ON THE RESONANCE PROPERTIES OF QUASI-LINEAR SECOND-ORDER DIFFERENTIAL-DIFFERENCE EQUATIONS

by

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ABSTRACT

Since very little has appeared in the literature regarding solutions of driven nonlinear differential-difference equations, it has been the purpose of this investigation to obtain approximate solutions to these equations and to investigate their resonance properties. The equations considered are second-order quasi-linear differential-difference equations.

Stability criteria are presented for equations having delayed damping and for equations having a delayed restoring force.

Application of the Ritz method leads to general equations which determine the constants in the assumed solution. The general equations for systems with odd nonlinearities are used to obtain the resonance properties for several specific examples. Unusual jump resonance phenomena are obtained when the input frequency is varied. Regions of the response curve occur which are not connected to each other.

The approximate solution is verified by an analog computer simulation employing track and store techniques to enable automatic plotting of the response curves. The Ritz-method results compare favourably with the analog-simulation results.

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LIST OF SYMBOLS

C	=	a constant
D_1, D_2	=	coefficients of damping
h	=	a positive constant
k	=	a constant
L	=	a positive constant
s	=	complex frequency
$\sup(\lvert x_1 \rvert, \ldots, \lvert x_n \rvert)$	=	the largest value in the set, $(\lvert x_1 \rvert, \ldots, \lvert x_n \rvert)$
t	=	time
T	=	a time-constant
$\tilde{x}(t)$	=	an approximation to $x(t)$
$\dot{x}(t)$	=	the first derivative of $x(t)$ with respect to t
$\ddot{x}(t)$	=	the second derivative of $x(t)$ with respect to t
α	=	a dummy variable of integration
$\Gamma(x)$	=	the Gamma function of x
ε	=	a positive constant
ϑ	=	a variable which is restricted to the interval, $-h \leq \vartheta \leq 0$
μ	=	a constant
ξ	=	a dummy variable of integration
$\emptyset$	=	a variable which is restricted to the interval, $-h \leq \emptyset \leq 0$
φ	=	a function
ω	=	angular frequency
ω_n^2	=	coefficient of restoring force

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1. INTRODUCTION

Ordinary differential equations (or systems of equations) have been and still are a useful tool in the analysis of a wide variety of physical phenomena. It must be kept in mind, however, that in using ordinary differential equations to describe phenomena, it is assumed that the future behavior of the system depends only on the present state and is independent of the past. In order to more accurately describe physical systems, therefore, it is necessary to consider the fact that their rate of change depends not only on their present state, but also on their past history. In place of the ordinary differential equation ,

$$\frac{dx}{dt} = f(x,t) \quad , \qquad x(t_o) = C, \tag{1.1}$$

we must write

$$\frac{dx}{dt} = f \left[x(t),\ x(\vartheta),\ t \right] \quad , \tag{1.2}$$

where ϑ ranges over a set of values less than t, and the initial function, $x_o(\vartheta)$, is specified for the range, ϑ, less than the initial time, t_o. Of the many equations of this type, perhaps the simplest is a differential - difference equation ,

$$\frac{dx}{dt} = f \left[x(t),\ x(t - t_1), \ldots, x(t - t_k),\ t \right], \tag{1.3}$$

where $\quad 0 < t_1 < t_2 < \ldots < t_k$.

In this case, only a certain finite interval of the immediate past history of the system is involved in the determination of the present (instead of the whole past as in an integro-differential equation).

The study of differential-difference equations was begun by John Bernoulli[1] in 1728, and theory regarding these equations has been developed in several hundred papers since then. Due to their application to control theory, this study has been greatly intensified in recent years. Many Russian authors, such as Krasovskii[2] and Khalanai[3], have considered the problem of stability. Bellman and Cooke[4] have developed theorems analogous to those for ordinary differential equations. Pinney[5] has also contributed to the general theory, and bibliographies by Choksy[6] and Weiss[7] give reference to 356 separate items all dealing with time delay systems and/or associated mathematics. Abundant references are also given in the previous works cited.

Differential-difference equations arise in the theory of elasticity[8], mathematical economics[9], population studies[10], the study of combustion in liquid-fuel rocket motors[11], mathematical biophysics[12], the theory of automatic control (13), (14), (15), (16) and in a number of other areas involving time delays.

1.1 Scope of the Present Work

Some theorems developed by Krasovskii pertaining to the stability of systems with delay are given in Chapter 2. These theorems are then applied to discuss the stability of two rather general classes of second-order nonlinear systems with delay.

Although much of the theory pertaining to existence, uniqueness, and stability of solutions to differential-difference equations has been developed, very little has been done to obtain solutions due to their complexity(17). Because of this complexity it is desirable to obtain an approximate solution. This approximate solution must approach the actual solution with some reasonable degree of accuracy and must also exhibit any qualitative behaviour characteristic of the system, such as jump resonance. Therefore, approximate solutions have been developed in Chapter 3 by employing the well known Ritz method(22). The general equations for the amplitude and phase of a one-term approximation to the solution are obtained and applied to several particular examples. The results thus obtained are then compared to those obtained by an analog computer simulation discussed in Chapter 4. Both sets of results show that rather unusual jump resonance phenomena are often obtained.

In Chapter 5 some general conclusions and ideas for future research are given.

2. STABILITY

Before investigating the solution of any differential equation, it is useful to determine beforehand which values of the coefficients of the equation (if any) lead to stable solutions. Then, only systems which are inherently stable in the undriven case need be studied to determine their resonance properties.

The stability of a linear differential-difference equation can be determined by examining the roots of its associated characteristic equation[18],[19]. If these roots have negative real parts, the solution of the equation is stable. Because of its transcendental nature, the characteristic equation, in general, possesses an infinite number of roots. The relation between the location of these roots and the coefficients of the differential-difference equation is necessarily much more complicated than for an ordinary differential equation. A graphical technique sometimes provides more useful information than a direct examination of the roots of the characteristic equation. One such technique, employing a dual Nyquist diagram, is given by Jones[20]. The algebraic part and the transcendental part of the characteristic equation can be plotted separately and the stability determined from their intersection points. The effect of the delay on the stability can then be easily determined.

For nonlinear systems with delay, the stability is most easily determined by applying Lyapunov's second (or direct) method. This method is described in most modern books on advanced control theory. Its great merit is that it can be applied to stability discussions for very general equations without finding explicit solutions.

2.1 Some Theorems and Definitions Concerning Lyapunov's Second Method for Equations with Time Delay

The Russian author, Krasovskii, has done a great deal of work concerning the stability of systems with time delay. Some of his theorems and definitions are given in this section; the interested reader is referred to Krasovskii's book[2] for the proofs and further information on the subject.

We are interested in equations of the form,

$$\frac{dx_i}{dt} = X_i\left[x_1(t-h_{i1}),\ldots, x_n(t-h_{in});\ x_1(t),\ldots, x_n(t),\ t\right] \tag{2.1}$$

$$(i = 1,\ldots,n;\ 0 \le h_{ij}(t) \le h),$$

where the right hand members, $X_i(y_1,\ldots, y_n;\ x_1,\ldots, x_n, t)$, are continuous functions of their arguments and are defined for

$$|x_i| < H, \qquad |y_i| < H \qquad (i = 1,\ldots, n), \tag{2.2}$$

where H is a constant (or infinity). We suppose further that these functions satisfy a Lipschitz condition with respect to the x_j and y_j in the region (2.2), i.e.,

$$\left|X_i(y_1'',\ldots, y_n'';\ x_1'',\ldots, x_n'', t) - X_i(y_1',\ldots, y_n';\ x_1',\ldots, x_n', t)\right|$$

$$\le L\left(\sum_{j=1}^{n} |x_j''-x_j'| + \sum_{j=1}^{n} |y_j''-y_j'|\right).$$

It will also be assumed that

$$X_i \left[x_j(\phi),t\right] = 0$$

on the entire arc,

$$x_j(\phi) = 0; \quad i = 1,\dots, n; \quad j = 1,\dots, n;$$

$$-h \le \phi \le 0.$$

The following notation is used:

$$\|x\| = \sup(|x_1|,\dots, |x_n|),$$

$$\|x\|_2 = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}},$$

$$\|x\|_2^h = \sup(|x_i(\phi)|) \qquad \text{for } -h \le \phi \le 0,$$

$$\|x\|_2^h = \left\{ \int_{-h}^{0} \left[\sum_{i=1}^{n} x_i^2(\phi)\right] d\phi \right\}^{\frac{1}{2}},$$

$$x_o(\phi_o) = \left[x_{io}(\phi_o)\right] \quad (i = 1,\dots, n \ ; \ -h \le \phi_o \le 0).$$

To emphasize the dependence of $x(t)$ on the initial curve and the initial value of t, we shall denote $\left[x_i(t)\right]$ by $x\left[x(\phi_o), t_o, t\right]$.

<u>Definition 1.</u> (a) The solution, $x = 0$, of equation (2.1) is called stable if for every positive number, $\varepsilon > 0$, we can find a positive number, $\delta > 0$, such that whenever the inequality,

$$\|x_o(\phi_o)\|^h \le \delta,$$

is satisfied, the relation,

$$\left\|x\left[x_o(\phi_o), t_o, t\right]\right\| < \varepsilon,$$

holds for $t \ge t_o$.

(b) If, whenever condition (a) is satisfied, the conditions,

$$\lim_{t \to \infty} \left\| x \left[x_o(\emptyset_o),\ t_o,\ t \right] \right\| = 0 \quad ,$$

$$\left\| x \left[x_o(\emptyset_o),\ t_o,\ t \right] \right\| < H_1 \quad \text{for all } t \geq t_o (H_1 = \text{const.}),$$

are satisfied for all initial curves, $x_o(\emptyset_o)$, satisfying the inequality ,

$$\left\| x_o(\emptyset_o) \right\|^h \leq H_o, \tag{2.3}$$

then the solution, $x = 0$, of equation (2.1) is asymptotically stable and the region (2.3) lies in the region of attraction of the unperturbed motion.

<u>Theorem 1.</u> Suppose there is a functional , $V\left[x(\emptyset),\ t\right]$, that satisfies the conditions,

$$\left| V \left[x(\emptyset),\ t \right] \right| \leq W_1(\ \left\| x(0) \right\| \ ) + W_2(\left\| x(\emptyset) \right\| \tfrac{h}{2}) \ ,$$

$$V \left[x(\emptyset),\ t \right] \geq \omega \left[\left\| x(0) \right\| \right],$$

$$\lim_{t \to 0+} \sup(\tfrac{\Delta V}{\Delta t}) = -\varphi\left[\left\| x(0) \right\| \right], \tag{2.4}$$

where $W_1(r)$ and $W_2(r)$ are functions that are continous and monotonic for $r > 0$, and $W_1(0) = W_2(0) = 0$; $\omega(r)$ is a function that is continous and positive for $r \neq 0$. Then the null solution, $x = 0$, of equation (2.1) is asymptotically stable.

If X_i and $h_{ij}(t)$ are periodic functions of the time, t, all with period, Θ, (or if X_i and h_{ij} are independent of the time, t), then condition (2.4) may be replaced by the following weaker hypothesis:

A sufficient condition that $\lim_{t \to 0^+} \sup(\frac{\Delta V}{\Delta t})$ should be nonpositive along a trajectory is that the equation ,

$$\lim_{t \to 0^+} \sup\left(\frac{\Delta V}{\Delta t}\right) = 0 ,$$

be valid for all $t \geq t_0$ only along the trajectory, $x = 0$.

<u>Definition 2.</u> The solution, $x = 0$, of equation (2.1) is called uniformly asymptotically stable with respect to the time, $t_0 \geq 0$, and with respect to the initial curve, $x_0(\emptyset_0)$, in region (2.3), if it satisfies condition (b) of Definition 1 and also satisfies the following conditions:

(i) the number, $\delta > 0$, of Definition 1 (a) may be chosen independent of $t_0 \geq 0$;

(ii) for arbitrary $\eta > 0$, there exists a number, $T(\eta)$, such that

$$\left\| x\left[x_0(\emptyset_0),\ t_0,\ t + \emptyset\right]\right\|^h < \eta$$

holds for every $t \geq t_0 + T(\eta)$, independent of the choice of a piecewise-continuous initial curve, $x_0(\emptyset_0)$, in region (2.3).

If the right hand member of equation (2.1) is a periodic function of time with period, θ, (or is independent of the time, t), then the null solution, $x = 0$, is always uniformly asymptotically stable in the sense of Definition 2.

Now consider the "perturbed" system of equations,

$$\frac{dx_i}{dt} = X_i\left\{x_1(t),\ldots,x_n(t);\ x_1\left[t-h_{i1}^*(t)\right],\ldots,x_n\left[t-h_{in}^*(t)\right],\ t\right\}$$
$$+ R_i\left\{x_1(t),\ldots,x_n(t);\ x_1\left[t-g_{i1}(t)\right],\ldots,x_n\ t-g_{in}(t)\ ,\ t\right\}$$

$$(i,\ j = 1,\ldots,n\ ;\ 0 \leq h_{ij}^*(t) \leq h\ ;\ 0 \leq g_{ij}(t) \leq h),$$

where the continuous functions, R_i, are not required to reduce to zero for $x_j = y_j = 0$.

