ELECTROMAGNETIC WAVES WITHIN NON-UNIFORM BOUNDARIES AND IN INHOMOGENEOUS ISOTROPIC MEDIA

by

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For an axially symmetric periodic structure with a slowly varying radius, an approximate wave equation is derived which is separable. The field problem is then reduced to finding the solution to Hill's equation.

A treatment of electromagnetic waves in media with characteristics possessing finite discontinuities in the direction of propagation is developed. The development avoids the use of explicit boundary conditions. To illustrate the method, three examples are given.

This method is extended to include media with characteristics possessing finite discontinuities in, and transverse to, the direction of propagation. Two examples are given. In the first an E-wave solution is found for a cylindrical waveguide loaded periodically with dielectric disc, the disc radius being smaller than the cavity radius. Two methods of solution are offered: one is a first mode approximation and the other is an approximate series solution.

A short comparison is made between experimental measurements made on dielectric loaded periodic structures of the forementioned type and theoretical predictions based on the first mode approximation.

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A treatment of electromagnetic waves in media with characteristics possessing finite discontinuities in the direction of propagation is developed. The development avoids the use of explicit boundary conditions. To illustrate the method, three examples are given.

This method is extended to include media with characteristics possessing finite discontinuities in, and transverse to, the direction of propagation. Two examples are given. In the first an E-wave solution is found for a cylindrical waveguide loaded periodically with dielectric discs, each with a centrally located hole. In the second example, an H-wave solution is found in a cylindrical resonant cavity containing a centrally located solid dielectric disc, the disc radius being smaller than the cavity radius. Two methods of solution are offered: one is a first mode approximation and the other is an approximate series solution.

A short comparison is made between experimental measure-

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1. INTRODUCTION

The topic of this thesis arose during an investigation into the wave solution in a dielectric loaded periodic structure of the type illustrated in Figure 1.1. A central hole is present

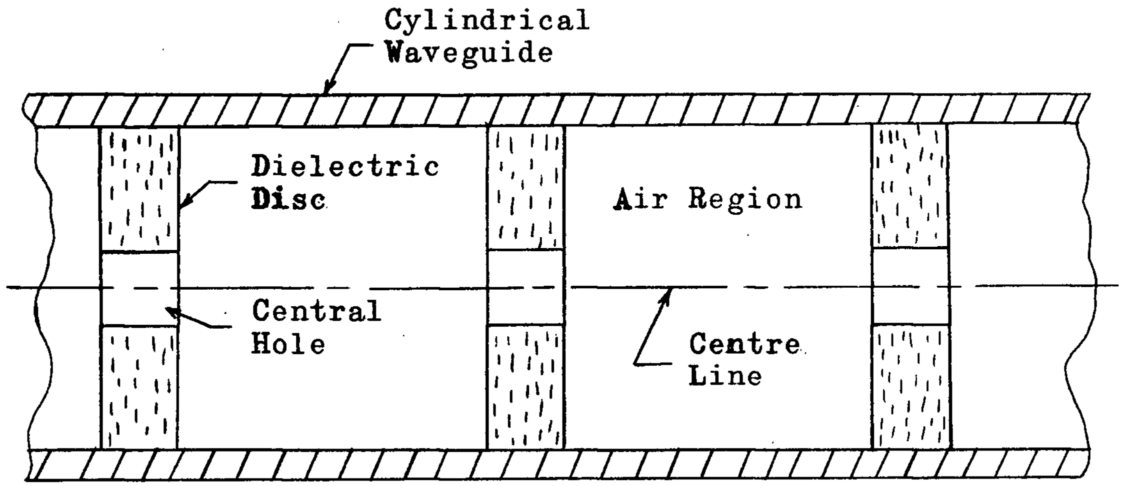

Fig. 1.1. Cross Section of a Dielectric Loaded Periodic Structure

in each disc to allow electrons to pass along the axis of the waveguide. As a result, the structure may be used in beam-couplers, i.e. linear accelerators, travelling wave tubes and backward wave oscillators.

In principle, the field problem in this type of structure can be solved by solving in each homogeneous region the differential wave equations developed from Maxwell's equations and by matching the solutions for the different regions at the boundaries.[1] Due to the excessive labour involved in any numerical work carried out to establish a match at all the boundaries, for practical purposes the method is not entirely satisfactory. Consequently, in design work it is customary to simplify the problem by using the solid disc model[1] and/or the anisotropic model[1] to approximate the structure.

These models oversimplify the problem and it was thought that a different approach might be useful. Instead of considering the medium inside the waveguide as being made up of homogeneous sections, it was decided that an investigation should be carried out with the emphasis shifted to the fact that the medium as a whole is inhomogeneous. In other words, the permittivity is a function of the spatial parameters.

This shift in emphasis changes the problem from solving a simple wave equation in each region and matching solutions at the boundaries to solving a wave equation which holds throughout the waveguide. Although the matching problem is explicitly removed by viewing the medium as a whole, the wave equation to be solved is more complex since it has spatially dependent coefficients. In fact, to add to the complexity, the coefficients have finite discontinuities in the axial and radial directions because the permittivity has such discontinuities. Also, since in some problems partial differentials of the permittivity occur in the coefficients, at points where the permittivity has finite discontinuities impulses may occur in the coefficients. Section 5 is devoted to finding approximate solutions to wave equations having coefficients which have finite discontinuities and impulses occurring in the axial and radial directions. Two solution methods are offered; one is a first mode approximation and the other is an approximate series solution.

In order to gain experience in treating wave equations with coefficients having finite discontinuities and impulses, some problems which have discontinuities of the permittivity only in the axial direction are examined in section 4. These problems are treated first because they are easier than the problems in which the discontinuities of the permittivity occur in the radial as well as the axial direction. Also, the problems in section 4 may be solved by other methods, thus providing a check on the answers found.

During the course of the investigations of dielectric loaded periodic structures, it was noticed that the wave equations have periodic coefficients and are satisfied by Floquet-type solutions.[2] Since in metal loaded periodic structures the field solution is in the Floquet form, the following question arose. Could a wave equation with periodic coefficients be found for metal loaded periodic structures? This question motivated the investigation of section 3 where such a wave equation is given for an axially symmetric periodic structure with a slowly varying radius.

While examining the field behaviour in the type of structure shown in Figure 1.1, it was established that before a pure E-wave could exist in the structure, the field behaviour must be considerably restricted as is shown in section 2. Further investigation led to some general E- and H-field conditions which result from the spatial dependence of the characteristics of the medium. These conditions are discussed in section 2. Since in an inhomogeneous medium the field behaviour often must be restricted before an E- or H-field can exist, the question was asked if analogous restrictions might exist for waveguides with varying cross sections, such as for example a circular-section waveguide having an axially dependent radius. These anticipated restrictions were found and are given in section 2.

The main purpose of this thesis is to present general methods for the analysis of field problems arising in periodic loaded waveguides. However, much of the theory is by no means restricted to such waveguides. In fact, the primary reason for examining (in section 5) resonant cavities which are partially filled with dielectric material is to demonstrate this point. In addition to the theoretical study, a short comparison is made in the final section between experimental measurements made on dielectric loaded periodic structures and theoretical predictions based on the first mode approximation.

2. ON E- AND H-FIELD CONDITIONS ON ELECTROMAGNETIC WAVES

2.1 General

As is very well known, it is convenient to classify electromagnetic waves which may be propagated in regular hollow waveguides into two categories, namely, waves in which there is no longitudinal component of electric intensity, H- or TE-waves, and waves in which there is no longitudinal component of magnetic intensity, E- or TM-waves. This classification arises naturally from the boundary value problem presented by the walls of the waveguide (which are normally assumed to be perfectly conducting). The discussion of wave propagation is reduced to an eigenvalue problem and it is found that certain eigenfunctions correspond to E-type waves and the rest to H-type waves. It can, also, be shown that the summation of all these E-type and H-type waves form a complete set[3] so that the general problem may be solved.

Very considerable practical use is made of the fact that the wave pattern corresponding to a single eigenfunction may be excited alone and it may be surmised that even if the postulation of E- and H-waves had not been so convenient in the mathematical analysis, it would still have been desirable to invent them.

The topics to be discussed in this section are the restrictions imposed upon other field components of a wave by the postulation that one field intensity component, such as E_z or H_z, is zero.

A number of authors[4-16] have investigated electromagnetic wave problems in waveguides with varying cross sections. For such problems, often the field must satisfy certain restrictions before an E-wave can exist and, likewise, before and H-wave can exist. In this regard, an example in section 2.41, utilizing the equations established in section 2.2, demonstrates the restrictions on an E-wave in a waveguide in which the walls are described by

$$r = f(z)$$

where r is the radial and z is the axial cylindrical coordinate. To within the knowledge of the writer, in previous work[5,8,11,12,14] these restrictions or similar ones are all assumed rather than derived.

Authors[17-20] have noted that pure E- or H-waves cannot exist in an inhomogeneous linear isotropic medium unless there are certain limitations imposed upon the field. In the thesis and paper by Adler[17,18] as well as in the report by Malinovsky and Angelakos[19] these limitations are noted for the case where the properties of the medium are functions of the transverse co-ordinates. Zucker and Cohen[20] discuss in their paper the condition imposed upon an H-wave when the permeability is constant, the conductivity is zero and the permittivity is a function of the spatial coordinates. In the present treatment, necessary conditions for E- and H-waves to exist in a linear inhomogeneous isotropic medium are derived for the case in which the permeability, permittivity and conductivity can all be functions of the spatial coordinates. Under some circumstances these conditions are, also, sufficient as is shown in the example discussed in section 2.44.

2.2 Conditions

If a rectangular coordinate system is oriented so that the z-coordinate is in the direction of propagation (along the axis of the waveguide), by definition the condition for an E-wave to exist is

$$H_z \equiv 0. \tag{2.1}$$

In treating certain E-wave problems, such as the one discussed in section 3, it is advantageous to refer to a system of orthogonal curvilinear coordinates and, then, to express the field vectors in terms of three components in the directions of the unit vectors of the curvilinear system. For such a situation a useful relation which is an equivalent statement to (2.1) is

$$\nabla z . \vec{H} = 0 \tag{2.2}$$

where $\vec{H}$ is the magnetic field intensity vector. The equivalence of (2.1) and (2.2) is obvious since

$$\nabla z = \vec{k}$$

where $\vec{k}$ is the unit vector in the z-direction.

In a similar manner, for H-waves

$$E_z \equiv 0$$

or

$$\nabla z . \vec{E} = 0 \tag{2.3}$$

where $\vec{E}$ is the electric field intensity vector.

For waveguide problems in which the properties of the medium are spatially dependent, the field is, in general, a hybrid of E- and H-waves. Before a pure E-wave can exist in a waveguide filled with a linear inhomogeneous isotropic medium, the necessary condition*

$$\nabla \cdot \left[(\varepsilon - j\frac{\sigma}{\omega})^{-1} \quad \frac{\partial \vec{H}}{\partial z} \right] = 0 \tag{2.4}$$

must be satisfied. In (2.4) ε is the permittivity and σ is the conductivity of the medium and ω is the angular frequency at which the wave oscillates.

Equation (2.4) can be established by employing the wave equation derived in Appendix 1 for $\vec{H}$, namely

$$\nabla(\nabla\cdot\vec{H}) - \nabla^2\vec{H} = \omega^2\mu(\varepsilon - j\frac{\sigma}{\omega})\vec{H}$$

$$+ (\varepsilon - j\frac{\sigma}{\omega})^{-1} \left[\left(\nabla(\varepsilon - j\frac{\sigma}{\omega}) \right) \cdot \left(\nabla \vec{H}\right) - \left(\nabla(\varepsilon - j\frac{\sigma}{\omega})\cdot\nabla\right)\vec{H} \right]$$

where μ is the permeability of the medium. When the coefficients of the component vector in the z-direction are equated, the result is

$$\frac{\partial}{\partial z} (\nabla\cdot\vec{H}) - \nabla^2 H_z = \omega^2\mu(\varepsilon - j\frac{\sigma}{\omega})H_z$$

$$+ (\varepsilon - j\frac{\sigma}{\omega})^{-1}\left[\left(\nabla(\varepsilon - j\frac{\sigma}{\omega})\right)\cdot\left(\frac{\partial \vec{H}}{\partial z}\right) - \left(\nabla(\varepsilon - j\frac{\sigma}{\omega})\cdot\nabla\right)H_z\right]. \tag{2.5}$$

---

* For some cases this condition is, also, sufficient as will be demonstrated in section 2.44.

Since for an E-wave

$$H_z \equiv 0,$$

then from (2.5)

$$\frac{\partial}{\partial z}(\nabla.\vec{H}) = (\varepsilon - j\frac{\sigma}{\omega})^{-1}\nabla(\varepsilon - j\frac{\sigma}{\omega}) \cdot \frac{\partial\vec{H}}{\partial z}$$

or

$$(\varepsilon - j\frac{\sigma}{\omega})^{-1}\left(\nabla\cdot\frac{\partial\vec{H}}{\partial z}\right) - (\varepsilon - j\frac{\sigma}{\omega})^{-2}\left(\nabla(\varepsilon - j\frac{\sigma}{\omega}) \cdot \frac{\partial\vec{H}}{\partial z}\right) = 0.$$

Therefore,

$$\nabla\cdot\left[(\varepsilon - j\frac{\sigma}{\omega})^{-1}\frac{\partial\vec{H}}{\partial z}\right] = 0.$$

Likewise, the necessary condition which must be satisfied before an H-wave can exist in a linear inhomogeneous isotropic medium is

$$\nabla\cdot\left[\mu^{-1}\frac{\partial\vec{E}}{\partial z}\right] = 0. \qquad (2.6)$$

Through the use of the wave equation

$$\nabla(\nabla.\vec{E}) - \nabla^2\vec{E} = \omega^2\mu(\varepsilon - j\frac{\sigma}{\omega})\vec{E} + \mu^{-1}\left[(\nabla\mu)\cdot(\nabla\vec{E}) - (\nabla\mu.\nabla)\vec{E}\right],$$

which is derived in Appendix 1, (2.6) can be established in exactly the same manner as (2.4).

2.3 Common Equation Form

As a point of curiosity, equations (2.2), (2.3), (2.4) and (2.6) can be expressed in the common form

$$\nabla\cdot\left[\eta^{-1}\frac{\partial\vec{F}}{\partial z}\right] = 0. \qquad (2.7)$$

(a) For (2.2) the form (2.7) can be realized in the following manner. If (2.2) is multiplied by μ, the result is

$$\nabla z \cdot \vec{B} = 0 \tag{2.8}$$

where

$$\vec{B} = \mu \vec{H}$$

and $\vec{B}$ is the magnetic flux density vector. Now, (2.8) can be differentiated with respect to z to give

$$\frac{\partial}{\partial z}(\nabla z \cdot \vec{B}) = \nabla z \cdot \frac{\partial \vec{B}}{\partial z} + \nabla \frac{\partial z}{\partial z} \cdot \vec{B} = \nabla z \cdot \frac{\partial \vec{B}}{\partial z} = 0. \tag{2.9}$$

Since no magnetic charges exist,

$$\nabla \cdot \vec{B} = 0. \tag{2.10}$$

Through the differentiation of (2.10) with respect to z, the result is

$$\frac{\partial}{\partial z}(\nabla \cdot \vec{B}) = \nabla \cdot \frac{\partial \vec{B}}{\partial z} = 0. \tag{2.11}$$

At this point, (2.9) is multiplied by z^{-2} and (2.11) is multiplied by z^{-1}; taking the difference of the resulting expressions gives

$$\nabla \cdot \left[z^{-1} \; \frac{\partial \vec{B}}{\partial z} \right] = 0. \tag{2.12}$$

Hence,

$$\eta = z; \; \vec{F} = \vec{B}.$$

It can be readily shown that (2.12) is equivalent to (2.2) or (2.8) by letting ∇ operate on each term of the product $z^{-1} \frac{\partial \vec{B}}{\partial z}$. If this is done,

$$z^{-1}\,\nabla\cdot\,\frac{\partial\vec{B}}{\partial z} - z^{-2}\,\nabla z\cdot\,\frac{\partial\vec{B}}{\partial z} = 0.$$

Since (2.11) must be satisfied,

$$\nabla z\cdot\,\frac{\partial\vec{B}}{\partial z} = 0.$$

Hence,

$$\frac{\partial}{\partial z}\,(\nabla z\cdot\vec{B}) = 0$$

and thus

$$\nabla z\cdot\hat{B} = f(x,y)e^{j\omega t}$$

or

$$B_z = f(x,y)e^{j\omega t}.$$

Since B_z has no z-dependence, this solution for B_z corresponds to a field component that is cut-off at all frequencies and thus cannot be part of a wave except for the case in which the trivial solution,

$$B_z \equiv 0,$$

holds. Therefore,

$$\nabla z\cdot\vec{B} = 0.$$

(b) For (2.3) the form (2.7) can be realized in the following manner. If (2.3) is multiplied by $(\varepsilon - j\frac{\sigma}{\omega})$, the result is

$$\nabla z\cdot\vec{D}' = 0$$

where

$$\vec{D}' = (\varepsilon - j\frac{\sigma}{\omega})\vec{E}.$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot\vec{J}$$

where ρ is the volume charge density and $\vec{J}$ is the current density vector. For a linear medium

$$\vec{J} = \sigma\vec{E}.$$

Consequently, ρ can be expressed as

$$\rho = \nabla\cdot j\frac{\sigma}{\omega}\vec{E}.$$

Since the divergence of the electric flux density vector, $\vec{D}$, gives

$$\nabla\cdot\vec{D} = \rho,$$

then

$$\nabla\cdot\vec{D} = \nabla\cdot(j\frac{\sigma}{\omega}\vec{E}).$$

For a linear medium

$$\vec{D} = \varepsilon\vec{E}.$$

Therefore,

$$\nabla\cdot(\varepsilon - j\frac{\sigma}{\omega})\vec{E} = 0$$

or

$$\nabla\cdot\vec{D}' = 0.$$

Now, it is easily seen that by the substitution of $\vec{D}'$ for $\vec{B}$ in the proof given in (a), the equation

$$\nabla \cdot \left[z^{-1} \ \frac{\partial \vec{D}'}{\partial z} \right] = 0$$

can be derived. Hence,

$$\eta = z, \ \vec{F} = \vec{D}' \ .$$

(c) For (2.4) it is obvious that

$$\eta = \varepsilon - j\frac{\sigma}{\omega} \ , \ \vec{F} = \vec{H}.$$

(d) For (2.6) it is obvious that

$$\eta = \mu, \ \vec{F} = \vec{E}.$$

2.4 Examples of Field Restrictions

In the examples to follow only the field restrictions on E-waves are discussed. Similar treatments can be carried out for H-wave problems.

2.41 A Waveguide with a Radially Varying Cross Section

The investigation is on the field behaviour of an E-wave in a perfectly conducting waveguide in which the walls are described by

$$r = f(z).$$

Furthermore, to simplify the problem, the waveguide is to be filled with a medium that has a constant permeability and permittivity and a zero conductivity.

This example was selected, partly, to demonstrate how the stipulation that only an E-wave is allowed in a waveguide can considerably restrict the behaviour of the field. A further reason was that the results to be obtained will be made use of in the treatment in section 3 of a waveguide with a periodically varying radius.

For the problem in hand, new orthogonal curvilinear co-ordinates (u_1, u_2, u_3) are chosen such that u_1 is constant at the walls of the waveguide. Consequently, the new coordinates can be expressed as

$$\left.\begin{aligned} u_1 &= u_1(r,z) \\ u_2 &= \phi \\ u_3 &= u_3(r,z) \end{aligned}\right\} \qquad (2.13)$$

where ϕ is the cylindrical angular coordinate and u_1, u_2 and u_3 are analytic functions of r, ϕ and z. It is assumed that (2.13) can be solved with respect to r, ϕ and z to give

$$\begin{aligned} r &= r(u_1,u_3) \\ \phi &= u_2 \\ z &= z(u_1,u_3). \end{aligned}$$

The differential elements of distance in the curvilinear system are[21]

$$ds_i = h_i du_i, \quad (i = 1,2,3)$$

where the Lamé coefficients are

$$h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2} \qquad (2.14)$$

with x, y and z being the rectangular coordinates. Since

$$x = r \cos \phi$$
$$y = r \sin \phi,$$

then

$$\frac{\partial x}{\partial u_1} = \frac{\partial r}{\partial u_1} \cos \phi, \quad \frac{\partial x}{\partial u_2} = -r \sin \phi, \quad \frac{\partial x}{\partial u_3} = \frac{\partial r}{\partial u_3} \cos \phi$$

$$\frac{\partial y}{\partial u_1} = \frac{\partial r}{\partial u_1} \sin \phi, \quad \frac{\partial y}{\partial u_2} = r \cos \phi, \quad \frac{\partial y}{\partial u_3} = \frac{\partial r}{\partial u_3} \sin \phi .$$

Therefore, from (2.14)

$$h_1 = \sqrt{\left(\frac{\partial r}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}$$

$$h_2 = r$$

$$h_3 = \sqrt{\left(\frac{\partial r}{\partial u_3}\right)^2 + \left(\frac{\partial z}{\partial u_3}\right)^2} .$$

Hence,

$$h_i = h_i(u_1, u_3). \tag{2.15}$$

From (2.2) the expression

$$\frac{1}{h_1} \frac{\partial z}{\partial u_1} H_1 + \frac{1}{h_3} \frac{\partial z}{\partial u_3} H_3 = 0$$

is obtained where H_1 is the component of $\hat{H}$ in the u_1 direction and H_3 is the component in the u_3 direction. Throughout the remainder of this example it will be assumed that $\frac{\partial z}{\partial u_1}$ and $\frac{\partial z}{\partial u_3}$ exist and are not identically zero. Hence,

$$H_1 = -\frac{h_1}{h_3} \xi H_3 \tag{2.16}$$

where

$$\xi = \frac{\partial z}{\partial u_3} \left(\frac{\partial z}{\partial u_1}\right)^{-1} .$$

Maxwell's equations are

$$\nabla \times \vec{E} = -j\omega\vec{B} \tag{2.17}$$

$$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D} . \tag{2.18}$$

For the present example the conductivity is zero. Consequently, if the curl of (2.18) is taken and (2.17) is used to eliminate $\vec{E}$, $\vec{H}$ must satisfy

$$\omega^2\mu\varepsilon\ \vec{H} = \nabla \times (\nabla \times \vec{H}). \tag{2.19}$$

From (2.19) the scalar equation obtained by equating the coefficients of the component vector in the u_3 direction is

$$\omega^2\mu\varepsilon\ H_3 = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}\left[\frac{h_2}{h_1 h_3}\left(\frac{\partial}{\partial u_3}(h_1H_1) - \frac{\partial}{\partial u_1}(h_3H_3)\right)\right]\right.$$

$$\left. - \frac{\partial}{\partial u_2}\left[\frac{h_1}{h_2 h_3}\left(\frac{\partial}{\partial u_2}(h_3H_3) - \frac{\partial}{\partial u_3}(h_2H_2)\right)\right]\right] \tag{2.20}$$

where H_2 is the component of $\vec{H}$ in the u_2 direction. Since h_1, h_2 and h_3 are not functions of u_2, as is seen from (2.15), the term involving H_2 in (2.20) can always be operated on first of all by u_2. If this operation is done first, then $\frac{\partial H_2}{\partial u_2}$ can be considered in place of H_2. Consequently, as will be shown, H_2 can be eliminated from (2.20) by using

$$\nabla \cdot \vec{H} = 0 . \tag{2.21}$$

From (2.21)

$$\frac{\partial H_2}{\partial u_2} = - \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial u_1} (h_2 h_3 H_1) + \frac{\partial}{\partial u_3} (h_1 h_2 H_3)\right]. \tag{2.22}$$

Now, if (2.16) and (2.22) are used to eliminate H_1 and H_2 from (2.20), the resulting differential equation is

$$\omega^2 \mu\varepsilon\, H_3 = \frac{1}{h_1 h_2}\left[- \frac{\partial}{\partial u_1}\left[\frac{h_2}{h_1 h_3} \frac{\partial}{\partial u_1} (h_3 H_3)\right] - \frac{h_1}{h_2} \frac{\partial^2 H_3}{\partial u_2{}^2} \right.$$

$$- \frac{h_1}{h_2 h_3} \frac{\partial}{\partial u_3}\left[\frac{h_2}{h_1 h_3} \frac{\partial}{\partial u_3} (h_1 h_2 H_3)\right] - \frac{\partial}{\partial u_1}\left[\frac{h_2}{h_1 h_3} \frac{\partial}{\partial u_3}\left(\frac{h_1{}^2}{h_3} \xi H_3\right)\right]$$

$$\left. + \frac{h_1}{h_2 h_3} \frac{\partial}{\partial u_3}\left[\frac{h_2}{h_1 h_3} \frac{\partial}{\partial u_1} (h_1 h_2 \xi H_3)\right] \right]. \tag{2.23}$$

At this point the boundary conditions will be introduced. They are[22]

$$\hat{n} \times \vec{E} = 0 \tag{2.24}$$

$$\hat{n} \cdot \vec{B} = 0 \tag{2.25}$$

$$\hat{n} \cdot \vec{D} = \rho_s \tag{2.26}$$

$$\hat{n} \times \vec{H} = \vec{K} \tag{2.27}$$

where

$\hat{n}$ = the unit vector normal to the surface of the conductor and directed into the region where the field exists,

ρ_s = the surface charge density,

$\vec{K}$ = the surface current density.

If E_2 is the component of $\hat{E}$ in the u_2 direction and E_3 is the component in the u_3 direction, from (2.24) on the boundary

$$E_2 = 0, \; E_3 = 0.$$

From (2.25)

$$H_1 = 0 \tag{2.28}$$

and thus from (2.16)

$$H_3 = 0. \tag{2.29}$$

Equation (2.18) yields the scalar equations[23]

$$j\omega\varepsilon \, E_1 = \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (h_3 H_3) - \frac{\partial}{\partial u_3} (h_2 H_2) \right] \tag{2.30}$$

$$j\omega\varepsilon \, E_2 = \frac{1}{h_1 h_3} \left[\frac{\partial}{\partial u_3} (h_1 H_1) - \frac{\partial}{\partial u_1} (h_3 H_3) \right] \tag{2.31}$$

$$j\omega\varepsilon \, E_3 = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (h_2 H_2) - \frac{\partial}{\partial u_2} (h_1 H_1) \right]$$

where E_1 is the component of $\hat{E}$ in the u_1 direction. Since H_1, H_3, E_2 and E_3 are zero on the boundary, all of their derivatives in the u_2 and u_3 directions are zero on the boundary. Hence, at the boundary (2.30) gives

$$j\omega\varepsilon \, E_1 = - \frac{1}{h_2 h_3} \frac{\partial}{\partial u_3} (h_2 H_2) \tag{2.32}$$

and (2.31) gives

$$\frac{\partial}{\partial u_1}(h_3 H_3) = 0$$

or with the use of (2.29)

$$\frac{\partial H_3}{\partial u_1} = 0. \qquad (2.33)$$

According to (2.16), (2.29) and (2.33)

$$\frac{\partial H_1}{\partial u_1} = 0.$$

Hence, (2.22) shows that on the boundary

$$\frac{\partial H_2}{\partial u_2} = 0. \qquad (2.34)$$

If (2.32) is differentiated with respect to u_2, the result is

$$\frac{\partial E_1}{\partial u_2} = -\frac{1}{j\omega\varepsilon\ h_2 h_3}\left[h_2 \frac{\partial}{\partial u_3}\left(\frac{\partial H_2}{\partial u_2}\right) + \frac{\partial h_2}{\partial u_3}\ \frac{\partial H_2}{\partial u_2}\right]$$

and thus from (2.34)

$$\frac{\partial E_1}{\partial u_2} = 0. \qquad (2.35)$$

If $\hat{i}_1$, $\hat{i}_2$ and $\hat{i}_3$ are the unit vectors in the u_1, u_2 and u_3 directions respectively, then

$$\hat{n} = -\hat{i}_1$$

and from (2.27), (2.28) and (2.29)

$$\hat{K} = K_1\hat{i}_1 + K_2\hat{i}_2 + K_3\hat{i}_3 = \begin{vmatrix} \hat{i}_1 & \hat{i}_2 & \hat{i}_3 \\ -1 & 0 & 0 \\ 0 & H_2 & 0 \end{vmatrix} = -H_2\hat{i}_3.$$

Therefore,

$$K_1 \equiv 0, \; K_2 \equiv 0$$

and

$$K_3 = -H_2.$$

Consequently, the surface current on the waveguide walls must flow in the u_3 direction and from (2.34) it can be seen that K_3 cannot have any u_2 dependence.

According to (2.26)

$$\rho_s = -D_1 = -\varepsilon E_1$$

and from (2.35) it can be noted that ρ_s is independent of u_2.

Now, it is convenient to return to (2.23). H_3 must satisfy (2.23) and the boundary conditions (2.29) and (2.33). Further boundary conditions can be established by using the form of (2.23)

$$\frac{\partial^2 H_3}{\partial u_1{}^2} = a_0 \frac{\partial^2 H_3}{\partial u_2{}^2} + a_1 \frac{\partial^2 H_3}{\partial u_3{}^2} + a_2 \frac{\partial^2 H_3}{\partial u_3 \partial u_1} + a_3 \frac{\partial H_3}{\partial u_1} + a_4 \frac{\partial H_3}{\partial u_3} + a_5 H_3 \tag{2.36}$$

where

$$a_i = a_i(u_1, u_3), \; (i = 0, 1, \ldots, 5).$$

In fact, it will be shown that on the boundary

$$\frac{\partial^n H_3}{\partial u_1{}^n} = 0, \quad (n = 0, 1, 2, \ldots) . \tag{2.37}$$

Equation (2.37) can be established by induction. First of all, through the use of (2.29) and (2.33) and the fact that u_2 and u_3 lie on the boundary, (2.36) gives

$$\frac{\partial^2 H_3}{\partial u_1{}^2} = 0 .$$

The assumption is, now, made that

$$\frac{\partial^n H_3}{\partial u_1{}^n} = 0, \quad (n = 0, 1, \ldots, N - 1).$$

If (2.36) is differentiated N-2 times with respect to u_1, then

$$\frac{\partial^N H_3}{\partial u_1{}^N} = \frac{\partial^{N-2}}{\partial u_1{}^{N-2}}\left[a_0 \frac{\partial^2 H_3}{\partial u_2{}^2}\right] + \frac{\partial^{N-2}}{\partial u_1{}^{N-2}}\left[a_1 \frac{\partial^2 H_3}{\partial u_3{}^2}\right] + \frac{\partial^{N-2}}{\partial u_1{}^{N-2}}\left[a_2 \frac{\partial^2 H_3}{\partial u_3 \partial u_1}\right]$$
$$+ \frac{\partial^{N-2}}{\partial u_1{}^{N-2}}\left[a_3 \frac{\partial H_3}{\partial u_1}\right] + \frac{\partial^{N-2}}{\partial u_1{}^{N-2}}\left[a_4 \frac{\partial H_3}{\partial u_3}\right] + \frac{\partial^{N-2}}{\partial u_1{}^{N-2}}\left[a_5 H_3\right] . \tag{2.38}$$

Since only derivatives up to N-1 with respect to u_1 occur on the right side of (2.38),

$$\frac{\partial^N H_3}{\partial u_1{}^N} = 0$$

and thus (2.37) is established.

Therefore, on the boundary

$$\frac{\partial^n H_3}{\partial u_1{}^i \, \partial u_2{}^j \, \partial u_3{}^k} = 0, \ (n = i + j + k = 0, 1, 2, \ldots).$$

This result is easily verified since from (2.37) $\dfrac{\partial^i H_3}{\partial u_1{}^i}$ is zero and thus any change in $\dfrac{\partial^i H_3}{\partial u_1{}^i}$ on the boundary must, also, be zero.

Since H_3 and all of its derivatives are zero on the boundary, from the three dimensional Taylor series expansion the only analytic solution to (2.23) is

$$H_3 \equiv 0. \tag{2.39}$$

Hence, from (2.16)

$$H_1 \equiv 0. \tag{2.40}$$

As a consequence, from (2.31)

$$E_2 \equiv 0. \tag{2.41}$$

Through the use of (2.22), (2.39) and (2.40)

$$\frac{\partial H_2}{\partial u_2} \equiv 0.$$

Equation (2.17) yields the scalar equations

$$j\omega\mu H_1 = -\frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3E_3) - \frac{\partial}{\partial u_3}(h_2E_2)\right] \qquad (2.42)$$

$$j\omega\mu H_2 = -\frac{1}{h_1 h_3}\left[\frac{\partial}{\partial u_3}(h_1E_1) - \frac{\partial}{\partial u_1}(h_3E_3)\right]$$

$$j\omega\mu H_3 = -\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2E_2) - \frac{\partial}{\partial u_2}(h_1E_1)\right]. \qquad (2.43)$$

By means of (2.40), (2.41) and (2.42)

$$\frac{\partial E_3}{\partial u_2} \equiv 0$$

and from (2.39), (2.41) and (2.43)

$$\frac{\partial E_1}{\partial u_2} \equiv 0.$$

Therefore, it can be concluded that before an analytic E-wave solution can exist in the structure under examination, it is necessary that the fields be restricted to having no u_2 dependence and

$$H_1 \equiv 0,\ E_2 \equiv 0,\ H_3 \equiv 0.$$

2.42 A Medium with a Radial and Axial Dependence

The example to be discussed is the case in which an E-wave exists in a cylindrical waveguide which is filled with a medium having a spatially dependent effective permittivity and a constant permeability. In particular, the effective permittivity satisfies the equation

$$\varepsilon' = \varepsilon - j\,\frac{\sigma}{\omega} = f(r,z)$$

with $\frac{\partial \varepsilon'}{\partial r}$ and $\frac{\partial \varepsilon'}{\partial z}$ existing and being not identically zero.

In an inhomogeneous medium, if an E-wave exists, some rather restricting requirements might have to be met. The present example was selected, partly, to show what type of restrictions might be expected. However, a more immediate reason was that the results will be utilized in section 5, which deals with dielectric loaded structures.

For a waveguide filled with a medium which behaves in the forementioned manner, since

$$\nabla \cdot \vec{H} = 0,$$

from (2.4)

$$\nabla \varepsilon' \cdot \frac{\partial \vec{H}}{\partial z} = 0. \tag{2.44}$$

In cylindrical coordinates (2.44) is

$$\frac{\partial \varepsilon'}{\partial r}\frac{\partial H_r}{\partial z} + \frac{1}{r}\frac{\partial \varepsilon'}{\partial \phi}\frac{\partial H_\phi}{\partial z} + \frac{\partial \varepsilon'}{\partial z}\frac{\partial H_z}{\partial z} = 0 \tag{2.45}$$

where H_r is the radial component and H_ϕ is the angular component of $\vec{H}$. Since the effective permittivity does not have angular dependence and for an E-wave

$$H_z \equiv 0, \tag{2.46}$$

then from (2.45)

$$\frac{\partial \varepsilon'}{\partial r} \; \frac{\partial H_r}{\partial z} = 0.$$

As stated

$$\frac{\partial \varepsilon'}{\partial r} \neq 0,$$

consequently, for an E-wave to exist

$$\frac{\partial H_r}{\partial z} \equiv 0. \tag{2.47}$$

If (2.47) is integrated

$$H_r = f(r,\phi)e^{j\omega t}. \tag{2.48}$$

The solution in (2.48) corresponds to a field component that is cut-off at all frequencies and thus cannot be part of a wave except for the trivial solution,

$$H_r \equiv 0. \tag{2.49}$$

As a result of (2.49), further restrictions on the field can be found through the use of Maxwell's equations, (2.17) and (2.18). In cylindrical coordinates Maxwell's equations are

$$\frac{1}{r} \; \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = - j\omega\mu \; H_r \tag{2.50}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j\omega\mu \; H_\phi \tag{2.51}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \, (rE_\phi) - \frac{1}{r} \frac{\partial E_r}{\partial \phi} = -j\omega\mu \; H_z \tag{2.52}$$

$$\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = j\omega\varepsilon' \ E_r$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = j\omega\varepsilon' \ E_\phi \tag{2.53}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi} = j\omega\varepsilon' \ E_z.$$

The substitution of (2.46) and (2.49) into (2.53) gives

$$E_\phi \equiv 0. \tag{2.54}$$

By means of (2.49), (2.50) and (2.54)

$$\frac{\partial E_z}{\partial \phi} \equiv 0 \tag{2.55}$$

and from (2.46), (2.52) and (2.54)

$$\frac{\partial E_r}{\partial \phi} \equiv 0. \tag{2.56}$$

Furthermore, if (2.51) is differentiated with respect to ϕ, from (2.55) and (2.56)

$$\frac{\partial H_\phi}{\partial \phi} \equiv 0.$$

Therefore, it can be concluded that before an E-wave can exist in the medium under examination, it is necessary that the field has no angular dependence and

$$H_r \equiv 0, \ E_\phi \equiv 0, \ H_z \equiv 0.$$

A point to note is that the restrictions imposed upon the field components, other than

$$H_z \equiv 0,$$

are initially caused by the radial dependence of the effective permittivity. If the effective permittivity is only a function of z, no restrictions result from (2.4) because in (2.45)

$$H_z \equiv 0$$

which forces

$$\frac{\partial \varepsilon'}{\partial z} \frac{\partial H_z}{\partial z} = 0$$

and thus all terms in (2.45) vanish.

2.43 Case of Equation (2.4) Being Not Sufficient

A simple example which demonstrates that (2.4) is not sufficient in general is where

$$\varepsilon = \text{constant}$$

$$\mu = f(y)$$

$$\sigma = 0$$

$$\rho = 0$$

and the magnetic field satisfies the condition

$$\nabla \cdot \vec{H} \Big|_{z=z_0} = 0 \qquad (2.57)$$

at the plane $z=z_0$.

