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STEADY-STATE OSCILLATIONS AND STABILITY OF ON-OFF FEEDBACK SYSTEMS

ABSTRACT

Methods for studying the behaviour of on-off feedback systems, with the emphasis on steady-state periodic phenomena, are presented in this thesis. The two main problems analyzed are (1) the determination of the periods of self and forced oscillations in single-, double-, and multiloop systems containing an arbitrary number of on-off elements; and (2) the investigation of the asymptotic stability in the small of single-loop systems containing one on-off element which may or may not have a linear region of operation.

To study the periodic phenomena in on-off systems, methods of determining the steady-state response of a single on-off element are first described. Concepts pertaining to the steady-state behaviour are then introduced: in this respect it has been found that generalizations of the concepts of the Hamel and Tsypkin loci and also of the phase characteristic of Neimark are useful in the study of self and forced oscillations.

Both the Tsypkin loci and the phase characteristic concepts are used to determine the possible periods of self and forced oscillations in single- and double-loop systems containing an arbitrary number of on-off elements; these concepts are also applied to multiloop systems.

On-off elements containing a linear region of operation, called a proportional band, are then described; both the transient and periodic responses are presented. An approximate method for determining the periodic response is given. The concept of the Tsypkin loci is used to determine the possible periods of self and forced oscillations in a single-loop system containing one on-off element with a proportional band.

The asymptotic stability in the small, or local stability, of the periodic states of single-loop systems containing one ideal on-off element has been considered by Tsypkin. In this thesis, Tsypkin's results have been generalized to include the cases on on-off elements containing a proportional band. The stability of such systems is determined by the stability of equivalent sampled-data systems with samplers having finite pulse widths. Finally, this stability problem is solved by a

direct approach, one that makes use of the physical definition of local stability; the results obtained by this method agree with those derived by the sampled-data approach.

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PUBLICATIONS

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1. INTRODUCTION

The study of on-off feedback control systems having a single loop with one on-off element has been developed by many authors during the last three decades. Many of the techniques for investigating the steady-state behaviour of such systems resort to approximate methods, of which the best known is that of the describing function.[1,2,3] On the other hand, the best known exact methods are those of D.A. Kahn,[4] B. Hamel,[5] J.Z. Tsypkin,[6] and E.V. Bohn.[7]

Concerning the determination of the periods of self oscillations in a single-loop feedback control system containing two symmetric relays, Tu Syui-Yan and Tei-Lui-Vy[8] gave both an exact solution, using the method of the Tsypkin Loci, and an approximate solution, using the method based on harmonic balance. Also, Yu.I. Neimark and L.P. Shilnikov[9] studied the symmetric periodic motions of a multistage relay system by means of Neimark's concept of the phase characteristic.

Nevertheless, to the knowledge of the author, no study of multiloop automatic control systems containing an arbitrary number of on-off elements has been attempted. The main purpose of the first two parts of this thesis is to investigate the complicated forms of oscillation in a single-loop system containing a single on-off element and the simple symmetric modes of self and forced oscillations in single-, and double-loop control systems having an arbitrary number of on-off elements.

Part I of this thesis gives the fundamental concepts and formulae required in the study of the various systems considered in Part II. The working principle, classification, and equations of on-off elements are reviewed in Chapter 2. The response of these elements to an arbitrary input and to the general periodic input, and the methods of calculating the response are given in Chapter 3. Next, in Chapter 4, the concepts pertaining to the periodic response of on-off elements, namely, the concepts of the Hamel and Tsypkin loci (or hodograph), are reformulated so as not only to make evident the relationships existing among these concepts, but also to facilitate the study of self and forced oscillations in the multiloop systems considered in Part II. The conditions for the existence of self and forcedoscillations for the various multiloop systems are then determined with the help of these concepts. Methods of solving for the simple symmetric modes of oscillation in single-, and double-loop systems are given in Chapters 5, 6, and 7.

Feedback control systems with proportional bands are considered in Part III. The problem of determining the periodic states of feedback control systems having a single nonlinear element with arbitrary piecewise linear characteristic has received rigorous attention in the last few years. M.A. Aizerman and F.R. Gantmakher[10,11] studied the piecewise linear characteristic consisting of segments parallel to two given straight lines, whereas L.A. Gusev[12] dealt with an arbitrary piecewise linear characteristic. Their methods of solving the problem differ, but in both cases the solutions take into account all the harmonics. Part III deals with an exact method for the determination of the transient state in a system containing one nonlinear element having the saturation characteristic with hysteresis. A simple method of solving the simple symmetric oscillations in such a system is presented. The method is approximate, but sufficiently accurate for systems possessing linear parts with a filtering action. An exact solution is then formulated in the form of a set of linear Volterra integral equations of the second kind.

Finally, Part IV of the thesis deals with the stability of the periodic states in control systems having one on-off element with or without a proportional band. An exact solution shows that the "asymptotic stability in the small" of such systems reduces to a consideration of the stability of finite pulse width sampling systems with feedback. The results obtained are a generalization of those of Tsypkin.[6] An approximate method applicable to systems with nonlinear elements having characteristics other than the on-off type, with or without a proportional band, is also presented. In contrast to the sampled-data approach, a direct method of investigating the stability of self and forced oscillations in single-loop systems having one on-off element is presented. This method is directly related to the physical definition of stability: a disturbance is applied, and the ensuing deviation from the state of equilibrium is studied.

PART I

FUNDAMENTAL CONCEPTS OF ON-OFF ELEMENTS

2. ON-OFF ELEMENTS

According to their working principle, on-off control systems are essentially nonlinear. Therefore it is evidently impossible to analyze their behaviour by the well-known linear methods of the theory of feedback control systems. Nevertheless, the specific peculiarity of on-off systems, namely that they are piecewise linear, permits their investigation by comparatively simple mathematical methods.

In general, the on-off or relay element may be regarded as consisting of the on-off component followed by a linear part, which is composed of the actual linear part of the relay plus the linear part following the relay. Figure 2.1 gives the convention and notations for the relay element. The symbol N represents the on-off (nonlinear) component, whereas

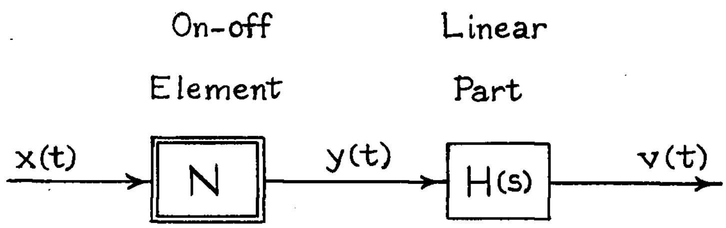


Figure 2.1. Conventions and notations for the relay element.

H(s) denotes the transfer function of the linear part, where s is the complex frequency variable. The quantities x(t), y(t), and v(t) are respectively the input to the on-off element, the input to the linear part, and the output of the linear part, and are all functions of the time variable t.

In the field of automatic control $x(t)$ is referred to as the control signal, and $y(t)$ as the correction signal.

In on-off control systems the correction signal $y(t)$ changes by jumps at every instant when the control signal $x(t)$ passes through certain fixed values known as the threshold values. Hence the linear part of the system $H(s)$ is subjected to rectilinear pulses of fixed height, the sign, duration and relative distribution of which depend both upon the external excitation and upon the initial conditions existing in the linear part of the system.

In general, on-off elements may be classified as symmetric or asymmetric with respect to the origin of the coordinate axes x and y, where $x = x(t)$ is the control signal, and $y = y(t)$ is the correction signal. Furthermore, in each of these two classes a dead zone may or may not be present. In addition these elements may or may not possess hysteresis, that is, $y(t)$ may be a single or multivalued function of $x(t)$. Table I gives this classification of on-off elements.

Equations and characteristics of on-off elements

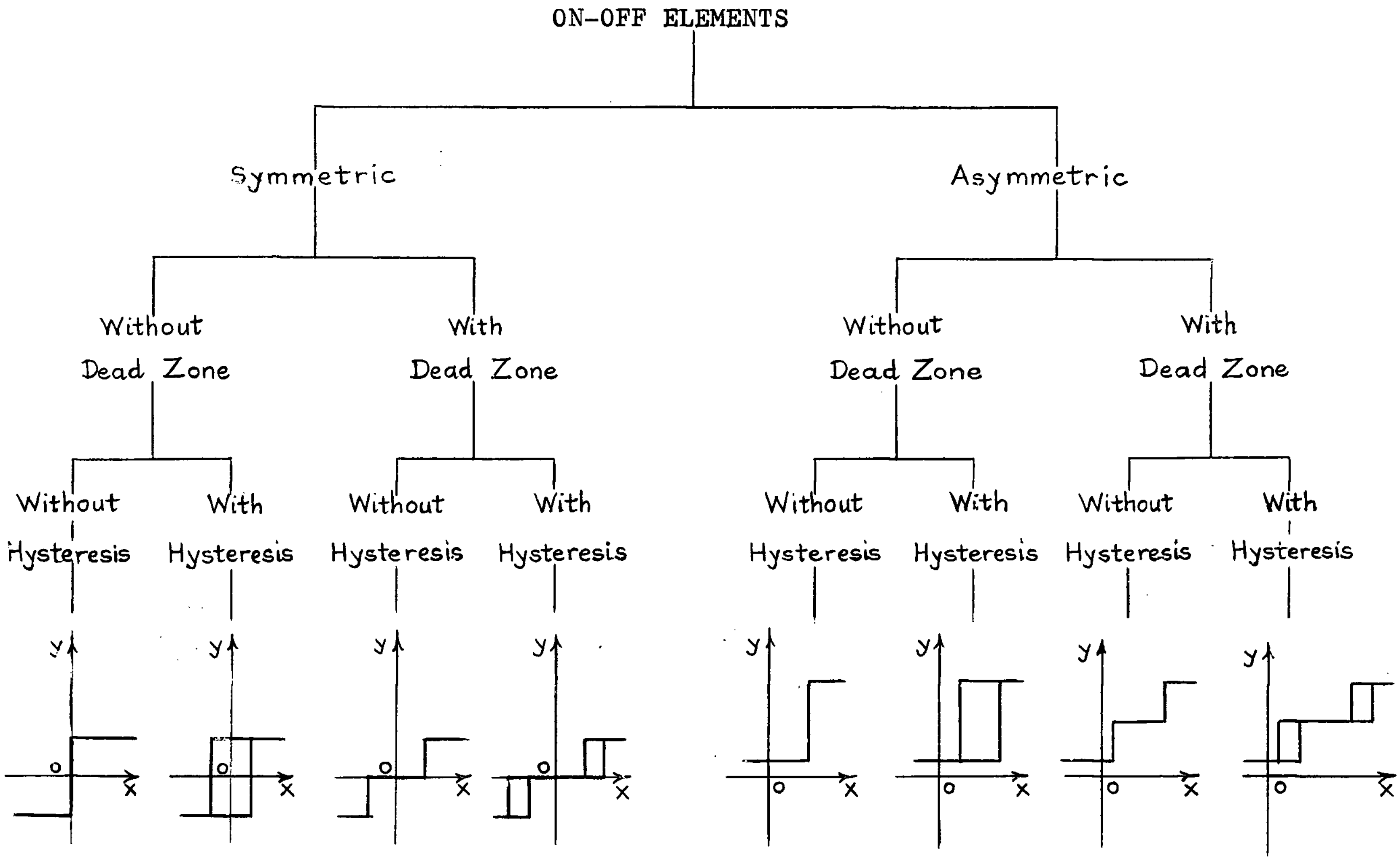
The output $y(t)$ of the on-off symmetric component N is a function both of $x(t)$ and $\dot{x}(t)$, where $\dot{x}(t) \triangleq \frac{dx(t)}{dt}$. Consequently, the equation of the on-off symmetric component can be written in the form

$$y(t) = \Phi(x(t), \dot{x}(t))$$

where $\Phi(x(t),\dot{x}(t))$ is a nonlinear function. For simplicity we will use the notation

$$y = \Phi(x) \tag{2.1}$$

TABLE I. CLASSIFICATION OF ON-OFF ELEMENTS



The plot of y vs. x is called the characteristic of the on-off component N.

In the case of asymmetric on-off elements the characteristic can be expressed in the form

$$y = y_a + \Phi(x - x_a), \tag{2.2}$$

that is, $\Phi(x - x_a)$ is symmetric with respect to the point (x_a, y_a). The characteristics and corresponding equations for asymmetric on-off components are given in Table II. If the elements are symmetric we merely put $x_a = y_a = 0$.

From Table II we observe that the first three characteristics can be regarded as special cases of the fourth. In fact,

$$\Phi_4(x - x_a)\Big]_{\lambda=1} = \Phi_3(x - x_a),$$

$$\Phi_4(x - x_a)\Big]_{\lambda=-1} = \Phi_2(x - x_a),$$

and finally

$$\Phi_4(x - x_a)\Big]_{x_o = 0} = \Phi_1(x - x_a).$$

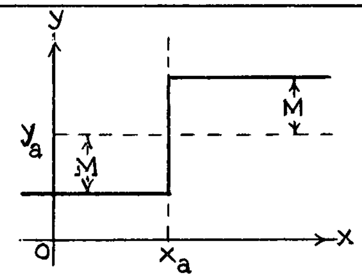
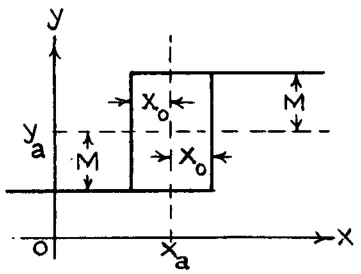
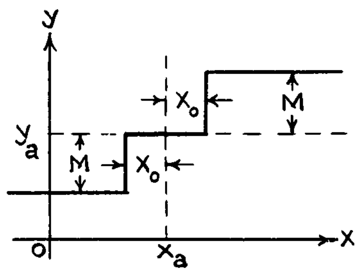
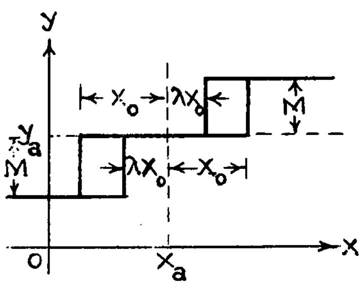
The linear part of the system can best be analyzed by means of the Laplace transform. In the case of zero initial conditions, the output of the linear part is determined by

$$V(s) = H(s)\,Y(s) \tag{2.3}$$

where

$$V(s) = \mathcal{L}\Big(v(t)\Big) \quad \text{and} \quad Y(s) = \mathcal{L}\Big(y(t)\Big) .$$

TABLE II. CHARACTERISTICS AND EQUATIONS OF ON-OFF COMPONENT N

Characteristic	Equation
	$y - y_a = \Phi_1(x-x_a) = M \text{ sign } (x-x_a)$
	$y - y_a = \Phi_2(x-x_a) = \begin{cases} M \text{ sign}(x-x_a-x_o), & \text{for } \dot{x} > o \\ M \text{ sign}(x-x_a+x_o), & \text{for } \dot{x} < o \end{cases}$
	$y - y_a = \Phi_3(x-x_a) = \frac{M}{2}\left[\text{sign}(x-x_a-x_o) + \text{sign}(x-x_a+x_o)\right]$
	$y-y_a = \Phi_4(x-x_a) = \begin{cases} \frac{M}{2}\left[\text{sign}(x-x_a-x_o)+\text{sign}(x-x_a+\lambda x_o)\right] & \text{for } \dot{x} > o \\ \frac{M}{2}\left[\text{sign}(x-x_a+x_o)+\text{sign}(x-x_a-\lambda x_o)\right] & \text{for } \dot{x} < o \end{cases}$

Remarks:

1. $x \longrightarrow \boxed{N} \longrightarrow y$

2. $$\text{sign } (x-a) = \begin{cases} 1, & \text{for } x > a, \\ o, & \text{for } x = a, \\ -1, & \text{for } x < a. \end{cases}$$

3. In the case of a symmetric characteristic put $x_a = y_a = o$.

Equation (2.3) may be rewritten as

$$V(s) = H(s)\ \mathcal{L}\left(y_a + \Phi(x - x_a)\right) .$$

Now suppose that non-zero initial conditions exist within the linear part H(s). By means of the Laplace transform, the output V(s) can always be expressed as

$$V(s) = H(s)Y(s) + V_o(s),$$

where $V_o(s)$ is the output resulting from the initial conditions within H(s). Consequently, the effect of the initial conditions may conveniently be referred to the output of the linear part in the manner shown in Figure 2.2. Similarly, any external influence f(t) applied to the system may be referred

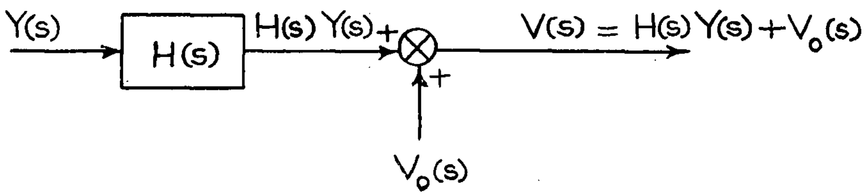


Figure 2.2. Initial conditions in the linear part referred to the output.

to the output of the linear part.

3. RESPONSE OF ON-OFF ELEMENTS

In on-off elements the correction signal $y(t)$ changes by jumps at every instant when the control signal $x(t)$ passes through the threshold values with $\dot{x}(t) > o$ in certain cases and $\dot{x}(t) < o$ in others. Consequently, the investigation of the response of on-off control systems is reduced to the investigation of the behaviour of the linear parts of the system to a sequence of rectilinear pulses, the parameters of which depend upon the form of the control signal and upon the threshold values of the on-off elements. Hence, the basic method of determining the response of the system is through the application of the superposition principle to the linear parts. For any one on-off element, the response is determined by the equation

$$V(s) = H(s)\, \mathcal{L}\left(y_a + \Phi(x - x_a)\right) + V_o(s) \quad .$$

3.1 THE RESPONSE FOR AN ARBITRARY INPUT

The most general on-off characteristic, that is, the case of the asymmetric on-off element with hysteresis and dead zone is represented by the equation:

$$y - y_a = \Phi_4(x - x_a) \quad .$$

Without loss of generality, and for definiteness, we will assume that the control signal $x(t)$ passing through the first threshold value at the instant τ_1 is decreasing, that is $\dot{x}(\tau_1) < o$. The general forms of the control and correction signals, together with the on-off characteristic are shown in Figure 3.1.

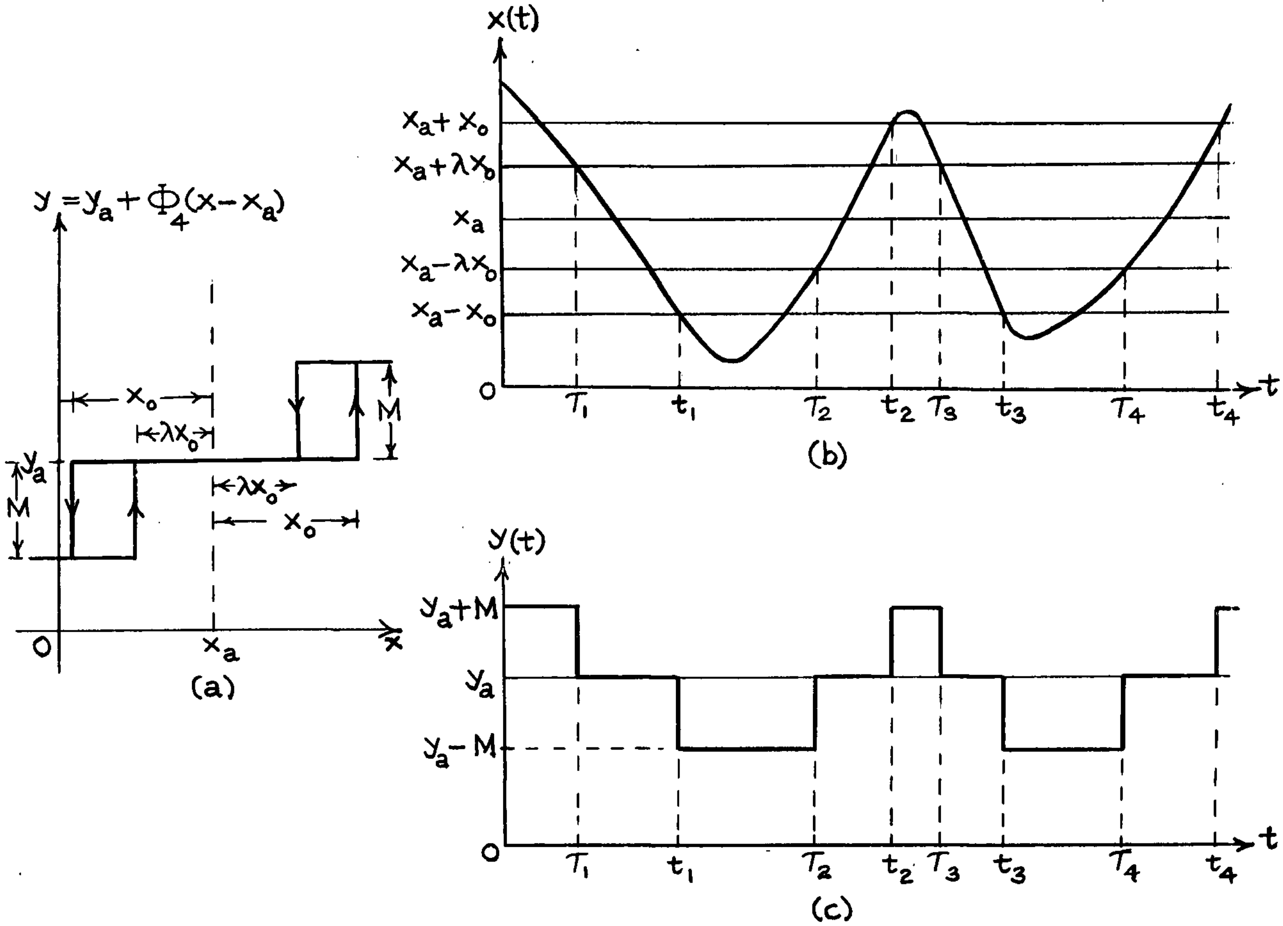


Figure 3.1. (a) On-off characteristic with dead zone and hysteresis;
(b) Control signal x(t); (c) Correction signal y(t).

The switching conditions

$$\left.\begin{aligned} x(t_k) &= x_a + (-1)^k x_o, \\ \dot{x}(t_k)(-1)^k &> o \end{aligned}\right\} (k = 1,2,\ldots) \qquad (3.1)$$

correspond to the switching instants $t_1, t_2, \ldots,$ along the threshold values $x_a + (-1)^k x_o$; whereas the switching conditions

$$\left.\begin{aligned} x(\tau_k) &= x_a + (-1)^{k+1} \lambda x_o \\ \dot{x}(\tau_k)(-1)^k &> o \end{aligned}\right\} \quad (k = 1,2,\ldots) \quad (3.2)$$

correspond to the switching instants $\mathcal{T}_1, \mathcal{T}_2, \ldots$ along the threshold values $x_a + (-1)^{k+1}\lambda x_o$. It may happen that the switching instant t_m is absent, in which case the switching instant $\mathcal{T}_{m+1}$ will also be absent.

The input to the linear part is given by

$$y(t) = y_a u(t) + M \sum_{k=1}^{n} (-1)^{k-1} \left[u(t-t_{k-1}) - u(t - \mathcal{T}_k)\right], (\mathcal{T}_n \leq t < t_n) \tag{3.3}$$

$$= \text{Right-hand side of (3.3)} + M(-1)^n u(t - t_n), (t_n \leq t < \mathcal{T}_{n+1}) \tag{3.4}$$

where $t_o = o$, and $u(t-a)$ is the unit step function initiated at the time $t = a$.

Let $g(t - a)$ be the response of the linear part to the unit step $u(t - a)$, that is

$$\mathcal{L}\left(g(t - a)\right) = \frac{H(s)}{s} e^{-sa}$$

with the understanding that

$$g(t - a) = o \text{ for } t < a .$$

Then the expression for the response of the on-off element to an arbitrary input with switching instants $\mathcal{T}_1, t_1, \mathcal{T}_2, t_2, \ldots$ is

$$v(t) = \begin{cases} v_o(t) + y_a g(t) + M \sum_{k=1}^{n} (-1)^{k-1} \left[g(t-t_{k-1}) - g(t-\mathcal{T}_k)\right], & (\mathcal{T}_n \leq t < t_n) \quad (3.5) \\ \text{Right-hand side of (3.5)} + M(-1)^n g(t-t_n), & (t_n \leq t < \mathcal{T}_{n+1}) \quad (3.6) \end{cases}$$

where $v_o(t)$ represents the response due to the initial conditions; that is

$$v(t) = \begin{cases} v_o(t) + y_a g(t) + Mg(t) & (o \leq t < \tau_1) \\ v_o(t) + y_a g(t) + M\left[g(t) - g(t-\tau_1)\right] & (\tau_1 \leq t < t_1) \\ v_o(t) + y_a g(t) + M\left[g(t) - g(t-\tau_1) - g(t-t_1)\right] & (t_1 \leq t < \tau_2) \\ v_o(t) + y_a g(t) + M\left[g(t) - g(t-\tau_1) - g(t-t_1) \right. & (\tau_2 \leq t < t_2) \\ \qquad \left. + g(t - \tau_2)\right] & \\ v_o(t) + y_a g(t) + M\left[g(t) - g(t-\tau_1) - g(t-t_1) \right. & (t_2 \leq t < \tau_3) \\ \qquad \left. + g(t - \tau_2) + g(t-t_2)\right], & \\ \ldots & \end{cases}$$

In general, the response may be constructed graphically by means of the superposition principle.

3.2 THE STEADY STATE RESPONSE

Various methods of evaluating the steady-state output response of the linear part of the system are now presented for the general case of an on-off characteristic represented by

$$y - y_a = \Phi_4(x - x_a) \quad .$$

In the case of complicated forms of oscillations, self or forced, the input to the linear part of the system $y(t)$ repeats itself, in general, after 2n commutations, where n is an even integer. In the absence of a dead zone there are, in general, n commutations, where n is even. The general forms of the periodic control signal $x(t)$ and of the periodic correction signal $y(t)$, corresponding to the on-off characteristic under consideration, are shown in Figures 3.2(a) and 2(b), respectively.

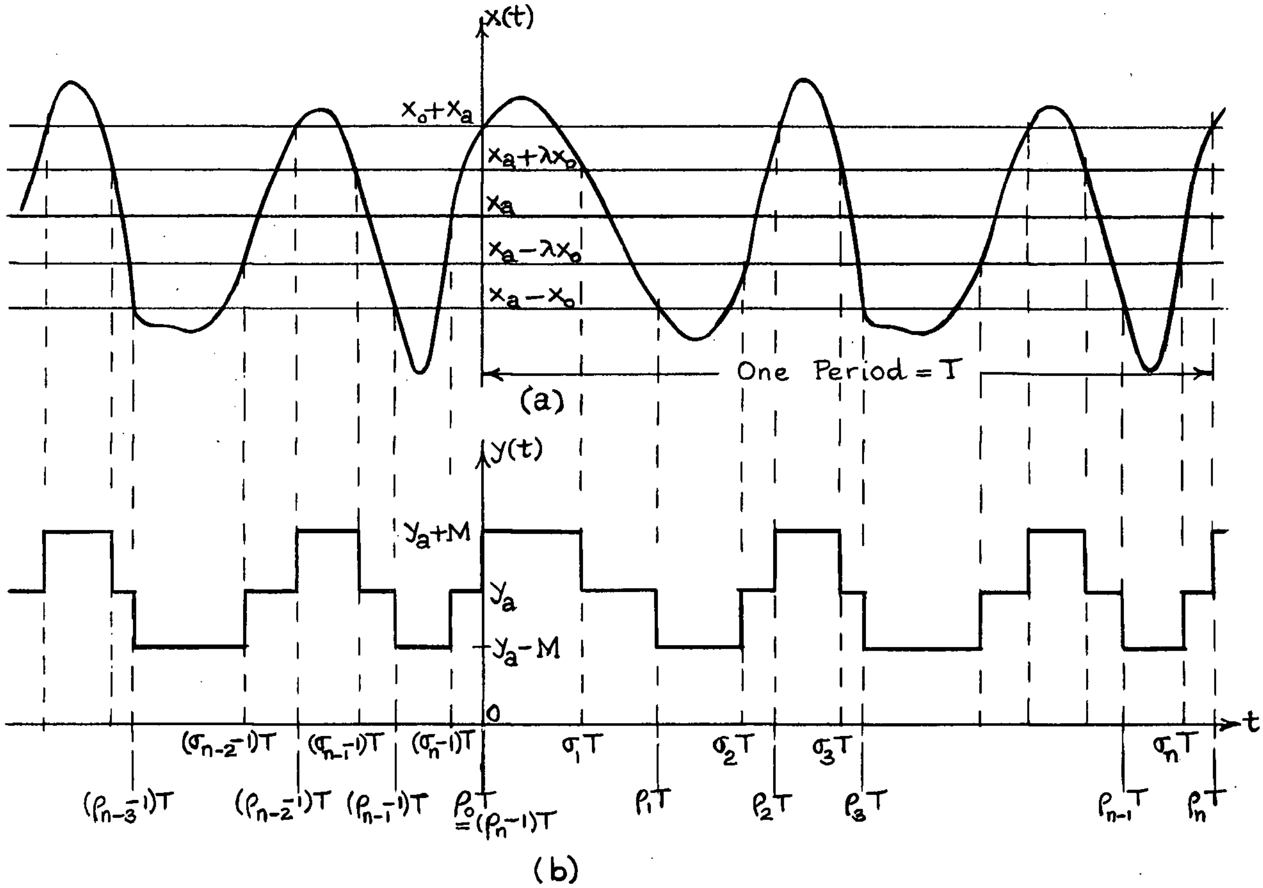


Figure 3.2. (a) General Form of Control Signal x(t)

(b) General Form of Correction Signal y(t), in the case of complicated oscillations.

