} & 0 & M_{fd} & 0 \\ +M_{fg} & L & 0 & M_{gd} & 0 \\ 0 & 0 & L_h & 0 & M_{hq} \\ +M_{fd} & M_{gd} & 0 & L_d & 0 \\ 0 & 0 & M_{hq} & 0 & L_q \end{bmatrix}$$

$$(T)_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{hq} & 0 & M_q \\ -M_{fd} & -M_{gd} & 0 & -M_d & 0 \end{bmatrix} \qquad (5\text{-}50)$$

(L)(i), gives the flux in each coil due to excitation of all coils in the same axis. $(T)_r(i)$, gives the flux in each coil due to excitation of all coils at right angles to it and hence is a cross-flux.

After elimination of (i'), the rotor currents, (5-49) can be written with the zero sequence added [6]

$$
\begin{aligned}
e_d &= -ri^d + p\psi_d - \psi_q p\theta \\
e_q &= -ri^q + p\psi_q + \psi_d p\theta \\
e_o &= -ri_o + p\psi_o
\end{aligned}
\qquad (5\text{-}51)
$$

Here

$$
\begin{aligned}
\psi_q &= -x_q(p)\, i^q \\
\psi_d &= G(p)E - x_d(p)\, i^d
\end{aligned}
\qquad (5\text{-}52)
$$

In this form the torque equations (5-48) is

$$
t_e^r = i^q \psi_d - i^d \psi_q \qquad (5\text{-}53)
$$

For an alternator at steady state p=0 and $p\theta^r=1$ in the per unit system Also $x_d(p) = x_d$ and $x_q(p) = x_q$, $G(p) = \frac{M_{fd}}{R_f}$. If $E = E_f \frac{R_f}{M_{fd}}$ where E_f is the "internal generated voltage", then $G(p)E = E_f$. Assuming the alternator is connected to an infinite bus whose field angle lags that of the generator by δ so that

$$
\begin{bmatrix} e_f \\ e_d \\ e_q \end{bmatrix} = \begin{bmatrix} E \\ e \sin\delta \\ e\cos\delta \end{bmatrix} \qquad (5\text{-}54)
$$

the equations of motion become

$$
\begin{bmatrix} e\sin\delta \\ e\cos\delta - E_f \end{bmatrix} = \begin{bmatrix} -r & x_q \\ -x_d & -r \end{bmatrix} \begin{bmatrix} i^d \\ i^q \end{bmatrix} \qquad (5\text{-}55)
$$

$$
t_e^r = i^d i^q (x_q - x_d) + E_f i^q \qquad (5\text{-}56)
$$

The hunting equations for a synchronous machine can be obtained from (4-23) by the following changes (c.f. 3-33; 3-34).

$$y^n \longrightarrow Y^n$$
$$x^n \longrightarrow X^n$$
$$\eta^m_\beta \longrightarrow \sigma^m_\beta$$

In the representation used here the reference axes remain tied to the field axes and thus oscillate ($\Delta Y^n = 0$). The hunting equations become

$$\begin{bmatrix}(\Delta e)\\(\Delta t)\end{bmatrix} = \begin{bmatrix}(z_{mn}(p)) & , \left(\frac{\partial \Phi_m}{\partial X^n}\sigma^n_\beta\right) + \left(\frac{\partial e_m}{\partial Y^n}\sigma^n_\beta\right)\frac{1}{p}\\ +(\nabla_i \Phi_n \sigma^i_\beta) & , \quad (Z_{\alpha\beta}(p))\end{bmatrix}\begin{bmatrix}(\Delta i)\\(\Delta p\theta)\end{bmatrix} \tag{5-57}$$

For these equations the references of Fig. 5D will be used.