From (2.4)

$$\nabla \cdot \frac{\partial \vec{H}}{\partial z} = \frac{\partial}{\partial z} (\nabla \cdot \vec{H}) = 0.$$

Therefore,

$$\nabla \cdot \vec{H} = f(x,y)e^{j\omega t}.$$

Consequently, to satisfy (2.57),

$$\nabla \cdot \vec{H} = 0 \qquad (2.58)$$

throughout the waveguide. Also, from (2.10)

$$\nabla \cdot \vec{B} = 0$$

and thus

$$\nabla\mu \cdot \vec{H} + \mu\nabla \cdot \vec{H} = \nabla\mu \cdot \vec{H} = 0.$$

Hence,

$$\frac{d\mu}{dy} H_y = 0.$$

Since

$$\frac{d\mu}{dy} \neq 0,$$

then

$$H_y \equiv 0. \qquad (2.59)$$

From (2.58) and (2.59)

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_z}{\partial z} = 0 \qquad (2.60)$$

and if $\vec{E}$ is eliminated from (2.17) and (2.18), the result is

$$\nabla^2 H_x + \omega^2\mu\varepsilon\, H_x = 0 \qquad (2.61)$$

$$\nabla^2 H_z + \omega^2\mu\varepsilon\, H_z = 0. \qquad (2.62)$$

Equations (2.60), (2.61) and (2.62) form a self-consistent set from which H_x and H_z can be solved. In fact, the solution,

$$H_z \equiv 0,$$

is only the trivial solution for H_z. Hence, (2.4) is not sufficient in this case.

2.44 Case of Equation (2.4) Being Sufficient

A simple example which demonstrates that (2.4) may be sufficient is where

$$\varepsilon = \text{constant}$$

$$\mu = f(z)$$

$$\sigma = 0$$

$$\rho = 0$$

and

$$\nabla \cdot \hat{H} \Big|_{z=z_0} = 0.$$

As in section 2.43

$$\nabla\mu \cdot \hat{H} = 0.$$

Hence,

$$\frac{d\mu}{dz} H_z = 0.$$

Since

$$\frac{d\mu}{dz} \neq 0,$$

then

$$H_z \equiv 0.$$

3. AN APPROXIMATE WAVE EQUATION FOR AN AXIALLY SYMMETRIC PERIODIC STRUCTURE WITH A SLOWLY VARYING RADIUS

3.1 General

Basically, there are two types of periodic structures, those with a periodically changing boundary and those with a periodically varying medium. Examples of the latter type will be treated in sections 4 and 5. The present section is devoted to an axially symmetric periodic structure with a slowly varying radius.

In many beam-couplers such as linear accelerators, O-type travelling wave tubes and backward wave oscillators, an E_z field component is necessary. For these devices E-wave solutions are of special interest and for this reason the treatment to follow will be restricted to an examination of E-wave fields. Mention should be made that for H-waves an analogous approach can be followed.

One advantage of structures with angular symmetry is that they have a minimum surface area for any fixed volume and thus have a high intrinsic Q. Besides this, axially symmetric structures are of interest because in such guides E-wave solutions may exist in which all the field components except E_z go to zero on the axis. Consequently, in these structures the electron beam defocusing problem is not as great as in structures such as the sinuous waveguides discussed by Cullen[24], in which field components other than E_z exist along the z-axis.

3.2 Theory

An approximate wave equation is derived which is separable and, as a consequence, it turns out that the field problem can be reduced to finding the solution to Hill's equation[25,26].

The radius of the periodic structure to be investigated varies as

$$r = u_1\left(1 + b\,\Theta\left(\frac{2\pi}{p}z\right)\right) \qquad (3.1)$$

where $\Theta(\frac{2\pi}{p}z)$ is a periodic function with a period p. Also, $0 \le u_1$, $0<b<1$ and $\left|\Theta(\frac{2\pi}{p}z)\right| \le 1$.

A new system of orthogonal curvilinear coordinates (u_1, u_2, u_3) are chosen such that u_1 is constant at the waveguide wall. Also, u_3 is regarded as a function that is perturbed from z. Consequently, the new system to be introduced is

$$\left.\begin{aligned} u_1 &= \frac{r}{1 + b\,\Theta(\frac{2\pi}{p}z)} \\ u_2 &= \phi \\ u_3 &= z + \Delta(r,z)\ . \end{aligned}\right\} \qquad (3.2)$$

If $\Theta(\frac{2\pi}{p}z)$ is expanded in a Taylor series about u_3,

$$\Theta\left(\frac{2\pi}{p}z\right) = \Theta\left(\frac{2\pi}{p}u_3\right) - \frac{2\pi}{p}\Theta'\left(\frac{2\pi}{p}u_3\right)\Delta(r,z) + \ldots$$

where

$$\Theta'\left(\frac{2\pi}{p}u_3\right) = \frac{d\,\Theta(\frac{2\pi}{p}u_3)}{d(\frac{2\pi}{p}u_3)}\ .$$

Therefore,

$$r = u_1\left(1 + b\,\Theta(\tfrac{2\pi}{p}z)\right) = u_1\left(1 + b\,\Theta(\tfrac{2\pi}{p}u_3)\right) - \tfrac{2\pi}{p}u_1 b\ \ \Theta'(\tfrac{2\pi}{p}u_3)\Delta(r,z) + \cdots \tag{3.3}$$

or

$$r = u_1\left(1 + b\,\Theta(\tfrac{2\pi}{p}u_3)\right) - \Delta r$$

where Δr is the perturbation of r in going from z to u_3 in the argument of $\Theta(\frac{2\pi}{p}z)$. For the types of problems to be considered, $\Theta(\frac{2\pi}{p}z)$ is to vary slowly enough to insure that

$$u_1\left(1 + b\,\Theta(\tfrac{2\pi}{p}z)\right) \cong u_1\left(1 + b\,\Theta(\tfrac{2\pi}{p}u_3)\right). \tag{3.4}$$

Hence,

$$|\Delta r| \ll r$$

and

$$\left|\tfrac{2\pi}{p}u_1 b\ \Theta'(\tfrac{2\pi}{p}u_3)\Delta(r,z)\right| \ll u_1\left(1 + b\,\Theta(\tfrac{2\pi}{p}u_3)\right). \tag{3.5}$$

Also, according to (3.4)

$$r \cong u_1\left(1 + b\,\Theta(\tfrac{2\pi}{p}u_3)\right). \tag{3.6}$$

A further restriction is that the approximation

$$\Delta(r,z) \cong \Delta\left[u_1\left(1 + b\,\Theta(\tfrac{2\pi}{p}u_3)\right),\ u_3\right] \tag{3.7}$$

must be satisfied. In other words, $\Delta(r,z)$ must vary slowly with respect to z. From a Taylor series expansion,

$$\Delta(r,z) = \Delta\left[u_1\left(1+b\Theta(\tfrac{2\pi}{p}u_3)\right), u_3\right] + (z-u_3)\left.\frac{\partial\Delta}{\partial z}\right|_{\substack{z=u_3\\ r=u_1\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)}}$$

$$+\left[r - u_1\left(1+b\Theta(\tfrac{2\pi}{p}u_3)\right)\right]\left.\frac{\partial\Delta}{\partial r}\right|_{\substack{z=u_3\\ r=u_1\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)}} + \ldots. \tag{3.8}$$

To within a first order approximation from (3.3)

$$r - u_1\left(1+b\Theta(\tfrac{2\pi}{p}u_3)\right) \cong -\frac{2\pi}{p}u_1 b\,\Theta'(\tfrac{2\pi}{p}u_3)\Delta(r,z).$$

Therefore, (3.8) becomes

$$\Delta(r,z) = \Delta\left[u_1\left(1+b\Theta(\tfrac{2\pi}{p}u_3)\right), u_3\right] - \Delta(r,z)\left.\frac{\partial\Delta}{\partial z}\right|_{\substack{z=u_3\\ r=u_1\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)}}$$

$$-\frac{2\pi}{p}u_1 b\,\Theta'(\tfrac{2\pi}{p}u_3)\Delta(r,z)\left.\frac{\partial\Delta}{\partial r}\right|_{\substack{z=u_3\\ r=u_1\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)}} + \ldots .$$

As a result, in order to satisfy (3.7)

$$\left|\left.\frac{\partial\Delta}{\partial z}\right|_{\substack{z=u_3\\ r=u_1\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)}} + \frac{2\pi}{p}u_1 b\Theta'(\tfrac{2\pi}{p}u_3)\left.\frac{\partial\Delta}{\partial r}\right|_{\substack{z=u_3\\ r=u_1\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)}}\right| << 1$$

or

$$\left| \frac{\partial \Delta \left[u_1 \left(1+b\Theta(\frac{2\pi}{p}u_3) \right), u_3 \right]}{\partial u_3} \right| << 1. \qquad (3.9)$$

From (3.2), (3.6) and (3.7) the old coordinates can be approximately expressed in terms of the new coordinates as

$$\left.\begin{aligned} r &\cong u_1 \left(1+b\Theta(\tfrac{2\pi}{p}u_3) \right) \\ \phi &= u_2 \\ z &\cong u_3 - \Delta\left[u_1 \left(1+b\Theta(\tfrac{2\pi}{p}u_3) \right), u_3 \right] . \end{aligned}\right\} \qquad (3.10)$$

A differential change in a vector $\vec{R}$ can be written as

$$d\vec{R} = dx\ \vec{i} + dy\ \vec{j} + dz\ \vec{k} = du_1\vec{a}_1 + du_2\vec{a}_2 + du_3\vec{a}_3$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors in the x, y and z directions respectively and $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ are the unitary vectors in the u_1, u_2 and u_3 directions respectively. Therefore,

$$\vec{a}_i = h_i\vec{i}_i = \frac{\partial x}{\partial u_i}\ \vec{i} + \frac{\partial y}{\partial u_i}\ \vec{j} + \frac{\partial z}{\partial u_i}\ \vec{k},\ (i = 1, 2, 3) \qquad (3.11)$$

where, as already mentioned in section 2.51, $\vec{i}_i$ is the unit vector in the u_i direction and

$$h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2}\ .$$

Since

$$x = r \cos \phi$$
$$y = r \sin \phi,$$

then from (3.10)

$$\left.\begin{aligned} h_1 &\cong \sqrt{\left(1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right)^2 + \left(\frac{\partial\Delta}{\partial u_1}\right)^2} \\ h_2 &\cong u_1\left(1 + b\Theta\left(\frac{2\pi}{p}u_3\right)\right) \\ h_3 &\cong \sqrt{\left(1 - \frac{\partial\Delta}{\partial u_3}\right)^2 + \left(\frac{2\pi}{p}u_1 b\Theta'\left(\frac{2\pi}{p}u_3\right)\right)^2} \end{aligned}\right\} \quad (3.12)$$

and

$$\left.\begin{aligned} \vec{i}_1 &\cong \frac{\left(1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right)\left(\cos u_2\vec{i} + \sin u_2\vec{j}\right) - \frac{\partial\Delta}{\partial u_1}\vec{k}}{\sqrt{\left(1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right)^2 + \left(\frac{\partial\Delta}{\partial u_1}\right)^2}} \\ \vec{i}_2 &= -\sin u_2\vec{i} + \cos u_2\vec{j} \\ \vec{i}_3 &\cong \frac{\frac{2\pi}{p}u_1 b\Theta'\left(\frac{2\pi}{p}u_3\right)\left(\cos u_2\vec{i} + \sin u_2\vec{j}\right) + \left(1 - \frac{\partial\Delta}{\partial u_3}\right)\vec{k}}{\sqrt{\left(1 - \frac{\partial\Delta}{\partial u_3}\right)^2 + \left(\frac{2\pi}{p}u_1 b\Theta'\left(\frac{2\pi}{p}u_3\right)\right)^2}}. \end{aligned}\right\} (3.13)$$

Therefore, from (3.13)

$$\vec{i}_1\cdot\vec{i}_2 = 0$$

$$\vec{i}_2\cdot\vec{i}_3 = 0$$

$$\vec{i}_1\cdot\vec{i}_3 \cong \frac{\frac{2\pi}{p}u_1 b\Theta'\left(\frac{2\pi}{p}u_3\right)\left(1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right) - \frac{\partial\Delta}{\partial u_1}\left(1 - \frac{\partial\Delta}{\partial u_3}\right)}{\sqrt{\left(1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right)^2 + \left(\frac{\partial\Delta}{\partial u_1}\right)^2}\sqrt{\left(1 - \frac{\partial\Delta}{\partial u_3}\right)^2 + \left(\frac{2\pi}{p}u_1 b\Theta'\left(\frac{2\pi}{p}u_3\right)\right)^2}}.$$

Hence, $\vec{i}_1$ and $\vec{i}_3$ are approximately orthogonal to one another if

$$\frac{\partial\Delta}{\partial u_1}\left|1 - \frac{\partial\Delta}{\partial u_3}\right| = \frac{2\pi}{p}u_1 b\Theta'\left(\frac{2\pi}{p}u_3\right)\left|1 + b\Theta\left(\frac{2\pi}{p}u_3\right)\right| . \tag{3.14}$$

Due to (3.9), (3.14) can be simplified to

$$\frac{\partial\Delta}{\partial u_1} \cong \frac{2\pi}{p}u_1 b\Theta'\left(\frac{2\pi}{p}u_3\right)\left|1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right| . \tag{3.15}$$

Since along the axis of the waveguide Δ must be zero, integrating (3.15) gives

$$\Delta \cong \frac{1}{2}\left(\frac{2\pi}{p}\right)u_1{}^2 b\Theta'\left(\frac{2\pi}{p}u_3\right)\left|1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right| . \tag{3.16}$$

To insure that the wave equation is separable, the restriction

$$\left|\frac{2\pi}{p}u_1 b\Theta'\left(\frac{2\pi}{p}u_3\right)\right|^2 << 1 \tag{3.17}$$

is made. Consequently, from (3.12) along with (3.9)

$$\left.\begin{aligned} h_1 &\cong 1+b\Theta\left(\frac{2\pi}{p}u_3\right) \\ h_2 &\cong u_1\left|1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right| \\ h_3 &\cong 1 \end{aligned}\right\} \tag{3.18}$$

and, as will be shown, for these Lamē coefficients a separable solution exists.

Through the use of (3.16) and (3.17) it can be seen that (3.5) is satisfied. Now, (3.16) is differentiated with respect to u_3 and the resulting expression, when substituted into (3.9), gives the condition

$$\left| \frac{1}{2} \left(\frac{2\pi}{p}\right)^2 u_1{}^2 b^2 \Theta'\left(\frac{2\pi}{p}u_3\right)^2 + \frac{1}{2}\left(\frac{2\pi}{p}\right)^2 u_1{}^2 b \Theta''\left(\frac{2\pi}{p}u_3\right)\left(1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right)\right| << 1. \tag{3.19}$$

If (3.17) is kept in mind, then (3.19) is satisfied if

$$\left|\left(\frac{2\pi}{p}u_1\right)^2 b\ \Theta''\left(\frac{2\pi}{p}u_3\right)\left(1+b\Theta\left(\frac{2\pi}{p}u_3\right)\right)\right| << 1. \tag{3.20}$$

The conditions imposed on the waveguide parameters are not as restricting as they might appear. For example, a check will show that the parameters, given in (3.1), of a structure with a radius varying as

$$r = .36(1 + .5 \cos z)$$

easily satisfy the conditions. Such a waveguide has a noteworthy amount of loading, since b is a measure of the loading.

As already established in section 2.41, before an E-wave can exist in the type of structure being treated, the field can have no u_2 dependence and

$$H_1 \equiv 0,\ E_2 \equiv 0,\ H_3 \equiv 0.$$

Therefore, from Maxwell's equations

$$\left.\begin{aligned} j\omega\varepsilon_1\ E_1 &= -\frac{1}{h_2 h_3}\ \frac{\partial(h_2H_2)}{\partial u_3} \\ j\omega\varepsilon_1\ E_3 &= \frac{1}{h_1 h_2}\ \frac{\partial(h_2H_2)}{\partial u_1} \\ j\omega\mu_1\ H_2 &= -\frac{1}{h_1 h_3}\left[\frac{\partial(h_1E_1)}{\partial u_3} - \frac{\partial(h_3E_3)}{\partial u_1}\right]. \end{aligned}\right\} \tag{3.21}$$

where μ_1 and ε_1 are constant. Eliminating E_1 and E_3 from (3.21) and using the h_i's given in (3.18) yields

$$u_1 \frac{\partial}{\partial u_1}\left[\frac{1}{u_1}\frac{\partial}{\partial u_1}(h_2H_2^*)\right] + h_1{}^2\left[\frac{\partial^2}{\partial u_3{}^2}(h_2H_2^*) + \omega^2\mu_1\varepsilon_1 h_2H_2^*\right] = 0 \tag{3.22}$$

where H_2^* is an approximation of H_2 since the h_i's are approximate. Equation (3.22) is separable and $h_2H_2^*$ can be expressed (with time dependence suppressed) as

$$h_2H_2^* = R(u_1)T(u_3) \; . \tag{3.23}$$

If (3.23) is substituted into (3.22), the result is

$$\frac{u_1}{R}\frac{d}{du_1}\left[\frac{1}{u_1}\;\frac{dR}{du_1}\right] + h_1{}^2\left[\frac{1}{T}\frac{d^2T}{du_3{}^2} + \omega^2\mu_1\varepsilon_1\right] = 0 \; .$$

Therefore,

$$u_1\frac{d}{du_1}\left[\frac{1}{u_1}\frac{dR}{du_1}\right] + K^2R = 0 \tag{3.24}$$

$$\frac{d^2T}{du_3{}^2} + \left[\omega^2\mu_1\varepsilon_1 - \left(\frac{K}{h_1}\right)^2\right]T = 0 \; . \tag{3.25}$$

The solution to (3.24) is

$$R = u_1J_1(Ku_1) \; . \tag{3.26}$$

Since at the wall of the waveguide

$$E_3 = 0,$$

the boundary condition to be fulfilled is

$$\frac{1}{Ku_1}\frac{d}{du_1}\left[u_1 J_1(Ku_1)\right]\Bigg|_{u_1 = \bar{u}_1} = J_0\ (K\bar{u}_1) = 0\ .$$

If (3.18) is used to eliminate h_1 from (3.25), then

$$\frac{d^2T}{du_3{}^2} + \left[\omega^2\mu_1\varepsilon_1 - \frac{K^2}{\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)^2}\right]T = 0 \qquad (3.27)$$

and (3.27) is commonly known as Hill's equation. Extensive treatment of Hill's equation exists in the literature. For example, a general solution method is discussed by Whittaker and Watson[27] and, also, by Brillouin.[28]

Now, (3.27) is a linear second-order differential equation with a periodic coefficient. For this type of equation, provided the coefficient is single valued, Floquet's theorem[29] states that a particular solution has the form

$$e^{\gamma u_3}\ P(\tfrac{2\pi}{p}u_3)$$

where $P(\frac{2\pi}{p}u_3)$ is a periodic function with period p.

Since $P(\frac{2\pi}{p}u_3)$ can be expanded in a Fourier series, in the treatment put forth by the forementioned authors, T is expanded in the series

$$T = e^{-j\frac{\chi}{p}u_3}\sum_{n=-\infty}^{\infty} a_n e^{-j\frac{2n\pi}{p}u_3} = \sum_{n=-\infty}^{\infty} a_n e^{-j(\frac{\chi+2n\pi}{p})u_3}$$

(3.28)

where

$$\gamma = -j\frac{\chi}{p} \; .$$

Also, the coefficient of T in (3.27) is expanded in the Fourier series

$$\omega^2\mu_1\varepsilon_1 - \frac{K^2}{\left|1+b\Theta(\frac{2\pi}{p}u_3)\right|^2} = \sum_{n=-\infty}^{\infty} b_n e^{-j\frac{2n\pi}{p}u_3} \; . \tag{3.29}$$

Then, (3.28) and (3.29) are substituted into (3.27) and the expression obtained is

$$-\sum_{m=-\infty}^{\infty} \left(\frac{\chi+2m\pi}{p}\right)^2 a_m e^{-j\left(\frac{\chi+2m\pi}{p}\right)u_3} + \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} b_k a_n e^{-j\left(\frac{\chi+2(k+n)\pi}{p}\right)u_3}$$

$$= 0 \; . \tag{3.30}$$

By multiplying (3.30) by $\frac{1}{p}\, e^{j\left(\frac{\chi+2m\pi}{p}\right)u_3}$ and integrating the resulting expression from 0 to p, the equations

$$-\left(\frac{\chi+2m\pi}{p}\right)^2 a_m + \sum_{n=-\infty}^{\infty} b_{m-n}\, a_n = 0 \tag{3.31}$$

or

$$a_m - \frac{1}{\left(\frac{\chi+2m\pi}{p}\right)^2} \sum_{n=-\infty}^{\infty} b_{m-n}\, a_n = 0, \quad (m = \ldots,-2,-1,0,1,2,\ldots) \tag{3.32}$$

are determined. The determinantal equation for (3.32) is set equal to zero so that a non-trivial solution can be found for the a_m's.

For the case in which the series,

$$\sum_{n=-\infty}^{\infty} b_n ,$$

is absolutely convergent, Whittaker and Watson[30] evaluate the determinantal equation for the set of equations which result when (3.31) is divided by $b_0 - (\frac{\chi+2m\pi}{p})^2$. In this instance, the determinant is

$$\mathcal{D}_1(\chi) = \det \left| B_{mn} \right|$$

where

$$B_{mm} = 1$$

$$B_{mn} = \frac{b_{m-n}}{b_0 - (\frac{\chi+2m\pi}{p})^2} , \quad m \neq n .$$

From Whittaker and Watson's treatment

$$\mathcal{D}_1(\chi) = 1 + K\left[\cot\left(\frac{\chi+\sqrt{b_0}p}{2}\right) - \cot\left(\frac{\chi-\sqrt{b_0}p}{2}\right)\right]. \tag{3.33}$$

The constant K can be calculated at $\chi = 0$. Hence,

$$K = \left(\frac{\mathcal{D}_1(0)-1}{2}\right) \tan\sqrt{b_0}\,\frac{p}{2} . \tag{3.34}$$

Since

$$\mathcal{D}_1(\chi) = 0 ,$$

from (3.33) and (3.34)

$$\sin^2 \frac{\chi}{2} = \mathcal{D}_1(0) \sin^2 \frac{\sqrt{b_0}p}{2} . \tag{3.35}$$

An approximate solution to Hill's equation can be found by truncating the series (3.28) and (3.29). Solutions using perturbation theory have been given by Brillouin[31] as well as by McLachlan[32].

Once T is determined, from (3.21), (3.23) and (3.26) the field is known and

$$\left.\begin{aligned} E_1 &\cong \frac{j}{\omega\varepsilon_1} \frac{J_1(Ku_1)}{\left|1+b\Theta(\frac{2\pi}{p}u_3)\right|} \frac{dT}{du_3} \\ H_2 &\cong \frac{J_1(K\mu_1)T(u_3)}{\left|1+b\Theta(\frac{2\pi}{p}u_3)\right|} \\ E_3 &\cong -j \frac{K}{\omega\varepsilon_1} \frac{J_0(Ku_1)T(u_3)}{\left|1+b\Theta(\frac{2\pi}{p}u_3)\right|} \quad . \end{aligned}\right\} \quad (3.36)$$

The field component, E_z, can be determined from

$$E_z = \vec{k} \cdot \vec{E} = \vec{k} \cdot \vec{i}_1 E_1 + \vec{k} \cdot \vec{i}_3 E_3 \quad .$$

Through the use of (3.13) and (3.15)

$$E_z \cong -\frac{2\pi}{p}u_1 b\, \Theta'(\frac{2\pi}{p}u_3)\ E_1 + E_3$$

or

$$E_z \cong -\frac{j}{\omega\varepsilon_1\left|1+b\Theta(\frac{2\pi}{p}u_3)\right|}\left[\frac{2\pi}{p}\, u_1 b\Theta'(\frac{2\pi}{p}u_3)\ J_1(Ku_1)\ \frac{dT}{du_3} + K\ J_0(Ku_1)T(u_3)\right] .$$

Along the axis of the waveguide

$$\Delta = 0 \quad .$$

and thus

$$z = u_3.$$

Therefore,

$$E_z \cong -j \frac{K}{\omega\varepsilon_1} \frac{T(z)}{\left(1+b\Theta(\frac{2\pi}{p}z)\right)} .$$

For the limiting case in which $b \longrightarrow 0$,

$$u_1 \longrightarrow r$$

$$u_3 \longrightarrow z$$

and (3.36) becomes

$$\left.\begin{aligned} E_r &= \frac{j}{\omega\varepsilon_1} J_1(Kr) \frac{dT}{dz} \\ H_\phi &= J_1(Kr)\, T(z) \\ E_z &= -\frac{jK}{\omega\varepsilon_1} J_0(Kr)\, T(z) \end{aligned}\right\} \qquad (3.37)$$

where T satisfies the differential equation

$$\frac{d^2T}{dz^2} + (\omega^2\mu_1\varepsilon_1 - K^2)\, T = 0.$$

It can be recognized that (3.37) is an E-wave solution in a uniform circular-section waveguide. Hence, in the limit (3.36) is in agreement with the known solution.

Some comments can now be made with regard to periodically perturbing the radius of a circular-section waveguide. As can be seen from (3.36), the u_1 dependence is the same as the r dependence in a uniform waveguide and thus the variation of the cross section perturbs the argument of the Bessel functions. Also, according to (3.36) the amplitude of the field is modulated by the factor

$$\frac{1}{1+b\Theta(\frac{2\pi}{p}u_3)} .$$

With reference to (3.27) it is convenient to define an effective K^2 as

$$K^2_{effective} = K^2 + \Delta K^2$$

where

$$\Delta K^2 = K^2 \left[\frac{1}{\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)^2} - 1 \right]$$

Since $K^2_{effective}$ is an oscillatory perturbation of K^2, the effective propagation factor, defined as

$$\beta^2_{effective} = \omega^2\mu_1\varepsilon_1 - \frac{K^2}{\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)^2} , \tag{3.38}$$

oscillates. This result implies that the phase of the field is modulated.

In a vacuum from (3.38)

$$\beta^2_{effective} = \omega^2\mu_0\varepsilon_0 - \frac{K^2}{\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)^2}$$

and thus

$$\beta^2_{effective} < \omega^2\mu_0\varepsilon_0 = \left(\frac{\omega}{c}\right)^2 \tag{3.39}$$

where c is the speed of light. This situation is not necessarily true in dielectric loaded structures such as the one discussed in section 4 since

$$\beta^2_{effective} = \omega^2\mu_0\varepsilon - K^2$$

and, as a consequence, ε can be increased until

$$\omega^2\mu_0\varepsilon - K^2 > \left(\frac{\omega}{c}\right)^2 .$$

Hence, along the axis of such dielectric loaded structures it is possible to have regions where the phase velocity of the field is less than the speed of light whereas (3.39) indicates that for an empty metal structure with a slowly varying radius such low velocities are impossible.

If the space harmonic where n = 0 in (3.28) is to be used for beam-coupling, it will be shown in what follows that the radius of the waveguide wall must be restricted by the condition

$$u_1 > \left[\frac{M_m c}{v_{ph} X (1+b) \sqrt{\mu_r \varepsilon_r}} \right] p$$

where

M_m = the mth root of $J_0(x)$

v_{ph} = the phase velocity of the 0th space harmonic

μ_r = the relative permeability

ε_r = the relative permittivity.

For many purposes the 0th space harmonic is of greatest interest because in (3.28) a_0 is usually larger in magnitude than a_n for $n \neq 0$.

Since from the boundary condition $E_3 = 0$

$$J_0(Ku_1) = 0,$$

then

$$Ku_1 = M_m. \tag{3.40}$$

If a relation

$$Kp = N \tag{3.41}$$

can be established,

$$u_1 = \frac{M_m}{N} p. \tag{3.42}$$

To find a lower bound for u_1, an upper bound of N must be determined. The latter bound can be found by noting that the phase velocity of the 0th space harmonic is

$$v_{ph} = \frac{\omega}{\chi/p} = \frac{\omega p}{\chi} .$$

Hence,

$$\omega = \frac{\chi v_{ph}}{p} \tag{3.43}$$

and thus

$$\omega^2 \mu_0 \varepsilon_0 = \left(\frac{\chi v_{ph}}{cp}\right)^2 . \tag{3.44}$$

Substituting (3.44) into (3.38) yields

$$\beta^2_{effective} = \mu_r \varepsilon_r \left(\frac{\chi v_{ph}}{cp}\right)^2 - \left(\frac{K}{1+b\Theta(\frac{2\pi}{p}u_3)}\right)^2$$

or

$$\beta^2_{effective} = \frac{1}{p^2\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)^2}\left[\left(\frac{\sqrt{\mu_r\varepsilon_r}\,\chi v_{ph}\left(1+b\Theta(\frac{2\pi}{p}u_3)\right)}{c}\right)^2 - (Kp)^2\right].$$

If

$$Kp > \frac{\sqrt{\mu_r\varepsilon_r}\,\chi v_{ph}(1+b)}{c} ,$$

then

$$\beta^2_{effective} < 0$$

and the field is evanescent everywhere. Therefore, for propagating regions to exist in the waveguide, Kp must be less than

$$\frac{\sqrt{\mu_r \varepsilon_r}\,\chi v_{ph}(1+b)}{c} .$$

Hence, from (3.42)

$$\bar{u}_1 > \left[\frac{M_m c}{\sqrt{\mu_r \varepsilon_r}\,\chi v_{ph}(1+b)}\right] p . \qquad (3.45)$$

Since

$$\omega^2 \mu_0 \varepsilon_0 = \left(\frac{2\pi}{\lambda}\right)^2$$

where λ is the free space wavelength, by means of (3.44)

$$p = \frac{\chi v_{ph}}{2\pi c} \lambda .$$

Eliminating p from (3.45) gives

$$\bar{u}_1 > \left[\frac{M_m}{2\pi \sqrt{\mu_r \varepsilon_r}\,\chi(1+b)}\right] \lambda . \qquad (3.46)$$

If coupling is desired in the first pass band and on the axis, (3.45) can be relaxed since $\chi < \pi$ and $v_{ph} < c$. Even further relaxation can be achieved because $M_m \geq 2.405$ and $b \geq 0$. As a result,

$$\bar{u}_1 > \frac{2.405}{2\pi \sqrt{\mu_r \varepsilon_r}} p = \frac{.38}{\sqrt{\mu_r \varepsilon_r}} p \qquad (3.47)$$

$$\bar{u}_1 > \frac{2.405}{4\pi \sqrt{\mu_r \varepsilon_r}} \lambda = \frac{.16}{\sqrt{\mu_r \varepsilon_r}} \lambda . \qquad (3.48)$$

When the guide is empty, (3.47) and (3.48) become

$$u_1 > .38p \tag{3.49}$$

$$u_1 > .16\lambda.$$

Also, (3.49) can be written as

$$\frac{2\pi u_1}{p} > 2.405. \tag{3.50}$$

It can be seen that when (3.50) is satisfied, in general, (3.17) and (3.20) are not satisfied unless b is small. In other words, to keep the radius of the waveguide slowly varying, b must be a small number. Consequently, when an empty periodic structure with a slowly varying radius is used, the indication is that to achieve heavy loading the beam-coupler would have to be operated in a higher pass-band and/or utilizing a higher space harmonic. As can be surmised from (3.45), by going to a higher pass-band and/or employing a higher order space harmonic the lower bound for u_1 is reduced.

If the waveguide is filled with a dielectric material, such as titania, having a relative permittivity of 93.5, then (3.47) becomes

$$\frac{2\pi u_1}{p} > .25.$$

As a result, (3.17) and (3.20) can be satisfied without "b" necessarily being small. Therefore, when filled with a dielectric such as titania, the structure under study may be heavily loaded while operating in the first pass-band and using the 0th space harmonic.

Dielectric filled structures of this type may have applications in travelling wave tubes since in these tubes dielectric losses do not present a problem.

A structure which has not been examined but could be investigated by adapting the present treatment is illustrated in Figure 3.1. This structure is a combination of the two basic

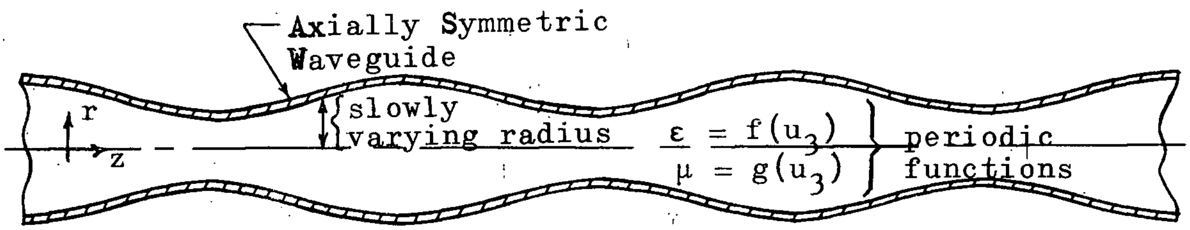

Fig. 3.1. A Structure with a Periodically Varying Boundary and Medium

types of periodic structures mentioned in section 3.1. One specific example is the case in which dielectric regions of constant permittivity are periodically spaced between constant u_3 surfaces.

3.3 Discussion

The theory just developed gives a relatively simple field solution for an axially symmetric periodic structure with a slowly varying radius and, as a consequence, should prove to be of some use in designing specific structures for beam-couplers.

For a structure in which the radius of the walls is not slowly varying, the applicability of the present development has not been examined. However, as a point of speculation, it might be quite meaningful to employ the expressions in (3.36) in order to obtain the field on the axis of the structure.

Section 3.2 has served to introduce linear second-order differential equations with periodic coefficients and has presented a classical treatment for finding the Floquet solution. As will be seen in sections 4 and 5, the same type of mathematical problem arises in the investigation of wave propagation through periodically varying media. In both sections, a modified approach is adopted for solving the differential equations in order to overcome difficulties caused by discontinuities in the media.

4. WAVES IN MEDIA WITH FINITE DISCONTINUITIES IN THE DIRECTION OF PROPAGATION

4.1 General

A treatment of electromagnetic waves in media with characteristics possessing finite discontinuities in the direction of propagation is developed. The development avoids the use of explicit boundary conditions at the discontinuities and in this respect is believed to be novel. To illustrate the method, three examples are given, for which solutions have previously been obtained by the use of explicit boundary conditions. The purpose in investigating these examples is to help clarify the issues involved before proceeding to more complex problems, as for example, problems in which the discontinuities occur transverse to as well as in the direction of propagation. Such problems are discussed in section 5.

In a linear isotropic medium having either the permeability or effective permittivity a function of z, certain general statements can be made about the wave solutions to Maxwell's equations. First of all, the differential wave equations are sufficiently separable for E-, H-and TEM-waves to allow the z-dependent part of the wave solution to be separated from the transverse dependent part[33]. The z-variation due to the permeability or effective permittivity is incorporated into the differential equation satisfied by the z-dependent part of the wave solution. Consequently, the partial differential equation that is satisfied by the transverse dependent part of the wave solution is identical to the corresponding equation that would arise for problems involving a homogeneous medium.

It is easily shown, as done in "Waves in Inhomogeneous Isotropic Media"[33] and in the examples discussed in section 4.5, (e.g. equation (4.27)), that the differential equations containing the z-dependent part of the wave solution can be expressed in the form

$$\frac{d}{dz}\left[a(z)^{-1}\,\frac{dT}{dz}\right] - b(z)\; T = 0 \qquad (4.1)$$

where the coefficients, a(z) and b(z), are specified in the chart shown in Figure 4.1.

Wave	a(z)	-b(z)
TEM	1	$\omega^2\mu\varepsilon'(z) - K^2$
	1	$\omega^2\mu(z)\varepsilon' - K^2$
E	$\varepsilon'(z)$	$\dfrac{\omega^2\mu\varepsilon'(z) - K^2}{\varepsilon'(z)}$
	1	$\omega^2\mu(z)\varepsilon' - K^2$
H	1	$\omega^2\mu\varepsilon'(z) - K^2$
	$\mu(z)$	$\dfrac{\omega^2\mu(z)\varepsilon' - K^2}{\mu(z)}$

K = separation constant

Fig. 4.1. Chart for Coefficients of Equation (4.1)

The following investigation is centered on equation (4.1) for problems in which the coefficients are finite functions and either one or both coefficients have finite discontinuities but otherwise are well behaved. Actually, the following treatment is, also, applicable to cases in which the discontinuities occur in the derivatives of the coefficients.

The Sturm-Liouville problem involving (4.1) has been dealt with for particular cases by Marcuvitz[34] and, also, Angulo[35,36] from an approach using Green's characteristic function. Collin[37] deals with this problem by applying the Rayleigh-Ritz method. Such approaches are applicable to the Sturm-Liouville problem because specific initial and final conditions are known. In the present treatment the situation is quite different since no initial or final conditions are given.

The standard approach for solving (4.1) is to find the solution in each region where the coefficients are well behaved and to use given boundary conditions at the points where the coefficients have finite discontinuities.[38,39,40] The boundary conditions relate the solutions of the different regions. It might be added that this approach may, also, be used for the Sturm-Liouville problem.

In this thesis the problem will be approached from the point of view that (4.1) holds for all z. Consequently, at points where the coefficients have discontinuities, boundary conditions are not explicitly needed. Hence, the solution to (4.1) will be sought without the use of stated boundary conditions. Brillouin[41] adopts the same point of view in his investigation of (4.1) for the case where

$$a(z) = 1$$

$$-b(z) = \text{a rectangular waveform.}$$

However, his method of solution, which will be discussed in section 4.523, leads to difficulties and as a result, his answer only holds for the limiting case in which the discontinuities disappear. These difficulties will be avoided in the discussion to follow.

For the standard approach, the condition on T at a boundary point is that T is continuous or has a given finite discontinuity; at all interior points T is continuous. Consequently, from the point of view adopted herein, it is reasonable to state that the allowable functions for the solution to (4.1) may have finite discontinuities at points and, everywhere else, the allowable functions are to be continuous. At a point where a function has a finite discontinuity, the function is not defined. This lack of definition is easily rectified by arbitrarily assigning the function a value at the point of discontinuity. For example, in Fourier series the mean of the left and right limiting values is given as the value of the function at the point of discontinuity.

