THE CURRENT DISTRIBUTION AND INPUT IMPEDANCE OF CYLINDRICAL ANTENNAS

by

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ABSTRACT

A critical discussion of various theories of cylindrical antennas is given. It is shown that the Hallen-King theory is of a semi-empirical nature. Both the current distribution and input impedance derived from this theory depend on a semi-empirical choice of an expansion parameter. It is also shown that the Hallen-King "slice generator" cannot be used to represent the effect of the transmission line. A theory which correctly accounts for this effect is developed and theoretical formulas are developed for both the current distribution and input impedance.

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THE CURRENT DISTRIBUTION AND INPUT IMPEDANCE OF CYLINDRICAL ANTENNAS

1. Introduction

A considerable amount of theoretical work has gone into developing formulas for the current distribution and input impedance of cylindrical antennas. The results of these theories are contained in King's monumental work [1].

It is a well known fact that the results of these theoretical works agree reasonably well with experiments. However, despite this agreement, there are serious theoretical objections to the methods used which consists of a combination of circuit and field concepts. These are discussed in more detail in Chaps. II and III. In Chap. II some of the existing methods are reviewed with an aim to give adequate guidance to the critical discussion of the methods contained in Chap. III. The following Chapter deals with the development of a correct theory and a comparison, as far as it is possible, with the results obtained by King.

The impedance concept is reviewed in Sec. 3-1. A formula for the input impedance in terms of the input power and input current in antennas is also developed in this section. This general formula is then used in Sec. 4-5 to obtain a formula for the input impedance of cylindrical antennas. Since analytical evaluation of the integrals (3-11) and (3-12) is very difficult a series expansion of the integrals is introduced. The resulting impedance formula is consequently also in series form. This is suitable for digital computer work.

2. Review of the Existing Methods

2.1 The Energy Method

In evaluating the input impedance of an antenna, the energy radiated from it must be known first. The energy equation can be derived from Maxwell's equations, and this can be found in any standard book on Electromagnetic Waves[2]. Therefore, only the result will be given here, which is

$$P_i = -\frac{1}{2}\int_V \vec{E}_i \cdot \vec{i}^* \, dv = P_r + j\omega(W_m - W_e) \qquad \ldots(2\text{-}1)$$

where

P_i = complex input power

$$W_m = \frac{\mu}{2}\int_V \vec{H} \cdot \vec{H}^* \, dv \qquad \ldots(2\text{-}2)$$

= twice the average stored magnetic energy.

$$W_e = \frac{\epsilon}{2}\int_V \vec{E} \cdot \vec{E}^* \, dv \qquad \ldots(2\text{-}3)$$

= twice the average stored electric energy.

$$P_r = \frac{1}{2}\int_S (\vec{E} \times \vec{H}^*) \cdot \vec{N} \, ds \qquad \ldots(2\text{-}4)$$

= the average radiated energy per unit time

$\vec{E}$ = the electric field intensity,

$\vec{i}$ = the current density,

$\vec{H}$ = the magnetic field intensity,

$\vec{N}$ = outward unit normal

S = a closed surface enclosing a volume V.

In order that P_r as given by Eq. (2-4) represents the total power radiated from the antenna, the surface S must enclose the antenna including the source driving it (Fig. 2-1).

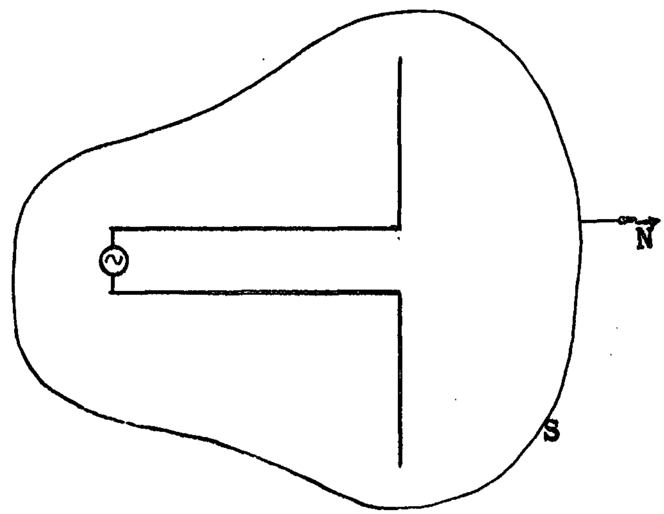


Fig. 2-1. Region of Integration for Eq.(2-1)

After evaluating the energy, the input impedance can be calculated using the ordinary definition

$$P_i = \frac{1}{2} Z_i \left|I_i\right|^2 \qquad \ldots(2\text{-}5)$$

where Z_i = the input impedance,

I_i = the input current of the antenna.

2.1.1 The Poynting Vector Method

In the Poynting vector method the radiated power is calculated using Eq. (2-4) where the surface of integration is a large spherical surface with the antenna located at its centre. Due to the simplicity of the analysis, only the real part of the power can be calculated and consequently only the radiation resistance of the antenna will be obtained. The current distribution along a thin linear antenna is assumed sinusoidal and has the form

$$I(z) = I_m \sin \beta(L - |z|) \qquad \ldots(2\text{-}6)$$

which is approximately the case for thin antennas[3]. L in Eq. (2-6) is the half length of the antenna and $\beta = \omega\sqrt{\mu\epsilon}$.

The distant field behaves like a spherical wave emerging from a point source located at the origin, which will be taken as the centre of the surface of integration and the centre of the antenna. The field components can be calculated using the relations

$$H_\varphi = \frac{1}{\mu} \operatorname{curl}_\varphi \bar{A} \qquad \ldots(2\text{-}7)$$

$$E_\theta = \eta H_\varphi \qquad \ldots(2\text{-}8)$$

$$|\bar{A}| = A_z = \frac{\mu}{4\pi} \int_A I(z) \frac{e^{-j\beta R}}{R} dz \qquad \ldots(2\text{-}9)$$

where η = 120π, the intrinsic impedance of free space,

R = the distance from a current element $I(z)dz$ to the point of observation.

The results are[4]

$$H_\varphi = j\frac{I_m e^{-j\beta R_o}}{2\pi R_o} \left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \right]$$

$$E_\theta = j\frac{60\, I_m e^{-j\beta R_o}}{R_o} \left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \right] \qquad \ldots(2\text{-}10)$$

where R_o is the distance from the origin to the point of observation. Since from Eq. (2-10) the field components are in phase, the real part of the radiated power can be obtained using Eq. (2-4) by substituting the magnitudes of the field components in the appropriate places. Z_i in Eq. (2-5) must now be replaced by the radiation resistance R_{rad}.

Because Eq. (2-5) gives an infinite value for the input resistance if $\sin \beta L = 0$, it is customary in this method to calculate the radiation resistance in terms of the maximum current along the antenna. This value is called the "loop resistance".

2.1.2 <u>The Induced-Emf Method</u>

In the induced-emf method, the antenna is assumed thin and the current distribution sinusoidal. This method gives the real as well as the imaginary part of the loop impedance of the antenna. Each small section of the antenna is considered to be a Herztian dipole, and each contributes to the total power radiated from it by an amount proportional to the current flowing through the element and the electric field intensity induced in the element. Since for thin linear antennas the current distribution is axial, the total power radiated from the antenna is then

$$P_{rad} = P_i = -\frac{1}{2}\int_A E_z \, I^*(z) \, dz \qquad \ldots(2\text{-}11)$$

where E_z is the tangential component of the electric field intensity on the surface of the antenna, and A implies that the integration must be carried out over the whole length of the antenna.

The loop impedance can now be calculated using Eq. (2-5) with I_o and Z_o replacing I_i and Z_i respectively. The reason for taking the "loop impedance" instead of the input impedance is similar to that of the previous method.

2.2 The Hallen-King Method[(1),(5),(6),(7)].

The Hallen-King method opens a way to a theoretical determination of the current distribution along the antenna, taking the boundary conditions into consideration, which is not attempted in the previous methods. The terminal driving condition is replaced by a "slice generator" and the current distribution is obtained by solving the boundary value problem exactly.

To be able to follow the essential steps in this method, consider a cylindrical antenna with its axis coincident with the z-axis of a cylindrical coordinate system and centred at the origin. Let the radius of the cylindrical part be a, the length 2L and the width of the gap 2b (Fig. 2-2). The dimensions are such that both a and b are $\ll$ L.

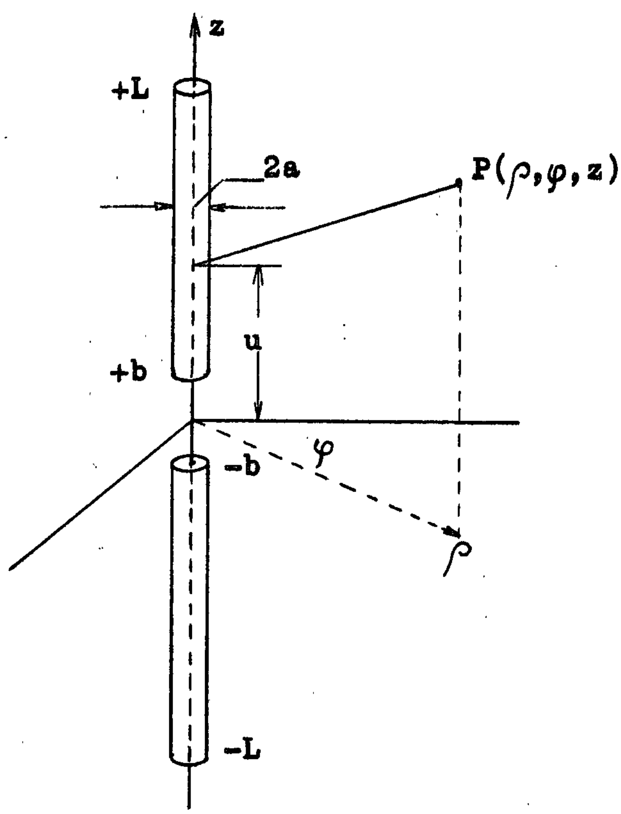


Fig. 2-2. Cylindrical Antenna

The electric field intensity $\vec{E}$ at an arbitrary point $P(\rho,\varphi,z)$ in space can be expressed in terms of the current and charge distributions along the antenna as

$$\vec{E} = -\operatorname{grad} U - j\omega\vec{A} \qquad \ldots(2\text{-}12)$$

where

$$U = \frac{1}{4\pi\epsilon_o}\int_A q(u)\,\Psi(z,u)\,du \qquad \ldots(2\text{-}13)$$

$$A = \frac{\mu_o}{4\pi}\int_A I(u)\,\Psi(z,u)\,du \qquad \ldots(2\text{-}14)$$

$$\Psi(z,u) = \frac{e^{-j\beta\sqrt{(z-u)^2+\rho^2}}}{\sqrt{(z-u)^2+\rho^2}} \qquad \ldots(2\text{-}15)$$

$I(z)$ is the current distribution along the antenna,

$q(z)$ is the linear charge density distribution along the antenna,

ϵ_o is the permittivity of free space,

μ_o is the permeability of free space.

The boundary condition requires that the tangential component of the electric field E_z vanishes on the surface of the antenna. Therefore evaluating (2-12) on the surface of the antenna and taking only the z-component of the result gives, after substituting

$$\operatorname{div}\vec{A} = -j\frac{\beta^2}{\omega}U, \qquad \ldots(2\text{-}16)$$

$$\frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z = 0 \qquad \ldots(2\text{-}17)$$

where $\beta = \frac{2\pi}{\lambda}$ with λ= the wave length of the signal. In the above equations, circular symmetry of the current and charge distributions are assumed.

The differential equation (2-17) can be easily solved for A_z and substituting Eq. (2-14) into this solution an integral equation for the current distribution is obtained.

$$A_z = \frac{\mu_o}{4\pi} \int_A I(u)\ \Psi(z,u)\ du = B \sin \beta \left| z \right| + C \cos \beta z \qquad \ldots(2\text{-}18)$$

The voltage applied by the "slice generator" at the base of the antenna determines one of the constants of integration in Eq. (2-18). This is done by equating the scalar potential difference across the gap produced by the charges on the antenna with that of the "slice generator". The scalar potential at any point along the surface of the antenna can be most conveniently determined using the relation (2-16) which in the one-dimensional case becomes

$$\frac{\partial}{\partial z} A_z = -j\ \frac{\beta^2}{\omega}\ U \qquad \ldots(2\text{-}19)$$

Then for $\beta b \ll 1$ one has

$$\left. \frac{\partial A_z}{\partial z} \right|_{z=b} = \beta B$$

$$\left. \frac{\partial A_z}{\partial z} \right|_{z=-b} = -\ \beta B$$

Hence

$$\frac{\partial A_z(b)}{\partial z} - \frac{\partial A_z(-b)}{\partial z} = 2\ \beta B = -\ \frac{j\beta^2}{\omega} \Big[U(b) - U(-b) \Big]$$

From here follows immediately

$$B = -\frac{j\beta}{\omega}\,\frac{V_\delta}{2} \qquad \ldots(2\text{-}20)$$

where $V_\delta = U(b) - U(-b)$. $\ldots(2\text{-}21)$

The other constant, C , cannot be evaluated directly. This is determined by the other boundary condition which specifies that the current must vanish at both ends of the antenna. Therefore this constant can only be determined after the integral equation,

$$\int_A I(u)\,\Psi(z,u)\,du = -j\,\frac{V_\delta}{60}\,\sin\beta\,|z| + C\cos\beta z \qquad \ldots(2\text{-}22)$$

which is obtained by simply substituting (2-20) in Eq. (2-18), has been solved.

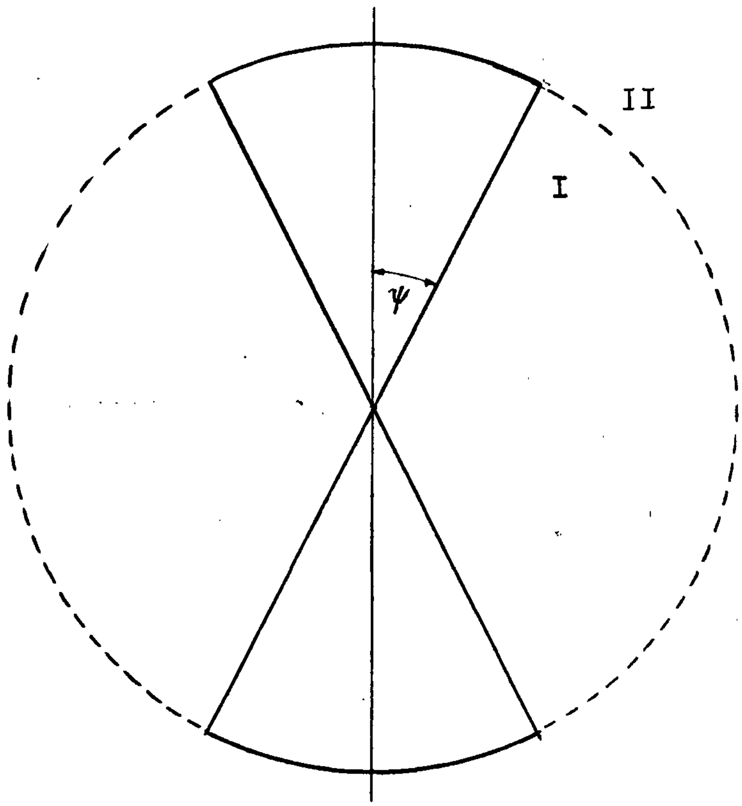
After the current distribution is known, the input impedance Z_i is calculated using the formula

$$Z_i = \frac{V_\delta}{I_i} \qquad \ldots(2\text{-}23)$$

where V_δ = the scalar potential difference applied at the base,
I_i = the input (base) current.

2.3 Mode Theory of Schelkunoff

Consider a biconical antenna with zero gap width, and let the caps be spherical surfaces (Fig. 2-3).



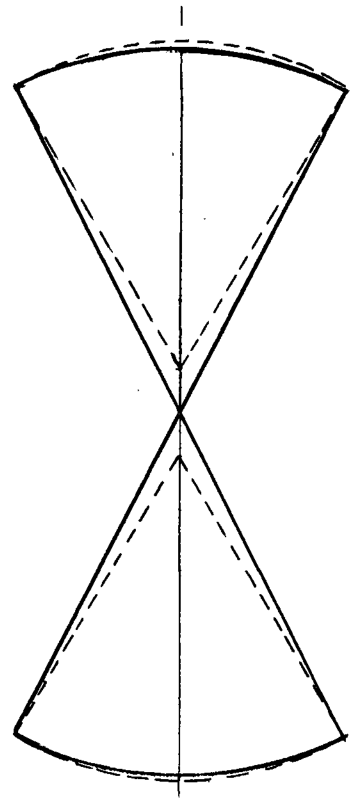


Fig. 2-3a. Idealized Biconical Antenna

Fig. 2-3b. Broken Line: Actual Solid Line: Idealized

Using spherical coordinates and the antenna configuration shown in Fig. 2-3a, Maxwell's equations with the appropriate boundary conditions can be solved exactly. This is possible since, as can be easily seen, any one of the boundaries can be made to coincide with one of the spherical coordinate surfaces. In practice, the gap is not zero; but if it is small compared with the length of the antenna, Fig. 2-3b suggests that the deviation from the ideal case is very small indeed and hence the solution based on the ideal configuration will be a very good approximation to the actual one.

Because the boundary conditions imposed on the solution in region I are different from those on the solution in region II (Fig.2-3a), it is necessary to construct two sets of solutions, one for region I and the other for region II in such a way as to satisfy the appropriate boundary conditions. The continuity of the solution is maintained by matching the solutions at the surface dividing these two regions. For the details of matching these solutions, the reader is referred to the listed references[(8),(9),(10)]. It will be appropriate to point out here, however, that the solutions contain enough integration constants to make this matching possible.

Of particular interest here is how the input impedance of the antenna can be evaluated. It is found that the solution in region I consists of two parts: the principal and the complementary parts. Accordingly the current along the antenna consists of a principal part $I_o(r)$ associated with the principal or the TEM wave and a complementary current $\bar{I}(r)$, associated with all other waves, and hence

$$I(r) = I_o(r) + \bar{I}\,(r) \qquad \ldots(2\text{-}24)$$

The input admittance is defined as

$$Y_i = \frac{I(r_i)}{V(r_i)} = \frac{I_o\,(r_i) + \bar{I}\,(r_i)}{V(r_i)} \qquad \ldots(2\text{-}25)$$

where V(r) is the transverse voltage defined as

$$V(r) = \int_{\psi}^{\pi-\psi} r\, E_\theta\, d\theta \qquad \ldots(2\text{-}26)$$

and ψ is the half-angle of the cone. It turns out, that only the TEM component of the wave contributes to the integral (2-26). It is further known that the complementary current vanishes with the width of the gap and hence for a small gap its effect can be neglected. The expression for the input impedance is therefore simplified into

$$Z_i = \frac{V_o(r_i)}{I_o(r_i)} \qquad \ldots(2\text{-}27)$$

which means that the input impedance can be calculated in terms of the TEM wave alone.

The principal current and the transverse voltage along the antenna satisfy the ordinary transmission line equations

$$K\, I_o(r) = V_o(L) \left[j \sin \beta(L - r) + K\, Y_t \cos \beta(L - r) \right] \ldots(2\text{-}28)$$

$$V_o(r) = V_o(L) \left[\cos \beta(L - r) + j\, K\, Y_t \sin \beta(L - r) \right] \ldots(2\text{-}29)$$

where $K = 120 \log. \operatorname{cotg} \psi$ ...(2-30)

is the characteristic impedance of the antenna,

and $Y_t = I_o(L)/V_o(L)$ ...(2-31)

is the terminal admittance of the antenna.

From Eqs. (2-27),(2-28) and (2-29) it follows immediately that

$$Z_i = \frac{1}{Y_i} = K \frac{Z_a \sin \beta L - j\, K \cos \beta L}{K \sin \beta L - j\, Z_a \cos \beta L} \qquad \ldots(2\text{-}32)$$

with $Z_a = K^2\, Y_t$. ...(2-33)

Note, however, that I(L) must be evaluated from the general solution and not from Eq. (2-28). Similarly V(L) must be calculated from Eq.(2-26).

This theory treats a cylindrical antenna as a limiting case of a biconical antenna, that is, by letting the cone angle ψ approach zero. The input impedance of an infinitely thin cylindrical antenna can therefore be calculated from Eq.(2-32) by the same limiting process.

2.4 Storer's Variational Method(11),(12)

To facilitate comparison, consider the following relations

$$P'_n = \frac{1}{2} V'_n I_n = -\frac{1}{2}\int_A \vec{E}_n \cdot \vec{i}_n \, dv \qquad \ldots(2\text{-}34)$$

$$Z'_n = \frac{V'_n}{I_n} = \frac{2P'_n}{I_n^2} \qquad \ldots(2\text{-}35)$$

where V'_n is defined by Eq. (2-34).

The index n refers to a set of possible field configurations obtained from a set of current distribution I_n. Note that although dimensionally the same, P'_n and Z'_n do not represent power and impedance in the conventional sense.

Basically Storer uses a differential equation similar to that of Hallen and King. The antenna is assumed to be driven at the base, which has an infinitely small gap, by a potential discontinuity defined as

$$\lim_{e \to 0} \int_{-e}^{e} (E_z)_{r=a} \, dz = -V \qquad \ldots(2\text{-}36)$$

which means that

$$(E_z)_{r=a} = V\ \delta(z)$$

with $$\lim_{e \to 0} \int_{-e}^{e} \delta(z)\, dz = 1 \qquad \ldots(2\text{-}37)$$

His differential equation is

$$V\ \delta(z) = \frac{j\omega}{\beta^2}\left(\frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z\right) \qquad \ldots(2\text{-}38)$$

assuming that the antenna is a perfect conductor.

The input impedance of the antenna is defined as

$$Z_i = \frac{V}{I_i} \qquad \ldots(2\text{-}39)$$

Multiplying the differential equation (2-38) on both sides by $\frac{1}{2}I(z)$ and then integrating the result with respect to z over the whole length of the antenna gives

$$\tfrac{1}{2} V\ I_i = \frac{j\omega}{2\beta^2}\int_A \left(\frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z\right) I(z)\, dz \qquad \ldots(2\text{-}40)$$

Finally dividing both sides of Eq. (2-40) by $\frac{1}{2}I_i^2$ and substituting the result in Eq. (2-39) gives

$$Z_i = \frac{j\omega\mu}{4\pi\ I_o^2}\int_A\int_A K(z-z')\ I(z')\ I(z)\ dz'\ dz \qquad \ldots(2\text{-}41)$$

where $K(z-z') = K(z'-z)$

$$= \frac{1}{\beta^2}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)\frac{e^{-j\beta R}}{R} \qquad \ldots(2\text{-}42)$$

It can be shown that the "impedance" function (2-41) has an extremal property, in the sense that the first variation of the impedance for small variation in the current distribution I(z) along the antenna can be made zero. Therefore, it can be expected that a good result for the "input impedance" will be obtained if a suitable current distribution is used in Eq. (2-41).