From (5-54)

$$\frac{\partial e_m}{\partial Y^n} = \begin{bmatrix}0\\0\\0\\e \cos\delta \frac{\partial \delta}{\partial Y^n}\\-e \sin\delta \frac{\partial \delta}{\partial Y^n}\end{bmatrix} \tag{5-58}$$

For a machine connected to an infinite bus

$$\delta = y^\alpha - y^{\alpha b} = y^\alpha - y^d - (y^{\alpha b} - y^d)$$

where $y^{\alpha b}$ is the rotor angle of the bus with respect to the chosen reference. Using this

$$\frac{\partial \delta}{\partial Y^n} = -1$$

Hence

$$\frac{\partial e_m}{\partial Y^n} = \begin{bmatrix}0\\0\\0\\-e \cos\delta\\e \sin\delta\end{bmatrix} \tag{5-59}$$

The voltage variation Δe is

$$\Delta e = \Delta\begin{bmatrix}e_f\\0\\0\\e \sin\delta\\e \cos\delta\end{bmatrix} = \begin{bmatrix}0\\0\\0\\e \cos\delta\\-e \sin\delta\end{bmatrix}\Delta\delta + \begin{bmatrix}\Delta e_f\\0\\0\\\Delta e \sin\delta\\\Delta e \cos\delta\end{bmatrix} \tag{5-60}$$

where $\Delta\delta$ is the variation of δ other than Lie, the latter having already been included in the equations of motion. Here

$$\Delta\delta = \Delta(y^{\alpha} - y^{d}) - \Delta(y^{\alpha\delta} - y^{\alpha}) = \Delta y^{\alpha} = \Delta\theta^{r} \qquad (5\text{-}61)$$

The generalized incidence matrix is given by

$$\begin{bmatrix} pX^{f} \\ pX^{g} \\ pX^{h} \\ pX^{d} \\ pX^{q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} (p\theta^{r}) \qquad (5\text{-}62)$$

Therefore the remaining terms of (5-57) are

$$\frac{\partial e_m}{\partial Y^n}\sigma^n_{\beta} = \frac{\partial e_m}{\partial Y^n}\sigma^n_{r} = \begin{bmatrix} 0 & & & & 0 \\ & 0 & & & 0 \\ & & 0 & & 0 \\ & & & -e\cos\delta & -1 \\ & & & e\sin\delta & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e\cos\delta \\ -e\sin\delta \end{bmatrix} \qquad (5\text{-}63)$$

$$\frac{\partial \Phi_m}{\partial X^n}\sigma^n_{r} = (T)_r\, i^n \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{hq} & 0 & M_q \\ -M_{fd} & -M_{gd} & 0 & -M_d & 0 \end{bmatrix} \begin{bmatrix} i^f \\ i^g \\ i^h \\ i^d \\ i^q \end{bmatrix} \qquad (5\text{-}64)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ M_{hq}i^h + M_q i^q \\ -M_{fd}i^f - M_{gd}i^g - M_d i^d \end{bmatrix} \qquad (5\text{-}65)$$

$$\nabla_i \Phi_h \sigma^i_r = (i)^t((T)^t_r + (T)_r)$$

$$= \begin{bmatrix} -M_{fd}i^q \\ -M_{gd}i^q \\ M_{hq}i^d \\ M_{hq}i^h + (M_q - M_d)\, i^q \\ -(M_{fd}i^f + M_{gd}i^g) + (M_q - M_d)\, i^d \end{bmatrix}^t \qquad (5\text{-}66)$$

Substituting these in (5-54) the oscillations equations with the polarity of Fig. 5D are on the following page.

Although this derivation is for a single machine, the approach can be extended to a multi-machine system.

$$
\begin{bmatrix} \Delta e_f \\ 0 \\ 0 \\ \Delta e \sin\delta \\ \Delta e \cos\delta \\ \Delta t \end{bmatrix}
=
\begin{bmatrix}
R_f+pL_f, & M_{fg}p, & 0, & M_{fd}p, & 0, & 0 \\
M_{fg}p, & R_g+pL_g, & 0, & M_{gd}p, & 0, & 0 \\
0, & 0, & R_h+pL_h, & 0, & M_{hq}p, & 0 \\
M_{fd}p, & M_{gd}p, & M_{hq}p\theta^r, & R_d+L_dp, & M_qp\theta^r, & M_{hq}i^h+M_qi^q \\
-M_{fd}p\theta^r, & -M_{gd}p\theta^r, & M_{hq}p, & -M_dp\theta, & R_q+L_qp, & -M_{fd}i^f-M_{gd}i^g-M_di^d \\
-M_{fd}i^q, & -M_{gd}i^q, & M_{hq}i^d, & M_{hq}i^h+(M_q-M_d)i^q, & -(M_{fd}i+M_{gd}i^g)+(M_q-M_d)\,i^d, & Dp+Jp^2
\end{bmatrix}
\begin{bmatrix} \Delta i^f \\ \Delta i^g \\ \Delta i^h \\ \Delta i^d \\ \Delta i^q \\ \Delta\theta^r \end{bmatrix}
$$