An example might be useful to clarify the type of function involved. If T is continuous over the interval $[z_0, z_2]$ but has a kink at z_1, then the derivative of T has a finite discontinuity occurring at z_1. The derivative can be symbolically represented by

$$\frac{dT}{dz} = \begin{cases} f_1(z), & z_0 \le z < z_1 \\ f_2(z), & z_1 < z \le z_2 \end{cases} .$$

Since $\frac{dT}{dz}$ has a finite discontinuity at z_1, then $\frac{d^2T}{dz^2}$ must have an impulse at z_1. In the same vein, higher order derivatives can be discussed.

4.2 Continuity Properties of the Solution

Equation (4.1) contains the information that if T exists, then T and, also, $a(z)^{-1} \frac{dT}{dz}$ are continuous for all z. These continuity properties can be established by integrating (4.1) twice. On the first integration of (4.1) the result is

$$a(z)^{-1} \frac{dT}{dz} = \int^{z} b(\tau)\ T(\tau)\ d\tau \qquad (4.2)$$

and on the second integration of (4.1) the result is

$$T(z) = \int^{z} a(\tau)\ d\tau \int^{\tau} b(t)\ T(t)\ dt. \qquad (4.3)$$

If the assumption is made that T(t) has finite discontinuities, the integral

$$\int^{\tau} b(t)\ T(t)\ dt \qquad (4.4)$$

is continuous. Therefore, from (4.3) T(z) is continuous. Hence, a contradiction exists and thus T(t) cannot have finite discontinuities. If, instead, the assumption is made that T(t) is continuous, the integral (4.4) is, once more, continuous. Therefore, from (4.3) T(z) is continuous. Hence, no contradiction exists and thus

$$T = \text{a continuous function.} \qquad (4.5)$$

Since T is a continuous function, from (4.2)

$$a(z)^{-1} \frac{dT}{dz} = \text{a continuous function.} \qquad (4.6)$$

From (4.6) it is seen that if $a(z)$ has finite discontinuities, $\frac{dT}{dz}$ must also have finite discontinuities. As a result, T will have kinks.

4.3 Boundary Conditions

At the points where the coefficients have discontinuities, the boundary conditions, if desired, can be obtained from (4.5) and (4.6). Hence, provided (4.1) holds for all z, the boundary conditions are contained in (4.1). Consequently, the problem posed when (4.1) holds for all z is identical to the standard problem in which (4.1) holds in the regions where the coefficients are well behaved and in which the given boundary conditions are equal to the boundary conditions obtained from (4.5) and (4.6). This statement can be justified by the following argument. From both points of view (4.1) holds everywhere except at the boundaries; at the boundaries (4.5) and (4.6) show that (4.1) restrains T in the same manner as given equivalent boundary conditions would. Hence, the identity holds.

4.4 Existence and Uniqueness

Through the use of the method of successive approximations, the existence of a solution to (4.1) can be established for the initial value problem,

$$T(z_0) = T_0$$

$$a(z)^{-1} \left.\frac{dT}{dz}\right|_{z=z_0} = S_0.$$

The proof is started by considering the system of two linear equations that are equivalent to (4.1),

$$\frac{dS}{dz} = b(z)\ T$$

$$\frac{dT}{dz} = a(z)\ S.$$

The development of the proof from this point is somewhat standard and is given, along with a proof of the uniqueness of the solution, by E. L. Ince.[42]

A more general existence theorem that might be consulted was proved by Caratheodory[43] in 1927.

4.5 Example Solutions

4.51 An Electrostatic Field Solution in a Periodic Medium

This section will deal with an electrostatic problem arising in a periodic medium loaded with infinite dielectric slabs. The cross section of such a medium is illustrated in Figure 4.2. The field is to be set up by a positive charge on an infinite metal plate at $z = -d$ and by a negative charge on a similar plate at $z = d$. In the problem to be investigated $d \rightarrow \infty$.

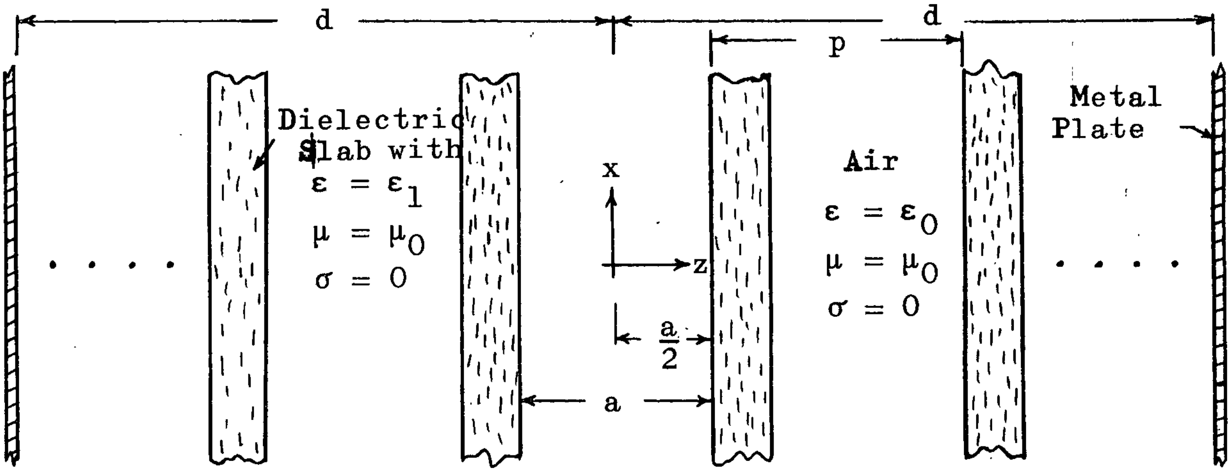

Fig. 4.2. Cross Section of the Medium for Example 4.51

For a medium behaving in the manner shown in Figure 4.2, the permittivity varies in the z-direction as a rectangular waveshape. This variation is described in Figure 4.3.

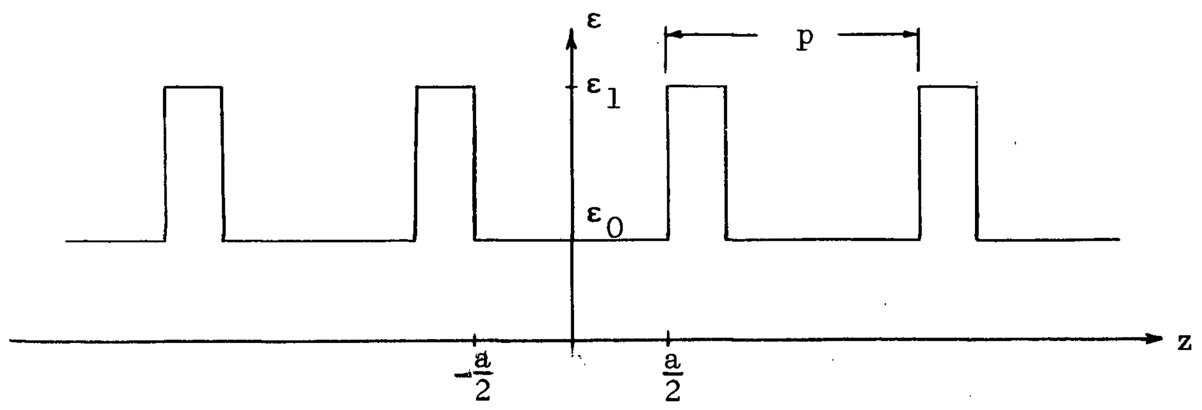

Fig. 4.3. The Functional Variation of the Permittivity for Example 4.51

In the region between the metal plates, since no free charge is present,

$$\nabla \cdot \vec{D} = 0. \tag{4.7}$$

The field is taken to have only z-dependence and thus from (4.7)

$$\frac{dD_z}{dz} = 0. \tag{4.8}$$

Since

$$\omega = 0$$

and

$$K = 0,$$

(4.1) can be reduced to (4.8) by letting

$$a(z)^{-1} \frac{dT}{dz} = D_z .$$

In this case, from (4.6) it is seen that D_z is continuous and, to be more specific, from (4.2) it is seen that

$$D_z = \text{constant.}$$

If the constant is evaluated at $z = \frac{a}{2}$,

$$D_z = D_1 = \varepsilon_0 E_1^- = \varepsilon_1 E_1^+ \tag{4.9}$$

where for $\delta > 0$

$$D_1 = D_z \Big|_{z = \frac{a}{2}}$$

$$E_1^- = \lim_{\delta \to 0} E_z \Big|_{z = \frac{a-\delta}{2}}$$

$$E_1^+ = \lim_{\delta \to 0} E_z \Big|_{z = \frac{a+\delta}{2}} .$$

Actually, at this point the problem is completely solved since

$$D_z = \varepsilon E_z \tag{4.10}$$

and thus

$$E_z = \frac{\varepsilon_0}{\varepsilon} E_1^- . \tag{4.11}$$

From (4.11) it can be seen that E_z has a rectangular waveshape which is inversely proportional to the permittivity.

A Fourier series solution for E_z will now be found from (4.8). This approach, although it is for the present problem somewhat more laborious than the preceding method, demonstrates a method of solution which is more applicable to complex problems.

If D_z is eliminated from (4.8) by using (4.10), it follows that

$$\frac{dE_z}{dz} + \frac{1}{\varepsilon}\frac{d\varepsilon}{dz} E_z = 0. \tag{4.12}$$

where $\frac{d\varepsilon}{dz}$ is a periodic impulse function. On the assumption that E_z is periodic with period p, the required solution may be expanded in the Fourier series

$$E_z = \sum_{n=-\infty}^{\infty} a_n e^{-j\frac{2n\pi}{p}z} . \tag{4.13}$$

As a first step for finding a_n for $n \neq 0$, (4.12) is multiplied by $e^{j\frac{2n\pi}{p}z}$ and is integrated from $-p/2$ to $p/2$. Hence,

$$\int_{-p/2}^{p/2} e^{j\frac{2n\pi}{p}z} \frac{dE_z}{dz}\, dz + \int_{-p/2}^{p/2} e^{j\frac{2n\pi}{p}z} E_z \frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\, dz = 0$$

or after integrating the left hand integral by parts and eliminating E_z from the right hand integral through the use of (4.10),

$$e^{j\frac{2n\pi}{p}z} E_z \Big|_{-p/2}^{p/2} - j\frac{2n\pi}{p}\int_{-p/2}^{p/2} e^{j\frac{2n\pi}{p}z} E_z dz - \int_{-p/2}^{p/2} e^{j\frac{2n\pi}{p}z} D_z \frac{d}{dz}\left[\frac{1}{\varepsilon}\right] dz$$

$$= 0. \tag{4.14}$$

Since E_z is periodic, the first term from the left is zero. Also, from (4.6) D_z is continuous. Hence, (4.14) gives

$$j\frac{2n\pi}{p}\int_{-p/2}^{p/2} e^{j\frac{2n\pi}{p}z} E_z dz = -\left[e^{-j\frac{n\pi a}{p}} D_{-1}\Delta\left[\frac{1}{\varepsilon}\right]\Big|_{z=-\frac{a}{2}} + e^{j\frac{n\pi a}{p}} D_1\Delta\left[\frac{1}{\varepsilon}\right]\Big|_{z=\frac{a}{2}} \right]$$

$$\tag{4.15}$$

where

$$\Delta\left[\frac{1}{\varepsilon}\right] = \text{difference in } \frac{1}{\varepsilon}$$

$$D_{-1} = D_z \Big|_{z=-\frac{a}{2}}.$$

If (4.13) is substituted into (4.15), a_n is found to be

$$a_n = -\frac{1}{j2n\pi}\left[e^{-j\frac{n\pi a}{p}} D_{-1}\left[\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right] + e^{j\frac{n\pi a}{p}} D_1\left[\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_0}\right]\right]$$

or

$$a_n = \frac{1}{j2n\pi}\left[1 - \frac{1}{\varepsilon_r}\right]\left[e^{j\frac{n\pi a}{p}} E_1^- - e^{-j\frac{n\pi a}{p}} E_{-1}^+\right] \qquad (4.16)$$

where

$$\varepsilon_r = \varepsilon_1/\varepsilon_0$$

$$E_{-1}^+ = \lim_{\delta \to 0} E_z \Big|_{z = \frac{-a+\delta}{2}}.$$

Hence,

$$E_z = a_0 + \left[1 - \frac{1}{\varepsilon_r}\right] \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\left[e^{j\frac{n\pi a}{p}} E_1^- - e^{-j\frac{n\pi a}{p}} E_{-1}^+\right]}{j2n\pi} e^{-j\frac{2n\pi}{p}z}$$

(4.17)

If (4.17) is evaluated for

$$E_z = E_1^-,$$

then

$$E_1^- = a_0 + \left[1 - \frac{1}{\varepsilon_r}\right] \lim_{\delta \to 0} \left[E_1^- \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{j\frac{n\pi\delta}{p}}}{j2n\pi} - E_{-1}^+ \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{-j\frac{2n\pi}{p}(a-\frac{\delta}{2})}}{j2n\pi} \right]$$

$$= a_0 + \left[1 - \frac{1}{\varepsilon_r}\right] \lim_{\delta \to 0} \left[E_1^- \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi\delta}{p}}{n\pi} + E_{-1}^+ \sum_{n=1}^{\infty} \frac{\sin \frac{2n\pi}{p}(a-\frac{\delta}{2})}{n\pi} \right].$$

Since[44]

$$\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi\theta}{p}}{n\pi} = \frac{p-\theta}{2p}, \quad 0 < \theta < 2p \quad ,$$

$$E_1^- = a_0 + \left[1 - \frac{1}{\varepsilon_r}\right] \left[\frac{1}{2} E_1^- + \left(\frac{p-2a}{2p}\right) E_{-1}^+\right] . \tag{4.18}$$

In the same way, it can be shown that

$$E_{-1}^+ = a_0 + \left[1 - \frac{1}{\varepsilon_r}\right] \left[\left(\frac{p-2a}{2p}\right) E_1^- + \frac{1}{2} E_{-1}^+\right] . \tag{4.19}$$

By evaluating E_z in such a manner, no new unknowns are introduced and two homogeneous linear equations in a_0, E_1^- and E_{-1}^+ are obtained. Consequently, from (4.18) and (4.19), a_0 and E_{-1}^+ can be found in terms of E_1^- and are

$$a_0 = \left[\frac{a}{p} + \frac{1}{\varepsilon_r} \left(\frac{p-a}{p}\right)\right] E_1^-$$

$$E_{-1}^+ = E_1^- . \tag{4.20}$$

Hence, from (4.16) and (4.20)

$$a_n = \left[1 - \frac{1}{\varepsilon_r}\right] \frac{\sin \frac{n\pi a}{p}}{n\pi} E_1^- .$$

Therefore,

$$E_z = E_1^- \left[\frac{a}{p} + \frac{1}{\varepsilon_r} \left(\frac{p-a}{p}\right) + 2\left[1 - \frac{1}{\varepsilon_r}\right] \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi a}{p}}{n\pi} \cos \frac{2n\pi}{p} z\right]$$

and this is the Fourier series expression for the rectangular wave-shape found in (4.11).

4.52 A Steady State E-Field Solution in a Cylindrical Waveguide Loaded with Solid Dielectric Discs

4.521 Derivation of the Differential Equations

The problem to be investigated is the steady state E-field behaviour in a perfectly conducting cylindrical waveguide loaded with solid dielectric discs. The differential equations to be solved can be arrived at through Maxwell's equations which for the steady state case are given by (2.17) and (2.18).

For the problem in hand, the medium is linear. The permittivity is a function of z, the permeability is constant, μ_0, and the conductivity is zero.

If the curl of (2.17) is taken, then

$$\nabla \times (\nabla \times \vec{E}) = -j\omega\mu_0 \nabla \times \vec{H} . \qquad (4.21)$$

Now, $\nabla \times \vec{H}$ can be eliminated from (4.21) by using (2.18). Therefore,

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \omega^2 \mu_0 \vec{D} . \qquad (4.22)$$

It is more convenient to consider $\vec{D}$ instead of $\vec{E}$ since D_z is continuous. In terms of $\vec{D}$ alone, (4.22) becomes

$$\nabla\left(\nabla\cdot\left(\frac{\vec{D}}{\varepsilon}\right)\right) - \nabla^2\left(\frac{\vec{D}}{\varepsilon}\right) = \omega^2\mu_0\ \vec{D}\ .$$

The scalar equation obtained when the coefficients of the component vector in the z-direction are equated is

$$\frac{\partial}{\partial z}\left(\nabla\cdot\left(\frac{\vec{D}}{\varepsilon}\right)\right) - \nabla^2\left(\frac{D_z}{\varepsilon}\right) = \omega^2\mu_0 D_z\ . \qquad (4.23)$$

With the help of (4.7), (4.23) can be reduced to

$$\nabla^2 D_z - \frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\frac{\partial D_z}{\partial z} + \omega^2\mu_0\varepsilon D_z = 0\ . \qquad (4.24)$$

Once D_z is found from (4.24), the remaining field components can be found through the use of (2.17), (2.18) and the linearity relationships

$$\vec{D} = \varepsilon\ \vec{E}$$

$$\vec{B} = \mu_0\ \vec{H}\ .$$

The method of separation of variables can now be used to solve (4.24). First of all, (4.24) can be written as

$$\nabla_t^{\,2} D_z + \frac{\partial^2 D_z}{\partial z^2} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\frac{\partial D_z}{\partial z} + \omega^2\mu_0\varepsilon\ D_z = 0 \qquad (4.25)$$

where $\nabla_t^{\,2}$ denotes the part of ∇^2 which operates in the transverse plane. By letting

$$D_z = F(r,\phi)\ T(z),$$

(4.25) gives

$$-\frac{1}{F}\nabla_t^2 F = \frac{1}{T}\left[\frac{d^2T}{dz^2} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\frac{dT}{dz} + \omega^2\mu_0\varepsilon\ T\right] = K^2 .$$

Therefore,

$$\nabla_t^2 F + K^2 F = 0 \tag{4.26}$$

$$\frac{d^2T}{dz^2} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\frac{dT}{dz} + (\omega^2\mu_0\varepsilon - K^2)\ T = 0. \tag{4.27}$$

On the boundary of the waveguide

$$r = b$$

$$D_z = 0.$$

Hence,

$$F(b,\phi) = 0.$$

The solution, F, to this boundary value problem is quite straight forward and can be shown to be[45]

$$F = J_n(Kr)\begin{pmatrix}\cos n\phi \\ \sin n\phi\end{pmatrix}$$

where

$$n = 0,\ 1,\ 2,\ \ldots\ldots$$

and K is found from the roots of

$$J_n(Kb) = 0.$$

As for (4.27), it is a differential equation with the same form as (4.1) and, consequently, (4.27) has the same properties as (4.1). In sections 4.522 and 4.523 solutions for (4.27) will be given.

4.522 A Field Solution for the Matched Case

For the case where the dielectric discs are matched into the air regions, the wave solution can be completed by solving for T in the manner to be described. The solution will be found for the field in a medium in which the distance between adjacent discs as well as thickness of the discs is arbitrary. The cross section of such a medium is illustrated in Figure 4.4.

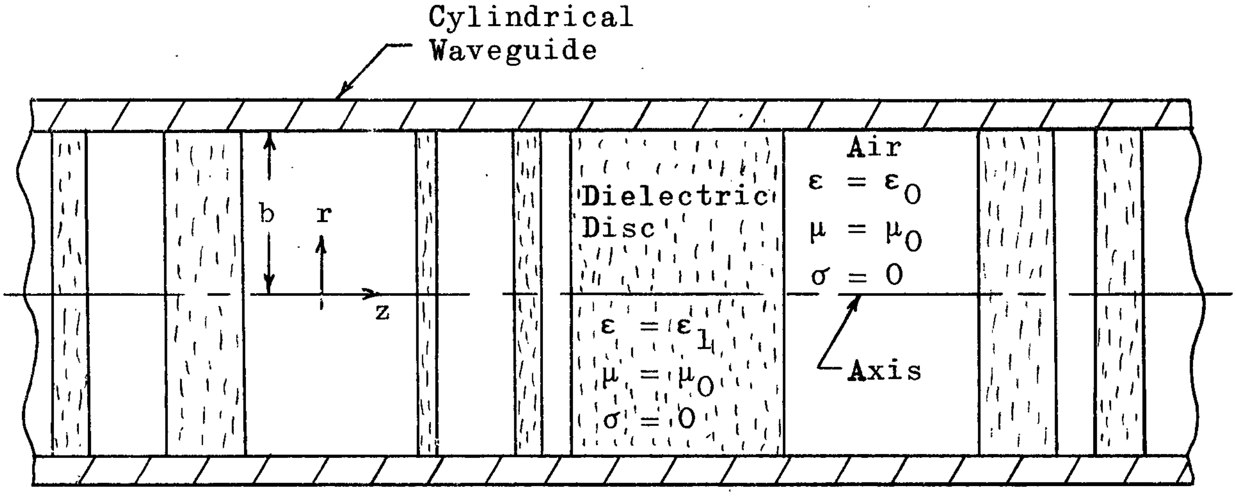

Fig. 4.4. Cross Section of a Dielectric Disc-Loaded Structure

Since the permittivity can only have the value ε_0 or ε_1, as shown in Figure 4.5, it follows that $\omega^2\mu_0\varepsilon - K^2$ can only have the value $\omega^2\mu_0\varepsilon_0 - K^2$ or $\omega^2\mu_0\varepsilon_1 - K^2$ and, similarly, $C^2\varepsilon^2$, C being a constant, can only have the value $C^2\varepsilon_0^{\ 2}$ or $C^2\varepsilon_1^{\ 2}$. The dependence of both $\omega^2\mu_0\varepsilon - K^2$ and $C^2\varepsilon^2$ is shown in Figure 4.5.

For the matched case, the wave impedance[46] in the air regions

$$Z_0 = \frac{\sqrt{\omega^2\mu_0\varepsilon_0 - K^2}}{\omega\varepsilon_0}$$

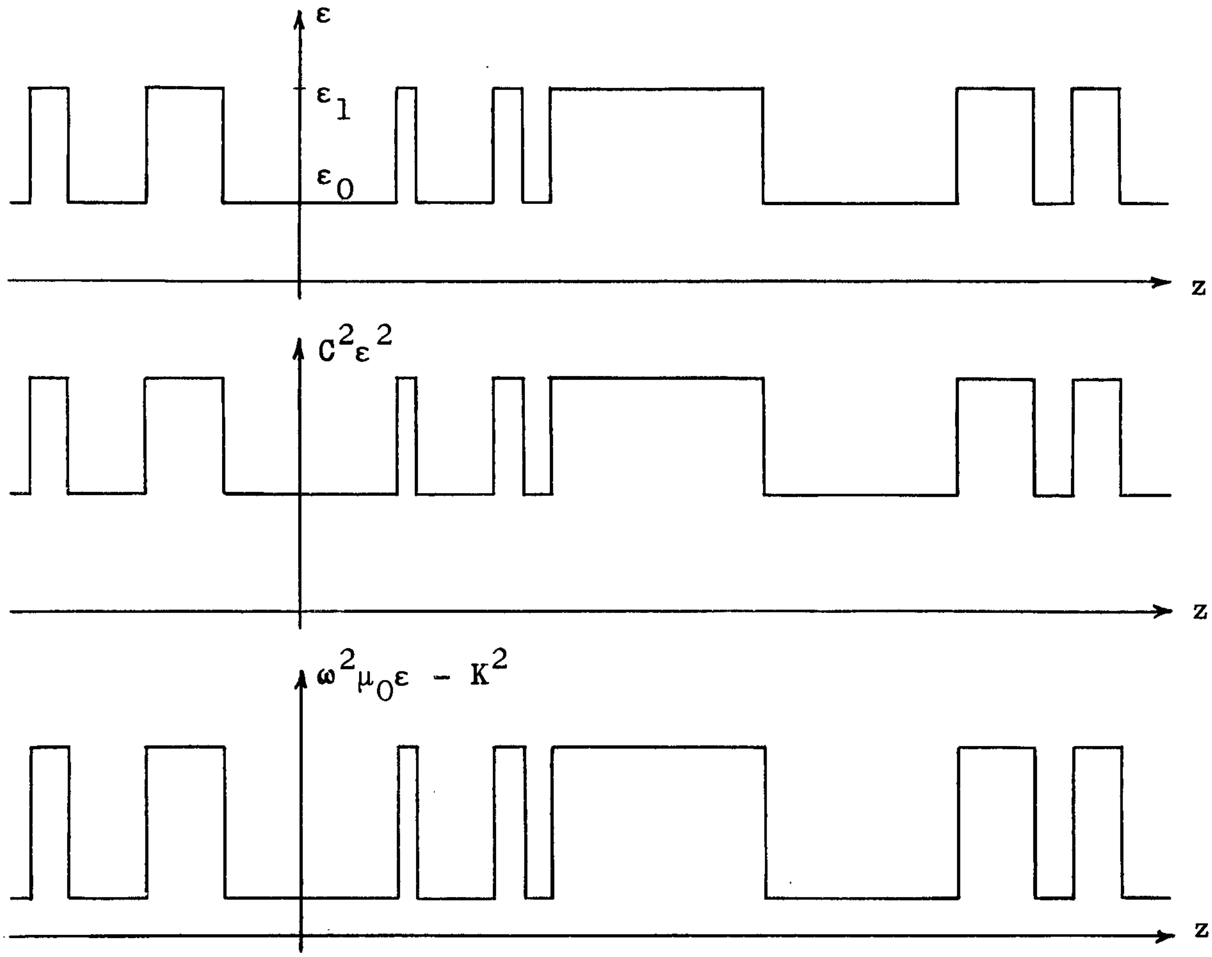

Fig. 4.5. Diagram of ε, $C^2\varepsilon^2$ and $\omega^2\mu_0\varepsilon - K$ Versus z

must equal the wave impedance in the dielectric regions

$$Z_1 = \frac{\sqrt{\omega^2\mu_0\varepsilon_1 - K^2}}{\omega\varepsilon_1} .$$

Hence,

$$\frac{\omega^2\mu_0\varepsilon_0 - K^2}{\varepsilon_0{}^2} = \frac{\omega^2\mu_0\varepsilon_1 - K^2}{\varepsilon_1{}^2} = C^2 . \tag{4.28}$$

Consequently, when a match exists, the identity

$$\omega^2\mu_0\varepsilon - K^2 \equiv C^2\varepsilon^2 \tag{4.29}$$

holds. As a result, (4.27) can be expressed as

$$\frac{d^2T}{dz^2} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\frac{dT}{dz} + C^2\varepsilon^2\, T = 0. \tag{4.30}$$

The solution to (4.30) and thus (4.27) is

$$T = A_1 e^{-j\int_0^z C\varepsilon(\tau)d\tau} + A_2 e^{j\int_0^z C\varepsilon(\tau)d\tau}$$

where A_1 and A_2 are arbitrary constants. Through the use of (4.29), T becomes

$$T = A_1 e^{-j\int_0^z \sqrt{\omega^2\mu_0\varepsilon - K^2}\, d\tau} + A_2 e^{j\int_0^z \sqrt{\omega^2\mu_0\varepsilon - K^2}\, d\tau} . \tag{4.31}$$

The integral,

$$\int_0^z \sqrt{\omega^2\mu_0\varepsilon - K^2}\, d\tau ,$$

can be evaluated graphically as is shown in Figure 4.6. If it is remembered that no reflections occur at the interfaces between the air and dielectric material, a quick check will show that the answer obtained using the standard approach is in agreement with (4.31).

From (4.31) T must be continuous and thus (4.5) is satisfied. If (4.31) is differentiated, the result is

$$\frac{1}{\varepsilon}\frac{dT}{dz} = -jC\left[A_1 e^{-j\int_0^z \sqrt{\omega^2\mu_0\varepsilon - K^2}\, d\tau} - A_2 e^{j\int_0^z \sqrt{\omega^2\mu_0\varepsilon - K^2}\, d\tau}\right]$$

and the right hand side is continuous. Hence, (4.6) is satisfied.

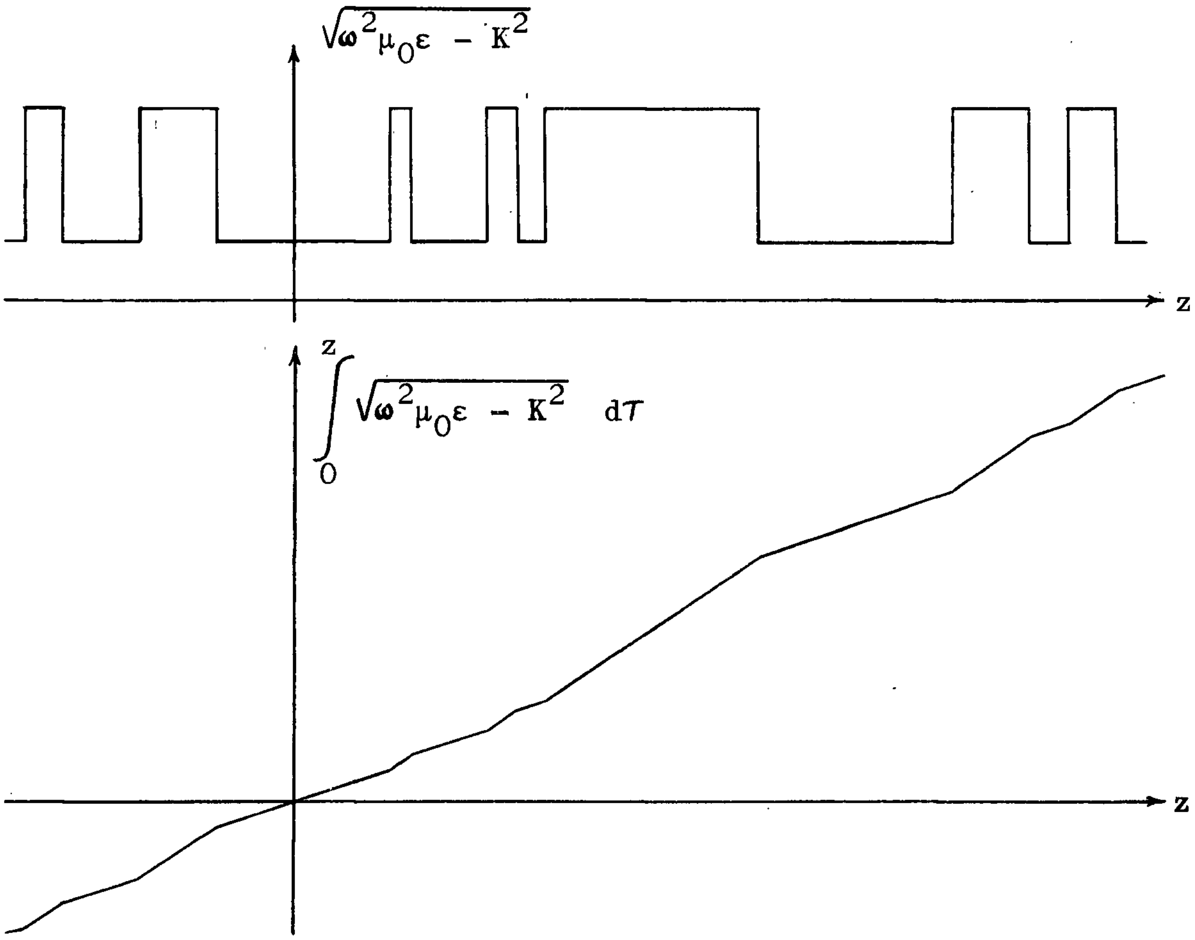

Fig. 4.6. Diagram of $\int_0^z \sqrt{\omega^2\mu_0\varepsilon - K^2}\, d\tau$ Versus z

4.523 A Field Solution for a Periodic Loaded Structure

For the problem to be solved, identical dielectric discs are placed at periodic intervals in the cylindrical waveguide. Figure 4.7 illustrates the cross section of such a structure. This section will only deal with the solution of (4.27) since the remainder of field solution has been outlined in section 4.521.

The z-dependence of the permittivity is shown in Figure 4.8. The permittivity can be expressed as

$$\varepsilon = \varepsilon_0 + c_0 h(z) \tag{4.32}$$

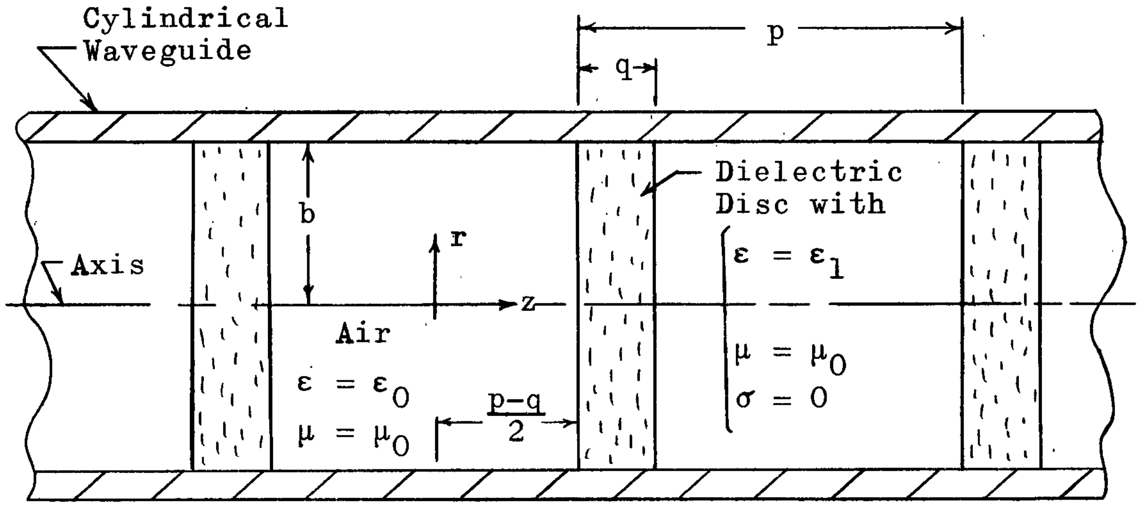

Fig. 4.7. Cross Section of a Dielectric Disc-Loaded Periodic Structure

where

$$c_0 = \varepsilon_1 - \varepsilon_0$$

and h(z) is the unit rectangular waveshape shown in Figure 4.8.

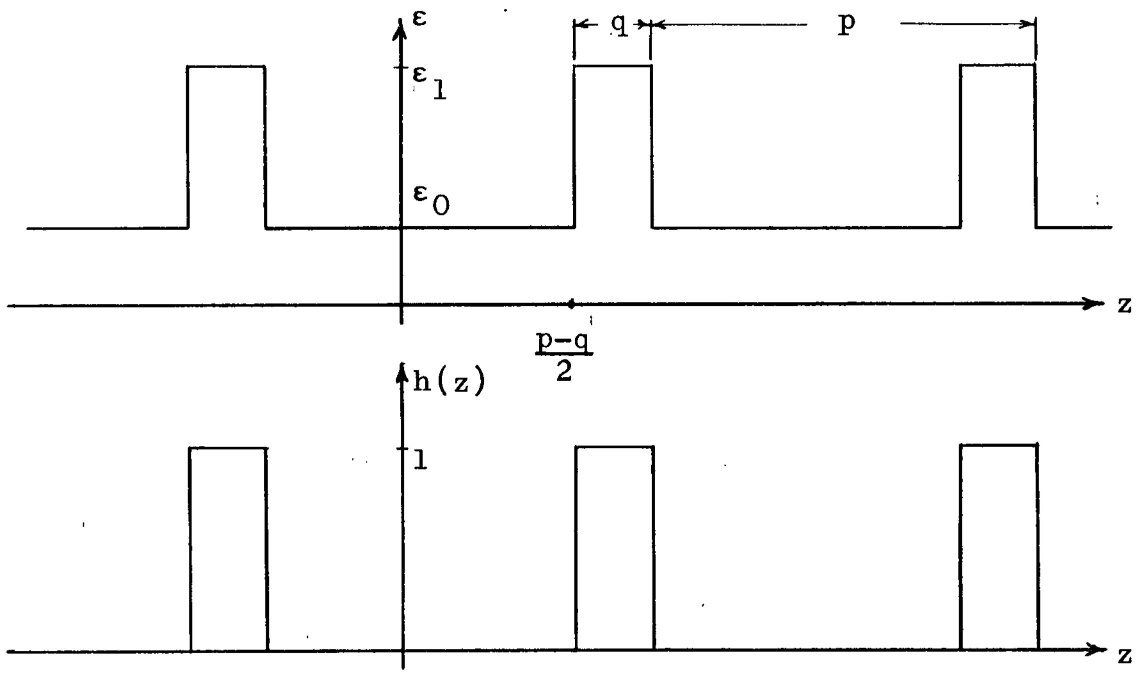

Fig. 4.8. Diagram of ε and h(z) Versus z

Through the use of (4.32), (4.27) can be expressed as

$$\frac{d^2T}{dz^2} + (\omega^2\mu_0\varepsilon_0 - K^2)\ T = \frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\frac{dT}{dz} - \omega^2\mu_0 c_0 h(z)\ T\ . \tag{4.33}$$

Since (4.33) is a linear second-order differential equation with periodic coefficients, according to Floquet's theorem[47], T can be expressed as

$$T = e^{-j\frac{\chi}{p}z} \sum_{n=-\infty}^{\infty} a_n e^{-j\frac{2\pi n}{p}z} = \sum_{n=-\infty}^{\infty} a_n e^{-j\left(\frac{\chi+2n\pi}{p}\right)z}\ .$$

For convenience, the definition

$$s_n = j\left(\frac{\chi+2n\pi}{p}\right)$$

is made. As a result,

$$T = \sum_{n=-\infty}^{\infty} a_n e^{-s_n z}\ . \tag{4.34}$$

At this point, (4.33) is multiplied by $\frac{1}{p}\ e^{s_n z}$ and the resulting expression is integrated from 0 to p. Hence,

$$\frac{1}{p}\int_0^p e^{s_n z}\frac{d^2T}{dz^2}\,dz + \frac{(\omega^2\mu_0\varepsilon_0 - K^2)}{p}\int_0^p e^{s_n z}\,T\,dz$$

$$= \frac{1}{p}\int_0^p e^{s_n z}\frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\frac{dT}{dz}\,dz - \frac{\omega^2\mu_0 c_0}{p}\int_0^p e^{s_n z}\,h(z)\,T\,dz$$

or after integrating the first integral on the left by parts,

$$\frac{1}{p} e^{s_n z} \frac{dT}{dz}\Bigg|_0^p - \frac{s_n}{p} e^{s_n z} T\Bigg|_0^p + \frac{s_n^2}{p}\int_0^p e^{s_n z} T\,dz$$

$$+ \frac{(\omega^2\mu_0\varepsilon_0 - K^2)}{p}\int_0^p e^{s_n z} T\,dz$$

$$= \frac{1}{p}\int_0^p e^{s_n z} \frac{1}{\varepsilon}\frac{dT}{dz}\frac{d\varepsilon}{dz}\,dz - \frac{\omega^2\mu_0 c_0}{p}\int_0^p e^{s_n z} h(z)\,T\,dz. \tag{4.35}$$

The functions $e^{s_n z}\frac{dT}{dz}$ and $e^{s_n z}T$ are periodic and thus

$$\frac{1}{p} e^{s_n z} \frac{dT}{dz}\Bigg|_0^p = 0$$

$$\frac{s_n}{p} e^{s_n z}\, T\Bigg|_0^p = 0.$$

Through the use of this fact and the recollection of the functional behaviour of $h(z)$, (4.35) can be reduced to

$$\frac{s_n^2 + \beta_0^2}{p}\int_0^p e^{s_n z} T\,dz = \frac{1}{p}\int_0^p e^{s_n z}\frac{1}{\varepsilon}\frac{dT}{dz}\frac{d\varepsilon}{dz}\,dz - \frac{\omega^2\mu_0 c_0}{p}\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_n z} T\,dz \tag{4.36}$$

where

$$\beta_0^2 = \omega^2\mu_0\varepsilon_0 - K^2.$$

If (4.34) is substituted into the left hand integral in (4.36), the resulting expression for a_n is

$$a_n = \frac{1}{s_n^2 + \beta_0^2} \left[\frac{1}{p} \int_0^p e^{s_n z} S \frac{d\varepsilon}{dz} dz - \frac{\omega^2 \mu_0 c_0}{p} \int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_n z} T \, dz \right] \tag{4.37}$$

where

$$S = \frac{1}{\varepsilon} \frac{dT}{dz} .$$

From (4.6) S is a continuous function.