For his analysis Storer takes the current distribution to be

$$I(z) = V' \left[A \sin \beta(L - |z|) + B \left(1 - \cos \beta(L - |z|) \right) \right] \qquad \ldots(2\text{-}43)$$

The constants A and B in Eq. (2-43) are determined by imposing two conditions, namely:

(i) $I(0) = I_i = \dfrac{V'}{Z_i}$

(ii) The first variation of the input impedance is zero.

The first condition is satisfied by taking

$$B = \frac{1/Z_i - A \sin \beta L}{1 - \cos \beta L} \qquad \ldots(2\text{-}44)$$

which can be proved by simply substituting Eq.(2-44) into Eq. (2-39).

The current distribution is therefore given by

$$I(z) = V' \left[A \sin \beta(L - |z|) + \frac{1/Z_i - A \sin \beta L}{1 - \cos \beta L} \left(1 - \cos \beta(L - |z|) \right) \right] \qquad \ldots(2\text{-}45)$$

Application of the second condition which requires that

$$\frac{\partial Z_i}{\partial A} = 0 \qquad \ldots(2\text{-}46)$$

will determine the value of the constant A. Finally the "input impedance" of the antenna is determined using the relation (2-41).

3. A Discussion of the Existing Methods[13]

3.1 Introduction

Within their respective limitations, all methods described in the foregoing sections give results that agree to some extent with experiments. The simplifying assumptions and the mathematical model chosen in the theories, however, are subject to severe criticisms[8],[14]. It is therefore necessary to make a closer study of these theories.

However, before attempting any critical discussion, the impedance concept applicable to the system being considered must be fully understood. For this purpose, consider a black-box B (Fig. 3-1) which has two input terminals. Let I_i be input current, V_i the scalar potential difference between terminals 2 and 1, P_i the complex input power, and Z_i the input impedance.

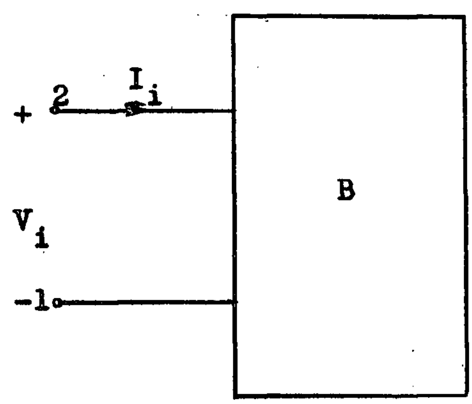


Fig. 3-1. Black Box

Then the following relations hold[15]

$$Z_{i(V,I)} = \frac{V_i}{I_i} \qquad \ldots(3\text{-}1)$$

$$Z_{i(P,I)} = \frac{2P_i}{I\ I^*} \qquad \ldots(3\text{-}2)$$

$$Z_{i(P,V)} = \frac{V\ V^*}{2\ P_i^*} \qquad \ldots(3\text{-}3)$$

where the capital letter indices indicate to what quantities the impedance is referred.

At low frequencies, there is no question about the meaning of the potential difference "V_i", the input current "I_i" and the input power "P_i". These quantities are uniquely defined, and consequently all three formulas above will give the same result, although the first one is more commonly used at these frequencies.

In antenna theories and generally in high-frequency networks, this is not the case. The potential difference "V" between two points is no longer uniquely defined. Power, on the other hand, is uniquely defined. Current distribution along cylindrical antennas is also clearly defined. A correct definition for the input impedance must therefore be based on these quantities. Relation (3-2) meets this requirement. If desired a scalar potential difference between the input terminals may then be defined as

$$V_i = \frac{2\ P_i}{I^*} = Z_i\ I_i \qquad \ldots(3\text{-}4)$$

To be useful the definition of impedance as given by Eq.(3-2) must be related to experimentally defined quantities. Impedances are measured indirectly by first measuring the standing-wave ratio along the driving line far from the load terminals together with the positions of the field's maxima and minima. From these the reflection coefficient Γ and hence the relative impedance ζ of the load with respect to the characteristic impedance Z_o of the transmission line can be found using the relation

$$\zeta = \frac{1 + \Gamma}{1 - \Gamma} = \frac{Z_L}{Z_o} \qquad \ldots(3\text{-}5)$$

where Z_L is the impedance of the load. The assumption made here is that the higher order modes of the field have little effect on the apparent impedance of the load.

To derive a formula for antenna impedance, the scalar potential U, vector potential $\vec{A}$ and current distribution $\vec{i}$ are most appropriate. Consider for example Fig. 3-2, where a cylindrical antenna is shown driven by a two-wire transmission line.

From Eq. (2-12) it follows immediately that

$$\vec{E} \cdot \vec{i}^* = - j\omega\vec{A} \cdot \vec{i}^* - \text{grad } U \cdot \vec{i}^* \qquad \ldots(3\text{-}6)$$

The tangential component of the electric field vanishes on the surface of the conductor. Hence integrating over all current filaments F bounded by the terminal planes gives

$$0 = - j\omega \int_F \vec{A} \cdot \vec{i}^* \, d\tau - \int_F \text{grad } U \cdot \vec{i}^* \, d\tau \ \ldots(3\text{-}7)$$

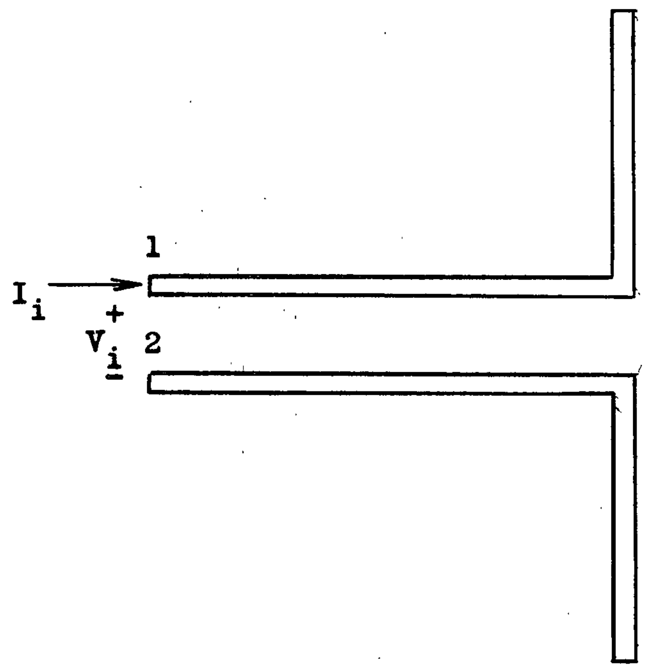


Fig. 3-2. Antenna Driven by a Two-Wire Line

Using the vector identity

$$\operatorname{div}\,(U\vec{i}^{\,*}) = \operatorname{grad} U \cdot \vec{i}^{\,*} + U \operatorname{div} \vec{i}^{\,*} \qquad \ldots(3\text{-}8)$$

and the equation of continuity

$$\operatorname{div} \vec{i}^{\,*} = j\omega q^{*} \qquad \ldots(3\text{-}9)$$

Eq. (3-7) becomes, after some simple manipulations

$$\tfrac{1}{2} V_i\, I_i^{*} = j\omega(W'_M - W'_e) = P_i \qquad \ldots(3\text{-}10)$$

where

$$W'_M = \frac{1}{2}\int_F \vec{A}\cdot\vec{i}^{\,*}\, d\tau \qquad \ldots(3\text{-}11)$$

$$W'_e = \frac{1}{2}\int_F U\, q^{*}\, d\tau \qquad \ldots(3\text{-}12)$$

I_i = the input current,

V_i = the scalar potential difference between terminals 1 and 2.

By bringing the terminals 1 and 2 to the input terminals of the antenna, the complex input power of the antenna is obtained. Eq. (3-2) then gives a suitable formula for the input impedance as seen from the terminals 1 and 2. That is

$$Z_i = \frac{2j\omega}{|I_i|^2} (W_M' - W_e') \qquad \ldots(3\text{-}13)$$

3.2 <u>The Energy Method</u>

The Poynting vector and the induced-emf methods, by the nature of the assumed current distribution, fail to give satisfactory results whenever the half length L of the antenna satisfies the condition $\beta L = n\pi$; for these cases the input impedance referred to the base current is infinite or very large if βL is in the neighborhood of $n\pi$, where n is an integer. To get around this difficulty, "loop impedance" is more commonly used.

The failure of the induced-emf method to take account of the boundary condition is also a subject of controversy[(8),(16),(17)]. The method is based on the experimentally determined fact that for thin antennas the current distribution is nearly sinusoidal. The boundary condition is not considered. Due to the mathematical difficulties involved, approximations are necessary and these cannot satisfy all physical conditions imposed on the problem. However, no <u>valid</u> explanation for the success of this method has been given.

Since $\vec{E}.\vec{i} = 0$ over the antenna, this means that, according to the induced-emf method, the input power P_i is zero. For actual antennas, this is not the case and consequently the question arises as to the source of the radiated power. In the usual method of explanation this source is considered to be concentrated at the gap and the input power is then given by

$$P_i = -\frac{1}{2}\int_A E \cdot I^* \, dz = -\frac{1}{2} I_o^* \int_{-b}^{b} E_z \, dz = \frac{1}{2} V_o I_o^* = \frac{1}{2} Z_o \left| I_o \right|^2 \quad \text{...(3-14)}$$

where A indicates that the integration must be carried out over the antenna including the gap and 2b is the width of the gap, which is sufficiently narrow such that the current may be assumed constant. A "generator" located at the gap is therefore considered to drive the current against the field.

Actually the energy is guided to the antenna from a distant source by the transmission line. The power radiated from the antenna can be found using Eq. (2-4). It must be stressed however, that in using this equation the surface of integration must enclose the source. Otherwise the result of the integration is zero. This does not mean that there is no radiated power, but rather that there is an area S_t where the time average of the energy flow is into the volume and a surface S_∞ where there is an equal flow out of the volume (see Fig. 3-3).

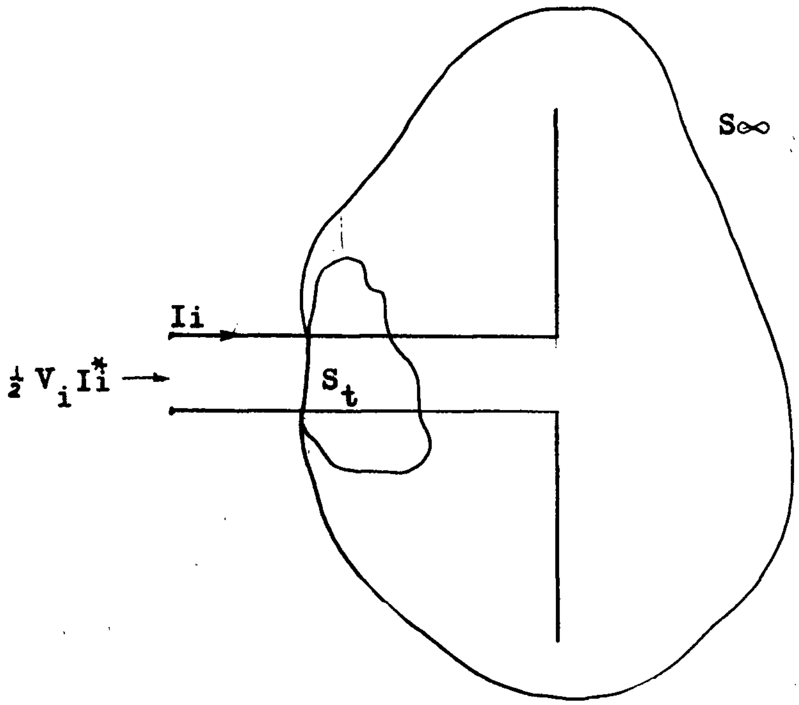


Fig. 3-3. Integration Surfaces for Eqs.(3-15) and (3-16)

Hence one can write

$$P_i = -\frac{1}{2}\int_{S_t} \vec{E} \times \vec{H}^* \cdot \vec{N}\, ds \qquad \text{...(3-15)}$$

and

$$P_{rad} = -\frac{1}{2}\int_{S_\infty} \vec{E} \times \vec{H}^* \cdot \vec{N}\, ds \qquad \text{...(3-16)}$$

This separation is not natural especially if the feeder is a two-wire line. A more natural separation of the input and output power is obtained using the scalar and vector potential as outlined in Sec. 3.1. Knowing the input or the maximum current the input impedance of the antenna can be computed using Eq. (3-13).

It is a simple matter to see, however, that Eq. (3-14) is in no way related to the derivation of the input impedance of cylindrical antennas using the induced-emf method. It is also not related to Eq. (3-10).

In the induced-emf method the source of excitation is considered to be the distributed generators along the antenna. The concept of a distributed generator must be clearly defined, since otherwise the impedance Z_o as determined from Eq.(3-14) will have no physical significance.

A distributed generator can be specified by means of two conditions:

1) a constraint on the current flow.
   In antennas, due to the skin effect, the current flows largely near the surface. We can then speak of a surface constraint.

2) a mechanism that will balance the force on the current and charges due to the electromagnetic field.

In antenna theory all electromagnetic forces have been accounted for in Maxwell's equations. From the macroscopic point of view any additional mechanism for introducing forces must appear to be non-electrical. A convenient mechanism is therefore a mechanical force.

Now consider a linear radiator as shown in Fig. 3-4. Energy may be assumed delivered to this radiator from some mechanical source as was assumed above. Referring to Eq.(2-1), one can now interpret the left-hand side of this equation as the complex power input due to the mechanical forces.

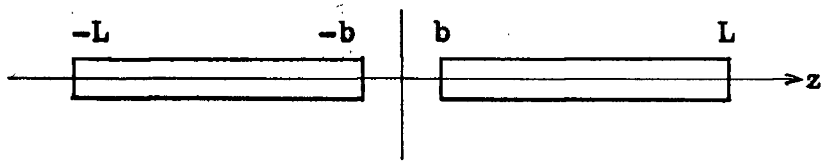


Fig. 3-4. Linear Radiator

The usefulness of this method can best be seen by considering the problem of finding the self-inductance of a thin closed current filament at low frequencies (Fig. 3-5). This problem will first be solved using the induced-emf method, assuming that the current I_o is constant over the filament. The divergence of the current density and hence the charge density are therefore everywhere zero. The electric field intensity at a point P on the filament is therefore given by

$$(\vec{E})_P = - j\omega(\vec{A})_P$$

$$= - \frac{j\omega\mu}{4\pi} \oint_L I_o \frac{e^{-j\beta r}}{r} \, d\vec{R}_1 \qquad \ldots(3\text{-}17)$$

where L implies that the integration must be carried out over the whole current filament.

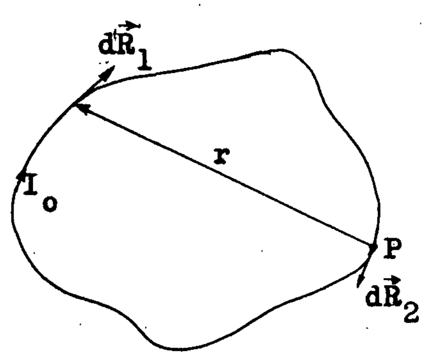


Fig. 3-5. Current Carrying Loop

The mechanical input power is given by (see Eq. 2-1)

$$P_{mech} = -\tfrac{1}{2}\oint_L \vec{E} \cdot I_o^* \, d\vec{R}_2$$

$$= \frac{j\omega\mu \, I_o I_o^*}{8\pi} \int_L \int_L \frac{e^{-j\beta r}}{r} \, d\vec{R}_1 \cdot d\vec{R}_2$$

Therefore

$$Z_{rad} = \frac{j\omega\mu}{4\pi} \int_L \int_L \left[\frac{\cos \beta r}{r} - j \, \frac{\sin \beta r}{r}\right] d\vec{R}_1 \cdot d\vec{R}_2$$

Hence

$$X = \frac{\omega\mu}{4\pi} \int_L \int_L \frac{\cos \beta r}{r} \, d\vec{R}_1 \cdot d\vec{R}_2$$

$$R_{rad} = \frac{\omega\mu}{4\pi} \int_L \int_L \frac{\sin \beta r}{r} \, d\vec{R}_1 \cdot d\vec{R}_2$$

If the loop is small one has approximately

$$X = \frac{\omega\mu}{4\pi} \int_L \int_L \frac{d\vec{R}_1 \cdot d\vec{R}_2}{r} = \omega L \qquad \ldots(3\text{-}18)$$

$R_{rad} = 0$

This gives Neumann's formula for the self-inductance of a thin loop.

The same problem will now be solved using the boundary-value method. To do this the loop must be broken at one place and a transmission line connected to the terminals. This will supply the required energy (Fig. 3-6).

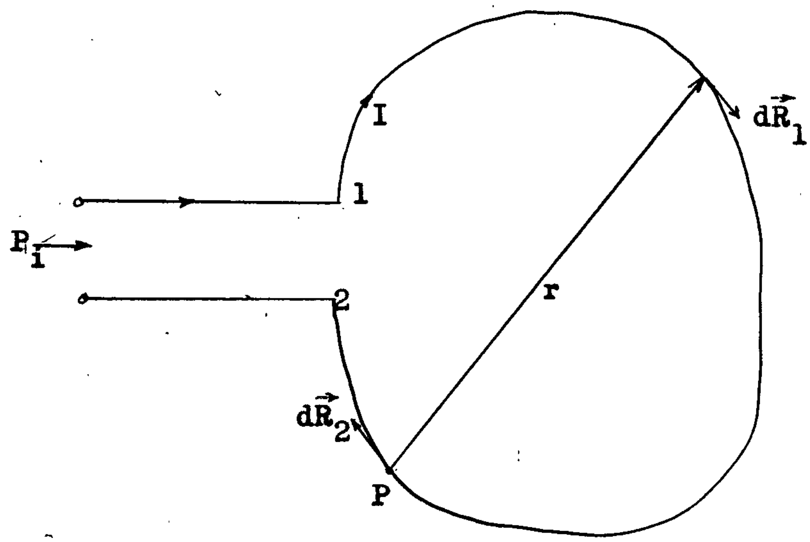


Fig. 3-6. A Loop Driven by a Two-Wire Line

Let the current distribution meeting the boundary condition be $I = I_o\,(1 + \delta)$ ...(3-19)

where δ is a small variation and let the charge density be q. Then from Eqs. (3-10),(3-11) and (3-12) follows that

$$\frac{1}{2} V_i\ I_i^* = \frac{1}{2}\, j\omega \int_1^2 \vec{A}_1 \cdot I_1^*\ d\vec{R}_1 - \frac{1}{2}\, j\omega \int_1^2 U_1\ q_1^* \left|d\vec{R}_1\right|$$

$$= \frac{j\omega\mu}{8\pi}\int_1^2\!\!\int_1^2 I_2 I_1^*\, \psi\ d\vec{R}_1 \cdot d\vec{R}_2 - \frac{j\omega}{8\pi\epsilon}\int_1^2\!\!\int_1^2 q_1\ q_2^*\ \psi \left|d\vec{R}_1\right|\left|d\vec{R}_2\right|$$

$$= \frac{j\omega\mu\ I_o I_o^*}{8\pi}\int_1^2\!\!\int_1^2 \psi\,(1 + \delta_2)(1 + \delta_1^*) \left|d\vec{R}_1\right|\left|d\vec{R}_2\right|$$

$$- \frac{j\omega}{8\pi\epsilon}\int_1^2\!\!\int_1^2 q_1\ q_2^*\ \psi \left|d\vec{R}_1\right|\left|d\vec{R}_2\right|$$

$$\tfrac{1}{2}V_i I_i^* = \frac{j\omega\mu\ I_o I_o^*}{8\pi}\int_1^2\int_1^2 \psi\ d\vec{R}_1 . d\vec{R}_2$$

$$+ \frac{j\omega\mu\ I_o I_o^*}{8\pi}\int_1^2\int_1^2 \psi\ (\delta_1^* + \delta_2 + \delta_2\delta_1^*)\ d\vec{R}_1 . d\vec{R}_2$$

$$- \frac{j\omega}{8\pi\epsilon_o}\int_1^2\int_1^2 \psi\ q_1 q_2^* \left|d\vec{R}_1\right|\left|d\vec{R}_2\right| \qquad \ldots(3\text{-}20)$$

At low frequencies the variation δ and charge density q are small. The first integral in Eq.(3-20) is therefore dominant and this is the same as the mechanical power input as calculated using the induced-emf method.

One now understands that in evaluating the input impedance of cylindrical antennas, the induced-emf method avoids calculating the energy integrals (3-11) and (3-12) and calculates the approximate values from the mechanical input power. Since it is known that for thin antennas the current distribution is almost sinusoidal, the energy integrals obtained are good approximations.

However the significance of the mechanical force is still open to questions, since its existence on antennas is not apparent. To answer this, one has only to recall the evaluation of the capacitance of charged conductors in an electrostatic field. It is a well known result of elementary electromagnetic theory that Eq. (3-12) can be obtained by considering the charges to be brought into an assumed position from infinity by means of mechanical forces. If the difference between the assumed and actual charge distributions is small, the error in the computed capacitance will also be small.

3.3 Hallen-King Method

In the Hallen-King method the current distribution along the antenna is determined from Eq.(2-22). This is an integral equation of the first kind. Hallen, by a suitable choice of an expansion parameter Ψ solved the equation by successive approximations. The convergence of the iteration, due to the complexity of the higher order terms, is not known. It turns out that the second or third order solution is sufficient for practical purposes. The effect of the choice of the expansion parameter on the convergence of the iteration is discussed by King[18] and by Gray[19].

The input impedance and the current distribution calculated using this theory agree very well with experiments. This alone does not justify the theory. To investigate this, one recalls that the following assumptions were made:

1) The antenna is excited at the gap by a scalar potential difference V_δ defined as

$$V_\delta = \lim_{b \to 0} \left[U(b) - U(-b) \right] \qquad \ldots(3\text{-}21)$$

where U is the scalar potential of the antenna. This is called a "slice generator" and replaces the transmission line.