(5-57)

(5-67)

6. CONCLUSIONS

By selecting a set of independent coordinates, i.e., a holonomic system, and applying Hamilton's Principle, the voltage and torque equations applicable to any physical machine were derived. Contributions arising from interactions between mechanical rotations and magnetic fluxes were shown to be due to Lie variations.

Once the basic equations were derived, the hunting equations and transformation properties followed readily. Of importance is the independance of the voltage transformation matrix, ($\mathcal{C}$), and the current transformation matrix, (C), for a general transformation of the system equations. Restrictions arising when the transformation is from one true system to another leads to the invariance properties of power.

The table following compares the conventional ideas, principally those of Kron, with those presented in the present thesis.

The thesis indicates several possible areas of investigation: stability and optimization studies using the Hamiltonian approach; the realizability of a true system, which may not necessarily be a physical system although the converse is always true; and the investigation of non-linear problems.

	Conventional	Present Thesis
Holonomic system	Slip ring machines only	Any physical machine
Lagrange's equations	Slip ring machines only	Any physical machine taking into account Lie variations
Quasi-holonomic	Fixed commutator axis machines	No such system
Non-holonomic	Anything which is not included in the above, hypothetical or physical	Any hypothetical machine for which the transformation matrices (C) and (€) from a physical machine satisfy $\frac{dC}{dt} \neq 0; \frac{d€}{dt} \neq 0$
Direct application of loop or Nodal analysis	Slip ring machines	Any true system; includes all physical systems
Moving saliency	Non-holonomic	Holonomic
Invariance of tensor products	- - -	Follows from (C) and (€) properties
Polarity convention	No standard references	Follows from the basic equations
General comment	General equations derived by Kron apply only to a linear system, i.e., $\Phi_m = L_{mn}(x^m y^n) i^n$	System equations obtained are completely general, including non-linearities, and are less complicated in form.

Table 2. Comparison of Conventional Ideas with those of the Present Thesis

APPENDIX I CONDITIONS ON THE FLUX TENSOR TO GUARANTEE AN EXTREMUM

The existance of an extremum of the action is guaranteed by three tests which can be applied to the Lagrangian. These extend the results of reality tests for mutual inductance in static circuit theory.

The three tests will be dealt with in turn.

a) The Euler Equations are satisfied.

b) The Legendre Test. If $S = \int_0^t L(t, q^i, \dot{q}^i)dt$ is an extremum the quadratic form whose matrix is $(a_{ij}) = (E_{\dot{q}^i, \dot{q}^j})$ must be positive definite. Thus the matrix

$$\begin{bmatrix} \frac{\partial \Phi_1}{\partial i^1}, & \frac{\partial \Phi_1}{\partial i^2} & \cdots \\ \frac{\partial \Phi_2}{\partial i^1}, & \frac{\partial \Phi_2}{\partial i^2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

is positive definite. A necessary and sufficient condition for this to be true is that the minor determinants be greater than or equal to zero. That is

$$\frac{\partial \Phi_j}{\partial i^j} \geqslant 0$$

$$\begin{bmatrix} \frac{\partial \Phi_1}{\partial i^1}, & \frac{\partial \Phi_1}{\partial i^2} \\ \frac{\partial \Phi_2}{\partial i^1}, & \frac{\partial \Phi_2}{\partial i^2} \end{bmatrix} \geqslant 0$$