Definition 3. The null solution, $x = 0$, of the system (2.5) is called stable for persistent disturbances if for every $\varepsilon > 0$ there exist positive numbers, δ_0, η, Δ, such that the solution, $x\left[x_0(\phi_0),\ t_0,\ t\right]$, of the system (2.5) satisfies the inequality,

$$\left\| x\left[x_0(\phi_0),\ t_0,\ t\right]\right\| < \varepsilon ,$$

for all $t \geq t_0$, $t_0 \geq 0$, whenever the initial curve, $x_0(\phi_0)$, satisfies the inequality ,

$$\left\| x_0(\phi_0)\right\|^h \leq \delta_0 \quad ,$$

and the perturbed time delays, h^*_{ij}, and the functions, R_i, satisfy the inequalities,

$$\left|R_i(x_1, \ldots, x_n\ ;\ y_1, \ldots, y_n,\ t)\right| < \eta\ , (i = 1, \ldots, n),$$

for $\|x\| < \varepsilon$, $\|y\| < \varepsilon$, and

$$\left|h_{ij}(t) - h^*_{ij}(t)\right| < \Delta \quad (i,\ j = 1, \ldots, n).$$

Theorem 2. Suppose that the null solution, $x = 0$, of equation (2.1) is asymptotically stable uniformly with respect to time, t_0, and the initial curve, $x_0(\phi_0)$, in the sense of Definition 3., then the null solution, $x = 0$, is stable for persistent disturbances.

2.2 Stability of Systems with Time Delay in Particular Cases

2.2.1. Systems with Delayed Damping

Consider the second-order nonlinear equation,

$$\frac{d^2x}{dt^2} = X\left[x(t), y(t)\right] + \varphi\left[y(t-h), t\right] , \qquad (2.6)$$

($y = \frac{dx}{dt}$; h is a positive constant),
where the functions X and φ satisfy the requirements,

$$\frac{X(x,y) - X(x,0)}{y} < -a , \quad \frac{X(x,0)}{x} < -b \quad \text{for } x \neq 0, y \neq 0, \qquad (2.7)$$

where a and b are positive constants, and

$$\left|\varphi(y,t)\right| \leq L|y| . \qquad (2.8)$$

We write equation (2.6) in the equivalent form,

$$\frac{dx}{dt} = y, \qquad (2.9)$$

$$\frac{dy}{dt} = X(x,0) + \left[X(x,y) - X(x,0)\right] + \varphi\left[y(t-h), t\right] .$$

Krasovskii[2] has shown the following:
Define the functional V by

$$V\left[x(\vartheta), y(\vartheta)\right] = -\int_0^x X(\xi, 0)d\xi + \frac{y^2}{2} + \frac{a}{2}\int_{-h}^0 y^2(\xi)\, d\xi . \qquad (2.10)$$

To estimate the derivative, $\frac{dV}{dt}$, along a trajectory of the system (2.9), write

$$\frac{dV}{dt} = \left[X(x,y) - X(x,0)\right] y + y\varphi\left[y(t-h),\ t\right] + \frac{ay^2}{2} - \frac{ay^2}{2}(t-h).$$

Conditions (2.7) and (2.8) give the estimates ,

$$\frac{dV}{dt} \leq \frac{-ay^2(t)}{2} + L\left|y(t)y(t-h)\right| - \frac{a}{2}\, y^2(t-h). \tag{2.11}$$

The functional, V, satisfies the hypotheses of Theorem 1 if the right hand member of inequality (2.11) is a negative-definite function of the arguments, $y(t)$ and $y(t-h)$. This condition is satisfied if the numbers, a and L, are related by

$$a > L. \tag{2.12}$$

Thus, inequality (2.12) is a condition sufficient for the asymptotic stability of the null solution, $x = y = 0$, of the system (2.9).

If the right hand member of equation (2.6) is a periodic function of time with period, θ, (or is independent of time), then the null solution is always uniformly asymptotically stable in the sense of Definition 2 and hence the system will be stable for persistent disturbances.

The stability criterion expressed by equation (2.12) can also be obtained heuristically. Due to the delay, h, there will be some frequencies of oscillation, ω, such that ωh is an odd multiple of π. At these frequencies, the delayed damping term, $y(t-h)$, will be out of phase with the damping term, $y(t)$. Since a and L are, effectively, the coefficients of the damping and delayed damping terms, then a must be greater than L in order that the system should have no negative damping at any frequency.

2.2.2 Systems with Delayed Restoring Force

Consider the linear equation with delayed restoring force ,

$$\frac{d^2x(t)}{dt^2} + a\,\frac{dx(t)}{dt} + bx(t) - cx(t-h) = 0 \ ,$$

(a, b, and c are constants; h is a positive constant). This can be written in the equivalent form ,

$$\frac{dx}{dt} = y \ , \tag{2.14}$$

$$\frac{dy}{dt} = -bx - ay + cx(t-h).$$

We wish to find α_{11}, α_{12} and α_{22} such that the quadratic form,

$$v(x,y) = \alpha_{11}x^2 + 2\alpha_{12}xy + \alpha_{22}y^2 \ , \tag{2.15}$$

is positive-definite and satisfies the condition ,

$$\frac{\partial v}{\partial x}\,y + \frac{\partial v}{\partial y}\,(-bx - ay) = -x^2 - y^2 \ . \tag{2.16}$$

Substituting equation (2.15) in equation (2.16), we obtain

$$\alpha_{11} = \frac{a^2 + b(b+1)}{2ab} \ , \tag{2.17}$$

$$\alpha_{12} = \frac{1}{2b} \tag{2.18}$$

and

$$\alpha_{22} = \frac{b+1}{2ab} \ . \tag{2.19}$$

$v(x,y)$ is a positive-definite function of the arguments, $x(t)$ and $y(t)$, if $\alpha_{11} > 0$ and $\alpha_{11}\alpha_{22} > \alpha_{12}^2$ (see Appendix A). A sufficient condition for this is that a and b be greater than zero.

The functional, V, can be taken in the form,

$$V\left[x(\vartheta),\ y(\vartheta)\right] = \alpha_{11}x^2 + 2\alpha_{12}xy + \alpha_{22}y^2 + \mu \int_{-h}^{0} x^2(\xi)d\xi, \quad (2.20)$$

$$(\mu > 0) .$$

The value of the derivative, $\frac{dV}{dt}$, along a trajectory of the system (2.14) is

$$\lim_{\Delta t \to 0^+} \left(\frac{\Delta V}{\Delta t}\right) = -\left[(1-\mu)x^2(t) + y^2(t) + \mu x^2(t-h) - 2\alpha_{12}cx(t)x(t-h) - 2\alpha_{22}cy(t)x(t-h)\right] . \quad (2.21)$$

The right hand member of equation (2.21) is a negative-definite function of the arguments, $x(t)$, $x(t-h)$ and $y(t)$, if

$$(1 - \mu) > 0 \quad (2.22)$$

and

$$(1 - \mu)\ (\mu - \alpha_{22}^2c^2) - \alpha_{12}^2\ c^2 > 0. \quad (2.23)$$

Maximizing the left hand member of inequality (2.23) with respect to μ, we find

$$\mu = \frac{1 + \alpha_{22}^2c^2}{2} . \quad (2.24)$$

Substituting this value of μ in inequality (2.23), we obtain

$$\left[\frac{1 - \alpha_{22}^2c^2}{2}\right]^2 - \alpha_{12}^2c^2 > 0 . \quad (2.25)$$

Since we have chosen $b > 0$, then $\alpha_{12} > 0$ (see equation (2.18), and hence inequality (2.25) may be written

$$\alpha_{22}^2c^2 < 1 - 2\alpha_{12}|c| . \quad (2.26)$$

From inequality (2.22) and equation (2.24), we find that $\alpha_{22}^2c^2$ must be less than unity. Thus if inequality (2.26) is satisfied (and hence (2.23)), then inequality (2.22) is satisfied as well, and the right hand member of equation (2.21) is negative-definite. The functional, V, thus satisfies the hypotheses of Theorem 1 and hence inequality (2.26) is a condition sufficient for the asymptotic stability of the null solution, $x = y = 0$, of the system (2.14).

Having determined the stability criterion for the linear equation (2.13), let us now consider the nonlinear equation,

$$\frac{d^2x}{dt^2} = X\left[x(t),\ y(t)\right] + \varphi\left[x(t-h),\ t\right] \tag{2.27}$$

$$\left(y = \frac{dx}{dt}\ ;\ h \text{ is a positive constant}\right),$$

where the functions X and φ satisfy the requirements,

$$\frac{X(x,y) - X(x,0)}{y} = -a, \quad \frac{X(x,0)}{x} < -b \quad \text{for } x \neq 0,\ y \neq 0, \tag{2.28}$$

where a and b are positive constants, and

$$\left|\varphi\left[x(t-h),\ t\right]\right| \leq L\ \left|x(t-h)\right| \quad . \tag{2.29}$$

The linear equation (2.13) will now be a special case of the general equation (2.27). Equation (2.27) can be written in the equivalent form,

$$\frac{dx}{dt} = y \quad , \tag{2.30}$$

$$\frac{dy}{dt} = X(x,0) + \left[X(x,y) - X(x,0)\right] + \varphi\left[x(t-h),\ t\right] \quad .$$

Define the functional, V, by

$$V\left[x(\vartheta), y(\vartheta)\right] = -2\,\alpha_{22} \int_0^x X(\xi,0)d\xi + (\alpha_{11} - \alpha_{22}b)x^2$$

$$+ 2\alpha_{12}xy + \alpha_{22}y^2 + \mu \int_{-h}^{0} x^2(\xi)d\xi \quad , \tag{2.31}$$

$(\mu > 0)$.

To estimate the derivative, $\frac{dV}{dt}$, along a trajectory of the system (2.30), we write

$$\frac{dV}{dt} = \left[-2\alpha_{22}X(x,0) + 2(\alpha_{11} - \alpha_{22}b)x + 2\alpha_{12}y\right] y$$
$$+ 2(\alpha_{12}x + \alpha_{22}y)\ \left\{X(x,0) + \left[X(x,y) - X(x,0)\right] + \varphi\left[x(t-h),\ t\right]\right\}$$
$$+ \mu x^2 - \mu x^2(t-h) \quad . \tag{2.32}$$

Conditions (2.28) and (2.29) give the estimates,

$$\frac{dV}{dt} \leq -2\left[\alpha_{12}bx^2 + (\alpha_{12}a + \alpha_{22}b - \alpha_{11})xy + (\alpha_{22}a - \alpha_{12})y^2\right]$$
$$+ 2\alpha_{12}Lx\ \left|x(t-h)\right| + 2\alpha_{22}Ly\ \left|x(t-h)\right|$$
$$+ \mu x^2 - \mu x^2(t-h) \quad . \tag{2.33}$$

If α_{11}, α_{12}, α_{22} and μ satisfy equations (2.17), (2.18), (2.19) and (2.24) respectively, then the right hand member of inequality (2.33) is equivalent to the right hand member of equation (2.21) with $cx(t-h)$ replaced by $L\left|x\ (t-h)\right|$. Thus, the right hand member of inequality (2.33) is a negative-definite function of the arguments, $x(t)$, $x(t-h)$ and $y(t)$, if

$$\alpha_{22}^2L^2 < 1 - 2\alpha_{12L} \quad . \tag{2.34}$$

The functional, V, satisfies the hypotheses of Theorem 1 and hence inequality (2.34) is a condition sufficient for the asymptotic stability of the null solution, $x = y = 0$, of the system (2.30).

By use of the simple transformation, $t_1 = b^{\frac{1}{2}}t$, in equation (2.27), we may replace b by unity, a by

$$\frac{a}{b} = 2D_1$$

and L by

$$\frac{L}{b} = k \quad , \quad (k > 0) \quad .$$

The stability criterion (inequality (2.34)) then becomes

$$(2D_1)^2 > \frac{k^2}{1 - k} \quad . \tag{2.35}$$

The stable region defined by inequality (2.35) is indicated in Fig. 2.1.

Again, if the right hand member of equation (2.27) is a periodic function of time with period, Θ, (or is independent of time), then the null solution is always uniformly asymptotically stable in the sense of Definition 2, and hence the system will be stable for persistent disturbances.

Having determined the stability criteria for the two equations (2.6) and (2.27), we may now proceed with the development of the approximate solutions.

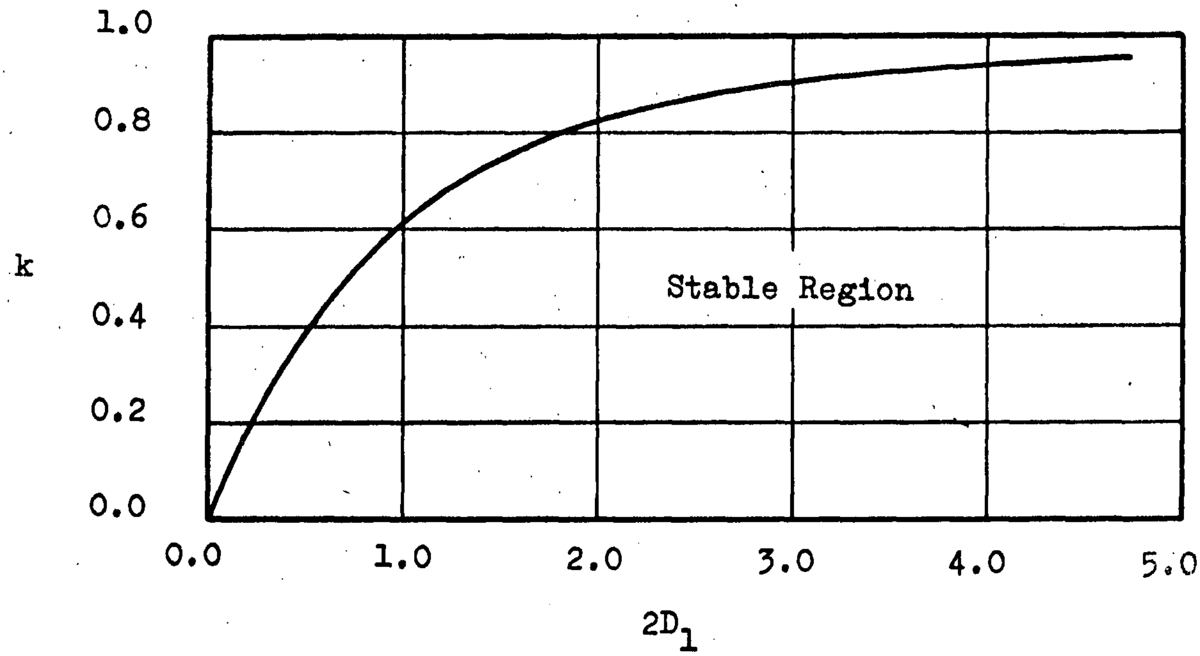

Figure 2.1 Region of Stability

3. APPROXIMATE ANALYTICAL SOLUTIONS

In the study of linear systems it is convenient to deal with sinusoidal inputs and the resulting sinusoidal outputs. The ratio between the complex amplitudes of output and input is known as the "transfer function". Although the information represented by these transfer functions seems to be very specific, the property of superposition, inherent in linear systems, makes these functions the basis for a complete description of system behavior.