It may happen that $\rho_i T$ is absent. In such a case it follows from the characteristic of the on-off element that $\sigma_{i+1}T$ is also absent.

The correction signal y(t) can be expressed as the sum of a fixed component y_a, and a sequence of rectilinear pulses relative to y_a and denoted by $y_1(t)$; that is

$$y(t) = y_a + y_1(t), \tag{3.7}$$

where, letting

$$\Delta u_{k,i} = (-1)^i \left(u\left[t-(k+\rho_i)T\right] - u\left[t-(k+\sigma_{i+1})T\right]\right), \tag{3.8}$$

$$y_1(t) = M\left((-1)^{\ell} u\left[t-(m+\rho_\ell)T\right] + \sum_{i=o}^{\ell-1}\Delta u_{m,i} + \sum_{k=m-1}^{-\infty}\ \sum_{i=o}^{n-1}\Delta u_{k,i}\right) \tag{3.9a}$$

$$(m+\rho_\ell)T \le t < (m+\sigma_{\ell+1})T,$$
$$m=o,\ \pm 1,\ \pm 2,\ \dots ,$$
$$\ell=o,\ 1,\ \dots,\ n-1\ ;$$

$$y_1(t) = M\left(\sum_{i=o}^{\ell-1}\Delta u_{m,i} + \sum_{k=m-1}^{\infty}\ \sum_{i=o}^{n-1}\Delta u_{k,i}\right) \tag{3.9b}$$

$$(m+\sigma_\ell)T \le t < (m+\rho_\ell)T,$$
$$m=o,\ \pm 1,\ \pm 2,\ \dots ,$$
$$\ell=1,\ 2,\ \dots,\ n.$$

Alternatively, expressions (3.9a) and (3.9b) can be written as

$$y_1(t) = M\left((-1)^{\ell} u\left[t-(m+\rho_\ell)T\right] + \sum_{k=m}^{-\infty}\left(\sum_{i=o}^{\ell-1}\Delta u_{k,i} + \sum_{i=\ell}^{n-1}\Delta u_{k-1,i}\right)\right) \tag{3.10a}$$

$$(m+\rho_\ell)T \le t < (m+\sigma_{\ell+1})T;$$

$$y_1(t) = M\sum_{k=m}^{-\infty}\left(\sum_{i=o}^{\ell-1}\Delta u_{k,i} + \sum_{i=\ell}^{n-1}\Delta u_{k-1,i}\right), \tag{3.10b}$$

$$(m+\sigma_\ell)T \le t < (m+\rho_\ell)T,$$

respectively.

In the case of dead-zone only, the above expressions retain the same form, except that the σ's change values, whereas in the absence of a dead zone we have $\lambda = -1$ and $x_o \ge o$, so that we simply replace σ_i by ρ_i for all i.

The output v(t) of the linear part of the system is determined as follows. Let $g(t)$ be the response to a unit step input initiated at time $t = o$:

$$g(t) = \begin{cases} \mathcal{L}^{-1}\left(\frac{H(s)}{s}\right), & t \geq o \\ o & , t < o \end{cases} \tag{3.11}$$

Then the response of the linear part to the input $y_1(t)$ is given by

$$v_1(t) = M\left[(-1)^{\ell} \; g\left[t-(m+\rho_\ell)T\right] + \sum_{k=m}^{-\infty}\left(\sum_{i=o}^{\ell-1} \Delta g_{k,i} + \sum_{i=\ell}^{n-1} \Delta g_{k-1,i}\right)\right] \tag{3.12a}$$

$$(m+\rho_\ell)T \leq t < (m+\sigma_{\ell+1})T,$$
$$m = o, \pm 1, \pm 2, \ldots$$
$$\ell = 0, 1, \ldots, n-1 \;;$$

$$v_1(t) = M\sum_{k=m}^{-\infty}\left(\sum_{i=o}^{\ell-1} \Delta g_{k,i} + \sum_{i=\ell}^{n-1} \Delta g_{k-1,i}\right), \tag{3.12b}$$

$$(m+\sigma_\ell)T \leq t < (m+\rho_\ell)T,$$
$$m = o, \pm 1, \pm 2, \ldots$$
$$\ell = 1, 2, \ldots, n,$$

where

$$\Delta g_{k,i} = (-1)^i \left(g\left[t-(k+\rho_i)T\right] - g\left[t-(k+\sigma_{i+1})T\right]\right) \tag{3.13}$$

Since
$$\mathcal{L}\left(g(t-\tau)\right) = \frac{H(s)}{s}\, e^{-s\tau}$$

then

$$\mathcal{L}\left(\Delta g_{k,i}\right) = (-1)^i \frac{H(s)}{s}\left(e^{-s\rho_i T} - e^{-s\sigma_{i+1}T}\right)e^{-skT},$$

so that

$$\mathcal{L}\left(v_1(t)\right) = Me^{-smT}\,\frac{H(s)}{s}\left[\frac{\sum_{i=o}^{\ell-1}\xi_i + e^{sT}\sum_{i=\ell}^{n-1}\xi_i}{1-e^{sT}}\right], \tag{3.14a}$$

$$(m+\sigma_\ell)T \le t < (m+\rho_\ell)T,$$

$$\mathcal{L}\left(v_1(t)\right) = Me^{-smT}\,\frac{H(s)}{s}\left[(-1)^{\ell}\,e^{-s\rho_\ell T} + \frac{\sum_{i=o}^{\ell-1}\xi_i + e^{sT}\sum_{i=\ell}^{n-1}\xi_i}{1-e^{sT}}\right] \tag{3.14b}$$

$$(m+\rho_\ell)T \le t < (m+\sigma_{\ell+1})T\ ,$$

where $$\xi_i = (-1)^i\,(e^{-s\rho_i T} - e^{-s\sigma_{i+1}T}) \tag{3.14c}$$

The response of the linear part to a fixed component y_a in the steady state is

$$v_a = y_a g(\infty) = y_a H(o)\ , \tag{3.15}$$

which is finite if the linear part of the system is stable. Consequently, the total output of the linear part of the system can be expressed as

$$\begin{aligned} v(t) &= v_a + v_1(t) \\ &= y_a\,H(o) + \frac{M}{2\pi j}\oint_{C_1 \text{ or } C_2}\frac{H(s)}{s}\,I_1(s)e^{-smT}e^{st}\,ds\ , \end{aligned} \tag{3.16}$$

$$(m+\sigma_\ell)T \le t < (m+\rho_\ell)T,$$

where

$$I_1(s) = \frac{\sum_{i=o}^{\ell-1}\xi_i + e^{sT}\sum_{i=\ell}^{n-1}\xi_i}{1-e^{sT}}\ , \tag{3.17}$$

where C_1 is a path enclosing only the poles of $H(s)/s$, where C_2 is a path enclosing only the poles of $I_1(s)$, and where the contour integrals along C_1 and C_2 are taken in the mathematically positive and negative sense respectively; whereas

$$v(t) = y_a H(o) + \frac{M}{2\pi j} \oint_{C_1 \text{ or } C_2} \frac{H(s)}{s} I_2(s)\, e^{-smT} e^{st}\, ds \tag{3.18}$$

$$(m+\rho_\ell)T \leq t < (m+\sigma_{\ell+1})T ,$$

where

$$I_2(s) = (-1)^\ell e^{-s\rho_\ell T} + I_1(s) \tag{3.19}$$

In general, $v_1(t)$ is asymmetric, and

$$v_1(t+T) = v_1(t) . \tag{3.20}$$

If, however, the condition

$$v_1(t + \frac{T}{2}) = -v_1(t)$$

is satisfied, then the function $v_1(t)$ is said to be symmetric. This necessarily means that

$$\left.\begin{aligned} \frac{n}{2} &= \text{odd integer} , \\ \rho_{\frac{n}{2}} &= \frac{1}{2} , \\ \rho_{\frac{n}{2}+k} &= \frac{1}{2} + \rho_k , \quad (k=1, 2, \ldots, \tfrac{n}{2}) \\ \sigma_{\frac{n}{2}+k} &= \frac{1}{2} + \sigma_k , \quad (k=1, 2, \ldots, \tfrac{n}{2}) \end{aligned}\right\} \tag{3.21}$$

Thus, if we are considering the response $v_1(t)$ for $mT \leq t < (m+ \frac{1}{2})T$, then, substituting conditions (3.21) into (3.17), we get

$$I_1(s) = \frac{\sum_{i=o}^{\ell-1} \xi_i + e^{sT}\sum_{i=\ell}^{n-1} \xi_i}{1 - e^{sT}} = \frac{\left[\sum_{i=o}^{\ell-1} \xi_i - e^{s\frac{T}{2}} \sum_{i=\ell}^{\frac{n}{2}-1} \xi_i\right](1-e^{sT/2})}{1 - e^{sT}}$$

$$= \frac{\sum_{i=o}^{\ell-1} \xi_i - e^{sT/2}\sum_{i=\ell}^{\frac{n}{2}-1} \xi_i}{1 + e^{sT/2}} . \tag{3.22}$$

Consequently, in the case of symmetric but complicated forms of oscillations, the response of the linear part of the system is given by

$$v(t) = y_a H(o) + \frac{M}{2\pi j} \oint_{C_1 \text{ or } C_2} \frac{H(s)}{s} I_1(s) e^{-smT} e^{st} ds \tag{3.23a}$$

$$(m+\sigma_\ell)T \leq t < (m+\rho_\ell)T,$$
$$m=o, \pm 1, \ldots; \ \ell = 1, 2, \ldots, \frac{n}{2}$$

$$v(t) = y_a H(o) + \frac{M}{2\pi j} \oint_{C_1 \text{ or } C_2} \frac{H(s)}{s} \left[(-1)^\ell e^{-s\rho_\ell T} + I_1(s)\right] e^{-smT} e^{st} ds \tag{3.23b}$$

$$(m+\rho_\ell)T \leq t < (m+\sigma_{\ell+1})T,$$
$$m=o, \pm 1, \ldots; \ \ell = o, 1, \ldots, \frac{n}{2} - 1$$

where $I_1(s)$ is now given by (3.22).

Methods of Calculating the Periodic Output Waveform

So far we have set up very general expressions for the periodic output v(t) of the linear part of the system. Let us now turn our attention to the various methods of calculating the shape of the periodic state. We will classify these methods as follows:

1. The g-Method, which uses the unit step response $g(t)$ of the linear part of the system;

2. The C_1-Method: We derived an integral representation of $v_1(t)$ in the form

$$v_1(t) = \frac{M}{2\pi j} \oint_{C_1} \frac{H(s)}{s} I(s) e^{st} ds , \tag{3.24}$$

where C_1 is a contour enclosing only the poles of $H(s)/s$. By the residue theorem, of the theory of functions of a complex variable,

$$v_1(t) = M \sum_{\text{at Poles of } \frac{H(s)}{s}} \text{Residues of } \frac{H(s)}{s} I(s)e^{st} \tag{3.25}$$

Thus, this method uses the transfer function, $H(s)$, of the linear part of the system.

3. The C_2-Method: An alternate integral representation of $v_1(t)$ was found to be

$$v_1(t) = \frac{M}{2\pi j} \oint_{C_2} \frac{H(s)}{s} I(s) e^{st} ds , \tag{3.26}$$

where C_2 is a contour enclosing only the poles of $I(s)$. Thus, by the residue theorem,

$$v_1(t) = -M \sum_{\text{Poles of } I(s)} \text{Residues of } \frac{H(s)}{s} I(s)e^{st} \tag{3.27}$$

Since the poles of $I(s)$ all lie along the imaginary axis of the complex s-plane, we are essentially using $H(j\omega)$, the so-called frequency response of the linear part of the system, in the evaluation of $v_1(t)$. For this purpose we will find it more convenient to rewrite $H(j\omega)$ as

$$H(j\omega) = H_o(\omega)\, e^{j\theta(\omega)}$$

where $\quad H_o(\omega) = |H(j\omega)|$, and $\theta(\omega) = \arg H(j\omega)$.

<u>The g-Method of Determining the Periodic Output Waveform</u>

Recalling that $\quad v_a(t) = y_a g(\infty)$

we find the total output $v(t)$, in terms of $g(t)$, to be

$$v(t) = y_a g(\infty) + (\ 3.12a), \quad (m+\rho_\ell)T \le t < (m+\sigma_{\ell+1})T, \quad \ell = o,\ 1, \ldots,\ n-1; \tag{3.28}$$

$$v(t) = y_a g(\infty) + (\ 3.12b), \quad (m+\sigma_\ell)T \le t < (m+\rho_\ell)T, \quad \ell = 1,\ 2, \ldots,\ n \tag{3.29}$$

$$(m = o,\ \pm 1,\ \pm 2, \ldots)\quad .$$

Hence the construction of the periodic state reduces to the superposition of the responses of the linear part of the system to pulses of height $(-1)^i M$ and of duration $(\sigma_i - \rho_{i-1})$, $i=1,\ldots,\ n$, plus the steady component $y_a g(\infty)$.

This method is convenient if $\Delta g_{k,i} \longrightarrow o$ as $k \longrightarrow \infty$, that is, in those cases where the linear part of the system is stable.

<u>The C_1-Method of Determining the Periodic Output Waveform</u>

Let us suppose that the transfer function $H(s)$ is a fractional rational function, i.e.

$$H(s) = \frac{P(s)}{Q(s)} \quad ,$$

and that the degree of the numerator does not exceed that of the denominator. Furthermore, let us assume that H(s) has poles at

$$s_o = o \quad \text{of multiplicity } r_o - 1,$$
$$s_\nu \neq o \quad \text{of multiplicity } r_\nu, \; (\nu = 1, 2, \ldots, p) \; .$$

The sum of the multiplicities of the poles is equal to the degree of the denominator of H(s), i.e.

$$r_o - 1 + r_1 + r_2 + \ldots + r_p = N, \text{ say } .$$

Let us put

$$C_{\nu\mu} = \frac{1}{(r_\nu - \mu - 1)!} \frac{d^{r_\nu - \mu - 1}}{ds^{r_\nu - \mu - 1}} \left[\frac{P(s)}{Q(s)s} (s - s_\nu)^{r_\nu} \right]_{s=s_\nu} \tag{3.30}$$

Recalling that

$$v_a(t) = y_a g(\infty) = y_a H(o) \; ,$$

and using Eqs. (3.30) and (3.25), we get the total output of the linear part of the system in the form

$$v(t) = y_a C_{oo} + M \sum_{\nu=o}^{p} \sum_{\mu=o}^{r_\nu - 1} \frac{C_{\nu\mu}}{\mu!} \frac{d^\mu I(s_\nu) e^{s_\nu t}}{ds_\nu^{\mu}} \tag{3.31}$$

We now evaluate special cases of (3.31).

Suppose that H(s) has only simple poles, all different from zero. Then

$$r_o = r_1 = \ldots = r_N = 1, \; p=N, \; \mu=o \quad ,$$

so that (3.31) becomes

$$v(t) = y_a C_{oo} + M \sum_{\nu=o}^{N} C_{\nu o} I(s_\nu) e^{s_\nu t} \tag{3.32}$$

where

$$C_{oo} = \frac{P(o)}{Q(o)}\ , \text{ and } C_{\nu o} = \frac{P(s_\nu)}{Q'(s_\nu)s_\nu} \quad .$$

Therefore, in the case where y(t) is asymmetric, we have from (3.16), (3.17) and (3.32)

$$v(t) = y_a C_{oo} + M \sum_{\nu=1}^{N} C_{\nu o}\ I_1(s_\nu)\ e^{s_\nu t}\ , \qquad (3.33)$$

$$(\sigma_\ell T \leq t < \rho_\ell T;\ \ell = 1,\ 2, \ldots,\ n),$$

and from (3.18), (3.19) and (3.32)

$$v(t) = (y_a + M(-1)^{\ell})C_{oo} + M \sum_{\nu=1}^{N} C_{\nu o}\ I_2(s_\nu)\ e^{s_\nu t} \qquad (3.34)$$

$$(\rho_\ell T \leq t < \sigma_{\ell+1} T;\ \ell = o,\ 1, \ldots, n-1) \quad .$$

In the simplest case where n = 2, and ρ_1 and σ_2 are absent, i.e. the input has the shape shown in Figure 3.3, we obtain:

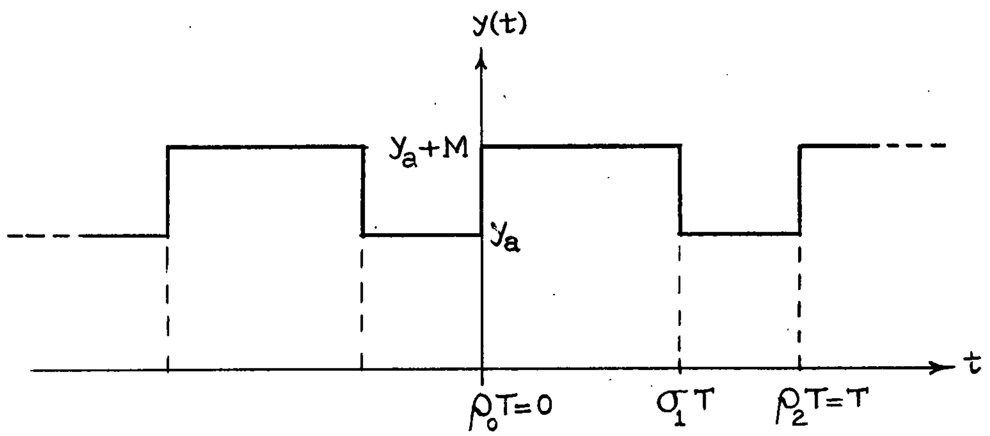


Figure 3.3. Form of y(t) for n = 2, with ρ_1 and σ_2 absent.

$$v(t) = (y_a + M)C_{oo} + M\sum_{\nu=1}^{N} C_{\nu o}\,\frac{1 - e^{s_\nu(1-\sigma_1)T}}{1 - e^{s_\nu T}}\,e^{s_\nu t} \qquad (3.35)$$

$$(o \leq t < \sigma_1 T),$$

$$v(t) = y_a C_{oo} + M\sum_{\nu=1}^{N} C_{\nu o}\,\frac{1 - e^{-s_\nu \sigma_1 T}}{1 - e^{s_\nu T}}\,e^{s_\nu t} \qquad (3.36)$$

$$(\sigma_1 T \leq t < T) \quad .$$

In the other simple case where dead zone is absent and $n = 2$ we have

$$\sigma_1 = \rho_1, \quad \sigma_2 = \rho_2 = 1,$$

so that equation (3.34) reduces to

$$v(t) = (y_a + M)C_{oo} + 2M\sum_{\nu=1}^{N} C_{\nu o}\,\frac{1 - e^{s_\nu(1-\rho_1)T}}{1 - e^{s_\nu T}}\,e^{s_\nu t} \qquad (3.37)$$

$$(o \leq t < \rho_1 T) \quad ,$$

$$v(t) = (y_a - M)C_{oo} + 2M\sum_{\nu=1}^{N} C_{\nu o}\,\frac{1 - e^{-s_\nu \rho_1 T}}{1 - e^{s_\nu T}}\,e^{s_\nu t} \qquad (3.38)$$

$$(\rho_1 T \leq t < T) \quad .$$

Let us now consider the complicated forms of symmetric oscillations, the general formulas of which are given by (3.23a) and (3.23b). Special cases of these follow.

Case 1: $H(s)$ has simple poles all distinct from zero, so that

$$r_o = r_1 = \dots = r_N, \; p = N, \; \mu = o \; .$$

In this case we get

$$v(t) = y_a C_{oo} + M \sum_{\nu=1}^{N} C_{\nu o} \left[\frac{\sum_{i=o}^{\ell-1} \xi_i(s_\nu) - e^{s_\nu \frac{T}{2}} \sum_{i=\ell}^{\frac{n}{2}-1} \xi_i(s_\nu)}{1 + e^{s_\nu \frac{T}{2}}} \right] e^{s_\nu t} \tag{3.39}$$

$$(\sigma_\ell T \leq t < \rho_\ell T;\ \ell = 1, 2, \ldots, \tfrac{n}{2}) ,$$

and

$$v(t) = M(-1)^\ell C_{oo} + M \sum_{\nu=1}^{N} C_{\nu o} (-1)^\ell e^{-s_\nu \rho_\ell T} e^{s_\nu t}$$

$$+ \text{Right-hand side of Eqn. (3.39)} \tag{3.40}$$

$$(\rho_\ell T \leq t < \sigma_{\ell+1} T;\ \ell = o, 1, \ldots, \tfrac{n}{2} - 1) .$$

In the simplest case when $\frac{n}{2} = 1$ (recall that $\frac{n}{2}$ must be an odd number for symmetric oscillations), equations (3.39) and (3.40) reduce to

$$v(t) = y_a C_{oo} + M \sum_{\nu=1}^{N} C_{\nu o} \frac{1 - e^{-s_\nu \sigma_1 T}}{1 + e^{s_\nu \frac{T}{2}}} e^{s_\nu t} \tag{3.41}$$

$$(\sigma_1 T \leq t < \tfrac{1}{2} T = \rho_1 T),$$

and

$$v(t) = (y_a + M) C_{oo} + M \sum_{\nu=1}^{N} C_{\nu o} \frac{1 + e^{s_\nu (\frac{1}{2} - \sigma_1) T}}{1 + e^{s_\nu \frac{T}{2}}} e^{s_\nu t} \tag{3.42}$$

$$(o \leq t < \sigma_1 T),$$

respectively.

Case 2: H(s) has one pole equal to zero, and the other N-1 poles are simple, i.e.

$$H(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{Q_1(s)s} , \quad Q_1(o) \neq o .$$

Then $r_o = 2$, $r_1 = r_2 = \dots = r_{N-1} = 1$, so that from Eq. (3.31) we obtain

$$v(t) = (y_a + I(o))C_{oo} + C_{o1} \left.\frac{dI(s)e^{st}}{ds}\right]_{s=o} + \sum_{\nu=1}^{N-1} C_{\nu o} I(s_\nu) e^{s_\nu t} \tag{3.43}$$

Computing Eq. (3.43) in the case of (3.23a) and (3.23b), i.e. for complicated but symmetric oscillations, we obtain

$$v(t) = y_a C_{oo} + C_{o1} \frac{MT}{2} \left[\sum_{i=o}^{\ell-1} (-1)^i (\sigma_{i+1} - \rho_i) - \sum_{i=\ell}^{\frac{n}{2}-1} (-1)^i (\sigma_{i+1} - \rho_i) \right]$$

$$+ M \sum_{\nu=1}^{N-1} C_{\nu o} \left[\frac{\sum_{i=o}^{\ell-1} \xi_i(s_\nu) - e^{s_\nu \frac{T}{2}} \sum_{i=\ell}^{\frac{n}{2}-1} \xi_i(s_\nu)}{1 + e^{s_\nu \frac{T}{2}}} \right] e^{s_\nu t} \tag{3.44}$$

$$(\sigma_\ell T \leq t < \rho_\ell T; \ \ell = 1, 2, \dots, \tfrac{n}{2}) ,$$

$$v(t) = M \left[(-1)^\ell C_{oo} + C_{o1} (-1)^\ell (t - \rho_\ell T) + \sum_{\nu=1}^{N-1} C_{\nu o} (-1)^\ell e^{-s_\nu \rho_\ell T} e^{s_\nu t} \right] \tag{3.45}$$

$$+ \text{Right-hand side of Eq. (3.44)}$$

$$(\rho_\ell T \leq t < \sigma_{\ell+1} T; \ \ell = o, 1, \dots, \tfrac{n}{2} - 1),$$

where

$$C_{oo} = \frac{d}{ds} \left[\frac{P(s)}{Q_1(s)} \right]_{s=o} , \quad C_{o1} = \frac{P(o)}{Q_1(o)}, \text{ and } C_{\nu o} = \frac{P(s_\nu)}{Q'(s_\nu) s_\nu} . \tag{3.46}$$

Furthermore, if $\frac{n}{2} = 1$, that is we have simple symmetric oscillations, Equations (3.44) and (3.45) reduce to

$$v(t) = y_a C_{oo} + C_{o1} \frac{MT}{2} \sigma_1 + M \sum_{\nu=1}^{N-1} C_{\nu o} \frac{1 - e^{-s_\nu \sigma_1 T}}{1 + e^{s_\nu \frac{T}{2}}} e^{s_\nu t} \quad (3.47)$$

$$(\sigma_1 T \leq t < \tfrac{T}{2}),$$

and

$$v(t) = (y_a + M) C_{oo} + C_{o1} M(t - \tfrac{T}{2}\sigma_1) + M \sum_{\nu=1}^{N-1} C_{\nu o} \frac{1 + e^{s_\nu (\frac{1}{2} - \sigma_1 T)}}{1 + e^{s_\nu \frac{T}{2}}} e^{s_\nu t}$$

$$(o \leq t < \sigma_1 T) \quad . \quad (3.48)$$

Case 3: H(s) has two poles equal to zero, whereas the other N - 2 poles are simple, i.e.

$$H(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{Q_2(s)s^2} , \quad Q_2(o) \neq o \quad .$$

Then $r_o = 3$, $r_1 = r_2 = \dots = r_{N-2} = 1$. Equation (3.31) then becomes

$$\frac{v(t)}{M} = (\frac{y_a}{M} + I(o))C_{oo} + C_{o1} \left.\frac{dI(s)e^{st}}{ds}\right]_{s=o}$$

$$+ \frac{C_{o2}}{2!} \left.\frac{d^2 I(s)e^{st}}{ds^2}\right]_{s=o} + \sum_{\nu=1}^{N-2} C_{\nu o} I(s_\nu)\, e^{s_\nu t} \quad (3.49)$$

The computation of (3.49) in the case of (3.23a) and (3.23b), i.e. for complicated but symmetric oscillations, yields

$$\frac{v(t)}{M} = \frac{y_a}{M} C_{oo} + C_{o1} \frac{T}{2} \left[\sum_{i=o}^{\ell-1} (-1)^i (\sigma_{i+1} - \rho_i) - \sum_{i=\ell}^{\frac{n}{2}-1} (-1)^i (\sigma_{i+1} - \rho_i) \right] +$$

$$+ \frac{C_{o2}}{2} \left(tT \left[\sum_{i=o}^{\ell-1} (-1)^i (\sigma_{i+1} - \rho_i) - \sum_{i=\ell}^{\frac{n}{2}-1} (-1)^i (\sigma_{i+1} - \rho_i) \right] \right.$$

$$+ \left(\frac{T}{2}\right)^2 \left[\sum_{i=o}^{\ell-1} (-1)^i (\rho_i - \sigma_{i+1})(2\rho_i + 2\sigma_{i+1} + 1) \right.$$

$$\left. \left. - \sum_{i=\ell}^{\frac{n}{2}-1} (-1)^i (\rho_i - \sigma_{i+1})(2\rho_i + 2\sigma_{i+1} - 1) \right] \right)$$

$$+ \left(\sum_{\nu=1}^{N-2} C_{\nu o} \left[\sum_{i=o}^{\ell-1} \xi_i(s_\nu) - e^{s_\nu \frac{T}{2}} \sum_{i=\ell}^{\frac{n}{2}-1} \xi_i(s_\nu) \right] e^{s_\nu t} / (1 + e^{s_\nu \frac{T}{2}}) \right)$$

$$(\sigma_\ell T \leq t < \rho_\ell T; \ \ell = 1, 2, \ldots, \tfrac{n}{2}) , \qquad (3.50)$$

whereas

$$\frac{v(t)}{M} = (-1)^\ell \left[C_{oo} + C_{o1}(t - \rho_\ell T) + \frac{C_{o2}}{2}(t - \rho_\ell T)^2 + \sum_{\nu=1}^{N-2} C_{\nu o} e^{-s_\nu \rho_\ell T} e^{s_\nu t} \right]$$

$$+ \text{Right-hand side of Eq. (3.50)} \qquad (3.51)$$

$$(\rho_\ell T \leq t < \sigma_{\ell+1} T; \ \ell = o, 1, \ldots, \tfrac{n}{2} - 1),$$

where

$$C_{oo} = \frac{1}{2} \frac{d^2}{ds^2} \left[\frac{P(s)}{Q_2(s)} \right]_{s=o} , \quad C_{o1} = \frac{d}{ds} \left[\frac{P(s)}{Q_2(s)} \right]_{s=o} ,$$

$$(3.52)$$

$$C_{o2} = \frac{P(o)}{Q_2(o)} , \qquad \text{and} \qquad C_{\nu o} = \frac{P(s_\nu)}{Q'(s_\nu) s_\nu}$$

In the case of simple symmetric oscillations, i.e. $\frac{n}{2} = 1$, $\rho_{\frac{n}{2}} T = \rho_1 T = \frac{T}{2}$, equations (3.50) and (3.51) reduce to

$$\frac{v(t)}{M} = \frac{y_a}{M} C_{oo} + C_{o1} \frac{T}{2}\sigma_1 + \frac{C_{o2}}{2} \frac{T}{2}\sigma_1\left[2t - \frac{T}{2}(2\sigma_1 + 1)\right]$$

$$+ \sum_{\nu=1}^{N-2} C_{\nu o} \frac{1 - e^{-s_\nu \sigma_1 T}}{1 + e^{s_\nu \frac{T}{2}}} e^{s_\nu t} , \qquad (3.53)$$

$$(\sigma_1 T \leq t < \tfrac{T}{2}) ,$$

and

$$\frac{v(t)}{M} = (\frac{y_a}{M} + 1)C_{oo} + C_{o1}(t - \frac{T}{2}\sigma_1) + \frac{C_{o2}}{2}\left[t^2 - tT\sigma_1 - (\tfrac{T}{2})^2\sigma_1(1-2\sigma_1)\right]$$

$$+ \sum_{\nu=1}^{N-2} C_{\nu o} \frac{1 + e^{s_\nu(\frac{1}{2} - \sigma_1)T}}{1 + e^{s_\nu \frac{T}{2}}} e^{s_\nu t} \qquad (3.54)$$

$$(o \leq t < \sigma_1 T) .$$

Cases 1, 2 and 3 dealt with above are the ones usually encountered in practice. Other cases may be similarly evaluated by an application of equation (3.31).