$$\vdots$$

For the unsaturated case this is

$$L_{jj} \geq 0$$

$$\begin{bmatrix} L_{11} , & L_{12} \\ L_{21} , & L_{22} \end{bmatrix} \geq 0$$

$$\vdots \qquad \vdots$$

c) The Jacobi Test: All terms of the form

$$\left(\frac{\partial^2 E}{\partial i^r \partial i^r}\right)\left(\frac{\partial^2 E}{\partial i^s \partial i^s}\right) - \left(\frac{\partial^2 E}{\partial i^s \partial i^r}\right)^2$$

must be greater than or equal to zero. That is

$$\left(\frac{\partial \Phi_r}{\partial i^r}\right)\left(\frac{\partial \Phi_s}{\partial i^s}\right) - \left(\frac{\partial \Phi_r}{\partial i^s}\right)^2 \geq 0$$

For no saturation this is

$$(L_{rr})(L_{ss}) - (L_{rs})^2 \geq 0$$

This is included in the above since $L_{rs} = L_{sr}$ by energy considerations or as can be proved directly since

$$\frac{\partial E}{\partial i^r} = \Phi_r$$

$$\frac{\partial^2 E}{\partial i^s \partial i^r} = \frac{\partial \Phi_r}{\partial i^s} = \frac{\partial^2 E}{\partial i^r \partial i^s} = \frac{\partial \Phi_s}{\partial i^r} \quad .$$

APPENDIX II THE HAMILTONIAN; ENERGY AND CO-ENERGY

The Lagrangian of an electromechanical system is an implicit function of time and hence if dissipation is neglected (closed system) the Hamiltonian is conserved. If H is the Hamiltonian for the electrical part of the system and L_e the Lagrangian

$$H = \frac{\partial L_e}{\partial i^{i}} i^{i} - L_e$$

$$= \Phi_i i^{i} - E_e$$

H is this case is not equal to E_e generally and hence the energy is generally not a constant under free oscillations of the machine.[14] It will fluctuate so as to maintain H a constant.

An interpretation of the Hamiltonian can be made by generalizing the case of a single coil. (Fig. IIA).

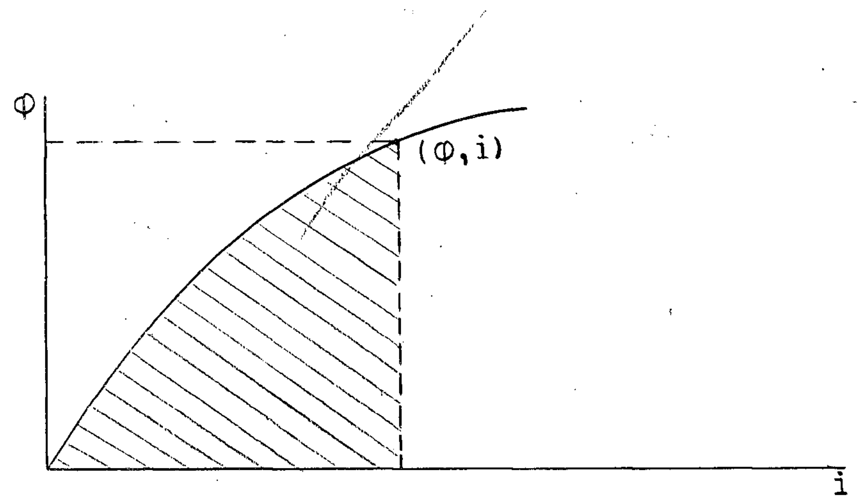


Fig. IIA Single Coil B H Curve

In this case the energy is $\int_0^i \Phi \, di$ which is equal to the shaded area. The co-energy is defined as

$$C_E = \Phi i - E = \Phi i - \int_0^i \Phi(i) di$$

which is the unshaded area of the rectangle, and is the Hamiltonian.

Thus as a generalization, the co-energy of the system is defined as

$$C_E = \Phi_i i^i - E_e = H$$

and the Hamiltonian is equal to the generalized co-energy.

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