The second integral from the right is found to be

$$\int_0^p e^{s_n z} S \frac{d\varepsilon}{dz} dz = e^{s_n(\frac{p-q}{2})} S(\tfrac{p-q}{2}) \int_{\frac{p-q}{2}-\delta}^{\frac{p-q}{2}+\delta} \frac{d\varepsilon}{dz} dz + e^{s_n(\frac{p+q}{2})} S(\tfrac{p+q}{2}) \int_{\frac{p+q}{2}-\delta}^{\frac{p+q}{2}+\delta} \frac{d\varepsilon}{dz} dz$$

$$= \left[e^{s_n(\frac{p-q}{2})} S(\tfrac{p-q}{2}) - e^{s_n(\frac{p+q}{2})} S(\tfrac{p+q}{2}) \right] (\varepsilon_1 - \varepsilon_0). \tag{4.38}$$

The integral

$$I = \int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_n z} T \, dz$$

is determined from (4.33). Over the interval $\frac{p-q}{2}$ to $\frac{p+q}{2}$, (4.33) simplifies to

$$\frac{d^2T}{dz^2} + (\omega^2 \mu_0 \varepsilon_1 - K^2) T = 0. \tag{4.39}$$

If (4.39) is multiplied by $\frac{1}{p} e^{s_n z}$ and the resulting expression is integrated from $\frac{p-q}{2}$ to $\frac{p+q}{2}$, then

$$\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_n z} \frac{d^2T}{dz^2}\, dz + (\omega^2\mu_0\varepsilon_1 - K^2) \int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_n z}\, T\, dz = 0.$$

Consequently,

$$(\omega^2\mu_0\varepsilon_1 - K^2)\ I = - \int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_n z} \frac{d^2T}{dz^2}\, dz$$

$$= -e^{s_n z} \frac{dT}{dz}\Bigg|_{\frac{p-q}{2}}^{\frac{p+q}{2}} + s_n \int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_n z} \frac{dT}{dz}\, dz$$

$$= -e^{s_n(\frac{p+q}{2})} \varepsilon_1 S(\tfrac{p+q}{2}) + e^{s_n(\frac{p-q}{2})} \varepsilon_1 S(\tfrac{p-q}{2})$$

$$+ s_n\left[e^{s_n(\frac{p+q}{2})} T(\tfrac{p+q}{2}) - e^{s_n(\frac{p-q}{2})} T(\tfrac{p-q}{2}) - s_n I\right].$$

Hence,

$$I = \frac{s_n\left[e^{s_n(\frac{p+q}{2})} T(\frac{p+q}{2}) - e^{s_n(\frac{p-q}{2})} T(\frac{p-q}{2})\right] - \varepsilon_1\left[e^{s_n(\frac{p+q}{2})} S(\frac{p+q}{2}) - e^{s_n(\frac{p-q}{2})} S(\frac{p-q}{2})\right]}{s_n^2 + \beta_1^2}$$

(4.40)

where

$$\beta_1^2 = \omega^2\mu_0\varepsilon_1 - K^2.$$

From the substitution of (4.38) and (4.40) into (4.37)

$$a_n = \frac{(\varepsilon_1 - \varepsilon_0)}{p}\left[\frac{e^{s_n(\frac{p-q}{2})}\left[\omega^2\mu_0 s_n\ T(\frac{p-q}{2}) + (s_n{}^2 - K^2)\ S(\frac{p-q}{2})\right]}{(s_n^2 + \beta_0^2)(s_n^2 + \beta_1^2)} - \frac{e^{s_n(\frac{p+q}{2})}\left[\omega^2\mu_0 s_n\ T(\frac{p+q}{2}) + (s_n{}^2 - K^2)\ S(\frac{p+q}{2})\right]}{(s_n^2 + \beta_0^2)(s_n^2 + \beta_1^2)}\right]. \tag{4.41}$$

Another way to express a_n is in the partial fraction expansion,

$$a_n = \frac{1}{p}\left[\frac{C_1}{s_n - j\beta_0} + \frac{C_2}{s_n + j\beta_0} + \frac{C_3}{s_n - j\beta_1} + \frac{C_4}{s_n + j\beta_1}\right] e^{s_n(\frac{p-q}{2})}$$

$$- \frac{1}{p}\left[\frac{D_1}{s_n - j\beta_0} + \frac{D_2}{s_n + j\beta_0} + \frac{D_3}{s_n - j\beta_1} + \frac{D_4}{s_n + j\beta_1}\right] e^{s_n(\frac{p+q}{2})} \tag{4.42}$$

where

$$\left.\begin{aligned}
C_1 &= \frac{1}{2}\left[T(\tfrac{p-q}{2}) + j\frac{\varepsilon_0}{\beta_0}\ S(\tfrac{p-q}{2})\right] \\
C_2 &= \frac{1}{2}\left[T(\tfrac{p-q}{2}) - j\frac{\varepsilon_0}{\beta_0}\ S(\tfrac{p-q}{2})\right] \\
C_3 &= \frac{1}{2}\left[-T(\tfrac{p-q}{2}) - j\frac{\varepsilon_1}{\beta_1}\ S(\tfrac{p-q}{2})\right] \\
C_4 &= \frac{1}{2}\left[-T(\tfrac{p-q}{2}) + j\frac{\varepsilon_1}{\beta_1}\ S(\tfrac{p-q}{2})\right]
\end{aligned}\right\} \tag{4.43}$$

$$\left.\begin{aligned} D_1 &= \frac{1}{2}\left[T(\tfrac{p+q}{2}) + j\frac{\varepsilon_0}{\beta_0} S(\tfrac{p+q}{2})\right] \\ D_2 &= \frac{1}{2}\left[T(\tfrac{p+q}{2}) - j\frac{\varepsilon_0}{\beta_0} S(\tfrac{p+q}{2})\right] \\ D_3 &= \frac{1}{2}\left[-T(\tfrac{p+q}{2}) - j\frac{\varepsilon_1}{\beta_1} S(\tfrac{p+q}{2})\right] \\ D_4 &= \frac{1}{2}\left[-T(\tfrac{p+q}{2}) + j\frac{\varepsilon_1}{\beta_1} S(\tfrac{p+q}{2})\right] . \end{aligned}\right\} \qquad (4.44)$$

From (4.43) it can be seen that

$$C_1 + C_2 + C_3 + C_4 = 0 \qquad (4.45)$$

$$\frac{\beta_0}{\varepsilon_0} C_1 - \frac{\beta_0}{\varepsilon_0} C_2 + \frac{\beta_1}{\varepsilon_1} C_3 - \frac{\beta_1}{\varepsilon_1} C_4 = 0 \qquad (4.46)$$

and from (4.44), similarly,

$$D_1 + D_2 + D_3 + D_4 = 0 \qquad (4.47)$$

$$\frac{\beta_0}{\varepsilon_0} D_1 - \frac{\beta_0}{\varepsilon_0} D_2 + \frac{\beta_1}{\varepsilon_1} D_3 - \frac{\beta_1}{\varepsilon_1} D_4 = 0. \qquad (4.48)$$

The next step is to find the four relationships between the C's and D's. Once these relationships are determined, four of the unknowns can be eliminated from (4.45), (4.46), (4.47) and (4.48). As a result, the problem is reduced to finding the solution to four linear homogeneous equations.

Since the original four unknowns are $T(\frac{p-q}{2})$, $S(\frac{p-q}{2})$, $T(\frac{p+q}{2})$ and $S(\frac{p+q}{2})$, the four relationships between the C's and the D's will be found by summing the series for T(z) and S(z) at the points $\frac{p-q}{2}$ and $\frac{p+q}{2}$ and by eliminating $T(\frac{p-q}{2})$, $S(\frac{p-q}{2})$, $T(\frac{p+q}{2})$ and and $S(\frac{p+q}{2})$ with the equations

$$\left.\begin{aligned}
C_1 &= \frac{1}{2}\left[T(\tfrac{p-q}{2}) + j\frac{\varepsilon_0}{\beta_0} S(\tfrac{p-q}{2})\right] \\
C_2 &= \frac{1}{2}\left[T(\tfrac{p-q}{2}) - j\frac{\varepsilon_0}{\beta_0} S(\tfrac{p-q}{2})\right] \\
D_3 &= \frac{1}{2}\left[-T(\tfrac{p+q}{2}) - j\frac{\varepsilon_1}{\beta_1} S(\tfrac{p+q}{2})\right] \\
D_4 &= \frac{1}{2}\left[-T(\tfrac{p+q}{2}) + j\frac{\varepsilon_1}{\beta_1} S(\tfrac{p+q}{2})\right] \quad .
\end{aligned}\right\} \qquad (4.49)$$

In this way, no new unknowns are introduced. The equations resulting from (4.49) are four linear nonhomogeneous equations in which the C's can be expressed in terms of the D's. If the determinant of the coefficients of the C's is non-zero, the solution for the C's in terms of the D's is unique.

In Appendix 2 the series are summed and the resulting expressions are

$$\left.\begin{aligned}
T(\tfrac{p-q}{2}) &= y_1C_1+y_2C_2+y_3C_3+y_4C_4+e^{j(\chi-2\theta_0)}y_1^{*}D_1+e^{j(\chi+2\theta_0)}y_2^{*}D_2 \\
&\quad -e^{j2\theta_1}y_3D_3-e^{-j2\theta_1}y_4D_4 \\
S(\tfrac{p-q}{2}) &= -j\frac{\beta_0}{\varepsilon_0}y_1C_1+j\frac{\beta_0}{\varepsilon_0}y_2C_2-j\frac{\beta_1}{\varepsilon_0}y_3C_3+j\frac{\beta_1}{\varepsilon_0}y_4C_4-j\frac{\beta_0}{\varepsilon_0}e^{j(\chi-2\theta_0)}y_1^{*}D_1 \\
&\quad +j\frac{\beta_0}{\varepsilon_0}e^{j(\chi+2\theta_0)}y_2^{*}D_2+j\frac{\beta_1}{\varepsilon_0}e^{j2\theta_1}y_3D_3-j\frac{\beta_1}{\varepsilon_0}e^{-j2\theta_1}y_4D_4
\end{aligned}\right\} \qquad (4.50)$$

* denotes complex conjugate

$$T\left(\frac{p+q}{2}\right)=e^{-j(\chi-2\theta_0)}y_1C_1+e^{-j(\chi+2\theta_0)}y_2C_2-e^{-j2\theta_1}y_3^{*}C_3-e^{j2\theta_1}y_4^{*}C_4$$

$$-y_1D_1-y_2D_2-y_3D_3-y_4D_4$$

$$S\left(\frac{p+q}{2}\right)= -j\frac{\beta_0}{\varepsilon_1}e^{-j(\chi-2\theta_0)}y_1C_1+j\frac{\beta_0}{\varepsilon_1}e^{-j(\chi+2\theta_0)}y_2C_2+j\frac{\beta_1}{\varepsilon_1}e^{-j2\theta_1}y_3^{*}C_3$$

$$-j\frac{\beta_1}{\varepsilon_1}e^{j2\theta_1}y_4^{*}C_4+j\frac{\beta_0}{\varepsilon_1}y_1D_1-j\frac{\beta_0}{\varepsilon_1}y_2D_2+j\frac{\beta_1}{\varepsilon_1}y_3D_3-j\frac{\beta_1}{\varepsilon_1}y_4D_4$$

where

$2\theta_0 = \beta_0(p-q)$ = the phase change in the air region

$2\theta_1 = \beta_1 q$ = the phase change in the dielectric region

$$y_1 = \frac{e^{j\left(\frac{\chi-\beta_0 p}{2}\right)}}{2j\sin\left(\frac{\chi-\beta_0 p}{2}\right)}$$

$$y_2 = \frac{e^{j\left(\frac{\chi+\beta_0 p}{2}\right)}}{2j\sin\left(\frac{\chi+\beta_0 p}{2}\right)}$$

$$y_3 = \frac{e^{j\left(\frac{\chi-\beta_1 p}{2}\right)}}{2j\sin\left(\frac{\chi-\beta_1 p}{2}\right)}$$

$$y_4 = \frac{e^{j\left(\frac{\chi+\beta_1 p}{2}\right)}}{2j\sin\left(\frac{\chi+\beta_1 p}{2}\right)} .$$

The substitution of (4.50) into (4.49) yields

$$
\left.
\begin{aligned}
& b_1^{*}\,x_1 + 0\,x_2 + \tfrac{1}{2}\left(1+\tfrac{\beta_1}{\beta_0}\right)b_3x_3 + \tfrac{1}{2}\left(1+\tfrac{\beta_1}{\beta_0}\right)b_4x_4 = 0 \\
& 0\,x_1 + b_2^{*}\,x_2 + \tfrac{1}{2}\left(1-\tfrac{\beta_1}{\beta_0}\right)b_3x_3 + \tfrac{1}{2}\left(1-\tfrac{\beta_1}{\beta_0}\right)b_4x_4 = 0 \\
& \tfrac{1}{2}\left(1+\tfrac{\beta_0}{\beta_1}\right)b_1e^{-j(\chi-2\theta_0)}x_1 + \tfrac{1}{2}\left(1+\tfrac{\beta_0}{\beta_1}\right)b_2e^{-j(\chi+2\theta_0)}x_2 + e^{j2\theta_1}b_3^{*}x_3 + 0\,x_4 = 0 \\
& \tfrac{1}{2}\left(1-\tfrac{\beta_0}{\beta_1}\right)b_1e^{-j(\chi-2\theta_0)}x_1 + \tfrac{1}{2}\left(1-\tfrac{\beta_0}{\beta_1}\right)b_2e^{-j(\chi+2\theta_0)}x_2 + 0\,x_3 + e^{-j2\theta_1}b_4^{*}x_4 = 0
\end{aligned}
\right\}
\tag{4.51}
$$

where

$$x_1 = \frac{C_1 - e^{j(\chi-2\theta_0)}D_1}{\sin\frac{\chi-\beta_0 p}{2}}, \qquad b_1 = e^{j\left(\frac{\chi-\beta_0 p}{2}\right)}$$

$$x_2 = \frac{C_2 - e^{j(\chi+2\theta_0)}D_2}{\sin\frac{\chi+\beta_0 p}{2}}, \qquad b_2 = e^{j\left(\frac{\chi+\beta_0 p}{2}\right)}$$

$$x_3 = \frac{C_3 - e^{j2\theta_1}D_3}{\sin\frac{\chi-\beta_1 p}{2}}, \qquad b_3 = e^{j\left(\frac{\chi-\beta_1 p}{2}\right)}$$

$$x_4 = \frac{C_4 - e^{-j2\theta_1}D_4}{\sin\frac{\chi+\beta_1 p}{2}}, \qquad b_4 = e^{j\left(\frac{\chi+\beta_1 p}{2}\right)}.$$

The determinant of the coefficients of x_i's $(i = 1,2,3,4)$ in (4.51) is given by

$$\mathcal{D} = e^{j\chi}\left(e^{-j\chi} + \frac{\beta_1}{\beta_0}\cos\beta_1(p+q)\cos\beta_0 q - \frac{\beta_0}{\beta_1}\sin\beta_1(p+q)\sin\beta_0 q - j\sin\left[\beta_0 q + \beta_1(p+q)\right]\right).$$

Therefore, provided either

$$\cos\chi \neq \frac{\beta_0}{\beta_1}\sin\beta_1(p+q)\sin\beta_0 q - \frac{\beta_1}{\beta_0}\cos\beta_1(p+q)\cos\beta_0 q$$

or

$$\sin\chi \neq -\sin\left[\beta_0 q + \beta_1(p+q)\right] ,$$

then

$$\mathcal{D} \neq 0$$

and the solution for (4.51) is

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0.$$

Hence,

$$\left.\begin{aligned} C_1 &= e^{j(\chi-2\theta_0)} D_1 \\ C_2 &= e^{j(\chi+2\theta_0)} D_2 \\ C_3 &= e^{j2\theta_1} D_3 \\ C_4 &= e^{-j2\theta_1} D_4 . \end{aligned}\right\} \qquad (4.52)$$

From (4.45) and (4.46) and from (4.47) and (4.48), after the D's are eliminated by using (4.52),

$$\left.\begin{aligned}
&C_1 + C_2 + C_3 + C_4 = 0\\
&\frac{\beta_0}{\varepsilon_0} C_1 - \frac{\beta_0}{\varepsilon_0} C_2 + \frac{\beta_1}{\varepsilon_1} C_3 + \frac{\beta_1}{\varepsilon_1} C_4 = 0\\
&e^{-j(\chi-2\theta_0)} C_1 + e^{-j(\chi+2\theta_0)} C_2 + e^{-j2\theta_1} C_3 + e^{j2\theta_1} C_4 = 0\\
&\frac{\beta_0}{\varepsilon_0} e^{-j(\chi-2\theta_0)} C_1 - \frac{\beta_0}{\varepsilon_0} e^{-j(\chi+2\theta_0)} C_2 + \frac{\beta_1}{\varepsilon_1} e^{-j2\theta_1} C_3 - \frac{\beta_1}{\varepsilon_1} e^{j2\theta_1} C_4 = 0.
\end{aligned}\right\} \tag{4.53}$$

Solving the determinant of the coefficients in (4.53) gives

$$\cos\chi = \cos 2\theta_0 \cos 2\theta_1 - \frac{1}{2}\left[\frac{\beta_0\varepsilon_1}{\beta_1\varepsilon_0} + \frac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1}\right] \sin 2\theta_0 \sin 2\theta_1. \tag{4.54}$$

From (4.53) C_1, C_2 and C_3 are found in terms of C_4 to be

$$\left.\begin{aligned}
\frac{C_1}{C_4} &= \frac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1} e^{j\chi}\left\{\frac{-\cos 2\theta_1 - j\frac{\beta_0\varepsilon_1}{\beta_1\varepsilon_0}\sin 2\theta_1 + e^{-j(\chi+2\theta_0)}}{\cos 2\theta_0 + j\frac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1}\sin 2\theta_0 - e^{j(\chi-2\theta_1)}}\right\}\\
\frac{C_2}{C_4} &= \frac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1} e^{j\chi}\left\{\frac{\cos 2\theta_1 - j\frac{\beta_0\varepsilon_1}{\beta_1\varepsilon_0}\sin 2\theta_1 - e^{-j(\chi-2\theta_0)}}{\cos 2\theta_0 + j\frac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1}\sin 2\theta_0 - e^{j(\chi-2\theta_1)}}\right\}
\end{aligned}\right\} \tag{4.55}$$

$$\frac{C_3}{C_4} = \left\{ \frac{-\cos 2\theta_0 + j\frac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1}\sin 2\theta_0 + e^{j(\chi+2\theta_1)}}{\cos 2\theta_0 + j\frac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1}\sin 2\theta_0 - e^{(\chi-2\theta_1)}} \right\} \cdot$$

If the series (4.34) is summed in the manner shown in Appendix 2, over the interval, $-(\frac{p-q}{2}) \leq z \leq (\frac{p-q}{2})$,

$$T(z) = \frac{(C_1 - D_1 e^{j\beta_0 q}) e^{j(\frac{\chi-\beta_0 p}{2})}}{2j \sin(\frac{\chi-\beta_0 p}{2})} e^{-j\beta_0(z - \frac{p-q}{2})}$$

$$+ \frac{(C_2 - D_2 e^{-j\beta_0 q}) e^{j(\frac{\chi+\beta_0 p}{2})}}{2j \sin(\frac{\chi+\beta_0 p}{2})} e^{j\beta_0(z - \frac{p-q}{2})}$$

$$+ \frac{(C_3 - D_3 e^{j\beta_1 q}) e^{j(\frac{\chi-\beta_1 p}{2})}}{2j \sin(\frac{\chi-\beta_1 p}{2})} e^{-j\beta_1(z - \frac{p-q}{2})}$$

$$+ \frac{(C_4 - D_4 e^{-j\beta_1 q}) e^{j(\frac{\chi+\beta_1 p}{2})}}{2j \sin(\frac{\chi+\beta_1 p}{2})} e^{j\beta_1(z - \frac{p-q}{2})}$$

and over the interval, $\frac{p-q}{2} \leq z \leq \frac{p+q}{2}$,

$$T(z) = \frac{\left(C_1 - e^{j\left[\chi - \beta_0(p-q)\right]} D_1\right) e^{-j\left(\frac{\chi - \beta_0 p}{2}\right)}}{2j \sin\left(\frac{\chi - \beta_0 p}{2}\right)} e^{-j\beta_0\left(z - \frac{p-q}{2}\right)}$$

$$+ \frac{\left(C_2 - e^{j\left[\chi + \beta_0(p-q)\right]} D_2\right) e^{-j\left(\frac{\chi + \beta_0 p}{2}\right)}}{2j \sin\left(\frac{\chi + \beta_0 p}{2}\right)} e^{j\beta_0\left(z - \frac{p-q}{2}\right)}$$

$$+ \frac{\left(C_3 - e^{j\left[\chi - \beta_1(p-q)\right]} D_3\right) e^{-j\left(\frac{\chi - \beta_1 p}{2}\right)}}{2j \sin\left(\frac{\chi - \beta_1 p}{2}\right)} e^{-j\beta_1\left(z - \frac{p-q}{2}\right)}$$

$$+ \frac{\left(C_4 - e^{j\left[\chi + \beta_1(p-q)\right]} D_4\right) e^{-j\left(\frac{\chi + \beta_1 p}{2}\right)}}{2j \sin\left(\frac{\chi + \beta_1 p}{2}\right)} e^{j\beta_1\left(z - \frac{p-q}{2}\right)}.$$

By means of (4.52), for $-\left(\frac{p-q}{2}\right) \le z \le \frac{p-q}{2}$

$$T(z) = C_1 e^{-j\beta_0\left(z - \frac{p-q}{2}\right)} + C_2 e^{j\beta_0\left(z - \frac{p-q}{2}\right)} \tag{4.56}$$

and for $\frac{p-q}{2} \le z \le \frac{p+q}{2}$

$$T(z) = -C_3 e^{-j\beta_1\left(z - \frac{p-q}{2}\right)} - C_4 e^{j\beta_1\left(z - \frac{p-q}{2}\right)}. \tag{4.57}$$

With this done, the problem is completely solved. The answer is in total agreement with the answer obtained by using the standard approach for solving (4.33)[48].

As an aside, for the matched case if only an incident wave is present, the coefficients of terms with $e^{j\beta_0 z}$ or $e^{j\beta_1 z}$ must be zero. Hence,

$$C_2 = 0, \; C_4 = 0 \tag{4.58}$$

and thus from (4.55)

$$\cos 2\theta_1 - j\frac{\beta_0\varepsilon_1}{\beta_1\varepsilon_0}\sin 2\theta_1 - e^{-j(\chi-2\theta_0)} = 0$$

$$\cos 2\theta_0 + j\frac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1}\sin 2\theta_0 - e^{j(\chi-2\theta_1)} = 0.$$

Therefore,

$$\frac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1} = 1 \tag{4.59}$$

$$\chi = 2\theta_0 + 2\theta_1. \tag{4.60}$$

As already mentioned in the general discussion, Brillouin[49] attempted to solve, from the same point of view, a differential equation similar to (4.33). Actually, the equation was of the type

$$\frac{d^2T}{dz^2} + (\omega^2\mu_0\varepsilon_0 - K^2)\,T = -\omega^2\mu_0 c_0 h(z)\,T.$$

As has been done in the problem just solved, he expanded T in the series

$$T = \sum_{m=-\infty}^{\infty} a_m e^{-j\left(\frac{\chi+2m\pi}{p}\right)z} .$$

However, he also expanded the coefficient

$$C(z) = \omega^2\mu_0\left[\varepsilon_0 + c_0 h(z)\right] - K^2$$

in a series and this expansion led him into some difficulties.

Brillouin expanded C(z) in the Fourier series

$$C(z) = \sum_{n=-\infty}^{\infty} b_n e^{-j\frac{2n\pi}{p}z}$$

and then proceeded in the manner outlined in section 3 for solving Hill's equation.

Brillouin demonstrated that for the problem under consideration the determinantal equation given in (3.35) gives the wrong answer except in the limiting case where the discontinuity of the rectangular waveform approaches zero. As he noted, the reason for this error is that the b_n's for a rectangular waveform are not absolutely convergent.

The result is that he was not able to complete his solution from this approach.

In the method of solution just established in this section, the difficulties Brillouin encountered are avoided because C(z) as well as $\frac{1}{\varepsilon}\frac{d\varepsilon}{dz}$ are not expanded. Instead, the meaning of C(z) and $\frac{1}{\varepsilon}\frac{d\varepsilon}{dz}$ is interpreted in the definite integrals that finally contain these terms.

5. WAVES IN MEDIA WITH FINITE DISCONTINUITIES IN, AND TRANSVERSE TO, THE DIRECTION OF PROPAGATION

5.1 General

An investigation of electromagnetic waves in media with characteristics possessing finite discontinuities in, and transverse to, the direction of propagation is carried out for two examples. In the first example an E-wave solution is sought in a cylindrical waveguide loaded periodically with dielectric discs which have a centrally located hole. For the second example, an H-wave solution is found in a cylindrical resonant cavity containing a centrally located solid dielectric disc, the disc radius being smaller than the cavity radius.

The standard approach[50] for solving wave problems in these media is similar to the one mentioned in section 4. The solution is determined in each region where the characteristics of the medium are well behaved and the boundary conditions are applied at the points where the characteristics are discontinuous. This involves matching infinite series with infinite series. Although in principle this approach gives an exact solution, in practice the amount of labour involved in any numerical work makes it desirable to truncate the series after the first harmonic[50].

Collin[51] applies the Rayleigh-Ritz method to get a truncated series expression which approximates the field for a slotted dielectric interface. This method can be adapted for finding the solutions to the examples to be discussed. Once again numerical difficulties become the limiting factor. For example, if two harmonics are used, a six by six determinantal equation must be solved.

An approach which could be taken would make use of the generalized telegraphist's equations[52] for waveguides and give infinite series solutions. If these equations were employed, the permittivity would not be differentiated. Consequently, the interpretation of the derivative of the permittivity at discontinuities would not be required. Such a requirement is necessary in the treatment in this thesis since a wave equation which holds throughout the medium is employed. However, if the telegraphist's equations were used, three infinite series would need to be determined directly; one corresponding to the transverse electric field, another to the transverse magnetic field and the final one to the longitudinal field. For the method to be suggested, only one infinite series needs to be determined directly. Consequently, the price that would be paid for not differentiating the permitttivity is the introduction of three times as many unknowns.

For the periodic structure to be discussed, an anisotropic dielectric approximation[53] can be made if the period of the loading is small compared with the wavelength of the field. However, in this approximation the periodic nature of the structure is lost.

A variational method which gives upper and lower limits for the field solution has been developed by Chu and Hansen[54]. One limit is found by matching E_z along a cylinder of radius equal to the hole radius. As a result, H_ϕ is, in general, discontinuous. However, a second equation is obtained by equating Poynting vectors at the surface of the cylinder. Similarly, the other limit is established by reversing the role of

E_z and H_ϕ. For this method, due to computational difficulties, it is desirable to use only the first harmonic of the field in regions exterior to the forementioned cylinder.[54]

A variational method which can be applied to the cavity problem is established by Nikel'skiy.[55] Although the field expression is quite inaccurate, a good upper and lower bound for the resonant frequency is offered.

Approximate solutions with varying degrees of usefulness for specific situations can be found by these methods. The investigation to follow offers further alternative solutions but no attempt is made to compare the methods given by the forementioned authors and the methods to follow.

The treatment in this section is an extension of the approach taken in section 4. Each medium discussed is considered inhomogeneous and the derivatives of the permittivity at the discontinuities are regarded as impulse functions. For each example, a first mode approximation is given and then an approximate series solution is suggested. The second solution gives, for the lowest order mode, an iterative answer in which each coefficient of the infinite series is approximated. To obtain a first iteration, the field needs only to be guessed inside the dielectric disc. No guess is needed in the air regions.

5.2 A Periodic Structure Loaded with Dielectric Discs, Each Having a Central Hole

In the E-wave problem now to be discussed, identical dielectric discs, each with a central hole, are placed at periodic intervals in a cylindrical waveguide. The cross section of this structure is illustrated in Figure 5.1.

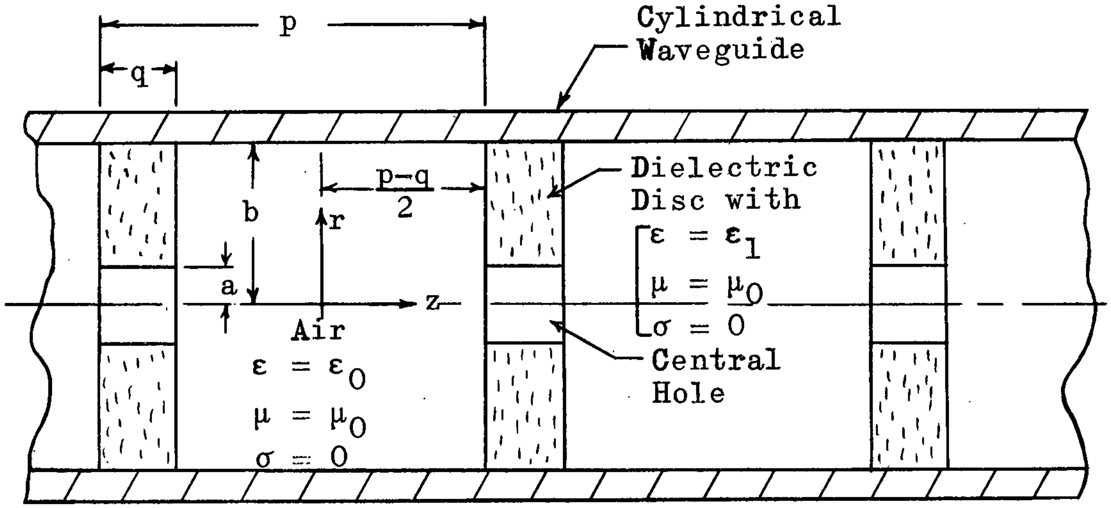

Fig. 5.1. Cross Section of a Dielectric Disc-Loaded Periodic Structure with a Central Hole

An easy check will show that the functional behaviour of the permittivity in the structure can be expressed as

$$\varepsilon = \varepsilon_0 + c_0 h(z)\, g(r) \tag{5.1}$$

where

$c_0 = \varepsilon_1 - \varepsilon_0$

$h(z)$ = the unit rectangular waveshape shown in Figure 4.8

$g(r)$ = the unit step function shown in Figure 5.2.

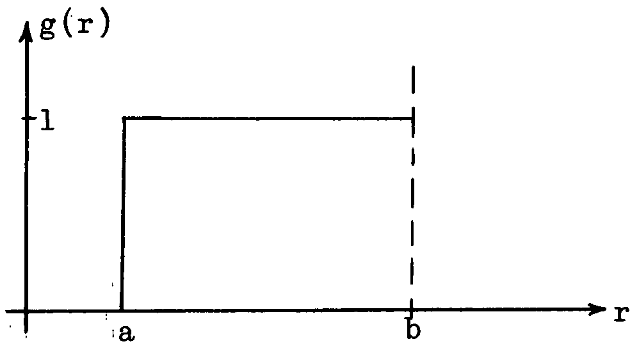

Fig. 5.2. The Unit Step Function

As shown in section 2.42, before an E-wave can exist in a region where the permittivity satisfies (5.1), it is necessary that the field has no angular dependence and

$$H_r \equiv 0, \; E_\phi \equiv 0, \; H_z \equiv 0.$$

Therefore, from (A1.10) in Appendix 1, H_ϕ must satisfy the partial differential equation

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\right] + \frac{\partial^2 H_\phi}{\partial z^2} = -\omega^2\mu_0\varepsilon H_\phi + \frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) + \frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial z}\frac{\partial H_\phi}{\partial z}\,. \tag{5.2}$$

The remainder of the treatment will be concentrated on the solution of (5.2) for H_ϕ. If other field components are wanted, they can be found by using Maxwell's equations and the linearity relations between the field densities and intensities. The reason neither E_z nor E_r is determined directly from the differential equations instead of H_ϕ is that each differential equation for the electric field components contains both E_z and E_r. As a result, the equations for the electric field are more difficult to handle than (5.2).

5.21 First Mode Approximation

The set of eigenfunctions, $\left\{ J_1(K_m r) \right\}$, in which the K_m's are determined from

$$J_0(Kb) = 0 \tag{5.3}$$

is complete and orthogonal over the open interval, (0,b). This statement[56] can be supported since the eigenfunctions are solutions to the Sturm-Liouville problem

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ry)\right] + K^2 y = 0$$

$y(0)$ is infinite

$$\left.\frac{d}{dr}(ry)\right|_{r=b} = 0.$$

Consequently, H_ϕ can be expanded, first of all, by the series

$$H_\phi = j\omega \sum_{m=1}^{\infty} \frac{1}{K_1} T_m(z)\, J_1(K_m r) \tag{5.4}$$

where $K_m b$ is the mth root of (5.3).

Substituting (5.1) into (5.2) yields

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\right] + \frac{\partial^2 H_\phi}{\partial z^2} + \omega^2\mu_0\varepsilon_0 H_\phi = -\omega^2\mu_0 c_0 h(z)g(r)\, H_\phi$$

$$+ c_0 h(z)\frac{dg}{dr}\frac{1}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) + c_0\frac{dh}{dz}g(r)\frac{1}{\varepsilon}\frac{\partial H_\phi}{\partial z}\,. \tag{5.5}$$

Through the multiplication of (5.5) by $rJ_1(K_m r)$ and the integration of the resulting expression from 0 to b, the equation obtained is found to be

$$\int_0^b r\, J_1(K_m r) \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\right] dr + \int_0^b r\, J_1(K_m r)\frac{\partial^2 H_\phi}{\partial z^2}\, dr$$

$$+ \omega^2\mu_0\varepsilon_0 \int_0^b r\, J_1(K_m r) H_\phi dr$$

$$= -\omega^2\mu_0 c_0 h(z)\int_0^b r\, J_1(K_m r) g(r) H_\phi dr + c_0 h(z)\int_0^b r\, J_1(K_m r)\frac{dg}{dr}\frac{1}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) dr$$

$$+c_0 \frac{dh}{dz}\int_0^b r\, J_1(K_m r) g(r)\frac{1}{\varepsilon}\frac{\partial H_\phi}{\partial z}\, dr. \qquad (5.6)$$

After two integrations by parts,

$$\int_0^b r\, J_1(K_m r)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\right] dr = -K_m^2\int_0^b r\, J_1(K_m r)\, H_\phi dr.$$

Consequently, (5.6) becomes

$$\frac{d^2}{dz^2}\int_0^b r\, J_1(K_m r)\, H_\phi dr + (\omega^2\mu_0\varepsilon_0 - K_m^2)\int_0^b r\, J_1(K_m r)\, H_\phi dr$$

$$= -\omega^2\mu_0 c_0 h(z)\int_a^b r\, J_1(K_m r)\, H_\phi dr + \frac{c_0}{\varepsilon^*} h(z) a\, J_1(K_m a)\left[\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\right]\Bigg|_{r=a+}$$

$$+ \frac{c_0}{\varepsilon^*}\frac{dh}{dz}\frac{d}{dz}\int_a^b r\, J_1(K_m r)\, H_\phi dr. \qquad (5.7)$$

where

$$\varepsilon^{*} = \varepsilon_0 + c_0 h(z)$$

$$a+ = \lim_{\delta \to 0} (a + \delta).$$

The derivative $\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)$ can be expanded in the series

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = j\frac{\omega}{K_1} \sum_{m=1}^{\infty} K_m\, T_m^{*}(z)\, J_0(K_m r). \tag{5.8}$$

Since

$$j\frac{\omega}{K_1} K_m \frac{b^2}{2} J_1{}^2(K_m b)\, T_m^{*}(z) = \int_0^b r\, J_0(K_m r)\, \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\, dr$$

$$= K_m \int_0^b r\, J_1(K_m r)\, H_\phi dr,$$

the substitution of (5.4) gives

$$T_m^{*}(z) = T_m(z).$$

By the use of the series (5.4) and (5.8), H_ϕ and its derivative are eliminated from (5.7). The result is

$$\left[\frac{d^2T_m}{dz^2} + \beta_{0m}{}^2 \quad T_m\right] \int_0^b r\ J_1{}^2(K_m r)dr$$

$$= -\omega^2\mu_0 c_0 h(z) \sum_{n=1}^{\infty} T_n \int_a^b r\ J_1(K_m r)J_1(K_n r)dr$$

$$+ \frac{c_0}{\varepsilon_1} h(z)\ a\ J_1(K_m a) \sum_{n=1}^{\infty} K_n\ T_n\ J_0(K_n a+)$$

$$+ \frac{c_0}{\varepsilon^*} \frac{dh}{dz} \sum_{n=1}^{\infty} \frac{dT_n}{dz} \int_a^b r\ J_1(K_m r)J_1(K_n r)dr \tag{5.9}$$

where

$$\beta_{0m}{}^2 = \omega^2\mu_0\varepsilon_0 - K_m{}^2. \tag{5.10}$$

Since[57]

$$\int_0^b r\ J_1{}^2(K_m r)dr = \frac{b^2}{2} J_1{}^2(K_m b)$$

$$\int_a^b r\ J_1(K_m r)\ J_1(K_n r)dr = \frac{b^2}{2} J_1{}^2(K_m b)\ \delta_{mn}$$

$$- \frac{a}{K_m{}^2 - K_n{}^2}\left[K_n J_1(K_m a)J_0(K_n a) - K_m J_0(K_m a)J_1(K_n a)\right]$$

where δ_{mn} is the Kronecker delta function, (5.9) becomes

$$\frac{d^2T_m}{dz^2} + \beta_{0m}^{\ 2}\ T_m = -\ \omega^2\mu_0 c_0 h(z)\ T_m + \frac{c_0}{\varepsilon^*}\frac{dh}{dz}\frac{dT_m}{dz}$$

$$+ \frac{2h(z)}{b^2 J_1^{\ 2}(K_m b)} \sum_{n=1}^{\infty} \left[\frac{c_0}{\varepsilon_1} K_n a\ J_1(K_m a)\ J_0(K_n a+) \right.$$

$$\left. + \frac{\omega^2\mu_0 c_0 a}{K_n^{\ 2} - K_m^{\ 2}} \left[K_m J_1(K_n a) J_0(K_m a) - K_n J_0(K_n a) J_1(K_m a)\right]\right] T_n$$

$$-2\frac{c_0}{\varepsilon^*}\frac{dh}{dz} \sum_{n=1}^{\infty} \frac{a\left[K_m J_1(K_n a) J_0(K_m a) - K_n J_0(K_n a) J_1(K_m a)\right]}{b^2\left[K_n^{\ 2} - K_m^{\ 2}\right] J_1^{\ 2}(K_m b)} \frac{dT_n}{dz} \ . \tag{5.11}$$

The solution to this infinite set of differential equations, once substituted into (5.4), provides an exact solution for H_ϕ. As can be easily surmised, in practice, only an approximate solution to the set can be anticipated.