The C_2-Method (or Frequency Response Method) of Determining the Periodic Output Waveform

Here we apply formula (3.27) to equations (3.14a) and (3.14b). The poles of $I_1(s)$ and $I_2(s)$, given by equations (3.17) and (3.19), are the same, and occur at

$$s = j\,\frac{2k\pi}{T} = jk\omega, \quad (k = o, \pm 1, \pm 2, \ldots; \ \omega = \frac{2\pi}{T}) .$$

Consequently,

$$v_1(t) = -M \sum_{k=-\infty}^{+\infty} \frac{H(jk\omega)}{jk\,\frac{2\pi}{T}} \left[\left[\frac{\sum_{i=o}^{\ell-1} \xi_i + e^{sT} \sum_{i=\ell}^{n-1} \xi_i}{-T\, e^{sT}} \right] e^{st} \right]_{s=j\frac{2k\pi}{T}} .$$

Now

$$\left[\frac{\sum_{i=o}^{\ell-1} \xi_i + e^{sT} \sum_{i=\ell}^{n-1} \xi_i}{-T\ e^{sT}}\right]_{s=jk\frac{2\pi}{T}} = -\frac{1}{T} \sum_{i=o}^{n-1} (-1)^i (e^{-jk2\pi\rho_i} - e^{-jk2\pi\sigma_{i+1}}).$$

Let us put

$$\frac{M}{jk\pi} \sum_{i=o}^{n-1} (e^{-jk2\pi\rho_i} - e^{-jk2\pi\sigma_{i+1}}) = C_k = |C_k|\ e^{-j\phi_k}, \tag{3.55}$$

and substitute

$$H(j\omega) = H_o(\omega)\ e^{j\theta(\omega)} \tag{3.56}$$

where

$$H_o(\omega) = |H(j\omega)|, \text{ and } \theta(\omega) = \arg H(j\omega) \ .$$

Then

$$v_1(t) = \sum_{k=-\infty}^{+\infty} \frac{|C_k|}{2} H_o(k\omega) e^{j\left[k\omega t - \phi_k + \theta(k\omega)\right]},$$

which can be rewritten as

$$v_1(t) = \frac{1}{2} C_o H_o(o) + \sum_{k=1}^{\infty} |C_k|\ H_o(k\omega) \cos\left[k\omega t - \phi_k + \theta(k\omega)\right]. \tag{3.57}$$

If $v_1(t)$ has the additional property of symmetry, then from equation (3.22)

$$\sum_{i=o}^{\ell-1} \xi_i + e^{sT} \sum_{i=\ell}^{n-1} \xi_i = \left[\sum_{i=o}^{\ell-1} \xi_i - e^{s\frac{T}{2}} \sum_{i=\ell}^{\frac{n}{2}-1} \xi_i\right] (1 - e^{s\frac{T}{2}}),$$

so that the poles at

$$s = j\frac{2k\pi}{T}, \quad k = o, \pm 2, \pm 4, \ldots$$

are eliminated. Hence, in the case of symmetric oscillations $v_1(t)$ becomes

$$v_1(t) = \sum_{k=1}^{\infty}{}' |C_k| \; H_o(k\omega) \cos\left[k\omega t - \phi_k + \theta(k\omega)\right] \tag{3.58}$$

where $\sum'$ means the summation with respect to odd numbers only. Also C_k is now given by

$$C_k = \frac{M}{jk\pi} \sum_{i=o}^{\frac{n}{2}-1} (-1)^i \left(e^{-jk2\pi\rho_i} - e^{-jk2\pi\sigma_{i+1}}\right) \quad .$$

Equation (3.58) may be conveniently rewritten as

$$v_1(t) = \sum_{k=1}^{\infty} |C_{2k-1}| \; H_o((2k-1)\omega) \cos\left[(2k-1)\omega t - \phi_{2k-1} + \theta((2k-1)\omega)\right] \tag{3.59}$$

4. CONCEPTS PERTAINING TO THE STEADY-STATE RESPONSE OF ON-OFF ELEMENTS

Before proceeding to the study of self and forced oscillations in on-off feedback control systems, we will first introduce concepts pertaining to the steady-state response of such systems. In this respect, the Hamel and Tsypkin loci (or hodograph, or characteristic)[5,6] have been formulated to facilitate the solutions of periodic oscillations in single-loop systems containing one on-off element. Furthermore, Neimark[9] used the concept of the phase characteristic to determine the simple symmetric self-oscillations in a single-loop system containing an arbitrary number of on-off elements, but no mention was made as to how it may be adapted to the problem of forced oscillations.

In this chapter we redefine the above-mentioned concepts in order (i) to include the effects of initial conditions and of external influences, (ii) to show the relationships existing among these concepts, but moreso (iii) to extend their sphere of application to the solution of the possible periodic motions in multi-loop control systems, containing an arbitrary number of on-off elements.

For this purpose it will be convenient to regard any given system as a composition of simple unit systems, or sub-systems, shown in Figure 4.1(a), the characteristics of which can be readily ascertained. Let us assume that the characteristic of the on-off element in Figure 4.1(a) is symmetric with hysteresis and dead zone, as depicted in Figure 4.1(b). The initial conditions are referred to the output of the linear part and are designated by $v_0(t)$,

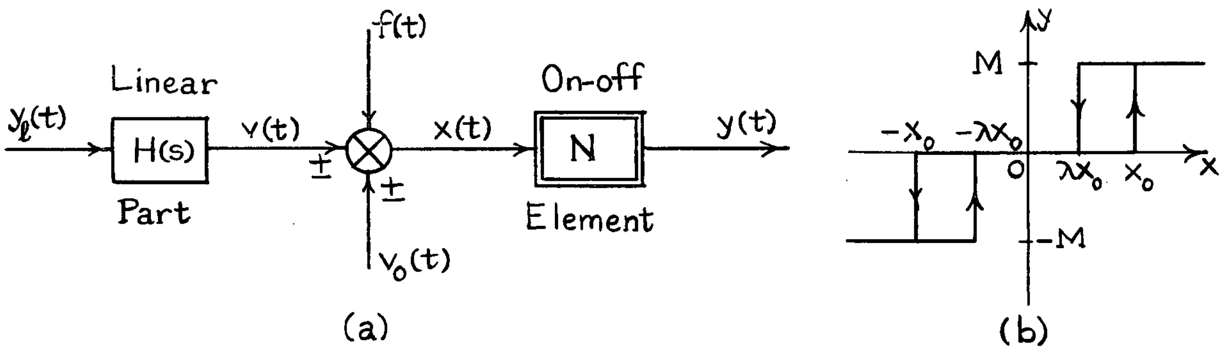


Figure 4.1(a) Block diagram of unit system

(b) Characteristic of on-off element

whereas f(t) accounts for any external action.

Let the input to the linear part of the system be a steady periodic waveform of symmetric rectangular pulses as shown in Figure 4.2(a). Then the output v(t) of the linear part will also be a periodic waveform with the same periodicity as the input $y_\ell(t)$.

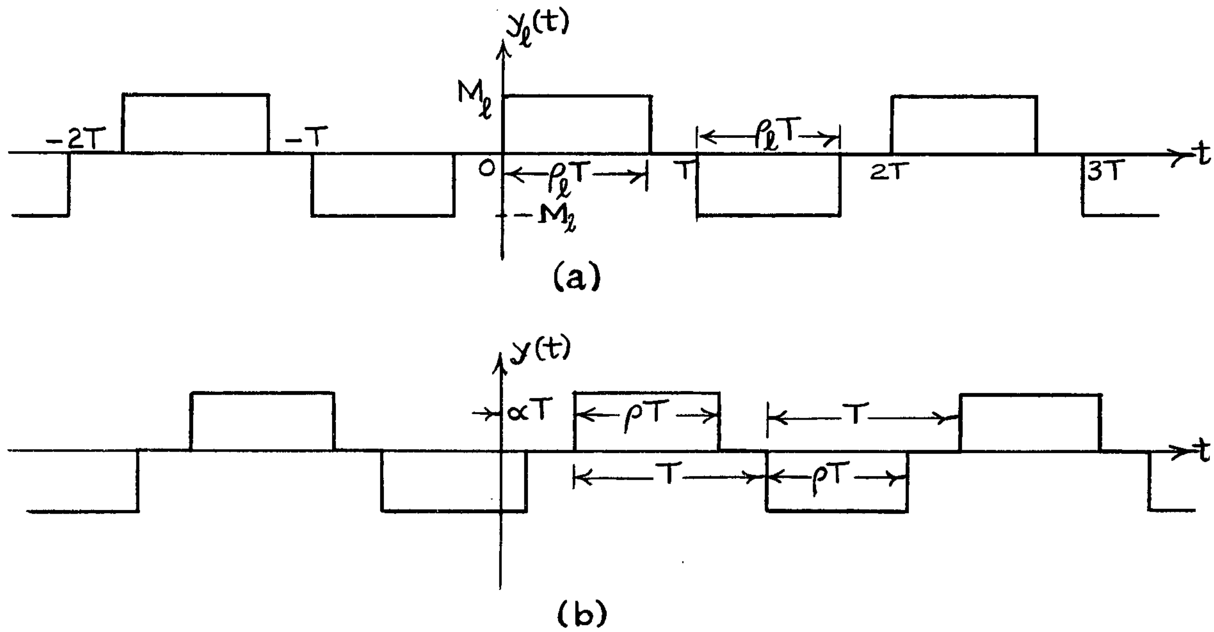


Figure 4.2(a) Input to linear part of Fig. 4.1(a),

(b) Output of on-off element of Fig. 4.1(a).

In fact

$$v(t) = \begin{cases} \dfrac{M}{2\pi j} \displaystyle\oint_{C_1 \text{ or } C_2} \dfrac{H(s)}{s} \dfrac{1 + e^{s(1-\rho_\ell)T}}{1 + e^{sT}} e^{st} ds, & (o \leq t < \rho_\ell T) \\ \\ \dfrac{M}{2\pi j} \displaystyle\oint_{C_1 \text{ or } C_2} \dfrac{H(s)}{s} \dfrac{1 - e^{-s\rho_\ell T}}{1 + e^{sT}} e^{st} ds, & (\rho_\ell T \leq t < T) \end{cases} \tag{4.1}$$

where C_1 is a contour enclosing only the poles of $H(s)/s$, where C_2 is a contour enclosing only the poles of $1/(1+e^{sT})$, and where the contour integrals along C_1 and C_2 are taken in a mathematically positive and negative sense respectively.

Now the input $x(t)$ to the on-off element is given by

$$x(t) = f(t) \pm v(t) \pm v_o(t) \tag{4.2}$$

In the case of simple symmetric periodic responses, that is $y(t+T) = -y(t)$, the only switching conditions are

$$x\left[(\alpha+k)T\right] = (-1)^k x_o = \frac{1}{\lambda}\, x\left[(\alpha+\rho+k)T\right] \tag{4.3}$$

$$\dot{x}\left[(\alpha+k)T\right](-1)^k > o > \dot{x}\left[(\alpha+\rho+k)T\right](-1)^k \tag{4.4}$$

$$(k=o,\ \pm 1,\ \pm 2,\ \ldots)$$

where α is taken as ≥ 0 and $0<\rho\leq 1$. Consequently, the output of the on-off element is also periodic with half period T ; it has a pulse duration ρT which is in general different from the pulse duration $\rho_\ell T$ of the input $y_\ell(t)$; and it is shifted to the right by an amount αT. The condition expressed by Eq.(4.3) is referred to as the condition for the proper switching instants, whereas that given by Eq.(4.4) is the condition for the proper direction of switching.

If a dead zone is absent then we put $\lambda = -1$, $\rho = 1$ so that the switching conditions reduce simply to

$$\left.\begin{aligned} x\left[(\alpha + k)T\right] &= (-1)^k x_o \qquad (4.5) \\ \dot{x}\left[(\alpha + k)T\right] & (-1)^k > o \qquad (4.6) \end{aligned}\right\} \quad (k = o, \pm 1, \pm 2, \ldots)$$

Furthermore, if hysteresis is absent then x_o is set equal to zero.

4.1 GENERALIZED CONCEPTS OF THE HAMEL AND TSYPKIN LOCI

From the above we note that the quantities $x(\alpha T)$ and $\dot{x}(\alpha T)$, together with $x\left[(\alpha + \rho)T\right]$ and $\dot{x}\left[(\alpha + \rho)T\right]$ in the presence of a dead zone, completely charaterize the parameters $\frac{\pi}{T} = \omega$, the frequency of the periodic response, ρ the relative pulse duration, and α the shift to the right relative to $y_{\ell}(t)$ of the output of the unit system. Hence we are led to the following concepts of a "characteristic" of a unit system of the type shown in Figure 4.1:

1. Generalized Hamel Loci. The generalized Hamel Loci are defined by

$$\mathcal{H}(\alpha,\omega) = x(\alpha\tfrac{\pi}{\omega}) + j\,\dot{x}(\alpha\tfrac{\pi}{\omega}) \qquad (4.7a)$$

and

$$\mathcal{H}(\alpha,\rho,\omega) = x\left[(\alpha + \rho)\tfrac{\pi}{\omega}\right] + j\,\dot{x}\left[(\alpha + \rho)\tfrac{\pi}{\omega}\right] \qquad (4.7b)$$

2. Generalized Tsypkin Loci. The generalized Tsypkin Loci are defined by

$$\mathcal{J}(\alpha,\omega) = \frac{1}{\omega}\,\dot{x}(\alpha\tfrac{\pi}{\omega}) + j\,x(\alpha\tfrac{\pi}{\omega}) \qquad (4.8a)$$

and

$$\mathcal{J}(\alpha,\rho,\omega) = \frac{1}{\omega}\dot{x}\left[(\alpha + \rho)\tfrac{\pi}{\omega}\right] + jx\left[(\alpha + \rho)\tfrac{\pi}{\omega}\right] \qquad (4.8b)$$

where $\mathcal{H}(\alpha,\rho,\omega)$ and $\mathcal{J}(\alpha,\rho,\omega)$ are required in addition to $\mathcal{H}(\alpha,\omega)$ and $\mathcal{J}(\alpha,\omega)$ in the case of a dead zone. It is interesting to note that for a given ω as α varies from 0 to 1, the quantity $\mathrm{Im}\,\mathcal{J}(\alpha,\omega)$ or $\mathrm{Re}\,\mathcal{H}(\alpha,\omega)$ determines the periodic waveform $x(t)$, since t in $x(t)$ takes on all values between 0 and T; similarly, the quantity $\mathrm{Re}\,\mathcal{J}(\alpha,\omega)$ weighted by the factor $1/\omega$ or $\mathrm{Im}\,\mathcal{H}(\alpha,\omega)$ determines the derivative $\dot{x}(t)$.

The Hamel and Tsypkin loci are convenient graphical representations of the input signal conditions at the switching instants. They are therefore useful in the study of periodic phenomena in on-off systems.

Sketches of the general form of the Hamel and Tsypkin loci are shown in Fig. 4.3.

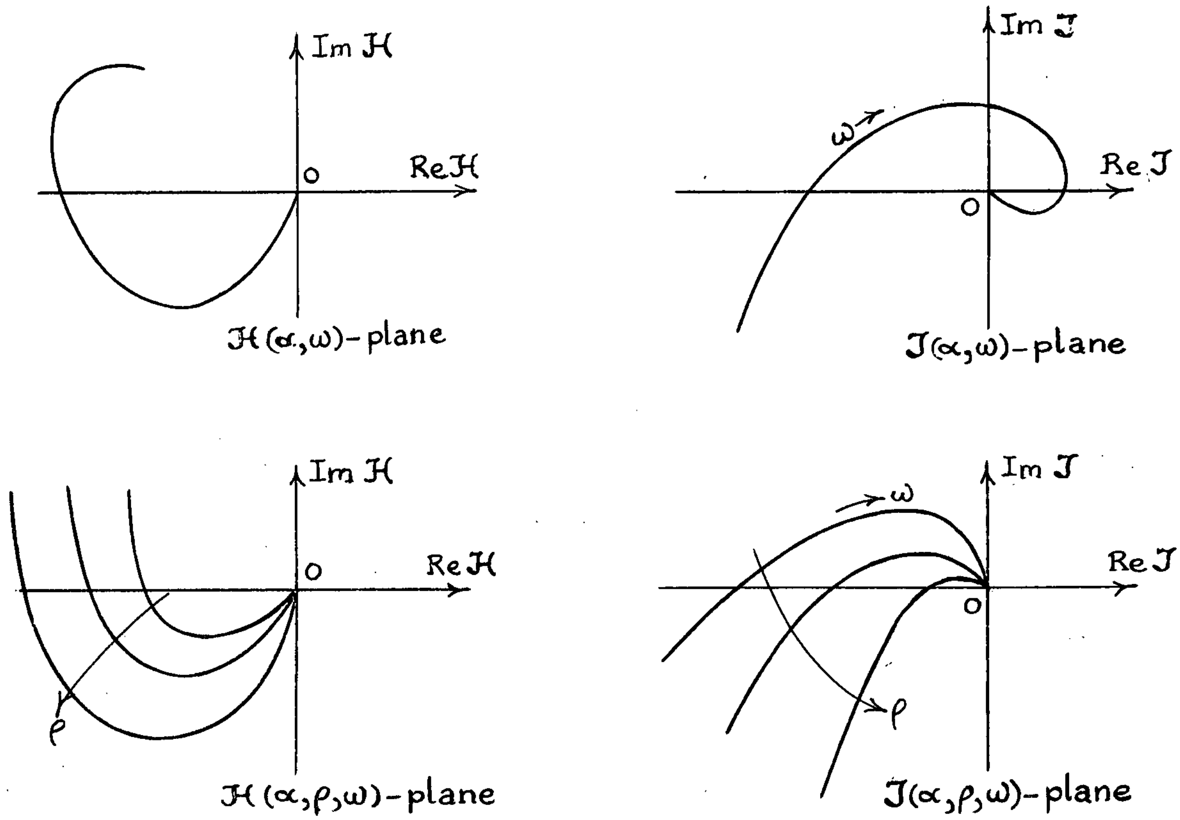


Figure 4.3. Sketches of general form of the Hamel and Tsypkin Loci.

Quite obviously, the Hamel and Tsypkin loci are equivalent except that Hamel's $\dot{x}$ is replaced by $\frac{1}{\omega}\dot{x}$ in the case of Tsypkin and that the coordinates are interchanged.

Hamel's characteristic is advantageous from the point of view that (i) it uses the phase-plane variables x and $\dot{x}$ which describe the system's behaviour, and (ii) a derivative control introduced into the system is very easily studied. On the other hand, the Tsypkin representation is generally very close to the transfer locus $H(j\omega)$ in the high frequency region.

4.2 CONCEPT OF THE PHASE CHARACTERISTIC

In the preceding section we observed that the output has the same general features as the input $y_\ell(t)$. In fact, it has the same periodicity, but it is shifted to the right by an amount αT as shown in Figure 4.2. The curve αT vs T will be referred to as the phase characteristic of the unit system. To emphasize the fact that αT is a function of T, we will denote it by $\Theta(T)$.

Clearly the instant $\Theta(T)$ of switching from $-M$ to $+M$ that is closest to the instant $t = 0$ is a non-negative root of the equation

$$x(t) = x_o \tag{4.9}$$

Obviously, the phase characteristic represents the information concerning the switching instants given by the intersection of the Hamel loci with the straight line x_o, or, alternatively, by the intersection of the Tsypkin loci with the straight line jx_o.

The Hamel and Tsypkin loci are very convenient concepts in the study of the single-loop system containing one on-off element, but are very cumbersome in the case of single or multiloop systems with more than one on-off element. It will be seen later that the phase characteristic is better suited for determining the periodic modes of oscillations in multiloop systems containing an arbitrary number of on-off elements. The investigation is considerably simplified in those cases where an analytic expression for the phase characteristic is available.

In the case of on-off elements with dead zone it is necessary to know ρT, the duration of the output pulse corresponding to a fixed input pulse duration $\rho_\ell T$. Consequently, in such cases the concept of the pulse duration characteristic, which is a curve of ρT vs T with ρ_ℓ as the parameter, has to be introduced.

We now proceed to the computation of the phase characteristic $\Theta(T)$ for a few simple systems, in which a dead zone is absent. We first list formulas for v(t), the output of the linear part of the system, for commonly encountered special cases of H(s):

Case 1: H(s) has simple poles, all distinct from zero. Then

$$v(t) = 2M_\ell\left[\frac{C_{oo}}{2} + \sum_{\nu=1}^{N} C_{\nu o}\,\frac{e^{s_\nu t}}{1+e^{s_\nu T}}\right] \qquad (4.10)$$

$$(o \leq t < T)$$

where

$$C_{oo} = \frac{P(o)}{Q(o)}\ , \text{ and } C_{\nu o} = \frac{P(s_\nu)}{Q'(s_\nu)s_\nu}\ .$$

Case 2: H(s) has one pole at the origin, and the remaining N-1 poles are simple, that is,

$$H(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{s\,Q_1(s)} \ .$$

Then

$$v(t) = M_\ell\left[C_{oo} + C_{o1}\left(t - \frac{T}{2}\right) + 2\sum_{\nu=1}^{N-1} C_{\nu o}\,\frac{e^{s_\nu t}}{1+e^{s_\nu T}}\right] \qquad (4.11)$$

$$(o \leq t < T)$$

where

$$C_{oo} = \frac{d}{ds}\left[\frac{P(s)}{Q_1(s)}\right]_{s=o},\quad C_{o1} = \frac{P(o)}{Q_1(o)},\ \text{and}\ C_{\nu o} = \frac{P(s_\nu)}{Q'(s_\nu)s_\nu} \ .$$

<u>Case 3</u>: H(s) has a second order pole at the origin, and the remaining N − 2 poles are simple, i.e.

$$H(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{s^2 Q_2(s)} \ .$$

Then

$$v(t) = M_\ell\left[C_{oo} + C_{o1}\left(t - \frac{T}{2}\right) + C_{o2}t(t-T)\right.$$

$$\left. + \sum_{\nu=1}^{N-2} C_{\nu o}\,\frac{2\,e^{s_\nu t}}{1+e^{s_\nu T}}\right] \qquad (4.12)$$

$$(o \leq t < T)$$

where

$$C_{oo} = \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{P(s)}{Q_2(s)}\right]_{s=o},\quad C_{o1} = \frac{d}{ds}\left[\frac{P(s)}{Q_2(s)}\right]_{s=o},\quad C_{o2} = \frac{P(o)}{Q_2(o)} \ ,$$

and

$$C_{\nu o} = \frac{P(s_\nu)}{Q'(s_\nu)s_\nu} .$$

Case 4: H(s) has a second order pole at $s_1 (\neq o)$, and the remaining N − 2 poles are simple and distinct from zero, i.e.

$$H(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s-s_1)^2 Q_3(s)}$$

Then

$$v(t) = M_\ell \Big[C_{oo} + (C_{1o} + C_{11}t - C_{11}\frac{T\,e^{s_1 T}}{1+e^{s_1 T}}) \frac{2e^{s_1 t}}{1+e^{s_1 T}}$$

$$+ \sum_{\nu=2}^{N-1} C_{\nu o} \frac{2\,e^{s_\nu t}}{1+e^{s_\nu T}} \Big] \qquad (4.13)$$

$$(o \leq t < T)$$

where

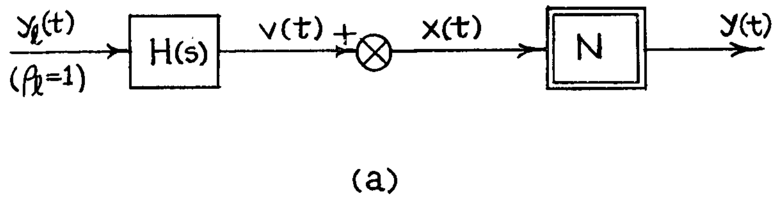
$$C_{oo} = \frac{P(o)}{Q(o)} ,\ C_{1o} = \frac{d}{ds}\left[\frac{P(s)}{s\ Q_3(s)}\right]_{s=s_1} ,\ C_{11} = \frac{P(s_1)}{s_1 Q_3(s_1)} ,$$

$$C_{\nu o} = \frac{P(s_\nu)}{s_\nu Q'(s_\nu)} \ (\nu = 2,\ldots,N-1)$$

We now turn our attention to the computation of the phase characteristic Θ(T) for a few systems.

System I: x (t) = +v(t): hysteresis and dead zone absent in N

This system is shown in Figures 4-4 (a) & (b).



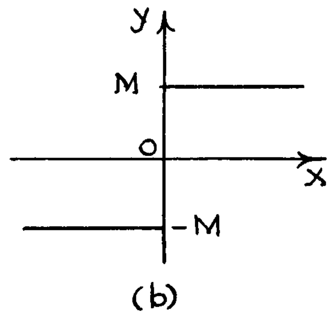


Figure 4.4 (a) Block diagram of System I: $x(t) = v(t)$

(b) Characteristic of N in Fig. 4.4(a).

Let us consider the following representations for H(s).

