WAVES IN INHOMOGENEOUS ISOTROPIC MEDIA

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ABSTRACT

For the case of a lossless medium containing no free charges and possessing a continuous and sufficiently differentiable spatially dependent permeability and permittivity, two vectorial differential wave equations, one for the electric and one for the magnetic field, are derived through the use of Maxwell's equations. From these two equations necessary conditions for E- and H-modes to exist in a waveguide are established.

The field equations for the case of constant permeability and z-dependent permittivity as well as the interchanged case are investigated. A test is developed which, if met, assures that the solutions are oscillatory for the ordinary differential equations containing the z-dependent part of the wave function.

For the dielectric loaded periodic structure the theory for inhomogeneous isotropic media is used to determine the restrictions on the field components which are necessary before E-modes can exist and to find the E-mode wave solutions for the solid disc case when the dielectric regions are matched into the air regions.

An investigation is carried out into the behaviour of plane waves in a medium with the permeability constant and the permittivity varying in the direction of propagation.

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WAVES IN INHOMOGENEOUS ISOTROPIC MEDIA

1. INTRODUCTION

The purpose of this thesis is to investigate theoretically the behaviour of electromagnetic waves in lossless inhomogeneous isotropic media containing no free charges.

Throughout this thesis inhomogeneous isotropic media will be called inhomogeneous media.

Inhomogeneous media are of interest because they may be used to make slow wave structures which in turn can be used in linear accelerators, traveling wave tubes, backward-wave oscillators, and microwave filters. Besides this, inhomogeneous media may be used in pre-accelerator designs.

At Stanford University, G.S. Kino has considered using a waveguide filled with a plasma of uniform cross-sectional density and with an axial density variation of the form $\rho_0 (1+\alpha \sin \gamma z)$ for a slow wave structure.[1] The axial variation in plasma density is to be achieved by propagating sound waves down the waveguide. In this case

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1. G.S. Kino, A Proposed Millimeter-wave Generator, Microwave Laboratory, W.W. Hanson Laboratories of Physics, Stanford University, Stanford, California.

the plasma forms an inhomogeneous medium.

The topic of this thesis arose during an investigation into the solution for the wave functions in a dielectric loaded periodic structure. Such a structure is shown in Figure 1.
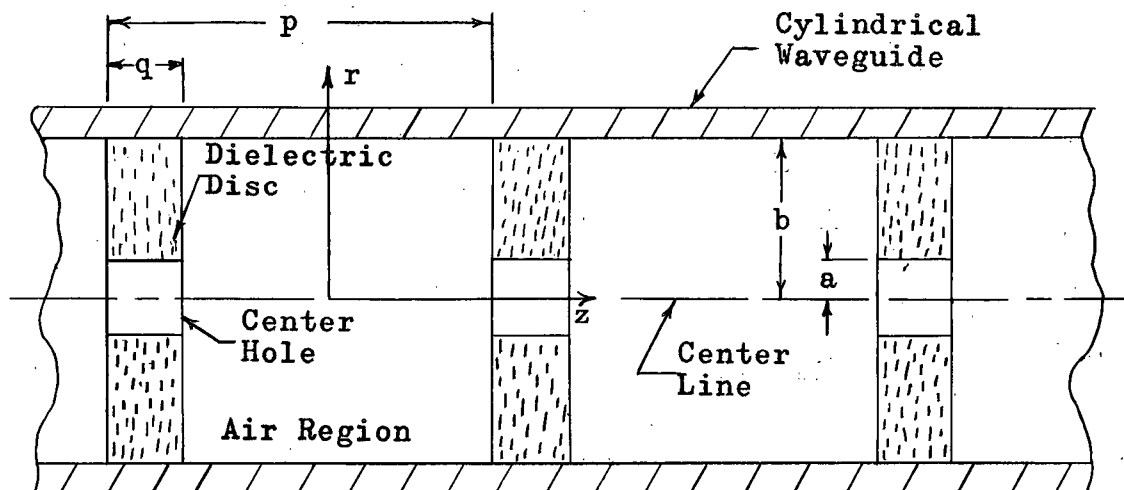
ε_1 = Permittivity of dielectric material

ε_0 = Permittivity of air

z = Longitudinal Coordinate ⎤

r = Radial coordinate ⎬ Cylindrical Coordinates

ϕ = Angular coordinate ⎦

Note: Discs are solid when a equals 0.

Fig. 1. <u>Cross Section of Dielectric Loaded Periodic Structure</u>

For a periodic structure similar to the one shown in Figure 1, the exact wave functions can be readily determined provided the dielectric discs are solid. These wave functions are derived in Appendix 1 for the E_{01}-mode. If the periodic structure is used in the usual fashion for

accelerator or traveling wave tube applications, electrons
must pass along the axis of the waveguide. To make this
possible, a hole must exist through the center of each disc.
With the hole present the problem of finding the wave
functions becomes exceedingly complex. In principle, this
problem can be solved by solving in each homogeneous region
the differential wave equations developed from Maxwell's
equations and by matching the solutions for the different
regions at the boundaries.[2] Also, it should be noted that
Floquet's Theorem must be used in the same manner as it is
used in Appendix 1. Due to the excessive labour involved
in any numerical work carried out to establish a match at
all the boundaries, the results have only formal
significance. Consequently, it has been necessary to use
the solid disc theory and/or the anisotropic theory
approximations in the design of periodic structures having
center holes in the dielectric discs.[2]

Since previous techniques used in attempting to
solve the problem in which the dielectric discs have center
holes are not entirely satisfactory, it was thought that a
different approach might be useful. Instead of placing the
emphasis on the medium inside the waveguide being made up
of homogeneous sections, it was decided that an investiga-
tion should be carried out with the emphasis shifted to the

2. R.B.R.-Shersby-Harvie et al, "A Theoretical and
   Experimental Investigation of Anisotropic-Dielectric-
   Loaded Linear Electron Accelerator", Proc. of I.E.E.,
   vol. 104, Part B, 1957.

fact that the medium as a whole is inhomogeneous. In other words, the permittivity is a function of the spatial parameters.

Three reasons can be advanced for following this approach. One reason is that a different approach at times reveals new information about a problem, and a second reason is that only one vectorial differential wave equation has to be solved, as will be shown in section 2.2, to obtain a field solution which holds throughout the waveguide. Also, since the permittivity in the neighbourhood of the boundaries between the air and dielectric regions as well as elsewhere is assumed continuous and sufficiently differentiable, the theory developed to attack the problem in which the dielectric discs have center holes can be expanded to include inhomogeneous media in general. The continuity and differentiability assumptions will be discussed in section 4.2.

The one vectorial differential wave equation, which is valid throughout the waveguide, yields three scalar partial differential equations. These scalar partial differential equations will hereafter be referred to as the unified differential equations. As it turned out, when the dielectric discs have center holes, no technique was devised to find the general solution for any of the unified differential equations. Consequently, the original objective was not achieved. However, this problem initiated the following work in this thesis.

For the case where both the permittivity and permeability are continuous and sufficiently differentiable functions of the spatial parameters, the electric and magnetic vectorial differential wave equations are derived. Through the use of these equations, necessary conditions for E- and H-modes to exist in a waveguide are found. An example of a use of the E-mode condition is shown.

For the case where either the permittivity or permeability is a function only of the axial parameter z and the remaining characteristic of the medium is constant, the pertinent unified differential equations are separated into ordinary differential equations. A test is developed which, if met, assures that the solutions are oscillatory for the ordinary differential equations containing the axial dependent portion of the wave function.

For the E-mode case certain limitations which must be imposed upon the field components in the dielectric loaded periodic structure are investigated using the theory for inhomogeneous media. Also, when the dielectric discs are solid, provided the dielectric regions are matched into the air regions, for the E-mode case a solution for the pertinent unified differential equation is given.

To provide a better physical understanding of the behaviour of electromagnetic waves in an inhomogeneous medium, an investigation is carried out into the behaviour of plane waves in a medium with the permeability constant

and with the permittivity varying in the direction of propagation.

In this thesis the behaviour of E-modes is investigated far more thoroughly than the behaviour of H-modes. The reason for this is that the dielectric loaded periodic structure discussed in this thesis is primarily used for linear accelerator and traveling wave tube applications and in these applications E-modes and not H-modes are excited.

2. GENERAL THEORY

2.1 Introduction

Through the use of Maxwell's equations, the following wave theory will be developed for the case of a lossless medium containing no free charges. To begin with, the situation where the permeability and permittivity of the medium are general functions of the spatial coordinates will be considered. Following this, the case with the permeability constant and permittivity a function of z will be treated along with the interchange of this case.[*]

Maxwell's equations in a medium containing no free charges and with zero conductivity are

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \tag{1}$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}, \tag{2}$$

$$\nabla \cdot \bar{D} = 0, \tag{3}$$

$$\nabla \cdot \bar{B} = 0 \tag{4}$$

where

$\bar{E}$ = Electric field intensity vector,

$\bar{D}$ = Electric flux density vector,

$\bar{H}$ = Magnetic field intensity vector,

$\bar{B}$ = Magnetic flux density vector.

Also, $\bar{D}$ and $\bar{E}$ are related by the equation

$$\bar{D} = \varepsilon \bar{E}, \tag{5}$$

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[*] Interchanged case is that of constant permittivity and of z-dependent permeability.

and $\bar{B}$ and $\bar{H}$ are related by the equation

$$\bar{B} = \mu\bar{H} \qquad\qquad (6)$$

where ϵ is the permittivity and μ is the permeability.

2.2 Permeability and Permittivity, Functions of the Space Coordinates

To obtain an expression for the electromagnetic field in a homogeneous medium, the differential wave equation which has to be solved is the standard equation

$$\nabla^2\phi = \mu\epsilon\frac{\partial^2\phi}{\partial t^2}$$

where $\mu\epsilon$ is constant.

When the medium is not homogeneous, the permeability and permittivity being continuous and sufficiently differentiable functions of the space coordinates, the differential wave equations from which the field expressions can be obtained are somewhat more complex. These more complex differential equations can be arrived at in the following manner.

For the electric field the vectorial differential wave equation can be derived by first taking the curl of equation (1).

$$\nabla \times (\nabla \times \bar{E}) = -\frac{\partial}{\partial t}\nabla \times \bar{B}.$$

Since

$$\bar{B} = \mu\bar{H}$$

and

$$\nabla \times (\nabla \times \bar{E}) = \nabla(\nabla \cdot \bar{E}) - \nabla^2\bar{E},$$

then

$$\nabla(\nabla.\bar{E}) - \nabla^2\bar{E} = -\frac{\partial}{\partial t}(\nabla \times \mu\bar{H})$$

or

$$\nabla(\nabla.\bar{E}) - \nabla^2\bar{E} = -\frac{\partial}{\partial t}\left[\mu\frac{\partial}{\partial t}(\epsilon\bar{E}) + \nabla\mu \times \bar{H}\right].$$

Since it has been assumed that the permeability and permittivity are not functions of time,

$$\nabla(\nabla.\bar{E}) - \nabla^2\bar{E} = -\mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} - \nabla\mu \times \frac{\partial\bar{H}}{\partial t}$$

$$= -\mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} - \frac{1}{\mu}\nabla\mu \times \frac{\partial\bar{B}}{\partial t}$$

$$= -\mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} + \frac{1}{\mu}\nabla\mu \times (\nabla \times \bar{E})$$

$$= -\mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} + \frac{1}{\mu}\left[\nabla\mu.\nabla\bar{E} - (\nabla\mu.\nabla)\bar{E}\right]$$

where in rectangular coordinates

$$\nabla\mu.\nabla\bar{E} = \nabla\mu.\frac{\partial\bar{E}}{\partial x}\bar{i} + \nabla\mu.\frac{\partial\bar{E}}{\partial y}\bar{j} + \nabla\mu.\frac{\partial\bar{E}}{\partial z}\bar{k}$$

with $\bar{i}$, $\bar{j}$ and $\bar{k}$ being the unit vectors in the x,y, and z directions respectively. From equations (3) and (5)

$$\nabla.\bar{D} = \nabla.\epsilon\bar{E} = 0.$$

Consequently,

$$\nabla\epsilon.\bar{E} + \epsilon\nabla.\bar{E} = 0$$

and thus

$$\nabla.\bar{E} = -\frac{1}{\epsilon}\nabla\epsilon.\bar{E}.$$

Therefore, the vectorial differential wave equation for the electric field is

$$\nabla^2\bar{E} + \nabla\left(\frac{1}{\epsilon}\nabla\epsilon.\bar{E}\right) = \mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} - \frac{1}{\mu}\left[\nabla\mu.\nabla\bar{E} - (\nabla\mu.\nabla)\bar{E}\right]. \quad (7)$$

Similarly, the vectorial differential wave equation for the magnetic field is

$$\nabla^2 \bar{H} + \nabla(\frac{1}{\mu} \nabla\mu \cdot \bar{H}) = \mu\varepsilon\frac{\partial^2 \bar{H}}{\partial t^2} - \frac{1}{\varepsilon}\left[\nabla\varepsilon \cdot \nabla\bar{H} - (\nabla\varepsilon \cdot \nabla)\bar{H}\right].$$

(8)

At this point sufficient theory has been developed to establish a necessary condition for the existence of an E-mode in a waveguide and a dual condition for the H-mode case.

2.21 E-Mode Condition

If the permeability and permittivity are continuous and sufficiently differentiable for all interior points in a waveguide, then for an E-mode to exist

$$\frac{\partial}{\partial z}\left(\frac{1}{\mu} \nabla\mu \cdot \bar{H}\right) + \frac{1}{\varepsilon} \nabla\varepsilon \cdot \frac{\partial \bar{H}}{\partial z} = 0$$

(9)

where z is the coordinate in the direction of propagation.
Proof: From equation (8) it can be seen that the scalar equation obtained when the coefficients of the component vector in the z direction are equated is

$$-\nabla^2 H_z + \mu\varepsilon\frac{\partial^2 H_z}{\partial t^2} + \frac{1}{\varepsilon} (\nabla\varepsilon \cdot \nabla)H_z = \frac{\partial}{\partial z}\left(\frac{1}{\mu} \nabla\mu \cdot \bar{H}\right) + \frac{1}{\varepsilon}\nabla\varepsilon \cdot \frac{\partial \bar{H}}{\partial z}$$

where H_z is the z component of $\bar{H}$.

Since for an E-mode

$$H_z = 0,$$

then

$$\frac{\partial}{\partial z}\left(\frac{1}{\mu} \nabla\mu \cdot \bar{H}\right) + \frac{1}{\varepsilon} \nabla\varepsilon \cdot \frac{\partial \bar{H}}{\partial z} = 0.$$

The dual for this condition is the following one.