Provided $a \ll b$,[58]

$$\left.\begin{aligned} J_0(K_1 a) &\cong 1 - \tfrac{1}{4}(K_1 a)^2 \\ J_1(K_1 a) &\cong \tfrac{1}{2}(K_1 a) - \tfrac{1}{16}(K_1 a)^3 \\ J_2(K_1 a) &\cong \tfrac{1}{8}(K_1 a)^2 \quad , \end{aligned}\right\} \tag{5.12}$$

and thus if T_1 is regarded as the driving function, for the smaller values of $m > 1$ from (5.11)

$$T_m \mathrel{\tilde{\propto}} \left(\frac{a}{b}\right)^2 T_1.$$

Through the use of the asymptotic expressions[58]

$$J_0(K_m a) \sim \sqrt{\frac{2}{\pi K_m a}} \cos\left(K_m a - \frac{\pi}{4}\right)$$

$$J_1(K_m a) \sim \sqrt{\frac{2}{\pi K_m a}} \cos\left(K_m a - \frac{3\pi}{4}\right),$$

for the larger values of m

$$T_m \mathrel{\tilde{\propto}} \begin{bmatrix} \sin K_m a \\ \cos K_m a \end{bmatrix} \left(\frac{1}{(K_m b)^3} \frac{a}{b}\right)^{\frac{1}{2}} T_1.$$

Therefore, for a first approximation of the field, all terms for $m > 1$ will be neglected. On making such an approximation, (5.11) gives

$$\frac{d^2 T_1}{dz^2} + \beta_{01}{}^2 T_1 = -\Bigg[\omega^2 \mu_0 c_0 - \frac{2}{b^2 J_1^2(K_1 b)} \frac{c_0}{\varepsilon_1}\Big[K_1 a J_1(K_1 a) J_0(K_1 a)$$

$$+\omega^2 \mu_0 \varepsilon_1 \frac{a^2}{2}\left(J_1{}^2(K_1 a) - J_0(K_1 a)\, J_2(K_1 a)\right)\Big]\Bigg] h(z) T_1$$

$$+\left[1 - \left(\frac{a}{b}\right)^2 \frac{1}{J_1^2(K_1 b)}\left[J_1{}^2(K_1 a) - J_0(K_1 a) J_2(K_1 a)\right]\right] \frac{c_0}{\varepsilon^*} \frac{dh}{dz} \frac{dT_1}{dz}. \tag{5.13}$$

By means of (5.12), if higher order quantities are neglected, (5.13) can be simplified into the expression

$$\frac{d^2T_1}{dz^2} + \beta_{01}^{\;2}\, T_1 = -\left[\omega^2\mu_0 c_0 - \Delta K_1^{\;2}\right] h(z)\, T_1 + \left[1-\alpha\right]\frac{1}{\varepsilon^*}\,\frac{d\varepsilon^*}{dz}\,\frac{dT_1}{dz} \tag{5.14}$$

where

$$\left.\begin{aligned}\Delta K_1^{\;2} &= K_1^{\;2}\,\frac{c_0}{\varepsilon_1}\left(\frac{a}{b}\right)^2 \frac{1}{J_1^{\;2}(K_1 b)}\left[1 + (\omega^2\mu_0\varepsilon_1 - 2K_1^{\;2})\,\frac{a^2}{8}\right]\\ \alpha &= \frac{1}{8}\left(\frac{K_1 a}{J_1(K_1 b)}\right)^2 \left(\frac{a}{b}\right)^2 .\end{aligned}\right\} \tag{5.15}$$

From (5.14) when

$$h(z) = 1,$$

$$\frac{d^2T_1}{dz^2} + \beta_{11}^{\;2}T_1 = 0$$

where

$$\beta_{11}^{\;2} = \omega^2\mu_0\varepsilon_1 - (K_1^{\;2} + \Delta K_1^{\;2}). \tag{5.16}$$

Before (5.14) is a valid approximation, in (5.16) $K_1^{\;2}$ should only be perturbed a small amount by the presence of the hole. Hence

$$\Delta K_1^{\;2} << K_1^{\;2}$$

and thus "a" must satisfy the condition

$$a^2\left[1 + (\omega^2\mu_0\varepsilon_1 - 2K_1^{\;2})\,\frac{a^2}{8}\right] << \frac{\varepsilon_1}{c_0}\, b^2 J_1^{\;2}(K_1 b). \tag{5.17}$$

A point worth mentioning is that $\Delta K_1^{\;2}$ is always positive since

$$K_1 a \ll K_1 b = 2.404.$$

If both T_1 and T_2 are taken into account, it can be shown that

$$T_2 \ll T_1$$

under the more relaxed condition

$$a^2\left[1 + (\omega^2\mu_0\varepsilon_1 - 2K_1^{\;2})\,\frac{a^2}{8}\right] \ll 4\frac{\varepsilon_1}{c_0}\, b^2 J_1^{\;2}(K_1 b). \tag{5.18}$$

In place of (5.17), (5.18) may be used.

Transforming (5.14) yields

$$\frac{d}{dz}\left[\frac{1}{\varepsilon^*(1-\alpha)}\,\frac{dT_1}{dz}\right] = -\left[\frac{\beta_{01}^{\;2} + (\omega^2\mu_0 c_0 - \Delta K_1^{\;2})h(z)}{\varepsilon^*(1-\alpha)}\right] T_1.$$

This equation is of the form given in (4.1) and, as a consequence, from (4.5) and (4.6)

$$T_1(z) = \text{a continuous function} \tag{5.19}$$

$$S_1(z) = \frac{1}{\varepsilon^*(1-\alpha)}\,\frac{dT_1}{dz} = \text{a continuous function.} \tag{5.20}$$

From (5.19) and (5.20) the boundary conditions for T_1 are known.

At this point, (5.14) can be solved either by the standard method using boundary conditions or by the method developed in section 4.523 for the solid disc problem. Since only minor adjustments to the answer in section 4.523 need to be made, the second method will be utilized.

If (4.33) is compared with (5.14), the solid disc solution can be made equal to the solution for (5.14) by making the following replacements:

$$T(z) \longrightarrow T_1(z)$$

$$S(z) \longrightarrow S_1(z)$$

$$K \longrightarrow K_1$$

$$\varepsilon_0 \longrightarrow \varepsilon_0^{1-\alpha}$$

$$\varepsilon_1 \longrightarrow \varepsilon_1^{1-\alpha}$$

$$\beta_0 \longrightarrow \beta_{01}$$

$$\beta_1 \longrightarrow \beta_{11} \ .$$

Once these replacements are made,

$$T_1(z) = \sum_{n=-\infty}^{\infty} a_n^* e^{-s_n z} \qquad (5.21)$$

$$a_n^* = \frac{1}{p}\left[\frac{C_1^*}{s_n - j\beta_{01}} + \frac{C_2^*}{s_n + j\beta_{01}} + \frac{C_3^*}{s_n - j\beta_{11}} + \frac{C_4^*}{s_n + j\beta_{11}}\right] e^{s_n\left(\frac{p-q}{2}\right)}$$

$$- \frac{1}{p}\left[\frac{D_1^*}{s_n - j\beta_{01}} + \frac{D_2^*}{s_n + j\beta_{01}} + \frac{D_3^*}{s_n - j\beta_{11}} + \frac{D_4^*}{s_n + j\beta_{11}}\right] e^{s_n\left(\frac{p+q}{2}\right)}$$

(5.22)

$$\left.\begin{aligned}
C_1^* &= \frac{1}{2}\left[T_1\left(\frac{p-q}{2}\right) + j\frac{\varepsilon_0^{1-\alpha}}{\beta_{01}} S_1\left(\frac{p-q}{2}\right)\right] \\
C_2^* &= \frac{1}{2}\left[T_1\left(\frac{p-q}{2}\right) - j\frac{\varepsilon_0^{1-\alpha}}{\beta_{01}} S_1\left(\frac{p-q}{2}\right)\right] \\
C_3^* &= \frac{1}{2}\left[-T_1\left(\frac{p-q}{2}\right) - j\frac{\varepsilon_1^{1-\alpha}}{\beta_{11}} S\left(\frac{p-q}{2}\right)\right] \\
C_4^* &= \frac{1}{2}\left[-T_1\left(\frac{p-q}{2}\right) + j\frac{\varepsilon_1^{1-\alpha}}{\beta_{11}} S_1\left(\frac{p-q}{2}\right)\right] \\
D_1^* &= \frac{1}{2}\left[T_1\left(\frac{p+q}{2}\right) + j\frac{\varepsilon_0^{1-\alpha}}{\beta_{01}} S_1\left(\frac{p+q}{2}\right)\right] \\
D_2^* &= \frac{1}{2}\left[T_1\left(\frac{p+q}{2}\right) - j\frac{\varepsilon_0^{1-\alpha}}{\beta_{01}} S_1\left(\frac{p+q}{2}\right)\right] \\
D_3^* &= \frac{1}{2}\left[-T_1\left(\frac{p+q}{2}\right) - j\frac{\varepsilon_1^{1-\alpha}}{\beta_{11}} S_1\left(\frac{p+q}{2}\right)\right] \\
D_4^* &= \frac{1}{2}\left[-T_1\left(\frac{p+q}{2}\right) + j\frac{\varepsilon_1^{1-\alpha}}{\beta_{11}} S_1\left(\frac{p+q}{2}\right)\right] .
\end{aligned}\right\} \quad (5.23)$$

Also,

$$\begin{aligned}
C_1^* &= e^{j(\chi-2\theta_0^*)} D_1^* \\
C_2^* &= e^{j(\chi+2\theta_0^*)} D_2^* \\
C_3^* &= e^{j2\theta_1^*} D_3^* \\
C_4^* &= e^{-j2\theta_1^*} D_4^*
\end{aligned}$$

where

$2\theta_0^* = \beta_{01}(p-q)$ = the phase change in the air region

$2\theta_1^* = \beta_{11}q$ = the phase change in the disc.

The dispersion relation is given by

$$\cos\chi = \cos 2\theta_0^* \cos 2\theta_1^* - \frac{1}{2}\left[\frac{\beta_{01}}{\beta_{11}}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^{1-\alpha} + \frac{\beta_{11}}{\beta_{01}}\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1-\alpha}\right]\sin 2\theta_0^* \sin 2\theta_1^* \tag{5.24}$$

and C_1^*, C_2^* and C_3^* are found in terms of C_4^* to be

$$\left.\begin{aligned}
\frac{C_1^*}{C_4^*} &= \frac{\beta_{11}}{\beta_{01}}\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1-\alpha} e^{j\chi}\left\{\frac{-\cos 2\theta_1^* - j\frac{\beta_{01}}{\beta_{11}}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^{1-\alpha}\sin 2\theta_1^* + e^{-j(\chi+2\theta_0^*)}}{\cos 2\theta_0^* + j\frac{\beta_{11}}{\beta_{01}}\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1-\alpha}\sin 2\theta_0^* - e^{j(\chi-2\theta_1^*)}}\right\} \\
\frac{C_2^*}{C_4^*} &= \frac{\beta_{11}}{\beta_{01}}\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1-\alpha} e^{j\chi}\left\{\frac{\cos 2\theta_1^* - j\frac{\beta_{01}}{\beta_{11}}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^{1-\alpha}\sin 2\theta_1^* - e^{-j(\chi-2\theta_0^*)}}{\cos 2\theta_0^* + j\frac{\beta_{11}}{\beta_{01}}\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1-\alpha}\sin 2\theta_0^* - e^{j(\chi-2\theta_1^*)}}\right\} \\
\frac{C_3^*}{C_4^*} &= \left\{\frac{-\cos 2\theta_0^* + j\frac{\beta_{11}}{\beta_{01}}\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1-\alpha}\sin 2\theta_0^* + e^{j(\chi+2\theta_1^*)}}{\cos 2\theta_0^* + j\frac{\beta_{11}}{\beta_{01}}\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1-\alpha}\sin 2\theta_0^* - e^{j(\chi-2\theta_1^*)}}\right\}.
\end{aligned}\right\} \tag{5.25}$$

From (4.56) and (4.57), for $-(\frac{p-q}{2}) \le z \le \frac{p-q}{2}$,

$$T_1(z) = C_1^* \, e^{-j\beta_{01}\left(z - (\frac{p-q}{2})\right)} + C_2^* \, e^{j\beta_{01}\left(z - (\frac{p-q}{2})\right)}$$

and for $\frac{p-q}{2} \le z \le \frac{p+q}{2}$

$$T_1(z) = -C_3^{*}\, e^{-j\beta_{11}\left(z - \left(\frac{p-q}{2}\right)\right)} - C_4^{*}\, e^{j\beta_{11}\left(z - \left(\frac{p-q}{2}\right)\right)} .$$

The new matched condition is

$$\frac{\beta_{11}}{\beta_{01}}\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{1-\alpha} = 1 \tag{5.26}$$

which gives

$$\chi = 2\theta_0^{*} + 2\theta_1^{*} . \tag{5.27}$$

Some points can, now, be made with regard to how the lowest order mode in the solid disc structure is affected by the introduction of the hole. From (5.20)

$$\left.\frac{1}{\varepsilon_0^{1-\alpha}}\frac{dT_1}{dz}\right|_{z=\frac{p-q}{2}-\delta} = \left.\frac{1}{\varepsilon_1^{1-\alpha}}\frac{dT_1}{dz}\right|_{z=\frac{p-q}{2}+\delta} \tag{5.28}$$

where $\delta > 0$ and $\delta \rightarrow 0$. Equation (5.28) can be rewritten as

$$\left.\frac{1}{\varepsilon_0}\frac{dT_1}{dz}\right|_{z=\frac{p-q}{2}-\delta} = \left.\frac{1}{\varepsilon_1\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{\alpha}}\frac{dT_1}{dz}\right|_{z=\frac{p-q}{2}+\delta} . \tag{5.29}$$

For the solid disc problem

$$\left.\frac{1}{\varepsilon_0}\frac{dT}{dz}\right|_{z=\frac{p-q}{2}-\delta} = \left.\frac{1}{\varepsilon_1}\frac{dT_1}{dz}\right|_{z=\frac{p-q}{2}+\delta} . \tag{5.30}$$

By analogy between (5.30) and (5.29) an effective permittivity for the dielectric disc with a hole is

$$\varepsilon_{eff.} = \varepsilon_1 \left(\frac{\varepsilon_0}{\varepsilon_1}\right)^{\alpha} . \tag{5.31}$$

Inspection of (5.31) shows that since $a > 0$,

$$\varepsilon_{eff.} < \varepsilon_1 .$$

Consequently, from this point of view, the hole reduces the effective permittivity of the disc.

As can easily be seen,

$$\beta_{01}{}^2 = \beta_0{}^2 .$$

Therefore, in the air region the presence of a small hole does not appear to affect the propagation coefficient and, in turn, the phase change. However,

$$\beta_{11}{}^2 = \beta_1{}^2 - \Delta K_1{}^2 .$$

Hence, the propagation coefficient in the region of the disc is reduced. Because of this reduction, the phase change across the disc is decreased.

From the above statements it appears that the perturbation caused by the hole may be reduced by increasing the dielectric constant of the disc.

A more accurate treatment which includes the effect of T_2 and, if desired, even T_3 should be possible without undue difficulties.

5.22 An Approximate Series Solution

An approximate series solution is now developed for (5.5),

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\right] + \frac{\partial^2 H_\phi}{\partial z^2} + \omega^2\mu_0\varepsilon_0 H_\phi = -\omega^2\mu_0 c_0 h(z)g(r)H_\phi$$

$$+ c_0 h(z)\frac{dg}{dz}\frac{1}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) + c_0\frac{dh}{dz}g(r)\frac{1}{\varepsilon}\frac{\partial H_\phi}{\partial z}\,. \tag{5.32}$$

The series expansion for H_ϕ is in the Floquet form,

$$H_\phi = \sum_{n=-\infty}^{\infty} H_n(r)e^{-j\left(\frac{\hat{\chi}+2n\pi}{p}\right)z} = \sum_{n=-\infty}^{\infty} H_n(r)e^{-\hat{s}_n z} \tag{5.33}$$

where $H_n(r)$ is the coefficient of the nth space harmonic.

If (5.32) is multiplied by $\frac{1}{p}\,e^{\hat{s}_n z}$ and the ensuing expression is integrated from 0 to p, then replacing H_ϕ by the series (5.33) produces a double infinity of linear homogeneous differential equations of second order. The solution to the set of differential equations, once substituted into (5.33), provides an exact solution for H_ϕ. Clearly, in practice, only an approximate solution to the set can be anticipated.

The approximate solution to be developed in what follows is analogous to the Stodola and Vianello method[59] for determining approximate solutions to boundary value problems in ordinary differential equations. In (5.32) the field terms to the right of the equality sign are approximated by the solid disc field terms found in section 4.523. Equally, and no doubt with greater accuracy, the solution for the first mode approximation in section 5.21 could be used. The solid disc solution is chosen because it is thought that the nature of the series approximation can be more easily discussed relative to the solid disc solution than to the first mode approximation.

Once the solid disc field is utilized in (5.32), the terms to the right of the equality sign act as a driving function and H_ϕ is the boundary value solution to this new driven differential equation. Actually, what is found is a first iterative solution and the field for the solid disc problem is the guessed solution.

Since from (2.18)

$$\left.\begin{aligned} \frac{\partial H_\phi}{\partial z} &= -j\omega\varepsilon E_r \\ \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) &= j\omega\varepsilon E_z, \end{aligned}\right\} \tag{5.34}$$

(5.32) can be rewritten as

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\right] + \frac{\partial^2 H_\phi}{\partial z^2} + \omega^2\mu_0\varepsilon_0 H_\phi = -\omega^2\mu_0 c_0 h(z)g(r)H_\phi + j\omega c_0 h(z)\frac{dg}{dr}E_z - j\omega c_0\frac{dh}{dz}g(r)E_r. \tag{5.35}$$

At this point, H_ϕ in the terms of (5.35) to the left of the equality sign is replaced by the series (5.33). Once this is done, (5.35) is multiplied by $\frac{1}{p}e^{\hat{s}_n z}$ and the resulting expression is integrated from 0 to p. Hence,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rH_n)\right] + (\hat{s}_n^{\,2} + \omega^2\mu_0\varepsilon_0)H_n = \frac{1}{p}\left[-\omega^2\mu_0 c_0 g(r)\int_0^p e^{\hat{s}_n z}h(z)H_\phi dz - j\omega c_0 g(r)\int_0^p e^{\hat{s}_n z}\frac{dh}{dz}E_r dz + j\omega c_0\frac{dg}{dr}\int_0^p e^{\hat{s}_n z}h(z)E_z dz\right]$$

or

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rH_n)\right] - \hat{K}_n^{\,2}H_n = \frac{1}{p}\left[-\omega^2\mu_0 c_0 g(r)\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{\hat{s}_n z}\, H_\phi dz\right.$$

$$\left. -j\omega c_0 g(r)\int_0^p e^{\hat{s}_n z}\,\frac{dh}{dz}\,E_r dz + j\omega c_0\,\frac{dg}{dr}\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{\hat{s}_n z}\,E_z dz\right] \tag{5.36}$$

where

$$\hat{K}_n^{\,2} = -(\hat{s}_n^{\,2} + \omega^2\mu_0\varepsilon_0) = \left(\frac{\hat{\chi}+2n\pi}{p}\right)^2 - \omega^2\mu_0\varepsilon_0. \tag{5.37}$$

The expression,

$$g(r)\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{\hat{s}_n z}\, H_\phi dz, \tag{5.38}$$

is completely determined provided H_ϕ is known in the dielectric disc because $g(r)$ is zero over the interval, $0 \le r < a$, and the region of integration is only from $\frac{p-q}{2}$ to $\frac{p+q}{2}$. Similarly, the expression,

$$g(r)\int_0^p e^{\hat{s}_n z}\,\frac{dh}{dz}\,E_r dz, \tag{5.39}$$

is specified completely if E_r is known inside the dielectric region. The reason is that at $z = \frac{p-q}{2}$ and $z = \frac{p+q}{2}$, E_r is continuous across the interface between the air and dielectric material. Also, due to g(r), E_r needs only to be known in the region, $a<r<b$. Since the limits of integration of

$$\frac{dg}{dr}\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{\hat{s}_n z}\, E_z dz \tag{5.40}$$

are only from $\frac{p-q}{2}$ to $\frac{p+q}{2}$ and E_z is continuous at $r = a$, the determination of (5.40) can be made once E_z is known in the dielectric disc. Therefore, the first iterative solution can be found by guessing a field solution in the dielectric region only.

When the solid disc field is used in (5.38), (5.39) and (5.40), a guess for each space harmonic in the dielectric region is provided, since each field component is expressible as a sum of space harmonics. If the guessed space harmonic is $H_n^0\, e^{-s_n z}$, then

$$H_n e^{-\hat{s}_n z} = E_n H_n^0\, e^{-s_n z}.$$

As can be seen from the first mode approximation, for $a \rightarrow 0$, $E_n \rightarrow 1$.

From section 4.521 the solid disc solution is

$$D_z = J_0(K_1 r)T(z).$$

Therefore, in the dielectric region from (5.34)

$$\left.\begin{aligned} E_r &= -\frac{1}{K_1\varepsilon_1} J_1(K_1 r) \frac{dT}{dz} = -\frac{1}{K_1} J_1(K_1 r)\, S(z) \\ H_\phi &= j\frac{\omega}{K_1} J_1(K_1 r)\, T(z) \\ E_z &= \frac{1}{\varepsilon_1} J_0(K_1 r)\, T(z)\ . \end{aligned}\right\} \quad (5.41)$$

The substitution of (5.41) into (5.36) gives the differential equation for $H_n^{\ 1}$, the amplitude of the first iterative space harmonic. This equation is

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(rH_n^{\ 1}\right)\right] - \hat{K}_n^{\ 2} H_n^{\ 1} = e_n J_1(K_1 r)\, g(r) + j\omega f_n J_0(K_1 r) \frac{dg}{dr} \quad (5.42)$$

where

$$\left.\begin{aligned} e_n &= -j\frac{\omega}{K_1}\left(\frac{\omega^2 \mu_0 c_0}{p}\right)\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{\hat{s}_n z}\, T(z)\, dz + j\frac{\omega}{K_1}\left(\frac{c_0}{p}\right)\int_0^p e^{\hat{s}_n z}\, S(z) \frac{dh}{dz}\, dz \\ f_n &= \frac{1}{\varepsilon_1}\left(\frac{c_0}{p}\right)\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{\hat{s}_n z}\, T(z)\, dz. \end{aligned}\right\} (5.43)$$

From (4.38) and (4.40)

$$\left.\begin{aligned} e_n = j\frac{\omega}{K_1}\left(\frac{c_0}{p(\hat{s}_n^{\ 2}+\beta_1^{\ 2})}\right)\Bigg[& e^{\hat{s}_n\left(\frac{p-q}{2}\right)}\left[\omega^2\mu_0\hat{s}_n\, T\left(\tfrac{p-q}{2}\right)+(\hat{s}_n^{\ 2}-K_1^{\ 2})\, S\left(\tfrac{p-q}{2}\right)\right] \\ & -e^{\hat{s}_n\left(\frac{p+q}{2}\right)}\left[\omega^2\mu_0\hat{s}_n\, T\left(\tfrac{p+q}{2}\right)+(\hat{s}_n^{\ 2}-K_1^{\ 2})\, S\left(\tfrac{p+q}{2}\right)\right]\Bigg] \end{aligned}\right\} (5.44)$$

$$f_n = \frac{1}{\varepsilon_1}\left(\frac{c_0}{p(\hat{s}_n^{\,2}+\beta_1^{\,2})}\right)\left[e^{\hat{s}_n\left(\frac{p+q}{2}\right)}\left[\hat{s}_n\, T\left(\tfrac{p+q}{2}\right) - \varepsilon_1\, S\left(\tfrac{p+q}{2}\right)\right] - e^{\hat{s}_n\left(\frac{p-q}{2}\right)}\left[\hat{s}_n\, T\left(\tfrac{p-q}{2}\right) - \varepsilon_1\, S\left(\tfrac{p-q}{2}\right)\right]\right]$$

where $T(\frac{p-q}{2})$, $S(\frac{p-q}{2})$, $T(\frac{p+q}{2})$ and $S(\frac{p+q}{2})$ can be determined to within an arbitrary constant by means of (4.43), (4.44), (4.52), (4.54) and (4.55).

An examination of the terms in the driving function for (5.42) will now be carried out. These terms are

$$j\omega f_n\, J_0(K_1 r)\, \frac{dg}{dr} \tag{5.45}$$

$$e_n\, J_1(K_1 r)\, g(r). \tag{5.46}$$

Over the intervals, $0 \le r < a$ and $a < r < b$, (5.45) makes no contribution. However, (5.45) gives an impulse at the surface, $r = a$, and thus contributes to the boundary condition at this surface for the derivative, $\frac{1}{r}\frac{d}{dr}(rH_n^{\,1})$. In a sense, the effect of the discontinuity caused by the hole is averaged at the surface, $r = a$, over the complete period of the structure. The term giving the averaging action is in f_n and, in particular, from (5.43) is

$$\frac{1}{p}\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{\hat{s}_n z}\, T(z)\, dz.$$

The expression in (5.46) provides the coupling between the space harmonics. This feature can be noted by substituting into e_n the space harmonic expansions for T(z) and S(z). In the limit as $a \longrightarrow 0$,

$$g(r) = 1$$

for all r and (5.42) reduces to

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rH_n{}^1)\right] - \hat{K}_n{}^2 H_n{}^1 = e_n\, J_1(K_1 r). \tag{5.47}$$

If the condition that E_z is zero at the waveguide wall is kept in mind, the solution to (5.47) is

$$H_n{}^1 = \frac{e_n}{\hat{s}_n{}^2 + \beta_0{}^2}\, J_1(K_1 r)$$

which is the solid disc solution. By taking the other extreme where the waveguide is empty , a = b, (5.42) becomes

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rH_n{}^1)\right] - \hat{K}_n{}^2\, H_n{}^1 = 0. \tag{5.48}$$

Similarly, for $c_0 = \varepsilon_1 - \varepsilon_0 = 0$, (5.48) results. The solution to (5.48) is

$$H_n{}^1 = J_1(j\hat{K}_n r)$$

where

$$J_0(j\hat{K}_n b) = 0.$$

Hence, the space harmonic solution is equivalent to the mode solution for an empty waveguide. Consequently, as should be expected, from (5.42) for $a = b$ or for $c_0 = 0$ no coupling exists between the space harmonics, and for $a = 0$ the coupling region is a maximum. When a hole is present in the disc, (5.42) shows that the coupling region is reduced from the solid disc case by the factor $g(r)$. Hence, from a physical point of view (5.42) seems quite reasonable.

The general solution for (5.42) will now be treated. $H_n^{\,1}$ could be expanded in a series made up of a complete set of functions and the coefficients of the series could be determined from (5.42). However, for the sake of simplicity, $H_n^{\,1}$ will be solved in each region where (5.42) is well behaved and boundary conditions will be utilized to match the solutions at $r = a$.

Examining (5.42) shows that in the region, $0 \leq r < a$, $H_n^{\,1}$ must satisfy the differential equation

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(rH_n^{\,1}\right)\right] - \hat{K}_n^{\,2}\,H_n^{\,1} = 0 \qquad (5.49)$$

and in the region, $a < r < b$,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(rH_n^{\,1}\right)\right] - \hat{K}_n^{\,2}\,H_n^{\,1} = e_n\,J_1(K_1 r). \qquad (5.50)$$

The boundary conditions for $H_n^{\,1}$ can be established by integrating (5.42). On the first integration

$$\frac{1}{r}\frac{d}{dr}\left(rH_n^{\,1}\right) = \hat{K}_n^{\,2}\int^{r} H_n^{\,1}(\tau)d\tau + e_n\int^{r} J_1(K_1\tau)g(\tau)d\tau$$

$$+ j\omega f_n\int^{r} J_0(K_1\tau)\frac{dg}{d\tau}\,d\tau \qquad (5.51)$$

and on the second integration

$$rH_n^{\,1} = \hat{K}_n^{\,2}\int^{r}\tau\,d\tau\int^{\tau}H_n^{\,1}(s)ds + e_n\int^{r}\tau\,d\tau\int^{\tau}J_1(K_1 s)g(s)ds$$

$$+ j\omega f_n\int^{r}\tau\,d\tau\int^{\tau}J_0(K_1 s)\frac{dg}{ds}\,ds. \qquad (5.52)$$

The integral

$$\int^{r}J_1(K_1\tau)g(\tau)d\tau$$

is a continuous function of r and the integral

$$\int^{r}J_0(K_1\tau)\frac{dg}{d\tau}\,d\tau$$

is a function of r with a finite discontinuity at $r = a$. Consequently, both the integrals

$$\int^{r}\tau\,d\tau\int^{\tau}J_1(K_1 s)g(s)ds$$

$$\int^{r}\tau\,d\tau\int^{\tau}J_0(K_1 s)\frac{dg}{ds}\,ds$$

are continuous functions. Therefore, by an argument similar to the one advanced in section 4.2,

$$H_n^{\,1}(r) = \text{a continuous function.} \qquad (5.53)$$

Since the integral

$$\int^{r} J_0(K_1\tau)\frac{dg}{d\tau}\,d\tau$$

is zero over the interval $0 \le r < a$ and has a finite step of $J_0(K_1 a)$ at $r = a$,

$$\frac{1}{r}\frac{d}{dr}(rH_n^{\,1})\bigg|_{r\,=\,a-} = \hat{K}_n^{\,2}\int^{r\,=\,a-} H_n^{\,1}(\tau)d\tau + e_n\int^{r\,=\,a-} J_1(K_1\tau)g(\tau)d\tau$$

$$\frac{1}{r}\frac{d}{dr}(rH_n^{\,1})\bigg|_{r\,=\,a+} = \hat{K}_n^{\,2}\int^{r\,=\,a+} H_n^{\,1}(\tau)d\tau + e_n\int^{r\,=\,a+} J_1(K_1\tau)g(\tau)d\tau$$

$$+ j\omega f_n J_0(K_1 a)$$

where

$$a- = \lim_{\delta\to 0}(a - \delta)$$

$$a+ = \lim_{\delta\to 0}(a + \delta)\;.$$

Hence, the resulting boundary condition at $r = a$ is

$$\frac{1}{r}\frac{d}{dr}(rH_n^{\,1})\bigg|_{r\,=\,a-} = \frac{1}{r}\frac{d}{dr}(rH_n^{\,1})\bigg|_{r\,=\,a+} - j\omega f_n J_0(K_1 a). \tag{5.54}$$

In the region, $0 \le r \le a$, from (5.49) the solution for $H_n^{\,1}$

is

$$H_n^{\,1} = A_n I_1(\hat{K}_n r). \tag{5.55}$$

If $\hat{K}_n$ is imaginary, $I_1(\hat{K}_n r)$ becomes $J_1(j\hat{K}_n r)$.

In the region, $a \leq r \leq b$, the particular solution for (5.50) has the form

$$H_{np}^{\,1} = B_n\, J_1(K_1 r). \tag{5.56}$$

Through the substitution of (5.56) into (5.50),

$$(-K_1^{\,2} - \hat{K}_n^{\,2})B_n = e_n$$

or

$$B_n = \frac{e_n}{(-\hat{K}_n^{\,2} - K_1^{\,2})} = \frac{e_n}{(\hat{s}_n^{\,2} + \omega^2\mu_0\varepsilon_0 - K_1^{\,2})} = \frac{e_n}{\hat{s}_n^{\,2} + \beta_0^{\,2}} = \frac{j\omega}{K_1}\,\hat{a}_n. \tag{5.57}$$

The homogeneous solution for (5.50) is

$$H_{nh}^{\,1} = C_n'\; I_1(\hat{K}_n r) + D_n K_1(\hat{K}_n r).$$

When $\hat{K}_n$ is imaginary, $I_1(\hat{K}_n r)$ is replaced by $J_1(j\hat{K}_n r)$ and $K_1(\hat{K}_n r)$ is replaced by $Y_1(j\hat{K}_n r)$. The combined solution for (5.50) is

$$H_n^{\,1} = j\frac{\omega}{K_1}\,\hat{a}_n\, J_1(K_1 r) + C_n'\; I_1(\hat{K}_n r) + D_n K_1(\hat{K}_n r). \tag{5.58}$$

At $r = b$, since E_z is zero,

$$\left. \frac{1}{r} \frac{d}{dr} (rH_\phi) \right|_{r = b} = 0.$$

As a result, from the series in (5.33)

$$\left. \frac{1}{r} \frac{d}{dr} (rH_n) \right|_{r = b} = \left. \frac{1}{r} \frac{d}{dr} (rH_n{}^1) \right|_{r = b} = 0. \qquad (5.59)$$

Substituting (5.58) into (5.59) yields

$$j\omega\hat{a}_n\, J_0(K_1 b) + \hat{K}_n C_n{}'\, I_0(\hat{K}_n b) - \hat{K}_n D_n K_0(\hat{K}_n b) = 0.$$

Since

$$J_0(K_1 b) = 0,$$

then

$$D_n = \frac{I_0(\hat{K}_n b)}{K_0(\hat{K}_n b)}\; C_n{}' = I_0(\hat{K}_n b)\hat{C}_n.$$

Therefore,

$$H_n{}^1 = j\frac{\omega}{K_1}\, \hat{a}_n J_1(K_1 r) + \left[K_0(\hat{K}_n b)\, I_1\, (\hat{K}_n r) + I_0(\hat{K}_n b)\, K_1\, (\hat{K}_n r)\right] \hat{C}_n. \qquad (5.60)$$

At the boundary, $r = a$, from (5.53), (5.54), (5.55) and (5.60)

$$I_1(\hat{K}_n a)\, A_n = G_1(\hat{K}_n a)\, \hat{C}_n + j\frac{\omega}{K_1}\, \hat{a}_n\, J_1(K_1 a)$$

$$\hat{K}_n\, I_0(\hat{K}_n a)\, A_n = \hat{K}_n\, G_0(\hat{K}_n a)\, \hat{C}_n + j\omega(\hat{a}_n - f_n) J_0(K_1 a)$$

where

$$G_1(\hat{K}_n r) = K_0(\hat{K}_n b)\ I_1(\hat{K}_n r) + I_0(\hat{K}_n b)\ K_1(\hat{K}_n r)$$

$$G_0(\hat{K}_n r) = K_0(\hat{K}_n b)\ I_0(\hat{K}_n r) - I_0(\hat{K}_n b)\ K_0(\hat{K}_n r).$$

Hence, A_n is given by

$$A_n = -j\omega \frac{\left[\left(\frac{\hat{K}_n}{K_1}\right) \hat{a}_n\ J_1(K_1 a)\ G_0(\hat{K}_n a) - (\hat{a}_n - f_n)\ J_0(K_1 a)\ G_1(\hat{K}_n a)\right]}{\hat{K}_n\ I_0(\hat{K}_n b)\left[I_0(\hat{K}_n a)\ K_1(\hat{K}_n a) + I_1(\hat{K}_n a)\ K_0(\hat{K}_n a)\right]}.$$

The term,

$$I_0(\hat{K}_n a)\ K_1(\hat{K}_n a) + I_1(\hat{K}_n a)\ K_0(\hat{K}_n a),$$

is the Wronskian and is equal to[60]

$$I_0(\hat{K}_n a)\ K_1(\hat{K}_n a) + I_1(\hat{K}_n a)\ K_0(\hat{K}_n a) = \frac{1}{\hat{K}_n a}.$$

Consequently,

$$A_n = -j\omega a \frac{\left[\left(\frac{\hat{K}_n}{K_1}\right) \hat{a}_n\ J_1(K_1 a)\ G_0(\hat{K}_n a) - (\hat{a}_n - f_n) J_0(K_1 a)\ G_1(\hat{K}_n a)\right]}{I_0(\hat{K}_n b)}. \tag{5.61}$$

Similarly,

$$\hat{C}_n = -j\omega a \frac{\left[\left(\frac{\hat{K}_n}{K_1}\right) \hat{a}_n\ J_1(K_1 a)\ I_0(\hat{K}_n a) - (\hat{a}_n - f_n) J_0(K_1 a)\ I_1(\hat{K}_n a)\right]}{I_0(\hat{K}_n b)}. \tag{5.62}$$

Since the Wronskian appearing when $\hat{K}_n$ is imaginary is

$$J_1(j\hat{K}_n a)\; Y_0(j\hat{K}_n a) - J_0(j\hat{K}_n a)\; Y_1(j\hat{K}_n a) = \frac{2}{\pi j \hat{K}_n a},$$

for such a case careful attention should be given to include the $2/\pi$ term in A_n and C_n.