(1) $H(s) = \frac{1}{s}$: We use Eq. (4.11). Here $\frac{P(s)}{Q_1(s)} = 1$, so that

$$C_{oo} = o,\ C_{o1} = 1,\ C_{\nu o} = o \text{ (all } \nu)$$

Hence $x(t) = M_\ell(t - \frac{T}{2})$.

Setting $x(t) = x_o = o$ we get the phase characteristic

$$\theta(T) = \frac{T}{2}$$

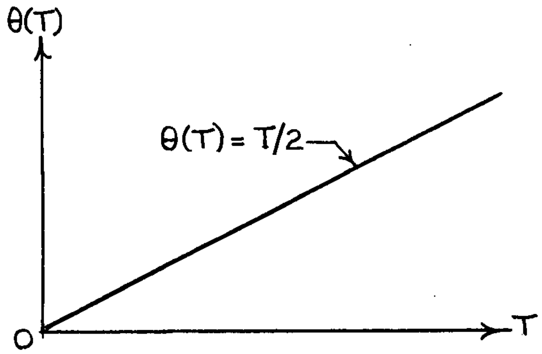


Figure 4.5. Phase characteristic for $H(s) = 1/s$.

(2) $H(s) = 1/s^2$: We use Eq. (4.12). The only non-zero coefficient is C_{o2} which is equal to 1.

Hence $x(t) = \frac{M_\ell}{2}(t-T)t$,

$(o \leq t < T)$

Thus

$\theta(T) = T$

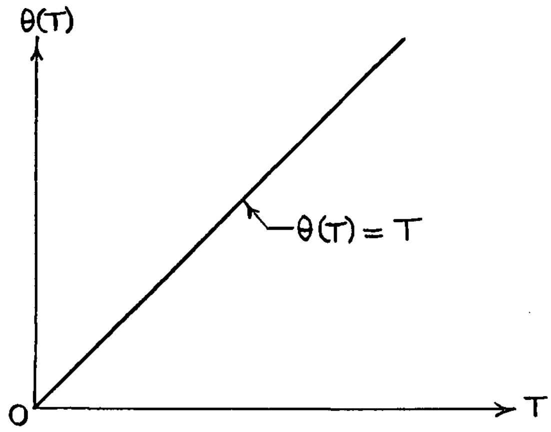


Figure 4.6. Phase characteristic for $H(s) = 1/s^2$

(3) $H(s) = 1/(\tau s+1)$: We use Eq. (4.10). Here $C_{oo} = 1$, $C_{10} = -1$, $s_1 = -1/\tau$.

Therefore

$$x(t) = M_\ell \left[1 - \frac{2e^{-t/\tau}}{1+e^{-T/\tau}}\right]$$

$(o \leq t < T)$

Setting $x(t) = o$ we get

$$\theta(T) = \tau \ln \frac{2}{1+e^{-T/\tau}}$$

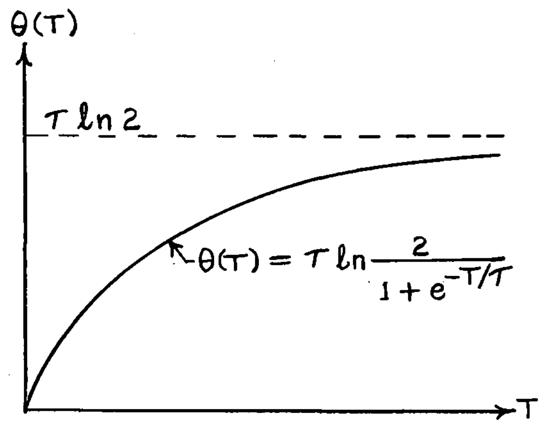


Figure 4.7. Phase characteristic for $H(s) = 1/(\tau s+1)$

(4) $H(s) = 1/(\tau s-1)$: Referring to case (3) above we simply replace τ by $-\tau$ and $H(s)$ by $-H(s)$ to get

$$x(t) = M_\ell \left[\frac{2e^{t/\tau}}{1+e^{T/\tau}} - 1\right],$$

$(o \leq t < T)$.

Therefore

$$\theta(T) = \tau \ln \frac{1+e^{T/\tau}}{2} \ .$$

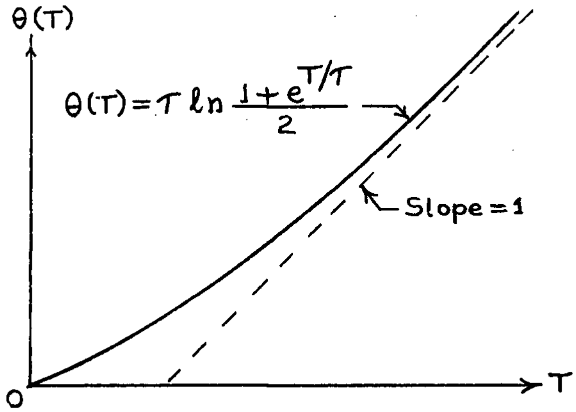


Figure 4.8. Phase characteristic for $H(s) = 1/(\tau s-1)$

(5) $\underline{H(s) = 1/[s(s+\alpha)]}$: We use Eq. (4.11). Here

$$C_{oo} = -\frac{1}{\alpha^2}\ ,\ C_{o1} = \frac{1}{\alpha}\ ,\ C_{1o} = \frac{1}{\alpha^2}\ ,\ s_1 = -\alpha \ .$$

Therefore

$$x(t) = M_\ell \left[-\frac{1}{\alpha^2} + \frac{1}{\alpha}\left(t - \frac{T}{2}\right) + \frac{1}{\alpha^2}\frac{2e^{-\alpha t}}{1+e^{-\alpha T}} \right]$$

$$(o \leq t < T)$$

No analytic expression can be found for $\theta(T)$.

But given α, we can solve for $\theta(T)$ graphically or numerically.

(6) $\underline{H(s) = \dfrac{1}{(s+\alpha)(s+\beta)}}$:

(i) Suppose $\alpha \neq \beta$, $\alpha \neq o$, $\beta \neq o$. Using Eq. (4.10) we get

$$x(t) = M_\ell \left[\frac{1}{\alpha\beta} + \frac{2}{\alpha-\beta}\left(\frac{1}{\alpha}\frac{e^{-\alpha t}}{1+e^{-\alpha T}} - \frac{1}{\beta}\frac{e^{-\beta t}}{1+e^{-\beta T}} \right) \right]$$

$$(o \leq t < T) \ .$$

(ii) Suppose $\alpha = \beta \neq o$. Then, using Eq. (4.13), we get

$$x(t) = M_{\ell}\left[\frac{1}{\alpha^2} + \left(-\frac{1}{\alpha^2} - \frac{t}{\alpha} + \frac{T}{\alpha}\,\frac{e^{-\alpha T}}{1+e^{-\alpha T}}\right)\frac{2e^{-\alpha t}}{1+e^{-\alpha T}}\right]$$

$$(o \leq t < T)$$

No analytic expression is available for $\theta(T)$ for this case. But, given α and β, we can solve for $\theta(T)$ either graphically or numerically.

(7) $H(s) = \frac{s}{(s+\alpha)(s+\beta)}$:

(i) Suppose $\alpha = \beta \neq o$. Then, using Eq. (4.13) we get

$$x(t) = M_{\ell}\left(t - \frac{Te^{-\alpha T}}{1+e^{-\alpha T}}\right)\frac{2e^{-\alpha t}}{1+e^{-\alpha T}}$$

$$(o \leq t < T).$$

Hence

$$\theta(T) = \frac{Te^{-\alpha T}}{1+e^{-\alpha T}}$$

$$= \frac{T}{1+e^{\alpha T}}\ .$$

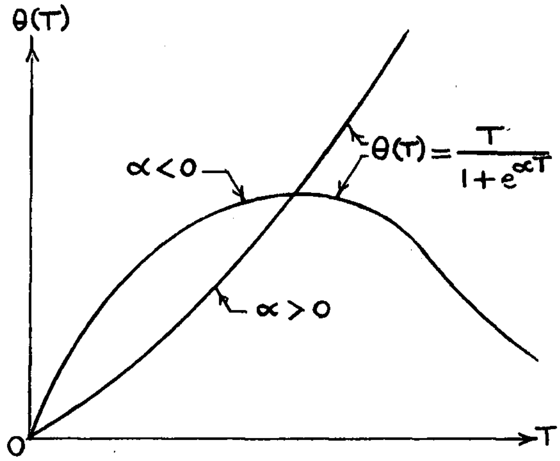


Figure 4.9. Phase characteristic for $H(s) = \frac{s}{(s+\alpha)^2}$, α and β pure reals, $\alpha = \beta \neq 0$.

(ii) Suppose $\alpha \neq \beta$, $\alpha \neq o$, $\beta \neq o$, & α and β pure reals. Then, using Eq. (4.10), we obtain

$$x(t) = \frac{2M_{\ell}}{\alpha - \beta}\left[\frac{e^{-\beta t}}{1+e^{-\beta T}} - \frac{e^{-\alpha t}}{1+e^{-\alpha T}}\right],$$

$$(o \leq t < T).$$

Putting $x(t) = o$ we obtain

$$\theta(T) = \frac{1}{\alpha - \beta} \ln \frac{1 + e^{-\beta T}}{1 + e^{-\alpha T}} ,$$

which may be rewritten as

$$\theta(T) = \frac{2}{\alpha - \beta} \tanh^{-1}\left[\sinh\frac{\alpha-\beta}{2}T \times \frac{1}{e^{\frac{\alpha + \beta}{2}T} + \cosh \frac{\alpha - \beta}{2}T}\right]$$

This phase characteristic is plotted in Fig. 4.10 for the case $\alpha > \beta > o$.

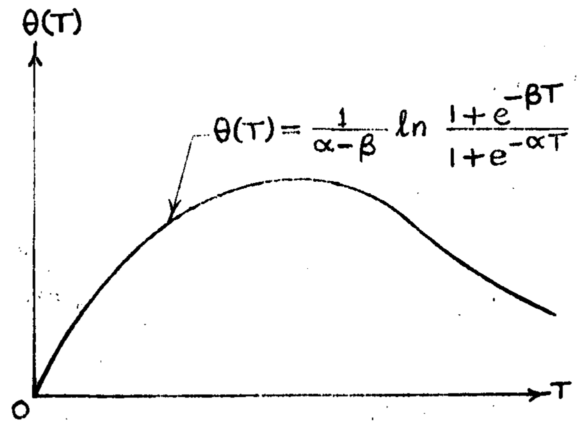


Figure 4.10. Phase characteristic for $H(s) = \frac{s}{(s + \alpha)(s + \beta)}$ where α, β are reals, $\alpha \neq \beta$, $\alpha > \beta > o$.

(iii) On the other hand, if α and β are complex then they are complex conjugates. Let

$\alpha = a+jb$, then $\beta = a-jb$,

and

$$\frac{\alpha - \beta}{2} = jb, \quad \frac{\alpha + \beta}{2} = a.$$

In this case we get

$$\theta(T) = \frac{1}{b}\tan^{-1}\left[\frac{\sin bT}{e^{\alpha T} + \cos bT}\right].$$

Fig. 4.11 shows a sketch of this phase characteristic.

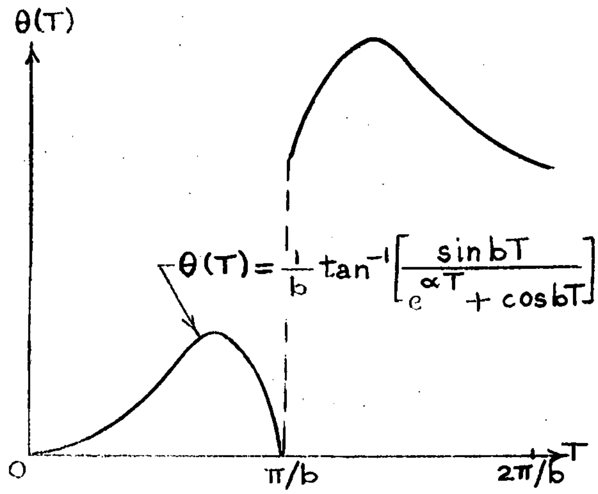


Figure 4.11. Phase characteristic for $H(s) = \frac{s}{(s + \alpha)(s + \beta)}$ where α and β are complex conjugates.

(8) $H(s) = e^{-s\mathcal{T}}$

Obviously $x(t) = y_\ell(t - \mathcal{T})$

Hence the phase characteristic is given by

$$\theta(T) = \mathcal{T} - \left[\!\left[\frac{\mathcal{T}}{2T}\right]\!\right] 2T$$

where $\left[\!\left[\frac{\mathcal{T}}{2T}\right]\!\right]$ denotes the integral part of $\mathcal{T}/(2T)$.

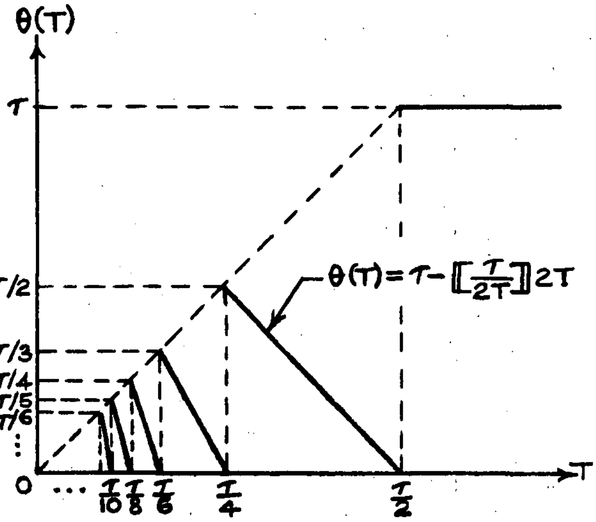


Figure 4.12. Phase Characteristic for $H(s) = e^{-s\mathcal{T}}$.

System II: $x(t) = -v(t)$; hysteresis and dead zone absent in N.

This system is shown in Figure 4.13.

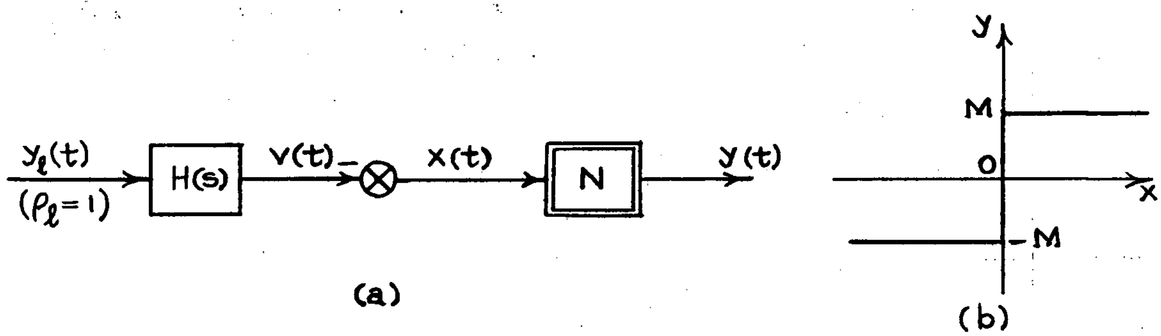


Figure 4.13. (a) Block diagram of system II.

(b) Characteristic of N in Fig. 4.13(a).

Let $\theta_I(T)$ be the phase characteristic of system I.

Let $\theta_{II}(T)$ be the phase characteristic of system II, corresponding to system I, i.e. same H(s) and same N but with the change $x(t) = -v(t)$. Then, in terms of the phase characteristic of system I, $\theta_I(T)$, the phase characteristic of system II is given by

$$\theta_{II}(T) = \theta_I(T) + T - \left[\!\left[\frac{\theta_I(T) + T}{2T}\right]\!\right] 2T , \tag{4.14}$$

where $[\![\ \ ]\!]$ denotes the integral part of its argument.

As illustrations consider the following cases:

(1) $H(s) = 1/s$: We obtained $\theta_I(T) = T/2$. Therefore, by Eq. (4.14),

$$\theta_{II}(T) = 3T/2$$

(2) $H(s) = 1/s^2$: In this case $\theta_I(T) = T$, so that

$$\theta_{II}(T) = 2T - \left[\!\!\left[\frac{2T}{2T}\right]\!\!\right] 2T = o.$$

<u>System III: $x(t) = +v(t)$; N has hysteresis, but no dead-zone.</u>

This system is shown in Figure 4.14.

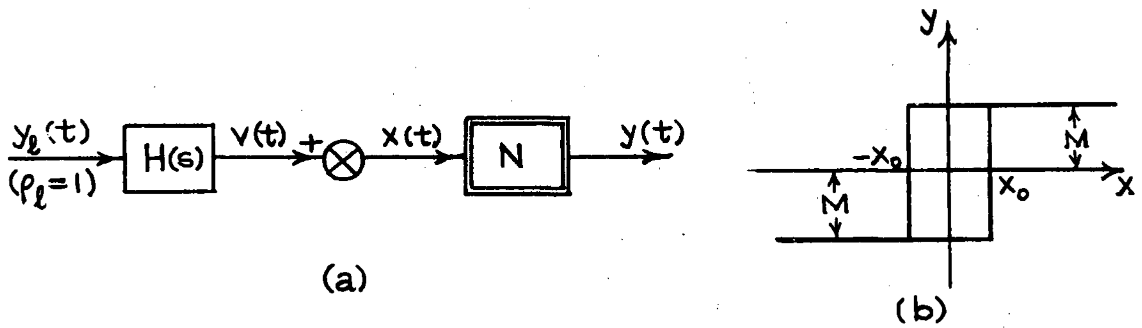


Figure 4.14. (a) Block diagram of System III; (b) characteristic of N.

For this particular system, the phase characteristic is found as the least positive root of the equation

$$v(t) = x_o.$$

We now compute $\theta(T)$ for the cases of H(s) considered in connection with system I.

(1) $H(s) = \frac{1}{s}$

Putting $v(t) = M_\ell(t - \frac{T}{2}) = x_o$

we get

$$\theta(T) = \frac{T}{2} + \frac{x_o}{M_\ell}$$

where it is understood that

$$x_o < M_\ell \frac{T}{2}$$

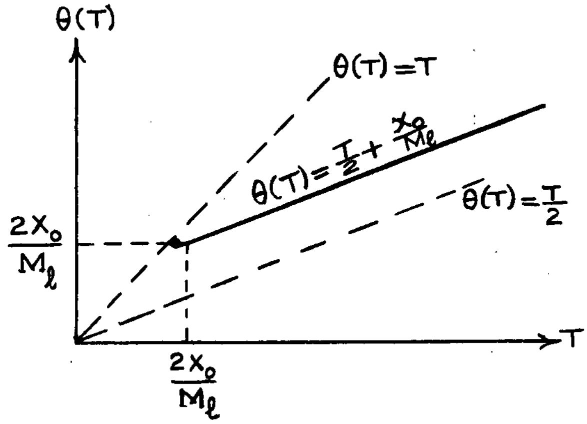


Figure 4.15. Phase characteristic for $H(s) = 1/s$.

(2) $H(s) = \frac{1}{s^2}$

In this instance we have

$$x(t) = \frac{M_\ell}{2} t (t - T) = -x_o$$

Provided that $x_o < x(t)_{max}$

i.e. $x_o < M_\ell \frac{T^2}{8}$

commutations will occur.

The phase characteristic is given by

$$\theta(T) = \frac{3T - \left[T^2 - \frac{8x_o}{M_\ell}\right]^{1/2}}{2}$$

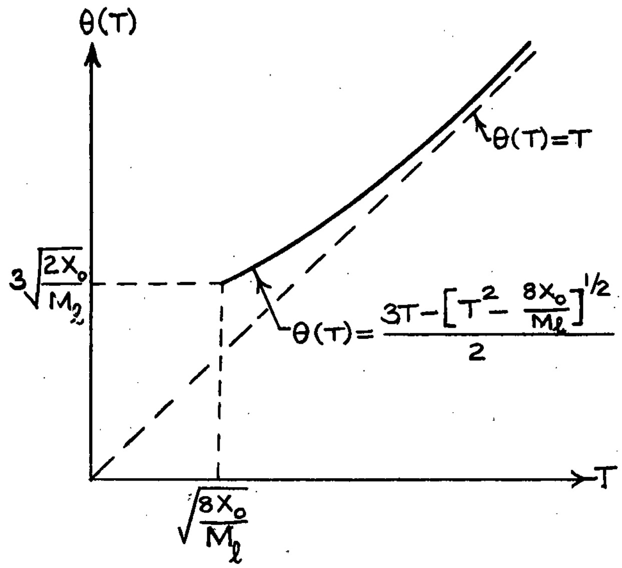


Figure 4.16. Phase characteristic for $H(s) = 1/s^2$.

(3) $H(s) = \frac{1}{\tau s + 1}$

Here

$$x(t) = M_\ell \left[1 - \frac{2e^{-t/\tau}}{1 + e^{-T/\tau}}\right]$$

$$(0 \leq t < T).$$

Provided that

$$x_0 < x(t)_{max} = M_\ell \tanh\frac{T}{2\tau}$$

commutations will occur. The phase characteristic is given by

$$\theta(T) = \tau \ln \frac{2}{(1 + e^{-T/\tau})(1 - \frac{x_0}{M_\ell})}$$

valid for $T > 2\tau \tanh^{-1} \frac{x_0}{M_\ell}$

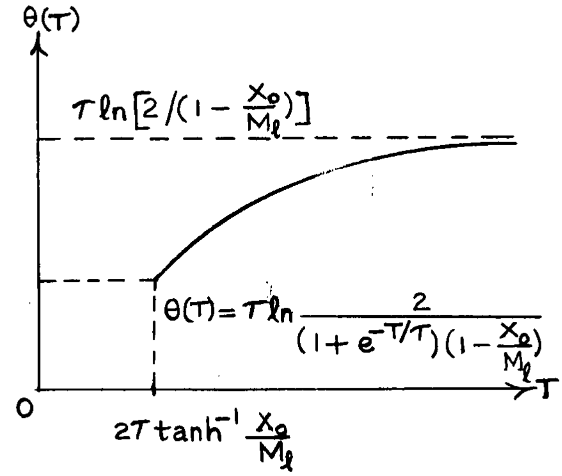
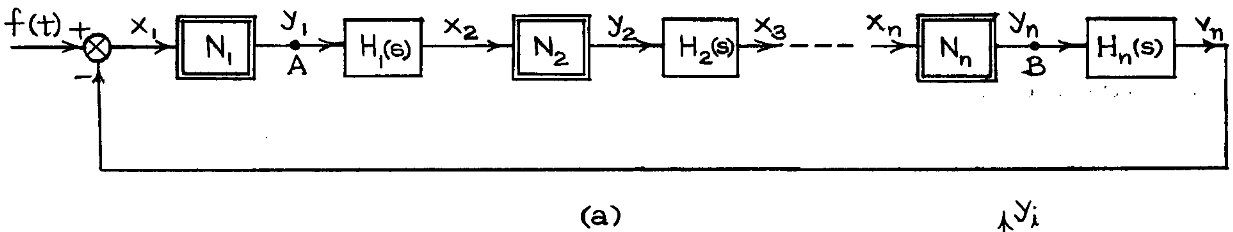


Figure 4.17. Phase characteristic for $H(s) = 1/(\tau s + 1)$.

4.3 CONDITIONS FOR THE EXISTENCE OF PERIODIC OSCILLATIONS IN SINGLE AND MULTILOOP SYSTEMS

Let us first examine a single-loop system containing an arbitrary number of n on-off elements. The system under consideration is shown in Figure 4.18.



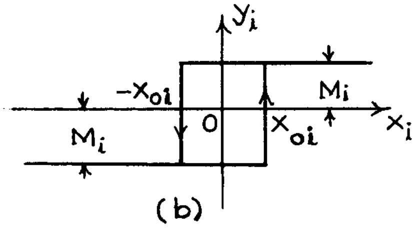


Figure 4.18. (a) Single loop system containing n on-off elements; (b) characteristics of N_i.

For the purpose of investigating the possible periods of oscillations, self or forced, we decompose the above system into n sub-systems or unit systems as shown in Figure 4.19. The

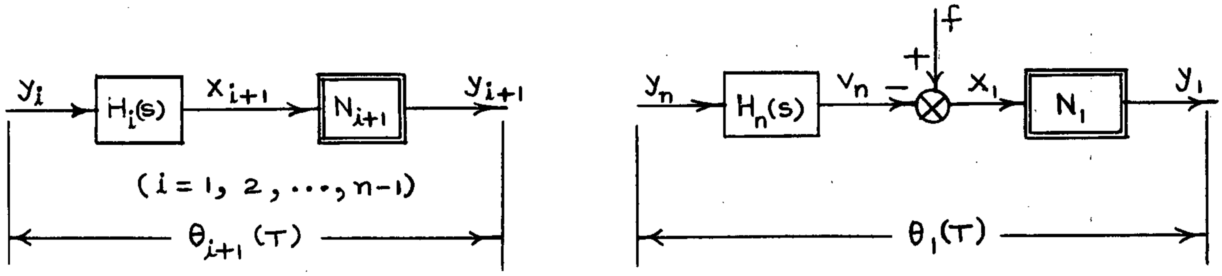


Figure 4.19. Decomposition of system in Fig. 4.18 into n sub-systems.

phase characteristic associated with the system containing the on-off element N_i is denoted by $\theta_i(T)$.

Let

$$\theta^*(T) \triangleq \sum_{i=1}^{n} \theta_i(T) - \left[\!\left[\frac{\sum_{i=1}^{n} \theta_i(T)}{2T} \right]\!\right] 2T \qquad (4.15)$$

The quantities $\sum_{i=1}^{n} \theta_i(T)$ and $\theta^*(T)$ will be referred to as the total phase characteristic and the reduced phase characteristic respectively of the open-loop system (opened at any connection between N_i and $H_i(s)$). Clearly, the closed-loop system will exhibit simple symmetric oscillations with half-period T if the reduced phase characteristic is equal to zero, that is,

$$\theta^*(T) = 0, \qquad (4.16)$$

and if

$$\left.\begin{aligned} x_i\left[\theta_i(T) + kT\right] &= (-1)^k x_{oi} \\ \dot{x}_i\left[\theta_i(T) + kT\right](-1)^k &> 0 \end{aligned}\right\} \quad (i = 1, \ldots, n;\ k = 0, \pm 1, \ldots) \qquad (4.17)$$

are the only switching conditions satisfied in the separate subsystems. Equations (4.16) and (4.17) are the conditions required for the existence of periodic oscillations in a single-loop system containing n on-off elements.

In the simplest case where $n = 1$, i.e. the single-loop system contains only one on-off element, the conditions for the existence of periodic oscillations reduce simply to the familiar expressions

$$\left.\begin{aligned} x_1(kT) &= (-1)^k x_{o1} \\ \dot{x}_1(kT)(-1)^k &> o \end{aligned}\right\} \quad (k = o, \pm 1, \ldots) \qquad (4.18)$$

In the more general case of multiloop systems the required conditions follow naturally from the above. Suppose that the system under consideration has ℓ loops, where the mth($m = 1, 2, \ldots, \ell$) loop contains an arbitrary number n_m of on-off elements. Some or all of these loops may have elements in common. Furthermore, assume that all the on-off elements are without dead zone. Let $x_{i,m}$ be the input to the ith nonlinear element ($i = 1, 2, \ldots, n_m$) in the mth loop ($m = 1, 2, \ldots, \ell$). We consider each loop in turn. Let $\theta_m^*(T)$ be the reduced phase characteristic of the mth open loop. Then the multiloop system will exhibit simple symmetric oscillations with half-period T if the reduced phase characteristics of all the loops are simultaneously zero, that is

$$\theta_m^*(T) = o, \ (m = 1, 2, \ldots, \ell) \qquad (4.19)$$

and if the proper switching instants and switching directions are also satisfied:

$$\left.\begin{aligned} x_{i,m}\left[\theta_{i,m}(T) + kT\right] &= (-1)^k x_{oi,m} \\ \dot{x}_{i,m}\left[\theta_{i,m}(T) + kT\right](-1)^k &> o \end{aligned}\right\} \quad \begin{aligned} &(i = 1,\ 2,\ \ldots,\ n_m; \\ &m = 1,\ 2,\ \ldots,\ \ell\ ; \\ &k = o,\ \pm 1,\ \ldots) \end{aligned} \tag{4.20}$$

where $\theta_{i,m}(T)$ is the phase characteristic associated with the subsystem containing the ith on-off element in the mth loop, and $x_{oi,m}$ is related to the hysteresis width of this on-off element. Another way of stating the conditions expressed by Eqs. (4.19) and (4.20) is that the existence conditions expressed by Eqs. (4.16) and (4.17) must hold simultaneously for each loop of the multiloop system.