2.22 H-Mode Condition

If the permeability and permittivity are continuous and sufficiently differentiable for all interior points in a waveguide, then for an H-mode to exist

$$\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon}\nabla\varepsilon\cdot\mathbf{E}\right) + \frac{1}{\mu}\nabla\mu\cdot\frac{\partial\mathbf{E}}{\partial z} = 0. \tag{10}$$

Proof: The proof for this condition is the same as for the previous condition except that equation (7) is used instead of equation (8), and also,

$$E_z = 0$$

instead of

$$H_z = 0$$

where E_z is the z component of $\mathbf{E}$.

2.23 Example of Use of a Mode Condition

The mode conditions can be used to determine certain restrictions which must be imposed upon the fields before E- and H-modes can exist in a waveguide filled with an inhomogeneous medium.

An example which demonstrates this use is the case where an E-mode exists in a waveguide which is filled with a medium having a permeability that is constant and a permittivity that is sufficiently well defined and satisfies the equation

$$\varepsilon = f(r,z) \tag{11}$$

where r is the radial parameter. For a waveguide filled with a medium which behaves in this manner, from the E-mode condition

$$\nabla \varepsilon \cdot \frac{\partial \bar{H}}{\partial z} = 0$$

or in cylindrical coordinates

$$\frac{\partial \varepsilon}{\partial r} \frac{\partial H_r}{\partial z} + \frac{1}{r} \frac{\partial \varepsilon}{\partial \phi} \frac{\partial H_\phi}{\partial z} + \frac{\partial \varepsilon}{\partial z} \frac{\partial H_z}{\partial z} = 0 \qquad (12)$$

where H_r is the radial component and H_ϕ is the angular component of $\bar{H}$. Since the permittivity does not have angular dependence,

$$\frac{\partial \varepsilon}{\partial \phi} = 0.$$

Also, for an E-mode

$$H_z = 0. \qquad (13)$$

Hence, from equation (12)

$$\frac{\partial \varepsilon}{\partial r} \frac{\partial H_r}{\partial z} = 0.$$

Since

$$\frac{\partial \varepsilon}{\partial r} \neq 0,$$

then for an E-mode to exist in the waveguide

$$\frac{\partial H_r}{\partial z} = 0. \qquad (14)$$

As a result of identity (14), further restrictions on the fields can be found through the use of equations (1) and (2).

From equations (1) and (2)

$$\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = -\frac{\partial B_r}{\partial t} \ , \tag{15}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\phi}{\partial t} \ , \tag{16}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,E_\phi) - \frac{1}{r}\frac{\partial E_r}{\partial \phi} = -\frac{\partial B_z}{\partial t} \ , \tag{17}$$

$$\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = \frac{\partial D_r}{\partial t} \ , \tag{18}$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \frac{\partial D_\phi}{\partial t} \ , \tag{19}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,H_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi} = \frac{\partial D_z}{\partial t} \tag{20}$$

where E_r is the radial component and E_ϕ is the angular component of $\bar{E}$. The substitution of identities (13) and (14) into equation (19) gives

$$\frac{\partial E_\phi}{\partial t} = 0 .$$

If the field varies in time as $e^{j\omega t}$, with ω being the frequency in radians per second, then

$$E_\phi = 0 . \tag{21}$$

Now, it can be seen from equations (15), (16), (17), (18), and (20) that

$$\frac{1}{r} \frac{\partial E_z}{\partial \phi} = -j\omega\mu H_r \ , \tag{22}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j\omega\mu H_\phi \ , \tag{23}$$

$$\frac{\partial E_r}{\partial \phi} = 0 \ , \tag{24}$$

$$-\frac{\partial H_\phi}{\partial z} = j\omega\epsilon E_r \ , \tag{25}$$

and

$$\frac{1}{r} \frac{\partial}{\partial r} (rH_\phi) - \frac{1}{r} \frac{\partial H_r}{\partial \phi} = j\omega\epsilon E_z . \tag{26}$$

If equation (22) is differentiated with respect to z and identity (14) is substituted into the resulting equation, then

$$\frac{\partial}{\partial z} \frac{\partial E_z}{\partial \phi} = 0. \tag{27}$$

Through the differentiation of equation (25) with respect to ϕ and the use of identity (24), it is found that

$$\frac{\partial}{\partial \phi} \frac{\partial H_\phi}{\partial z} = 0. \tag{28}$$

If identity (14) is integrated,

$$H_r = f_1(r,\phi), \tag{29}$$

and if identity (24) is integrated,

$$E_r = g_1(r,z).$$ (30)

Identity (27) can be integrated first with respect to z to give

$$\frac{\partial E_z}{\partial \phi} = \frac{\partial}{\partial \phi}\left[f_2(r,\phi)\right]$$

and then with respect to ϕ to give

$$E_z = f_2(r,\phi) + g_2(r,z).$$ (31)

Similarly, identity (28) can be integrated to give

$$H_\phi = f_3(r,\phi) + g_3(r,z).$$ (32)

From the substitution of expressions (29) and (31) into equation (26)

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = \frac{1}{r}\frac{\partial f_1(r,\phi)}{\partial \phi} + j\omega\varepsilon(r,z)f_2(r,\phi) + j\omega\varepsilon(r,z)g_2(r,z)$$

$$= s_1(r,\phi) + s_2(r,\phi,z) + s_3(r,z)$$ (33)

where

$$s_1(r,\phi) = \frac{1}{r}\frac{\partial f_1(r,\phi)}{\partial \phi},$$

$$s_2(r,\phi,z) = j\omega\varepsilon(r,z)f_2(r,\phi),$$ (34)

and

$$s_3(r,z) = j\omega\varepsilon(r,z)g_2(r,z).$$

However, from equation (32)

$$\frac{1}{r} \frac{\partial}{\partial r} (rH_\phi) = \frac{1}{r} \frac{\partial}{\partial r} \left[rf_3(r,\phi) \right] + \frac{1}{r} \frac{\partial}{\partial r} \left[rg_3(r,z) \right]$$

$$= t_1(r,\phi) + t_2(r,z) \qquad (35)$$

where

$$t_1(r,\phi) = \frac{1}{r} \frac{\partial}{\partial r} \left[rf_3(r,\phi) \right]$$

and

$$t_2(r,z) = \frac{1}{r} \frac{\partial}{\partial r} \left[rg_3(r,z) \right] .$$

Since H_ϕ must behave in the manner described in equation (32), the functional form given for

$$\frac{1}{r} \frac{\partial}{\partial r} (rH_\phi)$$

in equation (33) must be in agreement with equation (35). Such an agreement is fulfilled only if

$$s_2(r,\phi,z) = s_2(r,z) \qquad (36)$$

or if

$$s_2(r,\phi,z) = s_2(r,\phi) . \qquad (37)$$

Equation (37) cannot be satisfied because

$$\varepsilon = f(r,z) .$$

Hence, equation (36) must be satisfied. Therefore, from equation (34) it can be seen that it is necessary that

$$f_2(r,\phi) = f_2(r) .$$

With this being the case, from equation (31) E_z becomes

$$E_z = f_2(r) + g_2(r,z).\qquad(38)$$

Therefore,

$$\frac{\partial E_z}{\partial \phi} = 0,\qquad(39)$$

and thus from equation (22)

$$H_r = 0.\qquad(40)$$

Furthermore, if equation (23) is differentiated with respect to ϕ, from identities (24) and (39)

$$\frac{\partial H_\phi}{\partial \phi} = 0.\qquad(41)$$

It can be concluded from identities (24), (39), and (41) that the fields have no angular dependence.

A point to note is that the restrictions imposed upon the field components are initially caused by the radial dependence of the permittivity. If the permittivity is only a function of z, no restrictions result from the E-mode condition because in equation (12)

$$H_z = 0$$

which forces

$$\frac{\partial \varepsilon}{\partial z} \frac{\partial H_z}{\partial z} = 0,$$

and thus all terms in equation (12) vanish.

3. WAVE EQUATIONS FOR CASE OF CONSTANT PERMEABILITY AND OF z-DEPENDENT PERMITTIVITY AND THE INTERCHANGED CASE

3.1 General

At this point the problem where the permittivity is a function of z and the permeability is constant will be considered along with the interchanged case. For these problems the unified differential equations, which result from the vectorial differential wave equations, are sufficiently separable.

If the permittivity is a function of z only and the permeability is constant

$$\nabla \varepsilon = \frac{d\varepsilon}{dz} \, \bar{k}$$

and

$$\nabla \mu = 0.$$

Hence, equation (7) becomes

$$\nabla^2 \bar{E} + \nabla\left(\frac{1}{\varepsilon} \frac{d\varepsilon}{dz} E_z\right) = \mu\varepsilon\frac{\partial^2 \bar{E}}{\partial t^2} , \qquad (42)$$

and equation (8) becomes

$$\nabla^2 \bar{H} = \mu\varepsilon \frac{\partial^2 \bar{H}}{\partial t^2} + \frac{1}{\varepsilon} \frac{d\varepsilon}{dz} \frac{\partial \bar{H}}{\partial z} - \frac{1}{\varepsilon} \frac{d\varepsilon}{dz} \nabla H_z. \qquad (43)$$

Similarly, if the permeability is a function of z only and the permittivity is constant,

$$\nabla \varepsilon = 0$$

and

$$\nabla \mu = \frac{d\mu}{dz} \, \bar{k}.$$

Hence, equation (7) becomes

$$\nabla^2 \bar{E} = \mu\varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} + \frac{1}{\mu} \frac{d\mu}{dz} \frac{\partial \bar{E}}{\partial z} - \frac{1}{\mu} \frac{d\mu}{dz} \nabla E_z , \qquad (44)$$

and equation (8) becomes

$$\nabla^2 \bar{H} + \nabla\left(\frac{1}{\mu} \frac{d\mu}{dz} H_z\right) = \mu\varepsilon \frac{\partial^2 \bar{H}}{\partial t^2} . \qquad (45)$$

3.2 Differential Field Equations for Transverse Waves

For transverse waves

$$E_z = H_z = 0.$$

Therefore, equation (42) becomes

$$\nabla^2 \bar{E} = \mu\varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} , \qquad (46)$$

and equation (45) becomes

$$\nabla^2 \bar{H} = \mu\varepsilon \frac{\partial^2 \bar{H}}{\partial t^2} . \qquad (47)$$

For the case where the permittivity is dependent on z, the field equations can be determined by first solving for the two field components in equation (46) and then by using equations (1) and (2) in the usual manner. Hence, if the rectangular components of the field are to be determined, the partial differential equations

$$\nabla^2 E_x = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \qquad (48)$$

and

$$\nabla^2 E_y = \mu\varepsilon \frac{\partial^2 E_y}{\partial t^2} \tag{49}$$

have to be solved where E_x and E_y are the components of $\bar{\bar{E}}$ in the x and y directions respectively.

Similarly, for the case where the permeability is dependent on z, the field equations can be determined by first solving for the two field components in equation (47) and then by using equations (1) and (2). Hence, if the rectangular components of the field are to be determined, the partial differential equations

$$\nabla^2 H_x = \mu\varepsilon \frac{\partial^2 H_x}{\partial t^2} \tag{50}$$

and

$$\nabla^2 H_y = \mu\varepsilon \frac{\partial^2 H_y}{\partial t^2} \tag{51}$$

have to be solved where H_x and H_y are the components of $\bar{H}$ in the x and y directions respectively.

Equations (48), (49), (50), and (51) have the general form

$$\nabla^2 G = \mu\varepsilon \frac{\partial^2 G}{\partial t^2} \tag{52}$$

where $\mu\varepsilon$ is a function of z. Consequently, equation (52) will be considered for the remainder of this section.

Equation (52) can, also, be written as

$$\nabla_t^2 G + \frac{\partial^2 G}{\partial z^2} = \mu\varepsilon \frac{\partial^2 G}{\partial t^2}$$

where ∇_t^2 denotes the part of ∇^2 which operates in the transverse plane of a rectangular coordinate system. If the fields vary in time as $e^{j\omega t}$, then G can be expressed as

$$G = G_o(x,y,z)e^{j\omega t},$$

and thus

$$\nabla_t^2 G_o + \frac{\partial^2 G_o}{\partial z^2} = -\omega^2\mu\varepsilon G_o. \qquad (53)$$

The variables can be separated by letting

$$G_o = F(x,y)\ T(z). \qquad (54)$$

Once equation (54) is substituted into equation (53),

$$-\frac{1}{F}\nabla_t^2 F = \frac{1}{T}\frac{d^2 T}{dz^2} + \omega^2\mu\varepsilon = M^2$$

where M is the separation constant. Hence,

$$\nabla_t^2 F + M^2 F = 0. \qquad (55)$$

Equation (55) is the ordinary differential equation confronted when the transverse dependence of a transverse wave in a homogeneous medium is investigated. As well as equation (55), the differential equation

$$\frac{d^2 T}{dz^2} + (\omega^2\mu\varepsilon - M^2)T = 0 \qquad (56)$$

has to be solved to obtain the solution for the transverse waves under consideration.

For plane waves equation (56) is slightly simpler due to the fact that for plane waves

$$\nabla_t^2 E_x = \nabla_t^2 E_y = \nabla_t^2 H_x = \nabla_t^2 H_y = 0$$

which forces

$$M = 0.$$

Consequently, equation (56) becomes

$$\frac{d^2 T}{dz^2} + \omega^2 \mu\varepsilon T = 0. \qquad (57)$$

3.3 Differential Field Equations for Waves with Longitudinal Components

When the fields have longitudinal components, the wave solutions can be found by first solving for the longitudinal field components and then by using equations (1) and (2). This section is concerned with the differential equations arising in the solution of the longitudinal field components.

When the permittivity is dependent on z, from equation (45) the longitudinal component of the magnetic field must satisfy the differential equation

$$\nabla^2 H_z = \mu\varepsilon \frac{\partial^2 H_z}{\partial t^2} \qquad (58)$$

and from equation (44) the longitudinal component of the electric field must satisfy the differential equation

$$\nabla^2 E_z + \frac{\partial}{\partial z}\left(\frac{1}{\epsilon}\frac{d\epsilon}{dz} E_z\right) = \mu\epsilon \frac{\partial^2 E_z}{\partial t^2} \, . \qquad (59)$$

Equation (58) is the same type of differential equation as equation (52), and thus can be treated in a similar manner. However, the restriction that ∇_t^2 must operate in a rectangular coordinate system no longer applies.