According to (5.34) on the axis of the waveguide

$$E_z\Big|_{r=0} = \frac{1}{j\omega\varepsilon_0}\,\frac{1}{r}\,\frac{\partial}{\partial r}\,(rH_\phi)\Big|_{r=0}.$$

Through the use of the series in (5.33),

$$E_z\Big|_{r=0} = \frac{1}{j\omega\varepsilon_0}\sum_{n=-\infty}^{\infty}\frac{1}{r}\,\frac{d}{dr}\,(rH_n)\Big|_{r=0}\; e^{-s_n z}.$$

Hence, for this case the coefficient of the nth space harmonic of E_z is

$$b_n = \frac{1}{j\omega\varepsilon_0}\,\frac{1}{r}\,\frac{d}{dr}\,(rH_n)\Big|_{r=0}.$$

By means of (5.55) and (5.61)

$$b_n \cong \frac{\hat{K}_n}{j\omega\varepsilon_0}\,A_n\, I_0(\hat{K}_n r)\Big|_{r=0}$$

$$= -\hat{K}_n a\,\frac{\left[\left(\frac{\hat{K}_n}{K_1}\right)\hat{a}_n\, J_1(K_1 a)\, G_0(\hat{K}_n a) - (\hat{a}_n - f_n)\, J_0(K_1 a)\, G_1(\hat{K}_n a)\right]}{\varepsilon_0 I_0(\hat{K}_n b)}. \tag{5.63}$$

Over the interval, $0 \leq r \leq a$, from (5.33), (5.55) and (5.61) the first iterative solution can be expressed as

$$H_\phi^1 = j\frac{\omega}{K_1}a \sum_{n=-\infty}^{\infty} \frac{\left[K_1(\hat{a}_n - f_n)J_0(K_1 a)G_1(\hat{K}_n a) - \hat{K}_n \hat{a}_n J_1(K_1 a)G_0(\hat{K}_n a)\right]}{I_0(\hat{K}_n b)} I_1(\hat{K}_n r)e^{-\hat{s}_n z} \tag{5.64}$$

and over the interval, $a \leq r \leq b$, from (5.33), (5.60) and (5.62)

$$H_\phi'^1 = j\frac{\omega}{K_1} J_1(K_1 r) \sum_{n=-\infty}^{\infty} \hat{a}_n e^{-\hat{s}_n z} \tag{5.65}$$

$$+j\frac{\omega}{K_1}a \sum_{n=-\infty}^{\infty} \frac{\left[K_1(\hat{a}_n - f_n)J_0(K_1 a)I_1(\hat{K}_n a) - \hat{K}_n \hat{a}_n J_1(K_1 a)I_0(\hat{K}_n a)\right]}{I_0(\hat{K}_n b)} G_1(\hat{K}_n r)e^{-\hat{s}_n z}.$$

In (5.65) the first series corresponds to the solid disc solution and the second series is a perturbation term. This statement will become more evident as the discussion progresses.

The dispersion relation is found by equating at the point $(r = b,\ z = p/2)$ the guessed solution for H_ϕ in (5.41) and the first iterative solution in (5.65). This point was chosen because it is thought to be the point in the dielectric region where the guessed and first iterative solutions behave most like the exact solution. The justification for this statement is based on the fact that the chosen point is farthest removed from the perturbation caused by the hole and, in particular, from the perturbation near the corners introduced by the hole. By equating the guessed and first iterative solutions at a point where the field is relatively well-behaved, the hope is that elsewhere the first iterative solution is allowed to deviate more than otherwise from the guessed solution and in turn approach more closely to the exact solution.

From (5.41) and (5.65) at r = b, z = p/2

$$j\frac{\omega}{K_1} J_1(K_1 b) T(\tfrac{p}{2}) = j\frac{\omega}{K_1} J_1(K_1 b) \sum_{n=-\infty}^{\infty} \hat{a}_n e^{-\hat{s}_n \frac{p}{2}}$$

$$+j\frac{\omega}{K_1}(\tfrac{a}{b}) \sum_{n=-\infty}^{\infty} \frac{\left[K_1(\hat{a}_n - f_n) J_0(K_1 a) I_1(\hat{K}_n a) - \hat{K}_n \hat{a}_n J_1(K_1 a) I_0(\hat{K}_n a)\right]}{\hat{K}_n I_0(\hat{K}_n b)} e^{-\hat{s}_n \frac{p}{2}} \tag{5.66}$$

where

$$G_1(\hat{K}_n b) = \frac{1}{\hat{K}_n b}$$

has been employed. According to Appendix 3

$$\sum_{n=-\infty}^{\infty} \hat{a}_n e^{-\hat{s}_n \frac{p}{2}} = T(\tfrac{p}{2}) + \frac{e^{-j\frac{\hat{\chi}}{2}}}{2j}\left[\frac{C_1 e^{j\theta_0}}{\sin\left(\frac{\hat{\chi}-\beta_0 p}{2}\right)} + \frac{C_2 e^{-j\theta_0}}{\sin\left(\frac{\hat{\chi}+\beta_0 p}{2}\right)}\right]\left[1 - e^{j(\hat{\chi}-\chi)}\right].$$

Therefore, (5.66) becomes

$$\frac{J_1(K_1 b) e^{-j\frac{\hat{\chi}}{2}}}{2j}\left[\frac{C_1 e^{j\theta_0}}{\sin\left(\frac{\hat{\chi}-\beta_0 p}{2}\right)} + \frac{C_2 e^{-j\theta_0}}{\sin\left(\frac{\hat{\chi}+\beta_0 p}{2}\right)}\right]\left[e^{j(\hat{\chi}-\chi)} - 1\right]$$

$$= (\tfrac{a}{b}) \sum_{n=-\infty}^{\infty} \frac{\left[K_1(\hat{a}_n - f_n) J_0(K_1 a) I_1(\hat{K}_n a) - \hat{K}_n \hat{a}_n J_1(K_1 a) I_0(\hat{K}_n a)\right]}{\hat{K}_n I_0(\hat{K}_n b)} e^{-\hat{s}_n \frac{p}{2}} \tag{5.67}$$

and from this relation, since χ is known from (4.54), $\hat{\chi}$ can be determined as a function of ω for any particular structure.

Provided $(\hat{\chi}-\chi)$ is small, a simplified formula for obtaining $\hat{\chi}$ can be found from (5.67). To begin with,

$$\left[e^{j(\hat{\chi}-\chi)} - 1\right] \cong j(\hat{\chi}-\chi).$$

Therefore, from (5.67)

$$\hat{\chi}-\chi \cong R_e{}^{*}\left[\left(\frac{a}{b}\right)\frac{2\sin\left(\frac{\hat{\chi}-\beta_0 p}{2}\right)\sin\left(\frac{\hat{\chi}+\beta_0 p}{2}\right)\, e^{j\frac{\hat{\chi}}{2}}}{J_1(K_1 b)\left[\sin\left(\frac{\hat{\chi}+\beta_0 p}{2}\right)e^{j\theta_0}C_1+\sin\left(\frac{\hat{\chi}-\beta_0 p}{2}\right)e^{-j\theta_0}C_2\right]}\right.$$

$$\left. X \sum_{n=-\infty}^{\infty} \frac{\left[K_1(\hat{a}_n-f_n)J_0(K_1 a)I_1(\hat{K}_n a) - \hat{K}_n\hat{a}_n J_1(K_1 a)I_0(\hat{K}_n a)\right]}{\hat{K}_n I_0(\hat{K}_n b)}\, e^{-\hat{s}_n \frac{p}{2}}\right]. \tag{5.68}$$

The terms of the series in (5.67) which give a $(\hat{\chi}-\chi)$ contribution are of second order. This statement may be verified by investigating the relative magnitudes of the terms for the series in (5.67) compared with the terms of the series

$$J_1(K_1 b)\sum_{n=-\infty}^{\infty} \hat{a}_n e^{-\hat{s}_n \frac{p}{2}}.$$

As a result, for a first approximation the right side of (5.68) is evaluated for $\hat{\chi}=\chi$ and thus

---

* R_e equals real part.

$$\hat{\chi}-\chi \cong R_e\left[\left(\frac{a}{b}\right) \frac{2\sin\left(\frac{\chi-\beta_0 p}{2}\right)\sin\left(\frac{\chi+\beta_0 p}{2}\right)e^{j\frac{\chi}{2}}}{J_1(K_1 b)\left[\sin\left(\frac{\chi+\beta_0 p}{2}\right)e^{j\theta_0}C_1+\sin\left(\frac{\chi-\beta_0 p}{2}\right)e^{-j\theta_0}C_2\right]}\right.$$

$$\left. \times \sum_{\substack{n=-\infty \\ \hat{\chi}=\chi}}^{\infty} \frac{\left[K_1(\hat{a}_n-f_n)J_0(K_1 a)I_1(\hat{K}_n a)-\hat{K}_n\hat{a}_n J_1(K_1 a)I_0(\hat{K}_n a)\right]}{\hat{K}_n I_0(\hat{K}_n b)}e^{-\hat{s}_n\frac{p}{2}}\right]. \tag{5.69}$$

For sufficiently small "a"[61]

$$J_1(K_1 a) \longrightarrow \frac{1}{2}K_1 a$$

$$I_0(\hat{K}_n a) \longrightarrow 1$$

$$J_0(K_1 a) \longrightarrow 1$$

$$I_1(\hat{K}_n a) \longrightarrow \frac{1}{2}\hat{K}_n a$$

and thus from (5.62)

$$\hat{C}_n \longrightarrow -j\frac{\omega\hat{K}_n f_n}{2\,I_0(\hat{K}_n b)}\,a^2. \tag{5.70}$$

Therefore, as $a \longrightarrow 0$

$$\hat{C}_n \longrightarrow 0.$$

Consequently, the series in (5.67) approaches zero and, as a result,

$$\left[e^{j(\hat{\chi}-\chi)} - 1\right] \longrightarrow 0.$$

Hence,

$$\hat{\chi} \longrightarrow \chi$$

and thus

$$\hat{a}_n \longrightarrow a_n.$$

From (5.65) the first iterative solution approaches the solid disc solution. In other words, in the limit the two solutions are in agreement, as they should be.

Now, (5.69) will be examined for the case where the fields for the solid disc structure are matched. Such an investigation gives some indication of the behaviour of the effective propagation coefficient in the disc. For example, from the solid disc solution and the solution of the first mode approximation, if (4.60) and (5.27), respectively, are utilized,

$$\hat{\chi}-\chi \cong 2\theta_0^* + 2\theta_1^* - 2\theta_0 - 2\theta_1 = (\beta_{11} - \beta_1)q = (\Delta\beta_1)q \tag{5.71}$$

where $\Delta\beta_1$ is the perturbation of β_1 caused by the hole.

This examination will be restricted in that only frequencies in the first pass band will be considered. These frequencies are in the range of greatest interest in beam-coupler design since most higher order modes are still in cutoff.

The radius, a, is to be taken small enough so that $\hat{C}_n$

can be approximated by (5.70). In recollection from section 4.523 for a match to exist

$$C_2 = C_4 = 0$$

$$C_1 = -C_3.$$

Hence, from (4.57) inside the dielectric disc

$$T(z) = C_1 e^{-j\beta_1 \left| z - \left(\frac{p-q}{2}\right)\right|} .$$

As a consequence, (5.43) gives

$$f_n\Big|_{\hat{\chi}=\chi} = \frac{1}{p}\,\frac{c_0}{\varepsilon_1}\,C_1 e^{j\beta_1\left(\frac{p-q}{2}\right)} \int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_n z}\, e^{-j\beta_1 z}\, dz$$

$$= \frac{2}{p}\,\frac{c_0}{\varepsilon_1}\,C_1 e^{-j\beta_1\frac{q}{2}}\, e^{j\left(\frac{\chi+2n\pi}{2}\right)} \left[\frac{\sin\left(\frac{\chi+2n\pi}{p} - \beta_1\right)\frac{q}{2}}{\left(\frac{\chi+2n\pi}{p} - \beta_1\right)}\right]. \tag{5.72}$$

Substituting (5.72) into (5.70) yields

$$\hat{C}_n\Big|_{\hat{\chi}=\chi} \cong -j\frac{\omega a^2}{p}\,\frac{c_0}{\varepsilon_1}\, e^{-j\beta_1\frac{q}{2}}\, e^{j\left(\frac{\chi+2n\pi}{2}\right)} \left[\frac{\sin\left(\frac{\chi+2n\pi}{p} - \beta_1\right)\frac{q}{2}}{\left(\frac{\chi+2n\pi}{p} - \beta_1\right)}\right] \frac{\hat{K}_n C_1}{I_0(\hat{K}_n b)}\Bigg|_{\hat{\chi}=\chi} . \tag{5.73}$$

If (5.73) is employed in (5.69), the result is

$$\hat{\chi}-\chi \cong -K_1 b\left(\frac{a}{b}\right)^2\left(\frac{c_0}{\varepsilon_1}\right)\left(\frac{2}{p}\right)\frac{\sin\left(\frac{\chi-\beta_0 p}{2}\right)}{J_1(K_1 b)}\sum_{n=-\infty}^{\infty}\frac{\sin\left(\frac{\chi+2n\pi}{p}-\beta_1\right)\frac{q}{2}}{I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi}\left(\frac{\chi+2n\pi}{p}-\beta_1\right)} \, . \tag{5.74}$$

The series in (5.74) has been summed in Appendix 4. By means of this summation and the fact that

$$\chi = \beta_0(p-q) + \beta_1 q,$$

$$\hat{\chi}-\chi \cong -\frac{1}{2}\left(\frac{a}{b}\right)^2\frac{c_0}{\varepsilon_1}\frac{q}{\beta_1}\left(\frac{K_1}{J_1(K_1 b)}\right)^2 (\eta_1 + \eta_2 + \eta_3 S) \tag{5.75}$$

where S is the sum defined in Appendix 4 and

$$\eta_1 = \left(\frac{\beta_1 b}{K_1}\right)\left(\frac{\sin(\beta_1-\beta_0)\frac{q}{2}}{q/2}\right)\left(\frac{J_1(K_1 b)}{I_0\left(\sqrt{\beta_1^{\,2}-\beta_0^{\,2}-K_1^{\,2}}\,b\right)}\right)$$

$$\eta_2 = \left(\frac{\sin\beta_0(p-q)}{\beta_0}\right)\left(\frac{\beta_1}{\beta_1+\beta_0}\right)\left(\frac{\sin(\beta_1-\beta_0)\frac{q}{2}}{q/2\,\sin\left[\beta_0(p-q)+\left(\frac{\beta_1+\beta_0}{2}\right)q\right]}\right)$$

$$\eta_3 = J_1(K_1 b)\left(\frac{\beta_1}{K_1}\right)\left(\frac{\sin(\beta_1-\beta_0)\frac{q}{2}}{q/2}\right) \, .$$

The expression for $\hat{\chi}-\chi$ in (5.75) is now to be compared with the following expression obtained from the first mode approximation. For the first mode approximation

$$\beta_{11} = \sqrt{\beta_1^{\,2} - \Delta K_1^{\,2}} \, .$$

Hence, provided $\beta_1{}^2 \gg \Delta K_1{}^2$,

$$\beta_{11} \cong \beta_1 - \frac{1}{2}\,\frac{\Delta K_1{}^2}{\beta_1}\ .$$

Through the use of (5.71) and (5.15), for "a" very small

$$\hat{\chi} - \chi \cong -\frac{1}{2}\left(\frac{a}{b}\right)^2 \frac{c_0}{\varepsilon_1}\frac{q}{\beta_1}\left(\frac{K_1}{J_1(K_1 b)}\right)^2\ . \tag{5.76}$$

Since in the first pass band

$$0 \leq \chi \leq \pi, \tag{5.77}$$

the terms η_1 and η_2 are positive and this would indicate (5.75) and (5.76) have the same sign. As mentioned in Appendix 4, S is an oscillating series that is made up of the contributions of the higher order modes in the waveguide and most of the terms in S are evanescent. The contribution made by S is, in effect, ignored in deriving (5.76).

Owing to (5.77),

$$(\beta_1 - \beta_0)\,\frac{q}{2} \leq \frac{\pi}{2}\ .$$

Consequently, the approximation

$$\sin\,(\beta_1 - \beta_0)\,\frac{q}{2} \cong (\beta_1 - \beta_0)\,\frac{q}{2}$$

is not too unreasonable. Therefore, for both (5.75) and (5.76)

$$\hat{\chi} - \chi \,\tilde{\propto}\, q$$

and, also,

$$\hat{\chi}-\chi \propto \left(\frac{a}{b}\right)^2$$

$$\hat{\chi}-\chi \propto \frac{c_0}{\varepsilon_1} \, .$$

Furthermore, when $\beta_1 \longrightarrow \beta_0$, since

$$\frac{\sin\,(\beta_1 - \beta_0)\frac{q}{2}}{I_0\left(\sqrt{\beta_1{}^2 - \beta_0{}^2 - K_1{}^2}\,b\right)} \longrightarrow \frac{q\ K_1}{2b\beta_0 J_1(K_1 b)}$$

$$\sin\,(\beta_1 - \beta_0)\,\frac{q}{2} \longrightarrow 0,$$

then

$$\eta_1 \longrightarrow 1$$

$$\eta_2 \longrightarrow 0$$

$$\eta_3 \longrightarrow 0 \quad .$$

Therefore, for this case (5.75) and (5.76) approach exact agreement. At $\beta_1 = \beta_0$, η_1 is a maximum and η_1 decreases toward zero as β_1 increases. Since

$$\frac{\sin\,\beta_0(p-q)}{\beta_0} < (p-q)$$

$$\frac{\beta_1}{\beta_1 + \beta_0} < 1$$

$$\frac{\sin\,(\beta_1 - \beta_0)\,\frac{q}{2}}{\frac{q}{2}\,\sin\left[\beta_0(p-q) + (\frac{\beta_1 + \beta_0}{2})q\right]} < \frac{2}{q}\,, \tag{5.78}$$

an upper bound for η_2 is $2(\frac{p-q}{q})$. For a constant X, inequality (5.78) can be established since differentiating the left side by β_1 shows there is no local maximum. Consequently, the absolute maximum occurs when $\beta_1 >> \beta_0$ and thus

$$\frac{\sin\ (\beta_1 - \beta_0)\frac{q}{2}}{\sin\left[\beta_0(p-q)+(\beta_1+\beta_0)\frac{q}{2}\right]} \longrightarrow \frac{\sin\ \beta_1\ \frac{q}{2}}{\sin\ \beta_1\ \frac{q}{2}} = 1.$$

In conclusion, (5.75) and (5.76) appear to be in reasonable agreement.

5.3 A Cavity with a Solid Dielectric Disc Partially Filling the Central Region

The H-wave symmetric field solution in a cavity with a solid dielectric disc partially filling the central region is the topic of this section. An illustration of the cross section of the cavity is shown in Figure 5.3.

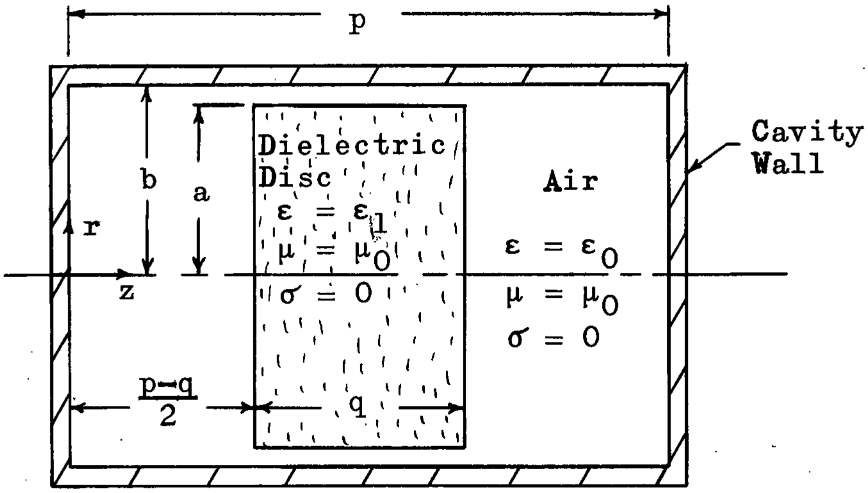

Fig. 5.3. Cross Section of a Cavity Filled Partially in the Central Region by a Dielectric Disc

In his investigation into the properties of dielectrics, Luthra[62] examined this problem. His treatment was initiated because in high power applications intense local fields may occur in the air gap between the disc and the cylindrical wall and, in turn, these fields often cause sparking and even the breakdown of the dielectric material. Luthra dealt with the case in which the air region is near cutoff and the dielectric disc is a half wavelength thick. As a result, he simplified the problem by collapsing the metal end walls to the dielectric surfaces and, then, analyzed this reduced problem.

Inside the cavity shown in Figure 5.3, the permittivity may be expressed as

$$\varepsilon = \varepsilon_0 + c_0 \hat{h}(z)\hat{g}(r) \tag{5.79}$$

where $\hat{h}(z)$ and $\hat{g}(r)$ are described in Figure 5.4.

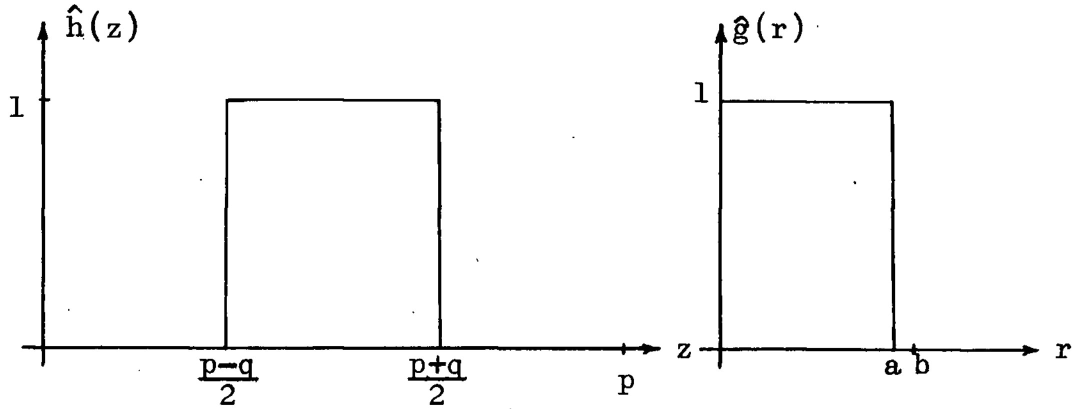

Fig. 5.4. The $\hat{h}(z)$ and $\hat{g}(r)$ Functions

The necessary restrictions on the field before an H-wave can exist in a region where the permittivity satisfies (5.79) can be arrived at in a manner similar to the treatment in section 2.42 for E-waves. The restrictions are that the field can have no angular dependence and

$$E_r \equiv 0, \; H_\phi \equiv 0, \; E_z \equiv 0.$$

Therefore, from (A1.13) in Appendix 1, E_ϕ must satisfy the partial differential equation

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right] + \frac{\partial^2 E_\phi}{\partial z^2} = -\omega^2\mu_0\varepsilon E_\phi. \qquad (5.80)$$

The reasons for solving for E_ϕ rather than H_r or H_z from the differential equations are the duals of the reasons advanced in section 5.2 for solving for H_ϕ.

If (5.79) is substituted into (5.80), the result is

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right] + \frac{\partial^2 E_\phi}{\partial z^2} + \omega^2\mu_0\varepsilon_0 E_\phi = -\omega^2\mu_0 c_0 \hat{h}(z)\hat{g}(r)\, E_\phi \;. \qquad (5.81)$$

The next section will deal with the first mode approximation of the solution to (5.81) and, immediately following, another section will establish an approximate series solution for (5.81).

5.31 First Mode Approximation

The set of eigenfunctions, $\left\{J_1(K_m r)\right\}$, in which the K_m's are determined from

$$J_1(Kb) = 0 \qquad (5.82)$$

is complete and orthogonal over the open interval, (0,b). As a result, E_ϕ can be expanded by the series

$$E_\phi = \sum_{m=1}^{\infty} T_m(z)\, J_1(K_m r) \qquad (5.83)$$

where $K_m b$ is the mth root of (5.82).

Multiplying (5.81) by $r\,J_1(K_m r)$ and integrating the expression which succeeds yields

$$\frac{d^2}{dz^2}\int_0^b rJ_1(K_m r)E_\phi dr + (\omega^2\mu_0\varepsilon_0 - K_m{}^2)\int_0^b rJ_1(K_m r)E_\phi dr$$

$$= -\omega^2\mu_0 c_0 \hat{h}(z)\int_0^a rJ_1(K_m r)\; E_\phi dr. \qquad (5.84)$$

Through the use of the series in (5.83), E_ϕ is eliminated from (5.84). As a consequence,

$$\left[\frac{d^2T_m}{dz^2} + \beta_{0m}^{\;2}\; T_m\right]\int_0^b rJ_1{}^2(K_m r)dr = -\omega^2\mu_0 c_0\hat{h}(z)\sum_{n=1}^{\infty} T_n\int_0^a rJ_1(K_m r)J_1(K_n r)dr$$

where

$$\beta_{0m}^{\;2} = \omega^2\mu_0\varepsilon_0 - K_m{}^2.$$

Therefore,

$$\frac{d^2T_m}{dz^2} + \beta_{0m}^{\;2}\; T_m = -\omega^2\mu_0 c_0\left(\frac{a}{b}\right)\frac{2}{b}\;\hat{h}(z)\sum_{n=1}^{\infty}\frac{\left[K_n J_1(K_m a)J_0(K_n a) - K_m J_0(K_m a)J_1(K_n a)\right]}{(K_m{}^2 - K^2{}_n)J_0{}^2(K_m b)}T_n \qquad (5.85)$$

since[63]

$$\int_0^b r\; J_1{}^2(K_m r)dr = \frac{b^2}{2}\; J_0{}^2(K_m b)$$

and

$$\int_0^a r\, J_1(K_m r) J_1(K_n r)\,dr = \frac{a}{K_m^2 - K_n^2}\left[K_n J_1(K_m a) J_0(K_n a) - K_m J_0(K_m a) J_1(K_n a)\right].$$

For the case in which

$$\Delta b = b - a \ll b, \tag{5.86}$$

$$\left.\begin{aligned} J_0(K_1 a) &\cong J_0(K_1 b)\left[1 - \tfrac{1}{2}(K_1 \Delta b)^2\right] \\ J_1(K_1 a) &\cong -J_0(K_1 b)(K_1 \Delta b) \\ J_2(K_1 a) &\cong -J_0(K_1 b)\left[1 + 2\,\frac{\Delta b}{b} + 2\left(\frac{\Delta b}{b}\right)^2 - \tfrac{1}{2}(K_1 \Delta b)^2\right] \end{aligned}\right\} \tag{5.87}$$

where the formulas[64,65]

$$J_n\left[K_1(b - \Delta b)\right] = \sum_{m=-\infty}^{\infty} J_m(K_1 b) J_{n-m}(-K_1 \Delta b)$$

$$J_2(K_1 a) = \frac{2}{K_1 a} J_1(K_1 a) - J_0(K_1 a)$$

have been utilized. If T_1 is regarded as the driving function, from (5.85) for the smaller values of $m > 1$

$$T_m \approx \left(\frac{\Delta b}{b}\right)^3 T_1.$$

By the use of the asymptotic expressions for $J_0(K_m a)$ and $J_1(K_m a)$, for the larger values of m

$$T_m \approx \begin{bmatrix} \sin K_m a \\ \cos K_m a \end{bmatrix} \left(\frac{1}{K_m}\right)^{\frac{5}{2}} T_1 \ .$$

Consequently, for a first approximation of the field, all terms for m > 1 will be neglected. From (5.85) this approximation yields

$$\frac{d^2T_1}{dz^2} + \beta_{01}{}^2 T_1 = -\omega^2 \mu_0 c_0 \left(\frac{a}{b}\right)^2 \left[\frac{J_1{}^2(K_1 a) - J_0(K_1 a) J_2(K_1 a)}{J_0{}^2(K_1 b)}\right] \hat{h}(z)\ T_1 . \tag{5.88}$$

Employing the approximations in (5.87) and neglecting higher order terms reduces (5.88) to

$$\frac{d^2T_1}{dz^2} + \beta_{01}{}^2 T_1 = -(\omega^2 \mu_0 c_0 - \Delta K_1{}^2)\hat{h}(z)\ T_1 \tag{5.89}$$

where

$$\Delta K_1{}^2 = \omega^2 \mu_0 c_0 \left(\frac{\Delta b}{b}\right)^2 \ . \tag{5.90}$$

When

$$\hat{h}(z) = 1,$$

(5.89) gives

$$\frac{d^2T_1}{dz^2} + \beta_{11}{}^2\ T_1 = 0$$

where

$$\beta_{11}^{\ 2} = \omega^2\mu_0\varepsilon_1 - (K_1^{\ 2} + \Delta K_1^{\ 2}). \tag{5.91}$$

To insure the air gap between the disc and the cylindrical wall only perturbs $K_1^{\ 2}$ by a small amount,

$$\Delta K_1^{\ 2} << K_1^{\ 2} .$$

Therefore, the condition,

$$\left(\frac{\Delta b}{b}\right)^2 << \left(\frac{K_1^{\ 2}}{\omega^2\mu_0 c_0}\right) , \tag{5.92}$$

must be satisfied. By taking T_2 into account as well as T_1, the stipulation,

$$T_2 << T_1,$$

is met provided

$$\left(\frac{\Delta b}{b}\right)^3 << .066 \left(\frac{K_1^{\ 2}}{\omega^2\mu_0 c_0}\right) . \tag{5.93}$$

If desired, (5.93) can be used in place of (5.92).

According to (5.89)

$$\left.\begin{aligned} T_1 &= \text{a continuous function} \\ \frac{dT_1}{dz} &= \text{a continuous function} \end{aligned}\right\} \tag{5.94}$$

and (5.94) establishes the boundary conditions.

Now, the symmetric solution to (5.89) can be determined in a manner analogous to the approach used in Appendix 5, for the case in which a = b. The symmetric solution is, in the region,

$0 \leq z \leq \frac{p-q}{2}$,

$$T_1 = \frac{\cos \beta_{11} \frac{q}{2}}{\sin \beta_{01}(\frac{p-q}{2})} A \sin \beta_{01} z,$$

in the region, $\frac{p-q}{2} \leq z \leq \frac{p+q}{2}$,

$$T_1 = A \cos \beta_{11}(z - \frac{p}{2})$$

and in the region, $\frac{p+q}{2} \leq z \leq p$,

$$T_1 = \frac{\cos \beta_{11} \frac{q}{2}}{\sin \beta_{01}(\frac{p-q}{2})} A \sin \beta_{01}(p-z).$$

The resonant frequencies are determined from the relation

$$\tan \beta_{11} \frac{q}{2} = \frac{\beta_{01}}{\beta_{11}} \cot \beta_{01}(\frac{p-q}{2}). \tag{5.95}$$

If this solution is compared with the solution in Appendix 5, it is easily seen that β_{01} and β_0 have the same form. Consequently, in the air region the presence of the air gap between the disc and the cylindrical wall does not appear to have any appreciable effect on the behaviour of the propagation coefficient. In the region of the disc, the square of the propagation coefficient, $\beta_{11}^{\;2}$, differs from $\beta_1^{\;2}$ by a factor $\Delta K_1^{\;2}$. This perturbation caused by the air gap can be adjusted for by increasing the permittivity of the disc to

$$\varepsilon_2 = \varepsilon_1 + c_0 \left(\frac{\Delta b}{b}\right)^2 .$$

Once this adjustment is made,

$$\beta_{11} = \beta_1 .$$

A more accurate solution which includes the effect of some of the higher order modes can be achieved by extending the above analysis.

The antisymmetric problem can be dealt with in a similar fashion to the treatment just given.

5.32 An Approximate Series Solution

The purpose of this section is to carry out an investigation to determine an approximate series solution for (5.81),

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right] + \frac{\partial^2 E_\phi}{\partial z^2} + \omega^2\mu_0\varepsilon_0 E_\phi = -\omega^2\mu_0 c_0 \hat{h}(z)\hat{g}(r)E_\phi . \tag{5.96}$$

Since the symmetric solution is desired, E_ϕ is expanded in the series

$$E_\phi = \sum_{n=1}^{\infty^*} E_n(r) \sin\frac{n\pi z}{p} \tag{5.97}$$

where "*" denotes that the sum is only over odd values of n.

Now, (5.96) is multiplied by $\sin \frac{n\pi z}{p}$ and the ensuing expression is integrated from 0 to p. Therefore, if to the left of the equality sign E_ϕ is replaced by the series (5.97), the result is

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rE_n)\right] - \hat{K}_n^{\,2} E_n = -\frac{2\omega^2\mu_0 c_0}{p}\,\hat{g}(r)\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} \sin\frac{n\pi z}{p}\,E_\phi dz \tag{5.98}$$

where

$$\hat{K}_n^{\,2} = \left(\frac{n\pi}{p}\right)^2 - \omega^2\mu_0\varepsilon_0 \; . \tag{5.99}$$

Determination of the expression,

$$\hat{g}(r)\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} \sin\frac{n\pi z}{p}\,E_\phi dz \;,$$

is complete provided E_ϕ is known inside the dielectric disc because $\hat{g}(r)$ is zero over the interval, $a < r < b$, and the region of integration is only from $\frac{p-q}{2}$ to $\frac{p+q}{2}$. As a consequence, the first iterative solution can be found by guessing E_ϕ in the dielectric region.

The solution found in Appendix 5, for the case in which a = b, will be used for the guess. Analogous to the guess made in section 5.22, the present conjecture gives an approximation inside the disc for each harmonic in (5.97). Since the solution in Appendix 5 is for the lowest order mode, the first iterative solution will be an approximation of the lowest order mode in the cavity illustrated in Figure 5.3.

From Appendix 5, inside the dielectric disc

$$E_\phi = AJ_1(K_1 r) \cos \beta_1 (z - \tfrac{p}{2}) \tag{5.100}$$

where

$$\tan \beta_1 \frac{q}{2} = \frac{\beta_0}{\beta_1} \cot \beta_0 (\tfrac{p-q}{2}) \tag{5.101}$$

$$\begin{aligned} \beta_0^{\,2} &= \omega_0^{\,2} \mu_0 \varepsilon_0 - K_1^{\,2} \\ \beta_1^{\,2} &= \omega_0^{\,2} \mu_0 \varepsilon_1 - K_1^{\,2} \,. \end{aligned} \tag{5.102}$$

The resonant frequency ω_0 is determined from (5.101).

Greater flexibility in the guess could be achieved by leaving K and ω undetermined in (5.100). However, another unknown, K, is introduced. Consequently, to complete the solution, the guessed E_ϕ and the first iterative E_ϕ would have to be equated at an extra point. As a result, any numerical work would be considerably increased.

By replacing E_ϕ in (5.98) by (5.100), a first iterative solution for E_n is found to satisfy

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\,(rE_n^{\,1})\right] - \hat{K}_n^{\,2}\, E_n^{\,1} = Ae_n\, J_1(K_1 r)\, \hat{g}(r) \tag{5.103}$$

where

$$e_n = -\frac{2\omega^2\mu_0 c_0}{p}\sin\frac{n\pi}{2}\left[\frac{\sin\left(\frac{n\pi}{p}+\beta_1\right)\frac{q}{2}}{\frac{n\pi}{p}+\beta_1}+\frac{\sin\left(\frac{n\pi}{p}-\beta_1\right)\frac{q}{2}}{\frac{n\pi}{p}-\beta_1}\right].$$

(5.104)

In (5.103) the driving function provides the coupling between the harmonics. The behaviour of this driving function can be explained in a manner similar to the explanation given in section 5.22 for the behaviour of the coupling driving function arising in (5.42).

Through arguments similar to those advanced in section 5.22, by means of (5.103) the continuity conditions,

$$\left.\begin{aligned} E_n^{\,1} &= \text{a continuous function} \\ \frac{1}{r}\frac{d}{dr}(rE_n^{\,1}) &= \text{a continuous function,} \end{aligned}\right\} \qquad (5.105)$$

can be established.