PART II

ON SELF AND FORCED OSCILLATIONS IN ON-OFF FEEDBACK CONTROL SYSTEMS

5. SINGLE-LOOP SYSTEM CONTAINING AN ARBITRARY NUMBER OF ON-OFF ELEMENTS

Let us first consider the system shown in Figure 4.18, that is a single loop system containing n on-off elements without dead zone, and investigate the possible half-periods of self and forced oscillations.

<u>Self-oscillations</u>

A simple graphical procedure for ascertaining the possible half-periods of self oscillation is as follows:

(i) the phase characteristics $\theta_i(T)$ vs T of the individual sub-systems ($i = 1, 2, \ldots, n$) are first evaluated;

(ii) the total phase characteristic, $\sum_{i=1}^{n} \theta_i(T)$ vs T, is then plotted;

(iii) finally, we apply the condition (4.16) that the reduced phase characteristic must equal zero; thus, the values of T at which the straight lines

$$\theta = 2kT, \ (k = o, 1, 2, \ldots)$$

intersect the total phase characteristic curve give the possible half-periods of self oscillation.

The construction is shown in Figure 5.1.

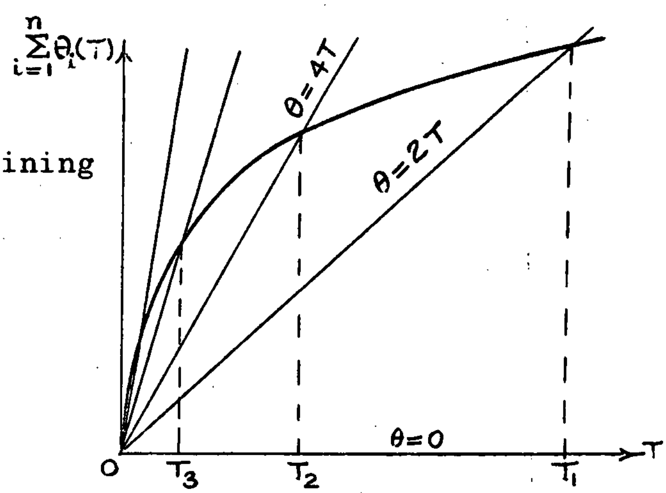


Figure 5.1.
Graphical procedure for determining possible half-periods of self oscillations.

T_1, T_2, T_3, ... represent the possible half-periods of self oscillation.

Forced oscillations

Let us assume that the input f(t) to the system, shown in Figure 4.18, is simple symmetric with half-period equal to T_o, i.e.

$$f(t) = -f(t + T_o).$$

Restricting ourselves to the consideration of simple symmetric oscillations, and excluding the case of sub-harmonics, the system variables

$$x_i, y_i \ (i = 1, \ldots, n), v_n$$

will eventually all be periodic with half-period T_o.

Consequently, the phase characteristics of the individual unit systems

$$\theta_{i+1}(T_o), \ (i = 1, 2, \ldots, n-1)$$

which are real non-negative quantities, are known (or can be calculated by the methods presented earlier). The only variable at our disposal is $\theta_1(T_o)$ which is a function both of the "amplitude" of f(t) and of the "phase shift" τ of f(t) relative to $v_n(t)$. Let us write

$$\left.\begin{array}{r} f(t) = Af_o(t - \tau) \\ A = \max \left| f(t) \right| \\ \max \left| f_o(t - \tau) \right| = 1 \end{array}\right\}, \quad o \leq \tau \leq 2T_o \qquad (5.1)$$

Thus, given A and $f_o(t)$, the sought-for quantity is the value (or values) of τ that will permit forced oscillations to occur in the system.

The procedure for determining the values of $\mathcal{T}$ that permit forced oscillations to occur is as follows:

(i) The total phase characteristic $\sum_{i=2}^{n} \theta_i(T_o)$ between points A and B (in Figure 4.18a) is computed.

(ii) The reduced phase characteristic between A and B, namely

$$\theta^*(T_o) = \sum_{i=2}^{n} \theta_i(T_o) - \left[\!\left[\frac{\sum_{i=2}^{n} \theta_i(T_o)}{2T_o} \right]\!\right] 2T_o \quad (5.2)$$

is evaluated. For forced oscillations to occur, the reduced phase characteristic of the entire loop must equal zero. Let us define the complementary phase characteristic of $\theta^*(T_o)$, with respect to $2T_o$, as

$$\theta_c^*(T_o) = \begin{cases} 2T_o - \theta^*(T_o) & , \text{ for } \theta^*(T_o) > o \\ o & , \text{ for } \theta^*(T_o) = o \end{cases}$$

Then forced oscillations may occur if the phase characteristic of the first sub-system (between B and A) is equal to the complementary phase characteristic $\theta_c^*(T_o)$ between A and B : that is,

$$\theta_1(T_o) = \theta_c^*(T_o).$$

(iii) The phase characteristic $\theta_1(T_o)$ is a function of $\mathcal{T}$ and will be denoted by $\theta_1(T_o, \mathcal{T})$: it is determined as the smallest non-negative root of the equation

$$x_1(t,\mathcal{T})\Big]_{T = T_o} = A_o f(t - \mathcal{T}) - v_n(t)\Big]_{T = T_o} = x_{o1} \quad (5.3)$$

(iv) The values of τ satisfying $\theta_1(T_o, \tau) = \theta_c^*(T_o)$ give rise to forced oscillations, provided that the only switching conditions are

$$\left.\begin{array}{l} x_1\left[\theta_c^*(T_o) + kT_o\right] = (-1)^k x_{o1} \\ \dot{x}_1\left[\theta_c^*(T_o) + kT_o\right](-1)^k > o \end{array}\right\} \quad (k = o, \pm 1, \ldots) \tag{5.4}$$

and

$$\left.\begin{array}{l} x_i\left[\theta_i(T_o) + kT_o\right] = (-1)^k x_{oi} \\ \dot{x}_i\left[\theta_i(T_o) + kT_o\right](-1)^k > o \end{array}\right\} \quad \begin{array}{l}(i = 2, 3, \ldots, n; \\ k = o, \pm 1, \ldots)\end{array} \tag{5.5}$$

and these can be verified from plots of $x_i(t)$ and $\dot{x}_i(t)$ as functions of t.

The construction corresponding to steps (iii) and (iv) above is shown in Figure 5.2.

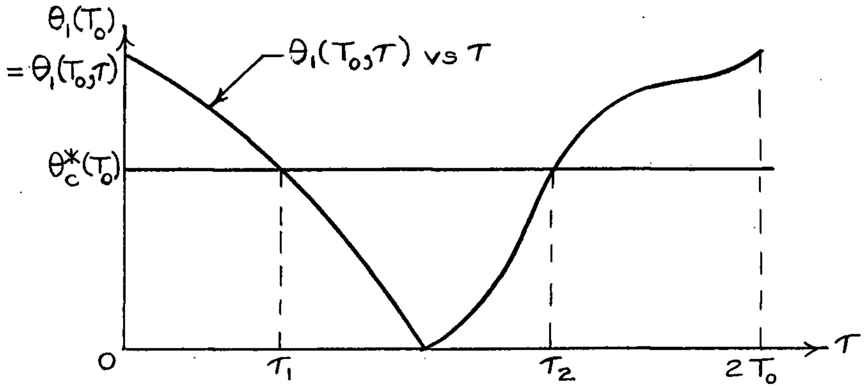


Figure 5.2. On the determination of possible values of τ that permit the occurrence of forced oscillations.

Another method for determining the values of τ that may permit forced oscillations utilizes the Tsypkin approach in the latter part of the procedure. The steps in the procedure are as follows:

(i) As above, the reduced phase characteristic $\theta^*(T_o)$ between points A and B (in Figure 4.18a) is first computed, and then the complementary phase characteristic $\theta_c^*(T_o)$ is found.

(ii) For forced oscillations to occur at a particular value of τ, two conditions must be satisfied: first,

$$x_1(t)\Big]_{t = \theta_c^*(T_o)} = Af_o(t - \tau) - v_n(t)\Big]_{t = \theta_c^*(T_o)} = x_{o1} \tag{5.6}$$

for the proper switching instants; then

$$\dot{x}_1(t)\Big]_{t = \theta_c^*(T_o)} > o$$

for the proper switching directions. The Tsypkin plane

$$\mathfrak{J} = \frac{1}{\omega}\dot{x} + jx$$

can be used to represent these two conditions graphically in the following manner.

(iii) The contributions $-\dot{v}_n\left[\theta_c^*(T_o)\right]$ and $-v_n\left[\theta_c^*(T_o)\right]$ to $\dot{x}_1$ and x, respectively, are first plotted on the $\mathfrak{J}$-plane; these are denoted as coordinates (a,b), as shown in Figure 5.3.

(iv) The remaining contributions $A\dot{f}_o\left[\theta_c^*(T_o) - \tau\right]$ and $Af_o\left[\theta_c^*(T_o) - \tau\right]$ to $\dot{x}_1$ and x_1, respectively, are added to those of part (iii). These contributions, however, are functions of τ and therefore, as τ varies between o and $2T_o$, they give rise to a curve $\mathfrak{F}\left[\theta_c^*(T_o), \tau\right]$, called the hodograph of f $\left[\theta_c^*(T_o)\right]$, about the point (a,b), where

$$\mathfrak{F}\left[\theta_c^*(T_o), \tau\right] = A\left[\frac{T_o}{\pi}\dot{f}_o(t - \tau) + jf_o(t - \tau)\right]_{t = \theta_c^*(T_o)} \tag{5.7}$$

(v) To satisfy the condition of the proper switching instant, the hodograph $\mathcal{F}$ must intersect the straight line jx_{01}. Also, to obtain the proper switching directions $\dot{x}_1\left[\theta_c^*(T_0)\right]>0$, the points of intersection must lie in the right-half $\mathcal{J}$-plane. Furthermore, the values of τ at these points of intersection (of $\mathcal{F}$ with jx_{01}) will allow forced oscillations to occur, provided that there are no additional commutations in the interval $\theta_c^*(T_0)<t<\theta_c^*(T_0) + T_0$.

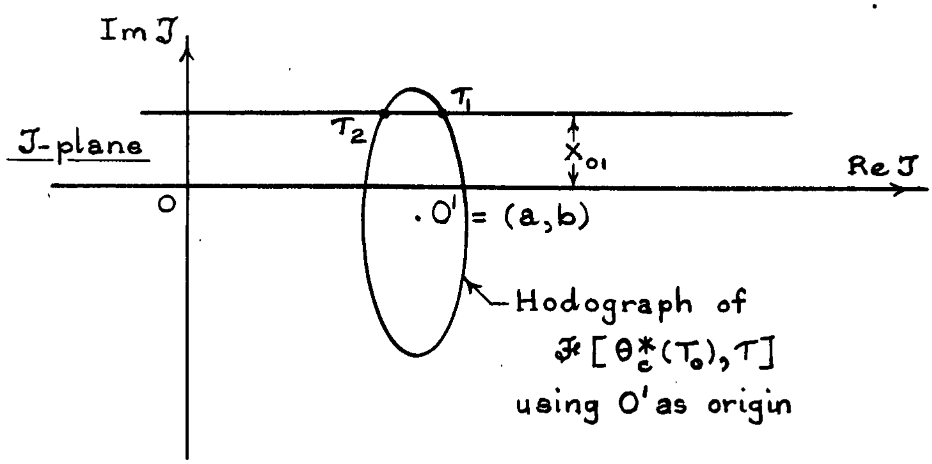


Figure 5.3. On the determination of possible values of τ that permit forced oscillations.

It is obvious from Eq. (5.7) that the non-negative real quantity A, called the "amplitude" of f(t), is a scale-factor for the hodograph of $\mathcal{F}\left[\theta_c^*(T_0), \tau\right]$: that is, the relative shape of this hodograph remains the same for various values of A, and an increase or decrease in the value of A merely magnifies or contracts the curve of $\mathcal{F}\left[\theta_c^*(T_0),\tau\right]$ about O' as origin. Hence the value of A, in general, determines the number of values of τ at which forced oscillations may occur.

The effect of varying A is illustrated in Figures 5.4 (a) to (f). In Figure 5.4 (a) the value of A is too small to allow forced oscillations with half-period equal to T_o. In this case sub-harmonic oscillations are possible. As A is increased to the critical value A_{1cr} the line jx_{o1} becomes tangent to the hodograph of $\mathfrak{F}\left[\theta_c^*(T_o),\tau\right]$ in the right-half $\mathfrak{J}$-plane. A further increase in A brings us to Figure 5.4 (c) for which forced oscillations may occur at $\tau = \tau_1, \tau_2$ (for the hodograph as drawn). For very large values of A forced oscillations will be possible at the one value of τ, namely $\tau = \tau_1$ in Figure 5.4 (d). In Figures 5.4 (e) and (f), O' lies in the left-half $\mathfrak{J}$-plane. At $A = A_{2cr}$ the hodograph of $\mathfrak{F}\left[\theta_c^*(T_o),\tau\right]$ passes through the intersection of the j Im$\mathfrak{J}$-axis and jx_{o1}, whereas a further increase in A may allow forced oscillations at the one value τ_1 as shown in Figure 5.4 (f).

For $A = A_{1cr}$, we have from Figure 5.4 (b):

$$\left| \operatorname{Im} \mathfrak{F}\left[\theta_c^*(T_o), \tau_{o1}\right] \right| = \left| b - x_{o1} \right| .$$

By using Eq. (5.7) the above equality can be written as

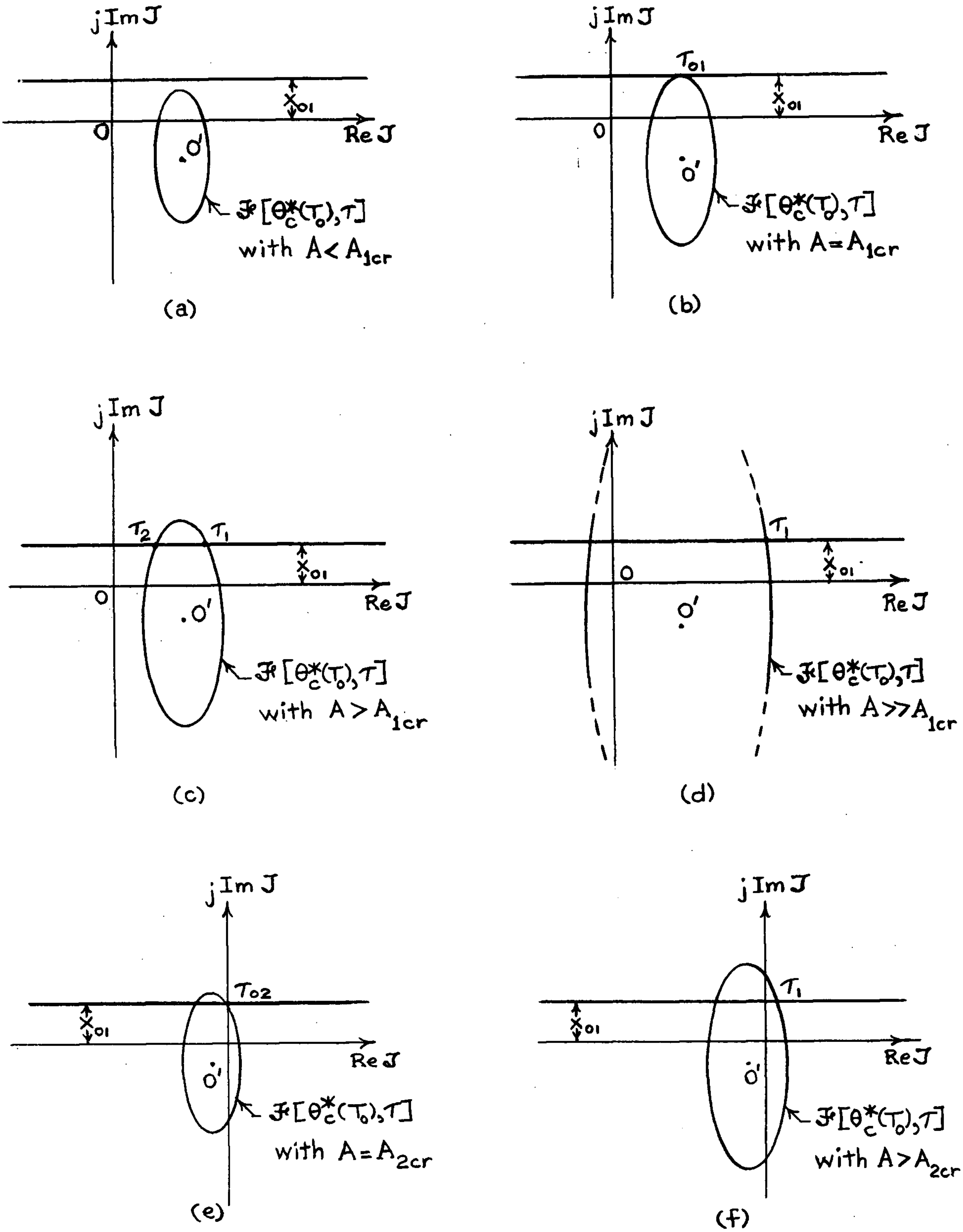
$$A_{1cr} = \frac{\left| b - x_{o1} \right|}{\left| f_o \left[\theta_c^*(T_o) - \tau_{o1}\right]\right|} \tag{5.8}$$

Similarly, from Figure 5.4 (e) we have

$$\left| \mathfrak{F}\left[\theta_c^*(T_o), \tau_{o2}\right] \right| = \sqrt{a^2 + (b - x_{o1})^2} ,$$

and by using Eq. (5.7) we obtain

$$A_{2cr} = \left[\frac{a^2 + (b - x_{o1})^2}{\left[\frac{T_o}{\pi} \dot{f}_o(\theta_c^*(T_o) - \tau_{o2})\right]^2 + \left[f_o(\theta_c^*(T_o) - \tau_{o2})\right]^2} \right]^{1/2} \tag{5.9}$$



Remarks: 0' = (a,b) : hodographs $\mathfrak{F}(\theta_c^*(T_o),\tau)$ drawn about 0' as origin.

Figure 5.4. Influence of A upon the number of values of τ that may permit forced oscillations.

Obviously, for $A > A_{1cr}$ or A_{2cr} the desired values of τ can be determined from the equality

$$\left| f_o \left[\theta_c^*(T_o) - \tau \right] \right| = \frac{|b - x_o|}{A} \tag{5.10}$$

or, by making use of Equation (5.8) and (5.9), this equality becomes

$$\left| f_o \left[\theta_c^*(T_o) - \tau \right] \right| = \frac{A_{1cr}}{A} \left| f_o \left[\theta_c^*(T_o) - \tau_{o1} \right] \right| \tag{5.11}$$

for $A > A_{1cr}$

and

$$\left| f_o \left[\theta_c^*(T_o) - \tau \right] \right| = \frac{A_{2cr}}{A} \left(\left[\frac{T_o}{\pi} \dot{f}_o(\theta_c^*(T_o) - \tau_{o2}) \right]^2 + \left[f_o(\theta_c^*(T_o) - \tau_{o2}) \right]^2 - \frac{a^2}{A_{2cr}^2} \right)^{1/2} \tag{5.12}$$

for $A > A_{2cr}$,

respectively.

In the special case where $f_o(t) = \sin \omega t$, we have

$$f_o(t - \tau) = \sin \omega(t - \tau) , \qquad \frac{1}{\omega} \dot{f}_o(t - \tau) = \cos \omega(t - \tau) ,$$

so that the hodograph of $\mathfrak{F}\left[\theta_c^*(T_o), \tau\right]$ is given by

$$\mathfrak{F}\left[\theta_c^*(T_o), \tau\right] = A \left(\cos \omega_o \left[\theta_c^*(T_o) - \tau \right] + j \sin \omega_o \left[\theta_c^*(T_o) - \tau \right] \right)$$
$$= A\, e^{j\omega_o \left[\theta_c^*(T_o) - \tau \right]} \tag{5.13}$$

where $\omega_o = \pi/T_o$. Hence the hodograph of $\mathcal{F}\left[\theta_c^*(T_o), \tau\right]$ is a circle of radius equal to A. By making use of Eq. (5.13), equalities (5.11) and (5.12) become

$$\left| \sin \omega_o \left[\theta_c^*(T_o) - \tau\right] \right| = \frac{A_{1cr}}{A} \left| \sin \omega_o \left[\theta_c^*(T_o) - \dot{\tau}_{o1}\right] \right| \tag{5.14}$$

for $A > A_{1cr}$,

and

$$\left| \sin \omega_o \left[\theta_c^*(T_o) - \tau\right] \right| = \frac{\sqrt{A_{2cr}^2 - a^2}}{A} \tag{5.15}$$

for $A > A_{2cr}$,

respectively.

6. DOUBLE-LOOP SYSTEM CONTAINING AN ARBITRARY NUMBER OF ON-OFF ELEMENTS

We mentioned earlier that in more complex systems the application of the Tsypkin method to the determination of the possible periods of simple symmetric oscillations becomes very cumbersome. In this chapter we first show that the Tsypkin approach can be used in the study of the double-loop system in which each loop contains one on-off element. This particular case points out the difficulties that would be encountered in any contemplated extension of the Tsypkin method to the study of systems with three or more on-off elements. We then indicate how the possible periods of simple symmetric oscillations in a double-loop system, containing an arbitrary number of on-off elements, may be determined by the phase characteristic method.

6.1 APPLICATION OF TSYPKIN'S METHOD TO A DOUBLE-LOOP SYSTEM WITH TWO ON-OFF ELEMENTS

The system under consideration is shown in Figure 6.1.

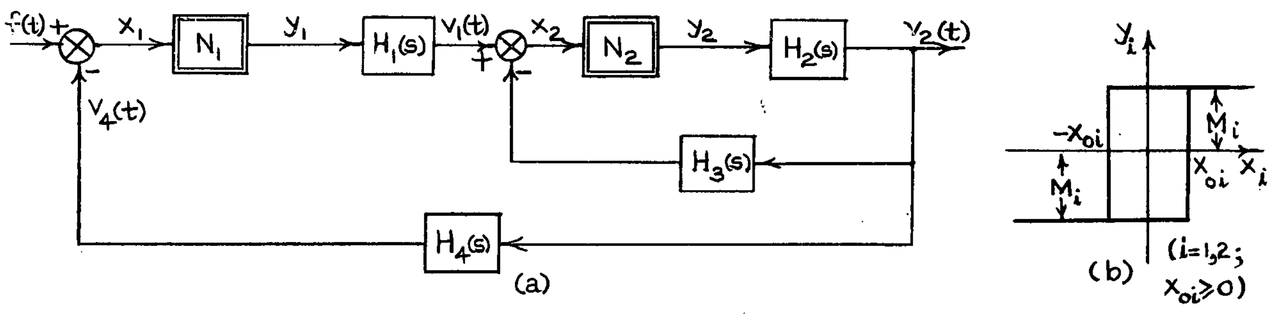


Figure 6.1 (a) Double-loop system containing two on-off elements.

(b) Characteristics of N_1 and N_2.

In the case of simple symmetric oscillations, the outputs of N_1 and N_2 are, in general, as shown in Figure 6.2. In fact, the expressions for $y_1(t)$ and $y_2(t)$ are

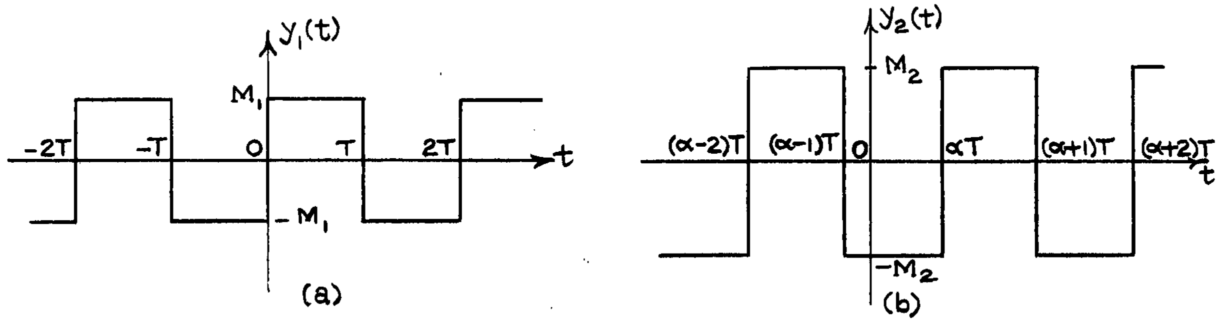


Figure 6.2. (a) and (b) Outputs of N_1 and N_2.

$$y_1(t) = 2M_1 \sum_{k=-\infty}^{n} (-1)^k \, u(t - kT), \text{ for } -\infty < t < (n+1)T$$

$$y_2(t) = 2M_2 \sum_{k=-\infty}^{n} (-1)^k \, u\left[t - (\alpha + k)T\right], \text{ for } -\infty < t < (n+1+\alpha)T,$$

where it is assumed that

$$\alpha \geq o \text{ and } o \leq \alpha < 2.$$

From the results of Chapter 3 the response of the linear part $H_1(s)$ is

$$v_1(t) = \frac{2M_1}{2\pi j} \oint_{C_1 or C_2} \frac{H_1(s)}{s} \frac{(-1)^n e^{-snT}}{1 + e^{sT}} e^{st} ds + K_1 \quad (6.1)$$

$$(nT \leq t < (n+1)T, \; n = o, \pm 1, \pm 2, \ldots)$$

where K_1 is a constant related to the initial conditions. Similarly, the outputs of the other linear parts are given by

$$v_i(t) = \frac{2M_2}{2\pi j} \oint_{C_1 or C_2} \frac{L_i(s)}{s} \frac{(-1) e^{-s(\alpha+n)T}}{1 + e^{sT}} e^{st} ds + K_i \quad (6.2)$$

for $(\alpha + n)T \leq t < (\alpha + n + 1)T$,

$i = 2, 3, 4; \; n = o, \pm 1, \pm 2, \ldots,$

where

$$L_2(s) = H_2(s)$$

$$L_3(s) = H_2(s)H_3(s)$$

$$L_4(s) = H_2(s)H_4(s),$$

and K_i are constants related to the initial conditions.

The conditions for symmetric oscillations of the above type are

$$\left.\begin{aligned} &x_1(o) = x_{o1},\ \dot{x}_1(o) > o \\ &x_1(t) > -x_{o1} \text{ for } o < t < T \end{aligned}\right\} \qquad (6.3)$$

and

$$\left.\begin{aligned} &x_2(\alpha T) = x_{o2},\ \dot{x}_2(\alpha T) > o \\ &x_2(t) > -x_{o2} \text{ for } \alpha T < t < (\alpha + 1)T \end{aligned}\right\} \qquad (6.4)$$

<u>Self Oscillations</u>

Following Tsypkin's method, we introduce the Tsypkin loci

$$\left.\begin{aligned} \mathcal{J}_1(\alpha,T) &= \frac{T}{\pi}\,\dot{x}_1(o) + j\,x_1(o) \\ \mathcal{J}_2(\alpha,T) &= \frac{T}{\pi}\,\dot{x}_2(\alpha T) + j\,x_2(\alpha T) \end{aligned}\right\} \qquad (6.5)$$

Using α as the parameter $(o \leq \alpha < 2)$ and T as the variable, we construct these loci as shown in Figure 6.3. The straight lines jx_{o1} and jx_{o2} are next inserted on the $\mathcal{J}_1$- and $\mathcal{J}_2$-planes, respectively. The points a_1, b_1, c_1,... of intersection of the $\mathcal{J}_1(\alpha,T)$ loci with the straight line jx_{o1} in the first quadrant of the $\mathcal{J}_1$-plane correspond to pairs of values (α,T) that satisfy the conditions $x_1(o) = x_{o1}$, $\dot{x}_1(o) > o$; similarly, the points

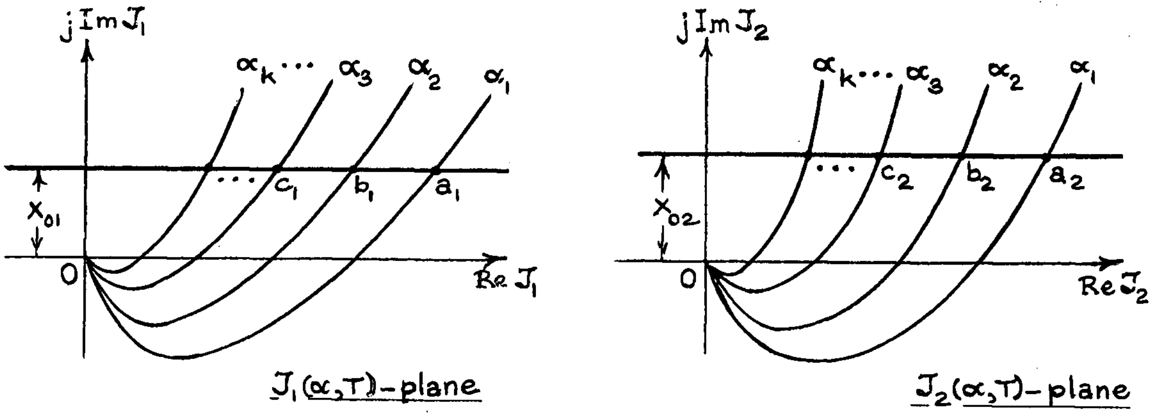


Figure 6.3. The Tsypkin loci $J_1(\alpha,T)$, $J_2(\alpha,T)$.

a_2, b_2, c_2,... in the first quadrant of the J_2-plane correspond to pairs of values (α,T) that satisfy the conditions $x_2(\alpha T) = x_{o2}$, $\dot{x}_2(\alpha T) > o$. We now plot these points of intersection as curves of $\alpha = f_1(T)$ corresponding to the points a_1, b_1, c_1,... of the J_1-plane, and $\alpha = f_2(T)$ corresponding to the points a_2, b_2, c_2,... of the J_2-plane. Any pair of values (α,T) at the intersection of the curves $f_1(T)$ and $f_2(T)$, such as (α^*, T^*) shown in Figure 6.4, may give rise to self oscillations.