Equation (59) can be simplified by replacing E_z by $\frac{1}{\epsilon}D_z$. In terms of D_z equation (59) becomes

$$\nabla^2\left(\frac{1}{\epsilon} D_z\right) + \frac{\partial}{\partial z}\left(\frac{1}{\epsilon^2}\frac{d\epsilon}{dz} D_z\right) = \mu \frac{\partial^2 D_z}{\partial t^2}$$

or

$$\nabla_t^2 D_z + \frac{\partial^2 D_z}{\partial z^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dz}\frac{\partial D_z}{\partial z} = \mu\epsilon \frac{\partial^2 D_z}{\partial t^2} \, . \qquad (60)$$

The variables can be separated by letting

$$D_z = F(x,y)T(z)e^{j\omega t} \, . \qquad (61)$$

If this expression for D_z is substituted into equation (60), the result is

$$\frac{1}{F}\nabla_t^2 F = -\frac{1}{T}\left(\frac{d^2 T}{dz^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dz}\frac{dT}{dz}\right) - \omega^2\mu\epsilon = -M^2$$

where once again M is the separation constant. The equation

$$\nabla_t^2 F + M^2 F = 0 \qquad (62)$$

is identical to the equation which contains the transverse dependent part of the longitudinal component of the field in a homogeneous medium and can be solved for a number of boundary value problems using well known techniques. The equation for the z-dependent part of D_z can be written as

$$\frac{d^2T}{dz^2} - \frac{1}{\varepsilon} \frac{d\varepsilon}{dz} \frac{dT}{dz} + (\omega^2 \mu\varepsilon - M^2)T = 0. \qquad (63)$$

When the permeability is dependent on z, from equation (46) the longitudinal component of the electric field must satisfy the differential equation

$$\nabla^2 E_z = \mu\varepsilon \frac{\partial^2 E_z}{\partial t^2}. \qquad (64)$$

and from equation (47) the longitudinal component of the magnetic field must satisfy the differential equation

$$\nabla^2 H_z + \frac{\partial}{\partial z} \left(\frac{1}{\mu} \frac{d\mu}{dz} H_z \right) = \mu\varepsilon \frac{\partial^2 H_z}{\partial t^2}. \qquad (65)$$

Equation (64) is the frequently occurring type of differential equation given in equation (52), and thus can be treated accordingly. The only change is that ∇_t^2 is no longer restricted to operate in a rectangular coordinate system.

In the same manner as equation (59) was simplified by using D_z, equation (65) can be simplified by using B_z. From equation (65) it can be seen that B_z can be expressed

in the form

$$B_z = F(x,y)T(z)e^{j\omega t}$$

with $F(x,y)$ satisfying equation (62) and $T(z)$ satisfying

$$\frac{d^2T}{dz^2} - \frac{1}{\mu}\frac{d\mu}{dz}\frac{dT}{dz} + (\omega^2\mu\varepsilon - M^2)T = 0.$$

3.4 Summary of z-Dependent Equations

If by definition

$$q(z) = \mu\varepsilon,$$

then the differential equations containing the z-dependent part of the solution for the wave functions considered in sections 3.2 and 3.3 can be summarized by the following three differential equations.

$$\frac{d^2T}{dz^2} + \omega^2 qT = 0, \tag{66}$$

$$\frac{d^2T}{dz^2} + \left(\omega^2 q - M^2\right)T = 0, \tag{67}$$

and

$$\frac{d^2T}{dz^2} - \frac{1}{q}\frac{dq}{dz}\frac{dT}{dz} + \left(\omega^2 q - M^2\right)T = 0. \tag{68}$$

3.5 Oscillation Theorem Due to Sturm

Equations (66) and (67) can be expressed in the form

$$\frac{d^2v}{dz^2} + h(z)v = 0, \tag{69}$$

and equation (68) can be transformed into the form of equation (69) by the following transformation. If T is expressed as

$$T = e^{\frac{1}{2}\int \frac{1}{q} \frac{dq}{dz} dz} \quad W = e^{\frac{1}{2}\ln q} \quad W = \sqrt{q}W \; ,$$

then equation (68) becomes

$$\frac{d^2W}{dz^2} + \left[\omega^2 q - M^2 + \frac{1}{2} \frac{1}{q} \frac{d^2q}{dz^2} - \frac{3}{4} \frac{1}{q^2} \left(\frac{dq}{dz}\right)^2 \right]W = 0 \qquad (70)$$

which is of the form of equation (69).

Owing to a theorem by Sturm, it is possible to show that for a physically realizable situation the solutions for equation (66) are oscillatory, and the solutions for equation (67) are oscillatory provided

$$\omega^2 \mu\epsilon - M^2 > 0.$$

Besides this, the theorem offers a possible test for showing whether or not the solutions for equation (68) or (70) are oscillatory.

Theorem:[3] The functions u(z) and v(z) are the respective solutions of the differential equations

$$\frac{d^2u}{dz^2} + g(z)u = 0 \qquad (71)$$

$$\frac{d^2v}{dz^2} + h(z)v = 0 \qquad (72)$$

---

3. L.R. Ford, _Differential Equations_, McGraw-Hill, Inc., New York, 1955, p. 169.

in an interval in which the coefficients of the equations are continuous. If a and b are consecutive roots of u(z) with a < b and if

$$h(z) \geqslant g(z)$$

$$h(z) \not\equiv g(z)$$

in the closed interval $[a,b]$, then there exists a root of v(z) between a and b.

Proof: First of all,

$$\frac{d}{dz}\left(v\,\frac{du}{dz} - u\,\frac{dv}{dz}\right) = v\,\frac{d^2u}{dz^2} - u\,\frac{d^2v}{dz^2} = (h - g)uv$$

which after integration becomes

$$\left(v\,\frac{du}{dz} - u\,\frac{dv}{dz}\right)\Bigg|_{z=a}^{z=b} = \int_a^b (h - g)uv\,dz. \qquad (73)$$

Now, the supposition that v(z) has no root between a and b is made. Without loss of generality u(z) and v(z) may both be considered positive in the interval a < z < b; either one can be replaced by its negative, if necessary. Consequently,

$$\frac{du(a)}{dz} > 0$$

and

$$\frac{du(b)}{dz} < 0.$$

Hence,

$$\left(v\,\frac{du}{dz} - u\,\frac{dv}{dz}\right)\Bigg|_{z=a}^{z=b} = v(b)\,\frac{du(b)}{dz} - v(a)\,\frac{du(a)}{dz} \leqslant 0.$$

However, the right hand side of equation (73) is positive.
Therefore, a contradiction exists, and thus the theorem is
proved.

From this theorem it follows that if the solution
to equation (71) is oscillatory over some interval, then,
provided that over this interval $g(z)$ and $h(z)$ are
continuous and

$$h(z) \geqslant g(z)$$

$$h(z) \not\equiv g(z),$$

the solution to equation (72) is, also, oscillatory over
the same interval.

For a physically realizable medium

$$\mu\varepsilon \geqslant k = \text{Constant} > 0$$

where normally

$$k = \mu_0 \varepsilon_0,$$

and for the non-trivial cases to be considered

$$\mu\varepsilon \not\equiv k.$$

Since the solution for

$$\frac{d^2 T}{dz^2} + \omega^2 k T = 0$$

is oscillatory, the solution for equation (66) is, also,
oscillatory.

In a similar manner, when

$$\omega^2\mu\varepsilon - M^2 \geq k' = \text{Constant} > 0, \qquad (74)$$

then since the solution for

$$\frac{d^2T}{dz^2} + k'T = 0$$

is oscillatory, the solution for equation (67) is oscillatory.

4. FIELD PROBLEM IN A PERIODIC STRUCTURE LOADED WITH DIELECTRIC DISCS

4.1 General

As mentioned in the introduction, the topic of this thesis arose from the problem of finding the wave functions for a periodic structure of the type shown in Figure 1. Section 4 will expand upon this problem through the use of the fact that inside the waveguide the medium as a whole is inhomogeneous. The permittivity is a function of the spatial parameters and the permeability is constant.

4.2 Functional Behaviour of the Permittivity

Before the unified differential wave equations can be dealt with, the functional behaviour of the permittivity must be specified. Since the permittivity in the interior of the air regions equals ε_0 and in the interior of the dielectric regions equals ε_1, the functional form of the permittivity minus ε_0 approaches the product of a rectangular wave variation in the z direction times a step variation in the radial direction. Consequently, if cylindrical coordinates are used, the permittivity can be expressed as

$$\varepsilon - \varepsilon_0 = h(r)g(z)$$

where $h(r)$ and $g(z)$ are sketched in Figure 2. The reasons that the curves in Figure 2 are shown as continuous and

smooth are given in the following paragraphs.

Before the theory so far developed can be applied to the field problem in a periodic structure loaded with dielectric discs, it must be assumed that the permittivity and all its first and second order derivatives are defined for all interior points in the waveguide. The following argument is given to justify this assumption.
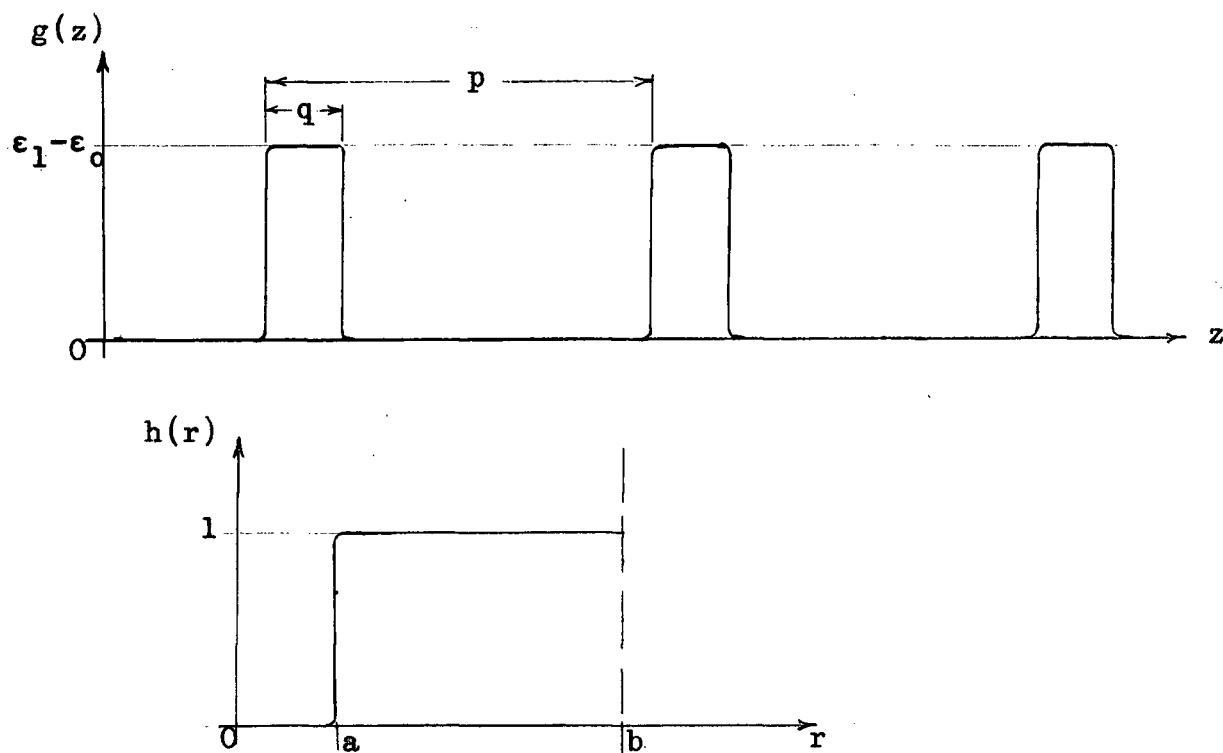

Fig. 2. Sketch of Functional Variations of Permittivity

At all points except those in the regions of the boundaries between the air and the dielectric medium, there is no doubt as to the existence of the permittivity and all its derivatives. If in the neighbourhood of the boundaries

the point of view of mathematical physics, which is that
matter is continuous, is taken, then the properties of
matter can also be regarded as continuous.[4] Therefore, the
permittivity can be considered continuous but changing very
rapidly through the boundaries. Consequently, a continuous
function can be used to describe the permittivity through
the boundaries which both approximates the situation as
closely as desirable and satisfies the assumptions made
about the behaviour of the permittivity. It is worth
noting that the step function used to describe the
boundaries in the standard approach for finding the wave
solutions is, also, an approximation, although a very good
one, of the actual situation. The step function is an
approximation because at the boundaries there exists not
one big discontinuity, but rather a large number of
discontinuities which arise from the discontinuities
between the atoms and, also, between the separate parts of
the atoms.

In the region of a boundary the permittivity could
be represented by the function

$$\varepsilon = \varepsilon_0 + \frac{\varepsilon_1 - \varepsilon_0}{e^{\left|\frac{a-s}{m}\right|} + 1} \qquad (75)$$

provided m is small but finite. For equation (75)

s = Coordinate in the normal direction to the boundary

4. A.G. Webster, *Partial Differential Equations of Mathematical Physics*, Dover Publications, New York, 1955, p.2.

and

a = s at the boundary.

As m decreases, for $s < a$, $\varepsilon \rightarrow \varepsilon_o$, and for $s > a$, $\varepsilon \rightarrow \varepsilon_1$. In the limit as $m \rightarrow 0$, with an increasing s, the permittivity changes discontinuously through the boundary from ε_o to ε_1.

4.3 Field Restrictions Due to E-Mode Condition

Now that it has been assumed that the permittivity behaves sufficiently well to ensure that the theory so far developed is applicable, the material in section 2 can be used to find the necessary conditions for the existence of E-modes in the periodic structure shown in Figure 1. When the dielectric discs have center holes,

$$\varepsilon = h(r)g(z) + \varepsilon_o = f(r,z).$$

Therefore, the restrictions on the field components found in section 2.23 must hold. If these restrictions are not met and a hybrid mode results, it is worth noting that the hybrid mode may very closely approximate an E-mode provided the longitudinal component of the magnetic field has only a secondary effect on the remaining field components. When no center hole exists, the permittivity has no radial dependence, and thus, the E-mode condition is satisfied without imposing any restrictions on the field components.