2) The input impedance is defined as

$$Z_i = \frac{V_\delta}{I_i} \qquad \ldots(3\text{-}22)$$

These are circuit concepts and they imply that the electromagnetic field can be determined entirely by the scalar quantity V_δ. This is of course impossible.

A closer examination of the theory will reveal the following:

a) The determination of the input impedance of the antenna requires first of all the determination of the current distribution satisfying the boundary condition. The first assumption suggests that the applied voltage V_δ determines the excitation level of the antenna. Actually this is determined by the complex power input from the transmission line and hence V_δ should be replaced by V_i, the scalar potential difference of the transmission line. In a linear system, I_i - the input current -, I_o-the maximum current - and V_i are porportional to each other. Hence any of these may be used as a proportionality constant to determine the excitation level. If the first two are used, the energy integrals (3-11) and (3-12) must be evaluated. Equating the results with the input power through Eq. (3-10) sets the excitation level of the antenna. The choice of I_o however, leads to the "loop impedance" and conversion into the physically more useful input impedance is necessary.

b) Here, however, V_δ is introduced as the "gap voltage" with the transmission line removed. This quantity reduces to zero with the width of the gap which can be seen by simply writing the expression for V_δ

$$V_\delta = \lim_{b \to 0} \left[U(b) - U(-b) \right]$$

$$= \lim_{b \to 0} \frac{1}{4\pi\epsilon} \int_A q(z') \left[\Psi(b,z') - \Psi(-b,z') \right] dz'$$

$$= 0$$

It will be shown in a later section that the transmission line alone will produce a scalar potential difference V_i across its terminals. V_i is the source of excitation of the antenna, and does not vanish as $b \to 0$. It is thus obvious that the "gap voltage" as it is defined, is not capable of representing the effect of the transmission line, since $V_i \neq V_\delta$.

c) The definition of the input impedance as given in the second assumption above, is made by King[7] on the ground of the convenience it offers. It is in no way related to the experimentally determined ξ (see Eq. 3-5). Taking V_δ as the proportionality constant (see part (a) above), the current distribution will have the following form

$$I(z) = V_\delta f(z) \qquad 0 \leq z \leq L \qquad \ldots(3\text{-}23)$$

$$I_i = I(b) = V_\delta f(b)$$

Adhering strictly to the original model, (see assumption #1), the method used by Hallen and King should give an input impedance that is zero, since it is a well known fact that for zero base separation the input impedance of the antenna vanishes[20]. Physically this is to be expected, since for a narrow gap, the field between the base surfaces will approximate that of a parallel plate capacitor. At zero gap width this capacitor has an infinite capacitance and hence the input impedance of the antenna is zero.

Since the Hallen-King method gives a finite input impedance, the question arises how this can possibly occur. The explanation of this paradox will now be given.

The zero-order current distribution as obtained by King (Reference 1, p.25,Eq.26) is given by

$$I(z) \doteq \frac{2\pi j V_\delta}{\zeta \Psi} \frac{\sin \beta(L - z)}{\cos \beta L} \quad \ldots(3\text{-}24)$$

At this stage Ψ can be regarded as an undetermined expansion parameter. In the Hallen and King method Ψ is obtained by an extensive semi-empirical method. Actually the amplitude of I(z) can only be determined from the power input to the antenna. This can be done approximately using the induced-emf method. This gives

$$P_i = -\frac{1}{2}\int_A \vec{E}_z \cdot \vec{i}^* \, d\tau = \frac{1}{2} V_\delta I_i^* = \frac{1}{2} Z_i I_i I_i^*$$

Finally substituting the current distribution (3-24) into this equation and rearranging the terms gives

$$\Psi = \frac{2\pi j \sin \beta L}{\zeta \cos \beta L} Z_i \quad \ldots(3\text{-}25)$$

where Z_i can be determined by the induced-emf method, since the current distribution is sinusoidal. Eq. (3-25) will then give an approximate value for the expansion parameter.

Using King's definition of input impedance $Z_i = \frac{V_\delta}{I_i}$ and Eq. (3-24) it is easy to obtain Eq. (3-25) once more. It is now apparent that in the Hallen-King method a judicious semi-empirical choice of Ψ must be made in order that this is the case. By choice of Ψ the induced-emf method is unwittingly applied to compute the power input and have $V_\delta = V_i$. King's original choice of Ψ was made on the basis of obtaining a rapidly converging solution of the integral equation for the current distribution. The history of the expansion parameter Ψ is a sequence of semi-empirical adjustment to overcome the inadequacies of the Hallen-King Model (Reference 10, p.p. 144-148).

3.4 Mode Theory of Antennas

The mode theory of antennas gives an exact formulation of the problem. Conditions that make this possible and the approximation involved have been discussed in Sec. 2.3. Note also that here attention is first focussed on the solution of Maxwell's equations instead of directly on the evaluation of the current distribution along the antenna.

The fact that a biconical antenna is a natural extension of a transmission line is also used to its full extent. This can be clearly seen in the expression for the input impedance [see Eq. (2-32)] which is a well-known expression for the impedance seen at the input terminals of the transmission line a distance L from the terminals to which the load Z_a is connected.

The evaluation of the terminal admittance Y_t is however, cumbersome, since this requires the exact evaluation of the field quantities at the ends of the antenna, where the solutions must be matched (see Sec. 2.3). Tai,[21] using variational methods has devised a method of finding Y_t without the necessity of matching the solutions.

From the point of view of obtaining the input impedance of a cylindrical antenna, the method is even less attractive, since it does not give the dependence of the reactive part of the impedance on the thickness of the antenna. In fact, to obtain an expression for the input impedance of cylindrical antenna, Schelkunoff has developed a theory of non-uniform transmission lines[22].

3.5 Storer's Variational Method

To understand the conditions underlying the variational expression for the input impedance of cylindrical antennas, Eqs. (2-34) and (2-35) in Sec. 2.4 are very useful. It follows directly from Eq. (2-34) that

$$P_2' = P_1' - \frac{1}{2}\int_A \left[\vec{E}_2 \cdot \vec{i}_2 - \vec{E}_1 \cdot \vec{i}_1\right] d\tau$$

$$= P_2' - \frac{1}{2}\int_A \left[(\vec{E}_2 - \vec{E}_1)\cdot(\vec{i}_2 - \vec{i}_1) - 2\,\vec{E}_1 \cdot \vec{i}_1\right] d\tau$$

$$- \frac{1}{2}\int_A \left[\vec{E}_1 \cdot \vec{i}_2 + \vec{E}_2 \cdot \vec{i}_1\right] d\tau$$

where the index 1 and 2 refer to two different field configurations.

Using the reciprocity theorem

$$\int_A \vec{E}_1 \cdot \vec{i}_2 \, d\tau = \int_A \vec{E}_2 \cdot \vec{i}_1 \, d\tau$$

in the last equation one obtains

$$P_2' = P_1' - \tfrac{1}{2}\left[\int_A (\vec{E}_2 - \vec{E}_1) \cdot (\vec{i}_2 - \vec{i}_1)\, d\tau - 2\int_A (\vec{E}_1 \cdot \vec{i}_1 - \vec{E}_1 \cdot \vec{i}_2) d\tau\right]$$

$$= P_1' - \tfrac{1}{2}\int_A \Delta\vec{E} \cdot \Delta\vec{i}\, d\tau - \int_A \vec{E}_1 \cdot \Delta\vec{i}\, d\tau \qquad \ldots(3\text{-}26)$$

where $\Delta\vec{E} = \vec{E}_2 - \vec{E}_1$ and $\Delta\vec{i} = \vec{i}_2 - \vec{i}_1$.

Finally by substituting Eq. (2-35) into Eq.(3-26) one obtains

$$Z_2' = Z_1' + Z_1'\left(\frac{I_1^2}{I_2^2} - 1\right) - \frac{1}{I_2^2}\int_A \Delta\vec{E} \cdot \Delta\vec{i}\, d\tau - \frac{2}{I_2^2}\int_A \vec{E}_1 \cdot \Delta\vec{i}\, d\tau \qquad \ldots(3\text{-}27)$$

This is a variational expression for the quantity $\Delta Z' = Z_2' - Z_1'$.

The approximate current distribution chosen by Storer is given by

$$I_k(z) = V_k'\left[A_k \sin\beta(L - |z|) + \frac{1/Z_k' - A_k \sin\beta L}{1 - \cos\beta L}\Big(1 - \cos\beta(L - |z|)\Big)\right]$$

$$(k = 1,2\ldots) \qquad \ldots(3\text{-}28)$$

Obviously the input current is V_k'/Z_k', and this does not depend directly on A_k. Hence setting I_1 equal to I_2 in Eq. (3-27) will give

$$Z_2' = Z_1' - \frac{1}{I_2^2}\int_A \Delta\vec{E} \cdot \Delta\vec{i}\, d\tau - \frac{2}{I_2^2}\int_A \vec{E}_1 \cdot \Delta\vec{i}\, d\tau \qquad \ldots(3\text{-}29)$$

It is obvious from Eq.(3-29) that the first variation of the "input impedance" is zero for small variation of the current distribution if the relation

$$\int_A \vec{E}_1 \cdot \Delta\vec{i}\, d\tau = 0 \qquad \ldots(3\text{-}30)$$

is satisfied. For arbitrary $\Delta\vec{i}$ this requires that $\vec{E}_1$ vanish over the antenna, except at the input where $\Delta\vec{i} = 0$. This is the boundary condition.

Eq.(3-28) determines A_k from $\frac{\partial Z_k}{\partial A_k} = 0$. This value of A_k will be called A_1. The tangential component of the electric field obtained from Eq. (3-28) with the appropriate A_k is not everywhere zero. Setting $\frac{\partial Z_k}{\partial A_k} = 0$ only makes the integral $\int_A \vec{E}_1 \cdot \Delta\vec{i}\, d\tau = \int_A E_z(z)\, \Delta I(z)\, dz = 0$.

The evaluation of the input impedance using relation (2-41) and current distribution (3-28) is therefore not correct since the boundary condition is not satisfied. Effectively, Storer, referring to Eq. (2-41), merely defines an "impedance" Z_1' as

$$Z_1' = \frac{j\omega\mu}{4\pi\, I_1^2} \int_A \int_A K(z,z')\, I_1(z')\, I_1(z)\, dz'\, dz \qquad \ldots(3\text{-}31)$$

and sets $Z_i = Z_1'$, the input impedance of the antenna.

There is no justification for this. However, it will be shown (see Eq. 4-16) that the field of the transmission line driving the antenna is large near the gap. The complex input power to the antenna is therefore given by

$$P_i = \frac{1}{2}\int_A \vec{E}_i \cdot \vec{i}^* \, d\tau = \frac{1}{2} V_i I_i^* = \frac{1}{2} Z_i I_i I_i^* \qquad \ldots(3\text{-}32)$$

where $\vec{E}_i$ is the impressed field from the transmission line and $\vec{i}$ is the current distribution along the antenna meeting the boundary condition.

Now if I(z) and $\vec{E}$ represent the current distribution and electric field intensity of the correct solution, then $\vec{E} = -\vec{E}_i$ and one has (see Eq. 2-34 and 2-35)

$$\frac{1}{2} Z' I_i^2 = -\frac{1}{2}\int_A \vec{E} \cdot \vec{I} \, d\tau = \frac{1}{2}\int_A E_i(z) \; I(z) \; dz$$

$$= \frac{1}{2} V_i I_i^2$$

$$= \frac{1}{2} Z_i I_i^2$$

From here it follows immediately that

$$Z' = Z_i \qquad \ldots(3\text{-}33)$$

There is no justification for setting $Z_1' = Z' = Z_i$. The stationary property for the particular current chosen applies only to Z_1' and not, as originally stated, to Z'. Despite this descrepancy, Storer's method gives a reasonably accurate formula for the input impedance of the cylindrical antenna. Storer's method is a semi-empirical extension of the induced-emf method.

4. Solution of the Cylindrical Antenna Problem

4.1 The Interaction Between the Transmission Line and the Antenna.

To obtain a valid solution of the cylindrical antenna problem, the effect of the transmission line must be correctly accounted for. This has previously never been done.

This effect appears as an interaction between the transmission line and the antenna. The current and charges on the transmission line produce a field which excites the antenna. Conversely the field of the antenna also affects the current distribution along the transmission line. In Hallen's model this interaction is represented by a "slice generator" that maintains a scalar potential difference V_δ across the gap.

The mechanism in this method is therefore clear. The field of the transmission line induces a current distribution along the antenna. This distribution must be such that the boundary conditions are satisfied. One of these requires that the tangential component of the total electric field vanishes on the surface of the antenna. This problem will be discussed in more detail in a later section.

It was mentioned earlier that the current distribution along the transmission line is affected by the antenna field. A question arises as to the possibility of taking this effect into account in view of the fact that the antenna current is unknown. This seemingly difficult situation can be accounted for if one first notes that the antenna induces higher order modes in the transmission line. From transmission line theory it is known that the effect of these higher order modes decreases with the spacing of the transmission line. This effect can be included in the impedance of the antenna.

Now let Z_i be the input impedance of the antenna including its coupling effect and Z_c be the characteristic impedance of the transmission line; then the principal current distribution along the transmission line is given by (see Fig. 4.1)

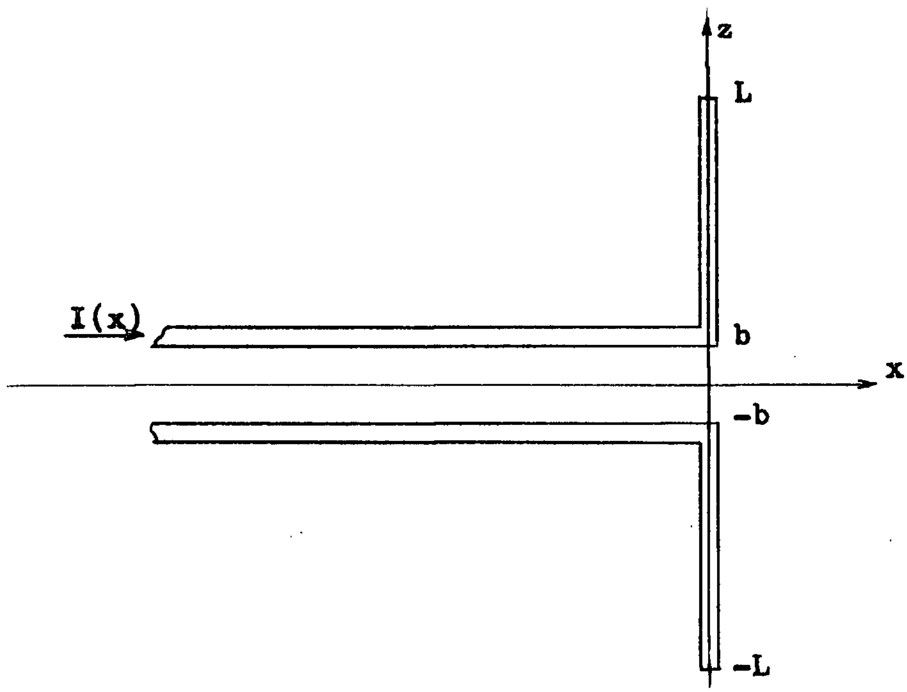


Fig. 4.1 Cylindrical Antenna Driven by a Two-Wire Line

$$I(x) = \frac{1}{Z_c}(V_o^+ e^{-j\beta x} - V_o^- e^{+j\beta x})$$

$$= \frac{V_o^+}{Z_c}(e^{-j\beta x} - \Gamma e^{j\beta x}) \qquad \ldots(4\text{-}1)$$

where V_o^+ = the scalar potential difference at the transmission line terminals (x=0) associated with the wave travelling in the positive x direction.

V_o^- = the scalar potential difference associated with the wave travelling in the negative x direction.

$\Gamma = \dfrac{V_o^-}{V_o^+}$ the voltage reflection coefficient.

For reasons of simplicity, the currents associated with the wave of higher modes are neglected. This can be done, since these waves are relatively small (for small b) and confined in a region near the junction of the transmission line and the antenna.

Since the current distribution along the antenna, as mentioned earlier, is determined by the boundary condition, it is necessary to know the tangential component of the electric field intensity on the surface of the antenna. Exact evaluation of this field is difficult and the result is very complicated. For thin antennas, the field evaluated on the z-axis, with the antenna removed, serves as a more useful approximate value of the field on the surface. This can be obtained using Eqs.(2-12) through (2-15). Since the vector potential does not contribute to this field, only the scalar potential function need be evaluated.

It is further found more convenient to evaluate the field of only one line first using a system of coordinates as shown in Fig. 4-2.

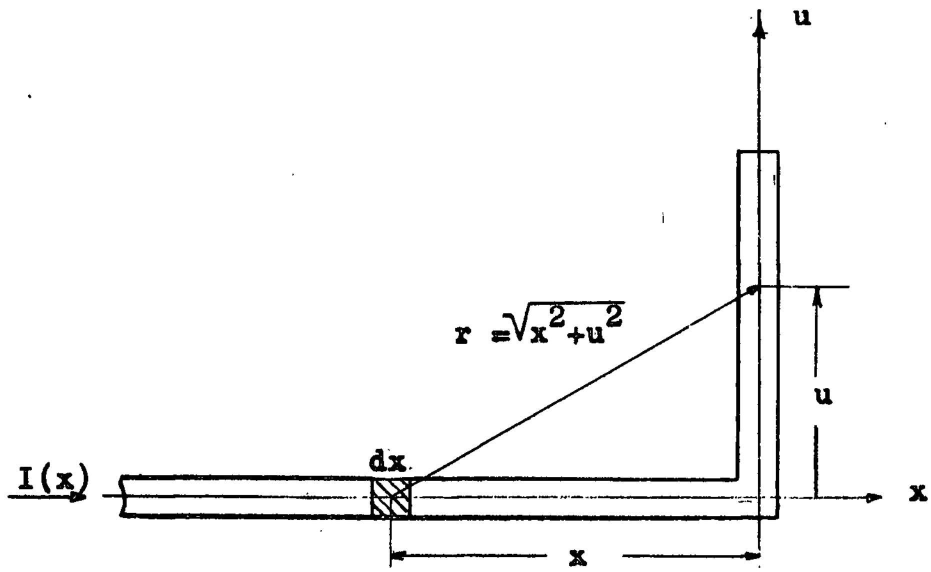


Fig. 4-2 Cylindrical Antenna Driven by a Two-Wire Line; Half Section.

The charge density distribution $Q(x)$ along the transmission line can be obtained from Eq. (4-1) using the equation of continuity

$$\frac{d}{dx} I(x) = - j\omega \, Q(x) \qquad \ldots(4\text{-}2)$$

Therefore

$$Q(x) = \frac{\beta}{\omega} \, \frac{V_o^+}{Z_c} \left[e^{-j\beta x} + \Gamma e^{j\beta x} \right] \qquad \ldots(4\text{-}3)$$

It was suggested in the previous chapter that it is better to use the input current I_i as proportionality factor than the voltage V_o^+ as is the case in Eq. (4-3). This can be done by first noting that

$$V_o^+ = Z_c \, I_o^+$$

and

$$V_o^- = -Z_c I_o^- \qquad \ldots(4\text{-}4)$$

Therefore $I = I_o^+ + I_o^- = \frac{V_o^+}{Z_c}\left(1 - \frac{V_o^-}{V_o^+}\right) = \frac{V_o^+}{Z_c}(1 - \Gamma)$

and $$\frac{V_o^+}{Z_c} = \frac{I_i}{1 - \Gamma} \qquad \text{...(4-5)}$$

Finally substitution of Eq.(4-5) into Eq. (4-3) gives

$$Q(x) = \frac{\beta}{\omega}\ \frac{I_i}{1-\Gamma}\left[e^{-j\beta x} + \Gamma\, e^{j\beta x}\right] \qquad \text{...(4-6)}$$

The scalar potential on an arbitrary point on the u-axis is given by

$$U(u) = \frac{1}{4\pi\epsilon_o}\int_{-L}^{0} Q(x)\,\psi(x,u)\,dx \qquad \text{...(4-7)}$$

where $\psi(x,u) = \dfrac{e^{-j\beta r}}{r} = \dfrac{e^{-j\beta\sqrt{x^2+u^2}}}{\sqrt{x^2+u^2}}$

and L is the length of the transmission line.