In nonlinear systems the property of superpostion does not hold. The outputs, in general, are no longer sinusoidal and the response to a sinusoidal input does not permit the response to an input of any other type to be foretold exactly. Nevertheless, the sinusoidal input functions are a convenient method for investigating certain representative features of system behaviour, such as the phenomena of jump resonance[22], [23]. The problem then, is to determine, with some reasonable degree of accuracy, the amplitude and phase of the system output corresponding to any amplitude and frequency of the sinusoidal input. The curves obtained for varying input frequency or amplitude will be called response curves. This problem has been solved by Klotter[24] for quasi-linear systems with no delay by employing the Ritz method. This method can also be applied to systems with delay as is done in section 3.2. It is interesting to note that if the delay is reduced to zero, the equations developed reduce to those developed by Klotter(as indeed they should).

3.1 The Ritz Method

Although the Ritz method is described in the literature[22], a brief discussion of it is given here.

The Ritz method postulates the existence of a function, $F(x,\dot{x},t)$, such that the Euler-Lagrange equation, obtained from the minimization of

$$I = \int_{t_a}^{t_b} F(x,\dot{x},t)\, dt \quad , \tag{3.1}$$

is the nonlinear equation we wish to solve, i.e.,

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left[\frac{\partial F}{\partial \dot{x}}\right] = E(x) = 0 \quad . \tag{3.2}$$

Consider then the minimization of the right hand member of equation (3.1) given

$$x(t) = \sum_{k=0}^{\infty} a_k \psi_k(t) \quad ,$$

where $x(t)$ is an exact solution to $E(x) = 0$ if the $\psi_k(t)$ form a complete, linearly independent set. We seek an approximate solution ,

$$\tilde{x}(t) = \sum_{k=0}^{n} a_k \psi_k(t) \, , \tag{3.3}$$

where n is arbitrary. The larger the n, the more accurate is the solution and the more work involved. Substituting equation (3.3) in equation (3.1), we obtain

$$I = \int_{t_a}^{t_b} F(a_0 \psi_0 + a_1 \psi_1 + \dots + a_n \psi_n, a_0 \dot{\psi}_0 + a_1 \dot{\psi}_1 + \dots + a_n \dot{\psi}_n, t)dt \; . \tag{3.4}$$

Since the set, ψ_k, is chosen beforehand (i.e., trigonometric functions if E(x) yields an oscillatory solution), equation (3.4) must be minimized with respect to a_k. Setting

$$\frac{\partial I}{\partial a_k} = 0$$

yields

$$\frac{\partial I}{\partial a_k} = \int_{t_a}^{t_b} \frac{\partial F}{\partial a_k}\, dt = \int_{t_a}^{t_b} \left[\frac{\partial F}{\partial \tilde{x}}\psi_k + \frac{\partial F}{\partial \dot{\tilde{x}}}\dot{\psi}_k\right] dt = 0 \quad ,$$

which finally becomes

$$\frac{\partial I}{\partial a_k} = \int_{t_a}^{t_b} \left[\frac{\partial F}{\partial \tilde{x}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\tilde{x}}}\right)\right] \psi_k\, dt + \frac{\partial F}{\partial \dot{\tilde{x}}}\psi_k \Big|_{t_a}^{t_b} = 0 \quad . \quad (3.5)$$

If we now specify that $\psi_k(t_a) = \psi_k(t_b) = 0$, or that ψ_k is periodic with period $(t_b - t_a)$, equation (3.5) becomes

$$\frac{\partial I}{\partial a_k} = \int_{t_a}^{t_b} \left[\frac{\partial F}{\partial \tilde{x}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\tilde{x}}}\right)\right] \psi_k\, dt = 0 \quad . \qquad (3.6)$$

Since $F(x,\dot{x},t)$ was so postulated that

$$\frac{\partial F}{\partial \tilde{x}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\tilde{x}}}\right) = E(\tilde{x}) \quad ,$$

equation (3.6) finally becomes

$$\frac{\partial I}{\partial a_k} = \int_{t_a}^{t_b} \tilde{x}\, E(\tilde{x})\, dt = 0 \quad . \qquad (3.7)$$

Equation (3.7) is known as the Ritz averaging integral and may be taken to mean that we are trying to satisfy the differential equation (3.2) in some "weighted" average.

Due to the orthogonality of the trigonometric functions with unity weighting, for oscillatory systems the Ritz method is equivalent to the Principle of Harmonic

Balance[22]. Cunningham[14] uses this method in his attack on nonlinear differential-difference equations.

3.2 Application of the Ritz Method to Steady-State Oscillations in Nonlinear Systems with Delay

The Ritz method will now be applied to the differential-difference equation,

$$E(x) = \ddot{x}(t) + 2D_1\omega_n g_1\left[\dot{x}(t)\right] + 2D_2\omega_n\, g_2\left[\dot{x}(t-h)\right] + \omega_n^2 f_1\left[x(t)\right]$$

$$+ k\omega_n^2 f_2\left[x(t-h)\right] - G\sin\omega t = 0 \quad . \qquad (3.8)$$

Equation (3.8) is the equation of motion of a fairly general nonlinear system with delay, subjected to a harmonic driving force. The functions, f_1 and f_2, describing the restoring and delayed restoring forces, and the functions, g_1 and g_2, describing the damping and delayed damping forces, are all assumed to be single-valued and integrable functions of their respective arguments.

If f_1, f_2, g_1 and g_2 are odd functions of their respective arguments, that is

$$\left.\begin{array}{ll} f_1\left[-x(t)\right] = -f_1\left[x(t)\right] \quad , & f_2\left[-x(t-h)\right] = -f_2\left[x(t-h)\right], \\ g_1\left[-x(t)\right] = -g_1\left[x(t)\right] \quad , & g_2\left[-x(t-h)\right] = -g_2\left[x(t-h)\right], \end{array}\right\} (3.9)$$

the resulting motion has the mean value zero.

If only terms with frequency, ω, are considered, an appropriate assumption for the approximate solution is

$$\tilde{x}(t) = X\sin(\omega t - \Theta) \quad . \qquad (3.10)$$

The Ritz conditions are

$$\int_0^{\frac{2\pi}{\omega}} E\left[\tilde{x}(t)\right] \sin \omega t \, dt = 0$$

and
$$\int_0^{\frac{2\pi}{\omega}} E\left[\tilde{x}(t)\right] \cos \omega t \, dt = 0 \quad .$$

When equations (3.9) are applied, the Ritz conditions become

$$\left[F_1 - \eta^2 + 2D_2 \eta G_2 \sin(\eta \omega_n h) + kF_2 \cos(\eta \omega_n h)\right]^2 + \left[2D_1 \eta G_1 + 2D_2 \eta G_2 \cos(\eta \omega_n h) - kF_2 \sin(\eta \omega_n h)\right]^2 = \left[\frac{S}{X}\right]^2 \quad (3.11)$$

and

$$\tan \theta = \frac{2D_1 \eta G_1 + 2D_2 \eta G_2 \cos(\eta \omega_n h) - kF_2 \sin(\eta \omega_n h)}{F_1 - \eta^2 + 2D_2 \eta G_2 \sin(\eta \omega_n h) + kF_2 \cos(\eta \omega_n h)} \quad , (3.12)$$

where

$$F_1 = \frac{4}{\pi X} \int_0^{\frac{\pi}{2}} f_1(X\sin\alpha)\sin\alpha \, d\alpha = \frac{4}{\pi X} \int_0^{\frac{\pi}{2}} f_1(X\cos\alpha)\cos\alpha \, d\alpha, (3.13)$$

$$F_2 = \frac{4}{\pi X} \int_0^{\frac{\pi}{2}} f_2(X\sin\alpha)\sin\alpha \, d\alpha = \frac{4}{\pi X} \int_0^{\frac{\pi}{2}} f_2(X\cos\alpha)\cos\alpha \, d\alpha, (3.14)$$

$$G_1 = \frac{4}{\pi \omega X} \int_0^{\frac{\pi}{2}} g_1(\omega X\sin\alpha)\sin\alpha \, d\alpha = \frac{4}{\pi \omega X} \int_0^{\frac{\pi}{2}} g_1(\omega X\cos\alpha)\cos\alpha \, d\alpha, (3.15)$$

$$G_2 = \frac{4}{\pi \omega X} \int_0^{\frac{\pi}{2}} g_2(\omega X\sin\alpha)\sin\alpha \, d\alpha = \frac{4}{\pi \omega X} \int_0^{\frac{\pi}{2}} g_2(\omega X\cos\alpha)\cos\alpha \, d\alpha, (3.16)$$

$$\eta = \frac{\omega}{\omega_n} \quad , \quad S = \frac{G}{\omega_n^2} \quad . \quad (3.17)$$

If the driving term in equation (3.8) has the form, $G \cos\omega t$, instead of $G \sin\omega t$, the assumed solution would be

$$\tilde{x}(t) = X \cos(\omega t - \Theta)$$

instead of equation (3.10). The resulting equations (3.11) to (3.17), however, are unchanged by these replacements.

If f_1, f_2, g_1 and g_2 are non-odd functions, that is they do not satisfy equations (3.9), the resulting motion does not have zero mean value. Therefore, a mean value, M, must be included in the assumption for x. Equation (3.10) is replaced by

$$\tilde{x} = M + X \sin(\omega t - \Theta) = M + A \sin\omega t - B \cos\omega t \;,$$

where $A = X \cos\Theta$ and $B = X \sin\Theta$.

Consequently, there will be three Ritz conditions for determining the three constants, M, A and B or M, X and Θ. These conditions are

$$\int_0^{\frac{2\pi}{\omega}} E\left[\tilde{x}(t)\right] \sin\omega t \, dt = 0 \;,$$

$$\int_0^{\frac{2\pi}{\omega}} E\left[\tilde{x}(t)\right] \cos\omega t \, dt = 0 \;,$$

$$\int_0^{\frac{2\pi}{\omega}} E\left[\tilde{x}(t)\right] dt = 0 \;.$$

If we let

$$F_{01} = \int_0^{\frac{2\pi}{\omega}} f_1\left[\tilde{x}(t)\right] dt, \qquad G_{01} = \frac{1}{\omega_n} \int_0^{\frac{2\pi}{\omega}} g_1\left[\dot{\tilde{x}}(t)\right] dt,$$

$$F_{02} = \int_0^{\frac{2\pi}{\omega}} f_2\left[\tilde{x}(t)\right] dt, \qquad G_{02} = \frac{1}{\omega_n} \int_0^{\frac{2\pi}{\omega}} g_2\left[\dot{\tilde{x}}(t)\right] dt,$$

$$F_{S1} = \frac{1}{\pi} \int_0^{\frac{2\pi}{\omega}} f_1\left[\tilde{x}(t)\right] \sin \omega t \, dt, \qquad G_{S1} = \frac{1}{\omega_n \pi} \int_0^{\frac{2\pi}{\omega}} g_1\left[\dot{\tilde{x}}(t)\right] \sin \omega t \, dt,$$

$$F_{S2} = \frac{1}{\pi} \int_0^{\frac{2\pi}{\omega}} f_2\left[\tilde{x}(t)\right] \sin \omega t \, dt, \qquad G_{S2} = \frac{1}{\omega_n \pi} \int_0^{\frac{2\pi}{\omega}} g_2\left[\dot{\tilde{x}}(t)\right] \sin \omega t \, dt,$$

$$F_{C1} = \frac{1}{\pi} \int_0^{\frac{2\pi}{\omega}} f_1\left[\tilde{x}(t)\right] \cos \omega t \, dt, \qquad G_{C1} = \frac{1}{\omega_n \pi} \int_0^{\frac{2\pi}{\omega}} g_1\left[\dot{\tilde{x}}(t)\right] \cos \omega t \, dt,$$

$$F_{C2} = \frac{1}{\pi} \int_0^{\frac{2\pi}{\omega}} f_2\left[\tilde{x}(t)\right] \cos \omega t \, dt, \qquad G_{C2} = \frac{1}{\omega_n \pi} \int_0^{\frac{2\pi}{\omega}} g_2\left[\dot{\tilde{x}}(t)\right] \cos \omega t \, dt,$$

the Ritz conditions become

$$F_{01} + kF_{02} + 2D_1G_{01} + 2D_2G_{02} = 0 \quad , \tag{3.18}$$

$$F_{S1} + kF_{S2} + 2D_1G_{S1} + 2D_2G_{S2} - \eta^2 A - S = 0 \tag{3.19}$$

and

$$F_{C1} + kF_{C2} + 2D\ G_{C1} + \eta^2 B = 0 \quad . \tag{3.20}$$

In general, equations (3.18), (3.19) and (3.20) represent a system of nonlinear algebraic equations for the three unknowns, M, A and B. The application is thus tedious, and hence only systems with odd nonlinearities are considered hereafter.

3.3 Illustrative Examples and Comparison of the Ritz-Method Results to the Analog-Simulation Results

There are numerous particular examples leading to equations with delay, e.g., differential-difference equations, which arise from many particular fields of interest, as mentioned in the Introduction.

A simple example is the equation describing the thickness of a sheet of metal coming from a rolling mill[14]:

$$\dot{x}(t) = -k\left[x(t-h) - x_0\right] \quad ,$$

where x is the thickness at any time, t, x_0 is the desired thickness, k is a constant determined by the control system and h is the delay due to the separation of the rolls and the measurement point.