4.4 Unified Differential Equations

When the discs have center holes, the unified

differential equations for an E-mode can be found from

equations (7) and (8). Since for an E-mode

$$H_z \equiv 0$$

and since from the E-mode condition

$$E_\phi \equiv 0 \; ,$$

$$H_r \equiv 0 \; ,$$

and the field components have no angular variation, the

unified differential equations are

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{\partial^2 E_z}{\partial z^2} + \frac{\partial}{\partial z}\left[\frac{1}{\epsilon}\left(\frac{\partial \epsilon}{\partial r}E_r + \frac{\partial \epsilon}{\partial z}E_z\right)\right] = \mu\epsilon\frac{\partial^2 E_z}{\partial t^2} \; ,$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right)\right] + \frac{\partial^2 E_r}{\partial z^2} + \frac{\partial}{\partial r}\left[\frac{1}{\epsilon}\left(\frac{\partial \epsilon}{\partial r}E_r + \frac{\partial \epsilon}{\partial z}E_z\right)\right] = \mu\epsilon\frac{\partial^2 E_r}{\partial t^2} \; ,$$

and

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rH_\phi\right)\right] + \frac{\partial^2 H_\phi}{\partial z^2} = \mu\epsilon\frac{\partial^2 H_\phi}{\partial t^2} + \frac{1}{\epsilon}\left[\frac{\partial \epsilon}{\partial r}\left(\frac{\partial H_\phi}{\partial r} + \frac{H_\phi}{r}\right) + \frac{\partial \epsilon}{\partial z}\frac{\partial H_\phi}{\partial z}\right].$$

As previously mentioned, the techniques applied so far to

these differential equations, as well as the ones arising

in the H-mode case, have not yielded general solutions.

However, numerical methods could be devised to calculate

specific solutions for these differential equations. Since

general solutions have not been found, these differential

equations will not be considered further in this thesis.

For the case where the discs are solid, the

permittivity is a function of z only. Consequently, the
unified differential equations for the longitudinal field
components can be separated as was seen in section 3.
Also, the complete wave solutions for this case can be
readily solved, as done in Appendix 1 for the E_{01}-mode,
by determining the fields in the dielectric regions and
in the air regions separately and by matching these field
solutions at the boundaries with the help of Floquet's
Theorem.

Consequently, if it could be shown that these
known solutions satisfy the unified differential equations
in the limit as the permittivity approaches a rectangular
waveshape, the viewpoint taken in this thesis would be
verified as applicable for finding the wave equations to
the solid disc case. However, this was not shown in general
because no method was devised to overcome two difficulties
simultaneously.

One difficulty is to find an infinite series
expression for the permittivity which converges absolutely
and yet has an nth order term which is manageable. This
difficulty is discussed by L. Brillouin in "Wave Propa-
gation in Periodic Structures".[5] Secondly, even if such
a series is obtained, since the known wave solutions are
expressed in terms of infinite series in z, single and

5. L. Brillouin, Wave Propagation in Periodic Structures,
   Dover Publications, Inc., New York, 1953, p. 186.

double infinite series appear in the z-dependent part
of the unified differential equations. This can readily
be seen from the equations derived in section 3. Consequent-
ly, the problem of establishing that the coefficients of the
z-dependent series in the known solutions satisfy the
unified differential equations is very awkward.

However, when the dielectric regions are matched
into the air regions, it is possible to find the wave
solution for an E-mode from the unified differential
equations. As will be shown, this solution agrees with
the solution obtained in Appendix 1.

4.5 E-Mode Solution of Unified Differential Equations for Matched Case

For the case where the dielectric discs are solid
and matched into the air regions, a complete wave solution
for an E-mode can be determined by first solving for the
field component D_z through the use of the pertinent
unified differential equation and then by finding the
other field components from equations (1) and (2).

Since the permittivity is a function of z and the
permeability is constant, the z-dependent portion of D_z can
be found from equation (63),

$$\frac{d^2 T}{dz^2} - \frac{1}{\varepsilon} \frac{d\varepsilon}{dz} \frac{dT}{dz} + (\omega^2 \mu \varepsilon - M^2) T = 0.$$

The cross section of a periodic structure with solid
dielectric discs is shown in Figure 3. For such a structure

the permeability equals μ_o.
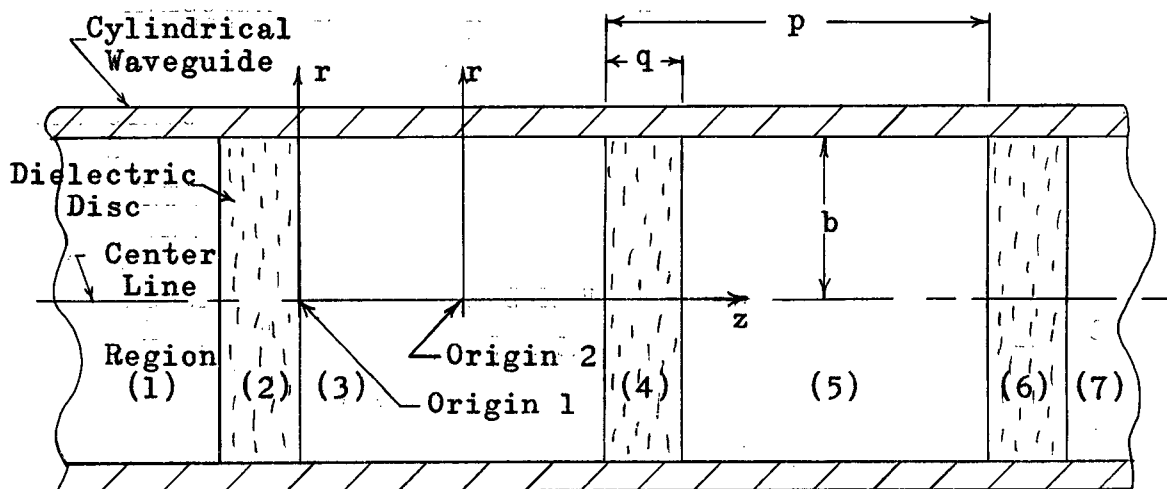
ε_1 = Permittivity of dielectric medium

ε_o = Permittivity of air

Note: For simplicity Origin 1 is used in Appendix 1 and
Origin 2 is used in the body of the thesis.

Fig. 3. <u>Cross Section of Solid Disc Periodic Structure</u>

Since the permittivity very closely approximates
a rectangular waveshape in the z direction, it follows that
$\omega^2 \mu_o \varepsilon - M^2$ and $k^2 \varepsilon^2$, k being a constant, also, very
closely approximate rectangular waveshapes as shown in
Figure 4. For the matched case

$$\frac{\omega^2 \mu_o \varepsilon_o - M^2}{\varepsilon_o^2} = \frac{\omega^2 \mu_o \varepsilon_1 - M^2}{\varepsilon_1^2} = k^2. \qquad (76)$$
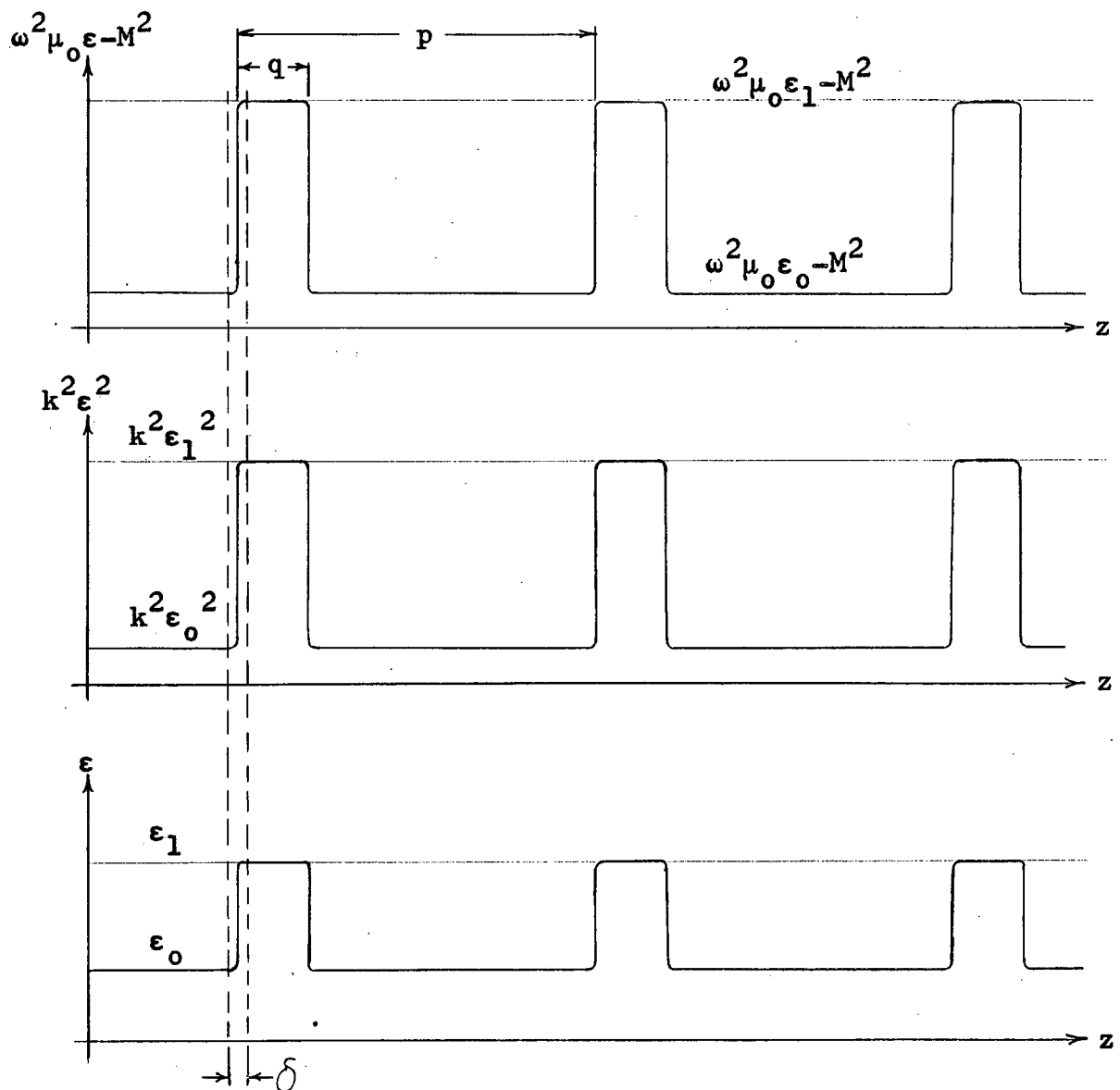
Fig. 4. Diagram of $\omega^2\mu_0\epsilon-M^2$ and $k^2\epsilon^2$ Versus z

Consequently, when a match exists, the identity

$$\omega^2\mu_0\epsilon-M^2 \stackrel{=}{=} k^2\epsilon^2 \tag{77}$$

holds except in the transition region at a boundary between

an air region and a dielectric region. The transition

region is defined by δ in Figure 4. As δ tends to zero, identity (77) tends to hold for all values of z. Consequently, the differential equation

$$\frac{d^2T}{dz^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dz}\frac{dT}{dz} + k^2\epsilon^2 T = 0 \qquad (78)$$

approximates equation (63) to any required degree of accuracy for the matched case. The solution for equation (78) is

$$T = A_1 e^{-j\int k\epsilon\ dz} + A_2 e^{j\int k\epsilon\ dz}$$

where A_1 and A_2 are arbitrary constants. Through the use of identity (77), T becomes

$$T = A_1 e^{-j\int\sqrt{\omega^2\mu_o\epsilon-M^2}\ dz} + A_2 e^{j\int\sqrt{\omega^2\mu_o\epsilon-M^2}\ dz}. \qquad (79)$$

As δ approaches zero, the solution for T given by equation (79) approaches the solution for the entire waveguide.

In the limit $\sqrt{\omega^2\mu_o\epsilon-M^2}$ approaches a rectangular waveform, and thus for the limiting case the integral

$$\int\sqrt{\omega^2\mu_o\epsilon-M^2}\ dz$$

can be evaluated graphically by integrating the rectangular waveform as illustrated in Figure 5.

From Figure 5 it can be seen that in the limiting case

$$\int\sqrt{\omega^2\mu_o\epsilon-M^2}\ dz = S_o z + \chi(z)$$

where

$$S_0 z = \text{Ramp function}$$

and

$$\chi(z) = \text{Periodic function oscillating about the ramp function.}$$
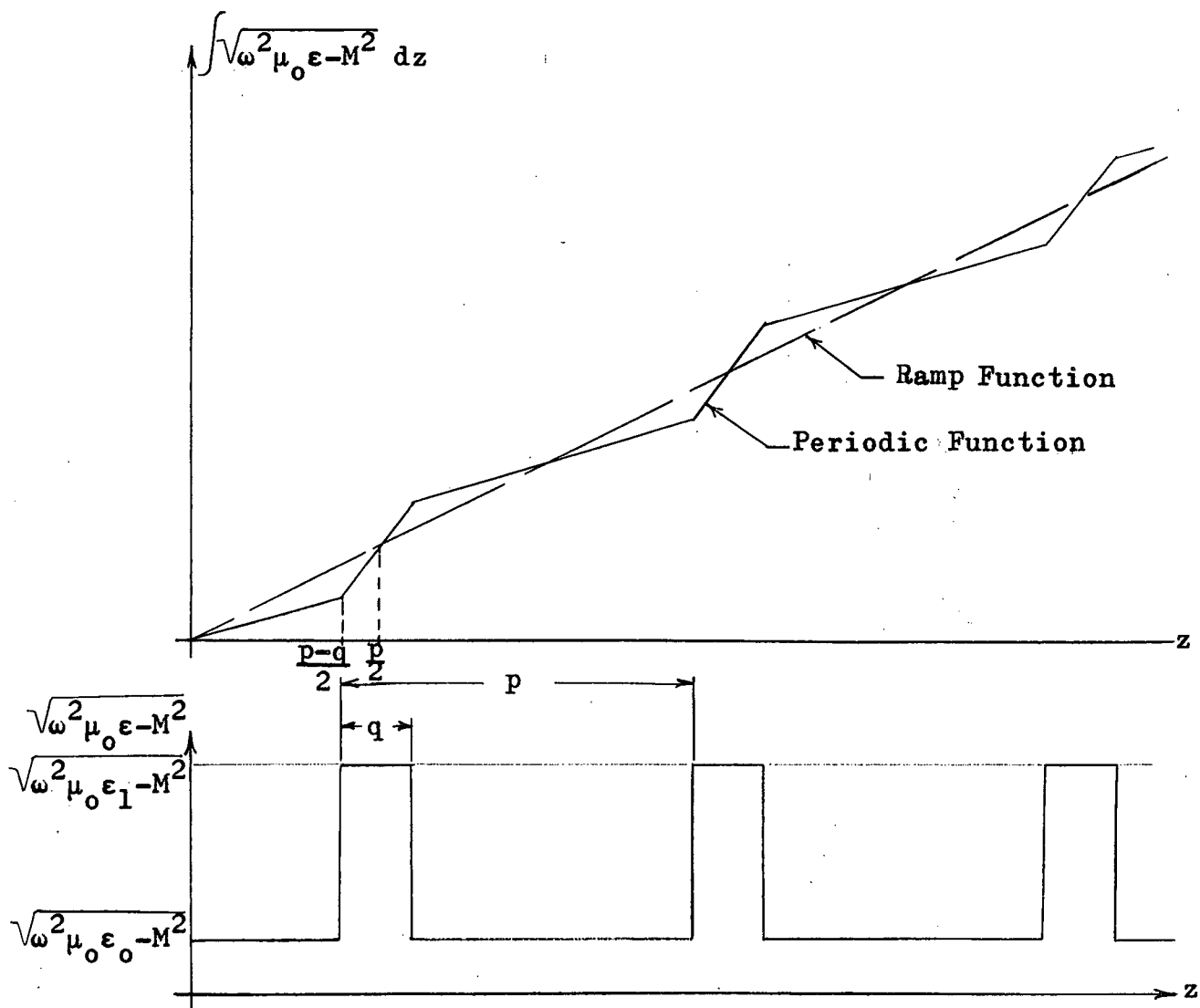