The u-component of the electric field intensity E(u) is given by

$$E(u) = -\frac{\partial}{\partial u}U(u)$$

$$= -\frac{\beta}{4\pi\omega\epsilon_o}\ \frac{I_i}{1-\Gamma}\int_{-L}^{0}\frac{\partial}{\partial u}\left[\frac{e^{-j\beta(r+x)}}{r} + \frac{e^{-j\beta(r-x)}}{r}\right]dx$$

$$\text{...(4-8)}$$

It can be shown that (see Appendix I)

$$\frac{\partial}{\partial u}\ \frac{e^{-j\beta(r+x)}}{r} = \frac{\partial}{\partial x}\left[\frac{r-x}{u}\ \frac{e^{-j\beta(r+x)}}{r}\right]$$

$$\frac{\partial}{\partial u}\ \frac{e^{-j\beta(r-x)}}{r} = -\frac{\partial}{\partial x}\left[\frac{r+x}{u}\ \frac{e^{-j\beta(r-x)}}{r}\right] \qquad \text{...(4-9)}$$

If Eq. (4-9) is substituted into Eq. (4-8) the integrand of the resulting integral becomes a perfect differential. Noting that $\frac{\beta}{4\pi\omega\epsilon_o} = 30$ the result is

$$E(u) = -\frac{30\ I_i}{1-\Gamma}\left[\frac{r-x}{u}\,\frac{e^{-j\beta(r+x)}}{r} - \Gamma\,\frac{r+x}{u}\,\frac{e^{-j\beta(r-x)}}{r}\right]\Bigg|_{x=-L}^{x=0}$$

$$= -\frac{30\ I_i}{1-\Gamma}\left[(1-\Gamma)\,\frac{e^{-j\beta u}}{u} - \frac{\sqrt{u^2+L^2}+L}{u}\,\frac{e^{-j\beta(\sqrt{u^2+L^2}-L)}}{\sqrt{u^2+L^2}}\right.$$

$$\left. + \Gamma\,\frac{\sqrt{u^2+L^2}-L}{u}\,\frac{e^{-j\beta(\sqrt{u^2+L^2}+L)}}{\sqrt{u^2+L^2}}\right] \quad \text{...(4-10)}$$

For large L the following approximate values may be substituted in Eq. (4-10)

$$\frac{\sqrt{u^2+L^2}+L}{\sqrt{u^2+L^2}} \cong 2$$

$$e^{-j\beta(\sqrt{u^2+L^2}-L)} \cong e^{-j\frac{\beta}{2}(u/L)^2} \cong 1 \quad \text{...(4-11)}$$

$$\frac{\sqrt{u^2+L^2}-L}{\sqrt{u^2+L^2}} \cong 0$$

The approximate value of Eq. (4-10) is therefore

$$E(u) \doteq -\frac{30\ I_i}{1-\Gamma}\left[(1-\Gamma)\,\frac{e^{-j\beta u}}{u} - \frac{2}{u}\right]$$

$$= -30\ I_i\ f(u) \quad \text{...(4-12)}$$

where $$f(u) = \frac{e^{-j\beta u}}{u} - \frac{2}{(1-\Gamma)u} \quad \text{...(4-13)}$$

Now the field of the transmission line can be obtained from Eq.(4-12) by simple transformations. Note however that for balanced condition

$$I_1(x) = - I_2(x)$$

$$Q_1(x) = - Q_2(x)$$

where the indices refer to the line concerned. To obtain the field of the line #1 (Fig. 4-1) substitute $u = z - b$ and for the second line $u = z + b$. The result is

$$E_i(z) = E_1(z) + E_2(z) = -30\ I_i\ \left[f(z - b) - f(z + b) \right] \qquad \ldots(4\text{-}14)$$

For small b, Eq. (4-14) can be approximated by

$$E_i(z) = 60\ I_i\ b \frac{\partial}{\partial z} f(z) \qquad \ldots(4\text{-}15)$$

where f(z) is obtained by simply substituting z for u in Eq. (4-13) and hence

$$E_i(z) = 60\ I_i\ b \frac{\partial}{\partial z} \left[\frac{e^{-j\beta z}}{z} - \frac{2}{(1 - \Gamma)\ z} \right] \qquad \ldots(4\text{-}16)$$

But $$\Gamma = \frac{Z_i/Z_c - 1}{Z_i/Z_c + 1} = \frac{\zeta - 1}{\zeta + 1}$$

hence $$\frac{2}{1 - \Gamma} = \zeta + 1 \qquad \ldots(4\text{-}17)$$

Finally using (4-17), Eq. (4-16) becomes, for small b

$$E_i(z) = 60\ I_i\ b \frac{\partial}{\partial z} \left[\frac{e^{-j\beta z} - 1}{z} - \frac{\zeta}{z} \right]$$

$$= -\ 60\ I_i\ b \zeta \frac{\partial}{\partial z} \left(\frac{1}{z}\right) \qquad \ldots(4\text{-}18)$$

As mentioned earlier in this section, the electric field $E_i(z)$ of the transmission line induces a current distribution $I(z)$ along the antenna such that the boundary condition is satisfied. Therefore there is energy transfer from the transmission line to the antenna; its average value is given by

$$P_i = \frac{1}{2}\int_A E_i(z)\ I^*(z)\ dz \qquad \ldots(4\text{-}19)$$

where A means that the integration must be carried out over the antenna only.

From previous analysis (see Eqs. 4-12 and 4-18) it can be seen that the impressed field $E_i(z)$ is concentrated near the gap. To obtain an approximate value of the power transfer, the input current $I_i = I(b)$ can therefore be substituted for $I(z)$ in the integrand of Eq. (4-19). The resulting error decreases with the base separation of the antenna. Hence for a small gap one has

$$P_i = -\ 30\ \left|I_i\right|^2 \xi \lim_{b\to 0} b \int_A \frac{\partial}{\partial z}\left(\frac{1}{z}\right) dz$$

$$= 60\,\xi \left|I_i\right|^2 \qquad \ldots(4\text{-}20)$$

The complex input power from the transmission line is given by Eqs. (2-5) and (3-10). Equating this to Eq. (4-20) gives the following relations

$$P_i = \frac{1}{2} Z_i\ \left|I_i\right|^2 = \frac{1}{2} V_i\ I_i^* = 60\,\xi \left|I_i\right|^2 \qquad \ldots(4\text{-}21$$

and

$$Z_i = 120\,\xi \qquad \ldots(4\text{-}22)$$

It is easy to see that

$$V_i = \int_A E_i \, dz \qquad \ldots(4\text{-}23)$$

with $b \to 0$ can then be interpreted as the "input voltage" to the antenna.

Eq. (4-22) would suggest that the characteristic impedance of the transmission line is 120 ohms. This value has no special significance and it is simply a result of taking $b \to 0$.

The result of this elementary analysis is important. It shows the dependence of the impressed field and the "gap voltage" on the input impedance of the antenna. It clears up the misleading statement often made in elementary antenna theory which states that the input impedance of the antenna can be obtained if the input current and the impressed field are known. The fallacy in this statement is that the impressed field is not known until the input impedance of the antenna is determined in the first place.

4.2 The Electric Field of the Antenna

Consider a cylindrical antenna with half length L, base separation 2b and diameter of the cylindrical surface 2a. Fig. 3-2 shows the antenna configuration.

The antenna field can be found from the retarded potentials. These are

$$\vec{A} = \frac{\mu_o}{4\pi} \int_{A_v} \vec{i} \, \frac{e^{-j\beta r}}{r} \, dv \qquad \ldots(4\text{-}24a)$$

$$U = \frac{1}{4\pi\varepsilon_o} \int_{A_v} q \frac{e^{-j\beta r}}{r} dv \qquad \text{...(4-24b)}$$

$$\vec{E} = - j\omega\vec{A} - \text{grad } U \qquad \text{...(4-25)}$$

where $\vec{i}$ and q are the current and charge densities of the antenna respectively and r is the distance from a volume element in the antenna to the point of observation. A_v implies that the integration must be carried out over the whole volume of the antenna.

The current and charge densities are related through the equation of continuity

$$\text{div } \vec{i} = - j\omega q \qquad \text{...(4-26)}$$

Due to the skin effect the current and charges on the antenna are confined within a thin layer on the surface of the antenna; by symmetry these distributions are independent of the angular positions and hence the current flow is axial. Therefore the volume integration in the above equations can be reduced into an integration over the surface of the antenna. The charge and current densities must be changed accordingly into q_s and i_s for surface densities. Since the current flow is axial the vector potential $\vec{A}$ has only a z-component.

Let the z-component of the electric field intensity of the antenna on its surface be disignated by E_{az} then

$$E_{az} = -\frac{j\omega\mu_o}{4\pi}\int_A i_s\Psi(r)\,ds - \frac{1}{4\pi\epsilon_o}\frac{\partial}{\partial z}\int_{A+C} q_s\Psi(r)\,ds$$

$$= -\frac{j\omega\mu_o}{4\pi}\int_A i_s\Psi(r)\,ds - \frac{1}{4\pi\epsilon_o}\frac{\partial}{\partial z}\int_A q_s\Psi(r)\,ds$$

$$-\frac{1}{4\pi\epsilon_o}\frac{\partial}{\partial z}\int_C q_s\Psi(r)\,ds \qquad \ldots(4\text{-}27)$$

where $\Psi(r) = \frac{e^{-j\beta r}}{r}$ with $r^2 = (z-u)^2 + 2a^2(1-\cos\varphi)$ ...(4-28)

The first two integrals in Eq. (4-27) must be carried out over the cylindrical surface and the last one over the caps and base surfaces of the antenna. The current flowing radially on the caps of the antenna does not contribute to the z-component of the electric field of the antenna.

For the integration over the cylindrical surface of the antenna, an average value of r over the angular position will be used instead of the exact value as given by Eq.(4-28). This is equal to $\sqrt{(z-u)^2 + a^2}$. King (Reference 1, p.16) shows that this choice introduces only a small error in the value of the integral.

Noting that $\frac{\partial}{\partial z}\Psi(r) = -\frac{\partial}{\partial u}\Psi(r)$, Eq. (4-27) then becomes

$$E_{az}(z) = -\frac{j\omega\mu_o}{4\pi}a\int_0^{2\pi}d\varphi\int_L i_s(u)\Psi(z,u)\,du +$$

$$\frac{a}{4\pi\epsilon_o}\int_0^{2\pi}d\varphi\int_L q_s(u)\frac{\partial}{\partial u}\Psi(z,u)du +$$

$$\frac{1}{4\pi\epsilon_o}\int_C q_s\frac{\partial}{\partial u}\Psi(r)\,ds$$

$$E_{az}(z) = -\frac{j\omega\mu_o}{4\pi} \int_L I(u)\Psi(z,u)\,du + \frac{1}{4\pi\epsilon_o}\int_L Q(u)\,\frac{\partial}{\partial u}\Psi(z,u)\,du$$

$$+ \frac{1}{4\pi\epsilon_o}\int_C q_s \frac{\partial}{\partial u}\Psi(r)\,ds \qquad \text{...(4-29}$$

where $2\pi a\, i(u) = I(u)$ and $2\pi a\, q(u) = Q(u)$ ...(4-30)

and $\int_L f(u)\,du$ stands for $\int_{-L}^{-b} f(u)\,du + \int_b^L f(u)\,du.$

The second integral in (4-29) can be integrated by parts and the result is

$$\frac{1}{4\pi\epsilon_o}\int_L Q(u)\,\frac{\partial}{\partial u}\Psi(z,u)\,du = E_s(z) - \frac{1}{4\pi\epsilon_o}\int_L \Psi(z,u)\,\frac{\partial}{\partial u}Q(u)\,du$$

From the equation of continuity it follows immediately that

$$\frac{\partial}{\partial u}Q(u) = -\frac{1}{j\omega}\,\frac{\partial^2}{\partial u^2}I(u) \qquad \text{...(4-31)}$$

Hence

$$\frac{1}{4\pi\epsilon_o}\int_L Q(u)\,\frac{\partial}{\partial u}\Psi(z,u)\,du = E_s(z) +$$

$$+ \frac{1}{4\pi j\omega\epsilon_o}\int_L \Psi(z,u)\,\frac{\partial^2}{\partial u^2}I(u)\,du \qquad \text{...(4-32)}$$

where $E_s(z) = \frac{1}{4\pi\epsilon_o}\Big[Q(L)\Psi(z,L) - Q(-L)\Psi(z,-L) +$

$$+ Q(-b)\Psi(z,-b) - Q(b)\Psi(z,b)\Big] \qquad \ldots(4\text{-}33)$$

Substituting Eq. (4-31) into Eq.(4-29) and noting that

$$\frac{j\omega\mu_o}{4\pi} = \frac{j}{4\pi\omega\epsilon_o}(\omega^2\mu_o\ \epsilon_o) = \frac{j\beta^2}{4\pi\omega\epsilon_o}$$

gives

$$E_{az}(z) = E_s(z) - \frac{j}{4\pi\omega\epsilon_o}\int_L\left[\frac{\partial^2 I}{\partial u^2} + \beta^2\ I(u)\right]\Psi(z,u)\ du + E_c(z) \qquad \ldots(4\text{-}34)$$

where

$$E_c(z) = \frac{1}{4\pi\epsilon_o}\int_C q_s \frac{\partial}{\partial u}\Psi(r)\ ds \qquad \ldots(4\text{-}35)$$

is the contribution to the tangential E-field due to the charges on the caps and base surfaces of the antenna.

For thin antennas the last term in Eq. (4-34) is small compared with the other terms and can be neglected.

To satisfy the boundary condition on the surface of the antenna, the total tangential electric field must vanish. Hence

$$E_{az}(z) + E_i\ (z) = 0$$

or

$$E_s(z) + E_i(z) = \frac{j}{4\pi\omega\epsilon_o}\int_L\left[\frac{\partial^2 I(u)}{\partial u^2} + \beta^2\ I(u)\right]\ \Psi\ (z,u)\ du \qquad \ldots(4\text{-}36)$$

It is interesting to note that $E_s(z)$ as given by (4-33) is related to the tangential component of the electric field of a sinusoidal current distribution. This is given by (see reference 4, p.323)

$$E_s(z) = -j\,30\,I_o \left[\frac{e^{-j\beta r_1}}{r_1} + \frac{e^{-j\beta r_2}}{r_2} - 2\cos\beta L\,\frac{e^{-j\beta r_o}}{r_o} \right] \qquad \ldots(4\text{-}37)$$

where

$$r_1^2 = (z - L)^2 + a^2$$

$$r_2^2 = (z + L)^2 + a^2$$

$$r_o^2 = z^2 + a^2 \qquad \ldots(4\text{-}38)$$

and the current distribution is

$$I(z) = I_o \sin\beta(L - |z|) \qquad \ldots(4\text{-}39)$$

In evaluating Eq. (4-37) the base separation b is assumed zero.

From Eq. (4-39) it follows that

$$Q(L) = -\frac{1}{j\omega}\frac{d}{dz}I(z)\Big|_{z=L} = \frac{\beta}{j\omega} I_o$$

$$Q(-L) = -\frac{1}{j\omega}\frac{d}{dz}I(z)\Big|_{z=-L} = -\frac{\beta}{j\omega} I_o$$

$$Q(-b)\Big|_{b=0} = Q(b)\Big|_{b=0} = -\frac{1}{j\omega}\frac{d}{dz}I(z)\Big|_{z=0}$$

$$= \frac{\beta}{j\omega} I_o \cos\beta L$$

Also $\qquad \dfrac{\beta}{4\pi\omega\epsilon_o} = 30$

Hence $$-j\ 30\ I_o = \frac{\beta}{4\pi j\omega\epsilon_o}\ I_o = \frac{Q(L)}{4\pi\epsilon_o} = -\frac{Q(-L)}{4\pi\epsilon_o} \quad \ldots(4\text{-}40)$$

$$j\ 60\ I_o \cos\beta L = \frac{j\beta}{4\pi\omega\epsilon_o}\ I_o \cos\beta L$$

$$= \lim_{b\to 0} -\frac{Q(b)}{4\pi\epsilon_o} = \lim_{b\to 0} \frac{Q(-b)}{4\pi\epsilon_o} \quad \ldots(4\text{-}41)$$

Substituting Eqs. (4-40) and (4-41) in Eq. (4-37) and noting that

$$\frac{e^{-j\beta r_1}}{r_1} = \Psi(z,L), \quad \frac{e^{-j\beta r_2}}{r_2} = \Psi(z,-L)$$

and $$\frac{e^{-j\beta r_o}}{r_o} = \lim_{b\to 0}\Psi(z,b)$$

$$= \lim_{b\to 0}\Psi(z,-b)$$

gives
$$E_z(z) = \frac{1}{4\pi\epsilon_o}\Big[Q(L)\ \Psi(z,L) - Q(-L)\ \Psi(z,-L) -$$

$$+ Q(b)\ \Psi(z,b) + Q(-b)\ \Psi(z,-b)\Big] \quad \ldots(4\text{-}42)$$

which is identical to Eq. (4-33). For a sinusoidal current distribution Eq. (4-33) is then identical to Eq. (4-37).

The current distribution along cylindrical antennas is known to be almost sinusoidal. Therefore the field of a sinusoidal distribution can be used to approximate $E_s(z)$ as given by Eq. (4-33).

4.3 The Current Distribution.

To obtain the current distribution along the antenna a differential equation for this distribution will be determined first. This can be done using Maxwell's equations and evaluating them on the surface of the antenna using an appropriate system of coordinates. Of particular interest here are the equations

$$\text{curl } \vec{E} = - j\omega\vec{B} = - j\omega\mu_o\vec{H} \qquad \ldots(4\text{-}43)$$

$$\text{curl } \vec{H} = \vec{i} + j\omega\vec{D} \qquad \ldots(4\text{-}44)$$

$$\text{div } \vec{D} = q \qquad \ldots(4\text{-}45)$$

In polar coordinates Eq. (4-43) becomes, taking only the φ component,

$$\text{curl}_\varphi \ \vec{E} = \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -j\omega\mu_o H_\varphi \qquad \ldots(4\text{-}46)$$

On the surface of the antenna, due to the fact that the current flow is axial, the magnetic field of the antenna at the surface has only a φ component. That this is consistent with the electric field intensity on the surface of the antenna can be seen from the following relations

$$\text{curl}_\rho \ \vec{E} = \frac{\partial E_z}{\rho\,\partial\varphi} - \frac{\partial E_\varphi}{\partial z} = - j\omega\mu_o H_\rho$$

By symmetry $\dfrac{\partial E_z}{\partial \varphi} = 0$

Symmetry and axial current flow gives $E_\varphi = 0$

Hence $\quad -j\omega\mu_o H_\rho = \dfrac{\partial E_z}{\rho\,\partial\varphi} - \dfrac{\partial E_\varphi}{\partial z} = 0$

Similarly the z-component of the magnetic field is related to the electric field through the relation

$$-j\omega\mu_o H_z = \text{curl}_z\ \vec{E} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho E_\varphi) - \frac{\partial E_\rho}{\partial\varphi}\right]$$

which can be proven equal to zero by the same method as above.

Integrating Eq. (4-44) over a typical meridian surface of the antenna gives

$$\int_S \text{curl}\ \vec{H}\ .\ \vec{ds} = \int_S \vec{i}\ .\ \vec{ds} \qquad \ldots(4\text{-}47)$$

where by a meridian surface S is meant the surface perpendicular to the axis of the antenna and bounded by the cylindrical surface of the antenna. Over this region the electric field intensity and hence D is everywhere zero.

Applying Stoke's Law to Eq. (4-47) gives

$$\int_S \text{curl}\ \vec{H}\ .\ \vec{ds} = \oint \vec{H}\ .\ \vec{dh} = \int_S \vec{i}\ .\ \vec{ds} = I_z$$

Hence $\qquad 2\pi a\ H_\varphi = I_z \qquad \ldots(4\text{-}48)$

Therefore combining (4-48) with (4-46) gives

$$2\pi a\ \left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial\rho}\right) = -\ j\omega\mu_o I_z \qquad \ldots(4\text{-}49)$$

From Eq. (4-45) it follows that

$$\int_V \text{div}\ \vec{D}\ dv = \int_V q\ dv$$

Using Gauss's theorem gives

$$E_\rho(z) = \frac{Q(z)}{2\pi a \epsilon_o} \qquad \ldots(4\text{-}50)$$

where $Q(z)$ is the charge density per unit length.

Hence
$$\frac{\partial E_\rho}{\partial z} = \frac{1}{2\pi a\epsilon_o}\frac{\partial}{\partial z} Q(z)$$

$$= -\frac{1}{j2\pi\omega\epsilon_o a}\frac{\partial^2 I(z)}{\partial z^2} \qquad \ldots(4\text{-}51)$$

The latter result follows from the equation of continuity.

Substitution of (4-51) into Eq. (4-49) gives

$$\frac{j}{\omega\epsilon_o}\frac{\partial^2 I(z)}{\partial z^2} + j\omega\mu_o\, I(z) = 2\pi a\frac{\partial E_z}{\partial \rho}$$

$$\frac{\partial^2 I(z)}{\partial z^2} + \beta^2 I(z) = -\, j2\pi\epsilon_o\omega\,\left(a\frac{\partial E_z}{\partial \rho}\right) \qquad \ldots(4\text{-}52)$$

The current distribution along the antenna must therefore satisfy the non-homogeneous second order differential equation (4-52). The solution of this differential equation is

$$I(z) = C\sin\beta(L - z + \theta) + \frac{j2\pi\omega\epsilon_o}{\beta}\int_z^L y(u)\sin\beta(z-u)\,du$$
$$\ldots(4\text{-}53)$$

where $y(z) = a\frac{\partial E_z}{\partial \rho}$ remains to be determined.

The function $y(z)$ can be obtained by first returning to Eq. (4-36) and Eq. (4-52). Combining these equations gives

$$\int_L y(u)\Psi(z,u)\,du = 2\left[E_s(z) + E_i(z)\right] \qquad \ldots(4\text{-}54)$$

This is an integral equation of the first kind if both functions on the right hand side are known. $E_s(z)$ is given by Eq. (4-32). It has been shown that this can be approximated by Eq. (4-37). Hence both terms on the right hand side of Eq. (4-54) are known, except for a proportionality constant I_o, that must be determined from the excitation level of the antenna.

One method of solving this integral equation is to expand both sides in Fourier series. The result is then a system of linear equations relating the Fourier coefficients of the known and the unknown functions (see Appendix II).

Another method is to use successive approximations and has been used successfully by Hallen. This method will be used here.

The zero order solution of the integral equation (4-54) can be found by first noting that the kernel $\Psi(z,u)$ has a very large value in the neighborhood of $z = u$.
Hence

$$\int_L y(u)\,\Psi(z,u)\,du \cong y(z)\int_L \Psi(z,u)\,du$$

$$\cong 2\left[E_s(z) + E_i(z)\right]$$

and the zero order solution for y(z) is given by

$$y(z) \cong \frac{2\,E_s(z) + E_i(z)}{\int_L \Psi(z,u)du}$$

$$\cong \frac{2\left[E_s(z) + E_i(z)\right]}{K(z)} \qquad \ldots(4\text{-}55)$$

where (Reference 7, p. 335, Eq. 48)

$$K(z) = -\,\mathrm{Cin}\ \beta(L - z) - \mathrm{Cin}\ \beta(L + z) + \ln \frac{L - z + \sqrt{(L - z)^2 + a^2}}{\sqrt{(L + z)^2 + a^2} - (L - z)} - j\left[\mathrm{Si}\ \beta(L - z) + \mathrm{Si}\ \beta(L + z)\right] \qquad \ldots(4\text{-}56)$$

To show the behaviour of the function K(z), a plot of this function for several lengths and thickness is reproduced in Figs. 4-3 and 4-4. The function is therefore almost constant over the antenna.