Studies in the field of population growth[10] lead to the equation,

$$\dot{x}(t) = rx(t)\left[1 - \frac{x(t-h)}{x_s}\right] \quad ,$$

where x is the population at any time, t, r is the reproduction rate, x_s is the steady-state population ultimately reached (or the average value thereof), and h is the delay due to the fact that the population does not react immediately to its increasing number.

An example from the field of economics is Goodwin's nonlinear model of the business cycle[9],

$$\gamma y(t+\theta) + (1 - \alpha)\, y(t+\theta) = \phi\left[y(t)\right] \quad ,$$

where y is the income at time, t, θ is the delay between investment decisions and corresponding outlays, γ is the time constant of the income-consumption relationship, α is the change in consumption per unit change in income, and $\emptyset(y)$ is the nonlinear induced investment.

The analysis of a microphone, amplifier and speaker combination with acoustic feedback[26] leads to the equation,

$$\ddot{I}(t) + \frac{R}{L}\,\dot{I}(t) + \frac{1}{LC}\,I(t) + \frac{Ak}{C}\,\dot{I}\left[t-\frac{2\delta}{c}\right] = \frac{Bk^3L^2}{C}\,\dot{I}^3\left[t-\frac{2\delta}{c}\right] \quad ,$$

where I is the plate current at time, t, A and B are constants related to the tube characteristics, c is the velocity of sound, δ is the distance to the reflecting object, k is an amplification constant, and R, L and C relate to the circuit elements.

A system with distributed parameters may sometimes be treated by approximating its transfer function by one of the form,

$$G = \frac{e^{-hs}}{Ts + 1} \quad ,$$

i.e., the transfer function of a delay (e^{-hs}) and a time-constant element.

The analysis of control systems[13] sometimes leads to equations of the form,

$$\ddot{x}(t) + a_1' \,\dot{x}(t) + a_1\,\dot{x}(t-h) + a_0\,x(t) = 0 \qquad (3.21)$$

and

$$\ddot{x}(t) + a_1\,\dot{x}(t) + a_0'\,x(t) + a_0\,x(t-h) = 0 \quad . \qquad (3.22)$$

Equation (3.21) contains a natural damping term, $a_1'\dot{x}(t)$, as well as a delayed damping term, $a_1\dot{x}(t-h)$. Equations of this type arise when an artificially produced damping is added to increase an insufficient natural damping, as in the stabilization of a rolling ship[13],[14]. Equation (3.22) contains a natural restoring force, $a_0'x(t)$, and a delayed restoring force, $a_0x(t-h)$. Equations of this type may arise, for example, in the guidance of an aircraft. The delay, h, could be due to the computation time of a computer in an autopilot or to a human operator who controls the rudder position and, therefore, the restoring force.

The simple equations (3.21) and (3.22) (at least simple in appearance) may be further complicated by nonlinear terms and the presence of a driving term or input, $f(t)$. The nonlinear terms may be due to hysteresis; backlash in gears; mechanical stops; clamping circuits; friction; saturating effects in amplifiers, inductors and capacitors; and a multitude of other sources. Equations of this type with $f(t)$ taken to be $G \cos \omega t$ or $G \sin \omega t$ are considered in this section using the approximate technique described in the previous section. The results are compared with the results obtained by an analog simulation. Before the approximate technique is applied to specific examples, however, a brief description of the phenomenon known as "jump resonance"(22),(23),(28) will be given.

The jump resonance phenomenon is peculiar to systems having a nonlinear restoring force, $f(x)$, and a sinusoidal input. The nonlinearity, $f(x)$, is assumed to be an odd function in the following discussion. If the input amplitude is held constant and the input frequency, ω, is increased, the response curves, ABCDE, are obtained (see Figs. 3.1 and 3.2); if the input frequency is decreased, the response curves, EDFBA, are obtained. If the damping is decreased, the resonant effect is more pronounced and the separation of the jump points, ω_1 and ω_2, is

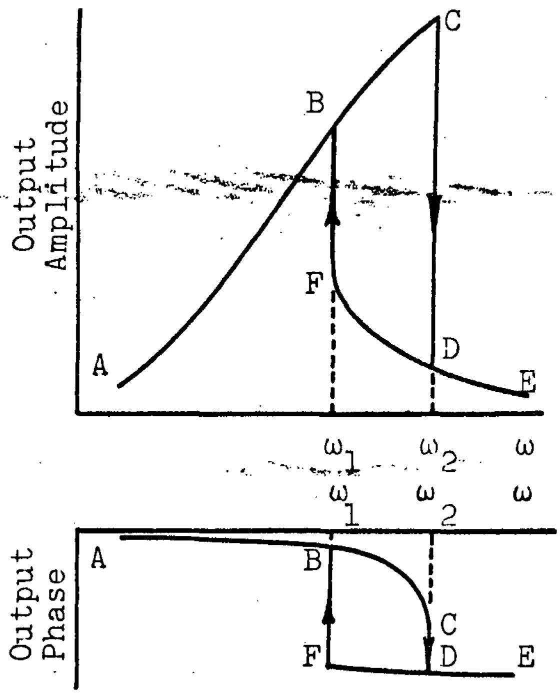

Figure 3.1 Frequency Response Curve for System with $|f(x)| \geq L$

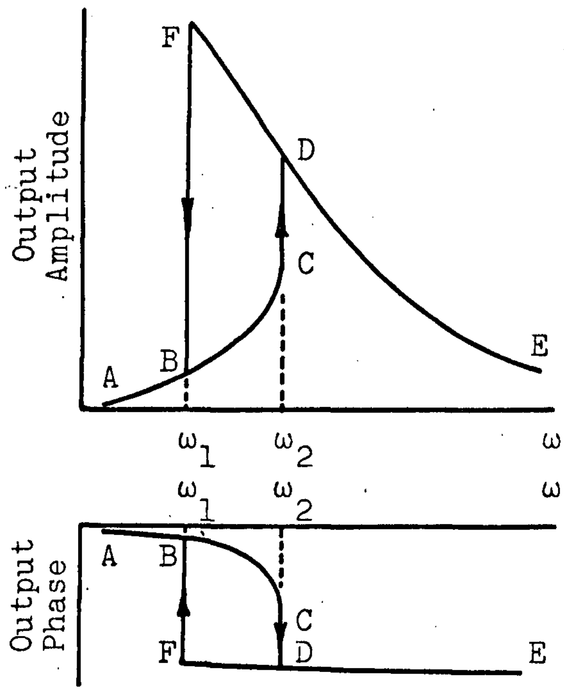

Figure 3.2 Frequency Response Curve for System with $|f(x)| \leq L$

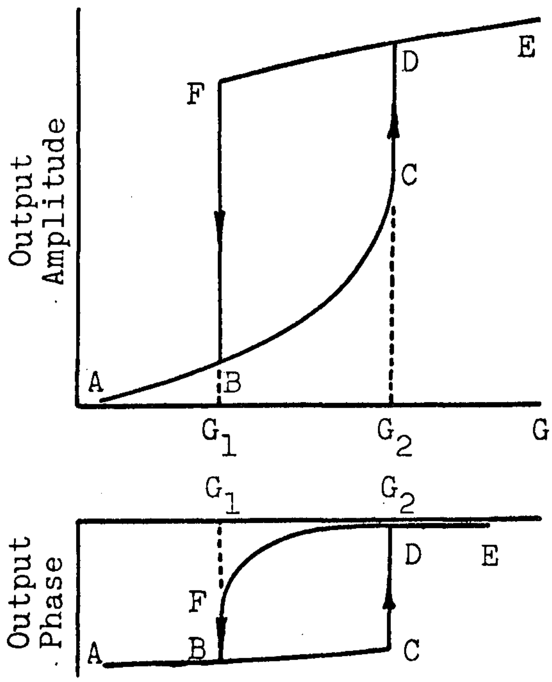

Figure 3.3 Amplitude Response Curve for System with $|f(x)| \geq L$

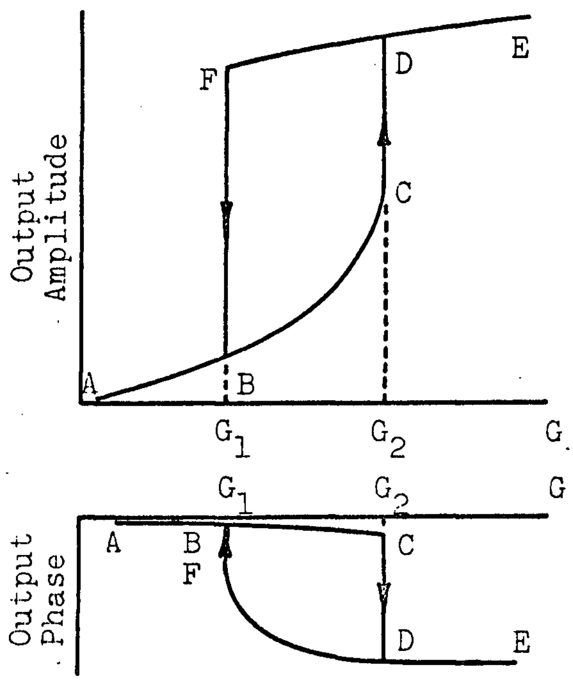

Figure 3.4 Amplitude Response Curve for System with $|f(x)| \leq L$

increased. If the damping is increased, the resonant effect is less pronounced and the separation of the jump points is decreased until some critical damping is reached beyond which no jumps are obtained[29]. If the input frequency is held constant and the input amplitude, G, is increased, the response curves, ABCDE, are obtained (see Figs. 3.3 and 3.4); if the input amplitude is decreased, the response curves, EDFBA, are obtained.

The resonance phenomenon previously described pertains to systems having no delay. Since no previous work on the jump resonance phenomenon for systems with delay has appeared in the literature, it is useful to apply the Ritz method to these systems and determine the effect of the delay on the response curves. This is done in the following examples using equations which stem from the important equations (3.21) and (3.22).

Example 1.

Consider the equation,

$$\ddot{x}(t) + 2D_1\omega_n\,\dot{x}(t) + 2D_2\omega_n\,\dot{x}(t-h) + \omega_n^2\left[x(t) + \mu^2 x^3(t)\right] - G\sin\omega t = 0 \quad , \tag{3.23}$$

which has a delayed damping term and a nonlinear restoring force of the type referred to in Figs. 3.1 and 3.3. This equation is of the same type as equation (2.6) and is, therefore, stable when $D_1 > |D_2|$.

Applying equations (3.13) to (3.16), we obtain

$$F_1 = 1 + \frac{3}{4}\mu^2 X^2 \quad , \tag{3.24}$$

$$G_1 = G_2 = 1 \tag{3.25}$$

and $$F_2 = 0 \quad . \tag{3.26}$$

After substituting equations (3.24), (3.25) and (3.26) into equation (3.11), we obtain

$$A^3 + a_2 A^2 + a_1 A + a_0 = 0 \quad , \tag{3.27}$$

where $A = \bar{X}^2$,

$$a_2 = 2(1 - \eta^2 + 2D_2\eta \sin \eta\omega_n h) \quad ,$$

$$a_1 = (1 - \eta^2 + 2D_2\eta \sin\eta\omega_n h)^2 + (2D_1\eta + 2D_2\eta\cos\eta\,\omega_n h)^2$$

and $a_0 = - \bar{S}^2$.

The quantities, X and S, have been replaced by dimensionless quantities, $\bar{X} = \frac{X}{L}$ and $\bar{S} = \frac{S}{L}$, where $L = \left[\sqrt{\frac{3}{4}\mu}\right]^{-1}$. After substituting equations (3.24), (3.25) and (3.26) into equation (3.12), we obtain

$$\tan \Theta = \frac{2D_1\eta + 2D_2\eta \cos\eta\omega_n h}{A + 1 - \eta^2 + 2D_2\eta \sin\eta\omega_n h} \quad . \tag{3.28}$$

The response curves can now be obtained by solving for the positive real roots of the cubic equation (3.27) in A and then substituting these values of A into equation (3.28). This computation has been done using an IBM 7040 digital computer which has a plotter available for recording output data. The response curves for any desired values of the coefficients can then be quickly obtained using a relatively simple computer program. Typical response curves are shown in Fig. 3.5 . These approximate curves are to be compared to those obtained by an analog simulation, where the amplitude, X, is taken to be the peak value of the output waveform and the phase, Θ, to be the