Fig. 5. **A Diagram of** $\int \sqrt{\omega^2 \mu_0 \epsilon - M^2} \, dz$ **Versus z**

The slope S_o of the ramp can be found in the following manner. From Figure 5 at z equals $\frac{p-q}{2}$

$$\int \sqrt{\omega^2 \mu_o \epsilon - M^2} \, dz = \frac{p-q}{2} \sqrt{\omega^2 \mu_o \epsilon_o - M^2}$$

and thus at z equals $\frac{p}{2}$

$$\int \sqrt{\omega^2 \mu_o \epsilon - M^2} \, dz = \frac{p-q}{2} \sqrt{\omega^2 \mu_o \epsilon_o - M^2} + \left(\frac{p}{2} - \frac{p-q}{2} \right) \sqrt{\omega^2 \mu_o \epsilon_1 - M^2}$$

$$= \frac{S_o p}{2} \, .$$

Hence,

$$S_o = \sqrt{\omega^2 \mu_o \epsilon_o - M^2} + \frac{q}{p} \left(\sqrt{\omega^2 \mu_o \epsilon_1 - M^2} - \sqrt{\omega^2 \mu_o \epsilon_o - M^2} \right) .$$

When only the incident wave is present,

$$T = A_1 e^{-j \left(S_o z + \chi(z) \right)} ,$$

and thus

$$D_z = A_1 F(r, \phi) e^{j \left(\omega t - S_o z - \chi(z) \right)} \tag{80}$$

where r is the radial and the ϕ the angular variable.

In the limit as the permittivity approaches a rectangular waveform, it can be seen from Figure 5 that in an air region

$$D_z \, \alpha \, e^{-j \int \sqrt{\omega^2 \mu_o \epsilon_o - M^2} \, dz} = e^{-j \left(\sqrt{\omega^2 \mu_o \epsilon_o - M^2} \right) z}$$

and in a dielectric region

$$D_z \, \alpha \, e^{-j \int \sqrt{\omega^2 \mu_o \epsilon_1 - M^2} \, dz} = e^{-j \left(\sqrt{\omega^2 \mu_o \epsilon_1 - M^2} \right) z} \, .$$

These results coincide with the results obtained by solving the wave equation in each of the homogeneous regions separately.

To check that the solution given by equation (80) has the same phase shift per section as found in Appendix 1, the expression for $T(z+p)$ should be considered, namely,

$$T(z+p) = A_1 e^{-j\left[S_0(z+p) + X(z+p)\right]}.$$

Since

$$X(z+p) = X(z),$$

then

$$T(z+p) = A_1 e^{-jS_0 p}\left[e^{-j\left(S_0 z + X(z)\right)}\right]$$

$$= e^{-jS_0 p}\, T(z).$$

Hence, the phase shift per section $\emptyset$ is given by

$$\emptyset = S_0 p = (p-q)\sqrt{\omega^2 \mu_0 \varepsilon_0 - M^2} + q\sqrt{\omega^2 \mu_0 \varepsilon_1 - M^2}.$$

This is the same value for $\emptyset$ as is found in Appendix 1.

Therefore, since in the limit the field in each section behaves in the same fashion as it was found to behave in Appendix 1 and the phase shift per section is identical to the value found in Appendix 1, the two approaches are in agreement.

When the limit has not been taken, then D_z has

the form to within any required degree of accuracy

$$A_1 F(r,\phi) e^{j\left(\omega t - S_0' z - \chi'(z)\right)}$$

where in the limit

$$S_0' \longrightarrow S_0$$

$$\chi'(z) \longrightarrow \chi(z).$$

The phase velocity for D_z can be approximately determined by differentiating

$$\omega t - S_0' z - \chi'(z) = \text{Constant}.$$

Hence, the phase velocity v_p is

$$v_p = \frac{dz}{dt} \cong \frac{\omega}{S_0' + \dfrac{d\chi'}{dz}}.$$

Consequently, the phase velocity is modulated by the periodic term $\dfrac{d\chi'}{dz}$.

5. PLANE WAVES IN A MEDIUM WITH PERMITTIVITY A CONTINUOUS FUNCTION OF z

5.1 Introduction

To obtain a better understanding of the behaviour of electromagnetic waves in an inhomogeneous medium, it was decided that an investigation should be made into the behaviour of plane waves in a medium with a permittivity which is a continuous and sufficiently differentiable function of z. This problem tends to be simpler than ones dealing with longitudinal field components. At the same time, the techniques used in solving the differential equation arising from plane wave considerations have only to be slightly modified for E- and H-mode problems in which equation (67) arises. This can easily be seen by comparing equation (67) with equation (66).

5.2 Problem

In principle a complete solution for a plane wave can readily be obtained for any sufficiently well behaved z-dependent functional form of the permittivity. Various particular forms were considered, and it was found that the form

$$\varepsilon = k_1 z^{2s-2} + \frac{k_2}{z^2} \tag{81}$$

where

$$k_1 = \text{Constant}$$
$$k_2 = \text{Constant}$$

and

$$s = \sqrt{1 - 4\omega^2\mu_o k_2} \quad , \quad 0 < s < 1, \qquad (82)$$

yields solutions which are easily interpreted in physical terms. Consequently, a permittivity satisfying equation (81) will be used.

A sketch of the permittivity expressed in equation (81) is given in Figure 6. Also, for the medium to be considered the permeability equals μ_o.
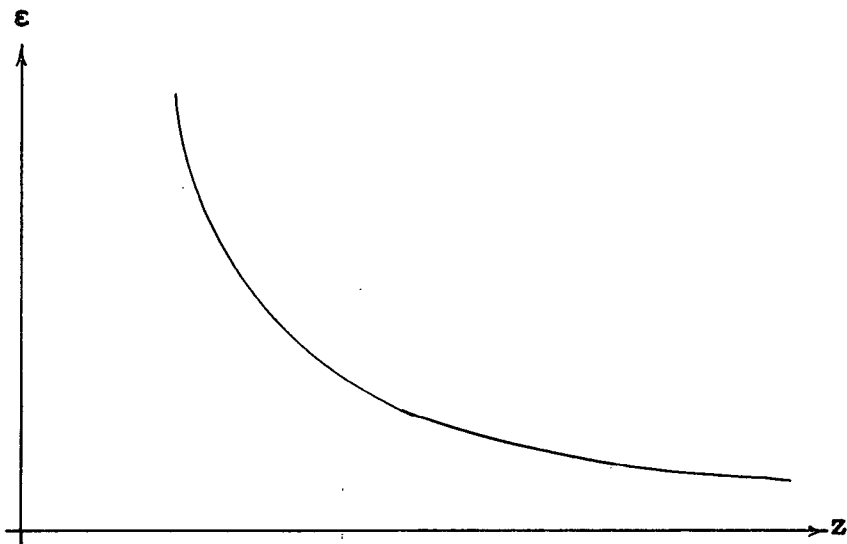
Fig. 6. Sketch of Permittivity Versus z

For a plane wave with

$$E_y = 0$$

and

$$H_x = 0$$

from equation (57), since

$$E_{xo} = T,$$

$$\frac{d^2 E_{xo}}{dz^2} + \omega^2 \mu_0 \left(k_1 z^{2s-2} + \frac{k_2}{z^2} \right) E_{xo} = 0 \qquad (83)$$

where E_{xo} is the component of the electric field in the x direction with time dependence suppressed.

Equation (83) can be solved in the following manner. First of all, the transformation

$$u = \frac{\omega \sqrt{\mu_0 k_1}}{s} z^s = \rho z^s, \; \rho = \frac{\omega \sqrt{\mu_0 k_1}}{s}, \qquad (84)$$

is made. From this transformation

$$\frac{dE_{xo}}{dz} = \rho s z^{s-1} \frac{dE_{xo}}{du}$$

and

$$\frac{d^2 E_{xo}}{dz^2} = \rho^2 s^2 z^{2s-2} \frac{d^2 E_{xo}}{du^2} + \rho s (s-1) z^{s-2} \frac{dE_{xo}}{du}$$

$$= -\omega^2 \mu_0 \left(k_1 z^{2s-2} + k_2 z^{-2} \right) E_{xo}. \qquad (85)$$

Consequently, by the substitution of expression (84) into equation (85)

$$s^2 u^2 \frac{d^2 E_{xo}}{du^2} + s(s-1) u \frac{dE_{xo}}{du} + \omega^2 \mu_0 \left(\frac{k_1 u^2}{\rho^2} + k_2 \right) E_{xo} = 0. \quad (86)$$

If E_{xo} is transformed into

$$E_{xo} = u^{\frac{1}{2s}} Y, \qquad (87)$$

then

$$\frac{dE_{xo}}{du} = u^{\frac{1}{2s}} \frac{dY}{du} + \frac{1}{2s} u^{\frac{1-2s}{2s}} Y \qquad (88)$$

and

$$\frac{d^2 E_{xo}}{du^2} = u^{\frac{1}{2s}} \frac{d^2 Y}{du^2} + \frac{1}{s} u^{\frac{1-2s}{2s}} \frac{dY}{du} + \frac{1}{2s}\left(\frac{1}{2s} - 1\right) u^{\frac{1-4s}{2s}} Y. \qquad (89)$$

If expressions (87), (88), and (89) are substituted into equation (86), the result is

$$u^2 \frac{d^2 Y}{du^2} + u \frac{dY}{du} + (u^2 - \frac{1}{4})Y = 0$$

which is Bessel's equation for $n = \frac{1}{2}$. Hence,

$$Y = C_1 J_{\frac{1}{2}}(u) + C_2 J_{-\frac{1}{2}}(u)$$

with C_1 and C_2 being arbitrary constants, and thus

$$E_{xo} = \sqrt{z}\left[N_1 J_{\frac{1}{2}}\left(\rho z^s\right) + N_2 J_{-\frac{1}{2}}\left(\rho z^s\right)\right]$$

where N_1 and N_2 are arbitrary constants.

Since

$$J_{\frac{1}{2}}(u) = \sqrt{\frac{2}{\pi u}} \sin u$$

and

$$J_{-\frac{1}{2}}(u) = \sqrt{\frac{2}{\pi u}} \cos u,$$

$$E_{xo} = \sqrt{z}\left(N_1\sqrt{\frac{2}{\pi\rho z^s}}\sin\rho z^s + N_2\sqrt{\frac{2}{\pi\rho z^s}}\cos\rho z^s\right)$$

or

$$E_{xo} = \sqrt{\frac{2}{\pi\rho}} \, z^{\frac{1-s}{2}}\left[A_1 e^{-j\rho z^s} + A_2 e^{j\rho z^s}\right] \quad (90)$$

where A_1 and A_2 are arbitrary constants. Hence,

$$E_x = E_{xo}e^{j\omega t} = \sqrt{\frac{2}{\pi\rho}} \, z^{\frac{1-s}{2}}\left[A_1 e^{j(\omega t - \rho z^s)} + A_2 e^{j(\omega t + \rho z^s)}\right] \quad (91)$$

Consequently, the incident electric wave is

$$E_{xi} = \sqrt{\frac{2}{\pi\rho}} \, z^{\frac{1-s}{2}} \, A_1 e^{j(\omega t - \rho z^s)},$$

and the reflected electric wave is

$$E_{xr} = \sqrt{\frac{2}{\pi\rho}} \, z^{\frac{1-s}{2}} \, A_2 e^{j(\omega t + \rho z^s)}.$$

Since

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t},$$

$$-\frac{\partial E_x}{\partial z} = \frac{\partial B_y}{\partial t} = j\omega\mu_0 H_y.$$

Therefore,

$$H_y = j\frac{e^{j\omega t}}{\omega\mu_0}\sqrt{\frac{2}{\pi\rho}}\left\{\frac{1-s}{2}z^{-\frac{1+s}{2}}\left[A_1 e^{-j\rho z^s} + A_2 e^{j\rho z^s}\right]\right.$$

$$\left. + z^{\frac{1-s}{2}}\left[-j\rho s z^{s-1} A_1 e^{-j\rho z^s} + j\rho s z^{s-1} A_2 e^{j\rho z^s}\right]\right\}$$

or

$$H_y = \frac{i}{\omega\mu_o} \sqrt{\frac{2}{\pi\rho}} \frac{1}{\sqrt{z}} \left\{ \left[\frac{1-s}{2} z^{-\frac{s}{2}} - j\rho s z^{\frac{s}{2}} \right] A_1 e^{j(\omega t - \rho z^s)} \right.$$

$$\left. + \left[\frac{1-s}{2} z^{-\frac{s}{2}} + j\rho s z^{\frac{s}{2}} \right] A_2 e^{j(\omega t + \rho z^s)} \right\}. \qquad (92)$$

Hence, the incident magnetic wave is

$$H_{yi} = \frac{i}{\omega\mu_o} \sqrt{\frac{2}{\pi\rho}} \frac{1}{\sqrt{z}} \left[\frac{1-s}{2} z^{-\frac{s}{2}} - j\rho s z^{\frac{s}{2}} \right] A_1 e^{j(\omega t - \rho z^s)},$$

and the reflected magnetic wave is

$$H_{yr} = \frac{i}{\omega\mu_o} \sqrt{\frac{2}{\pi\rho}} \frac{1}{\sqrt{z}} \left[\frac{1-s}{2} z^{-\frac{s}{2}} + j\rho s z^{\frac{s}{2}} \right] A_2 e^{j(\omega t + \rho z^s)}.$$

Consequently, the wave impedance seen in the medium by the incident wave is

$$Z_{oi} = \frac{E_{xi}}{H_{yi}} = \frac{\omega\mu_o z}{\rho s z^s + j\left(\frac{1-s}{2}\right)},$$

and by the reflected wave is

$$Z_{or} = -\frac{E_{xr}}{H_{yr}} = \frac{\omega\mu_o z}{\rho s z^s - j\left(\frac{1-s}{2}\right)} = Z_{oi}^*.$$

It can be noted that provided $z > 0$ the imaginary part of Z_{oi} is negative, implying that the reactance is capactive, while the imaginary part of Z_{or} is positive, implying that the reactance is inductive.