Hence the zero order solution of the current distribution is given by

$$I(z) = C \sin \beta(L - z + \theta) + \frac{j}{30} \int_z^L \frac{E_i(u) + E_s(u)}{K(u)} \sin \beta(z - u)\, du \qquad \ldots(4\text{-}57)$$

The evaluation of the integral in Eq. (4-57), due to the complexity of the integrand, cannot be done exactly. However, since the denominator, as shown above, is almost constant over the length of the antenna, the average value of the denominator can be used to approximate the exact function. This is evaluated in Appendix III and the result is

$$\Psi = 2 \ln \frac{2L}{a} + 2\left(\ln 2 - \mathrm{Cin}\ 2\beta L - \frac{\sin 2\beta L}{\beta L} + j\left[\frac{1 - \cos 2\beta L}{2\beta L} - \mathrm{Si}\ 2\beta L\right]\right) \qquad \ldots(4\text{-}58)$$

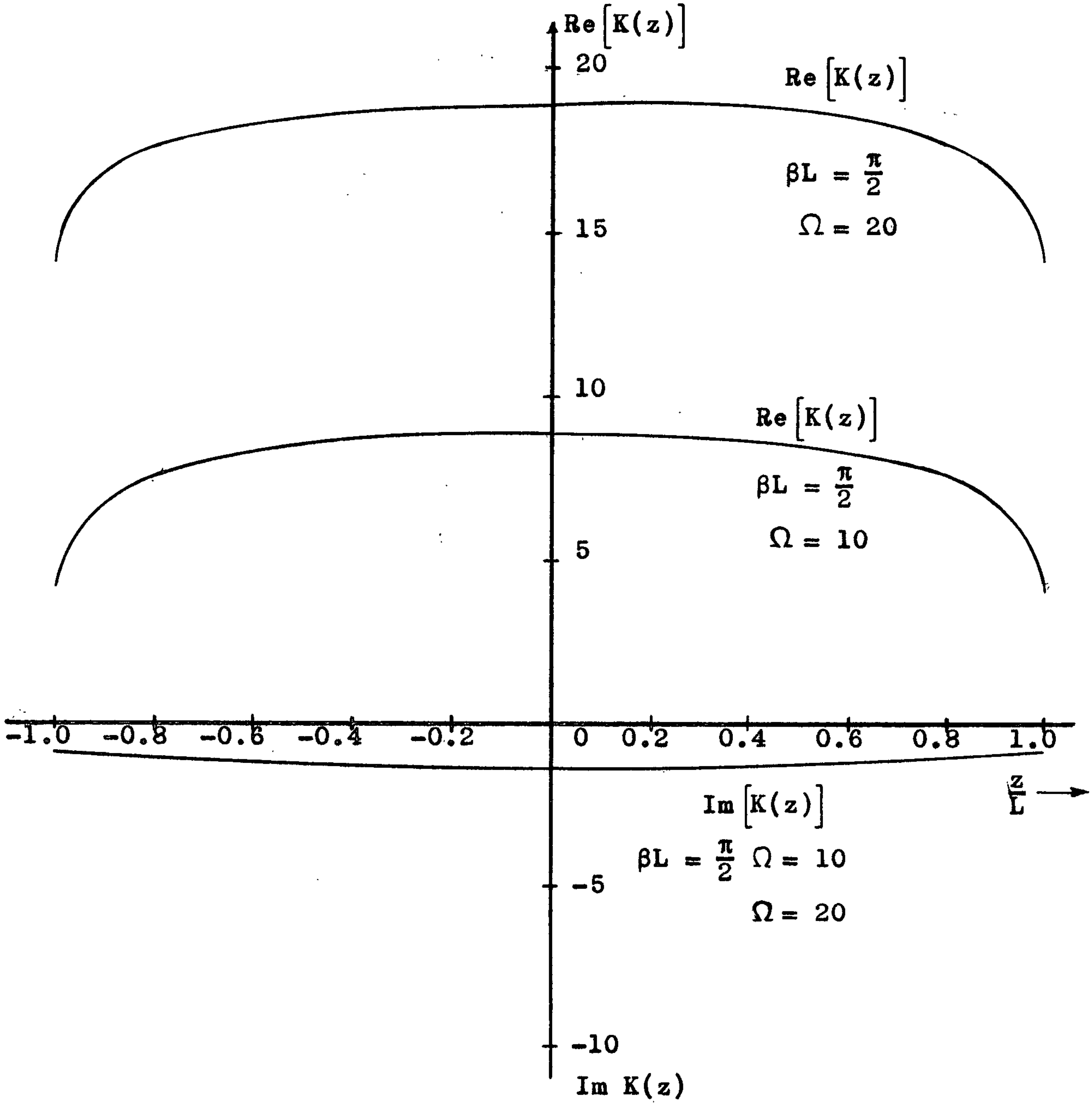


Fig. 4-3. Graphs of K(z) for $\beta L = \frac{\pi}{2}$, $\Omega = 10$ and $\beta L = \frac{\pi}{2}$, $\Omega = 20$

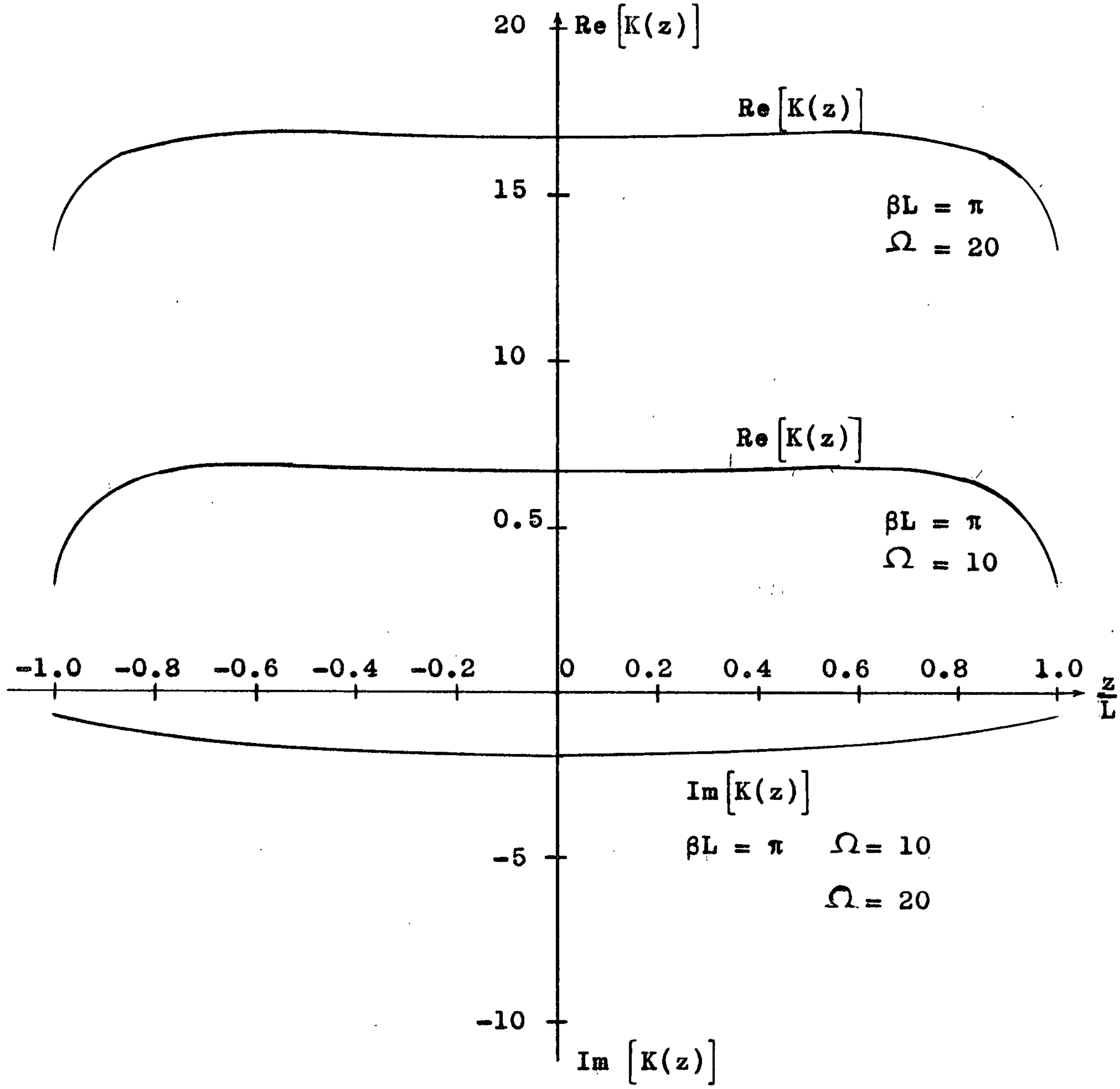


Fig. 4-4. Graphs of $K(z)$ for $\beta L = \pi, \Omega = 10$ and $\beta L = \pi$, $\Omega = 20$.

After taking this approximation, the integration is straight forward and the result is

$$I(z) = C \sin \beta(L - z + \theta) + j \frac{Q(L)}{120 \pi \epsilon_o} \frac{G(z)}{\Psi} - j \frac{I_i b}{\Psi} F(z) \qquad \ldots(4\text{-}59)$$

where G(z) and F(z) are defined and evaluated in Appendix IV and Appendix V respectively.

The constants C and θ can be found by considering the cap of the antenna (Fig. 4-5).

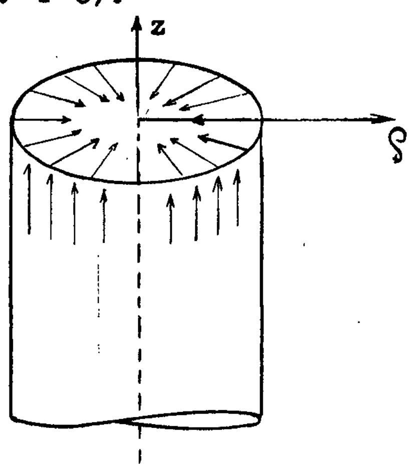


Fig. 4-5. Current Distribution on the End Surface

Due to symmetry the conduction current on the cap will flow radially, while on the cylindrical surface it is axial. To preserve continuity the axial flow on the boundary line between the cylindrical surface and the cap must equal the radial flow. Hence

$$\Big[I(z)\Big]_{z=L} = - \Big[I_\rho\Big]_{\rho=a} \qquad \ldots(4\text{-}60)$$

Positive direction of the current on the cap corresponds to the positive direction of ρ , as given in Fig. 4-5.

Evaluating the axial current distribution at z=L gives

$$I_z(L) = C \sin \beta\theta \qquad \ldots(4\text{-}61)$$

Due to rotational symmetry of the current and charge distribution on the caps, the equation of continuity is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\, I_\rho\right] = -j\omega Q(\rho) \qquad \ldots(4\text{-}62)$$

For thin antennas the charge density over the caps does not change very much with the radial distance ρ and hence

$$I_\rho = B\rho \qquad \ldots(4\text{-}63)$$

where B is a constant that can be determined by substituting Eq. (4-63) into Eq. (4-62) which gives

$$B = -\frac{j\omega Q(\rho)}{2} = -\frac{j\omega Q(a)}{2} \qquad \ldots(4\text{-}64)$$

Substituting (4-64) in Eq. (4-63) and using Eqs. (4-60) and (4-61) gives

$$C \sin \beta\theta = -\frac{j\omega Q(a)}{2}\, a = -\frac{j\omega a Q(L)}{2} \qquad \ldots(4\text{-}65)$$

Since $\left.\dfrac{d\, I(z)}{dz}\right|_{z=L} = -\, j\omega Q(L),$

it follows that $\beta C \cos \beta\theta = j\omega Q(L)$ $\qquad \ldots(4\text{-}66)$

Eq. (4-61) indicates to what extent the axial current distribution deviates from zero at the end of the antenna. For infinitely thin antennas, this value must be zero. It is known that the current on the cap of the antenna effectively increases the length of the antenna by an amount in the order of its radius.

Hence, for thin antennas $\sin \beta\theta = \beta\theta$ and $\cos \beta\theta = 1$

Therefore $$C = \frac{j\omega Q(L)}{\beta} \qquad \ldots(4\text{-}67)$$

and $$\theta = \frac{a}{2} \qquad \ldots(4\text{-}68)$$

The zero order current distribution is therefore given by

$$I(z) = \frac{j\omega Q(L)}{\beta}\left[\sin \beta(L - z + \tfrac{1}{2}a) + \frac{G(z)}{\Psi}\right] - j\,\frac{I_i\, b}{\Psi}\,F(z) \qquad \ldots(4\text{-}69)$$

For small gap width the contribution of the last term in Eq. (4-69) is very small since $F(z)$ increases as $\log\frac{L}{b}$ and consequently $\lim_{b\to 0} b\,F(z) = 0$. For practical purposes this can be neglected. This means that the coupling between the transmission line and the antenna has little effect on the antenna current distribution. Eq. (4-69) can then be simplified into

$$\begin{aligned} I(z) &= \frac{j\omega Q(L)}{\beta}\left[\sin \beta(L - z + \tfrac{1}{2}a) + \frac{G(z)}{\Psi}\right] \\ &= I_o\left[\sin \beta(L - z + \tfrac{1}{2}a) + \frac{G(z)}{\Psi}\right] \qquad \ldots(4\text{-}70 \end{aligned}$$

The constant I_o is determined from the complex power input to the antenna and making use of the fact that $I_i = I(b)$.

To facilitate comparison with the existing theories, the relative magnitudes of the current distributions for various lengths and thickness of the antenna are plotted in Figs. (4-6), (4-7), (4-8) and (4-9). In comparing the results it is important to note that the expansion parameter Ψ is not uniquely defined. The zero order current distribution plotted is obtained from Eq. (4-70) using Eq. (4-58) for Ψ. The first and second order plots of King

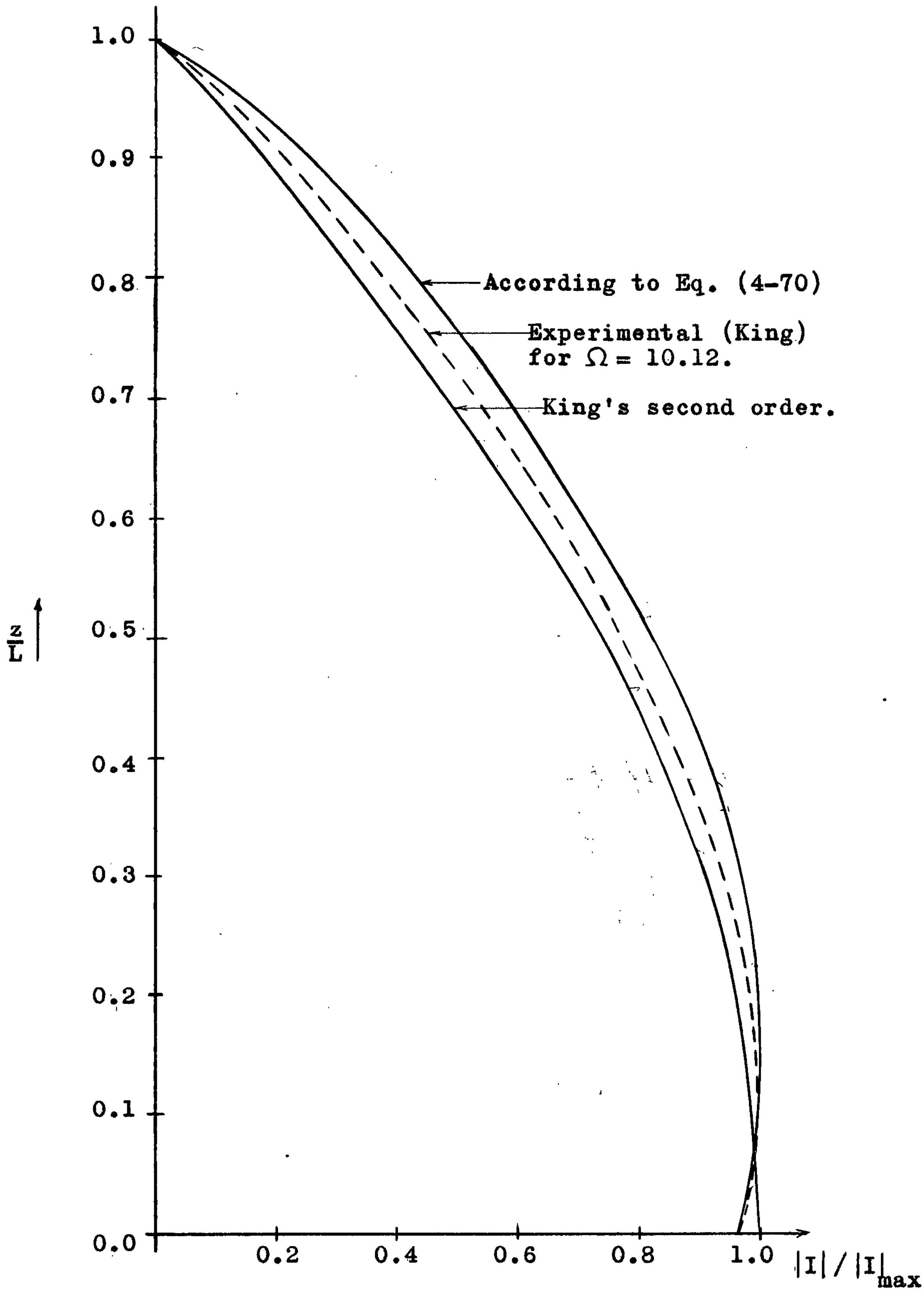


Fig. 4-6 Current Distribution Along Cylindrical Antenna for $\beta L = \pi/2$ and $\Omega = 10$.

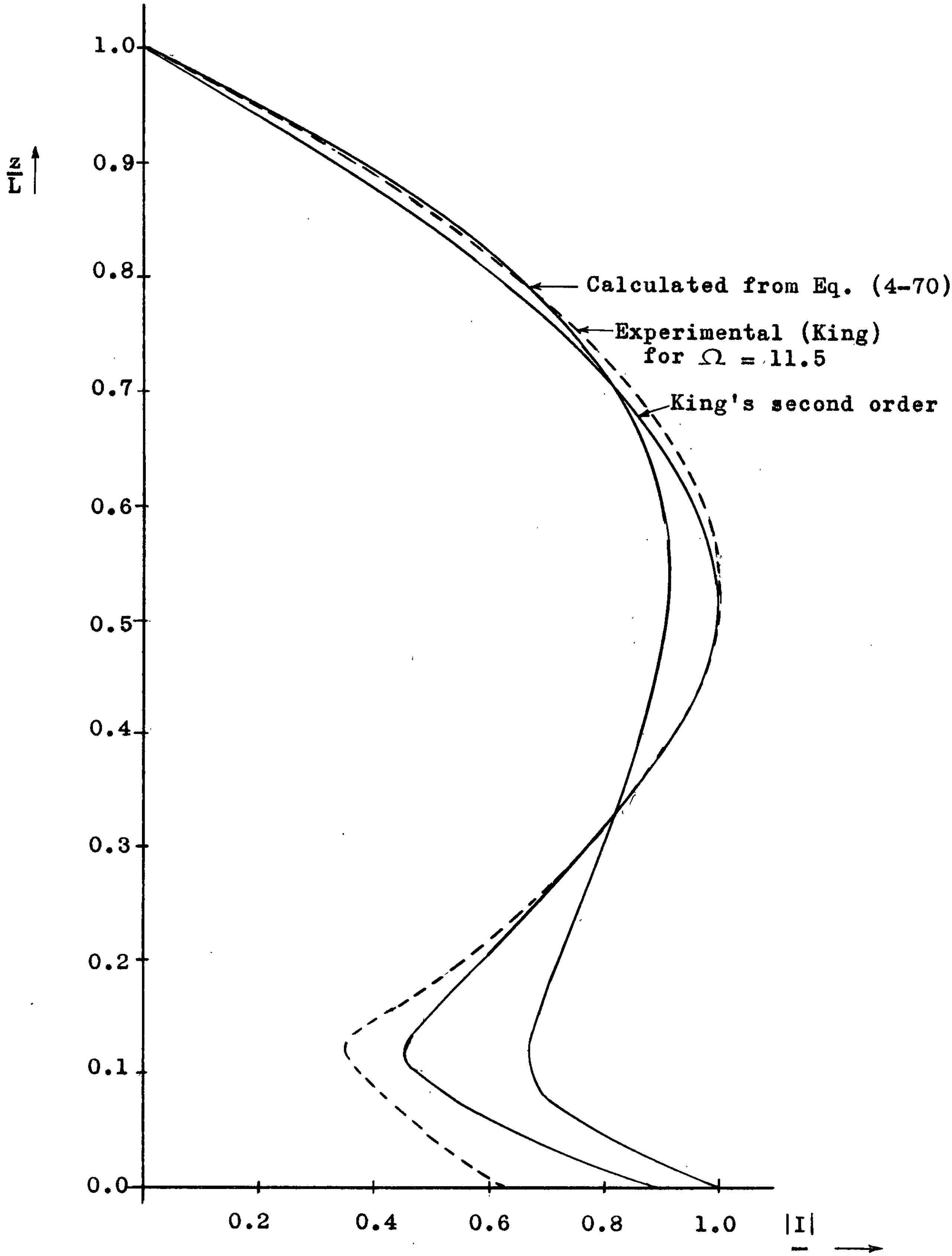


Fig. 4-7 Current Distribution Along Cylindrical Antenna for $\beta L=\pi$ and $\Omega=10$

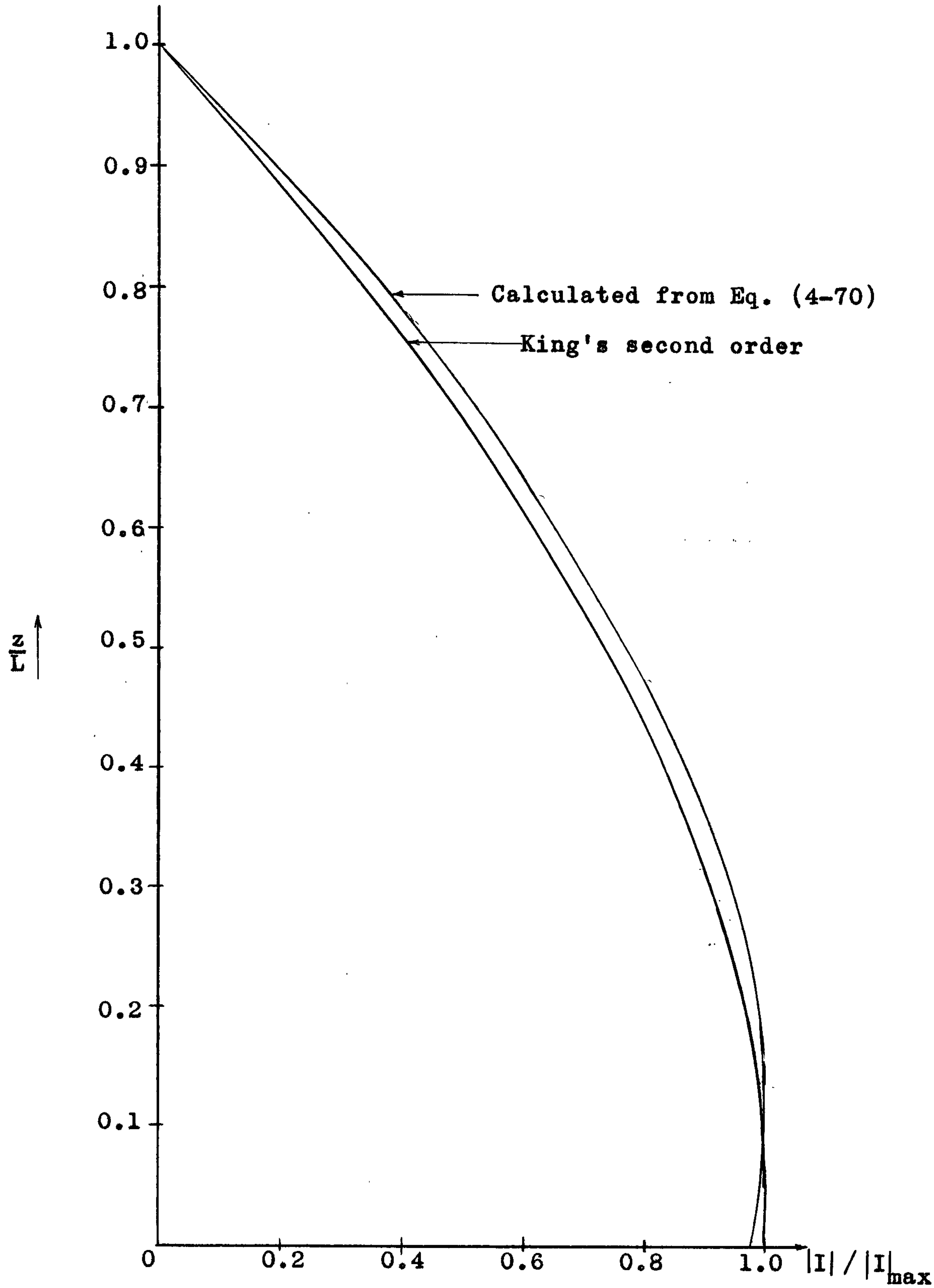


Fig. 4-8. Current Distribution Along Cylindrical Antenna for $\beta L = \pi/2$ and $\Omega = 20$.