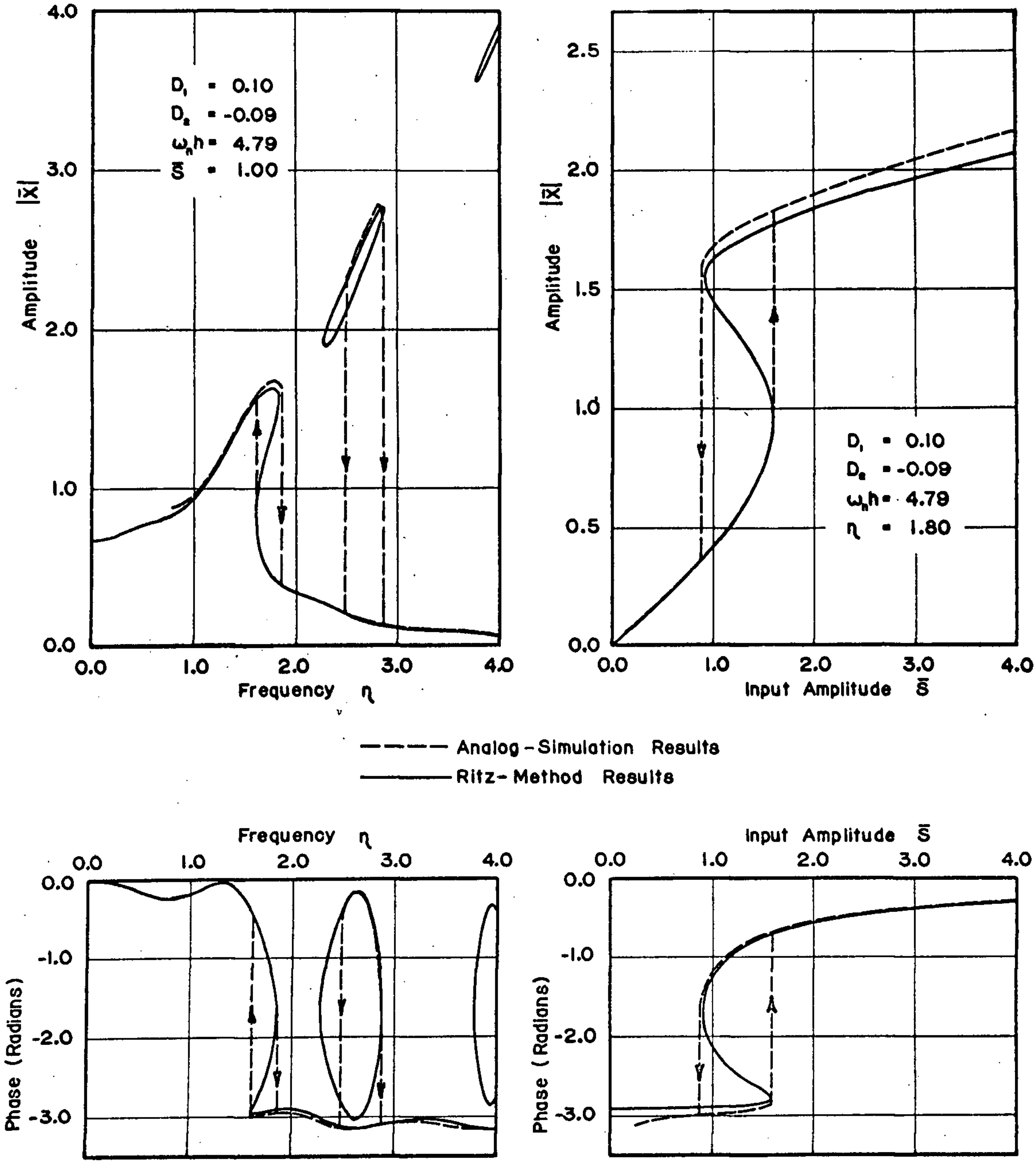

Figure 3.5 Response Curves, Example 1.

difference (in radians) between the zero crossings of the output and input waveforms.

As is the case for systems without delay, the Ritz-method results show that in some regions the output can exist in three states (corresponding to the cases where three positive real roots of equation (3.27) exist), whereas the analog-simulation results show only two states. This is due to the fact that the two extreme states are stable and the middle state is unstable[30] and, therefore, could never be obtained experimentally.

It is evident that the approximate results are quite close to the analog-simulation results (especially for low-output amplitudes where the effect of the nonlinearity is small), and also that the presence of the delayed damping produces isolated regions of the response curve when the input frequency is varied, whereas the response curve for varying input amplitude is similar to Fig. 3.3. At certain frequencies the delayed damping will be in phase with the natural damping , at certain other frequencies the delayed damping will be out of phase with the natural damping, while at intermediate frequencies the delayed damping will have a component in or out of phase with the natural damping and a component in or out of phase with the restoring force. Consequently, as the input frequency is increased or decreased the effective damping oscillates between two extremes. If the upper extreme is larger than the critical damping necessary for jump resonance, then isolated regions of the response curves are obtained as in Fig. 3.5.

These isolated regions can also be explained by considering the roots of equation (3.27). The nature of the roots depends on the quantity, $(q+r)$, where

$$q = \frac{\bar{S}^2 a(a^2 + 9b^2)}{27} \qquad \text{and} \qquad r = \frac{2b^2(a^2 + b^2)^2}{27} + \frac{\bar{S}^4}{4},$$

where $\qquad a = 1 - \eta^2 + 2D_2\eta \sin\eta\omega_n h$

and $\qquad b = 2D_1\eta + 2D_2\eta\cos\eta\omega_n h$.

There are two complex conjugate roots and one real root, three real unequal roots, or three real roots (two of which are equal), depending on whether $(q+r)$ is positive, negative, or zero respectively.

When there is damping but no delayed damping ($D_1 > 0$, $D_2 = 0$), q decreases monotonically from some positive value and finally becomes negative, while r increases monotonically from some positive value as η increases from zero. Due to the presence of η^6 in r, $(q+r)$ is positive for large η. Thus it is possible for $(q+r)$ to decrease from some positive value to some negative minimum and then increase, finally becoming positive as η increases from zero. Thus, as η increases, the number of real roots of equation (3.27) will be one, then three, and finally, one.

When there is a delay present, q and r are no longer monotonic due to the presence of the terms, $\sin\eta\omega_n h$ and $\cos\eta\omega_n h$. For large or small values of η, $(q+r)$ will be positive. For intermediate values of η, however, $(q+r)$ can oscillate about zero as η increases (the larger $\omega_n h$, the more oscillations occur). Thus, as η increases, the number of real roots of equation (3.27) will be one, then varying between one and three, and finally, one. Isolated regions of the response curves will then appear in the frequency range where the number of real roots of equation (3.27) varies between one and three. With the frequency in the proper interval, the isolated regions can be obtained by giving the system a sufficiently large initial condition or by increasing the input amplitude until a jump is obtained and then decreasing the input amplitude to its original value. The jumps associated with the isolated regions are always downward jumps in amplitude. Consequently, the isolated regions cannot be obtained simply by varying the input frequency.

The Ritz-method results show two isolated regions of the response curve for varying frequency, whereas the analog-simulation results show only one. The amplitude for the unstable portion of the isolated region is close to that for the stable portion. Thus the stable portion of the isolated region is probably unstable for small fluctuations in amplitude. This would also explain the difference between the Ritz-method results and the analog-simulation results for the isolated region that was obtained in the analog simulation.

The response curve for varying input amplitude is similar to that for a system without delay, because the effective damping remains constant if the frequency remains constant. If the frequency remains constant and the input amplitude is varied, the quantities, q and r, mentioned previously, are monotonic even with delay present and, therefore, only one region with three real roots is possible.

Example 2.

Consider the equation,

$$\ddot{x}(t) + 2D_1\omega_n \dot{x}(t) + 2D_2\omega_n \dot{x}(t-h) + \omega_n^2\left[x(t) + \mu^2 x^{1/3}(t)\right] - G \sin \omega t = 0 \quad , \tag{3.29}$$

which has a delayed damping term and a nonlinear restoring force of the type referred to in Figs. 3.2 and 3.4. Although the nonlinear function, $x^{1/3}$, is integrable, enabling the Ritz method to be applied, it does not satisfy a Lipschitz condition at the origin because of the infinite slope at this point. In any physical system, the slope of the nonlinearity could be large but never infinite. Since equation (3.29) is the mathematical model of some physical system, we may consider it to be an accurate model everywhere except for a small neighbourhood about the point, $x = 0$, where we assume the slope of the nonlinearity to be large but not infinite. The stability criterion is then the same as for equation (3.23), i.e., $D_1 > |D_2|$.

Applying equations (3.13) to (3.16), we obtain

$$F_1 = 1 + \mu^2 \frac{\Gamma(1/3)}{2^{1/3}\left[\Gamma(2/3)\right]^2} X^{-2/3} \quad , \tag{3.30}$$

$$G_1 = G_2 = 1 \tag{3.31}$$

and $$F_2 = 0 \quad , \tag{3.32}$$

where Γ is a Gamma function. After substituting equations (3.30), (3.31) and (3.32) into equation (3.11) we obtain the cubic equation (3.27), where now

$$A = \bar{X}^{2/3} \quad ;$$

$$a_1 = \left[(1 - \eta^2 + 2D_2\eta \sin\eta\omega_n h)^2 + (2D_1\eta + 2D_2\eta\cos\eta\omega_n h)^2\right]^{-1},$$

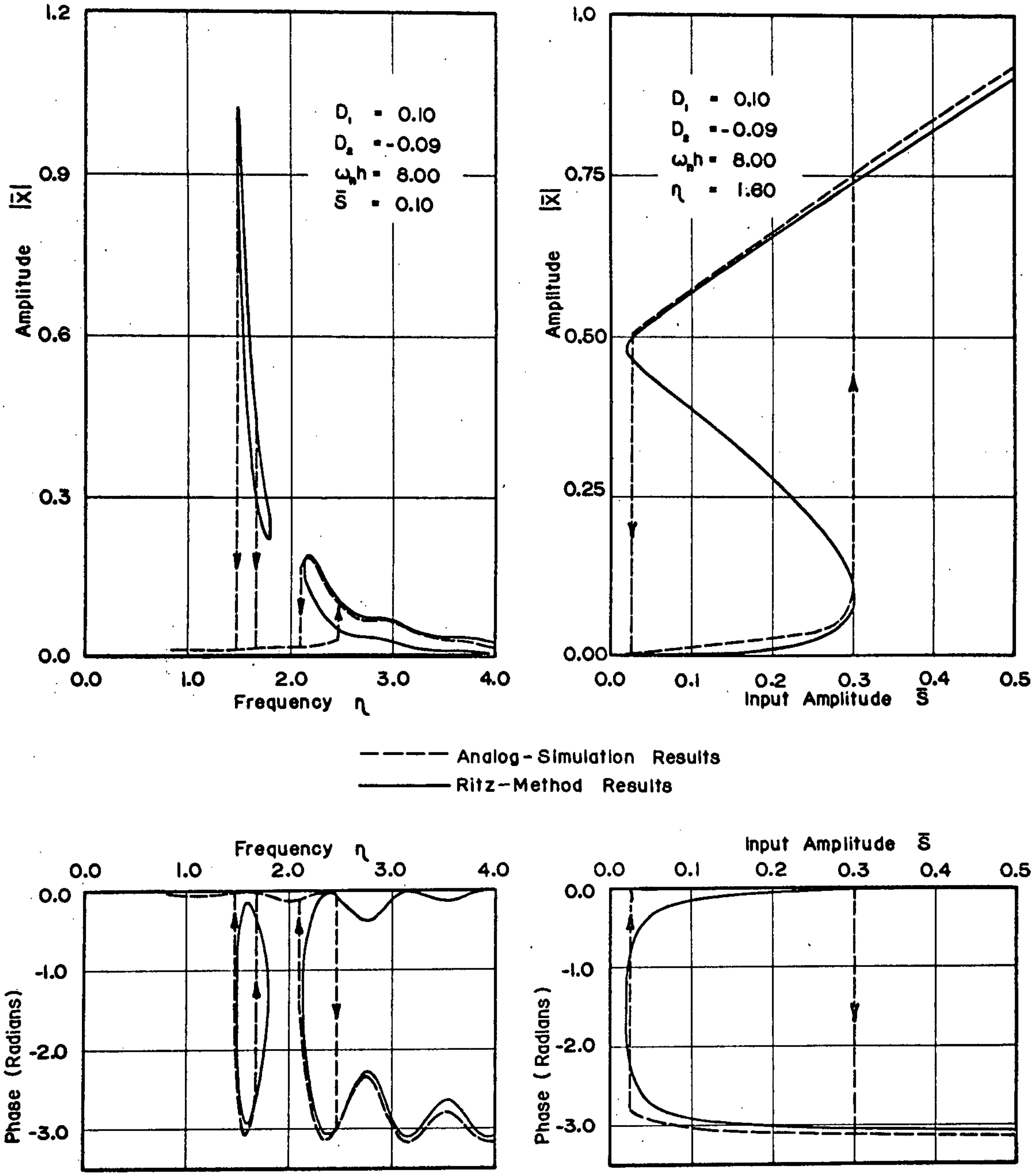

Figure 3.6 Response Curves, Example 2.

$$a_2 = 2(1 - \eta^2 + 2D_2\eta \sin \eta\omega_n h)\ a_1$$

and $\quad a_0 = -\bar{S}^2\ a_1 \quad .$

The quantities, X and S, have been replaced by dimensionless quantities, $\bar{X} = \frac{X}{L}$ and $\bar{S} = \frac{S}{L}$, where

$$L = \left[\frac{\Gamma(1/3)}{2^{1/3}\left[\Gamma(2/3)\right]^2}\right]^{3/2} \mu^3 = 1.2487\ \mu^3 \quad . \tag{3.33}$$

After substituting equations (3.30), (3.31) and (3.32) into equation (3.12), we obtain

$$\tan \Theta = \frac{2D_1\eta + 2D_2\eta \cos \eta\omega_n h}{A^{-1} + 1 - \eta^2 + 2D_2\eta \sin \eta\omega_n h} \quad .$$

The response curves can now be obtained as in Example 1. The curves shown in Fig. 3.6 are similar to those for a system without delay (see Figs. 3.2 and 3.4) except that isolated regions are obtained when the frequency is varied as in Example 1. The response curve for varying input amplitude is again similar to that for a system without delay. The approximate results are close to the analog-simulation results except for low-output amplitudes where the approximate results are least accurate, and the analog-simulation results are inaccurate because of the technique used to obtain the function, $x^{1/3}$.