The phase velocity of the incident field can be calculated by letting

$$\omega t - \rho z^s = \text{Constant} \qquad (93)$$

and by differentiating equation (93). If this is done,

$$\omega - \rho s z^{s-1} \frac{dz}{dt} = 0.$$

Hence, the phase velocity v_p is

$$v_p = \frac{dz}{dt} = \frac{\omega}{\rho s} z^{1-s}. \qquad (94)$$

Since

$$0 < s < 1,$$

the phase velocity increases as z increases.

If an E-mode field can be set up such as to have a phase velocity increasing with z, the field may possibly be very useful in pre-accelerator applications. Equation (94) tends to point towards the possibility of obtaining such an E-mode field through the use of an inhomogeneous medium of this type.

5.3 An Inhomogeneous Slab Between Two Homogeneous Media

An inhomogeneous slab can be used to effect a match between two different homogeneous semi-infinite media. The purpose of this section is to demonstrate this use for the case where the inhomogeneous slab has a permittivity that is functionally described in equation (81).

In Figure 7 the field is assumed to originate in medium 1, pass through the inhomogeneous medium, and enter medium 2. Also, it is assumed that the field in medium 2 is totally absorbed, none of the energy being reflected

back toward the source.

In medium 1 the field equations are

$$E_{x1} = N_1 e^{j(\omega t - \beta_1 z)} + N_2 e^{j(\omega t + \beta_1 z)}$$

and

$$H_{y1} = \sqrt{\frac{\epsilon_1}{\mu_0}} \left[N_1 e^{j(\omega t - \beta_1 z)} - N_2 e^{j(\omega t + \beta_1 z)} \right]$$

where

$$\beta_1 = \omega \sqrt{\mu_0 \epsilon_1}$$

and N_1 and N_2 are constants.[6] The equations representing
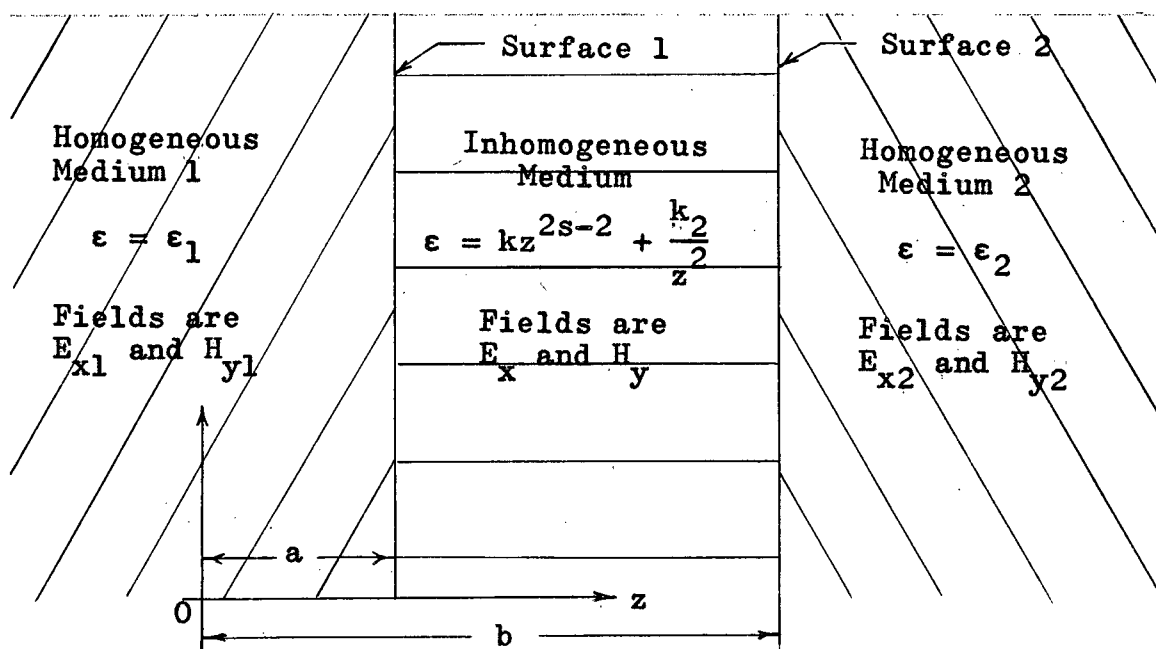
Fig. 7. **An Inhomogeneous Slab Between Two Homogeneous Media**

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6. S. Ramo and J.R. Whinnery, *Fields and Waves in Modern Radio*, John Wiley & Sons, Inc., New York, 1953, p.281.

the fields E_x and H_y in the inhomogeneous medium are given by equations (91) and (92). Since only the transmitted wave is present in medium 2, the field equations are

$$E_{x2} = C_1 e^{j(\omega t - \beta_2 z)}$$

and

$$H_{y2} = \sqrt{\frac{\varepsilon_2}{\mu_o}} \; C_1 e^{j(\omega t - \beta_2 z)}$$

where

$$\beta_2 = \omega \sqrt{\mu_o \varepsilon_2}$$

and C_1 is a constant.

At z equals a in Figure 7,

$$E_{x1} = E_x \tag{95}$$

and

$$H_{y1} = H_y. \tag{96}$$

At z equals b,

$$E_x = E_{x2} \tag{97}$$

and

$$H_y = H_{y2}. \tag{98}$$

From equations (95), (96), (97), and (98) the values for N_1 and N_2 can be determined, as done in Appendix 2, from the parameters of the media, the frequency, and the constant C_1. Once N_1 and N_2 are known, the reflection coefficient R at surface 1 can be readily found since

$$R = \frac{N_2}{N_1} \, .$$

<div style="text-align:right">(99)</div>

In Appendix 2 it is shown that

$$R = e^{-j2\beta_1 a}\left\{ -\left(\eta_a - \sqrt{\frac{\varepsilon_1}{\mu_o}}\right)\left(\eta^*_b + \sqrt{\frac{\varepsilon_2}{\mu_o}}\right) e^{j\rho(b^s - a^s)} \right.$$

$$\left. + \left(\eta^*_a + \sqrt{\frac{\varepsilon_1}{\mu_o}}\right)\left(\eta_b - \sqrt{\frac{\varepsilon_2}{\mu_o}}\right) e^{-j\rho(b^s - a^s)}\right\}$$

<div style="text-align:right">(100)</div>

$$\left(\eta_a + \sqrt{\frac{\varepsilon_1}{\mu_o}}\right)\left(\eta^*_b + \sqrt{\frac{\varepsilon_2}{\mu_o}}\right) e^{j\rho(b^s - a^s)} - \left(\eta^*_a - \sqrt{\frac{\varepsilon_1}{\mu_o}}\right)\left(\eta_b - \sqrt{\frac{\varepsilon_2}{\mu_o}}\right) e^{-j\rho(b^s - a^s)}$$

where

$$\eta_a = \frac{a^{\frac{s-2}{2}}}{\omega\mu_o}\left(\rho s a^{\frac{s}{2}} + j\,\frac{1-s}{2}\,a^{-\frac{s}{2}}\right)$$

$$\eta_b = \frac{b^{\frac{s-2}{2}}}{\omega\mu_o}\left(\rho s b^{\frac{s}{2}} + j\,\frac{1-s}{2}\,b^{-\frac{s}{2}}\right) .$$

If there is no reflection at surface 1 in Figure 7, all the energy can be transferred from medium 1 to medium 2. For this situation to occur,

$$R = 0 .$$

As shown in Appendix 2, in order that

$$R = 0 ,$$

two equations must be satisfied. These equations are

$$\omega\sqrt{k_1}\left(\sqrt{\varepsilon_2} \quad a^{s-1} - \sqrt{\varepsilon_1} \quad b^{s-1}\right)\cos\frac{\omega\sqrt{\mu_o k_1}}{s}(b^s - a^s)$$

$$= \frac{1-s}{2}\left(\frac{1}{a}\sqrt{\frac{\varepsilon_2}{\mu_o}} + \frac{1}{b}\sqrt{\frac{\varepsilon_1}{\mu_o}}\right)\sin\frac{\omega\sqrt{\mu_o k_1}}{s}(b^s - a^s) \tag{101}$$

and

$$\frac{1}{\omega}\sqrt{\frac{k_1}{\mu_o}}\frac{1-s}{2}\left(\frac{b^{s-1}}{a} - \frac{a^{s-1}}{b}\right)\cos\frac{\omega\sqrt{\mu_o k_1}}{s}(b^s - a^s)$$

$$= -\left[k_1(ab)^{s-1} + \frac{1}{\omega^2\mu_o ab}\left(\frac{1-s}{2}\right)^2 - \sqrt{\varepsilon_1\varepsilon_2}\right]\sin\frac{\omega\sqrt{\mu_o k_1}}{s}(b^s - a^s). \tag{102}$$

This means that two of the parameters must be determined by equations (101) and (102). Due to the periodicity of equations (101) and (102), these two parameters, excluding ε_1 and ε_2, have an infinite number of discrete values. Since equations (101) and (102) are not periodic with respect to ε_1 and ε_2, ε_1 and ε_2 have one solution each if they are determined by these equations.

If for a particular frequency two of the remaining parameters are evaluated by equations (101) and (102) and the others are assigned convenient values, the matched condition at surface 1 is established.

If the permittivity of the inhomogeneous medium is specified, the field solutions obtained for the inhomogeneous region only hold for one particular frequency. The reason is that once the permittivity is specified, k_1, k_2, and s have

fixed values, and thus from the equation

$$s = \sqrt{1-4\omega^2\mu_o k_2}$$

the frequency is determined. Therefore, when the permittivity
is specified, the frequency cannot be evaluated from
equations (101) and (102).

However, field solutions for an inhomogeneous
region which hold for any frequency after the permittivity
is specified can be obtained quite readily. For example,
the medium with a permittivity behaving as

$$\varepsilon = k_1 z^{2s-2}$$

where

$$k_1 = \text{Constant}$$

and

$$s = \frac{1}{n} \quad (n=1,3,5\ldots\ldots, 2n-1)$$

has such field solutions.

6. MANUFACTURING OF INHOMOGENEOUS DIELECTRIC MEDIA

Although not too much thought has been given to the possible ways of manufacturing a medium which has a permittivity that is a continuous function of z, three possible methods have been considered.

The first method is to construct the medium by using thin sheets of homogeneous dielectric material. If thin sheets having different values of permittivity are cemented together in some desired order, the resulting laminated dielectric medium varies functionally with z. Provided the thickness of the sheets is small compared to the wavelength of the field being propagated in the medium and provided the change in dielectric constant between adjacent sheets is small, it is believed that the medium as seen by the field effectively varies in the desired continuous manner with z.

Another possible method for manufacturing a medium with a permittivity which is a continuous function of z is by varying the density of the medium in the z direction. For example, the porosity of the medium could be varied in the z direction as in the case of some types of foam rubber. It might be possible to use a centrifugal process while the dielectric material is solidifying to establish a variable density.

Through the use of plasmas, a medium can be obtained with a permittivity which varies continuously in the z direction. For example, as already mentioned, at Stanford it has been proposed that sound waves be propagated down a waveguide filled with plasma to vary in a periodic fashion the density of the medium and thus the effective permittivity.

7. CONCLUSION

For the case of a lossless medium containing no free charges and possessing a continuous and sufficiently differentiable spatially dependent permeability and permittivity, two vectorial differential wave equations,

$$\nabla^2 \mathbf{E} + \nabla\left(\frac{1}{\epsilon}\nabla\epsilon\cdot\mathbf{E}\right) = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{1}{\mu}\left(\nabla\mu\cdot\nabla\mathbf{E} - (\nabla\mu\cdot\nabla)\mathbf{E}\right)$$

and

$$\nabla^2 \mathbf{H} + \nabla\left(\frac{1}{\mu}\nabla\mu\cdot\mathbf{H}\right) = \mu\epsilon\frac{\partial^2\mathbf{H}}{\partial t^2} - \frac{1}{\epsilon}\left(\nabla\epsilon\cdot\nabla\mathbf{H} - (\nabla\epsilon\cdot\nabla)\mathbf{H}\right),$$

were derived from Maxwell's equations. The magnetic vectorial differential wave equation was used to find the necessary condition,

$$\frac{\partial}{\partial z}\left(\frac{1}{\mu}\nabla\mu\cdot\mathbf{H}\right) + \frac{1}{\epsilon}\nabla\epsilon\cdot\frac{\partial\mathbf{H}}{\partial z} = 0,$$

for an E-mode to exist in a waveguide, and the electric vectorial differential wave equation was used to find the necessary condition,

$$\frac{\partial}{\partial z}\left(\frac{1}{\epsilon}\nabla\epsilon\cdot\mathbf{E}\right) + \frac{1}{\mu}\nabla\mu\cdot\frac{\partial\mathbf{E}}{\partial z} = 0,$$

for an H-mode to exist in a waveguide. Through the use of the E-mode condition, an investigation was carried out to determine certain restrictions which must be imposed upon the fields before E-modes can exist in a waveguide filled

with a medium whose permeability is constant and permittivity is

$$\varepsilon = f(r,z).$$

The restrictions were found to be that

$$H_r \cong 0,$$

$$E_\phi \cong 0,$$

and the fields have no angular variation. Also, it was noted that the E-mode condition did not impose any restriction upon the field components when the permittivity is a function of z only.

Through the use of the E-mode condition, an investigation into the restrictions imposed upon the fields when the permittivity has other functional variations is suggested for future study along with a complimentary investigation using the H-mode condition. It is worth pointing out that these restrictions are the duals to the restrictions on the fields for the interchanged cases, namely, the cases having the permeability spatially dependent and the permittivity constant. An investigation into the field restrictions using the mode conditions is recommended for cases where both the permeability and permittivity have various functional forms.