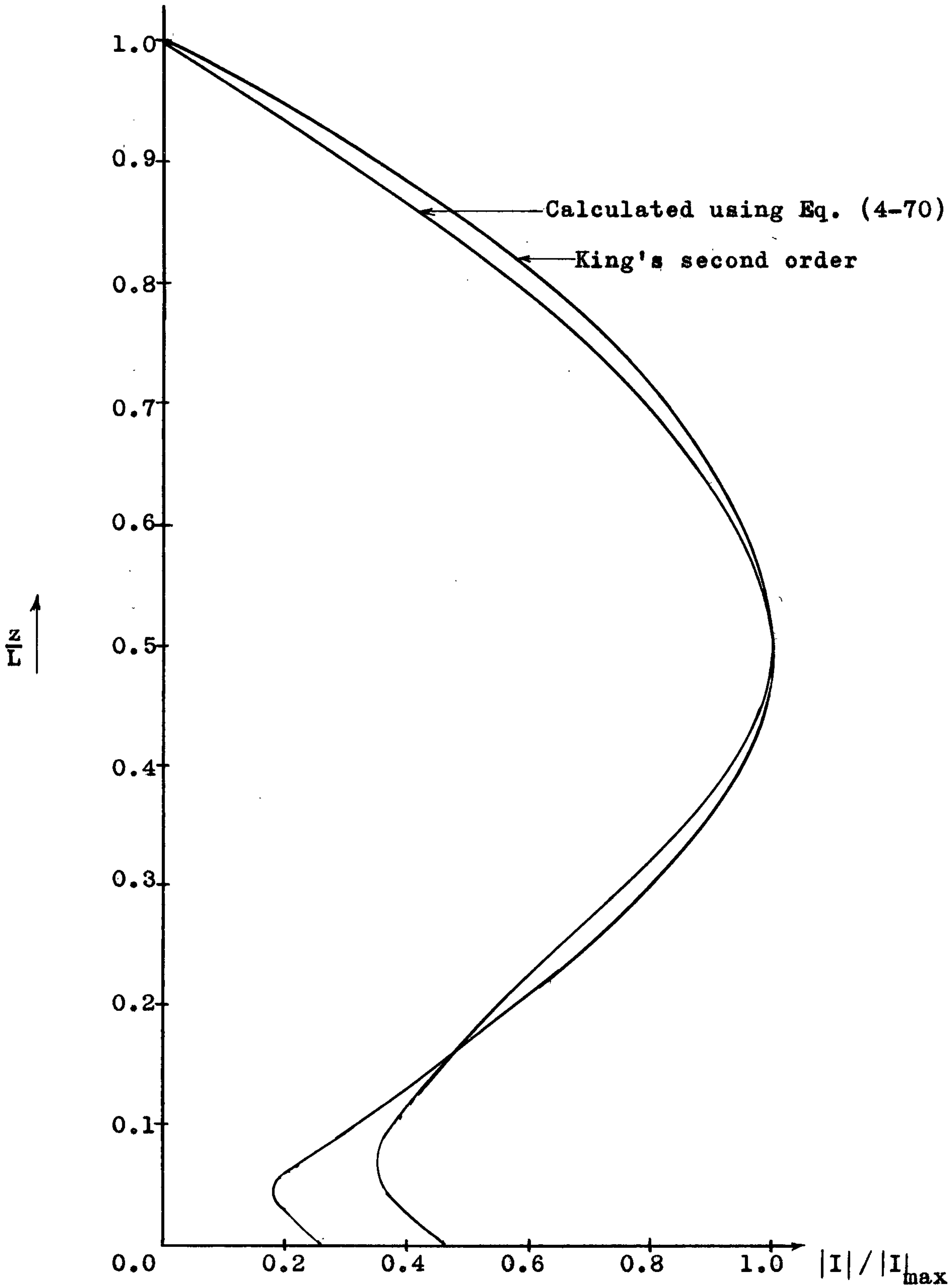


Fig. 4-9 Current Distribution Along Cylindrical Antenna for $\beta L = \pi$ and $\Omega = 20$.

use a more elaborate expansion parameter (reference 1, p. 107, Eq. 37).

4.4 Comparison of the Correction Terms

The current distribution along the antenna has been solved using the non-homogeneous differential equation (4-52) and the zero order solution is given in Eq. (4-70). This consists of a sinusoidal principal part and a correction term given by the first and second terms inside the bracket in Eq. (4-70) respectively.

Both Hallen and King solved the integral equation for the current (2-22) by successive approximations and obtained (see Reference 1, p. 85)

$$I(z) = \frac{j2\pi\, V_s}{\zeta\, \Psi} \; \frac{\sin \beta(L - z) + \sum\limits_{s=1}^{s=n} \dfrac{M_s(z)}{(\Psi)^s}}{\cos \beta L + \sum\limits_{s=1}^{s=n} \dfrac{A_s}{(\Psi)^s}} \qquad \ldots(4\text{-}71)$$

where $s = 1, 2, 3, \ldots\ldots\ldots n$, and n is equal to the required order of the solution.

Eq. (4-71) can be written as

$$I(z) = I_s \left[\sin \beta(L - z) + \sum_{s=1}^{s=n} \frac{M_s(z)}{(\Psi)^s} \right] \qquad \ldots(4\text{-}72)$$

where

$$I_s = \frac{j\,2\pi\,V\delta}{\zeta\,\Psi}\;\frac{1}{\sum_{s=1}^{s=n}\frac{A_s}{(\Psi)^s}} \qquad \ldots(4\text{-}73)$$

The Hallen/King first order solution is therefore given by

$$I(z) = I_1\left[\sin\beta(L - z) + \frac{M_1(z)}{\Psi}\right] \qquad \ldots(4\text{-}74)$$

A question now arises whether $M_1(z)$ is in some way related to $G(z)$ in Eq.(4-70), since both current distributions must satisfy the differential equation (4-52).

$G(z)$ is a particular solution of the differential equation (4-52). Hence applying the operator $D = \frac{d^2}{dz^2} + \beta^2$ to $G(z)$ gives

$$D\left[G(z)\right] = \Psi(z,L) + \Psi(z,-L) - 2\cos\beta L\,\Psi(z,0) \qquad \ldots(4\text{-}75)$$

It is shown in Appendix VI that

$$D\left[M_1(z)\right] = -\beta\left[\Psi(z,L) + \Psi(z,L) - 2\cos\beta L\,\Psi(z,0)\right] \ldots(VI\text{-}8)$$

This is similar to Eq. (4-75).

The consequence of the foregoing result is that

$$M_1(z) = G(z) + A\sin\beta(L - z) \qquad \ldots(4\text{-}76)$$

since $\quad M_1(L) = G(L) = 0$

Due to the complexity of $G(z)$ and $M_1(z)$ only graphical comparison of these functions will be made. Figs. (4-10), (4-11), (4-12) and (4-13) represent the function

$$h(z) = \mathrm{Re}\left[M_1(z) - G(z)\right] \qquad \ldots(4\text{-}77)$$

graphically, for various thickness of the antenna.

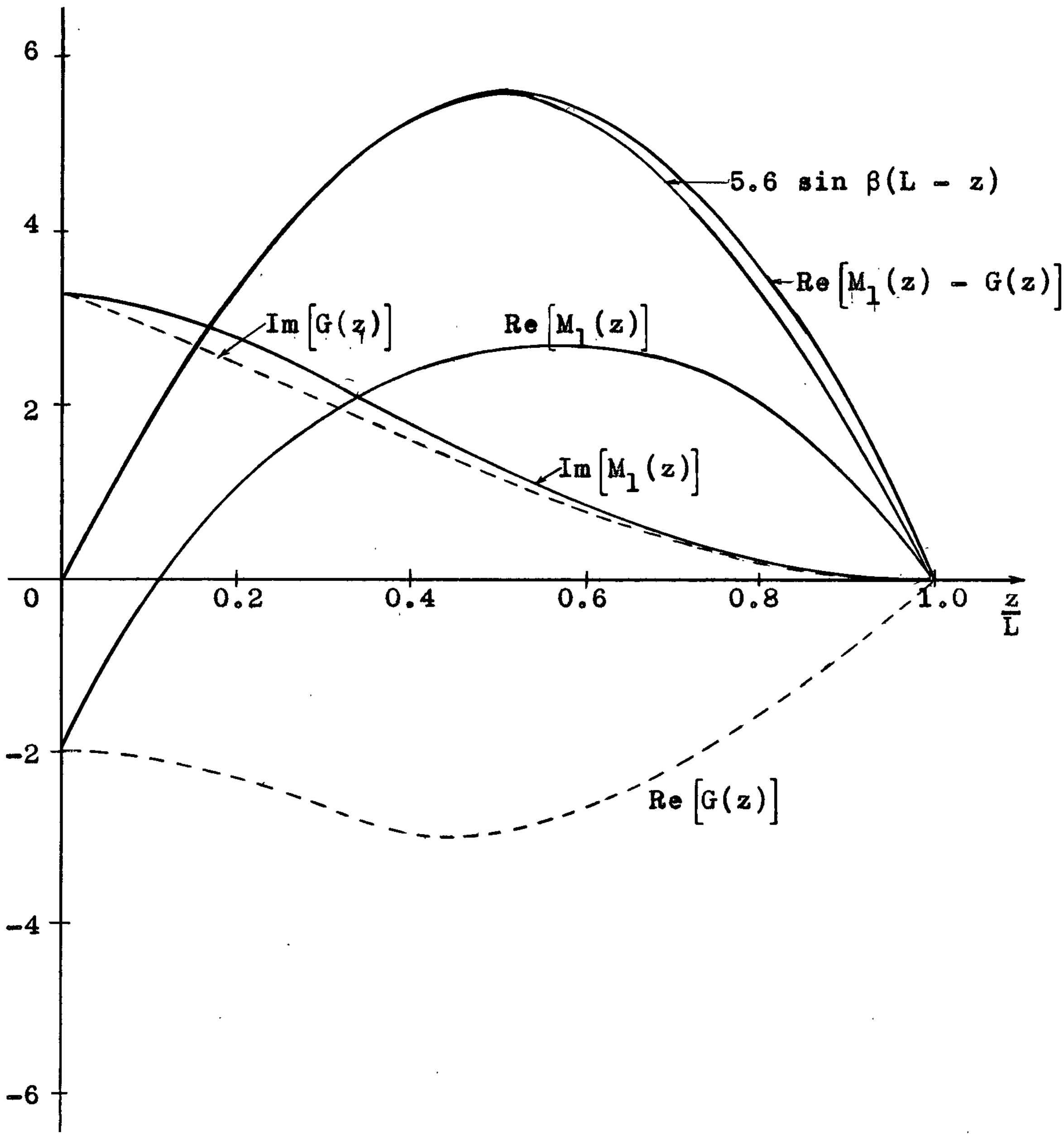


Fig. 4-1Q Comparison Between $M_{1H}(z)$ and $G(z)$ for $\beta L = \pi$ and $\Omega = 10$.

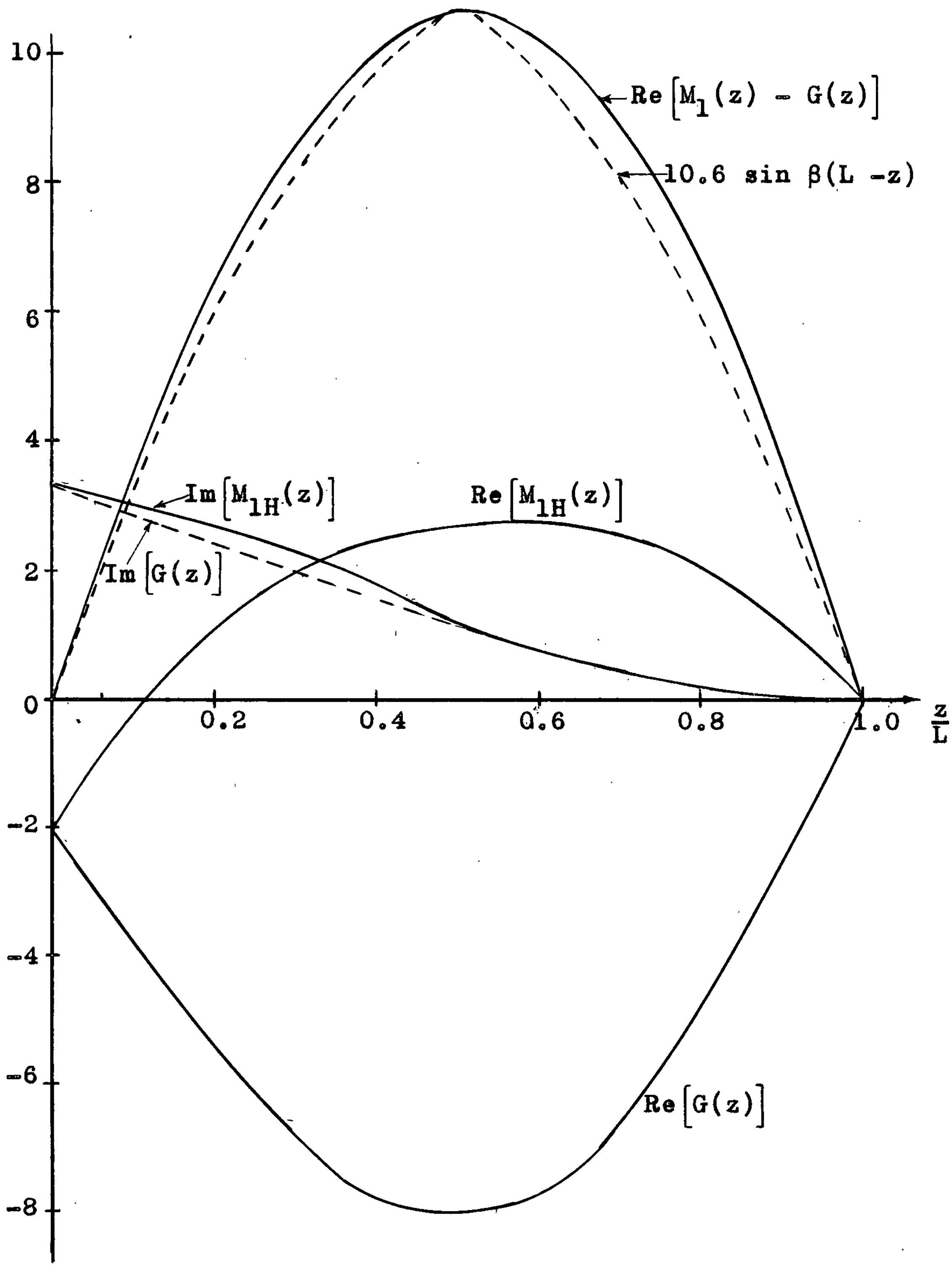


Fig. 4-11. Comparison Between $M_{1H}(z)$ and $G(z)$ for $\beta L = \pi$ and $\Omega = 20$

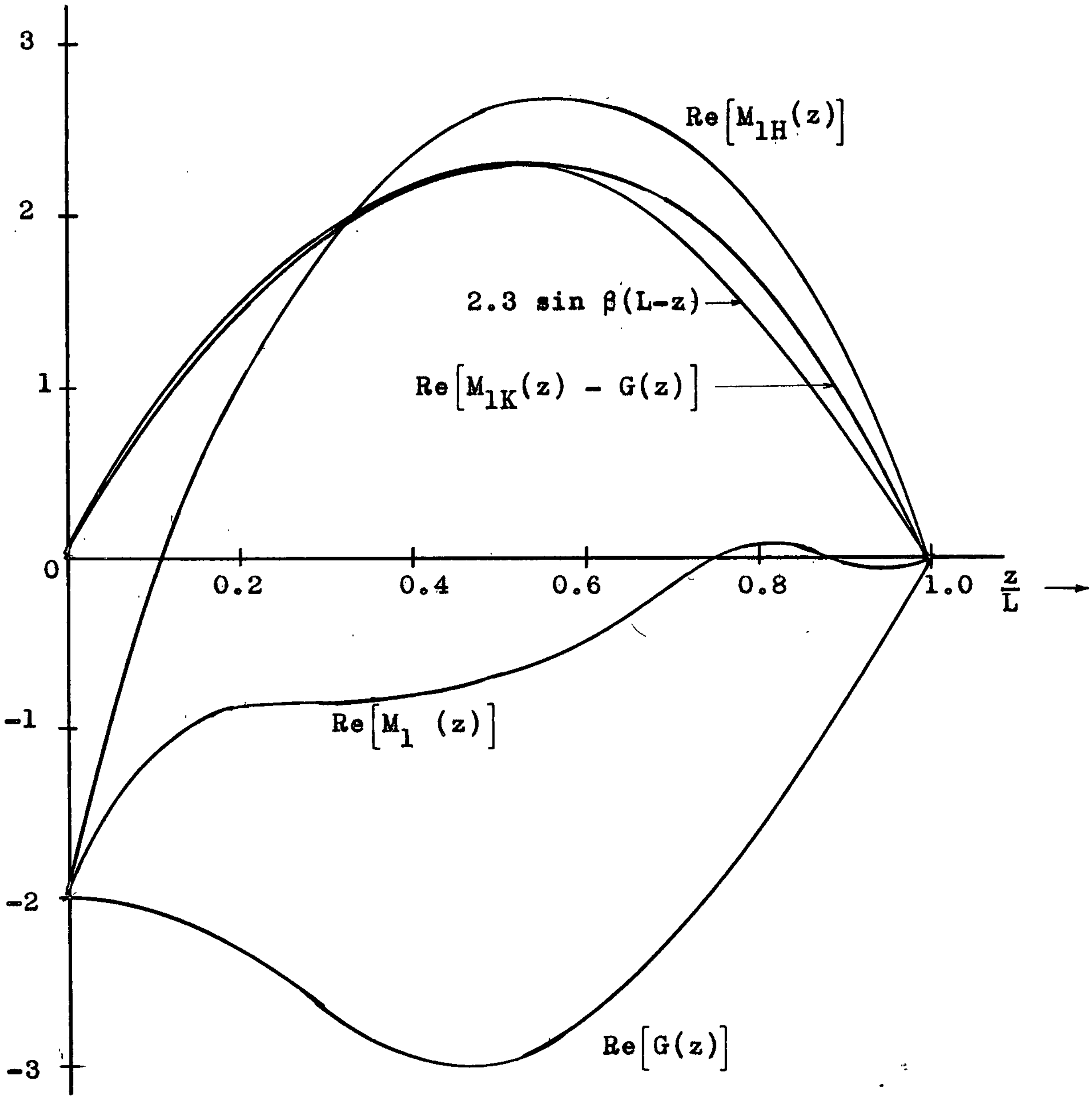


Fig. 4-12. Comparison Between $M_{1K}(z)$ and $G(z)$ for $\beta L=\pi$ and $\Omega = 10$.