<u>Example 3.</u>

Consider the equation,

$$\ddot{x}(t) + 2D_1\omega_n\ \dot{x}(t) + \omega_n^2\left[x(t) + \mu^2\ x^3(t)\right] + k\omega_n^2\ x(t-h)$$
$$- G \sin \omega t = 0 \quad , \tag{3.34}$$

which has a delayed restoring force and a nonlinear restoring force of the type referred to in Figs. 3.1 and 3.3. This equation is the same type as equation (2.27) and is, therefore, stable when

$$(2D_1)^2 > \frac{k^2}{1 - |k|} \quad . \tag{3.35}$$

Applying equations (3.13) to (3.16), we obtain

$$F_1 = 1 + \frac{3}{4}\mu^2 X^2 \quad , \tag{3.36}$$

$$G_1 = F_2 = 1 \tag{3.37}$$

and

$$G_2 = 0 \quad . \tag{3.38}$$

After substituting equations (3.36), (3.37) and (3.38) into equation (3.11), we obtain the cubic equation (3.27), where now

$$A = \bar{X}^2 \quad ,$$

$$a_2 = 2(1 - \eta^2 + k \cos \eta\omega_n h) \quad ,$$

$$a_1 = (1 - \eta^2 + k \cos \eta\omega_n h)^2 + (2D_1\eta - k \sin \eta\omega_n h)^2$$

and

$$a_0 = - \bar{S}^2 \quad ,$$

where the dimensionless quantities, $\bar{X}$ and $\bar{S}$, described in Example 1 have been used. After substituting equations (3.36), (3.37) and (3.38) into equation (3.12), we obtain

$$\tan \theta = \frac{2D_1\eta - k \sin \eta\omega_n h}{1 + A - \eta^2 + k \cos \eta\omega_n h} \quad .$$

The response curves can now be obtained as in Example 1. The response curves shown in Fig. 3.7 are similar to those for a system without delay (see Figs. 3.1 and 3.3), except that isolated regions are again obtained when the frequency is

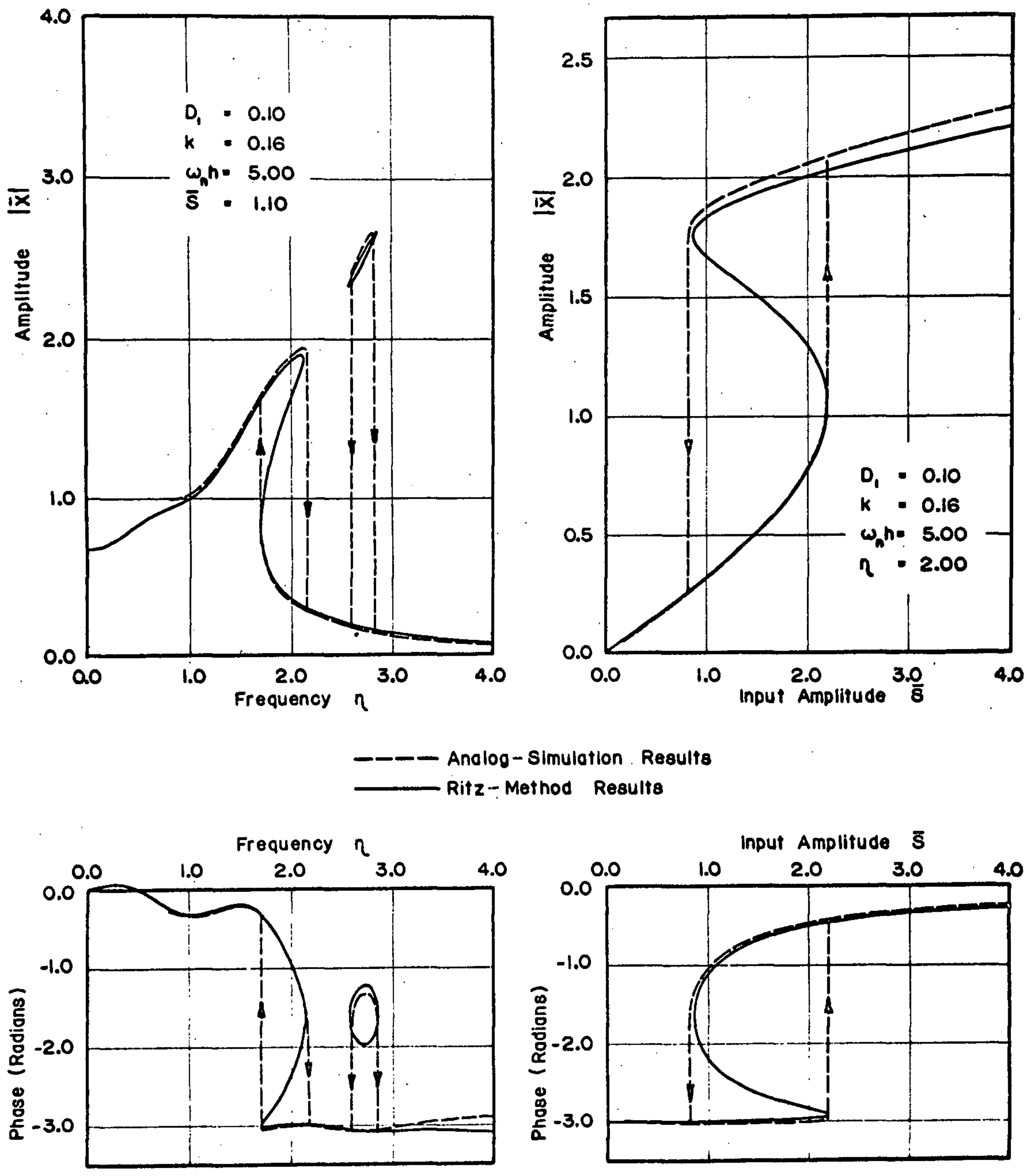

Figure 3.7 Response Curves, Example 3.

varied. The effective damping in this case changes with frequency because of the component of the delayed restoring force, which is in or out of phase with the natural damping. In other respects this example is similar to Example 1.

Example 4.

Consider the equation,

$$\ddot{x}(t) + 2D_1\omega_n \dot{x}(t) + \omega_n^2\left[x(t) + \mu^2 x^{1/3}(t)\right] + k\omega_n^2 x(t-h) - G \sin \omega t = 0 \quad , \tag{3.39}$$

which has a delayed restoring force and a nonlinear restoring force of the type referred to in Figs. 3.2 and 3.4. If we treat the nonlinearity, $x^{1/3}$, as in Example 2, the stability criterion for this equation is given by equation (3.35).

Applying equations (3.13) to (3.16), we obtain equation (3.30) for F_1,

$$G_1 = F_2 = 1 \tag{3.40}$$

and

$$G_2 = 0 \quad . \tag{3.41}$$

After substituting equations (3.30), (3.40) and (3.41) into equation (3.11), we obtain the cubic equation (3.27), where now

$$A = \bar{X}^{2/3} \quad ,$$

$$a_1 = \left[(1 - \eta^2 + k \cos\eta\omega_n h)^2 + (2D_1\eta - k \sin\eta\omega_n h)^2\right]^{-1} \quad ,$$

$$a_2 = 2(1 - \eta^2 + k \cos\eta\omega_n h)\, a_1$$

and $\quad a_0 = - \bar{S}^2 a_1 \quad ,$

where the dimensionless quantities, $\bar{X}$ and $\bar{S}$, described in Example 2 have been used. After substituting equations (3.30), (3.40) and (3.41) into equation (3.12), we obtain

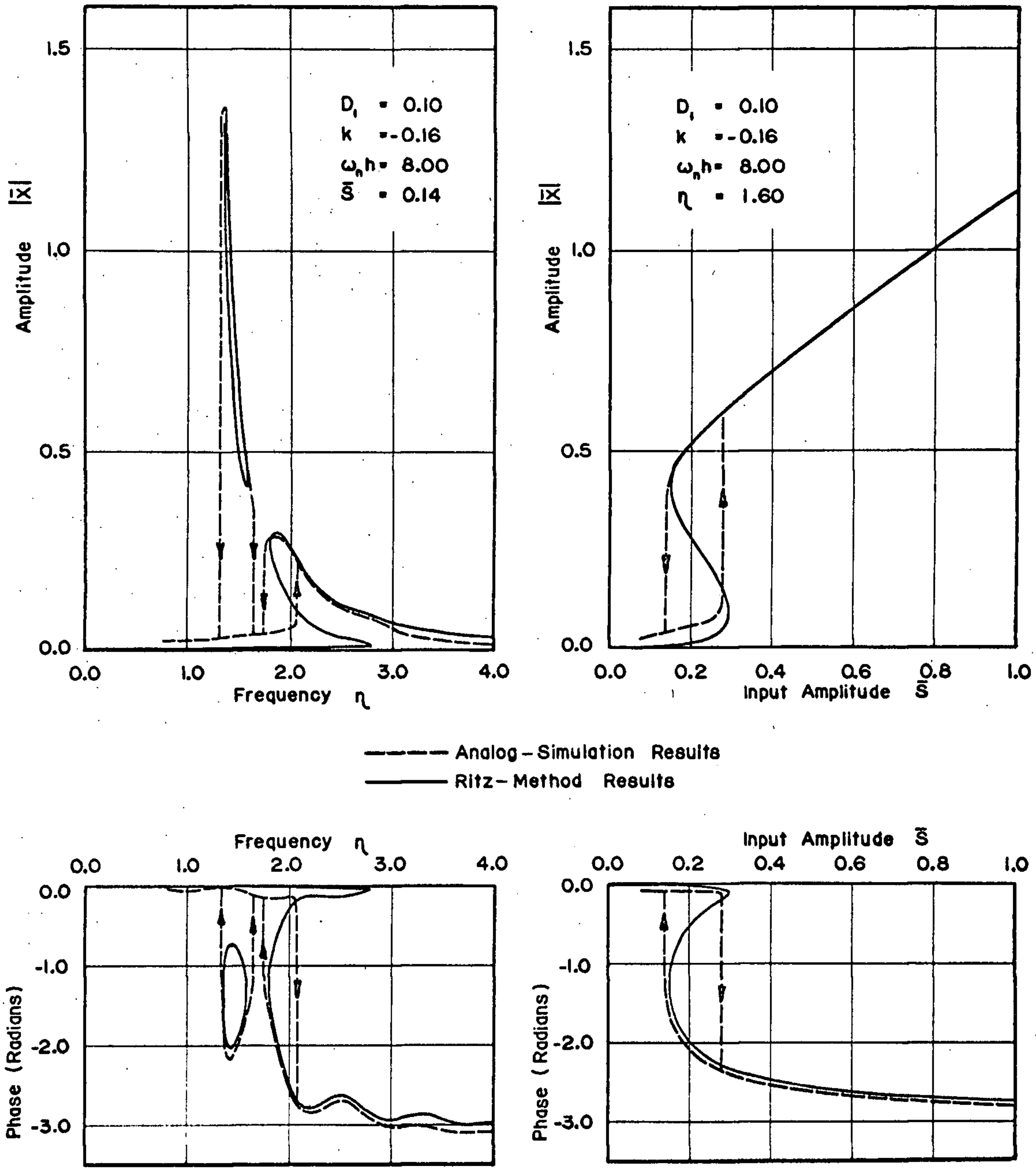

Figure 3.8 Response Curves, Example 4.

$$\tan \theta = \frac{2D_1\eta - k \sin\eta\omega_n h}{A^{-1} + 1 - \eta^2 + k \cos\eta\omega_n h} .$$

The response curves can now be obtained as in Example 1. The effective damping changes with frequency as in Example 3, otherwise the discussion for the response curves shown in Fig. 3.8 is the same as for Example 2.

The Ritz method can also be applied to systems which have nonlinear damping terms. These systems do not exhibit jump resonance, but it is of interest to obtain their response curves.

Example 5.

Consider the equation ,

$$\ddot{x}(t) + 2D_1\omega_n\left[\dot{x}(t) + \frac{\mu^2}{\omega_n^2}\dot{x}^3(t)\right] + 2D_2\omega_n\,\dot{x}(t-h) + \omega_n^2\,x(t) - G \sin \omega t = 0 \quad , \tag{3.42}$$

which has delayed damping and nonlinear damping. This equation is stable when $D_1 > |D_2|$.

Applying equations (3.13) to (3.16) and substituting the results into equations (3.11) and (3.12), we obtain for the cubic equation (3.27)

$$A = \bar{X}^2 \quad ,$$

$$a_2 = \frac{2D_1\eta + 2D_2\eta \cos\eta\omega_n h}{D_1\eta^3} \quad ,$$

$$a_1 = \frac{(1 - \eta^2 + 2D_2\eta \sin\eta\omega_n h)^2 + (2D_1\eta + 2D_2\eta \cos\eta\omega_n h)^2}{4D_1\,\eta^6}$$