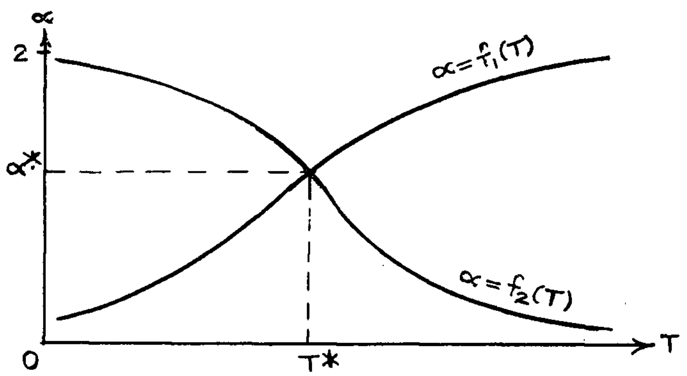


Figure 6.4. Curves of $\alpha = f_1(T)$ and $\alpha = f_2(T)$.

Forced oscillations.

Let the input to the system, $f(t)$, be periodic with half-period equal to $T_o = \pi/\omega_o$. The conditions for the existence of forced oscillations are again expressed by equations (6.3) and (6.4) with T set equal to T_o, but now, instead of

$$x_1(t) = -v_4(t) ,$$

we have

$$x_1(t) = f(t) - v_4(t).$$

Also, instead of α and T the sought-for quantities are α and ϕ where ϕ is the phase shift of $f(t)$ relative to some arbitrary reference phase ϕ_o. For convenience, we write

$$f(\omega_o t) = A_o f_o(\omega_o t - \phi)$$

or

$$f(t) = A_o f_o(t - \tau)$$

where

$$\tau = \phi/\omega_o ,$$

$$A_o = \max \left|f(t)\right| \quad \text{and} \quad \max \left|f_o(t)\right| = 1.$$

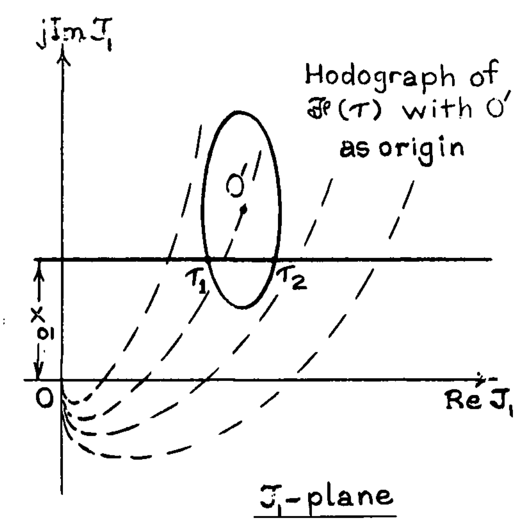
From the curve of $\alpha = f_2(T)$ we locate the value $\alpha = \alpha_o$ at which T is equal to T_o. We next insert the point

$$0' = -\frac{T_o}{\pi}\dot{v}_4(o) - v_4(o)\Big]_{\substack{\alpha = \alpha_o, \\ T = T_o}}$$

on the $\mathfrak{J}_1$-plane. With the point 0' as origin, we construct the hodograph of

$$\mathfrak{F}(\tau) = A_o \left[\frac{T_o}{\pi} \dot{f}_o (-\tau) + j\, f_o(-\tau) \right]$$

as τ varies from 0 to 2 T_o inclusively, as shown in Figure 6.5.



The value(s) of τ corresponding to the intersection of the $\mathfrak{F}(\tau)$ loci with the straight line jx_{01} and lying in the first quadrant of the $\mathfrak{J}_1$-plane, together with the value of α_0 determined above, are the sought-for value(s) of (α,τ) which may allow the occurrence of forced oscillations.

Figure 6.5. On the determination of the values of τ that permit forced oscillations.

The conditions $x_1(t) > -x_{01}$ for $o < t < T_0$ and $x_2(t) > -x_{02}$ for $\alpha_0 T_0 < t < (\alpha_0 + 1)T_0$ must be verified.

In principle, the Tsypkin approach can be applied to the study of the periods of oscillations in a double-loop system containing an arbitrary number of on-off elements. But the extension to cover the cases of more than two on-off elements is definitely awkward. Such complicated cases are best solved by the method of the phase characteristic.

6.2 APPLICATION OF THE PHASE CHARACTERISTIC METHOD TO A DOUBLE-LOOP SYSTEM CONTAINING AN ARBITRARY NUMBER OF ON-OFF ELEMENTS

Consider the double-loop system containing an arbitrary number of on-off elements as shown in Figure 6.6 (a). Assume

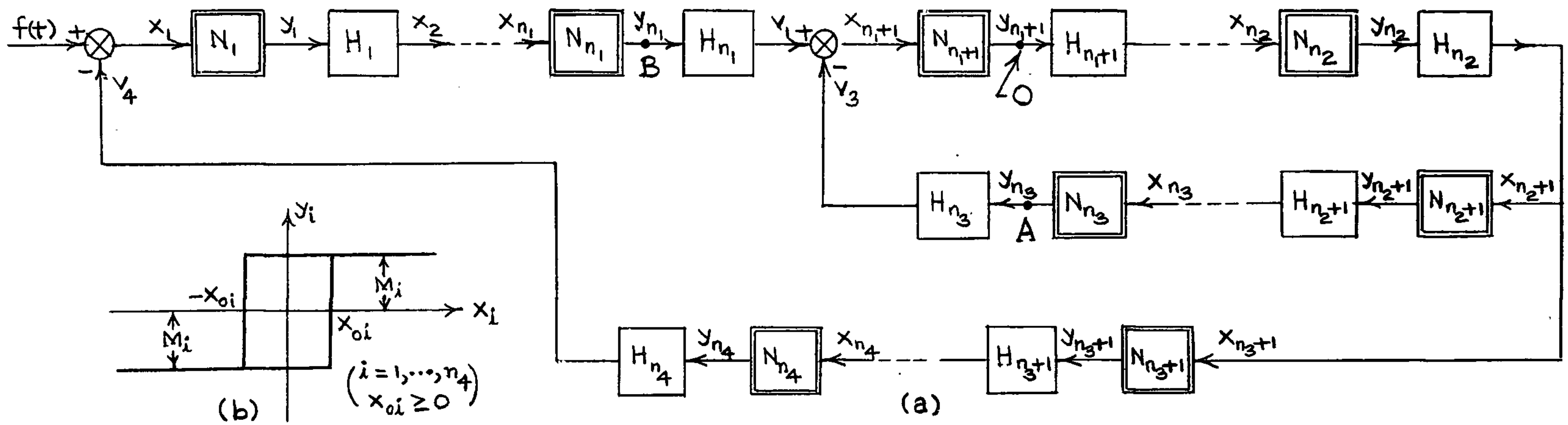


Figure 6.6 (a) Double-loop system containing an arbitrary number of on-off elements; (b) Characteristic of ith on-off element.

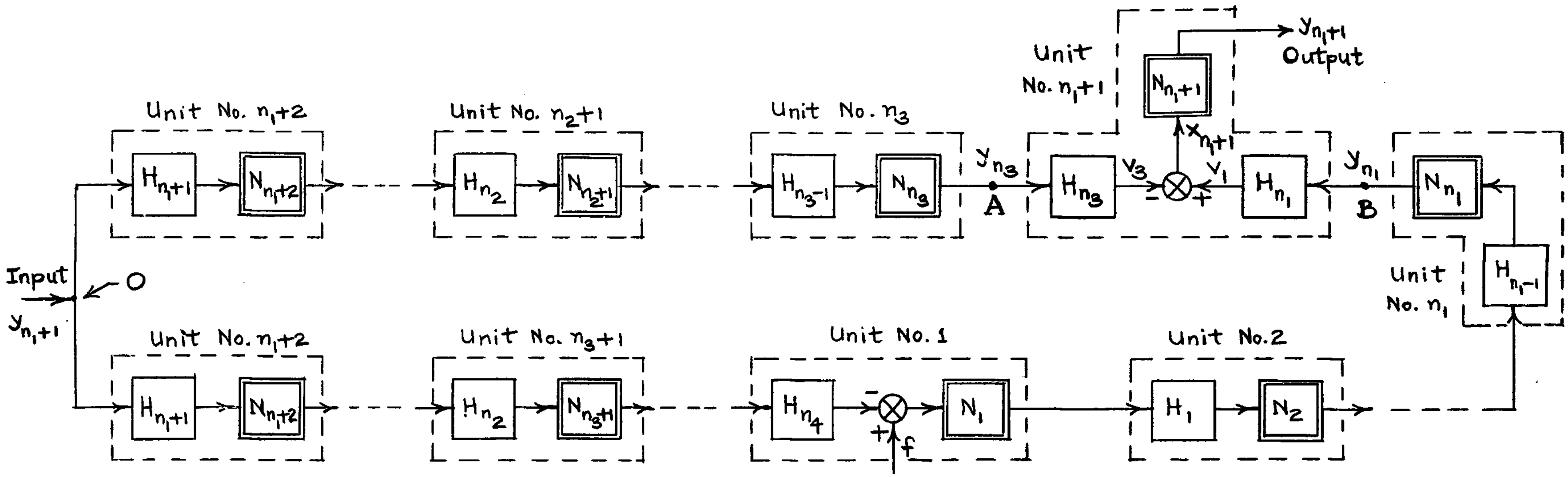


Figure 6.7. Open-loop system as a composition of unit systems.

that the characteristic of the ith on-off element has the form shown in Figure 6.6 (b), i.e. with or without hysteresis so that $x_{oi} \geq o$.

Self-oscillations.

In order to determine the possible periods of self-oscillation, we open the system in Figure 6.6 (a) at the point O. The resulting open-loop system can be regarded as a composition of unit systems as shown in Figure 6.7. The ith unit (or sub-system) consists of the ith on-off element and the linear system or systems immediately preceding it. Let $\theta_i(T)$ be the phase characteristic of the ith sub-system. The functions $\theta_i(T)$ for $i = 1, 2, \ldots, n_4$ except for $i = n_1 + 1$ (we are assuming a total of n_4 on-off elements in our system) are all known, or can be calculated by the methods indicated in Chapter 4.

We now evaluate the total phase characteristics, Θ_3 and Θ_1, between the points O and A and between O and B, respectively, in Figures 6.6 and 6.7:

$$\left.\begin{aligned} \Theta_3 &= \theta_{n_1+2} + \theta_{n_1+3} + \ldots + \theta_{n_2} + \theta_{n_2+1} + \ldots + \theta_{n_3} \\ \Theta_1 &= \theta_{n_1+2} + \theta_{n_1+3} + \ldots + \theta_{n_2} + \theta_{n_3+1} + \ldots + \theta_{n_4} \\ &\qquad + \theta_1 + \ldots + \theta_{n_1} \end{aligned}\right\} \quad (6.6)$$

where, for simplicity, we have written θ_i for $\theta_i(T)$. Next we determine the reduced phase characteristics

$$\Theta^*_3 = \Theta_3 - \left[\!\left[\frac{\Theta_3}{2T}\right]\!\right] 2T$$
$$\Theta^*_1 = \Theta_1 - \left[\!\left[\frac{\Theta_1}{2T}\right]\!\right] 2T \qquad (6.7)$$

Sketches of possible plots of Θ_1, Θ^*_1, Θ_3, Θ^*_3 as functions of T are given in Figure 6.8. Observe that $o \leq \Theta^*_i < 2T$ $(i = 1,3)$.

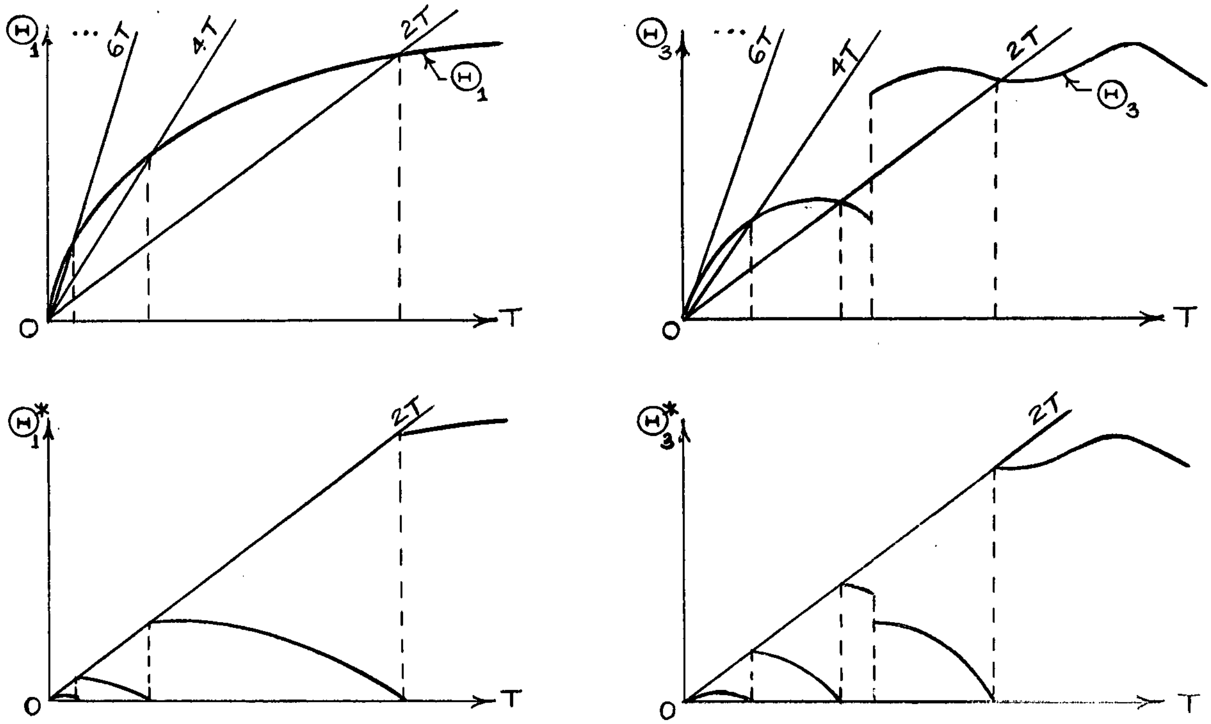


Figure 6.8. Sketches of possible plots of

Θ_1, Θ^*_1, Θ_3, Θ^*_3.

Consider now the $n_1 + 1$ th sub-system. Figure 6.9 illustrates the general forms of the inputs and output of this unit system. Since the functions Θ^*_1 and Θ^*_3 are known, we can therefore compute

$$v_i(t) = \frac{2M_i}{2\pi j} \oint_{C_1 or C_2} \frac{H_{n_i}(s)}{s} \frac{(-1)^k e^{-s(k+\frac{\Theta^*_i}{T})T}}{1 + e^{sT}} e^{st} ds \qquad (6.8)$$

$(i = 1,3;\ \Theta^*_i + kT \leq t < \Theta^*_i + (k + 1)T;\ k = o, \pm 1, \ldots)$.

Consequently,

$$x_{n_1+1}(t) = v_1(t) - v_3(t)$$

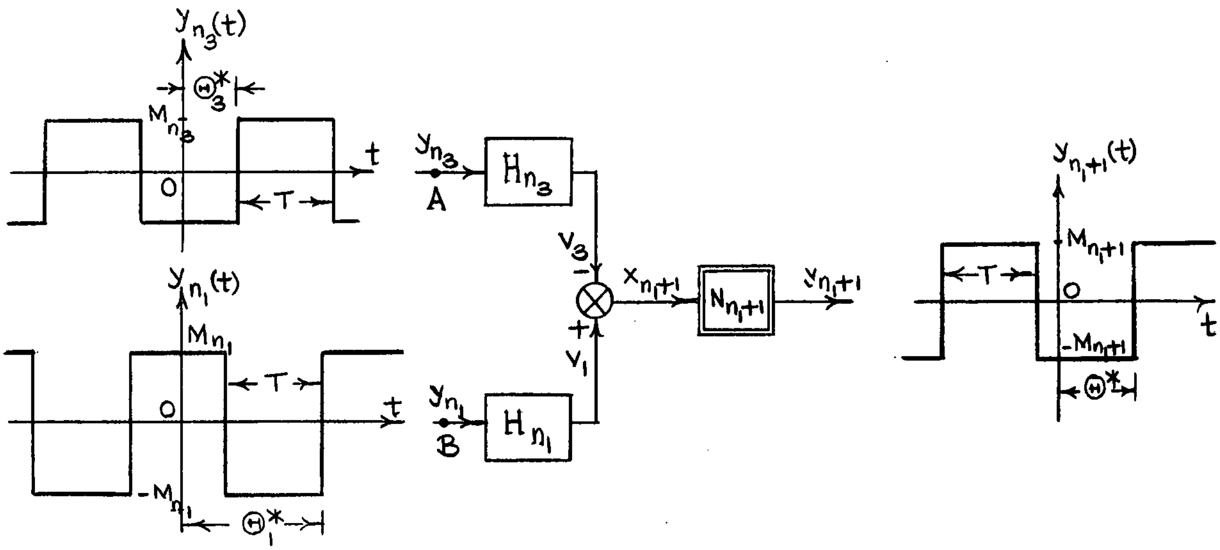


Figure 6.9. Relationships in the $n_1 + 1$ th sub-system.

can be determined for any time interval. In particular, we can determine the time $\Theta^* \equiv \Theta^*(T)$, $o \leq \Theta^* < 2T$, at which the output $y_{n_1+1}(t)$ of this unit system first jumps from $-M_{n_1+1}$ to M_{n_1+1}; in fact, Θ^* is the least positive root of the equation $x_{n_1+1}(t) = x_{o,n_1+1}$. The quantity $\Theta^* \equiv \Theta^*(T)$ is the reduced phase characteristic of the entire open-loop system in Figure 6.7. Hence the values of T for which $\Theta^*(T) = o$ are the possible half-periods of self oscillation of the closed-loop system.

The method described above automatically guarantees that the condition expressed by Eqs. (4.19) and (4.20) are satisfied: that is, that the reduced phase characteristic of each loop is zero.

Forced oscillations

The procedure for the determination of the conditions that permit forced oscillations is as follows:

Let $T = \pi/\omega_o$ be the half-period of forced oscillation. The total phase characteristic between 0 and B (in Figure 6.7), minus the contribution due to Unit No. 1, is denoted by $\Theta_2(T)$ and the corresponding reduced phase characteristic by $\Theta_2^*(T)$ so that

$$\left.\begin{aligned} \Theta_2(T) &= \Theta_1(T) - \theta_1(T) \\ \text{and} \quad & \\ \Theta_2^*(T) &= \Theta_2(T) - \left[\!\left[\frac{\Theta_2(T)}{2T}\right]\!\right] 2T \end{aligned}\right\} \qquad (6.9)$$

The reduced phase characteristic between 0 and A is denoted by $\Theta_3^*(T)$. For periodic phenomena of half-period T_o, the quantities $\Theta_2^*(T_o)$ and $\Theta_3^*(T_o)$ are fixed non-negative numbers less that $2T_o$.

Forced oscillations of half-period T_o may occur if the N_{n_1+1} element switches over at time $t = o$ and if the slope of the input to this element is positive at this instant: that is,

$$x_{n_1+1}(o) = x_{o,n_1+1} \text{ and } \dot{x}_{n_1+1}(o) > o$$

Because of the reduced phase shift of $\Theta_3^*(T_o)$ between 0 and A, the input to H_{n_3} at point A is shifted to the right by $\Theta_3^*(T_o)$ relative to the input at point 0. Referring to Unit

No. 1 to which the forcing function f(t) is applied, we let

$$f(t) = A_o f(t - \tau)$$

where τ is the phase shift of f(t) relative to the input to H_{n_4}. The phase characteristic of Unit No. 1 is a function of τ and is denoted by $\theta_1(T_o, \tau)$. Therefore the phase characteristic between O and B is also a function of τ; it is determined by

$$\Theta_1(T_o, \tau) = \Theta_2^*(T_o) + \theta_1(T_o, \tau)$$

Thus, relative to the input at O, the input to H_{n_1} is shifted to the right by $\Theta_1(T_o, \tau)$. Consequently, the output of $N_{n_1} + 1$ is

$$x_{n_1 + 1}(t) = -v_3(t - \Theta_3^*) + v_1(t - \Theta_1^*)$$

where $v_3(t)$ and $v_1(t)$ are the outputs of H_{n_3} and H_{n_1} when there is no phase shift of the waveforms between O and A and between O and B respectively.

The conditions that $x_{n_1 + 1}(o)$ and $\dot{x}_{n_1 + 1}(o)$ satisfy can be represented on the Tsypkin $\mathcal{J}_{n_1 + 1}$-plane

$$\mathcal{J}_{n_1 + 1} = \frac{T_o}{\pi} \dot{x}_{n_1 + 1} + jx_{n_1 + 1}$$

First the contribution due to $-v_3$ is plotted: it is the point

$$O' = -\frac{T_o}{\pi} \dot{v}_3(-\Theta_3^*(T_o)) - jv_3(-\Theta_3^*(T_o))$$

which is independent of τ. Next the contribution due to $-v_1$ is added to the point O'; this contribution depends on τ and therefore yields the curve

$$\mathfrak{F}_1(\tau) = \frac{T_o}{\pi}\dot{v}_1(-\Theta_1^*(T_o,\tau)) + jv_1(-\Theta_1^*(T_o,\tau))$$

with the point O' as its origin, as τ varies from 0 to $2T_o$. Figure 6.10 shows the $\mathcal{J}_{n_1+1}$-plane and the two contributions to $\dot{x}_{n_1+1}$ and x_{n_1+1}.

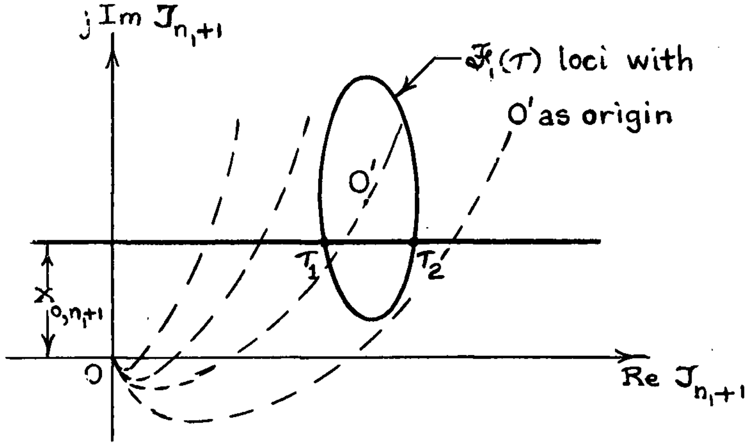


Figure 6.10. $\mathcal{J}_{n_1+1}$-plane.

The values of τ lying in the first quadrant of the $\mathcal{J}_{n_1+1}$-plane and corresponding to the points of intersection of the loci of $\mathfrak{F}_1(\tau)$ with the straight line jx_{o,n_1+1}, determine the conditions that are necessary for forced oscillations.

<u>Illustrative Example of the Application of the Phase Characteristic Concept to the Determination of the Periods of Self Oscillations in a Double-Loop System</u>

A double-loop system containing two N elements is shown in Figure 6.11. The method (of solving for the possible half-periods of self oscillation) given in section 6.2 is used: that is, the phase characteristic of the system is evaluated and the points of intersection with the straight lines $\theta = 2kT(k = o, 1, 2,...)$ give the possible half-periods of self oscillations.

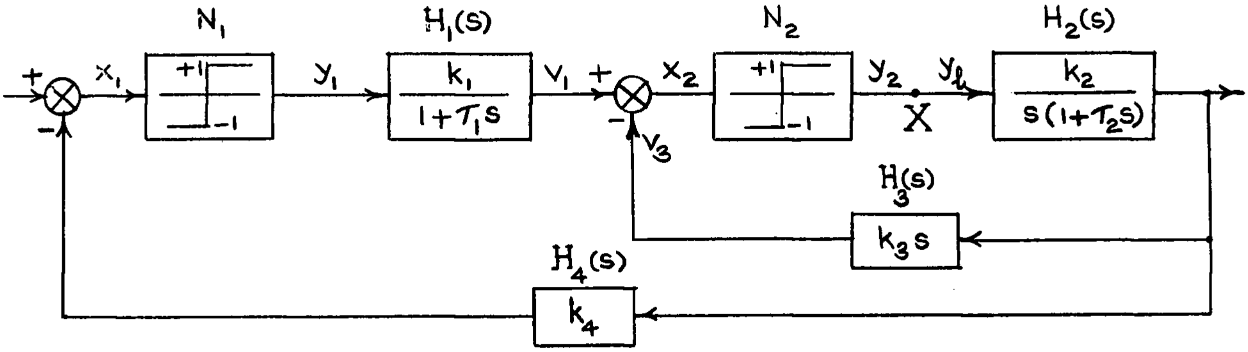


Figure 6.11. A double-loop system containing two N elements.

As indicated in Figure 6.6 and 6.7, the double-loop system is opened at the point X, and the system is redrawn as shown in Figure 6.12. The open-loop system consists of two unit systems, one (unit no.1) of the type shown in Figure 4.13 and

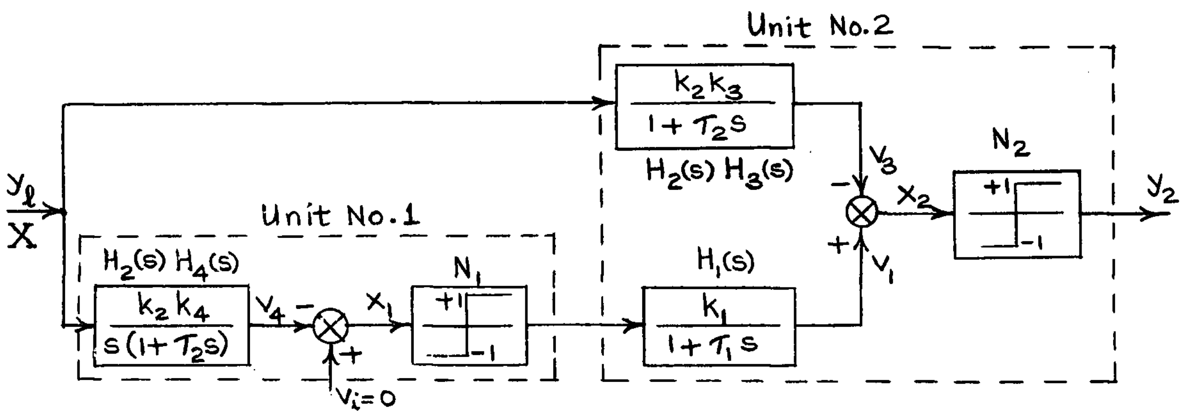


Figure 6.12. Open-loop system of Figure 6.11 showing unit systems.

the other (unit no.2) of the type shown in Figure 6.9.