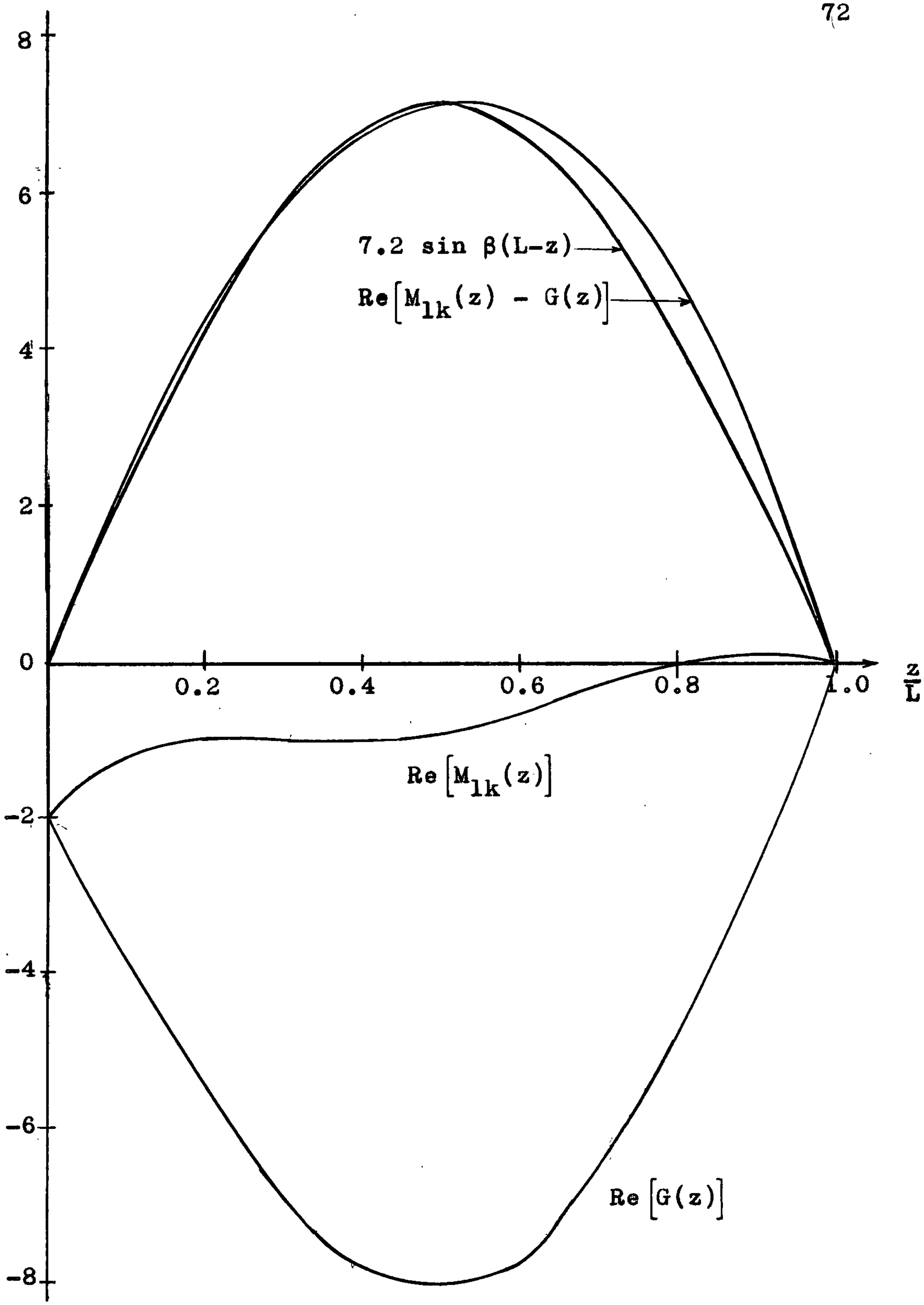


Fig.4-13. Comparison Between $M_{1k}(z)$ and $G(z)$ for $\beta L = \pi$ and $\Omega = 20$

Re$\left[f\right]$ means the real part of the function inside the bracket. Graphs of $g(z) = A \sin \beta(L - z)$ are also plotted on the same diagrams where the A's are appropriately chosen to match the maximum value of the corresponding $h(z)$.

It is thus seen that Hallen and King's correction terms contain a large sinusoidal component. This can be also seen by expanding $y(u)$ (Eq. 4-53) in powers of Ψ^{-1} which leads to the form

$$I(z) = I_o \left[\sin \beta(L - z) + \frac{G(z)}{\Psi} + \frac{H(z)}{\Psi^2}\right] \quad \ldots(4\text{-}78)$$

and comparing this with Eq. (4-71) which can be written as

$$I(z) = I_o \left[\sin \beta(L - z) + \frac{M_1(z)}{\Psi} + \frac{M(z)}{\Psi^2}\right] \quad \ldots(4\text{-}79)$$

where $H(z)$ and $M(z)$ are so defined that Eqs. (4-78) and (4-79) are exact representations of the current distribution.

Substituting Eq. (4-76) into Eq. (4-79) and comparing the result with Eq. (4-78) gives

$$M(z) = H(z) - A \Psi \sin \beta(L - z) \quad \ldots(4\text{-}80)$$

Evidently, King's higher order correction terms contain a large sinusoidal component. Graphical plotting of $M_2(z)$, see Fig. (4-14), indicates that this sinusoidal term must also be contained in the third and higher order terms which are not considered by King.

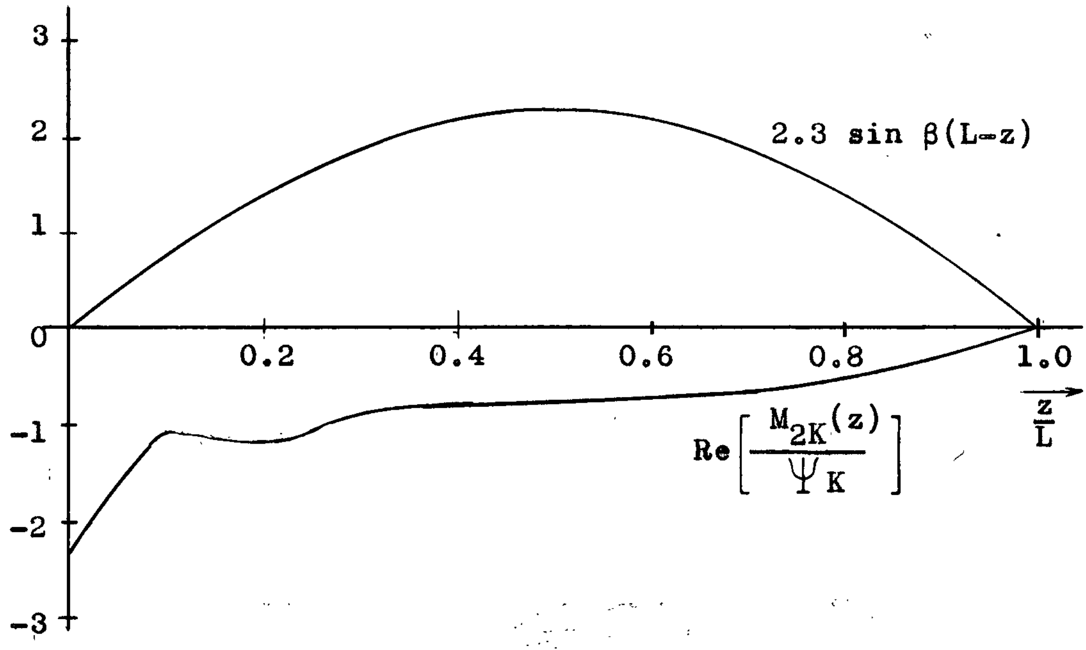


Fig. 4-14. King's Second Order Correction Term for $\beta L = \pi$ and $\Omega = 10$.

The agreement with the experimentally determined current distribution and input impedance is hence semi-empirical. This agreement is obtained by a suitable choice of Ψ and the fact that no terms higher than the second order are considered. The very peculiar fact that this same choice of Ψ gives a good approximation to the input impedance (see Eq.(3-25)), has, until now, completely obscured the empirical nature of the Hallen-King theory.

4.5 The Input Impedance

In Sec. 3-1 a formula for the input impedance applicable to cylindrical antennas is derived. Since the actual current distribution along a cylindrical antenna is very complicated, analytical evaluation of the integrals (3-41) and (3-42) is not possible. Evaluation of a formula suitable for numerical computations is therefore desirable. This can be done with the aid of a series expansion outlined in Appendix II.

Using this method one obtains

$$A(u) = \frac{\mu_o}{4\pi} \int_A i(u) \; \Psi\,(u,z,\varphi) \; a \; dz \; d\varphi$$

$$= \frac{\mu_o}{4\pi} \int_{-L}^{L} I(u) \; \frac{1}{2\pi} \int_{0}^{2\pi} \Psi\,(z,u,\;) \; d\varphi \; dz$$

$$= \frac{\mu_o}{2\pi} \sum_{n,m} Lk_{nm} \; I_m \; e^{(pu)/L} \qquad \ldots(4\text{-}81)$$

where

$$I_m = \frac{1}{2L} \int_{-L}^{L} I(u) \; e^{-(tz)/L} dz \qquad \ldots(4\text{-}82)$$

$$t = j\pi m, \quad p = j\pi n$$

and k_{nm} is as defined in Appendix II, Eq. II-9.

Hence

$$W'_M = \frac{1}{2} \int_A \vec{A}(u) \cdot \vec{i}^{\,*}(u) \; d\tau = \frac{1}{2} \int_{-L}^{L} A(u) \; I^*(u) \; du$$

$$= \frac{\mu_o}{4\pi} \sum_{n,m} Lk_{nm} \; I_m \int_{-L}^{L} I^*(u) \; e^{(pu)/L} \; du$$

$$= \frac{\mu_o}{2\pi} L \sum_{n,m} Lk_{nm} \; I_m \; I_n^* \qquad \ldots(4\text{-}83)$$

where

$$I_n^* = \frac{1}{2L} \int_{-L}^{L} I^*(u) \; e^{(pu)/L} \; dz \qquad \ldots(4\text{-}84)$$

Similarly

$$W_e' = \int_A U(u)\ q^*(u)\ d\tau$$

$$= \frac{L}{2\pi\epsilon_o} \sum_{n,m} Lk_{nm}\ Q_m\ Q_n^* \qquad \ldots(4\text{-}85)$$

But

$$Q_m = \frac{1}{2L} \int_{-L}^{L} Q(z)\ e^{-(tz)/L}\, dz$$

$$= -\frac{1}{2L} \int_{-L}^{L} \frac{1}{j\omega} \frac{d}{dz} I(z)\ e^{-(tz)/L}\ dz$$

$$= -\frac{1}{2j\omega L} \left[I(z)\ e^{-(tz)/L} \Big|_{-L}^{L} + \frac{t}{L} \int_{-L}^{L} I(z)\ e^{-(tz)/L}\ dz \right]$$

$$Q_m = -\frac{t}{j\omega L}\ I_m \qquad \ldots(4\text{-}86)$$

Similarly

$$Q_n^* = \frac{p}{j\omega L}\ I_n^* \qquad \ldots(4\text{-}87)$$

Hence

$$W_e' = \frac{1}{2\pi\omega^2\epsilon_o L} \sum_{n,m} Lk_{nm}\ (tp)\ I_m\ I_n^* \qquad \ldots(4\text{-}88)$$

Substituting Eqs.(4-80) and (4-85) into Eq. (3-13) and noting that

$$\frac{j\omega\mu_o L}{2\pi} = j\ 60\ \beta L$$

and

$$\frac{j}{2\pi\epsilon_o L} = j\ \frac{60}{\beta L},$$

gives after rearranging the terms

$$Z_i = j \frac{120}{|I_i|^2} (\beta L) \sum_{n,m} Lk_{nm} \left[1 - (\frac{\pi}{\beta L})^2 nm\right] I_m I_n^* \qquad \ldots(4\text{-}89)$$

The current distribution I(z) can be written as

$$I(z) = I_o f(z)$$

Consequently $I_i = I_o f(0) \qquad I_i^* = I_o^* f^*(0) \qquad \ldots(4\text{-}90)$

$$I_m = I_o f_m \qquad I_n^* = I_o^* f_n^* \qquad \ldots(4\text{-}91)$$

where $f_m = \frac{1}{2L} \int_{-L}^{L} f(z)\, e^{-(tz)/L}\, dz \qquad \ldots(4\text{-}92)$

$$f_n^* = \frac{1}{2L} \int_{-L}^{L} f^*(u)\, e^{(pu)/L}\, du \qquad \ldots(4\text{-}93)$$

Finally, substituting Eqs.(4-87) and (4-88) into Eq.(4-86) gives

$$Z_i = \frac{j\, 120\, (\beta L)}{|f(0)|^2} \sum_{n,m} Lk_{nm} \left[1 - (\frac{\pi}{\beta L})^2 nm\right] f_m f_n^* \qquad \ldots(4\text{-}94)$$

5. Conclusion

It has been shown that the Hallen-King theory of antennas is of a semi-empirical nature. Both the current distribution and input impedance depend on a semi-empirical choice of an expansion parameter and the number of terms taken into consideration. A correct solution of the problem must take into account the effect of the transmission line. This eliminates the inconsistencies existing in Hallen-King method and relates the theoretically determined input impedance to that determined experimentally.

The zero order current distribution obtained using the theory developed here (Eq. 4-70) compares favourably with King's second order distribution (see Figs. 4-6, 4-7, 4-8, and 4-9). Due to mathematical difficulties, the higher order solution cannot be found analytically. Expressions are available through which the current distribution (see Eqs. 4-53 and II-6) and the input impedance (see Eq. 4-94) can be determined exactly. These are in a form suitable for numerical computation with the use of digital computers.

APPENDIX I

Prove that

$$\frac{\partial}{\partial u} \frac{e^{-j\beta(r+x)}}{r} = \frac{\partial}{\partial x}\left[\frac{r-x}{u} \quad \frac{e^{-j\beta(r+x)}}{r}\right] \qquad \text{(I-1)}$$

Proof

Let

$$\frac{\partial}{\partial u} \frac{e^{-j\beta(r+x)}}{r} = \frac{\partial}{\partial x}\left[F(x,u)\, \frac{e^{-j\beta(r+x)}}{r}\right] \qquad \text{(I-2)}$$

Noting that $r^2 = x^2 + u^2$ and hence $\frac{\partial r}{\partial x} = \frac{x}{r}$ and $\frac{\partial r}{\partial u} = \frac{u}{r}$

the left hand side of (I-2) becomes

$$\frac{\partial}{\partial u} \frac{e^{-j\beta(r+x)}}{r} = -\left(j\,\frac{\beta u}{r} + \frac{u}{r^2}\right) \frac{e^{-j\beta(r+x)}}{r} \qquad \text{(I-3)}$$

Evaluating the right hand side of (I-2) gives

$$\frac{\partial}{\partial x}\left[F(x,u)\, \frac{e^{-j\beta(r+x)}}{r}\right] =$$

$$\left[\frac{\partial F(x,u)}{\partial x} - F(x,u)\left(j\,\beta\left(\frac{x}{r} + 1\right) + \frac{x}{r^2}\right)\right] \frac{e^{-j\beta(r+x)}}{r} \qquad \text{(I-4)}$$

since $\frac{\partial}{\partial x} \frac{e^{-j\beta(r+x)}}{r} = -\left[j\,\beta\left(\frac{x}{r} + 1\right) + \frac{x}{r^2}\right] \frac{e^{-j\beta(r+x)}}{r}$

Equating the imaginary parts of (I-3) and (I-4) gives

$$j\,\beta\left(\frac{x}{r} + 1\right) F(z,u) = j\,\beta\,\frac{u}{r}$$

Hence

$$F(x,u) = \frac{u}{r+x}$$

Multiplying the nominator and the denominator by $\frac{r - x}{r - x}$ gives

$$F(x,u) = \frac{u(r - x)}{r^2 - x^2} = \frac{r - x}{u} \qquad \text{(I-5)}$$

Substituting (I-5) into (I-2) gives (I-1). Q.e.d.

The real part of (I-4), omitting the factor $\frac{e^{-j\beta(r + x)}}{r}$ is

$$\frac{\partial F(x,u)}{\partial x} - x/r^2\, F(x,u) = -\frac{(r - x)x}{u\, r^2} + \frac{x - r}{ur} = -\frac{r^2 - x^2}{ur^2}$$
$$= -u/r^2$$

which is exactly the same as the corresponding term in (I-3)

<u>Prove that</u>

$$\frac{\partial}{\partial u} \frac{e^{-j\beta(r - x)}}{r} = -\frac{\partial}{\partial x}\left[\frac{r + x}{u} \frac{e^{-j\beta(r - x)}}{r}\right] \qquad \text{(I-6)}$$

<u>Proof</u>

Let $\frac{e^{-j\beta(r - x)}}{r} = E(x,u)$ and (I-7)

$$\frac{\partial}{\partial u} E(x,u) = \frac{\partial}{\partial x}\left[G(x,u)\, E(x,u)\right] \qquad \text{(I-8)}$$

Again noting $\frac{\partial r}{\partial x} = \frac{x}{r}$ and $\frac{\partial r}{\partial u} = \frac{u}{r}$, evaluation of (I-8) gives

$$-\left(j\frac{\beta u}{r} + \frac{u}{r^2}\right) = \frac{\partial G(x,u)}{\partial x} - G(x,u)\left[j\,\beta\left(\frac{x}{r} - 1\right) + \frac{x}{r^2}\right]$$

Equating the imaginary part gives

$$G(x,u) = \frac{u}{x - r} = -\frac{r + x}{u} \qquad \text{(I-9)}$$

Check on the real parts

$$\frac{\partial G(x,u)}{\partial x} - G(x,u)\,\frac{x}{r^2} = -\,\frac{x + r}{ur} + \frac{r + x}{u}\cdot\frac{x}{r^2}$$

$$= \frac{-\,xr - r^2 + xr + x^2}{u\,r^2}$$

$$= -\,\frac{u}{r^2}$$

Q.e.d.

APPENDIX II

Solution of the integral equation

$$E(u) = \frac{1}{4\pi} \int_{0}^{2\pi} \int_{A} y(z) \quad \Psi(z,u,\varphi) \; dz \; d\varphi \qquad \text{(II-1)}$$

with the aid of Fourier Series.

It is shown in reference (23) that

$$\int_{0}^{2\pi} \Psi(z,u,\varphi) \; d\varphi = \frac{2}{jL} \int_{C} K_o(s) \; I_o(s) \; e^{-\frac{s(u-z)}{L}} \; ds \qquad \text{(II-2)}$$

where $K_o(s)$ and $I_o(s)$ are modified Bessel's Functions, and the integration contour C lies along the imaginary axis of the complex plane S.

Substituting Eq. (II-2) in Eq. (II-1) gives

$$E(u) = \frac{1}{2} \int_{A} y(z) \left[\frac{1}{j\pi L} \int_{C} K_o(s) I_o(s) \; e^{-\frac{s(u-z)}{L}} ds \right] dz \qquad \text{(II-3)}$$

The term inside the square bracket, which is a function of u and z, can be expanded in a double Fourier Series of the form

$$\frac{1}{j\pi L} \int_{C} K_o(s) \; I_o(s) \; e^{-\frac{s(u-z)}{L}} \; ds = \sum_{nm} \left[k_{nm} \; e^{-(tz)/L} \; e^{(pu)/L} \right] \qquad \text{(II-4)}$$

with $t = j\pi m$, $m = 0, \pm 1, \pm 2, \ldots\ldots\ldots$

$p = j\pi n$, $n = 0, \pm 1, \pm 2, \ldots\ldots\ldots$

Substituting Eq.(II-4) in Eq.(II-3) gives

$$E(u) = \tfrac{1}{2}\sum_{nm} k_{nm}\, e^{(pu)/L} \int_A y(z)\, e^{-(tz)/L}\, dz$$

$$= \sum_{nm} Lk_{nm}\, e^{(pu)/L} \frac{1}{2L} \int_{-L}^{L} y(z)\, e^{-(tz)/L}\, dz \qquad \text{(II-5)}$$

Since $\int_A e^{(pu)/L}\, du = 0$, for $p \neq 0$

$= 2L$ for $p = 0$, multiplying Eq.(II-5) by

$\frac{e^{-(pu)/L}}{2L}$ du and then integrating the result over the same interval gives

$$E_n = \sum_{nm} Lk_{nm}\, y_m \qquad \text{(II-6)}$$

where

$$E_n = \frac{1}{2L} \int_{-L}^{L} E(u)\, e^{-(pu)/L}\, du \qquad \text{(II-7)}$$

$$y_m = \frac{1}{2L} \int_{-L}^{L} y(z)\, e^{-(tz)/L}\, dz \qquad \text{(II-8)}$$

The Fourier coefficient k_{nm} can be obtained by multiplying both sides of Eq. (II-4) by $e^{(tz-pu)/L}dz\, du$ and integrating the result over the antenna. This gives

$$k_{nm} = j\frac{1}{\pi L}\int_C I_o(s)K_o(s)\, \frac{\sinh\,(s-p)}{s-p}\, \frac{\sinh\,(s-t)}{s-t}\, ds \qquad \text{(II-9)}$$

The result of this integration is given in the Appendix of reference (23).

APPENDIX III

Evaluation of

$$\Psi = \frac{1}{2L} \int_{-L}^{+L} K(u)\, du \qquad \text{(III-1)}$$

Although it is possible to evaluate(III-1) by straight integration of K(z), it is more convenient to find Ψ using the original function

$$\Psi = \frac{1}{2L} \int_{-L}^{+L} du \int_{-L}^{+L} \Psi\,(u,z)\, dz \qquad \text{(III-2)}$$

To do this, transform $\Psi(u,z)$ into F(z,y) using the transformation

$$z = u + z$$
$$y = u - z \qquad \text{(III-3)}$$

and obtain

$$F(x,y) = \frac{e^{-j\beta\sqrt{(y^2 + a^2)}}}{\sqrt{(y^2 + a^2)}} \qquad \text{(III-4)}$$

The Jacobian of the transformation is

$$\left| J(u,z) \right| = \left| \frac{\partial(x,y)}{\partial(u,z)} \right| = \left\| \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right\| = 2$$

hence $\quad dx\ dy = 2\ du\ dy. \qquad$ (III-5)

Therefore using (III-2), (III-4) and (III-5) one obtains

$$\Psi = \frac{1}{2L} \iint_D \frac{e^{-j\beta\sqrt{(y^2 + a^2)}}}{2\sqrt{(y^2 + a^2)}}\, dx\ dy \qquad \text{(III-6)}$$

Where D implies the boundary of integration indicated by Fig. III-1b.

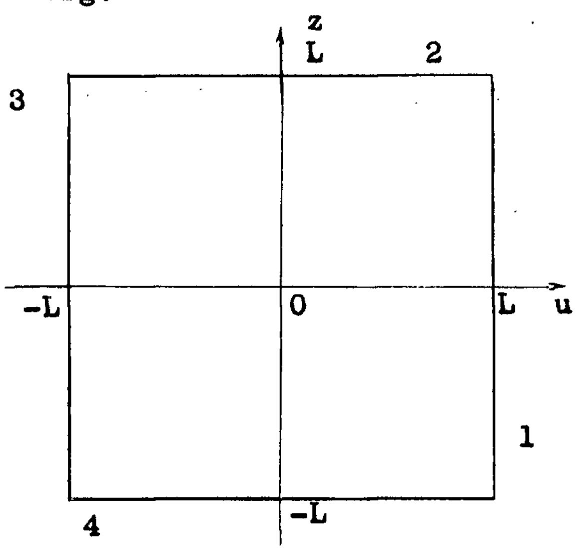


Fig. III-1a. Surface of Integration in (u,z) Plane.