The field equations were investigated for the case

where the permeability is constant and the permittivity
varies with z and for the case where the interchanged
situation is true. In particular, the field equations for
transverse waves and waves with longitudinal components
were considered. After the variables of the pertinent
unified differential field equations were separated, the
differential equations containing the transverse
dependent part of the field components were found to be
the same as the corresponding equations found for
homogeneous media. For the different cases considered,
the z-dependent part of the field components were found
to satisfy one of the following differential equations:

$$\text{i)} \quad \frac{d^2T}{dz^2} + \omega^2qT = 0,$$

$$\text{ii)} \quad \frac{d^2T}{dz^2} + (\omega^2q - M^2)T = 0,$$

$$\text{iii)} \quad \frac{d^2T}{dz^2} - \frac{1}{q}\frac{dq}{dz}\frac{dT}{dz} + (\omega^2q-M^2)T = 0$$

where M is a separation constant and

$$q = \mu\varepsilon.$$

Owing to a theorem by Sturm, it was possible to show that
for a physically realizable situation the solutions for the
first of these equations in T is oscillatory, and the
solutions for the second one is oscillatory provided

$$\omega^2\mu\varepsilon - M^2 > 0.$$

Besides this, the theorem offered a possible test for showing whether or not the solutions for the third equation are oscillatory.

The fields in a dielectric loaded periodic structure were considered from the viewpoint that the medium as a whole is inhomogeneous inside the waveguide. The assumption that the permittivity and all its first and second order derivatives are defined for all interior points in the waveguide was discussed from the point of view taken in mathematical physics, which is that all matter is continuous. After this assumption was made, an investigation into the restrictions on the fields when E-modes are present was carried out. For the case where the dielectric discs have center holes, since there is a radial variation in the permittivity as well as a longitudinal variation, the restrictions were recognized to be the same as those discovered for the example where

$$\varepsilon = f(r,z).$$

When the discs are solid, it was noted that the E-mode condition is satisfied without imposing any restrictions on the fields.

For a periodic structure with solid dielectric discs the theory developed for inhomogeneous dielectric media was used to find the E-mode field expressions when the dielectric regions are matched into the air regions.

In the limit as the permittivity approaches a rectangular waveshape, these field expressions were shown to be in agreement with the expressions derived by solving for the fields in each of the homogeneous regions and by matching the fields at the boundaries.

It is felt that further effort should be made to use the theory for inhomogeneous media to find the field equations for the solid dielectric disc case when a match does not exist. In that the behaviour of the field as the permittivity approaches its limit is known, there should be some method for showing that this known solution satisfies in the limit the differential wave equations resulting from the theory for inhomogeneous media. If this problem could be solved, it may shed some light on how to solve the field problem using the theory for inhomogeneous media when the dielectric discs have center holes. Furthermore, a second approach for finding the field solutions would be established, which is at least of academic interest.

Also, it is felt that a further attempt should be made to find accurate and manageable field solutions for the case where the dielectric discs have center holes. It has not yet been possible to attempt a thorough investigation of this problem.

An investigation was carried out into the behaviour of plane waves in a medium whose permittivity is

$$\epsilon = k_1 z^{2s-2} + \frac{k_2}{z^2} \qquad (103)$$

where k_1 and k_2 are constants and

$$s = \sqrt{1 - 4\omega^2 \mu_o k_2}, \quad 0 < s < 1.$$

The electric and magnetic fields were calculated to be

$$E_x = \sqrt{\frac{2}{\pi \rho}} \; z^{\frac{1-s}{2}} \left[A_1 e^{j(\omega t - \rho z^s)} + A_2 e^{j(\omega t + \rho z^s)} \right]$$

and

$$H_y = \frac{j}{\omega \mu_o} \sqrt{\frac{2}{\pi \rho}} \frac{1}{\sqrt{z}} \left\{ \left[\frac{1-s}{2} z^{-\frac{s}{2}} - j\rho s z^{\frac{s}{2}} \right] A_1 e^{j(\omega t - \rho z^s)} \right.$$

$$\left. + \left[\frac{1-s}{2} z^{-\frac{s}{2}} + j\rho s z^{\frac{s}{2}} \right] A_2 e^{j(\omega t + \rho z^s)} \right\}$$

where

$$\rho = \frac{\omega \sqrt{\mu_o k_1}}{s}.$$

The wave impedance seen in the medium by the incident wave was shown to be

$$Z_{oi} = \frac{\omega \mu_o z}{\rho s z^s + j\left(\frac{1-s}{2}\right)},$$

and the wave impedance seen in the medium by the reflected wave was shown to be

$$Z_{or} = \frac{\omega \mu_o z}{\rho s z^s - j\left(\frac{1-s}{2}\right)} = Z_{oi}^*.$$

The phase velocity was calculated to be

$$v_p = \frac{\omega}{\rho s} z^{1-s}.$$

Also, a slab of dielectric material having a permittivity satisfying equation (103) was placed between two different homogeneous semi-infinite regions and used to effect a match between these two regions.

A brief discussion is made on the possible methods of manufacturing inhomogeneous media with a constant permeability and a permittivity varying in the direction of propagation.

APPENDIX 1*

For a circular waveguide loaded with solid dielectric discs as shown in Figure 3, page 37, the field patterns for an E_{01}-mode can be determined by matching at the boundaries the fields found in each homogeneous region.

Through the use of Maxwell's equations, the field components for an E_{01}-mode are found to be

$$E_z = \left[A_1 e^{-j\beta_1 z} + A_2 e^{j\beta_1 z} \right] J_0(Mr) e^{j\omega t}, \tag{1}$$

$$E_r = \frac{j\omega}{Mv_1} \left[A_1 e^{-j\beta_1 z} - A_2 e^{j\beta_1 z} \right] J_1(Mr) e^{j\omega t}, \tag{2}$$

and

$$H_\phi = \frac{j\omega\varepsilon_1}{M} \left[A_1 e^{-j\beta_1 z} + A_2 e^{j\beta_1 z} \right] J_1(Mr) e^{j\omega t} \tag{3}$$

in the dielectric region (2), and

$$E_z = \left[C_1 e^{-j\beta_0 z} + C_2 e^{j\beta_0 z} \right] J_0(Mr) e^{j\omega t}, \tag{4}$$

$$E_r = \frac{j\omega}{Mv_0} \left[C_1 e^{-j\beta_0 z} - C_2 e^{j\beta_0 z} \right] J_1(Mr) e^{j\omega t}, \tag{5}$$

and

$$H_\phi = \frac{j\omega\varepsilon_0}{M} \left[C_1 e^{-j\beta_0 z} + C_2 e^{j\beta_0 z} \right] J_1(Mr) e^{j\omega t} \tag{6}$$

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* The treatment in this Appendix is based on the paper presented by G.B. Walker and C.G. Englefield, "Some Properties of Dielectric Loaded Slow Wave Structures", PGMTT Symposium, San Diego, May, 1960.

in the air region (3), where

$$\beta_1^2 = \omega^2 \mu_o \varepsilon_1 - M^2 ,$$ (7)

$$\beta_o^2 = \omega^2 \mu_o \varepsilon_o - M^2 ,$$ (8)

$$v_1 = \frac{\omega}{\beta_1} = \text{Phase velocity in dielectric region,}$$

$$v_o = \frac{\omega}{\beta_o} = \text{Phase velocity in air region,}$$

$$M = \frac{s_1}{b} ,$$

$$s_1 = \text{First root of } J_0(Mb) = 0 ,$$

and A_1, A_2, C_1, and C_2 are related constants.[7]

The field in region (4) can be determined by using Floquet's Theorem which states that in a given mode of oscillation of a periodic structure, at a specific frequency, the wave function is multiplied by a given complex phase constant when the field is observed a distance of one period down the structure.[8] Consequently, the field components in region (4) are given by the multiplication of the field components in equations (1),(2), and (3) by $e^{-j\emptyset}$,

7. G. Ramo and J.R. Whinnery, Fields and Waves in Modern Radio, John Wiley & Sons, Inc., New York, 1953, p. 375.

8. J.C. Slater, Microwave Electronics, D. van Nostrand Co., Inc., Princeton, New Jersey, Toronto, London, New York, 1950, p. 170.

where ϕ is the phase change per section.

If the fields are matched at z equals 0 and p-q, the equations obtained are

$$v_o A_1 - v_o A_2 - v_1 C_1 + v_1 C_2 = 0, \qquad (9)$$

$$\epsilon_1 A_1 + \epsilon_1 A_2 - \epsilon_o C_1 - \epsilon_o C_2 = 0, \qquad (10)$$

$$v_o e^{j(2\Theta_1 - \phi)} A_1 - v_o e^{-j(2\Theta_1 + \phi)} A_2 - v_1 e^{-j2\Theta_o} C_1 + v_1 e^{j2\Theta_o} C_2 = 0, \qquad (11)$$

and

$$\epsilon_1 e^{j(2\Theta_1 - \phi)} A_1 + \epsilon_1 e^{-j(2\Theta_1 + \phi)} A_2 - \epsilon_o e^{-j2\Theta_o} C_1 - \epsilon_o e^{j2\Theta_o} C_2 = 0 \qquad (12)$$

where the phase change in the air region is

$$2\Theta_o = \beta_o (p-q)$$

and the phase change in the dielectric region is

$$2\Theta_1 = \beta_1 q .$$

Equations (9),(10),(11), and (12) have unique solutions for three of the constants A_1, A_2, C_1, and C_2 in terms of the remaining constant only if

$$\begin{vmatrix} v_0 & -v_0 & -v_1 & v_1 \\[4pt] \varepsilon_1 & \varepsilon_1 & -\varepsilon_0 & -\varepsilon_0 \\[4pt] v_0 e^{j(2\Theta_1-\phi)} & -v_0 e^{-j(2\Theta_1+\phi)} & -v_1 e^{-j2\Theta_0} & v_1 e^{j2\Theta_0} \\[4pt] \varepsilon_1 e^{j(2\Theta_1-\phi)} & \varepsilon_1 e^{-j(2\Theta_1+\phi)} & -\varepsilon_0 e^{-j2\Theta_0} & -\varepsilon_0 e^{j2\Theta_0} \end{vmatrix} = 0. \tag{13}$$

From the expansion of equation (13) an expression for ϕ can be determined. This expression is

$$4\cos\phi = \left(\sqrt{\frac{Z_1}{Z_0}} + \sqrt{\frac{Z_0}{Z_1}}\right)^2 \cos(2\Theta_1+2\Theta_0) - \left(\sqrt{\frac{Z_1}{Z_0}} - \sqrt{\frac{Z_0}{Z_1}}\right)^2 \cos(2\Theta_1-2\Theta_0)$$

or

$$\cos\phi = \cos 2\Theta_0 \cos 2\Theta_1 - \frac{1}{2}\left(\frac{Z_0}{Z_1}+\frac{Z_1}{Z_0}\right)\sin 2\Theta_0 \sin 2\Theta_1$$

where Z_0 is the wave impedance for an E_{01}-mode in the air region,

$$Z_0 = \frac{1}{v_0 \varepsilon_0} \, ,$$

and Z_1 is the wave impedance for an E_{01}-mode in the dielectric region,

$$Z_1 = \frac{1}{v_1 \varepsilon_1} \, .$$

Through the use of any three of the equations (9),(10),(11), and (12), the constants A_1, A_2, and C_1 can be evaluated in terms of C_2. If this is done, the

relationships obtained are

$$\frac{A_1}{C_2} = \frac{\varepsilon_0}{\varepsilon_1} e^{j\emptyset} \frac{\cos 2\Theta_0 + j\frac{Z_0}{Z_1} \sin 2\Theta_0 - e^{-j(2\Theta_1+\emptyset)}}{e^{+j(\emptyset-2\Theta_0)} - \cos 2\Theta_1 - j\frac{Z_1}{Z_0} \sin 2\Theta_1} \quad ,$$

$$\frac{A_2}{C_2} = \frac{\varepsilon_0}{\varepsilon_1} e^{j\emptyset} \frac{\cos 2\Theta_0 - j\frac{Z_0}{Z_1} \sin 2\Theta_0 - e^{j(2\Theta_1-\emptyset)}}{e^{-j(\emptyset-2\Theta_0)} - \cos 2\Theta_1 - j\frac{Z_1}{Z_0} \sin 2\Theta_1} \quad ,$$

and

$$\frac{C_1}{C_2} = \frac{e^{j(\emptyset+2\Theta_0)} + j\frac{Z_1}{Z_0} \sin 2\Theta_1 - \cos 2\Theta_1}{e^{j(\emptyset-2\Theta_0)} - \cos 2\Theta_1 - j\frac{Z_1}{Z_0} \sin 2\Theta_1} \quad .$$

Now that the field components have been determined over one period p, expressions for the field components can be found which hold throughout the waveguide. For example, for E_z such an expression can be found by taking the following steps. The first step is to define the function

$$F(z) = \frac{E_z(r,z,t)e^{\frac{j\emptyset z}{p}}}{J_0(Mr)e^{j\omega t}} \quad . \tag{14}$$

From equation (14)

$$F(z+p) = \frac{E_z(r, z+p, t)e^{\frac{j\emptyset z}{p}} e^{j\emptyset}}{J_0(Mr)e^{j\omega t}} \quad ,$$

and from Floquet's Theorem

$$E_z(r, z+p, t) = E_z(r,z,t)e^{-j\emptyset} .$$

Hence,

$$F(z+p) = F(z) ,$$

and thus $F(z)$ is a periodic function in z with a period p. Consequently, $F(z)$ can be expressed as the Fourier sum

$$F(z) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{-j2\pi nz}{p}}$$

where

$$a_n = \frac{1}{p} \int_{-q}^{p-q} F(z) e^{\frac{j2\pi nz}{p}} dz .$$

Therefore,

$$E_z = J_0(Mr)e^{j\omega t} \sum_{n=-\infty}^{\infty} a_n e^{-j\left(\frac{\emptyset+2\pi n}{p}\right)z}$$

where

$$a_n = \frac{1}{p} \int_{-q}^{p-q} \frac{E_z(r,z,t)}{J_0(Mr)e^{j\omega t}} e^{j\left(\frac{\emptyset+2\pi n}{p}\right)z} dz .$$

When the dielectric regions are matched into the air regions,

$$A_2 = C_2 = 0.$$

Consequently, equations (1), (2), (3), (4), (5), and (6)

become respectively

$$E_z = A_1 J_0(Mr) e^{j(\omega t - \beta_1 z)} \; ,$$

$$E_r = \frac{j\omega}{Mv_1} A_1 J_1(Mr) e^{j(\omega t - \beta_1 z)} \; ,$$

$$H_\phi = \frac{j\omega\varepsilon_1}{M} A_1 J_1(Mr) e^{j(\omega t - \beta_1 z)} \; ,$$

$$E_z = C_1 J_0(Mr) e^{j(\omega t - \beta_0 z)} \; ,$$

$$E_r = \frac{j\omega}{Mv_0} C_1 J_1(Mr) e^{j(\omega t - \beta_0 z)} \; ,$$

and

$$H_\phi = \frac{j\omega\varepsilon_0}{M} C_1 J_1(Mr) e^{j(\omega t - \beta_0 z)} \; .$$