and $$a_0 = \frac{-\bar{S}^2}{4D_1\,\eta^6} \quad ,$$

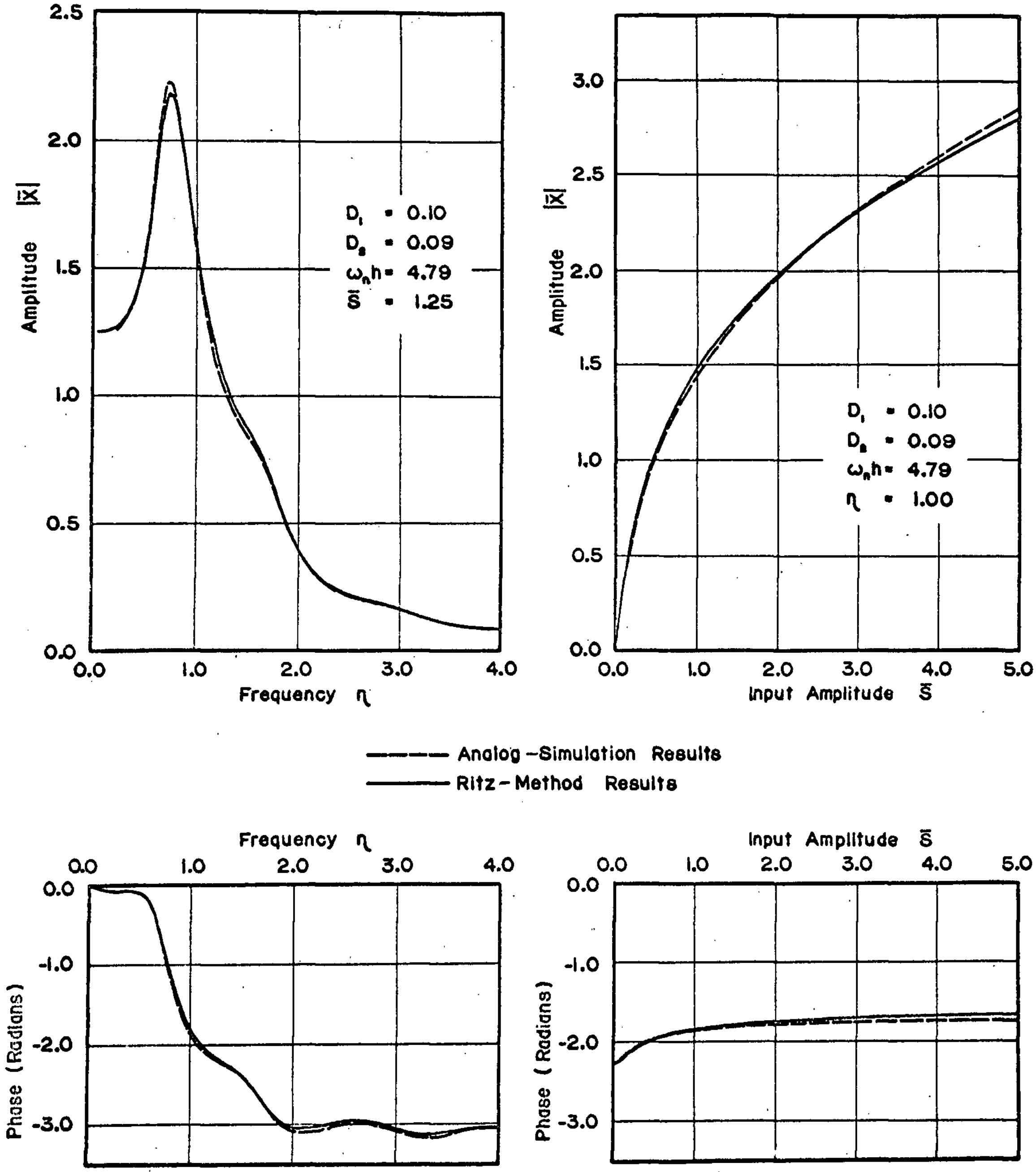

Figure 3.9 Response Curves, Example 5.

and for the phase

$$\tan \Theta = \frac{2D_1\eta\,(1 + \eta^2 A) + 2D_1\eta\,\cos\eta\omega_n h}{1 - \eta^2 + 2D_2\eta\,\sin\eta\omega_n h} ,$$

where the dimensionless quantities, $\bar{X}$ and $\bar{S}$, described in Example 1 have been used. Typical response curves are shown in Fig. 3.9. Since the effect of the nonlinear damping is small, the Ritz-method results and the analog-simulation results are quite close. The nonlinearity increases the damping as the output amplitude increases, causing the peak of the curve for varying frequency to be somewhat flattened and the curve for varying input amplitude to be concave down. The delayed damping causes slight humps in the frequency response curve due to the varying effective damping with frequency. This effect is somewhat diminished due to the nonlinear damping term.

Example 6.

Consider the equation,

$$\ddot{x}(t) + 2D_1\omega_n\left[\dot{x}(t) + \mu^2\,\omega_n^{2/3}\,\dot{x}^{1/3}(t)\right] + 2D_2\omega_n\,\dot{x}(t-h) + \omega_n^2\,x(t) - G\sin\omega t = 0 , \tag{3.43}$$

which has delayed damping and nonlinear damping. If the difficulties due to the presence of the cube root term are treated as in Example 2, the equation is stable when $D_1 > |D_2|$.

Applying equations (3.13) to (3.16) and substituting the results into equations (3.11) and (3.12), we obtain for the cubic equation (3.27)

$$A = \bar{X}^{2/3} ,$$

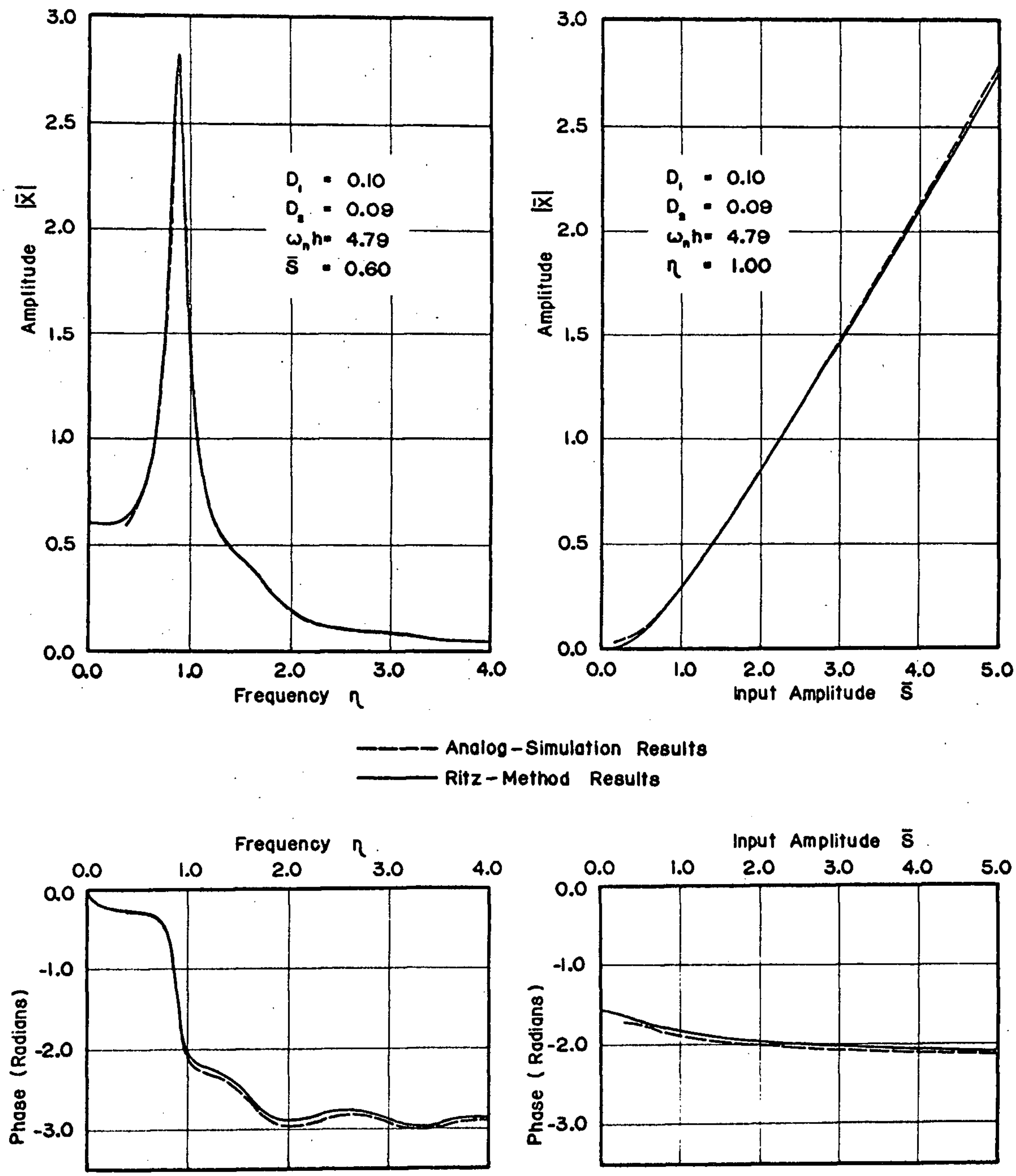

Figure 3.10 Response Curves, Example 6.

$$a_0 = \frac{-\bar{S}^2}{(1 - \eta^2 + 2D_2\eta \sin \eta\omega_n h)^2 + (2D_1\eta + 2D_2\eta \cos \eta\omega_n h)^2} ,$$

$$a_1 = -\frac{4D_1 \eta^{2/3} a_0}{\bar{S}^2}$$

and $$a_2 = -\frac{4D_1 \eta^{1/3} a_0}{\bar{S}^2} ,$$

and for the phase

$$\tan \Theta = \frac{2D_1\eta (1 + 1/[\eta^{2/3} A]) + 2D_2\eta \cos \eta\omega_n h}{1 - \eta^2 + 2D_2\eta \sin \eta\omega_n h} ,$$

where the dimensionless quantities, $\bar{\bar{X}}$ and $\bar{S}$, described in Example 2 have been used. Typical response curves are shown in Fig. 3.10. The Ritz-method results and the analog-simulation results are again quite close. The nonlinearity decreases the damping as the output amplitude increases, causing the peak of the curve for varying frequency to be sharply peaked and the curve for varying input amplitude to be concave up. This sharp peaking of the resonance curve would be useful where a high Q circuit is required. The effect of the delayed damping is the same as for Example 5.

4. VERIFICATION OF THE APPROXIMATE SOLUTION BY ANALOG SIMULATION

The validity of the approximate analytical method mentioned in section 3 depends on assuming the correct form of the solution. If the assumed form is incorrect, the results obtained by this method are completely meaningless.

In view of the quasi-linear nature of the systems considered, it has been assumed in section 3 that the response of the system to a sinusoidal input will be approximately sinusoidal and of the same frequency as the input. This assumption is easily verified by simulating the system on the PACE 231R analog computer. In order to compare the Ritz-method results to the analog results it is desirable to measure the amplitude and phase of the fundamental component of the output waveform. Since the system is nonlinear, the output waveform, in general, deviates somewhat from a true sinusoid. This deviation, however, is not large and, therefore, it is reasonable to base the measurement of the amplitude of the fundamental on the peak value of the output waveform and the phase on the zero-crossover.

The versatility of the PACE 231R enables automatic plotting of the system output amplitude and phase versus the frequency or amplitude of the sinusoidal system input. In view of the large number of examples considered, it is essential that the response curves be obtained automatically, otherwise the computing time and the time to plot the curves would be prohibitive.

The sinusoidal system input (A cosωt or A sinωt) is obtained by solving the nonlinear differential equation,

$$\ddot{x} - \varepsilon\left[A^2 - x^2 - \frac{\dot{x}^2}{\omega^2}\right]\dot{x} + \omega^2 x = 0 \ , \qquad (4.1)$$

which was suggested by Van der Pol and is discussed by Jackson(31), (see Fig. 4.1). Equation (4.1) has the limit cycle solution,

$$x = A\cos(wt + \theta)\ , \qquad (4.2)$$

which is easily verified by substituting equation (4.2) into equation (4.1). If the initial conditions are chosen as

$$x(0) = A \quad , \quad \dot{x}(0) = 0 \ ,$$

the solution begins at the limit cycle and the term, θ, in equation (4.2) becomes zero. With ε fairly large (say 10), the solution tends rapidly to the limit cycle if any disturbances occur. Therefore, if the signals corresponding to ω and A are varied reasonably slowly, the nonlinear oscillator of Fig. 4.1 will continuously yield the output, A cos ωt. The circuit described by Humo(32) does not function in this manner and hence his results are in error.

A control circuit (Fig. 4.2) enables the operator to hold A constant and automatically increase or decrease ω, or to hold ω constant and automatically increase or decrease A. With switches, S_{10}, S_{11}, S_{12} and S_{13}, in the left position, ω is swept by integrator 26 while A is set by pot. P59; with the switches in the right position, A is swept by integrator 26 while ω is set by pot P59. With switch, S_{02}, in the right