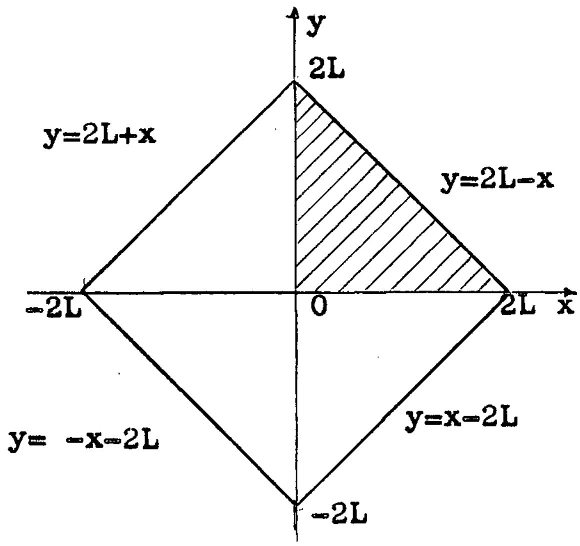


Fig. III-1b. Surface of Integration in (x,y) Plane.

The equation for the boundary of the surface of integration in (x,y) plane can be found in the following way:

The boundary corresponding to boundary (1) in the (u,z) plane, satisfies $u = L$.

Hence in the (x,y) plane this corresponds to

$$x = L + z$$

$$y = L - z$$

Eliminating z one obtains

$$y = 2L - x$$

By similar method the other boundaries can be obtained and the equations are as given on the diagram.

Therefore noting that the integrand is an even function of y and also that

$$\int_{y-2L}^{-y+2L} dx = 2\int_{0}^{-y+2L} dx$$

$$\int_{-2L-y}^{y-2L} dx = 2\int_{0}^{y-2L} dx,$$

integration over the shaded area only will give $\frac{1}{4}$ of the value of (III-6).

Hence

$$\Psi = \frac{4}{4\,L}\int_{0}^{2\,L} \frac{e^{-j\beta\sqrt{(y^2 + a^2)}}}{\sqrt{(y^2 + a^2)}}\,dy \int_{0}^{2L-y} dx$$

$$= \frac{1}{L}\int_{0}^{2\,L} (2L-y)\,\frac{e^{-\,j\beta\sqrt{(y^2 + a^2)}}}{\sqrt{(y^2 + a^2)}}\,dy$$

$$= -\,\frac{1}{L}\int_{0}^{2\,L} (y - 2L)\,\left(\frac{\cos\,\beta r}{r} - j\,\frac{\sin\,\beta r}{r}\right)\,dy \quad \text{(III-7)}$$

where $r = \sqrt{(y^2 + a^2)}$

But $$\int_{0}^{2\,L} (y - 2L)\,\frac{\cos\,\beta r}{r}\,dy \cong \int_{0}^{2\,L} y\,\frac{\cos\,\beta y}{y}\,dy$$

$$- \int_{0}^{2\,L} 2L\,\frac{\cos\,\beta r}{r}\,dy$$

$$\cong \frac{\sin\,2\beta L}{\beta} - 2L\int_{0}^{2\,L} \frac{1 - \cos\,\beta y}{y}\,dy - \int_{0}^{2L} \frac{dy}{r}$$

$$\int_0^{2L} (y-2L)\, \frac{\cos \beta r}{r}\, dy \cong \frac{\sin 2\beta L}{\beta} + 2L \text{ Cin } 2\beta L - 2L \ln \frac{4L}{a}$$

$$\int_0^{2L} (y - 2L)\, \frac{\sin \beta r}{r}\, dr \cong \int_0^{2L} (y - 2L)\, \frac{\sin \beta y}{y}\, dy$$

$$\cong \frac{1 - \cos 2\beta L}{\beta} - 2L \text{ Si } 2\beta L.$$

Therefore

$$\Psi \cong 2 \ln \frac{2L}{a} + 2 \left[\ln 2 - \text{Cin } 2\beta L - \frac{\sin 2\beta L}{\beta L} \right.$$

$$\left. + j\left(\frac{1 - \cos 2\beta L}{2\beta L} - \text{Si } 2\beta L \right) \right] \qquad \text{(III-8)}$$

APPENDIX IV

$$G\ (z) = \int_{z}^{L} (\Psi_1 + \Psi_4 - 2\cos\beta L\,\Psi_2)\ \sin\beta(z-u)\ du. \qquad \text{(IV-1)}$$

where $\Psi = \dfrac{e^{-j\beta R}}{R}$

$$R_1^2 = (L-u)^2 + a^2$$

$$R_2^2 = u^2 + a^2 \qquad \text{(IV-2)}$$

$$R_4^2 = (L+u)^2 + a^2$$

Using Eq.(IV-2) and noting that

$$\sin\beta(z-u) = \frac{e^{j\beta(z-u)} - e^{-j\beta(z-u)}}{2j}$$

Eq.(IV-1) becomes

$$G(z) = -\tfrac{1}{2}j \int_{z}^{L} \left[\frac{e^{-j\beta(R_1+u-z)}}{R_1} - \frac{e^{-j\beta(R_1-u+z)}}{R_1} \right.$$

$$+ \frac{e^{-j\beta(R_4+u-z)}}{R_4} - \frac{e^{-j\beta(R_4-u+z)}}{R_4}$$

$$- 2\cos\beta L\ \frac{e^{-j\beta(R_2+u-z)}}{R_2}$$

$$\left. + \cos\beta L\ \frac{e^{-j\beta(R_2-u+z)}}{R_2} \right] du \qquad \text{(IV-3)}$$

A typical integral in (IV -3) is of the form

$$G_1(z) = \int_{z}^{L} \frac{e^{-j\beta(R_1+u-z)}}{R_1}\ du$$

This integral can be easily evaluated using substitution

$$v = \beta(R_1 + u + L)$$

hence $\frac{dv}{v} = \frac{du}{R_1}$

$$u = z, \qquad v_1 = \beta\left[\sqrt{(L + z)^2 + a^2} + z + L\right]$$

$$u = L, \qquad v_2 = \beta\left[\sqrt{4L^2 + a^2} + 2L\right]$$

Therefore

$$G\ (z) = e^{j\beta(z + L)} \int_{v_1}^{v_2} \frac{e^{-j\beta v}}{v}\, dv$$

$$= e^{j\beta(z + L)}\left[\mathrm{Ei}\ v_2 - \mathrm{Ei}\ v_1\right]$$

The other integrals can be evaluated in a similar way and therefore only the answers and respective substitutions are given below.

The indices on G's are taken in successive order.

Second intrgral: $w = \beta(R_1 - u - L) \qquad \frac{dw}{w} = -\frac{du}{R_1}$

$$u = z, \quad w_1 = \beta\left[\sqrt{(L + z)^2 + a^2} - z - L\right]$$

$$u = L, \quad w_2 = \beta\left[\sqrt{4L^2 + a^2} - 2L\right]$$

$$G_2(z) = -e^{-j\beta(z + L)}\left[\mathrm{Ei}\ w_2 - \mathrm{Ei}\ w_1\right]$$

Third integral: $v = \beta(R_4 + u - L) \qquad \frac{dv}{v} = \frac{du}{R_4}$

$$u = z, \quad v_3 = \beta\left[\sqrt{(L - z)^2 + a^2} + z - L\right]$$

$$u = L, \quad v_4 = \beta a$$

$$G_3(z) = e^{j\beta(z - L)}\left[\mathrm{Ei}\ v_3 - \mathrm{Ei}\ v_4\right]$$

Fourth integral: $w = \beta(R_4 - u + L)$ $\qquad \dfrac{dw}{w} = -\dfrac{du}{R_4}$

$$u = z, \quad w_3 = \beta\left[\sqrt{(L - z)^2 + a^2} - z + L\right]$$

$$u = L, \quad w_4 = \beta a.$$

$$G_4(z) = -e^{-j\beta(z - L)}\left[\mathrm{Ei}\ w_3 - \mathrm{Ei}\ w_4\right]$$

Fifth integral: $v = \beta(R_2 + u)$ $\qquad \dfrac{dv}{v} = \dfrac{du}{R_2}$

$$u = z, \quad v_5 = \beta\left[\sqrt{z^2 + a^2} + z\right]$$

$$u = L, \quad v_6 = \beta\left[\sqrt{L^2 + a^2} + L\right]$$

$$G_5(z) = 2 \cos \beta L\ e^{j\beta z}\left[\mathrm{Ei}\ v_5 - \mathrm{Ei}\ v_6\right]$$

Sixth integral: $w = \beta(R_2 - u)$ $\qquad \dfrac{dw}{w} = \dfrac{du}{R_2}$

$$u = z, \quad w_5 = \beta\left[\sqrt{z^2 + a^2} - z\right]$$

$$u = L, \quad w_6 = \beta\left[\sqrt{L^2 + a^2} - L\right]$$

$$G_6(z) = -2 \cos \beta L\ e^{-j\beta z}\left[\mathrm{Ei}\ w_5 - \mathrm{Ei}\ w_6\right]$$

Therefore

$$\begin{aligned}
2j\ G(z) = {} & \cos \beta(z + L)\left[\mathrm{Ei}\ v_2 - \mathrm{Ei}\ v_1 + \mathrm{Ei}\ w_2 - \mathrm{Ei}\ w_1\right] \\
& + \cos \beta(z - L)\left[\mathrm{Ei}\ v_4 - \mathrm{Ei}\ v_3 + \mathrm{Ei}\ w_4 - \mathrm{Ei}\ w_3\right] \\
& - 2 \cos \beta L \cos \beta z\left[\mathrm{Ei}\ v_6 - \mathrm{Ei}\ v_5 + \mathrm{Ei}\ w_6 - \mathrm{Ei}\ w_5\right] \\
& + j \sin \beta(z + L)\left[\mathrm{Ei}\ v_2 - \mathrm{Ei}\ v_1 - \mathrm{Ei}\ w_2 + \mathrm{Ei}\ w_1\right] \\
& + j \sin \beta(z - L)\left[\mathrm{Ei}\ v_4 - \mathrm{Ei}\ v_3 - \mathrm{Ei}\ w_4 + \mathrm{Ei}\ w_3\right] \\
& - j\ 2 \cos \beta L \sin \beta L\left[\mathrm{Ei}\ v_6 - \mathrm{Ei}\ v_5 - \mathrm{Ei}\ w_6 + \mathrm{Ei}\ w_5\right]
\end{aligned} \tag{IV-4}$$

But $\mathrm{Ei}(v) = \mathrm{Ci}\ v - j\ \mathrm{Si}\ v$

$$\text{where} \qquad Ci(v) = -\int_{v}^{\infty} \frac{\cos x}{x}\, dx$$

$$Si(v) = \int_{0}^{v} \frac{\sin x}{x}\, dx \qquad \text{(IV - 5)}$$

and $v_4 = w_4 = u_o$

Finally substituting (IV-5) in (IV-4) and rearranging the terms gives

$$\begin{aligned}
G(z) = \tfrac{1}{2}\Big[& \sin\beta(z+L)\left(Ci\, v_2 - Ci\, v_1 - Ci\, w_2 + Ci\, w_1\right) \\
& - \cos\beta(z+L)\left(Si\, v_2 - Si\, v_1 + Si\, w_2 - Si\, w_1\right) \\
& + \sin\beta(L-z)\left(Ci\, v_3 - Ci\, w_3\right) \\
& - \cos\beta(L-z)\left(2\, Si\, u_o - Si\, v_3 - Si\, w_3\right) \\
& + 2\cos\beta L \cos\beta z\left(Si\, v_6 - Si\, v_5 + Si\, w_6 - Si\, w_5\right) \\
& - 2\cos\beta L \sin\beta z\left(Ci\, v_6 - Ci\, v_5 - Ci\, w_6 + Ci\, w_5\right)\Big] \\
- \tfrac{1}{2}\, j\, \Big[& \cos\beta(z+L)\left(Ci\, v_2 - Ci\, v_1 + Ci\, w_2 - Ci\, w_1\right) \\
& + \sin\beta(z+L)\left(Si\, v_2 - Si\, v_1 - Si\, w_2 + Si\, w_1\right) \\
& + \cos\beta(L-z)\left(2\, Ci\, u_o - Ci\, v_3 - Ci\, w_3\right) \\
& + \sin\beta(L-z)\left(Si\, v_3 - Si\, w_3\right) \\
& - 2\cos\beta L \cos\beta z\left(Ci\, v_6 - Ci\, v_5 + Ci\, w_6 - Ci\, w_5\right) \\
& - 2\cos\beta L \sin\beta z\left(Si\, v_6 - Si\, v_5 - Si\, w_6 + Si\, w_5\right)\Big]
\end{aligned}$$

(IV-6)

APPENDIX V

$$F_1(z) = \int_z^L \sin\beta(u - z)\, \frac{\partial}{\partial u}\left[\frac{e^{-j\ \beta u}}{u}\right] du \qquad \ldots(V\text{-}1)$$

$$= \int_z^L \sin\beta(u - z)\ d\ \frac{e^{-j\ \beta u}}{u}$$

Integrating by parts gives

$$F_1(z) = \sin\beta(u - z)\ \frac{e^{-j\ \beta u}}{u}\Bigg|_z^L - \beta\int_z^L \frac{e^{-j\ \beta u}}{u} \cos\beta(u - z)\ du$$

$$\ldots(V\text{-}2)$$

The last integral in (V-2) can be easily evaluated by first expanding the cosine function into exponential functions.

The result is

$$\int_z^L \frac{e^{-j\ \beta u}}{u} \cos\beta(u - z)\ du = \int_z^L \frac{e^{-j\ \beta u}}{u}\ \frac{e^{j\ \beta(u-z)} + e^{-j\ \beta(u-z)}}{2}\ du$$

$$= \int_z^L \frac{e^{-j\ \beta\ z}}{2u}\ du + \int_z^L \frac{e^{-j\ \beta(2u-z)}}{2u}\ du$$

$$= \frac{e^{-j\ \beta z}}{2} \ln\frac{L}{z} +$$

$$e^{j\ \beta\ z}\int_z^L \left[\frac{\cos 2\ \beta u}{2u} - j\frac{\sin 2\ \beta\ u}{2u}\right] du$$

$$= \frac{e^{-j\ \beta z}}{2} \ln\frac{L}{z} + \frac{e^{j\ \beta z}}{2}\Big(\ln\frac{L}{z} - \mathrm{Cin}\ 2\ \beta L$$

$$+ \mathrm{Cin}\ 2\ \beta z - j\ \mathrm{Si}\ 2\ \beta L + j\ \mathrm{Si}\ 2\ \beta z\Big)$$

$$\ldots(V\text{-}3)$$

Substituting (V-3) into (V-2) gives

$$F_1(z) = \frac{\sin\beta(L-z)\cos\beta L}{L} - \beta\cos(\beta z)\ln\frac{L}{z}$$

$$-\tfrac{1}{2}\beta\cos\beta z\,(\mathrm{Cin}\,2\,\beta z - \mathrm{Cin}\,2\,\beta L)$$

$$+\tfrac{1}{2}\beta\sin\beta z\,(\mathrm{Si}\,2\,\beta z - \mathrm{Si}\,2\,\beta L)$$

$$-j\left[\frac{\sin\beta(L-z)\sin\beta L}{L} + \tfrac{1}{2}\beta\sin\beta z\,(\mathrm{Cin}\,2\beta z - \mathrm{Cin}\,2\beta L)\right.$$

$$\left.+\tfrac{1}{2}\beta\cos\beta z\,(\mathrm{Si}\,2\beta z - \mathrm{Si}\,2\,\beta L)\right] \qquad \ldots(\text{V-4})$$

$$F_2(z) = \int_z^L \sin\beta(u-z)\frac{\partial}{\partial u}\left(\frac{1}{u}\right)du \qquad \ldots(\text{V-5})$$

$$= \int_z^L \sin\beta(u-z)\,d\left(\frac{1}{u}\right)$$

$$= \frac{\sin\beta(u-z)}{u}\Bigg|_z^L - \beta\int_z^L \frac{\cos\beta(u-z)}{u}\,du$$

$$= \frac{\sin\beta(L-z)}{L} - \beta\cos\beta z\left(\ln\frac{L}{z} - \mathrm{Cin}\,\beta L + \mathrm{Cin}\,\beta z\right)$$

$$-\beta\sin\beta z\,(\mathrm{Si}\,\beta L - \mathrm{Si}\,\beta z) \qquad \ldots(\text{V-6})$$

And $F(z) = F_1(z) + (\xi - 1)\,F_2(z)$ $\qquad \ldots(\text{V-7})$

APPENDIX VI

Hallen's first order correction term is given by (Reference 1, p. 85, Eq. 27)

$$M_1(z) = F_1(z) \sin \beta L - G_1(z) \cos \beta L +$$

$$+ G_1(L) \cos \beta z - F_1(L) \sin \beta z \qquad \ldots(\text{VI -1})$$

where (assuming perfect conductor)

$$F_1(z) = F_{oz}\Psi - \int_A F_{oz'} \, \Psi(z,z') \, dz'$$

$$F_{oz} = \cos bz - \cos \beta L$$

$$G_1(z) = G_{oz}\Psi - \int_A G_{oz'} \, \Psi(z,z') \, dz'$$

$$G_{oz} = \sin \beta|z| - \sin \beta L.$$

$$\Psi(z,z') = \frac{e^{-j\beta r}}{r}, \text{ with } r^2 = (z-u)^2 + a^2$$

Applying the operator $D = (\frac{\partial^2}{\partial z^2} + \beta^2)$ to Eq. (VI-1) and noting that $D\left[\cos \beta z\right] = D\left[\sin \beta z\right] = D\left[F_{oz}\right] = D\left[G_{oz}\right] = 0$ gives

$$D\left[M_1(z)\right] = - \sin \beta L \, D\left[\int_A F_{oz'} \, \Psi(z,z') \, dz'\right] +$$

$$\cos \beta L \, D\left[\int_A G_{oz'} \, \Psi(z,z') \, dz'\right] \qquad \ldots(\text{VI-2})$$

But

$$\frac{\partial^2}{\partial z^2} \int_A F_{oz'} \, \Psi(z,z') \, dz' = \frac{\partial}{\partial z} \int_A F_{oz'} \frac{\partial}{\partial z} \Psi(z,z') \, dz'$$

$$= - \frac{\partial}{\partial z} \int_{-L}^{L} F_{oz'} \frac{\partial}{\partial z'} \Psi(z,z') \, dz'$$

$$= - \frac{\partial}{\partial z} \left[\left(F_{oz'} \, \Psi(z,z') \right) \Big|_{-L}^{L} + \right.$$

$$- \int_{-L}^{L} \Psi(z,z') \frac{\partial}{\partial z'} F_{oz'} dz' \Big]$$

$$= - \beta \frac{\partial}{\partial z} \int_{-L}^{L} \sin \beta z' \; \Psi(z,z') \; dz'$$

$$= \beta \int_{-L}^{L} \sin \beta z' \frac{\partial}{\partial z'} \Psi(z,z') \; dz' \quad \ldots (VI-3)$$

Since $F_{o,L} = F_{o,-L} = 0$

Integrating Eq. (VI-3) by parts gives

$$\frac{\partial^2}{\partial z^2} \int_A F_{oz'} \Psi(z,z') \; dz' = \beta \sin \beta L \left[\Psi(z,L) + \Psi(z,-L) \right] - \beta^2 \int_{-L}^{L} \cos \beta z' \; \Psi(z,z') \; dz' \quad \ldots (VI-4)$$

Hence

$$\sin \beta L \; D \left[\int_A F_{oz'} \Psi(z,z') \; dz' \right] = \beta \sin^2 \beta L \left[\Psi(z,L) + \Psi(z,-L) \right] - \cos \beta L \sin \beta L \; \beta^2 \left[\int_A \Psi(z,z') dz' \right] \quad \ldots (VI-5)$$

Also

$$\frac{\partial^2}{\partial z^2} \int_A G_{oz'} \Psi(z,z') \; dz' = - \frac{\partial}{\partial z} \int_{-L}^{L} G_{oz'} \frac{\partial}{\partial z'} \Psi(z,z') \; dz'$$

$$= - \frac{\partial}{\partial z} \left[\left| G_{oz'} \Psi(z,z') \right| \Big|_{-L}^{L} \right.$$

$$\left. - \int_{-L}^{L} \Psi(z,z') \frac{\partial}{\partial z'} G_{oz'} \; dz' \right]$$

$$= \frac{\partial}{\partial z} \int_{-L}^{L} \Psi(z,z') \frac{\partial}{\partial z'} G_{oz'} \; dz' \quad \ldots (VI-6)$$

since $G_{oL} = G_{o(-L)} = 0$

Hence

$$\frac{\partial^2}{\partial z^2}\int_A G_{oz'}\,\Psi(z,z')\,dz' = -\beta\frac{\partial}{\partial z}\int_{-L}^{0}\cos\beta z'\,\Psi(z,z')\,dz'$$

$$+\beta\frac{\partial}{\partial z}\int_{0}^{L}\cos\beta z'\,\Psi(z,z')\,dz'$$

$$= \beta\left[\left(\cos\beta z'\,\Psi(z,z')\right)\Big|_{-L}^{0}\right.$$

$$\left.+\beta\int_{-L}^{0}\sin\beta z'\,\Psi(z,z')\,dz'\right]$$

$$-\beta\left[\left(\cos\beta z'\,\Psi(z,z')\right)\Big|_{0}^{L}\right.$$

$$\left.-\beta\int_{0}^{L}\sin\beta z'\,\Psi(z,z')\,dz'\right]$$

$$= \beta\Big[2\,\Psi(z,0)$$

$$-\cos\beta L\left[\Psi(z,L)+\Psi(z,-L)\right]\Big]$$

$$-\beta^2\int_{-L}^{L}\sin\beta|z'|\,\Psi(z,z')\,dz'$$

Therefore

$$\cos\beta L\,D\int_A G_{oz'}\,\Psi(z,z')\,dz' = 2\beta\cos\beta L\,\Psi(z,0)$$

$$-\beta\cos^2\beta L\left[\Psi(z,L)+\Psi(z,-L)\right]$$

$$-\cos\beta L\sin\beta L\,\beta^2\left[\int_A\Psi(z,z')\,dz'\right] \quad \ldots\text{(VI-7)}$$

Finally substituting Eqs. (VI-5) and (VI-7) into Eq. (VI-2) gives

$$D\left[M_1(z)\right] = -\beta\left[\Psi(z,L)+\Psi(z,-L)-2\cos\beta L\,\Psi(z,0)\right]$$

$$\ldots\text{(VI-8)}$$

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