From Eq. (4.11), the output of $H_2(s)\ H_4(s)$ is given by

$$v_4(t) = k_2k_4\left[-\tau_2 + \left(t - \frac{T}{2}\right) + 2\tau_2 \frac{e^{-t/\tau_2}}{1+e^{-T/\tau_2}}\right], \quad o \leq t < T \qquad (6.10)$$

Let the smallest non-negative value of t for which $v_4(t) = o$ be denoted by t_o. Therefore the phase characteristic $\theta_1(T)$ of unit no. 1 is

$$\theta_1(T) = \begin{cases} t_o & , \text{ if } \dot{v}_4(t_o) < o \\ t_o + T & , \text{ if } \dot{v}_4(t_o) > o \end{cases} \qquad (6.11)$$

From Eq. (4.10) and Figure 6.9, the output $v_1(t)$ of $H_1(s)$ is determined by

$$v_1(t) = \pm k_1 \left[1 - \frac{2e^{-(t+T-t_o)/\tau_1}}{1+e^{-T/\tau_1}}\right], \quad o \leq t < t_o \qquad (6.12a)$$

where the plus sign before k_1 is used when $\dot{v}_4(t_o) > o$, and the minus sign when $\dot{v}_4(t_o) < o$; and

$$v_1(t) = \mp k_1 \left[1 - \frac{2e^{-(t-t_o)/\tau_1}}{1+e^{-T/\tau_1}}\right], \quad t_o \leq t < T \qquad (6.12b)$$

where the minus sign before k_1 is used when $\dot{v}_4(t_o) > o$, and the plus sign when $\dot{v}_4(t_o) < o$. The output $v_3(t)$ of $H_2(s)H_3(s)$ is determined by Eq. (4.10):

$$v_3(t) = k_2k_3\left[1 - \frac{2e^{-t/\tau_2}}{1+e^{-T/\tau_2}}\right], \quad o \leq t < T \qquad (6.13)$$

The time $\Theta^*(T)$, $o \leq \Theta^*(T) < 2T$, at which the output of unit no.2 first jumps from -1 to +1 and at which $v_1(t) - v_3(t) = o$, is the phase characteristic of the open-loop system. The values of T for which $\Theta^*(T) = o$ are the possible half-periods of self oscillation of the closed-loop system.

For reasons of simplicity, the parameters k_1, k_2k_3, and τ_2 are kept fixed: the values used are

$$k_1 = 1 \ , \ k_2k_3 = 1 \ , \ \tau_2 = 1 \ .$$

Three different values of τ_1, $\tau_1 = 0.125$, 0.25, and 0.50, are used to illustrate the effect of the parameter τ_1 on the phase characteristic $\Theta^*(T)$ of the system. Figure 6.13 shows the plot of $\Theta^*(T)$ vs T for the above-mentioned values of k_1, k_2k_3, τ_2, and also shows the effect of varying τ_1. The possible half-periods of self oscillation are

$$T = 0.725 \text{ for } \tau_1 = 0.125,$$

$$T = 0.925 \text{ for } \tau_1 = 0.25,$$

and

$$T = 1.025 \text{ for } \tau_1 = 0.50.$$

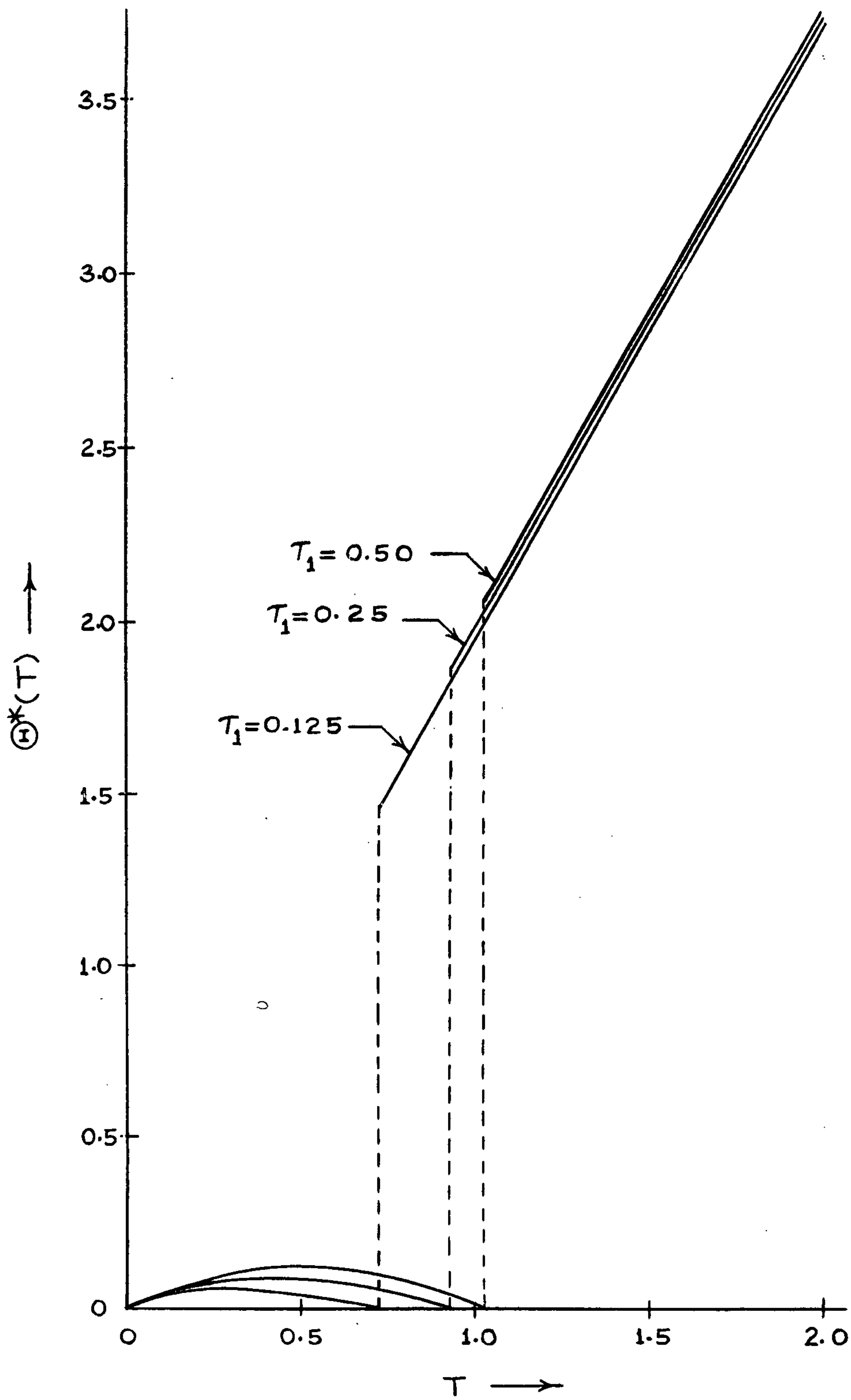


Figure 6.13. Phase characteristic of the system shown in Figure 6.12.

7. MULTILOOP SYSTEMS

In the preceding chapter we presented a method using the phase characteristic concept for the determination of the possible periods of symmetric oscillations in a double-loop system containing an arbitrary number of on-off elements. This method may also be applied to any multiloop system, containing any number of on-off elements, and in which all the loops can be opened simultaneously by opening the system at one point. If there exists no one point which can open all the loops simultaneously, then an entirely new method of attack must be developed.

This chapter will be devoted to systems composed of the three types of unit systems shown in Figure 7.1. Methods of finding the phase characteristic of the basic units designated

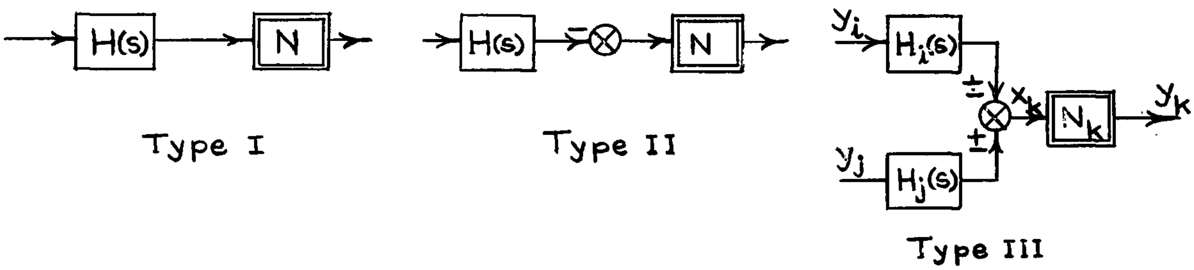


Figure 7.1. Basic unit systems under consideration.

type I and type II are indicated in Chapter 4. The manner of describing the phase characteristic patterns of the type III basic unit will now be discussed.

If all the on-off elements are without a dead zone, then the general forms of the inputs to and output of the type III unit system are as shown in Figure 7.2. Let γT be the phase lag of $y_j(t)$ with respect to $y_i(t)$. Clearly, as γ varies between the limits $0 \leq \gamma < 2$ we generate the possible situations that

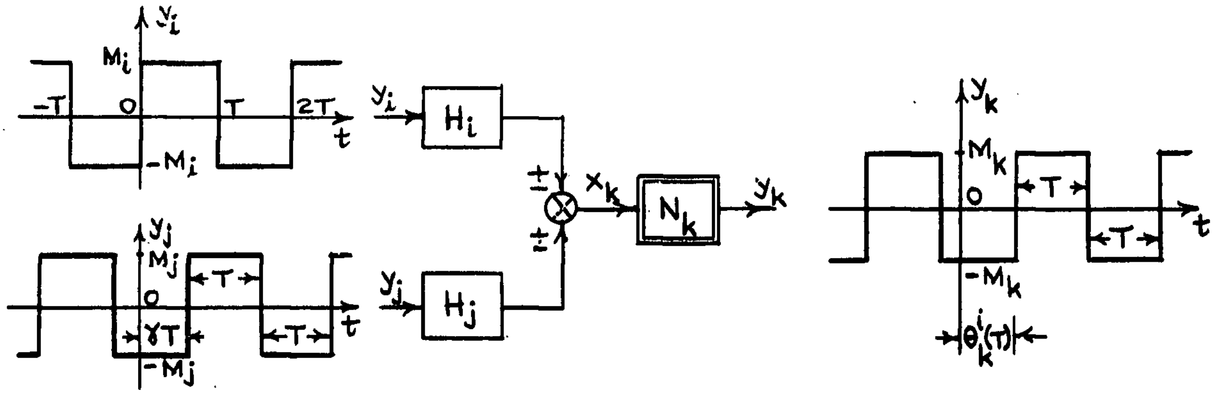


Figure 7.2. Phase characteristic notations and conventions for the type III unit system.

will occur in the presence of simple periodic phenomena with half-period T. Let $\theta_k^i(T,\gamma)$ be the phase characteristic of the output y_k of N_k relative to the input y_i to H_i; similarly, $\theta_k^j(T,\gamma)$ will denote the phase characteristic of y_k relative to y_j. For any fixed value of γ in $0 \leq \gamma < 2$ we can determine $\theta_k^i(T,\gamma)$. Since the phase relationship between y_i and y_j is given, this means that $\theta_k^j(T,\gamma)$ is known once $\theta_k^i(T,\gamma)$ has been determined. In fact,

$$\theta_k^j(T,\gamma) = \begin{cases} \theta_k^i(T,\gamma) - \gamma T, & \text{for } \theta_k^i(T,\gamma) \geq \gamma T \\ \theta_k^i(T,\gamma) - \gamma T + 2T, & \text{for } \theta_k^i(T,\gamma) < \gamma T \end{cases} \tag{7.1}$$

Consequently, by allowing γ to take on fixed values in the interval $o \leq \gamma < 2$ we can determine the phase characteristics for both $\theta_k^i(T,\gamma)$ and $\theta_k^j(T,\gamma)$ with γ as the parameter. For definiteness we will use the notation $\theta_k^i(T,\gamma_j^i)$ to represent the phase characteristic of y_k relative to y_i when y_j lags y_i by γT.

Having examined the phase relationships in the type III unit system, we can now determine the possible periods of self oscillation for the double-loop system in Figure 6.6 (a) by the following new approach.

For self oscillations of half-period T to occur, the reduced phase characteristic of each loop must be zero simultaneously. The new approach uses the information concerning the reduced phase characteristics of all the loops.

The system in Figure 6.6 (a) consists of basic units of type I and one basic unit of type III. (The various basic units are shown in Figure 6.1.) The phase characteristics of the individual units, namely

$$\theta_i(T) \text{ for all units except } i = n_1 + 1$$

and

$$\theta^{n_1}_{n_1+1}(T,\gamma^{n_1}_{n_3}) \text{ and } \theta^{n_3}_{n_1+1}(T,\gamma^{n_1}_{n_3}) \text{ for values of } \gamma$$

in the range $o \leq \gamma < 2$, are determined by the methods presented in Chapter 4.

With both the inner and outer loops (of Figure 5.6 (a)) open at A and B, the total phase characteristic of the inner loop is

$$\Theta_1(T,\gamma) = \theta_{n_1+1}^{n_3}(T,\gamma_{n_3}^{n_1}) + \sum_{i=n_1+2}^{n_3} \theta_i(T)$$

and that of the outer loop is (7.2)

$$\Theta_2(T,\gamma) = \theta_{n_1+1}^{n_1}(T,\gamma_{n_3}^{n_1}) + \sum_{i=n_1+2}^{n_2} \theta_i(T) + \sum_{i=n_3+1}^{n_4} \theta_i(T) + \sum_{i=1}^{n_1} \theta_i(T)$$

where $o \leq \gamma < 2T_o$. The corresponding reduced phase characteristics are then evaluated:

$$\Theta_i^*(T,\gamma) = \Theta_i(T,\gamma) - \left[\!\left[\frac{\Theta_i(T,\gamma)}{2T}\right]\!\right] 2T, \quad (i = 1,\ 2) \qquad (7.3)$$

The values of γ and T at which the reduced phase characteristics $\Theta_i^*(T,\gamma) = o$ are now plotted on a γ-T plane as curves of $\gamma = f_i(T)$, $(i = 1,\ 2)$, as shown in Figure 7.3. The reduced phase characteristics of the two loops are simultaneously zero for values of T at the intersection of the $f_1(T)$ and $f_2(T)$ curves. These values of T are possible half-periods of self oscillation for the closed-loop system.

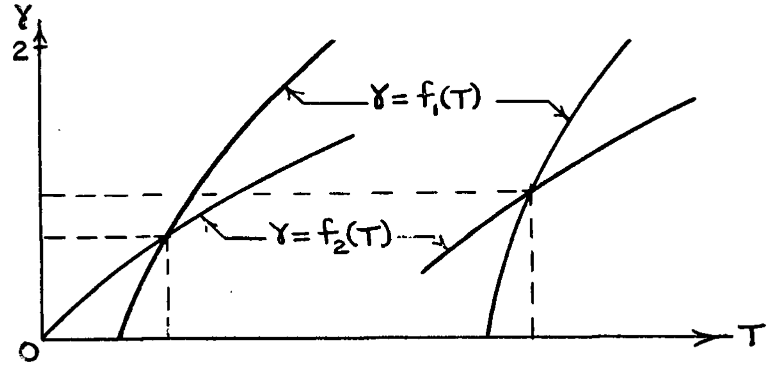


Figure 7.3. Curves of $\gamma = f_1(T)$ and $\gamma = f_2(T)$.

Possible periods of self oscillation in a more complex system

As the multiloop system increases in complexity, so does the procedure for the determination of the possible periods of oscillations. Nevertheless, a solution is possible in every case provided that we are willing to carry out the necessarily increased labor. For illustrative purposes we consider the four-loop system as shown in Figure 7.4.

The steps in the determination of the sought-for values of T are as follows:

(i) We first decompose the system into unit systems of the types I, II and III.

(ii) The phase characteristics of these unit systems are then evaluated. Let these be denoted by

$$\theta_i(T) \text{ for } i = 1, 2, \ldots, n_8 \text{ but } i \neq n_1+1,\ n_3+1,\ n_5+1\ ,$$

$$\theta^{n_1}_{n_1+1}(T, \gamma^{n_1}_{n_3}),\ \theta^{n_3}_{n_1+1}(T, \gamma^{n_1}_{n_3}),\ \theta^{n_2}_{n_3+1}(T, \gamma^{n_2}_{n_5}),\ \theta^{n_5}_{n_3+1}(T, \gamma^{n_2}_{n_5})\ ,$$

$$\theta^{n_4}_{n_5+1}(T, \gamma^{n_4}_{n_7}), \text{ and } \theta^{n_7}_{n_5+1}(T, \gamma^{n_4}_{n_7})\ .$$

Instead of a single value of γ (the quantity γ is the relative phase shift between the two inputs to a type III unit system), as in the case of the system of Figure 6.6 (a) with one type III unit system, we now have three values of γ because there are three type III unit systems. We therefore proceed thus:

(iii) We open loops 1, 2, and 3 at A, B, and C, as shown in Figure 7.4. The total phase characteristics $\Theta_1(T,\gamma)$, $\Theta_2(T,\gamma)$,

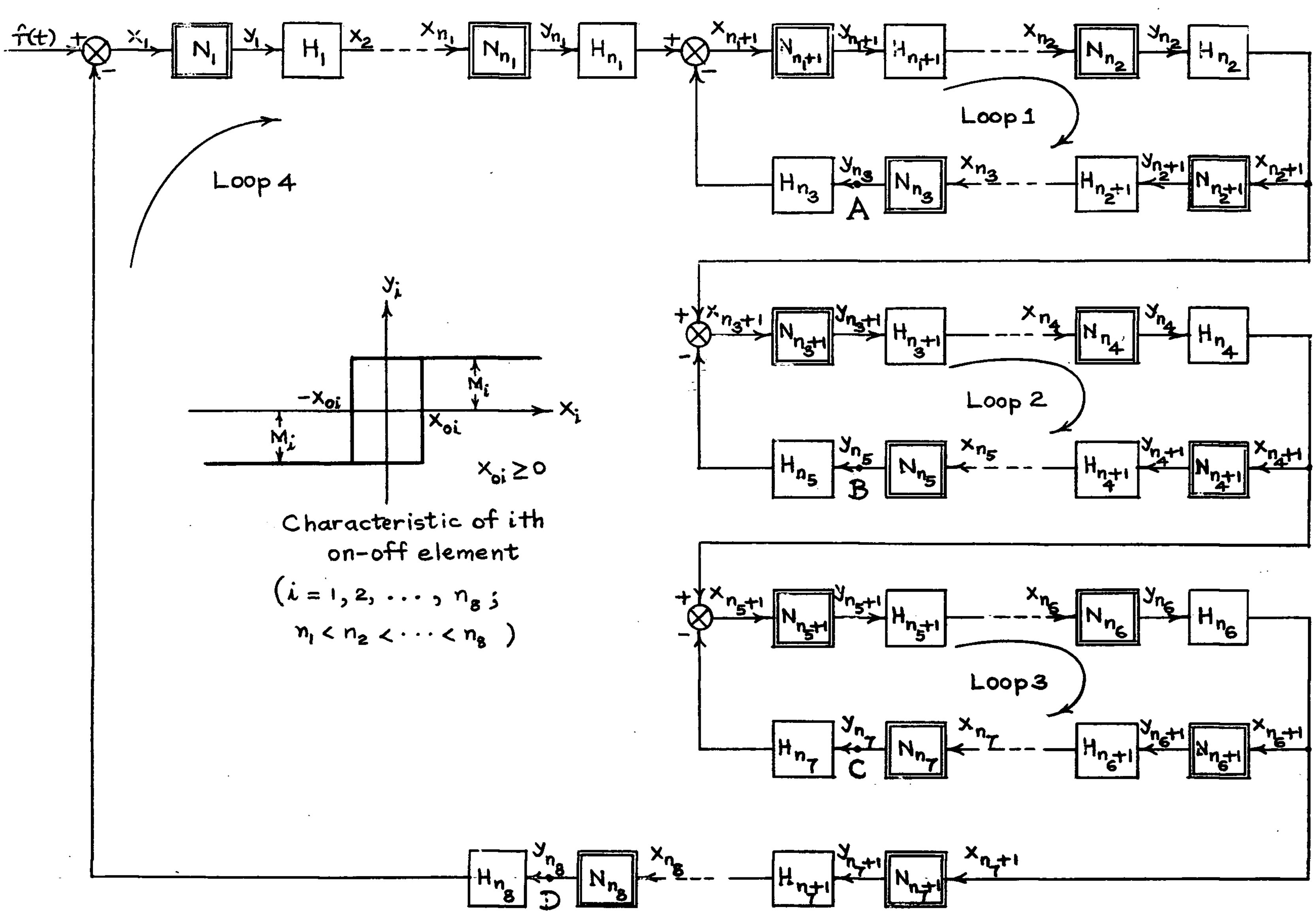


Figure 7.4. Four-loop system containing an arbitrary number of on-off elements.

and $\Theta_3(T,\gamma)$ of the open loops 1, 2, and 3, respectively, are determined, with the input phase shift variable γ ($0 \leq \gamma < 2$) as a parameter in each case:

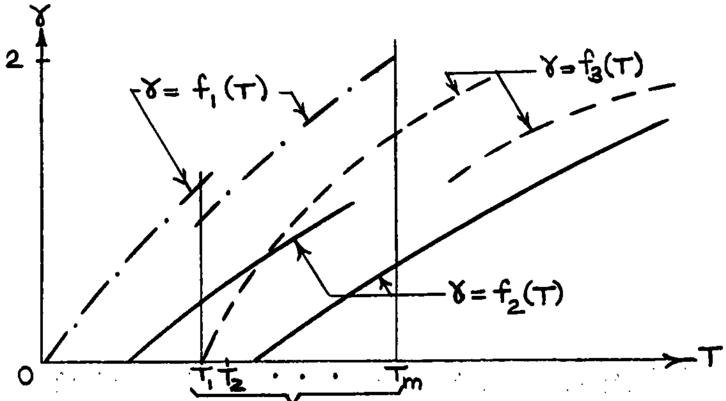
$$\left.\begin{aligned}
\Theta_1(T,\gamma) &= \theta_{n_1+1}^{n_3}(T,\gamma_{n_3}^{n_1}) + \sum_{i=n_1+2}^{n_3} \theta_i(T) \\
\Theta_2(T,\gamma) &= \theta_{n_3+1}^{n_5}(T,\gamma_{n_5}^{n_2}) + \sum_{i=n_3+2}^{n_5} \theta_i(T) \\
\Theta_3(T,\gamma) &= \theta_{n_5+1}^{n_7}(T,\gamma_{n_7}^{n_4}) + \sum_{i=n_5+2}^{n_7} \theta_i(T)
\end{aligned}\right\} \qquad (7.4)$$

From these we obtain the reduced phase characteristic for loops 1, 2, and 3:

$$\Theta_i^*(T,\gamma) = \Theta_i(T,\gamma) - \left[\!\left[\frac{\Theta_i(T,\gamma)}{2T}\right]\!\right] 2T \;, \quad (i = 1, 2, 3) \qquad (7.5)$$

If we now open loop 4 at D and close loops 1, 2, and 3, then the values of T corresponding to the zeros of $\Theta_i^*(T,\gamma)$, $(i = 1, 2, 3)$, may permit periodic oscillations to occur in loops 1, 2, and 3 simultaneously. The problem remaining is to determine from these values of T those that will allow periodic oscillations to occur simultaneously in all loops when loop 4 is closed. We solve this problem as follows:

(iv) The pairs of values (γ,T) corresponding to the zeros of the reduced phase characteristics $\Theta_i^*(T,\gamma)$ of loops 1, 2, and 3 are plotted as curves of $\gamma = f_i(T)$, $(i = 1, 2, 3)$, as shown in Figure 7.5.



An interval of T over which $\gamma = f_i(T)$ exists simultaneously for i = 1, 2, 3.

Figure 7.5. Curves of $\gamma = f_i(T)$ for i = 1, 2, 3 showing range of possible half-periods of oscillations in loops 1, 2, and 3.

(v) We consider only those intervals of T (in Figure 7.5) for which all $f_i(T)$ exist simultaneously; this means that on any vertical line through the γ-T plot, there exists a triplet of γ that determine a value of T such that oscillations are possible in loops 1, 2, and 3. However, if at a particular value T_o, the quantities $\gamma_1 = f_1(T_o)$, $\gamma_2 = f_2(T_o)$ exist but $\gamma_3 = f_3(T_o)$ does not, then oscillations of half-period T_o are possible in loops 1 and 2 but not in loop 3.

(vi) Sequences of values of T, say $T_1, T_2, \ldots, T_m$, covering the intervals of T in which $f_i(T)$ exist simultaneously for i = 1, 2, 3 are selected. At each value of T_i (i = 1,...,m) we read off the corresponding triplet $\gamma_{n_3}^{n_1}$, $\gamma_{n_5}^{n_2}$, and $\gamma_{n_7}^{n_4}$ from Figure 7.5. From the set of phase characteristics obtained in step (ii) we find the values of the phase characteristics of three type III units: namely $\theta_{n_1+1}^{n_1}(T,\gamma_{n_3}^{n_1})$, $\theta_{n_3+1}^{n_2}(T,\gamma_{n_5}^{n_2})$, and $\theta_{n_5+1}^{n_4}(T,\gamma_{n_7}^{n_4})$ for the above T_i and triplets of γ.

(vii) We now open loop 4 at D and form the total phase characteristic of this loop for the above T_i and triplets of γ:

$$\Theta_4(T_i) = \theta_{n_1+1}^{n_1}(T_i,\gamma_{n_3}^{n_1}) + \theta_{n_3+1}^{n_2}(T_i,\gamma_{n_5}^{n_2}) + \theta_{n_5+1}^{n_4}(T,\gamma_{n_7}^{n_4})$$

$$+\sum_{k=n_1+2}^{n_2}\theta_k(T_i) + \sum_{k=n_3+2}^{n_4}\theta_k(T_i) + \sum_{k=n_5+2}^{n_6}\theta_k(T_i) + \sum_{k=n_7+2}^{n_8}\theta_k(T_i) \tag{7.6}$$

At this stage we know that oscillations of half-period T (where T belongs to the above-chosen intervals) are possible in loops 1, 2, and 3. From among these values of T, we find those that will make the reduced phase characteristic $\Theta_4^*(T)$ of loop 4 equal to zero; self oscillations may occur at such values of T for which $\Theta_4^*(T) = o$, when loop 4 is closed.

Forced oscillations

The possible periods of forced oscillations are determined in precisely the same manner as the earlier indicated methods.

More complicated systems may be studied by the above-mentioned method or slight modifications of it.

PART III

ON-OFF ELEMENTS WITH PROPORTIONAL BAND

8. ON-OFF ELEMENTS WITH PROPORTIONAL BAND

In Parts I and II we considered ideal on-off elements. Let us now turn our attention to on-off elements with a proportional band. Examples of some of the characteristics of such elements are shown in Figure 8.1.

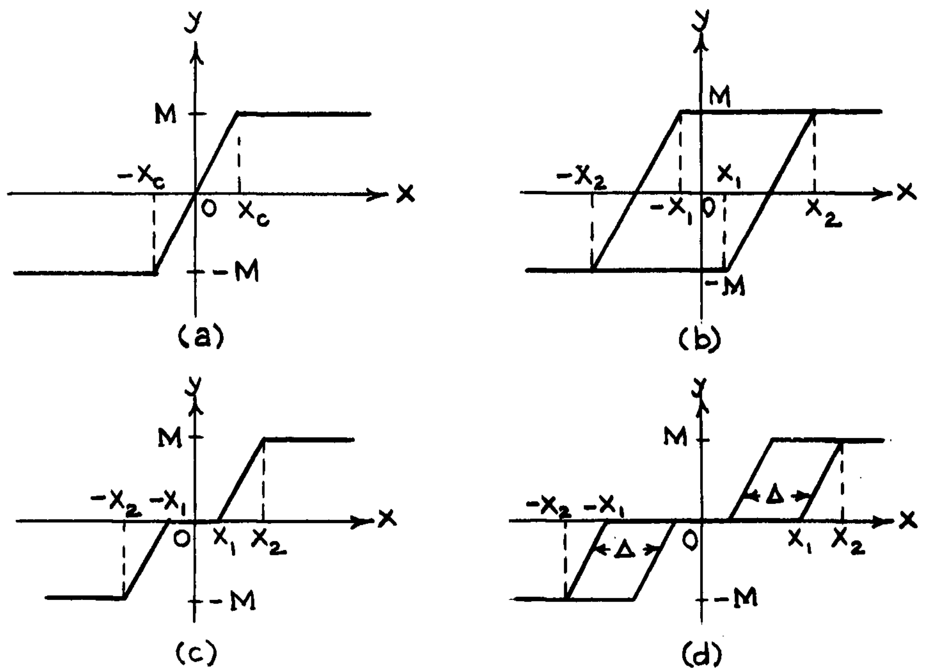


Figure 8.1. Characteristics of some on-off elements with proportional band.