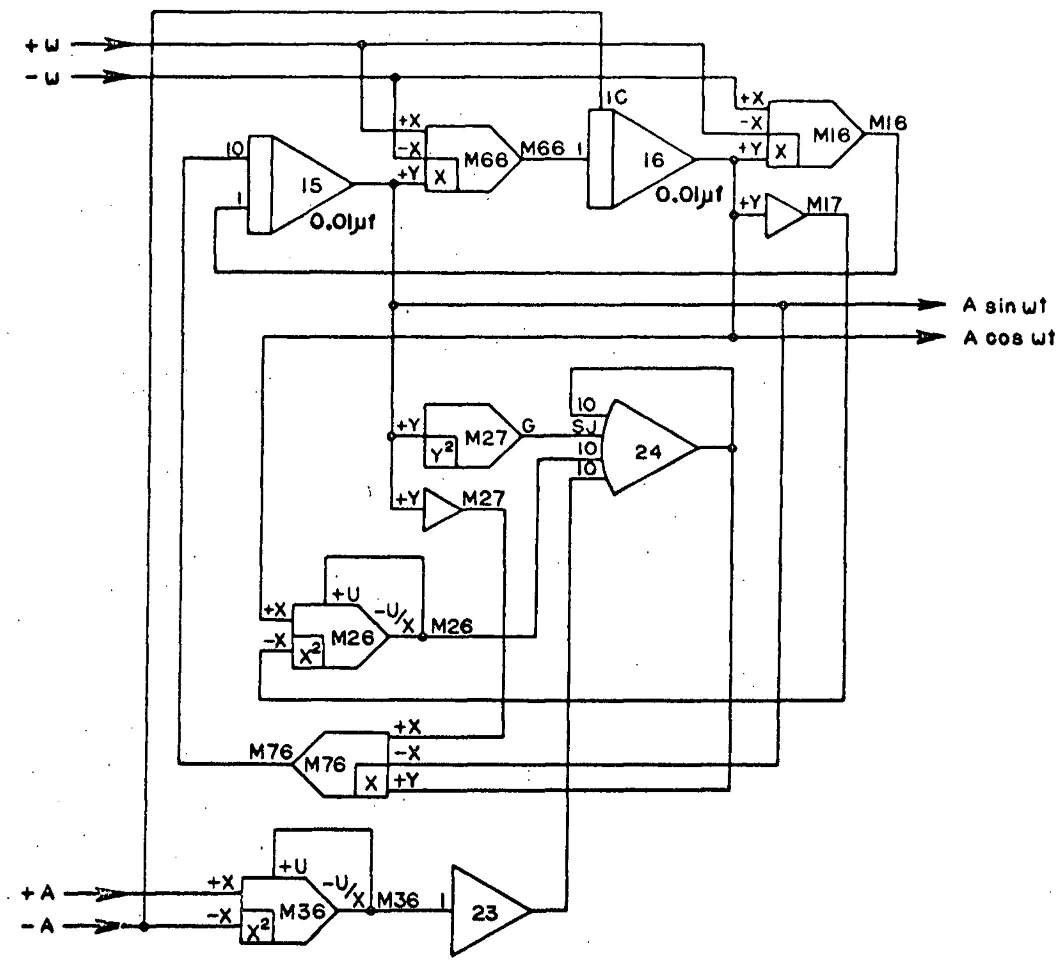

Figure 4.1 Nonlinear Oscillator

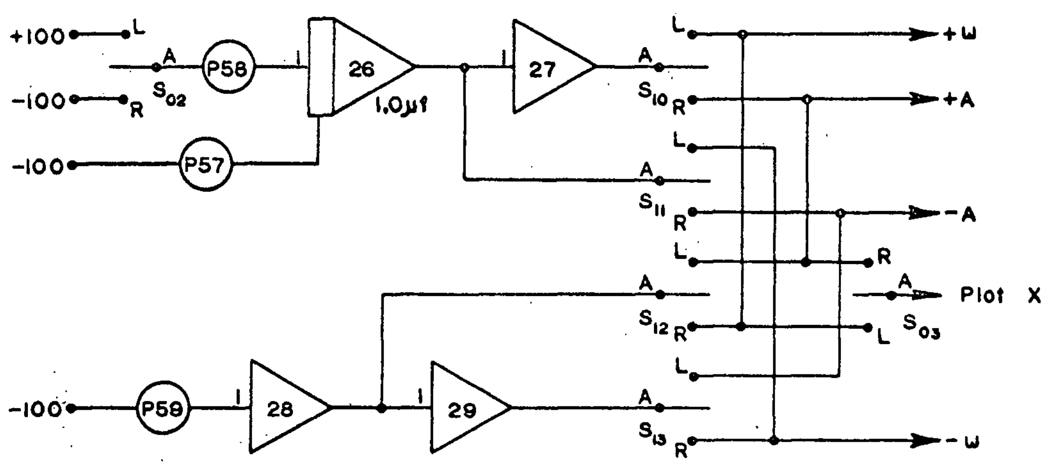

Figure 4.2 Control Circuit

position the output of integrator 26 increases; in the left position the output decreases. The sweep rate is controlled by pot. P58.

Fig. 4.3 shows the analog simulation for the system of example 1 (sec. 3.3) and the track and store circuits used to detect the system output amplitude or phase. The analog simulations of the other systems are similar to Fig. 4.3 and are therefore not shown. A twenty segment diode function generator provides a good approximation to the cube root quantity required for examples 2, 4 and 6 of section 3.3. The infinite slope at the origin for the cube root function cannot be obtained using the function generator and hence the results obtained for low amplitude inputs to the function generator are somewhat in error.

The delay element is simulated by means of an Ampex tape recorder (Model SP300 F.M. Direct). The tape speeds of 1 7/8 ($\pm$ 0.4%), 3 3/4 ($\pm$ 0.4%), 7 1/2 ($\pm$ 0.2%), and 15($\pm$ 0.2%) inches per second, provide delays of 1455, 728, 364, and 183 milliseconds respectively. The frequency response at a tape speed of 15 inches per second is from 0 to 2500 Hz. The maximum frequency is reduced by a factor of two each time the tape speed is reduced by a factor of two. The inputs and outputs to the various channels of the tape recorder are available as trunk line terminations at the analog patch panel. The resistors necessary to protect the tape recorder from overload and to provide the appropriate signal levels at the tape recorder and the analog patch panel are incorporated in these trunk lines.

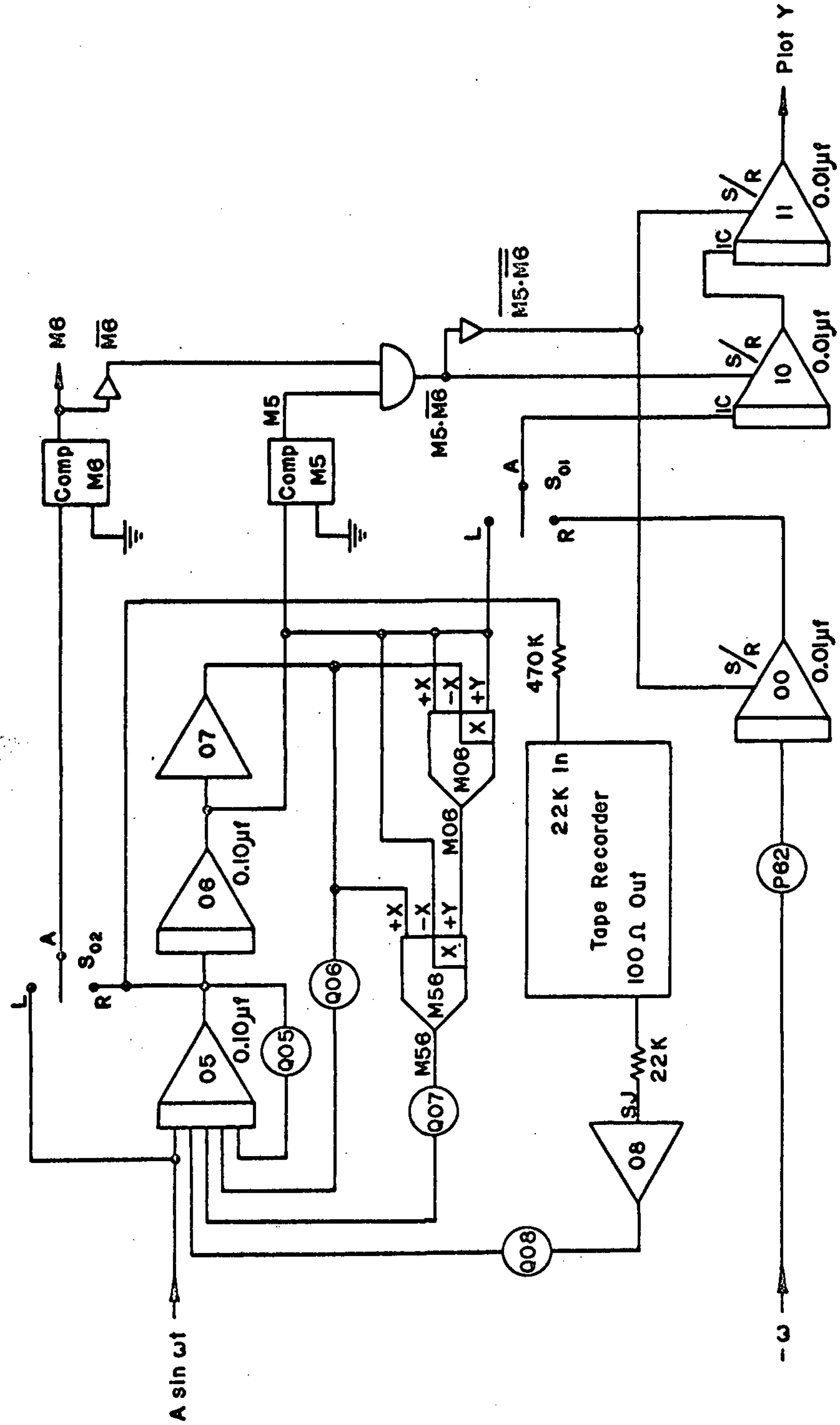

Figure 4.3 Analog Simulation of a System with Delay

The delay element could also be simulated by approximating the Laplace shift operator, e^{-hs}, which is the transfer function associated with a pure time delay, h. This simulation, however, requires a large number of integrators for as accurate an approximation as can be obtained with the tape recorder. The tape recorder has the added advantage that the delay can be changed merely by changing the tape speed.

With switches, S_{00} and S_{03}, in the left position, the output amplitude is plotted. The signal, x(t), is applied to comparator M5 and the signal, $-\dot{x}(t)$, is applied to comparator M6. Each signal is compared to zero volts. (The operation of the comparators, integrator mode control, and AND gates is described in Appendix B.) The normal digital output, M5, and the complementary output, $\overline{M6}$, are applied as inputs to an AND gate (see Fig. 4.3). The normal output ($M5 \cdot \overline{M6}$) of the AND gate is thus at a ONE level for the first half of each positive half-period of x(t) and at a ZERO level for the remainder of the period. The signal, $M5 \cdot \overline{M6}$, controls the mode of integrator 10, while the signal, $\overline{M5 \cdot \overline{M6}}$, controls the mode of integrator 11. Integrator 10 thus "tracks" the system output during the first half of each positive half-period, while integrator 11 "stores" and plots the maximum value of each positive half-period. The system output amplitude is thus obtained.

With switches, S_{00} and S_{03}, in the right position, the phase of the output is obtained. The system input is applied to comparator, M6, while integrators 10 and 11 track and store the output of integrator 00. The signal, $\overline{M5 \cdot \overline{M6}}$, which controls the mode of integrator 00, is now at a ONE level each time the system input goes positive until the time the system output goes positive. From the time the system input goes positive, therefore, integrator 00 integrates at a rate proportional to ω for a time equal to the time $x(t)$ lags the input and is then reset. The output of the track and store circuit is then proportional to the phase of the output.

If the signals corresponding to ω and A are varied slowly enough that transient effects are negligible, then continuous plots of the steady-state system response curves are obtained.

The results obtained by the analog simulation are given in sec 3.3, and are compared to the results obtained by the approximate analytical method.

5. CONCLUSIONS

The purpose of this investigation was to obtain approximate analytical solutions of quasi-linear differential-difference equations and to determine their resonance properties.

Stability criteria for these equations have been given prior to the approximate analytical solutions and the determination of the resonance properties. The stability criterion for equations with delayed damping is due to Krasovskii[21]; the stability criterion for equations with delayed restoring force has been developed by the author.

Approximate analytical solutions of a general second-order nonlinear differential-difference equation have been obtained by employing the Ritz method.

General equations which lead to the determination of the constants in the assumed solutions have been given for systems with odd nonlinearities and for systems with non-odd nonlinearities. The general equations for systems with odd nonlinearities have been used to obtain the resonance properties for several specific examples of such systems. It has been found that the response curves for varying input amplitude are similar to those for systems without delay, whereas the response curves for varying input frequency exhibit a rather peculiar jump phenomenon which is not obtained for systems without delay. When the input frequency is varied, isolated regions of the response curve occur. It has been found that these regions can be obtained physically by giving the system a sufficiently large initial condition, or by increasing the input amplitude sufficiently and then decreasing it to its original value. The isolated regions are attributed to a frequency-dependent effective damping caused by the interaction of the natural damping with the delayed damping or the delayed restoring force. This peculiar jump resonance phenomenon has not previously been mentioned in the literature.

The approximate solutions for the specific examples have been verified by an analog computer simulation. This simulation employs track and store techniques to enable automatic plotting of the response curves. The Ritz-method results compare favourably with the analog-simulation results.

In view of the success of the Ritz method for the examples considered, it would be useful to prove theoretically that the Ritz method is applicable to general nonlinear differential-difference equations. It would also be useful to extend other approximate techniques available for ordinary nonlinear differential equations to nonlinear differential-difference equations. It would then be possible to investigate transient behaviour and such phenomena as entrainment of frequency which occurs when an oscillator is subjected to a sinusoidal driving force.

In conclusion, approximate solutions to some quasi-linear differential-difference equations have been obtained and their resonance properties determined.

APPENDIX A

SOME DEFINITIONS AND PROPERTIES PERTAINING TO QUADRATIC FORMS

The following definitions and properties pertaining to quadratic forms are given by Ayres[33]:

A homogeneous polynomial of the type

$$q = X'AX = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j ,$$

whose coefficients, a_{ij}, are elements of F is called a quadratic form over F in the variables, $x_1, \ldots, x_n$.

The symmetric matrix, $A = [a_{ij}]$, $(a_{ij} = a_{ji})$, is called the matrix of the quadratic form and the rank of A is called the rank of the form. If the rank is $r < n$ the quadratic form is called singular; otherwise, non-singular.

A minor of matrix, A, is called principal if it is obtained by deleting certain rows and the same numbered columns of A. Thus, the diagonal elements of a principal minor of A are diagonal elements of A.

For a symmetric matrix, $A = [a_{ij}]$, over F, define the leading principal minors as

$$p_0 = 1,\ p_1 = a_{11},\ p_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\ \ldots,\ p_n = |A| .$$

A real quadratic form, X'AX, is positive-definite if, and only if, its rank is n and all leading principal minors are positive.

APPENDIX B

ON THE OPERATION OF SOME ANALOG COMPUTER COMPONENTS

The following specifications were obtained from the PACE 231R MLG System handbook(34):

B.1 Integrator Mode Control

The integrator is placed in the electronic switching (ES) mode by grounding its ES termination on the Memory and Logic Unit (MLU) pre-patch panel. In the ES mode the integrator is placed in "initial condition" by applying +5 volts (a ONE level) to one of the MLU panel IC terminations (designated S/R in Fig. 4.3). An input of zero volts (a ZERO level) switches the integrator to the "operate" mode.

B. 2 Electronic Compatators

The analog inputs are applied at the analog patch panel and provide the following digital outputs at the MLU pre-patch panel:

(1) When the analog input sum is negative (less than -10 mv.) the normal digital output is at a ZERO level;

(2) When the analog input sum is positive (greater than +10 mv.) the normal digital output is at a ONE level.

B. 3 AND Gates

If all of the inputs to an AND gate are at a ONE level the normal output is at a ONE level; if one or more of the inputs are at a ZERO level the normal output is at a ZERO level.

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