From (5.103) in the region, $0 \le r < a$,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rE_n^{\,1})\right] - \hat{K}_n^{\,2}\,E_n^{\,1} = Ae_n\,J_1(K_1 r)$$

(5.106)

and in the region, $a < r < b$,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rE_n^{\,1})\right] - \hat{K}_n^{\,2}\,E_n^{\,1} = 0. \qquad (5.107)$$

The solution to (5.106) is

$$E_n^{\,1} = -\frac{A\ e_n}{\hat{K}_n^{\,2} + K_1^{\,2}}\, J_1(K_1 r) + A_n I_1(\hat{K}_n r) \qquad (5.108)$$

and to (5.107) is

$$E_n^{\,1} = C_n'\ I_1(\hat{K}_n r) + D_n K_1(\hat{K}_n r). \qquad (5.109)$$

At $r = b$, E_ϕ is zero and thus from the series (5.97)

$$E_n^{\,1}\Big|_{r=b} = 0\ .$$

Hence, (5.109) gives

$$0 = C_n'\ I_1(\hat{K}_n b) + D_n K_1(\hat{K}_n b).$$

Consequently,

$$D_n = -\frac{I_1(\hat{K}_n b)}{K_1(\hat{K}_n b)}\, C_n' = -\, I_1(\hat{K}_n b)\ \hat{C}_n$$

and (5.109) becomes

$$E_n^{\,1} = \hat{C}_n\Big[K_1(\hat{K}_n b)\ I_1(\hat{K}_n r) - I_1(\hat{K}_n b)\ K_1(\hat{K}_n r)\Big]. \qquad (5.110)$$

By means of (5.105), (5.108) and (5.110) the boundary conditions at $r = a$ are

$$F_1(\hat{K}_n a)\ \hat{C}_n = I_1(\hat{K}_n a)\ A_n - \left(\frac{A\ e_n}{\hat{K}_n^{\ 2} + K_1^{\ 2}}\right) J_1(K_1 a)$$

$$\hat{K}_n\ F_0(\hat{K}_n a)\ \hat{C}_n = \hat{K}_n I_0(\hat{K}_n a)\ A_n - \left(\frac{A\ e_n K_1}{\hat{K}_n^{\ 2} + K_1^{\ 2}}\right) J_0(K_1 a)$$

where

$$F_1(\hat{K}_n r) = K_1(\hat{K}_n b)\ I_1(\hat{K}_n r) - I_1(\hat{K}_n b)\ K_1(\hat{K}_n r)$$

$$F_0(\hat{K}_n r) = K_1(\hat{K}_n b)\ I_0(\hat{K}_n r) + I_1(\hat{K}_n b)\ K_0(\hat{K}_n r)\ .$$

Solving for A_n and $\hat{C}_n$ yields

$$A_n = \frac{a e_n\left[\hat{K}_n J_1(K_1 a)\ F_0(\hat{K}_n a) - K_1 J_0(K_1 a)\ F_1(\hat{K}_n a)\right] A}{\left[\left(\frac{n\pi}{p}\right)^2 - \beta_{01}^{\ 2}\right] I_1(\hat{K}_n b)}$$

$$\hat{C}_n = \frac{a e_n\left[\hat{K}_n J_1(K_1 a)\ I_0(\hat{K}_n a) - K_1 J_0(K_1 a)\ I_1(\hat{K}_n a)\right] A}{\left[\left(\frac{n\pi}{p}\right)^2 - \beta_{01}^{\ 2}\right] I_1(\hat{K}_n b)} \qquad (5.111)$$

since

$$\hat{K}_n^{\ 2} + K_1^{\ 2} = \left(\frac{n\pi}{p}\right)^2 - \beta_{01}^{\ 2}\ .$$

Therefore, over the interval, $0 \leq r \leq a$, from (5.97), (5.108) and (5.111)

$$E_\phi{}^1 = -AJ_1(K_1 r) \sum_{n=1}^{\infty} \frac{e_n}{\left[(\frac{n\pi}{p})^2 - \beta_{01}{}^2\right]} \sin \frac{n\pi z}{p}$$

$$-a\ A \sum_{n=1}^{\infty} \frac{e_n\left[K_1 J_0(K_1 a)F_1(\hat{K}_n a) - \hat{K}_n J_1(K_1 a)F_0(\hat{K}_n a)\right]}{\left[(\frac{n\pi}{p})^2 - \beta_{01}{}^2\right] I_1(\hat{K}_n b)} I_1(\hat{K}_n r)\sin \frac{n\pi z}{p} \tag{5.112}$$

and over the interval, $a \leq r \leq b$, from (5.97), (5.110) and (5.111)

$$E_\phi{}^1 = -a\ A \sum_{n=1}^{\infty} \frac{e_n\left[K_1 J_0(K_1 a)I_1(\hat{K}_n a) - \hat{K}_n J_1(K_1 a)I_0(\hat{K}_n a)\right]}{\left[(\frac{n\pi}{p})^2 - \beta_{01}{}^2\right] I_1(\hat{K}_n b)} F_1(\hat{K}_n r)\sin \frac{n\pi z}{p}. \tag{5.113}$$

The star notation is dropped from the summation sign since e_n is zero for n even. Owing to the nature of e_n and the fact that only squared terms of n occur in $\hat{K}_n$, (5.112) and (5.113) can be rewritten respectively as

$$E_\phi{}^1 = \frac{2\omega^2\mu_0 c_0}{p} A \left[J_1(K_1 r) \sum_{n=-\infty}^{\infty} \frac{\sin \frac{n\pi}{2} \sin(\frac{n\pi}{p}+\beta_1)\frac{q}{2}}{\left[\frac{n\pi}{p}+\beta_1\right]\left[(\frac{n\pi}{p})^2 - \beta_{01}{}^2\right]} \sin \frac{n\pi z}{p} \right.$$

$$\left. +a \sum_{n=-\infty}^{\infty} \frac{\sin \frac{n\pi}{2} \sin(\frac{n\pi}{p}+\beta_1)\frac{q}{2}\left[K_1 J_0(K_1 a)F_1(\hat{K}_n a) - \hat{K}_n J_1(K_1 a)F_0(\hat{K}_n a)\right]}{(\frac{n\pi}{p}+\beta_1)\left[(\frac{n\pi}{p})^2 - \beta_{01}{}^2\right] I_1(\hat{K}_n b)} \times I_1(\hat{K}_n r)\sin \frac{n\pi z}{p} \right] \tag{5.114}$$

and

$$E_{\phi}{}^{1} = \frac{2\omega^2\mu_0 c_0 a A}{p} \sum_{n=-\infty}^{\infty} \frac{\sin\frac{n\pi}{2}\sin(\frac{n\pi}{p}+\beta_1)\frac{q}{2}\left[K_1 J_0(K_1 a) I_1(\hat{K}_n a) - \hat{K}_n J_1(K_1 a) I_0(\hat{K}_n a)\right]}{\left(\frac{n\pi}{p}+\beta_1\right)\left[(\frac{n\pi}{p})^2 - \beta_{01}{}^2\right] I_1(\hat{K}_n b)}$$

$$X\ F_1(\hat{K}_n r)\sin\frac{n\pi z}{p}\ . \tag{5.115}$$

The relation from which the resonant frequencies can be obtained is found by equating at a point the guessed solution in (5.100) and the first iterative solution in (5.114). The point chosen is at $r = \delta$, $z = \frac{p}{2}$ where $\delta \to 0$. Inside the dielectric disc, no other point is as far removed from the perturbation caused near the corners of the disc by the air gap.

Equating (5.100) to (5.114) at $r = \delta$, $z = \frac{p}{2}$ gives

$$J_1(K_1\delta) = \frac{2\omega^2\mu_0 c_0}{p}\left[J_1(K_1\delta)\sum_{n=-\infty}^{\infty}\frac{\sin^2\frac{n\pi}{2}\sin(\frac{n\pi}{2}+\beta_1)\frac{q}{2}}{(\frac{n\pi}{p}+\beta_1)\left[(\frac{n\pi}{p})^2-\beta_{01}{}^2\right]}\right.$$

$$\left.+a\sum_{n=-\infty}^{\infty}\frac{\sin^2\frac{n\pi}{2}\sin(\frac{n\pi}{p}+\beta_1)\frac{q}{2}\left[K_1 J_0(K_1 a)F_1(\hat{K}_n a)-\hat{K}_n J_1(K_1 a)F_0(\hat{K}_n a)\right] I_1(\hat{K}_n\delta)}{\left(\frac{n\pi}{p}+\beta_1\right)\left[\left(\frac{n\pi}{p}\right)^2-\beta_{01}{}^2\right] I_1(\hat{K}_n b)}\right].$$

Since

$$\lim_{\delta\to 0}\left[\frac{I_1(\hat{K}_n\delta)}{J_1(K_1\delta)}\right] = \frac{\hat{K}_n}{K_1}\ ,$$

$$1 = \frac{2\omega^2\mu_0 c_0}{p}\left[\sum_{n=-\infty}^{\infty}\frac{\sin^2\frac{n\pi}{p}\,\sin\left(\frac{n\pi}{p}+\beta_1\right)\frac{q}{2}}{\left(\frac{n\pi}{p}+\beta_1\right)\left[\left(\frac{n\pi}{p}\right)^2-\beta_{01}^{\;2}\right]}\right.$$

$$\left.+\frac{a}{K_1}\sum_{n=-\infty}^{\infty}\frac{\hat{K}_n\sin^2\frac{n\pi}{2}\sin\left(\frac{n\pi}{p}+\beta_1\right)\frac{q}{2}\left[K_1J_0(K_1a)F_1(\hat{K}_na)-\hat{K}_nJ_1(K_1a)F_0(\hat{K}_na)\right]}{\left(\frac{n\pi}{p}+\beta_1\right)\left[\left(\frac{n\pi}{p}\right)^2-\beta_{01}^{\;2}\right]I_1(\hat{K}_nb)}\right]. \tag{5.116}$$

In that

$$S = \sum_{n=-\infty}^{\infty}\frac{\sin^2\frac{n\pi}{2}\sin\left(\frac{n\pi}{p}+\beta_1\right)\frac{q}{2}}{\left(\frac{n\pi}{p}+\beta_1\right)\left[\left(\frac{n\pi}{p}\right)^2-\beta_{01}^{\;2}\right]} = \sum_{m=-\infty}^{\infty}\frac{\sin\left(\frac{(2m+1)\pi}{p}+\beta_1\right)\frac{q}{2}}{\left[\frac{(2m+1)\pi}{p}+\beta_1\right]\left[\left(\frac{(2m+1)\pi}{p}\right)^2-\beta_{01}^{\;2}\right]},$$

S can be summed in the same fashion as the series treated in Appendix 4. If the summation is carried out,

$$S = \frac{p}{2}\,\frac{1}{(\beta_1^{\;2}-\beta_{01}^{\;2})}\left[1-\frac{1}{\cos\beta_{01}\frac{p}{2}}\left(\cos\beta_1\frac{q}{2}\cos\beta_{01}\left(\frac{p-q}{2}\right)\right.\right.$$

$$\left.\left.-\frac{\beta_1}{\beta_{01}}\sin\beta_1\frac{q}{2}\sin\beta_{01}\left(\frac{p-q}{2}\right)\right)\right].$$

Therefore, (5.116) becomes

$$\frac{\omega^2-\omega_0^2}{\omega_0^2-\frac{\varepsilon_0}{\varepsilon_1}\omega^2} = \frac{1}{\cos\beta_{01}\frac{p}{2}}\left[\cos\beta_1\frac{q}{2}\cos\beta_{01}\left(\frac{p-q}{2}\right) - \frac{\beta_1}{\beta_{01}}\sin\beta_1\frac{q}{2}\sin\beta_{01}\left(\frac{p-q}{2}\right)\right]$$

$$\times\frac{\omega^2\mu_0c_0}{\left(\beta_1^2-\beta_{01}^2\right)}$$

$$+\frac{2\omega^2\mu_0c_0a}{K_1p}\sum_{n=-\infty}^{\infty}\frac{\hat{K}_n\sin^2\frac{n\pi}{2}\sin\left(\frac{n\pi}{p}+\beta_1\right)\frac{q}{2}\left[\hat{K}_nJ_1(K_1a)F_0(\hat{K}_na)-K_1J_0(K_1a)F_1(\hat{K}_na)\right]}{\left(\frac{n\pi}{p}+\beta_1\right)\left[\left(\frac{n\pi}{p}\right)^2-\beta_{01}^2\right]I_1(\hat{K}_nb)}.$$

This relation gives the resonant frequency ω.

A check of the limiting case when $a = b$ shows that the first iterative solution agrees with the solution given in Appendix 5.

5.4 Discussion

For the periodic structure loaded with dielectric discs having central holes, the dispersion equation found in the first mode approximation is much simpler than the dispersion equation found in the approximate series solution. This simpler dispersion equation is probably quite reliable when $a \ll b$ since the first mode approximation seems reasonable in all regions of the structure except possibly near the axis of the discs. In the vicinity of the holes in the discs the field is perturbed the most and thus higher order modes may become significant.

For the amplitude of the space harmonics, the approximate series solution gives an expression which is not unduly more complicated than the expression obtained by the first mode approximation. At the same time, along the axis of the waveguide, the amplitudes determined in the approximate series solution are probably somewhat more accurate. This statement would have greater force if the first mode approximation were employed for the guessed solution.

Consequently, in studying the properties of the periodic structure, in many instances possibly the best approach would be to use the dispersion equation given by the first mode approximation and to use in the vicinity of the axis the amplitudes of the space harmonics given by the approximate series solution.

In a like manner, in a study of the behaviour of the field in the resonant cavity discussed in section 5.3, possibly the most agreeable approach would be to find the resonant frequency from the first mode approximation and to use, at least over the interval, $a < r < b$, the field given by the approximate series solution.

6. SOME CALCULATIONS AND MEASUREMENTS FOR DIELECTRIC LOADED PERIODIC STRUCTURES

6.1 General

The dielectric loaded periodic structures to be examined are of the type shown in Figure 5.1. For a structure loaded with dielectric discs of polystyrene, the frequency and disc thickness required for π-mode confluence[66,67] are calculated from the first mode approximation theory developed in section 5.21. These results are compared with the corresponding values obtained experimentally by Walker and West[67] and with those determined from the solid disc theory of section 4.523. For discs of the thickness calculated from the first mode approximation theory, the dispersion curves given by the first mode approximation and the solid disc theory are plotted. An experimental dispersion curve for this case could not be readily obtained. However, the dielectric loaded periodic structure in the linear accelerator at The University of British Columbia could be set up for measurements without too much difficulty. As a consequence, this structure, which is loaded with titania discs, was used to find an experimental dispersion curve. This experimental curve is compared with the theoretical curves determined through the use of the first mode approximation and the solid disc theory.

For the accelerator structure at The University of British Columbia, the 0th order space harmonic of E_z is examined on the axis for the case in which its phase velocity is equal to the speed of light. A comparison is made between the harmonic coefficient found from the solid disc theory and from the approximate series solution discussed in section 5.22.

6.2 Polystyrene Loaded Structure

The discs in the polystyrene loaded structure have a relative dielectric constant of 2.550. Other parameters for the structure are

$$\left.\begin{aligned} b &= 3.620 \text{ cm} \\ a &= .6350 \text{ cm} \\ p &= 5.000 \text{ cm} . \end{aligned}\right\} \qquad (6.1)$$

The thickness, q, is to be determined so that a match takes place at $\chi = \pi$. Since $K_1 b$ is the first root of $J_0(Kb)$, then

$$K_1 b = 2.405. \qquad (6.2)$$

Hence,

$$J_1(K_1 b) = .5191 \qquad (6.3)$$

and from (6.1)

$$K_1 = .6644(10)^2. \qquad (6.4)$$

Substituting (6.1), (6.3) and (6.4) into (5.15) yields

$$\Delta K_1{}^2 = .6632(10)^{-1}\ K_1{}^2 + 1.544(10)^{-3}\omega^2\mu_0\varepsilon_1 \qquad (6.5)$$

$$\alpha = 2.540(10)^{-3} . \qquad (6.6)$$

From (5.16) and (6.5)

$$\beta_{11}{}^2 = \left(1 - 1.544(10)^{-3}\right)\omega^2\mu_0\varepsilon_1 - (1 + .6632(10)^{-1})K_1{}^2. \qquad (6.7)$$

According to (5.26) a match occurs when

$$\beta_{01}^{\ 2} = \left(\frac{\varepsilon_0}{\varepsilon_1}\right)^2 \left(\frac{\varepsilon_1}{\varepsilon_0}\right)^{2\alpha} \beta_{11}^{\ 2} . \tag{6.8}$$

From (6.6)

$$\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^{2\alpha} = (2.550)^{5.080(10)^{-3}} = 1 + .004752. \tag{6.9}$$

Since

$$\beta_{01}^{\ 2} = \omega^2\mu_0\varepsilon_0 - K_1^{\ 2} , \tag{6.10}$$

(6.8) can be rewritten as

$$\omega^2\mu_0\varepsilon_0 - K_1^{\ 2} = \left[1 + 3.201(10)^{-3}\right]\omega^2\mu_0\varepsilon_0 \frac{1}{\varepsilon_r} - (1 + .07139)\left(\frac{1}{\varepsilon_r}\right)^2 K_1^{\ 2}$$

or

$$\omega = K_1 c \sqrt{\frac{1 - \dfrac{1.07139}{\varepsilon_r^{\ 2}}}{1 - \dfrac{1.003201}{\varepsilon_r}}} = K_1 c \sqrt{1 + \frac{1}{\varepsilon_r}\ (.9591)} \tag{6.11}$$

where

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 2.998(10)^8 \text{ m/s}.$$

Therefore,

$$f = \frac{\omega}{2\pi} = 3718 \text{ Mc/s}.$$

Utilizing the matched condition from (4.59), for the solid disc theory, gives

$$f_{s.d.} = \frac{K_1 c}{2\pi} \sqrt{1 + \frac{1}{\varepsilon_r}} = 3740 \text{ Mc/s.}$$

From Walker and West's experimental measurements

$$f_{exp} = 3722 \text{ Mc/s.}$$

Consequently,

$$\left| f_{exp} - f \right| = 4 \text{ Mc/s}$$

$$\left| f_{exp} - f_{s.d.} \right| = 18 \text{ Mc/s}$$

and thus the difference between the experimental measurement and the theoretical calculation is improved by a factor of $4\frac{1}{2}$ if the first mode approximation is used instead of the solid disc theory.

Through the use of (6.11)

$$\omega^2 \mu_0 \varepsilon_0 = K_1^{\,2} \left[1 + \frac{1}{\varepsilon_r} (1 - .04088) \right].$$

Hence,

$$\beta_{01} = \sqrt{\omega^2 \mu_0 \varepsilon_0 - K_1^{\,2}} = K_1 \sqrt{\frac{(1 - .04088)}{\varepsilon_r}} = .4075(10)^2.$$

Also,

$$\omega^2 \mu_0 \varepsilon_1 - K_1^{\,2} = (\varepsilon_r - .04088) K_1^{\,2}$$

and from (6.5)

$$\Delta K_1^{\,2} = .07174 \; K_1^{\,2} \;. \tag{6.12}$$

As a result,

$$\beta_{11} = K_1 \sqrt{(\varepsilon_r - .1126)} = 1.037(10)^2 .$$

For a match to occur at $\chi = \pi$ from (5.27) the thickness of the disc is given by

$$q = \frac{\pi - \beta_{01}p}{\beta_{11} - \beta_{01}} = \frac{3.142 - 2.038}{(1.037 - .407)(10)^2} = 1.75 \text{ cm}$$

From the solid disc theory

$$\beta_0 = \frac{K_1}{\sqrt{\varepsilon_r}} = \frac{.6644(10)^2}{1.597} = .4160(10)^2$$

$$\beta_1 = K_1\sqrt{\varepsilon_r} = (.6644)(10)^2(1.597) = 1.061(10)^2.$$

By means of (4.60) the disc thickness is equal to the expression

$$q_{s.d.} = \frac{\pi - \beta_0 p}{\beta_1 - \beta_0} = \frac{3.142 - 2.080}{(1.061 - .416)(10)^2} = 1.65 \text{ cm}.$$

The measured value of the disc thickness is

$$q_{exp.} = 1.77 \text{ cm}.$$

Consequently,

$$q_{exp.} - q = .02 \text{ cm}$$

$$q_{exp.} - q_{s.d.} = .12 \text{ cm}.$$

Hence, in this case, the first mode approximation gives an answer which is considerably closer to the experimental results than the answer given by the solid disc theory.

For discs having a thickness of

$$q = 1.75 \text{ cm},$$

the dispersion curves given by the first mode approximation and the solid disc theory are plotted in Figure 6.1. These curves were determined from (5.24) and (4.54) respectively.

6.3 Titania Loaded Structure

The dispersion curve for the dielectric loaded periodic structure in the linear accelerator at The University of British Columbia was obtained from measurements of the resonances in the structure. The discs used for loading are made of titania which has a relative dielectric constant of 93.5. The other parameters of the structure are

$$\left.\begin{aligned} b &= 3.849 \text{ cm} \\ a &= 1.000 \text{ cm} \\ p &= 5.000 \text{ cm} \\ q &= .5766 \text{ cm} . \end{aligned}\right\} \qquad (6.13)$$

The experimental curve along with the curves determined from the first mode approximation and the solid disc theory is shown in Figure 6.2.

From Figure 6.2 it can be seen that the curve predicted by the first mode approximation is in close agreement with the experimental curve over a large part of the first pass band, i.e. over the interval $1 < \chi < \pi$ radians. For all points the curve given by the first mode approximation is an improvement on the dispersion curve given by the solid disc theory.

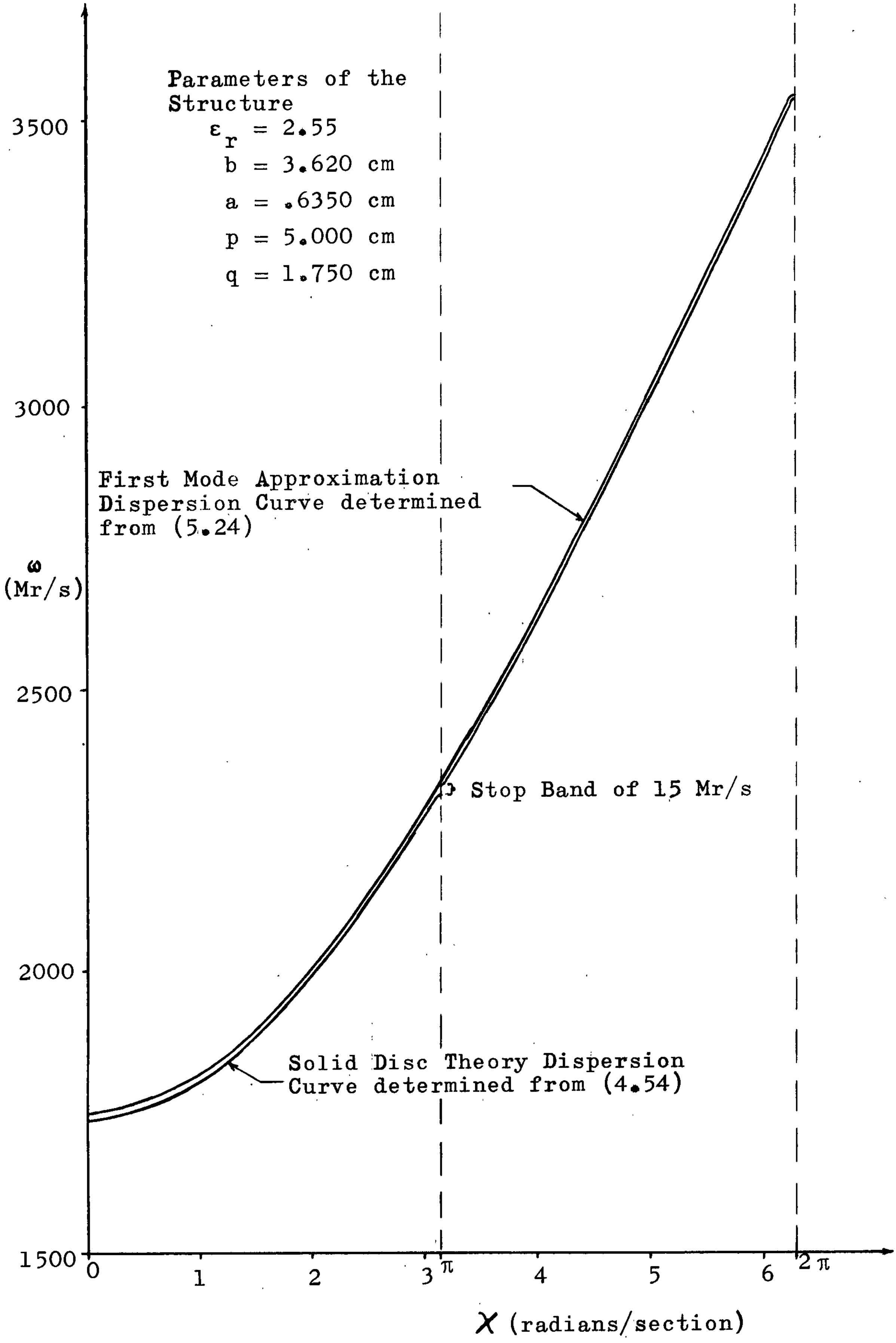

Fig. 6.1. <u>Dispersion Curves for the Polystyrene Loaded Structure</u>

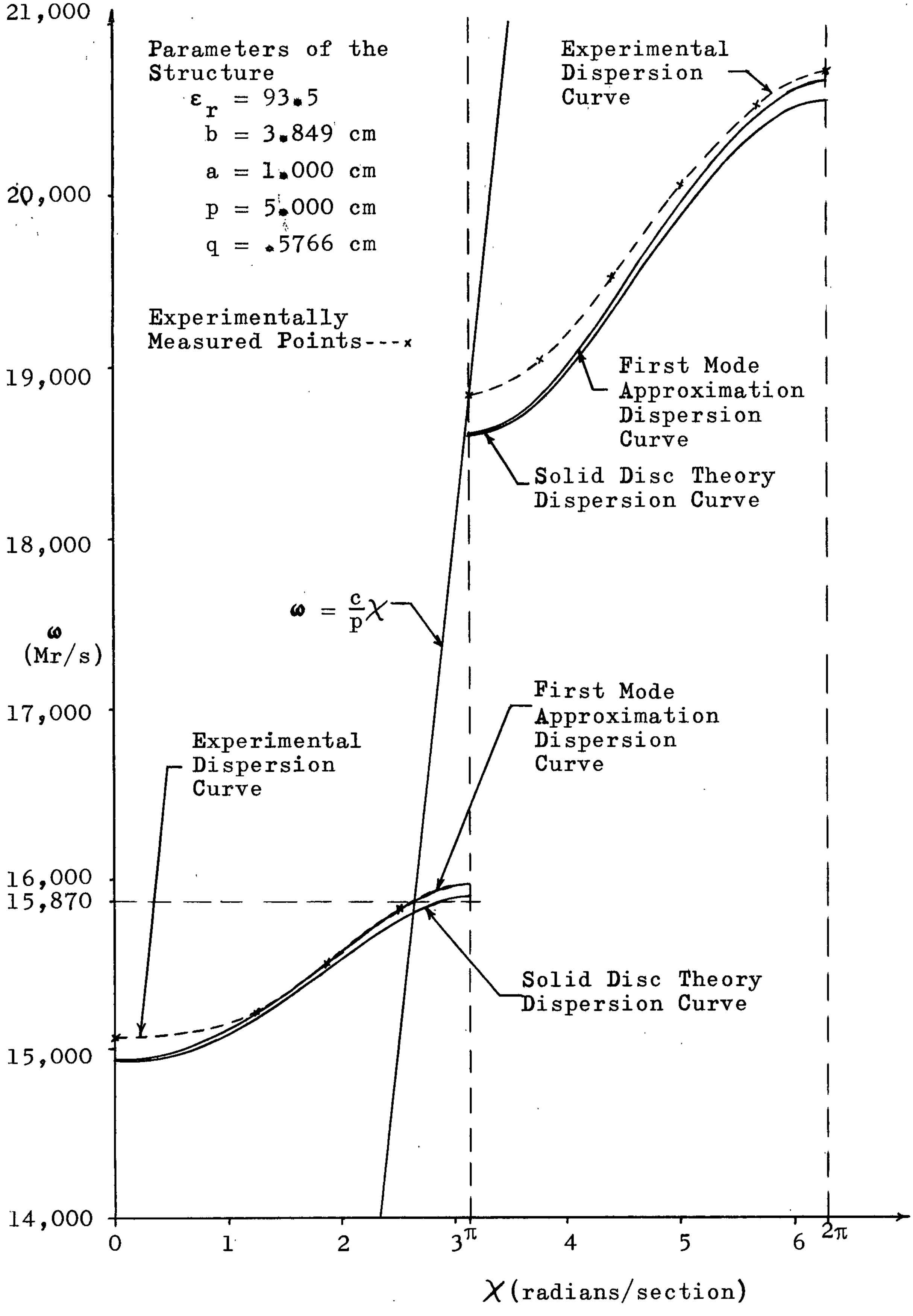

Fig. 6.2. <u>Dispersion Curves for the Titania Loaded Structure</u>

It should be pointed out that for the polystyrene loaded structure, since $\omega < 3.55(10)^9$ r/s, from (6.5)

$$\Delta K_1^{\ 2} < .0725 K_1^{\ 2}$$

and thus the condition (5.17) is easily satisfied. Hence, the first mode approximation should give results that are reasonably close to the experimental results. This statement is in accordance with the findings already discussed. However, for the titania loaded structure, since from Figure 6.2 $14.8(10)^9 < \omega < 20.7(10)^9$ r/s, then by means of (5.15), (6.2), (6.3) and (6.13)

$$.93\ K_1^{\ 2} < \Delta K_1^{\ 2} < 1.62\ K_1^{\ 2}.$$

Consequently, neither (5.17) nor (5.18) are truly satisfied. Therefore, it cannot be expected that the dispersion curve given by the first mode approximation will always be in close agreement with the experimental results. A better agreement can be expected by going to a second or third mode approximation.

For the case in which the phase velocity of the 0th space harmonic of E_z is the speed of light,

$$v_{phase} = c = \frac{\omega}{\gamma_p} \tag{6.14}$$

and thus from (5.37) in the approximate series solution

$$\hat{K}_0 = 0.$$

As a result, in (5.63)

$$\hat{K}_0^{\,2}\, G_0(\hat{K}_0 a) = 0$$

$$a\, \hat{K}_0\, G_1(\hat{K}_0 a) = 1.$$

Therefore, by means of (5.63) on the axis of the waveguide the coefficient of the 0th space harmonic is approximately

$$b_0 \cong J_0(K_1 a)\left(\frac{\hat{a}_0 - f_0}{\varepsilon_0}\right). \qquad (6.15)$$

From the solid disc theory of section 4.523

$$(b_0)_{s.d.} = \frac{1}{p}\int_0^p e^{s_0 z}\, E_z dz = \frac{1}{p}\int_0^p e^{s_0 z}\, \frac{1}{\varepsilon}\, T(z)\, dz$$

$$= \frac{1}{\varepsilon_0}\,\frac{1}{p}\int_0^p e^{s_0 z}\, T(z)dz + \frac{1}{p}\int_0^p e^{s_0 z}\left[\frac{1}{\varepsilon} - \frac{1}{\varepsilon_0}\right] T(z)dz$$

or

$$(b_0)_{s.d.} = \frac{a_0}{\varepsilon_0} - \frac{1}{\varepsilon_0}\left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{p}\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_0 z}\, T(z)dz. \qquad (6.16)$$

The operating point will be taken as the point in Figure 6.2 where the curve for (6.14) intersects in the first pass band the dispersion curve given by the first mode approximation. If the experimental dispersion curve is used instead, a similar operating point is obtained. For the present treatment $\hat{\chi}$ will be set equal to the value given by the first mode approximation for the phase shift per section. Consequently, from Figure 6.2 the operating point is

$$\begin{aligned} \omega &= 15{,}870 \text{ r/s} \\ \hat{\chi} &= 2.66 \text{ r} = 152.4^{\circ}. \end{aligned} \tag{6.17}$$

At this frequency from the dispersion curve given by the solid disc theory

$$\chi = 2.89 \text{ r} = 165.6^{\circ}. \tag{6.18}$$

Through the use of (5.57) and (5.43)

$$\hat{a}_0 - f_0 = -\frac{1}{p}\left(\frac{\omega^2 \mu_0 c_0}{\hat{s}_0^{\,2} + \beta_0^{\,2}} + \frac{c_0}{\varepsilon_1}\right)\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{\hat{s}_0 z}\, T(z)dz$$

$$+ \frac{1}{p}\left(\frac{c_0}{\hat{s}_0^{\,2} + \beta_0^{\,2}}\right)\int_0^p e^{\hat{s}_0 z}\, S(z)\, \frac{dh}{dz}\, dz$$

and from (4.37) and (6.16)

$$(b_0)_{s.d.} = \frac{1}{\varepsilon_0}\left[-\frac{1}{p}\left(\frac{\omega^2 \mu_0 c_0}{s_0^{\,2} + \beta_0^{\,2}} + \frac{c_0}{\varepsilon_1}\right)\int_{\frac{p-q}{2}}^{\frac{p+q}{2}} e^{s_0 z}\, T(z)dz\right.$$

$$\left. + \frac{1}{p}\left(\frac{c_0}{s_0^{\,2} + \beta_0^{\,2}}\right)\int_0^p e^{s_0 z}\, S(z)\, \frac{dh}{dz}\, dz\right].$$

By means of (4.57)

$$\hat{a}_0 - f_0 = \frac{1}{p}\left(\frac{\omega^2\mu_0 c_0}{\hat{s}_0{}^2+\beta_0{}^2} + \frac{c_0}{\varepsilon_1}\right)\left[\frac{C_3}{\hat{s}_0 - j\beta_1}\left[e^{\hat{s}_0\left(\frac{p+q}{2}\right)} e^{-j\beta_1 q} - e^{\hat{s}_0\left(\frac{p-q}{2}\right)}\right]\right.$$

$$\left. + \frac{C_4}{\hat{s}_0 + j\beta_1}\left[e^{\hat{s}_0\left(\frac{p+q}{2}\right)} e^{j\beta_1 q} - e^{\hat{s}_0\left(\frac{p-q}{2}\right)}\right]\right]$$

$$- \frac{1}{p}\left(\frac{j\beta_1 c_0}{(\hat{s}_0{}^2+\beta_0{}^2)\varepsilon_1}\right)\left[C_3\left[e^{\hat{s}_0\left(\frac{p-q}{2}\right)} - e^{\hat{s}_0\left(\frac{p+q}{2}\right)} e^{-j\beta_1 q}\right]\right.$$

$$\left. + C_4\left[e^{\hat{s}_0\left(\frac{p+q}{2}\right)} e^{j\beta_1 q} - e^{\hat{s}_0\left(\frac{p-q}{2}\right)}\right]\right] \qquad (6.19)$$

and

$$(b_0)_{s.d.} = \frac{1}{\varepsilon_0}\left\{\frac{1}{p}\left(\frac{\omega^2\mu_0 c_0}{s_0{}^2+\beta_0{}^2} + \frac{c_0}{\varepsilon_1}\right)\left[\frac{C_3}{s_0 - j\beta_1}\left[e^{s_0\left(\frac{p+q}{2}\right)} e^{-j\beta_1 q} - e^{s_0\left(\frac{p-q}{2}\right)}\right]\right.\right.$$

$$\left. + \frac{C_4}{s_0 + j\beta_1}\left[e^{s_0\left(\frac{p+q}{2}\right)} e^{j\beta_1 q} - e^{s_0\left(\frac{p-q}{2}\right)}\right]\right]$$

$$- \frac{1}{p}\left(\frac{j\beta_1 c_0}{(s_0{}^2+\beta_0{}^2)\varepsilon_1}\right)\left[C_3\left[e^{s_0\left(\frac{p-q}{2}\right)} - e^{s_0\left(\frac{p+q}{2}\right)} e^{-j\beta_1 q}\right]\right.$$

$$\left.\left. + C_4\left[e^{s_0\left(\frac{p+q}{2}\right)} e^{j\beta_1 q} - e^{s_0\left(\frac{p-q}{2}\right)}\right]\right]\right\}. \qquad (6.20)$$

Since

$$\hat{s}_0 = j\,\frac{\hat{\chi}}{p}$$

$$s_0 = j\,\frac{\chi}{p},$$

by the use of (6.13), (6.17) and (6.18)

$$\left.\begin{aligned} \hat{s}_0 &= j\ .532(10)^2 \\ s_0 &= j\ .578(10)^2\ . \end{aligned}\right\} \tag{6.21}$$

As a result of

$$\varepsilon_1 = 93.5\varepsilon_0, \tag{6.22}$$

it is seen that

$$c_0 = 92.5\varepsilon_0. \tag{6.23}$$

Substituting

$$b = 3.849 \text{ cm}$$

into (6.2) gives

$$K_1 = .6248(10)^2\ . \tag{6.24}$$

From (6.17)

$$\omega^2\mu_0\varepsilon_0 = \left(\frac{\omega}{c}\right)^2 = .2803(10)^4. \tag{6.25}$$

By means of (6.22), (6.24) and (6.25)

$$\left.\begin{aligned} \beta_0 &= j.3318(10)^2 \\ \beta_1 &= 5.081(10)^2\ . \end{aligned}\right\} \tag{6.26}$$

Employing the values given in (6.13), (6.17), (6.21), (6.22), (6.23), (6.25) and (6.26), yields, from (6.19) and (6.20)

$$\left.\begin{aligned} \hat{a}_0 - f_0 &= C_3(-10.37 + j1.407) + C_4(9.154 - j3.309) \\ (b_0)_{s.d.} &= \frac{1}{\varepsilon_0}\Big[C_3(-9.268 + j.1828) + C_4(8.337 - j1.968)\Big]. \end{aligned}\right\} \tag{6.27}$$

According to (4.55)

$$\frac{C_4}{C_3} = \frac{\cos 2\theta_0 + j\,\dfrac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1}\sin 2\theta_0 - e^{-j(2\theta_1 - \chi)}}{-\cos 2\theta_0 + j\dfrac{\beta_1\varepsilon_0}{\beta_0\varepsilon_1}\sin 2\theta_0 + e^{j(2\theta_1 + \chi)}} \,. \qquad (6.28)$$

Since

$$2\theta_0 = \beta_0(p-q)$$

$$2\theta_1 = \beta_1 q \;,$$

from (6.13) and (6.26)

$$\left.\begin{aligned} 2\theta_0 &= j\ 1.468 \\ 2\theta_1 &= 2.930. \end{aligned}\right\} \qquad (6.29)$$

The substitution of the values in (6.18), (6.22), (6.26) and (6.29) into (6.28) gives

$$\frac{C_4}{C_3} = -.9403 - j.1957 \,. \qquad (6.30)$$

Eliminating C_4 from (6.27) by means of (6.30) gives

$$\hat{a}_0 - f_0 = 19.8\ e^{j172.1^{\circ}}\ C_3 \qquad (6.31)$$

$$(b_0)_{s.d.} = 17.6\ e^{j178.6^{\circ}}\ \frac{C_3}{\varepsilon_0} \,. \qquad (6.32)$$

From the value of "a" given in (6.13) and the value of K_1 given in (6.24)

$$J_0(K_1 a) = J_0(.625) = .905. \qquad (6.33)$$

As a consequence of (6.15), (6.31) and (6.33), from the approximate series solution

$$b_0 = 17.9\ e^{j172.1^\circ}\ \frac{C_3}{\varepsilon_0}\ . \tag{6.34}$$

The magnitudes given by (6.32) and (6.34) differ by 1.7% and the arguments differ by 6.5°.