Since for the matched case

$$Z_0 = Z_1 \; ,$$

the phase shift per section is given by

$$\cos \phi = \cos 2\Theta_0 \cos 2\Theta_1 - \sin 2\Theta_0 \sin 2\Theta_1 .$$

Therefore,

$$\phi = 2\Theta_0 + 2\Theta_1 \; ,$$

$$= (p-q)\beta_0 + q\beta_1$$

$$= (p-q) \sqrt{\omega^2 \mu_0 \varepsilon_0 - M^2} + q \sqrt{\omega^2 \mu_0 \varepsilon_1 - M^2} \; .$$

APPENDIX 2

The fields for the different regions shown in Figure 7, page 51, are:

i) in medium 1

$$E_{x1} = \left[N_1 e^{-j\beta_1 z} + N_2 e^{j\beta_1 z} \right] e^{j\omega t}$$

and

$$H_{y1} = \sqrt{\frac{\varepsilon_1}{\mu_o}} \left[N_1 e^{-j\beta_1 z} - N_2 e^{j\beta_1 z} \right] e^{j\omega t}$$

ii) in the inhomogeneous medium

$$E_x = \sqrt{\frac{2}{\pi\rho}} \; z^{\frac{1-s}{2}} \left[A_1 e^{-j\rho z^s} + A_2 e^{j\rho z^s} \right] e^{j\omega t}$$

and

$$H_y = \frac{j}{\omega\mu_o} \sqrt{\frac{2}{\pi\rho}} \frac{1}{\sqrt{z}} \left\{ \left[\frac{1-s}{2} z^{-\frac{s}{2}} - j\rho s z^{\frac{s}{2}} \right] A_1 e^{-j\rho z^s} + \left[\frac{1-s}{2} z^{-\frac{s}{2}} + j\rho s z^{\frac{s}{2}} \right] A_2 e^{j\rho z^s} \right\} e^{j\omega t}$$

iii) in medium 2

$$E_{x2} = C_1 e^{-j\beta_2 z} e^{j\omega t}$$

and

$$H_{y2} = \sqrt{\frac{\varepsilon_2}{\mu_o}} \; C_1 e^{-j\beta_2 z} e^{j\omega t}$$

where

$$\beta_1 = \omega\sqrt{\mu_o \varepsilon_1} \; ,$$

$$\rho = \frac{\omega\sqrt{\mu_o k_1}}{s} \; ,$$

$$s = \sqrt{1 - 4\omega^2\mu_o k_2} \ , \quad 0 < s < 1 \ ,$$

$$\beta_2 = \omega\sqrt{\mu_o \epsilon_2} \ ,$$

and A_1, A_2, N_1, N_2, and C_1 are related constants.

At z equals a the boundary conditions are

$$E_{x1} = E_x$$

and

$$H_{y1} = H_y \ .$$

Therefore,

$$N_1 e^{-j\beta_1 a} + N_2 e^{j\beta_1 a} = \sqrt{\frac{2}{\pi\rho}} \ a^{\frac{1-s}{2}}\left[A_1 e^{-j\rho a^s} + A_2 e^{j\rho a^s}\right] \quad (1)$$

and

$$\sqrt{\frac{\epsilon_1}{\mu_o}}\left[N_1 e^{-j\beta_1 a} - N_2 e^{j\beta_1 a}\right] =$$

$$\frac{j}{\omega\mu_o}\sqrt{\frac{2}{\pi\rho}}\frac{1}{\sqrt{a}}\left\{\left[\frac{1-s}{2}a^{-\frac{s}{2}} - j\rho s a^{\frac{s}{2}}\right]A_1 e^{-j\rho a^s} + \left[\frac{1-s}{2}a^{-\frac{s}{2}} + j\rho s a^{\frac{s}{2}}\right]A_2 e^{j\rho a^s}\right\} .$$
$$(2)$$

At z equals b

$$E_x = E_{x2}$$

and

$$H_y = H_{y2}.$$

Consequently,

$$\sqrt{\frac{2}{\pi \rho}} \; b^{\frac{1-s}{2}} \left[A_1 e^{-j\rho b^s} + A_2 e^{j\rho b^s} \right] = C_1 e^{-j\beta_2 b} \tag{3}$$

and

$$\frac{j}{\omega \mu_o} \sqrt{\frac{2}{\pi \rho}} \frac{1}{b} \left\{ \left[\frac{1-s}{2} b^{-\frac{s}{2}} - j\rho s b^{\frac{s}{2}} \right] A_1 e^{-j\rho b^s} + \left[\frac{1-s}{2} b^{-\frac{s}{2}} + j\rho s b^{\frac{s}{2}} \right] A_2 e^{j\rho b^s} \right\}$$

$$= \sqrt{\frac{\epsilon_2}{\mu_o}} \; C_1 e^{-j\beta_2 b} \; . \tag{4}$$

At surface 1 the reflection coefficient R is defined as

$$R = \frac{N_2}{N_1} \; . \tag{5}$$

One approach to finding R is to express both N_1 and N_2 in terms of C_1. To do this, A_1 and A_2 must be found in terms of C_1.

From equations (3) and (4)

$$A_1 = \xi_b \, e^{j(\rho b^s - \beta_2 b)} \left(\eta_b^* + \sqrt{\frac{\epsilon_2}{\mu_o}} \right) C_1 \tag{6}$$

and

$$A_2 = \xi_b \, e^{-j(\rho b^s + \beta_2 b)} \left(\eta_b - \sqrt{\frac{\epsilon_2}{\mu_o}} \right) C_1 \tag{7}$$

where

$$\xi_b = \frac{\omega \mu_o}{2 \rho s b^{\frac{s}{2}}} \sqrt{\frac{\pi \rho b}{2}} \tag{8}$$

and

$$\eta_b = \frac{b^{\frac{s-2}{2}}}{\omega\mu_o}\left[\rho s b^{\frac{s}{2}} + j\left(\frac{1-s}{2}\right)b^{-\frac{s}{2}}\right]. \tag{9}$$

From equations (1) and (2) both N_1 and N_2 are solved for in terms of A_1 and A_2. The results are

$$N_1 = \xi_a e^{-j(\rho a^s - \beta_1 a)}\left(\eta_a + \sqrt{\frac{\epsilon_1}{\mu_o}}\right)A_1 - \xi_a e^{j(\rho a^s + \beta_1 a)}\left(\eta_a^* - \sqrt{\frac{\epsilon_1}{\mu_o}}\right)A_2 \tag{10}$$

and

$$N_2 = -\xi_a e^{-j(\rho a^s + \beta_1 a)}\left(\eta_a - \sqrt{\frac{\epsilon_1}{\mu_o}}\right)A_1 + \xi_a e^{j(\rho a^s - \beta_1 a)}\left(\eta_a^* + \sqrt{\frac{\epsilon_1}{\mu_o}}\right)A_2 \tag{11}$$

where

$$\xi_a = \frac{1}{2a^{\frac{s}{2}}}\sqrt{\frac{2a\mu_o}{\pi\rho\epsilon_1}} \tag{12}$$

and

$$\eta_a = \frac{a^{\frac{s-2}{2}}}{\omega\mu_o}\left[\rho s a^{\frac{s}{2}} + j\left(\frac{1-s}{2}\right)a^{-\frac{s}{2}}\right]. \tag{13}$$

Now, the values for A_1 and A_2 found respectively in equations (6) and (7) are substituted into equations (10) and (11) to give

$$N_1 = \xi_a \xi_b C_1 e^{j(\beta_1 a - \beta_2 b)}\left\{\left(\eta_a + \sqrt{\frac{\epsilon_1}{\mu_o}}\right)\left(\eta_b^* + \sqrt{\frac{\epsilon_2}{\mu_o}}\right)e^{j\rho(b^s - a^s)}\right.$$
$$\left. - \left(\eta_a^* - \sqrt{\frac{\epsilon_1}{\mu_o}}\right)\left(\eta_b - \sqrt{\frac{\epsilon_2}{\mu_o}}\right)e^{-j\rho(b^s - a^s)}\right\} \tag{14}$$

and

$$N_2 = \xi_a \xi_b C_1 e^{-j(\beta_1 a + \beta_2 b)} \left\{ -\left(\eta_a - \sqrt{\frac{\epsilon_1}{\mu_0}} \right) \left(\eta_b^* + \sqrt{\frac{\epsilon_2}{\mu_0}} \right) e^{j\rho(b^S - a^S)} \right.$$

$$\left. + \left(\eta_a^* + \sqrt{\frac{\epsilon_1}{\mu_0}} \right) \left(\eta_b - \sqrt{\frac{\epsilon_2}{\mu_0}} \right) e^{-j\rho(b^S - a^S)} \right\}. \tag{15}$$

If expressions (14) and (15) are substituted into equation (5), it is found that

$$R = e^{-j2\beta_1 a} \left\{ - \left(\eta_a - \sqrt{\frac{\epsilon_1}{\mu_0}} \right) \left(\eta_b^* + \sqrt{\frac{\epsilon_2}{\mu_0}} \right) e^{j\rho(b^S - a^S)} \right.$$

$$\left. + \left(\eta_a^* + \sqrt{\frac{\epsilon_1}{\mu_0}} \right) \left(\eta_b - \sqrt{\frac{\epsilon_2}{\mu_0}} \right) e^{-j\rho(b^S - a^S)} \right\} \tag{16}$$

$$\overline{ \left(\eta_a + \sqrt{\frac{\epsilon_1}{\mu_0}} \right) \left(\eta_b^* + \sqrt{\frac{\epsilon_2}{\mu_0}} \right) e^{j\rho(b^S - a^S)} - \left(\eta_a^* - \sqrt{\frac{\epsilon_1}{\mu_0}} \right) \left(\eta_b - \sqrt{\frac{\epsilon_2}{\mu_0}} \right) e^{-j\rho(b^S - a^S)} }$$

If there is no reflection at surface 1,

$$R = 0.$$

Therefore, from equation (16)

$$\left(\eta_a - \sqrt{\frac{\epsilon_1}{\mu_0}} \right) \left(\eta_b^* + \sqrt{\frac{\epsilon_2}{\mu_0}} \right) e^{j\rho(b^S - a^S)} = \left(\eta_a^* + \sqrt{\frac{\epsilon_1}{\mu_0}} \right) \left(\eta_b - \sqrt{\frac{\epsilon_2}{\mu_0}} \right) e^{-j\rho(b^S - a^S)}$$

or

$$\left(\eta_a \eta_b^* + \sqrt{\frac{\epsilon_2}{\mu_0}} \eta_a - \sqrt{\frac{\epsilon_1}{\mu_0}} \eta_b^* - \sqrt{\frac{\epsilon_1 \epsilon_2}{\mu_0^2}} \right) e^{j\rho(b^S - a^S)}$$

$$= \left(\eta_a \eta_b^* - \sqrt{\frac{\epsilon_2}{\mu_0}} \eta_a^* + \sqrt{\frac{\epsilon_1}{\mu_0}} \eta_b - \sqrt{\frac{\epsilon_1 \epsilon_2}{\mu_0^2}} \right) e^{-j\rho(b^S - a^S)}. \tag{17}$$

Now, if the following definitions are made:

i) $\mathcal{T} = \rho(b^S - a^S)$,

ii) $\eta_a = w_a + jv_a$,

iii) $\eta_b = w_b + jv_b$,

iv) $\Gamma_1 = w_a w_b + v_a v_b + \sqrt{\dfrac{\varepsilon_2}{\mu_o}}\, w_a - \sqrt{\dfrac{\varepsilon_1}{\mu_o}}\, w_b - \sqrt{\dfrac{\varepsilon_1 \varepsilon_2}{\mu_o^2}}$,

v) $\Gamma_2 = w_a w_b + v_a v_b - \sqrt{\dfrac{\varepsilon_2}{\mu_o}}\, w_a + \sqrt{\dfrac{\varepsilon_1}{\mu_o}}\, w_b - \sqrt{\dfrac{\varepsilon_1 \varepsilon_2}{\mu_o^2}}$,

vi) $\Psi_1 = v_a w_b - v_b w_a + \sqrt{\dfrac{\varepsilon_2}{\mu_o}}\, v_a + \sqrt{\dfrac{\varepsilon_1}{\mu_o}}\, v_b$,

vii) $\Psi_2 = -v_a w_b + v_b w_a + \sqrt{\dfrac{\varepsilon_2}{\mu_o}}\, v_a + \sqrt{\dfrac{\varepsilon_1}{\mu_o}}\, v_b$,

the substitution of these newly defined quantities into equation (17) yields

$$(\Gamma_1 + j\Psi_1)e^{j\mathcal{T}} = (\Gamma_2 + j\Psi_2)e^{-j\mathcal{T}} \,. \tag{18}$$

Once the real parts of equation (18) are equated, the resulting equation is

$$(\Gamma_1 - \Gamma_2)\cos\mathcal{T} = (\Psi_1 + \Psi_2)\sin\mathcal{T} \,. \tag{19}$$

Similarly, from the imaginary parts of equation (18)

$$(\Psi_1 - \Psi_2)\cos\mathcal{T} = -(\Gamma_1 + \Gamma_2)\sin\mathcal{T} \,. \tag{20}$$

In terms of the original parameters, equations (19) and (20) are respectively

$$\omega\sqrt{k_1}\left(\sqrt{\varepsilon_2}\ a^{s-1} - \sqrt{\varepsilon_1}\ b^{s-1}\right)\cos\frac{\omega\sqrt{\mu_o k_1}}{s}(b^s - a^s)$$

$$= \frac{1-s}{2}\left(\frac{1}{a}\sqrt{\frac{\varepsilon_2}{\mu_o}} + \frac{1}{b}\sqrt{\frac{\varepsilon_1}{\mu_o}}\right)\sin\frac{\omega\sqrt{\mu_o k_1}}{s}(b^s - a^s)$$

and

$$\frac{1}{\omega}\sqrt{\frac{k_1}{\mu_o}}\frac{1-s}{2}\left(\frac{b^{s-1}}{a} - \frac{a^{s-1}}{b}\right)\cos\frac{\omega\sqrt{\mu_o k_1}}{s}(b^s - a^s)$$

$$= -\left(k_1(ab)^{s-1} + \frac{1}{\omega^2\mu_o ab}\left(\frac{1-s}{2}\right)^2 - \sqrt{\varepsilon_1\varepsilon_2}\right)\sin\frac{\omega\sqrt{\mu_o k_1}}{s}(b^s - a^s)\ .$$

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