In calculating efficiencies such as the shunt impedance,

$$Z_{shunt} = \frac{|b_0|^2}{P_1}$$

where P_1 is the power dissipated per unit length, and the series impedance,

$$Z_{series} = \frac{|b_0|^2}{P_2}$$

where P_2 is the power flux, for the present example, the value for $|b_0|^2$ given by the solid disc theory would seem reasonably accurate for practical purposes since this value differs only by 3.4% from the value given by the approximate series solution.

The value for $|b_0|$ found in (6.34), where the effect of the hole is taken into account, and the value for $|b_0|$ found in (6.32), where the effect of the hole is not taken into account, differ slightly. Consequently, it appears that the presence of the hole does not greatly affect the value of $|b_0|$. Since the presence of the hole is taken into account in (6.34), it is not surprising that this equation gives a larger value for $|b_0|$ than (6.32). The reason this result might be expected is that in an empty waveguide all the energy of the field is in the 0th space harmonic and the hole case should lie somewhere between the solid disc case and the empty waveguide case.

7. CONCLUSION

In viewing a periodic structure as a whole rather than as a number of regions, wave equations have been found which hold throughout the waveguide. This point of view has led to a simple wave equation which gives an approximate field solution for an axially symmetric periodic structure with a slowly varying radius. This wave equation has been shown to be separable and the problem has been reduced to solving Hill's equation.

In dielectric loaded structures the wave equations derived have coefficients possessing finite discontinuities and in some cases impulses. These discontinuities are due to the discontinuities in the permittivity. For the examples treated with discontinuities only in the axial direction, the solutions obtained are in agreement with those found by other methods. When discontinuities occur in the radial as well as axial direction, a first mode approximation and an approximate series solution are derived. The first mode approximation is a relatively simple solution. For example, for the dielectric loaded periodic structure with each disc having a central hole, the first mode approximation is of the same order of complexity as the solid disc solution. From the comparison between experimental measurements and the theoretical predictions, to within limits, the first mode approximation gives reasonably good answers.

Although more complex than the first mode approximation, the approximate series solution provides a first iteration on any guessed solution that might be employed. As is done in section 5, the solid disc solution can be used for the initial guess but, no doubt, a better iterative solution could be achieved if, instead, the first mode approximation is utilized for the guessed solution.

APPENDIX 1

The vectorial differential wave equation for the magnetic field intensity can be arrived at in the following way. First of all, Maxwell's equations are

$$\nabla \times \vec{E} = - j\omega\vec{B} \tag{A1.1}$$

$$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D} \,. \tag{A1.2}$$

Now, if the curl of (A1.2) is taken, the result is

$$\nabla \times (\nabla \times \vec{H}) = \nabla \times \vec{J} + j\omega \nabla \times \vec{D}. \tag{A1.3}$$

Since the medium is linear and isotropic, then

$$\vec{D} = \varepsilon \, \vec{E} \tag{A1.4}$$

$$\vec{B} = \mu \, \vec{H} \tag{A1.5}$$

$$\vec{J} = \sigma \, \vec{E}; \tag{A1.6}$$

$\vec{D}$ and $\vec{J}$ can be eliminated from (A1.3) through the use of (A1.4) and (A1.6). Hence,

$$\nabla \times (\nabla \times \vec{H}) = j\omega \nabla \times (\varepsilon - j\tfrac{\sigma}{\omega})\vec{E}$$

$$= j\omega(\varepsilon - j\tfrac{\sigma}{\omega})\nabla \times \vec{E} + j\omega\nabla(\varepsilon - j\tfrac{\sigma}{\omega}) \times \vec{E}. \tag{A1.7}$$

From (A1.1), (A1.2), (A1.4), (A1.5) and (A1.6) it can be seen that

$$\nabla \times \vec{E} = - j\omega\mu\vec{H} \tag{A1.8}$$

$$\vec{E} = \frac{1}{j\omega(\varepsilon - j\frac{\sigma}{\omega})} \nabla \times \vec{H}. \tag{A1.9}$$

If (A1.8) and (A1.9) are used to eliminate terms with $\vec{E}$ in (A1.7), then (A1.7) becomes

$$\nabla \times (\nabla \times \vec{H}) = \omega^2\mu(\varepsilon - j\frac{\sigma}{\omega})\vec{H} + \frac{1}{(\varepsilon - j\frac{\sigma}{\omega})}\nabla(\varepsilon - j\frac{\sigma}{\omega}) \times (\nabla \times \vec{H})$$

or

$$\nabla(\nabla\cdot\vec{H}) - \nabla^2\vec{H} = \omega^2\mu(\varepsilon - j\frac{\sigma}{\omega})\vec{H} + \frac{1}{(\varepsilon - j\frac{\sigma}{\omega})}\Big[\nabla(\varepsilon - j\frac{\sigma}{\omega})\cdot\nabla\vec{H} - \Big(\nabla(\varepsilon - j\frac{\sigma}{\omega})\cdot\nabla\Big)\vec{H}\Big] \quad \text{(A1.10)}$$

where in rectangular co-ordinates

$$\nabla(\varepsilon - j\frac{\sigma}{\omega})\cdot\nabla\vec{H} = \nabla(\varepsilon - j\frac{\sigma}{\omega})\cdot\frac{\partial\vec{H}}{\partial x}\,\vec{i} + \nabla(\varepsilon - j\frac{\sigma}{\omega})\cdot\frac{\partial\vec{H}}{\partial y}\,\vec{j} + \nabla(\varepsilon - j\frac{\sigma}{\omega})\cdot\frac{\partial\vec{H}}{\partial z}\,\vec{k}.$$

In the case of the vectorial differential wave equation for the electric field intensity, the curl of (A1.1) is taken and the result is

$$\nabla \times (\nabla \times \vec{E}) = -j\omega\nabla \times \vec{B}. \quad \text{(A1.11)}$$

From (A1.5), (A1.11) can be expressed as

$$\nabla \times (\nabla \times \vec{E}) = -j\omega\mu\nabla \times \vec{H} - j\omega(\nabla\mu \times \vec{H}). \quad \text{(A1.12)}$$

If (A1.8) and (A1.9) are used to eliminate terms involving $\vec{H}$ in (A1.12), then (A1.12) becomes

$$\nabla \times (\nabla \times \vec{E}) = \omega^2\mu(\varepsilon - j\frac{\sigma}{\omega})\,\vec{E} + \frac{1}{\mu}\left[\nabla\mu \times (\nabla \times \vec{E})\right]$$

or

$$\nabla(\nabla.\vec{E}) - \nabla^2\vec{E} = \omega^2\mu(\varepsilon - j\frac{\sigma}{\omega})\ \vec{E} + \frac{1}{\mu}\left[\nabla\mu.\nabla\vec{E} - (\nabla\mu.\nabla)\ \vec{E}\right] .$$

(A1.13)

APPENDIX 2

In this appendix the series

$$T(z) = \sum_{n=-\infty}^{\infty} a_n e^{-s_n z} \tag{A2.1}$$

$$S(z) = \frac{1}{\varepsilon}\frac{dT}{dz} = -\frac{1}{\varepsilon} \sum_{n=-\infty}^{\infty} s_n\, a_n e^{-s_n z} \tag{A2.2}$$

where

$$s_n = j\left(\frac{\chi + 2n\pi}{p}\right)$$

$$a_n = \frac{1}{p}\left[\frac{C_1}{s_n - j\beta_0} + \frac{C_2}{s_n + j\beta_0} + \frac{C_3}{s_n - j\beta_1} + \frac{C_4}{s_n + j\beta_1}\right] e^{s_n\left(\frac{p-q}{2}\right)}$$

$$-\frac{1}{p}\left[\frac{D_1}{s_n - j\beta_0} + \frac{D_2}{s_n + j\beta_0} + \frac{D_3}{s_n - j\beta_1} + \frac{D_4}{s_n + j\beta_1}\right] e^{s_n\left(\frac{p+q}{2}\right)} \tag{A2.3}$$

are evaluated at the points $\frac{p-q}{2}$ and $\frac{p+q}{2}$. The only summations that are needed in this evaluation are[A1]

$$\left.\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{e^{s_n \alpha}}{s_n - j\beta} &= \frac{p\, e^{j\beta\alpha}\, e^{j\left(\frac{\chi-\beta p}{2}\right)}}{2j \sin\left(\frac{\chi-\beta p}{2}\right)} \\
\sum_{n=-\infty}^{\infty} \frac{e^{s_n \alpha}}{s_n + j\beta} &= \frac{p\, e^{-j\beta\alpha}\, e^{j\left(\frac{\chi+\beta p}{2}\right)}}{2j \sin\left(\frac{\chi+\beta p}{2}\right)}
\end{aligned}\right\} \tag{A2.4}$$

$$\sum_{n=-\infty}^{\infty} \frac{e^{-s_n\alpha}}{s_n - j\beta} = \frac{p\, e^{-j\beta\alpha}\, e^{-j\left(\frac{\chi-\beta p}{2}\right)}}{2j \sin\left(\frac{\chi-\beta p}{2}\right)}$$

$$\sum_{n=-\infty}^{\infty} \frac{e^{-s_n\alpha}}{s_n + j\beta} = \frac{p\, e^{j\beta\alpha}\, e^{-j\left(\frac{\chi+\beta p}{2}\right)}}{2j \sin\left(\frac{\chi+\beta p}{2}\right)}$$

for $0 < \alpha < p$.

As a starting point, the series for $T(\frac{p-q}{2})$ will be summed. Since the series in (A2.4) only hold for $0 < \alpha < p$, it is convenient first to consider $T(\frac{p-q}{2} - \delta)$ where $\delta > 0$ and then to obtain $T(\frac{p-q}{2})$ by taking the limit as $\delta \rightarrow 0$.

From (A2.1) and (A2.3)

$$T\left(\frac{p-q}{2} - \delta\right) = \frac{1}{p} \sum_{n=-\infty}^{\infty} \left[\frac{C_1}{s_n - j\beta_0} + \frac{C_2}{s_n + j\beta_0} + \frac{C_3}{s_n - j\beta_1} + \frac{C_4}{s_n + j\beta_1}\right] e^{s_n\delta}$$

$$- \frac{1}{p} \sum_{n=-\infty}^{\infty} \left[\frac{D_1}{s_n - j\beta_0} + \frac{D_2}{s_n + j\beta_0} + \frac{D_3}{s_n - j\beta_1} + \frac{D_4}{s_n + j\beta_1}\right] e^{s_n(q+\delta)}. \tag{A2.5}$$

It is easy to verify that

$$e^{j\chi}\, e^{-s_n p} = 1. \tag{A2.6}$$

Through the use of (A2.6), (A2.5) can be expressed more conveniently as

$$T(\frac{p-q}{2}-\delta) = \frac{1}{p}\sum_{n=-\infty}^{\infty}\left[\frac{C_1}{s_n-j\beta_0}+\frac{C_2}{s_n+j\beta_0}+\frac{C_3}{s_n-j\beta_1}+\frac{C_4}{s_n+j\beta_1}\right]e^{s_n\delta}$$

$$-\frac{1}{p}e^{j\chi}\sum_{n=-\infty}^{\infty}\left[\frac{D_1}{s_n-j\beta_0}+\frac{D_2}{s_n+j\beta_0}\right]e^{-s_n(p-q-\delta)}$$

$$-\frac{1}{p}\sum_{n=-\infty}^{\infty}\left[\frac{D_3}{s_n-j\beta_1}+\frac{D_4}{s_n+j\beta_1}\right]e^{s_n(q+\delta)} .$$

Hence, if the summations in (A2.4) are now used,

$$T(\frac{p-q}{2}-\delta) = \frac{e^{j\beta_0\delta}\,e^{j(\frac{\chi-\beta_0 p}{2})}}{2j\sin(\frac{\chi-\beta_0 p}{2})}C_1 + \frac{e^{-j\beta_0\delta}\,e^{j(\frac{\chi+\beta_0 p}{2})}}{2j\sin(\frac{\chi+\beta_0 p}{2})}C_2$$

$$+\frac{e^{j\beta_1\delta}\,e^{j(\frac{\chi-\beta_1 p}{2})}}{2j\sin(\frac{\chi-\beta_1 p}{2})}C_3 + \frac{e^{-j\beta_1\delta}\,e^{j(\frac{\chi+\beta_1 p}{2})}}{2j\sin(\frac{\chi+\beta_1 p}{2})}C_4 - \frac{e^{j\chi}\,e^{-j\beta_0(p-q-\delta)}\,e^{-j(\frac{\chi-\beta_0 p}{2})}}{2j\sin(\frac{\chi-\beta_0 p}{2})}D_1$$

$$-\frac{e^{j\chi}\,e^{j\beta_0(p-q-\delta)}\,e^{-j(\frac{\chi+\beta_0 p}{2})}}{2j\sin(\frac{\chi+\beta_0 p}{2})}D_2 - \frac{e^{j\beta_1(q+\delta)}\,e^{j(\frac{\chi-\beta_1 p}{2})}}{2j\sin(\frac{\chi-\beta_1 p}{2})}D_3$$

$$-\frac{e^{-j\beta_1(q+\delta)}\,e^{j(\frac{\chi+\beta_1 p}{2})}}{2j\sin(\frac{\chi+\beta_1 p}{2})}D_4 .$$

In the limit as $\delta \longrightarrow 0$

$$T\left(\frac{p-q}{2}\right) = y_1C_1+y_2C_2+y_3C_3+y_4C_4+e^{j(\chi-2\theta_0)}y_1^{*}D_1+e^{j(\chi+2\theta_0)}y_2^{*}D_2-e^{j2\theta_1}y_3D_3$$

$$-e^{-j2\theta_1}y_4D_4$$

where

$$2\theta_0 = \beta_0(p-q)$$

$$2\theta_1 = \beta_1 q$$

$$y_1 = \frac{e^{j\left(\frac{\chi-\beta_0 p}{2}\right)}}{2j\,\sin\left(\frac{\chi-\beta_0 p}{2}\right)}$$

$$y_2 = \frac{e^{j\left(\frac{\chi+\beta_0 p}{2}\right)}}{2j\,\sin\left(\frac{\chi+\beta_0 p}{2}\right)}$$

$$y_3 = \frac{e^{j\left(\frac{\chi-\beta_1 p}{2}\right)}}{2j\,\sin\left(\frac{\chi-\beta_1 p}{2}\right)}$$

$$y_4 = \frac{e^{j\left(\frac{\chi+\beta_1 p}{2}\right)}}{2j\,\sin\left(\frac{\chi+\beta_1 p}{2}\right)}\,.$$

As can be seen from Figure 4.8 in the body of the thesis, at $z = \frac{p-q}{2} - \delta$, $\varepsilon = \varepsilon_0$. Consequently, from (A2.2)

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* denotes complex conjugate

$$S\left(\frac{p-q}{2}-\delta\right) = -\frac{1}{p\varepsilon_0}\sum_{n=-\infty}^{\infty}\left[\frac{j\beta_0 C_1}{s_n-j\beta_0}-\frac{j\beta_0 C_2}{s_n+j\beta_0}+\frac{j\beta_1 C_3}{s_n-j\beta_1}-\frac{j\beta_1 C_4}{s_n+j\beta_1}\right]e^{s_n\delta}$$

$$+\frac{1}{p\varepsilon_0}\sum_{n=-\infty}^{\infty}\left[\frac{j\beta_0 D_1}{s_n-j\beta_0}-\frac{j\beta_0 D_2}{s_n+j\beta_0}+\frac{j\beta_1 D_3}{s_n-j\beta_1}-\frac{j\beta_1 D_4}{s_n+j\beta_1}\right]e^{s_n(q+\delta)}.$$

Hence,

$$S\left(\frac{p-q}{2}\right) = -j\frac{\beta_0}{\varepsilon_0}y_1C_1+j\frac{\beta_0}{\varepsilon_0}y_2C_2-j\frac{\beta_1}{\varepsilon_0}y_3C_3+j\frac{\beta_1}{\varepsilon_0}y_4C_4$$

$$-j\frac{\beta_0}{\varepsilon_0}e^{j(\chi-2\theta_0)}y_1^*D_1+j\frac{\beta_0}{\varepsilon_0}e^{j(\chi+2\theta_0)}y_2^*D_2+j\frac{\beta_1}{\varepsilon_0}e^{j2\theta_1}y_3D_3-j\frac{\beta_1}{\varepsilon_0}e^{-j2\theta_1}y_4D_4.$$

Similarly,

$$T\left(\frac{p+q}{2}\right) = e^{-j(\chi-2\theta_0)}y_1C_1+e^{-j(\chi+2\theta_0)}y_2C_2-e^{-j2\theta_1}y_3^*C_3-e^{j2\theta_1}y_4^*C_4$$

$$-y_1D_1-y_2D_2-y_3D_3-y_4D_4$$

$$S\left(\frac{p+q}{2}\right) = -j\frac{\beta_0}{\varepsilon_1}e^{-j(\chi-2\theta_0)}y_1C_1+j\frac{\beta_0}{\varepsilon_1}e^{-j(\chi+2\theta_0)}y_2C_2+j\frac{\beta_1}{\varepsilon_1}e^{-j2\theta_1}y_3^*C_3$$

$$-j\frac{\beta_1}{\varepsilon_1}e^{j2\theta_1}y_4^*C_4+j\frac{\beta_0}{\varepsilon_1}y_1D_1-j\frac{\beta_0}{\varepsilon_1}y_2D_2+j\frac{\beta_1}{\varepsilon_1}y_3D_3-j\frac{\beta_1}{\varepsilon_1}y_4D_4.$$

APPENDIX 3

The summation of the series

$$T_1\left(\frac{p}{2}\right) = \sum_{n=-\infty}^{\infty} \hat{a}_n \, e^{\frac{-\hat{s}_n p}{2}}$$

is carried out through the following steps. Through the use of (5.11) and (5.19) in the body of the thesis

$$T_1\left(\frac{p}{2}\right) = \frac{c_0}{p} \sum_{n=-\infty}^{\infty} \frac{e^{\frac{-\hat{s}_n p}{2}}}{(\hat{s}_n^{\,2}+\beta_0^{\,2})(\hat{s}_n^{\,2}+\beta_1^{\,2})} \left[e^{\hat{s}_n\left(\frac{p-q}{2}\right)} \left[\omega^2 \mu_0 \hat{s}_n T\left(\frac{p-q}{2}\right) \right.\right.$$

$$\left.\left. + (\hat{s}_n^{\,2}-K_1^{\,2})\, S\left(\frac{p-q}{2}\right)\right] - e^{\hat{s}_n\left(\frac{p+q}{2}\right)}\left[\omega^2\mu_0\hat{s}_n T\left(\frac{p+q}{2}\right)+(\hat{s}_n^{\,2}-K_1^{\,2})S\left(\frac{p+q}{2}\right)\right]\right]$$

and thus

$$T_1\left(\frac{p}{2}\right) = \frac{1}{p} \sum_{n=-\infty}^{\infty} \left[\left[\frac{C_1}{\hat{s}_n-j\beta_0} + \frac{C_2}{\hat{s}_n+j\beta_0} + \frac{C_3}{\hat{s}_n-j\beta_1} + \frac{C_4}{\hat{s}_n+j\beta_1}\right] e^{\frac{-\hat{s}_n q}{2}} \right.$$

$$\left. - \left[\frac{D_1}{\hat{s}_n-j\beta_0} + \frac{D_2}{\hat{s}_n+j\beta_0} + \frac{D_3}{\hat{s}_n-j\beta_1} + \frac{D_4}{\hat{s}_n+j\beta_1}\right] e^{\frac{\hat{s}_n q}{2}} \right] \tag{A3.1}$$

where the C's and D's are defined in (4.43) and (4.44) and related in (4.52). Summing the series in (A3.1) with the help of (A2.4) yields

$$T_1(\tfrac{p}{2}) = \frac{C_1 e^{-j\frac{\beta_0 q}{2}} e^{-j(\frac{\hat{\chi}-\beta_0 p}{2})}}{2j \sin(\frac{\hat{\chi}-\beta_0 p}{2})} + \frac{C_2 e^{j\frac{\beta_0 q}{2}} e^{-j(\frac{\hat{\chi}+\beta_0 p}{2})}}{2j \sin(\frac{\hat{\chi}+\beta_0 p}{2})} + \frac{C_3 e^{-j\frac{\beta_1 q}{2}} e^{-j(\frac{\hat{\chi}-\beta_1 p}{2})}}{2j \sin(\frac{\hat{\chi}-\beta_1 p}{2})}$$

$$+ \frac{C_4 e^{j\frac{\beta_1 q}{2}} e^{-j(\frac{\hat{\chi}+\beta_1 p}{2})}}{2j \sin(\frac{\hat{\chi}+\beta_1 p}{2})}$$

$$- \frac{D_1 e^{j\frac{\beta_0 q}{2}} e^{j(\frac{\hat{\chi}-\beta_0 p}{2})}}{2j \sin(\frac{\hat{\chi}-\beta_0 p}{2})} - \frac{D_2 e^{-j\frac{\beta_0 q}{2}} e^{j(\frac{\hat{\chi}+\beta_0 p}{2})}}{2j \sin(\frac{\hat{\chi}+\beta_0 p}{2})} - \frac{D_3 e^{j\frac{\beta_1 q}{2}} e^{j(\frac{\hat{\chi}-\beta_1 p}{2})}}{2j \sin(\frac{\hat{\chi}-\beta_1 p}{2})}$$

$$- \frac{D_4 e^{-j\frac{\beta_1 q}{2}} e^{j(\frac{\hat{\chi}+\beta_1 p}{2})}}{2j \sin(\frac{\hat{\chi}+\beta_1 p}{2})} .$$

By means of (4.52) the D's can be eliminated to give

$$T_1(\tfrac{p}{2}) = \frac{C_1\left[e^{-j(\frac{\hat{\chi}-2\theta_0}{2})} - e^{-j(\chi-2\theta_0)} e^{j(\frac{\hat{\chi}-2\theta_0}{2})}\right]}{2j \sin(\frac{\hat{\chi}-\beta_0 p}{2})}$$

$$+ \frac{C_2\left[e^{-j(\frac{\hat{\chi}+2\theta_0}{2})} - e^{-j(\chi+2\theta_0)} e^{j(\frac{\hat{\chi}+2\theta_0}{2})}\right]}{2j \sin(\frac{\hat{\chi}+\beta_0 p}{2})}$$

$$+ \frac{C_3\left[e^{-j\theta_1} e^{-j(\frac{\hat{\chi}-\beta_1 p}{2})} - e^{-j2\theta_1} e^{j(\frac{\hat{\chi}-\beta_1 p}{2})} e^{j\theta_1}\right]}{2j \sin(\frac{\hat{\chi}-\beta_1 p}{2})} +$$

$$+ \frac{C_4\left[e^{j\theta_1}e^{-j\left(\frac{\hat{\chi}+\beta_1 p}{2}\right)} - e^{j2\theta_1}e^{-j\theta_1}e^{j\left(\frac{\hat{\chi}+\beta_1 p}{2}\right)}\right]}{2j\,\sin\left(\frac{\hat{\chi}+\beta_1 p}{2}\right)}$$

or

$$T_1\left(\frac{p}{2}\right) = \frac{C_1 e^{-j\left(\frac{\hat{\chi}-2\theta_0}{2}\right)}}{2j\,\sin\left(\frac{\hat{\chi}-\beta_0 p}{2}\right)}\left[1 - e^{j(\hat{\chi}-\chi)}\right] + \frac{C_2 e^{-j\left(\frac{\hat{\chi}+2\theta_0}{2}\right)}}{2j\,\sin\left(\frac{\hat{\chi}+\beta_0 p}{2}\right)}\left[1 - e^{j(\hat{\chi}-\chi)}\right]$$

$$- C_3 e^{-j\theta_1} - C_4 e^{j\theta_1}\,. \qquad \text{(A3.2)}$$

From (A2.1) $T(\frac{p}{2})$ can be easily summed and shown equal to the expression

$$T\left(\frac{p}{2}\right) = -C_3 e^{-j\theta_1} - C_4 e^{j\theta_1}\,.$$

As a result, (A3.2) can be rewritten in the form

$$T_1\left(\frac{p}{2}\right) = T\left(\frac{p}{2}\right) + \frac{e^{-j\frac{\hat{\chi}}{2}}}{2j}\left[\frac{C_1 e^{j\theta_0}}{\sin\left(\frac{\hat{\chi}-\beta_0 p}{2}\right)} + \frac{C_2 e^{-j\theta_0}}{\sin\left(\frac{\hat{\chi}+\beta_0 p}{2}\right)}\right]\left[1 - e^{j(\hat{\chi}-\chi)}\right].$$

APPENDIX 4

The summation of the series,

$$S_t = \sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{2n\pi + \chi}{p} - \beta_1\right)\frac{q}{2}}{I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi} \left(\frac{2n\pi + \chi}{p} - \beta_1\right)},$$

will now be treated. If the definition,

$$\omega_n = \frac{\chi + 2n\pi}{p},$$

is made, then

$$S_t = \sum_{n=-\infty}^{\infty} \frac{\sin(\omega_n - \beta_1)\frac{q}{2}}{I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi} (\omega_n - \beta_1)} \qquad (A4.1)$$

and since $\hat{\chi}=\chi$,

$$\hat{K}_n^{\,2} = \left(\frac{\chi + 2n\pi}{p}\right)^2 - \omega^2\mu_0\varepsilon_0 = \omega_n^{\,2} - \beta_0^{\,2} - K_1^{\,2}.$$

For

$$\hat{K}_n^{\,2} = -\bar{K}_n^{\,2}$$

where $\bar{K}_n$ is real,

$$I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi} = J_0(\bar{K}_n b).$$

The zeros of $J_0(\bar{K}_n b)$ are $K_m b$ and thus poles occur in S_t at

$$\omega_n = \pm\sqrt{\beta_0^2 + K_1^2 - K_m^2} \,.$$

Expanding $I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi}$ around the zero, $\omega_n = \sqrt{\beta_0^2 + K_1^2 - K_m^2}$,

gives

$$I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi} = J_0(K_m b) - b\, J_1(K_m b)\, \frac{d\overline{K}_n}{d\omega_n}\Big|_{\overline{K}_n=K_m} (\omega_n - \sqrt{\beta_0^2 + K_1^2 - K_m^2}) + \cdots$$

or

$$I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi} = \frac{b\sqrt{\beta_0^2 + K_1^2 - K_m^2}}{K_m}\, J_1(K_m b)(\omega_n - \sqrt{\beta_0^2 + K_1^2 - K_m^2}) + \cdots.$$

Similarly, around $\omega_n = -\sqrt{\beta_0^2 + K_1^2 - K_m^2}$

$$I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi} = -\frac{b\sqrt{\beta_0^2 + K_1^2 - K_m^2}}{K_m}\, J_1(K_m b)(\omega_n + \sqrt{\beta_0^2 + K_1^2 - K_m^2}) + \cdots.$$

Now, the ratio,

$$\frac{1}{I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi} (\omega_n - \beta_1)} \,,$$

can be expanded in partial fractions as

$$\frac{1}{I_0(\hat{K}_n b)\Big|_{\hat{\chi}=\chi} (\omega_n - \beta_1)} = \frac{A_0}{\omega_n - \beta_1} + \sum_{m=1}^{\infty} \frac{A_m}{(\omega_n - \sqrt{\beta_0{}^2 + K_1{}^2 - K_m{}^2})}$$

$$+ \sum_{m=1}^{\infty} \frac{A_{-m}}{(\omega_n + \sqrt{\beta_0{}^2 + K_1{}^2 - K_m{}^2})}$$

where

$$A_0 = \frac{1}{I_0(\sqrt{\beta_1{}^2 - \beta_0{}^2 - K_1{}^2}\, b)}$$

$$A_m = \frac{K_m}{b\sqrt{\beta_0{}^2 + K_1{}^2 - K_m{}^2}\; J_1(K_m b)(\sqrt{\beta_0{}^2 + K_1{}^2 - K_m{}^2} - \beta_1)}$$

$$A_{-m} = \frac{K_m}{b\sqrt{\beta_0{}^2 + K_1{}^2 - K_m{}^2}\; J_1(K_m b)(\sqrt{\beta_0{}^2 + K_1{}^2 - K_m{}^2} + \beta_1)} \,.$$

Therefore,

$$S_t = \frac{p}{2\pi}\left[A_0 \sum_{n=-\infty}^{\infty} \frac{\sin\left[n + \frac{1}{2\pi}(\chi - \beta_1 p)\right]\pi\, \frac{q}{p}}{n + \frac{1}{2\pi}(\chi - \beta_1 p)}\right.$$

$$+ \sum_{m=1}^{\infty} A_m \sum_{n=-\infty}^{\infty} \frac{\sin\left[n + \frac{1}{2\pi}(\chi - \beta_1 p)\right]\pi\, \frac{q}{p}}{n + \frac{1}{2\pi}(\chi - \sqrt{\beta_0{}^2 + K_1{}^2 - K_m{}^2}\; p)} +$$

$$+ \sum_{m=1}^{\infty} A_{-m} \sum_{n=-\infty}^{\infty} \frac{\sin\left[n + \frac{1}{2\pi}(\chi - \beta_1 p)\right]\pi \frac{q}{p}}{n + \frac{1}{2\pi}(\chi + \sqrt{\beta_0^2 + K_1^2 - K_m^2}\; p)} \Bigg] .$$

Since[A2]

$$\sum_{n=-\infty}^{\infty} \frac{\cos n\theta}{n + a} = \frac{\pi \cos a(\pi - \theta)}{\sin a\pi}, \quad 0 < \theta < 2\pi$$

$$\sum_{n=-\infty}^{\infty} \frac{\sin n\theta}{n + a} = \frac{\pi \sin a(\pi - \theta)}{\sin a\pi}, \quad 0 < \theta < 2\pi$$

and

$$\pi \frac{q}{p} < \pi,$$

the sum becomes

$$S_t = \frac{p}{2} \left\{ \frac{1}{I_0(\sqrt{\beta_1^2 - \beta_0^2 - K_1^2}\; b)} \right.$$

$$+ \sum_{m=1}^{\infty} \frac{K_m}{b\sqrt{\beta_0^2 + K_1^2 - K_m^2}\; J_1(K_m b)} \left[\frac{\sin\left[\frac{\chi - \beta_1 q - \sqrt{\beta_0^2 + K_1^2 - K_m^2}\;(p-q)}{2}\right]}{(\sqrt{\beta_0^2 + K_1^2 - K_m^2} - \beta_1)\sin\left[\frac{\chi - \sqrt{\beta_0^2 + K_1^2 - K_m^2}\; p}{2}\right]} \right.$$

$$\left.\left. + \frac{\sin\left[\frac{\chi - \beta_1 q + \sqrt{\beta_0^2 + K_1^2 - K_m^2}\;(p-q)}{2}\right]}{(\sqrt{\beta_0^2 + K_1^2 - K_m^2} + \beta_1)\sin\left[\frac{\chi + \sqrt{\beta_0^2 + K_1^2 - K_m^2}\; p}{2}\right]} \right] \right\} . \qquad (A4.2)$$

The series in (A4.2) is oscillatory with $J_1(K_m b)$ being positive for m odd and negative for m even. In the series the term corresponding to $m = 1$ is due to a correction made by the lowest order mode, $J_1(K_1 r)$, to the solid disc solution. The terms for $m > 1$ are the contributions made by the higher order modes of the wave solution.

If the term for $m = 1$ is removed from the series, then for

$$\chi = \beta_0(p-q) + \beta_1 q$$

$$S_t = \frac{p}{2}\left[\frac{1}{I_0(\sqrt{\beta_1{}^2 - \beta_0{}^2 - K_1{}^2}\, b)} + \frac{K_1 \sin \beta_0(p-q)}{b\beta_0 J_1(K_1 b)(\beta_1+\beta_0)\sin\left[\frac{2\beta_0(p-q)+(\beta_1+\beta_0)q}{2}\right]} + \frac{S}{b}\right]$$

where

$$S = \sum_{m=2}^{\infty} \frac{K_m}{\sqrt{\beta_0{}^2 + K_1{}^2 - K_m{}^2}\, J_1(K_m b)}\left[\frac{\sin\left[\frac{\chi - \beta_1 q - \sqrt{\beta_0{}^2+K_1{}^2-K_m{}^2}(p-q)}{2}\right]}{(\sqrt{\beta_0{}^2+K_1{}^2-K_m{}^2}-\beta_1)\sin\left[\frac{\chi - \sqrt{\beta_0{}^2+K_1{}^2-K_m{}^2}\, p}{2}\right]}\right.$$

$$\left. + \frac{\sin\left[\frac{\chi-\beta_1 q+\sqrt{\beta_0{}^2+K_1{}^2-K_m{}^2}\,(p-q)}{2}\right]}{(\sqrt{\beta_0{}^2+K_1{}^2-K_m{}^2}+\beta_1)\sin\left[\frac{\chi+\sqrt{\beta_0{}^2+K_1{}^2-K_m{}^2}\, p}{2}\right]}\right].$$

Most of the elements in S are evanescent; they damp out asymptotically as

$$\frac{1}{\sqrt{K_m}}\, e^{-K_m \frac{q}{2}}.$$

APPENDIX 5

For the cavity shown in the Figure, when an H-wave exists with no angular dependence and

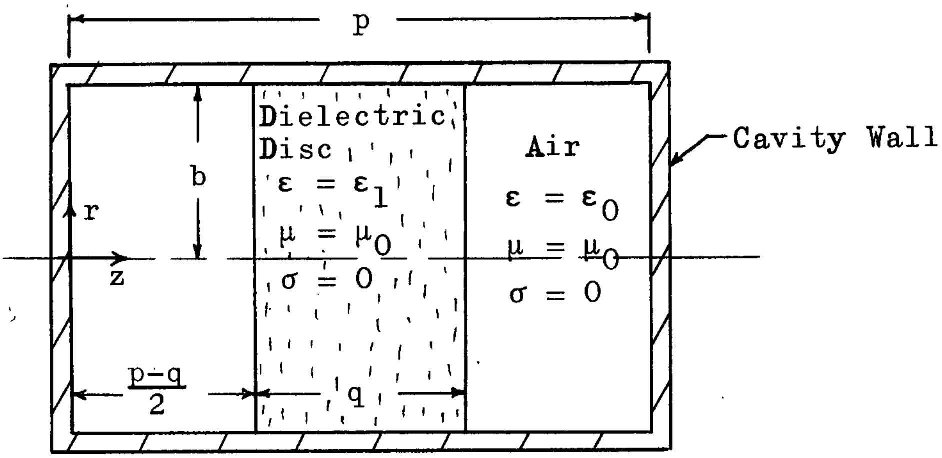

Fig. A5.1. Resonant Cavity Containing a Dielectric Disc

$$E_r \equiv 0, \; H_\phi \equiv 0, \; E_z \equiv 0,$$

in the air regions E_ϕ satisfies

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right] + \frac{\partial^2 E_\phi}{\partial z^2} + \omega_0{}^2\mu_0\varepsilon_0 E_\phi = 0$$

and in the dielectric region

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right] + \frac{\partial^2 E_\phi}{\partial z^2} + \omega_0{}^2\mu_0\varepsilon_1 E_\phi = 0 \; .$$

The boundary conditions that E_ϕ must obey are:

(i) at $z = 0$ and $z = p$

$$E_\phi = 0$$

(ii) at $r = 0$

E_ϕ is finite

(iii) at $r = b$

$E_\phi = 0$

(iv) at $z = \frac{p-q}{2}$ and $z = \frac{p+q}{2}$

E_ϕ is continuous

H_r and thus $\frac{\partial E_\phi}{\partial z}$ is continuous.

As a result, the solution for E_ϕ, which is symmetric about the plane $z = p/2$ and in the lowest order mode, is in the region, $0 \leq z \leq \frac{p-q}{2}$,

$$E_\phi = B\ J_1(K_1 r)\ \sin \beta_0 z$$

and in the region, $\frac{p-q}{2} \leq z \leq \frac{p+q}{2}$,

$$E_\phi = A\ J_1(K_1 r)\ \cos \beta_1 (z - \tfrac{p}{2})$$

and in the region, $\frac{p+q}{2} \leq z \leq p$,

$$E_\phi = B\ J_1(K_1 r)\ \sin \beta_0 (p-z)$$

where

$$\beta_0^{\,2} = \omega_0^{\,2}\mu_0\varepsilon_0 - K_1^{\,2}$$

$$\beta_1^{\,2} = \omega_0^{\,2}\mu_0\varepsilon_1 - K_1^{\,2}$$

$K_1 b$ = the first root of $J_1(Kb)$.

From conditions (iv)

$$B \sin \beta_0 (\frac{p-q}{2}) = A \cos \beta_1 \frac{q}{2}$$

$$\beta_0 B \cos \beta_0 (\frac{p-q}{2}) = \beta_1 A \sin \beta_1 \frac{q}{2} .$$

Therefore,

$$B = \frac{\cos \beta_1 \frac{q}{2}}{\sin \beta_0 (\frac{p-q}{2})} A$$

and

$$\tan \beta_1 \frac{q}{2} = \frac{\beta_0}{\beta_1} \cot \beta_0 (\frac{p-q}{2}) .$$

This last relation gives the resonant frequency, ω_0.

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