(a) Without hysteresis and dead zone.
(b) With hysteresis and without dead zone.
(c) Without hysteresis and with dead zone.
(d) With hysteresis and with dead zone.

8.1 TRANSIENT RESPONSE OF A SINGLE-LOOP SYSTEM CONTAINING ONE ON-OFF ELEMENT WITH PROPORTIONAL BAND

The system under consideration is shown in Figure 8.2.

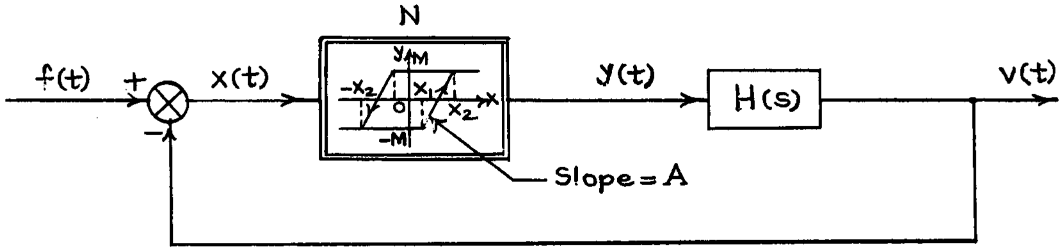


Figure 8.2. Block diagram of single-loop system containing one on-off element with proportional band.

Suppose that the error signal x(t) remains in the linear regions for all times t in the intervals

$$T_n \leq t \leq T_n + h_{n+1}, \quad (n = o, 1, 2, \ldots)$$

where, without loss of generality, we take $T_o = o$, and stays in the saturation regions for the remaining intervals

$$T_n + h_{n+1} \leq t \leq T_{n+1}, \quad (n = o, 1, 2, \ldots)$$

Let the transform of the initial conditions referred to the output of the linear part H(s) be denoted by $V_o(s)$. Then, an equivalent system, shown in Figure 8.3, consists of a number of samplers operating in parallel; the number of samplers depends on the number of times the error signal passes through the linear region of N. The samplers that correspond to operation in the linear regions have inputs denoted by $X_n(s)$, where $X_n(s) = X(s)$ for n = o, 1, 2, ...,;the sampler with input $X_n(s)$ is closed during the interval $T_n \leq t \leq T_n + h_{n+1}$, and open otherwise. The quantities $X_{np}(s)$ are the p-transforms of $X_n(s)$.[13] The sampler with input $\pm M$ is closed during the saturation intervals and open otherwise;

$A_{np}(s)$ is the p-transform of the output of this sampler.

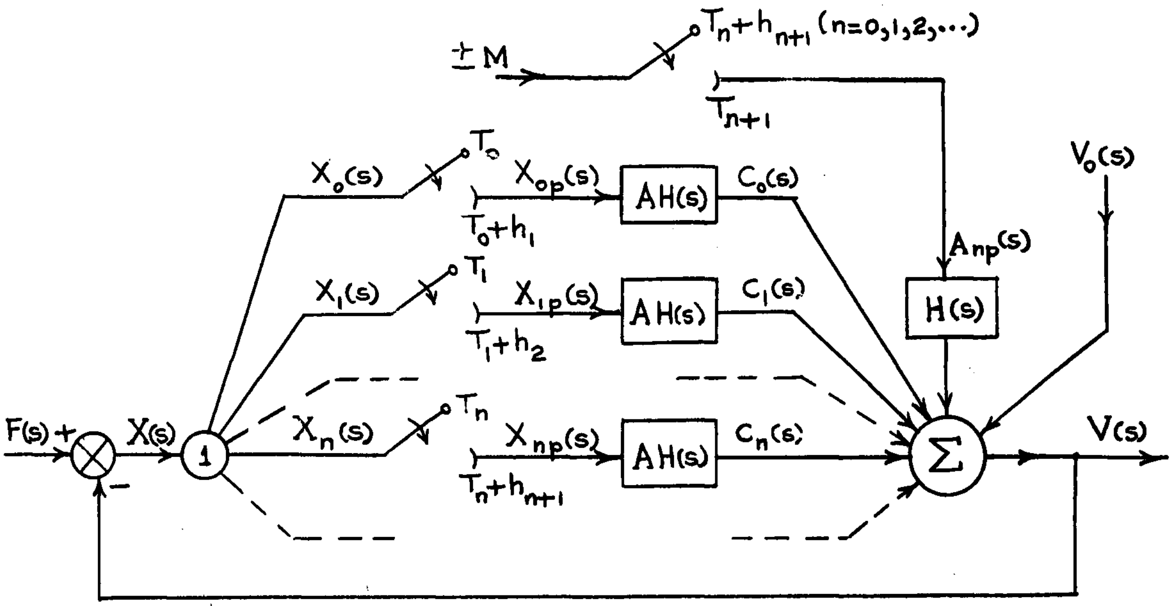


Figure 8.3. System equivalent to that of Figure 8.2.

Let us now evaluate the response of the above system for the different time intervals $(T_n, T_n + h_{n+1})$ and $(T_n + h_{n+1}, T_{n+1})$, $(n = o, 1, 2, \ldots)$. Figure 8.4 gives the equivalent system for the time interval $o \leq t < h_1$. (Note that $T_o = o$.) The input $X_o(s)$ to the sampler is given by

$$X_o(s) = F_o(s) - C_o(s)$$

$$= F_o(s) - X_o(s)AH(s);$$

that is,

$$X_o(s) = \frac{F_o(s)}{1 + AH(s)}, \quad o \leq t < h_1.$$

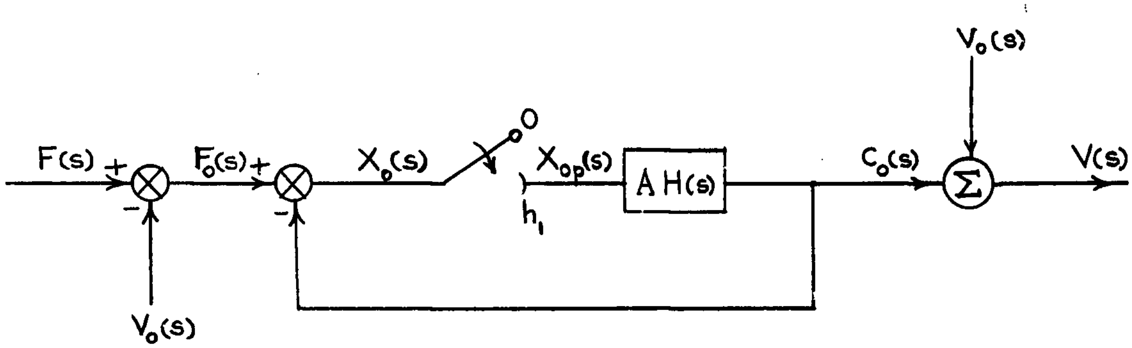


Figure 8.4. Equivalent system for the interval $o \leq t < h_1$.

Now at $t = h_1$ the sampler is opened and the input to H(s) is equal to zero for $t > h_1$, i.e. we can define

$$x_{op}(t) = \begin{cases} x_o(t) \ , & \text{for } o \leq t < h_1 \\ o \ , & \text{for } t > h_1 \end{cases}$$

Therefore

$$x_{op}(t) = x_o(t) \left[u(t) - u(t - h_1)\right], \text{ for } t > o.$$

Using the complex convolution integral we get the Laplace transform of $x_{op}(t)$:

$$X_{op}(s) = \frac{1}{2\pi j} \int_C X_o(\nu) \frac{1 - e^{-(s-\nu)h_1}}{s - \nu} d\nu$$

where C is a contour enclosing all the poles of $X_o(\nu)$ or $\left[1 - e^{-(s-\nu)h_1}\right]/(s-\nu)$ in a mathematically positive or negative sense respectively. Using the p-transform notation of the theory of sampled-data systems,[13] namely

$$P_{T_n}^{T_n+h_{n+1}}\left[E(s)\right] \triangleq \frac{1}{2\pi j}\int_C E(\nu)\,\frac{e^{-(s-\nu)T_n} - e^{-(s-\nu)(T_n+h_{n+1})}}{s-\nu}\,d\nu \qquad (8.1)$$

we get the transform of the component of the output from the first pulse:

$$C_o(s) = X_{op}(s)AH(s) = AH(s)\,P_o^{h_1}\left[\frac{F_o(s)}{1+AH(s)}\right],\ t \geq o.$$

Consequently, the output of the system is

$$V(s) = V_o(s) + AH(s)\,P_o^{h_1}\left[\frac{F_o(s)}{1+AH(s)}\right],\ \text{for } o \leq t < h_1.$$

For the duration $h_1 \leq t < T$, we have the additional component

$$B_o(s) = \pm M\,\frac{H(s)}{s}(e^{-sh_1} - e^{-sT_1}),\ \text{for } t > h_1$$

due to the saturation effect. Hence the total output of the system is

$$V(s) = V_o(s) + AH(s)\,P_o^{h_1}\left[\frac{F_o(s)}{1+AH(s)}\right] \pm M\,\frac{H(s)}{s}\,(e^{-sh_1} - e^{-sT_1}),$$

$$h_1 \leq t < T_1 \qquad (8.2)$$

or, in shorter notation,

$$V(s) = V_o(s) + C_o(s) + B_o(s)\ ,\ h_1 \leq t < T_1$$

$$= D_o(s)\ \text{say}.$$

Since $F(s)$, $H(s)$ are known and $V_o(s)$ is known or can be determined, the output $v(t)$ may be evaluated from the inverse of $V(s)$ for the interval in question.

Let us now consider the output for the duration $T_1 \leq t < T_1 + h_2$. The equivalent system for this period is shown in Figure 8.5.

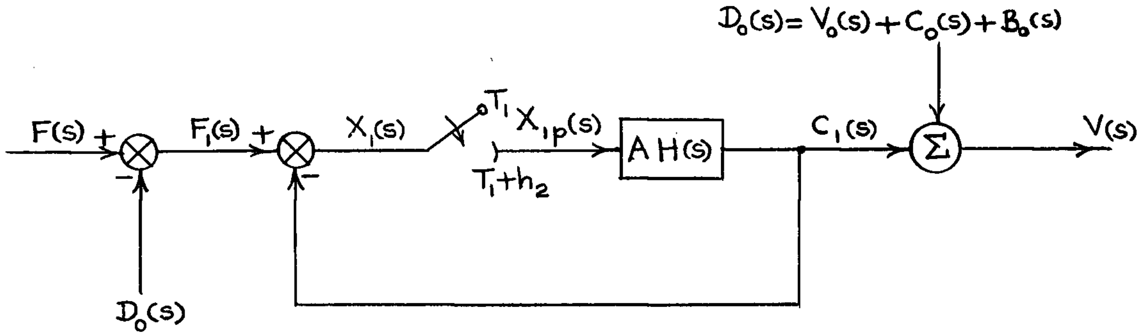


Figure 8.5. Equivalent system for the interval $T_1 \leq t < T_1 + h_2$.

Since the sampler in Figure 8.5 is open during $o < t < T_1$, the input $f_1(t) = \mathcal{L}^{-1}\left(F_1(s)\right)$ to this sampler has no effect on the output component $c_1(t)$ for $o < t < T_1$. We can therefore replace $f_1(t)$ by a new function $f_{11}(t)$:

$$f_{11}(t) = \begin{cases} o & , \text{ for } o < t < T_1, \\ f_1(t) & , \text{ for } t > T_1, \end{cases}$$

which may also be written as

$$f_{11}(t) = f_1(t)\ u(t - T_1).$$

In terms of the p-notation, the Laplace transform of $f_{11}(t)$ is

$$F_{11}(s) = \overset{\infty}{P}_{T_1}\left[F_1(s)\right] = \overset{\infty}{P}_{T_1}\left[F(s) - D_o(s)\right]$$

Consequently, the error for this duration, namely

$$x_1(t) = f_1(t) - c_1(t)$$

may likewise be replaced by

$$x_{11}(t) = f_{11}(t) - c_1(t),$$

which states that the effective error may be regarded as zero for the equivalent system during the interval $o < t < T_1$.

In order to calculate $X_{1p}(t)$ conveniently, we let t_1 represent a new time axis such that

$$t_1 = t - T_1.$$

Therefore

$$f_{11}(t) = f_{11}(t_1 + T_1) , \quad c_1(t) = c_1(t_1 + T_1) ,$$

$$x_{11}(t) = x_{11}(t_1 + T_1) , \quad x_{1p}(t) = x_{1p}(t_1 + T_1) .$$

The introduction of the new time axis t_1 renders the situation identical to that of the equivalent system for the interval $o < t < h_1$; that is, the input $f_{11}(t)$ is sampled for the period $o < t_1 < h_2$ and is fed to a system with zero initial conditions. Consequently,

$$\mathcal{L}\left(c_1(t)\right) = AH(s) \, \overset{h_2}{\underset{o}{P}} \left[\frac{\mathcal{L}\left(f_{11}(t)\right)}{1 + AH(s)} \right] .$$

By making use of the relationship

$$\mathcal{L}\left(g(t)\right) = \mathcal{L}\left(g(t_1 + T_1)\right) ,$$

that is, $$G(s) = e^{-sT_1} \mathcal{L}\left(g(t_1)\right) ,$$

and replacing $F_{11}(s)$ by $\overset{\infty}{\underset{T_1}{P}} \left[F(s) - V_o(s) - C_o(s) - B_o(s) \right] = \overset{\infty}{\underset{T_1}{P}} \left[F(s) - D_o(s) \right]$

we finally get the Laplace transform of the component $c_1(t)$ of the output to be

$$C_1(s) = e^{-sT_1} AH(s) \overset{h_2}{P_o^2} \left[\frac{e^{sT_1} \overset{\infty}{P}_{T_1} \left[F(s) - D_o(s)\right]}{1 + AH(s)} \right] .$$

The total output transform for the interval $T_1 \leq t < T_1 + h_2$ is

$$V(s) = V_o(s) + C_o(s) + B_o(s) + C_1(s) .$$

For the duration $T_1 + h_2 \leq t < T_2$ we have the additional component

$$B_1(s) = \pm M \frac{H(s)}{s} \left[e^{-s(T_1+h_2)} - e^{-sT_2} \right] , \quad t > T_1 + h_2$$

due to the saturation effect. Thus the total output transform is

$$V(s) = V_o(s) + C_o(s) + B_o(s) + C_1(s) + B_1(s),$$
$$\text{for } T_1 + h_2 \leq t < T_2 .$$

The generalization to the total output transform for any time interval is now obvious. In fact,

$$V(s) = V_o(s) + \begin{cases} \sum_{k=1}^{n} C_k(s) + \sum_{k=1}^{n-1} B_k(s) & , \text{ for } T_n \leq t < T_n + h_{n+1} \\ \\ \sum_{k=1}^{n} C_k(s) + B_k(s) & , \text{ for } T_n + h_{n+1} \leq t < T_{n+1} \end{cases} \tag{8.3}$$

where $V_o(s)$ represents the initial conditions referred to the output of the original system under consideration, where the component $C_k(s)$ is given by

$$C_k(s) = e^{-sT_k} AH(s)$$
$$\times P_o^{h_{k+1}} \left[\frac{e^{sT_k} \overset{\infty}{P}_{T_k} \left[F(s) - V_o(s) - C_o(s) - B_o(s) - \dots - C_{k-1}(s) - B_{k-1}(s)\right]}{1 + AH(s)} \right]$$

(8.4)

for $t > T_k$,

and where the saturation component $B_k(s)$ is given by

$$B_k(s) = \pm M \frac{H(s)}{s} \left[e^{-s(T_k + h_{k+1})} - e^{-sT_{k+1}} \right]$$

$$\text{for } t > T_k + h_{k+1} \qquad (8.5)$$

Analogous equations can be developed for the transient response in the case where the nonlinear element incorporates a dead-zone.

We have demonstrated above how the superposition principle (as applied to the linear part of the system) and some properties of the p-transform can be used to evaluate the exact response of the system under consideration by means of a step-by-step analysis.

8.2 PERIODIC OSCILLATIONS IN A SINGLE-LOOP SYSTEM CONTAINING ONE ON-OFF ELEMENT WITH PROPORTIONAL BAND

The determination of the periodic states in automatic control systems having a single nonlinear element with piecewise linear characteristic has already received wide attention in the literature.

M. A. Aizerman and F. R. Gantmakher [10, 11] determined the periodic states in nonlinear single-loop systems with a piecewise linear characteristic consisting of segments parallel to two given straight lines. In making use of this method it is necessary to integrate the equations of all the linear systems, into which the system under consideration can be decomposed.

The periodic solutions are then constructed with the help of these integrals.

L. A. Gusev[12] also dealt with the determination of the periodic states of a broader class of single-loop nonlinear control systems, namely, those with nonlinear elements having an arbitrary piecewise linear characteristic. His method does not require the integration of the respective linear equations into which the system may be decomposed. The periodic solutions are determined in the form of a complete Fourier series without neglecting harmonics. The problem here is reduced to solving a set of simultaneous transcendental equations that determine the behaviour in each segment of the characteristic.

In this section we will restrict our attention to a consideration of simple symmetric oscillations in the system as shown in Figure 8.2. We will present two new methods of solving the periodic states in such systems:

1. an approximate method which is valid for the sufficiently large class of systems in which there is some filtering action by the linear part of the system. It has the advantages of being just as simple as but much more accurate than the describing function method in the majority of cases of practical interest.
2. the second method is through the solution of linear Volterra integral equations. Reasonably accurate solutions may be found by the method of successive approximations.

1. The "Trapezoidal" Approximation.

Assume that the system in Figure 8.2 has attained a simple symmetric steady state such that x(t) is in the linear regions of the saturation characteristic (with or without hysteresis) for durations of length hT as shown in Figure 8.6 (a).

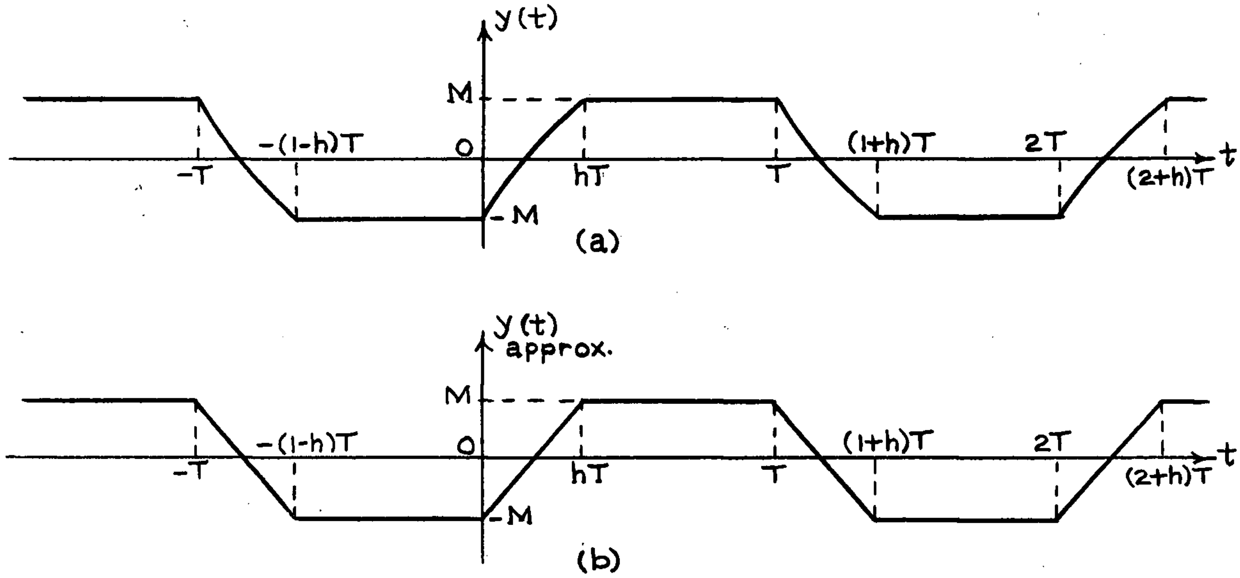


Figure 8.6. (a) Exact output of N in Figure 8.2 in the case of simple symmetric oscillations;

(b) Corresponding approximation when H(s) has a filtering action.

If the filtering action of the linear part of the system H(s) is good and the system input f(t) has a predominant fundamental component, then we can replace the portions of the waveform y(t) in the intervals $nT \leq t \leq nT + h (n = o, \pm 1, \ldots .)$ by straight line segments as shown in Figure 8.6 (b).

The precision of this approximation can be best judged by comparing it with that made by the describing function method.[3] For this purpose, assume that the input to the nonlinear element is sinusoidal. Then the typical output y(t) is a clipped sinusoid as shown in Figure 8.7, where it is assumed that $M<1$. The exact output of N is

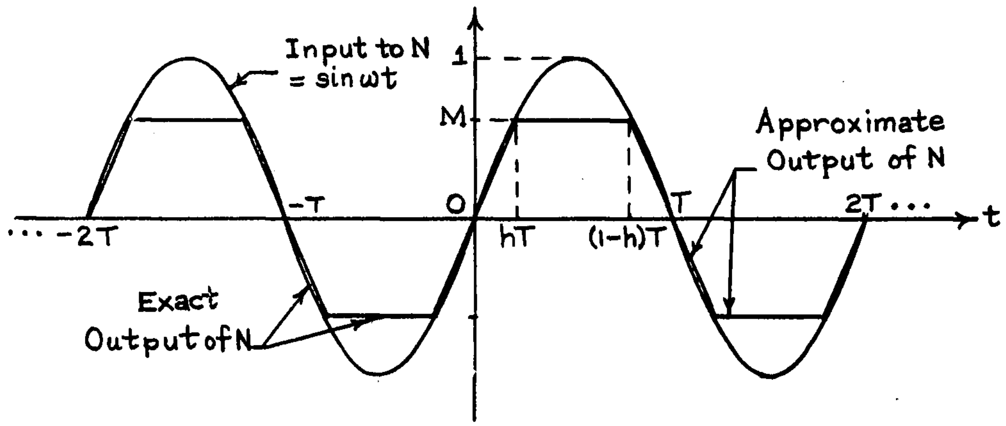


Figure 8.7. Exact and approximate outputs of N for a sinusoidal input.

$$y = \begin{cases} \sin \omega t, \text{ for } (n - h)\, T \leq t \leq (n + h)\, T \\ \qquad\qquad\qquad\qquad\qquad (n = 0, \pm 1, \pm 2, \ldots) \\ (-1)^n M = (\sin \pi h)(-1)^n, \text{ for } (n + h)T \leq t \leq (n + 1 - h)\, T \end{cases}$$

and its Fourier series expansion is

$$y = \frac{2}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left[\frac{\sin(n-1)\pi h}{n-1} + \frac{\sin(n+1)\pi h}{n+1} \right] \frac{\sin n\omega t}{n} \tag{8.6}$$

The approximate output, using straight line segments, is described by

$$y_{ap} = \begin{cases} (-1)^n \dfrac{\sin \pi h}{hT} (t-nT) \text{ , for } (n-h)\, T \leq t \leq (n+h)\, T \\ \qquad\qquad\qquad\qquad\qquad\qquad (n = o, \pm 1, \ldots) \\ (-1)^n M = (-1)^n \sin \pi h \text{ , for } (n+h)\, T \leq t \leq (n+1-h)\, T \end{cases}$$

and its Fourier series expansion is

$$y_{ap} = \frac{4 \sin \pi h}{\pi^2 h} \sum_{\substack{n=1 \\ n \text{ odd}}} \frac{\sin n\pi h}{n} \frac{\sin n\omega t}{n} \tag{8.7}$$

The first few terms of the expansions (8.6) and (8.7) for various values of h are

$$\left.\begin{aligned} y &= 0.944 \sin \omega t + 0.046 \sin 3\omega t - 0.028 \sin 5\omega t + \ldots \\ y_{ap} &= 0.916 \sin \omega t + 0.000 \sin 3\omega t - 0.036 \sin 5\omega t + \ldots \end{aligned}\right\} h = \frac{1}{3}$$

$$\left.\begin{aligned} y &= 0.817 \sin \omega t + 0.106 \sin 3\omega t - 0.021 \sin 5\omega t + \ldots \\ y_{ap} &= 0.814 \sin \omega t + 0.091 \sin 3\omega t - 0.032 \sin 5\omega t + \ldots \end{aligned}\right\} h = \frac{1}{4}$$

$$\left.\begin{aligned} y &= 0.475 \sin \omega t + 0.128 \sin 3\omega t + 0.047 \sin 5\omega t + \ldots \\ y_{ap} &= 0.475 \sin \omega t + 0.128 \sin 3\omega t + 0.046 \sin 5\omega t + \ldots \end{aligned}\right\} h = \frac{1}{8} \tag{8.8}$$

The describing function method ignores all the harmonics and considers only the fundamental component. The trapezoidal approximation, however, takes all the harmonics into consideration. An inspection of Equations (8.8) indicates that the latter approximation is superior to that of the describing function method for inputs clipped to about eighty-seven percent of their amplitudes.

Let us now analyse the periodic states of the system for the shape of the periodic output and the possible periods of oscillation. Consider $y(t)_{approx}$ as shown in Figure 8.6 (b).

Let

$$y_o(t) = \left(\frac{2t}{hT} - 1\right) M \left[u(t) - u(t - hT)\right]$$

$$y_1(t) = M \left[u(t - hT) - u(t - T)\right]$$

Then

$$y_{approx} = \begin{cases} \sum_{n=o}^{\infty} y_o(t + nT)(-1)^n + \sum_{n=1}^{\infty} y_1(t + nT)(-1)^n \\ \qquad\qquad \text{for } + o \leq t < hT \,, \\ \\ \sum_{n=o}^{\infty} (-1)^n \left[y_o(t + nT) + y_1(t + nT)\right] \\ \qquad\qquad \text{for } + hT \leq t < T \end{cases}$$

Now

$$\left.\begin{aligned} \mathcal{L}\left(y_o(t)\right) &= Y_o(s) = \frac{M}{Ths^2}\left[4 - (2 + shT)(1 + e^{-shT})\right] \\ \text{and} \\ \mathcal{L}\left(y_1(t)\right) &= Y_1(s) = \frac{M}{s}(e^{-shT} - e^{-sT}), \end{aligned}\right\} (8.9)$$

so that the output v(t) is given by

$$\left.\begin{aligned} v(t) &= \frac{1}{2\pi j} \oint_{C_1 or C_2} H(s)\, \frac{Y_o(s) - Y_1(s)e^{sT}}{1 + e^{sT}}\, e^{st}\, ds, \text{ for } o \leq t < hT \\ v(t) &= \frac{1}{2\pi j} \oint_{C_1 or C_2} H(s)\, \frac{Y_o(s) + Y_1(s)}{1 + e^{sT}}\, e^{st}\, ds, \text{ for } hT \leq t < T \end{aligned}\right\} (8.10)$$

where C_1 encloses only the poles of $H(s)\left[Y_o(s) - Y_1(s)e^{sT}\right]$ or

$H(s)\left[Y_o(s) + Y_1(s)\right]$, and C_2 encloses only the poles of $1/(1 + e^{sT})$. The contour integrals along C_1 and C_2 are evaluated in a mathematically positive and negative sense respectively. This will be implied for all contour integrals occurring in this chapter. Since $Y_o(s)$ and $Y_1(s)$ are known (Eq. (8.9)), and H(s) is given, the periodic output is determined by (8.10).

Consider the characteristics in Figures 8.1 (a) and 8.1 (b). The conditions for the existence of periodic oscillations are, under the assumption that t = o as shown in Figure 8.6 (a),

$$x\left[(n + h)T\right] = (-1)^n x_c = x\left[(n + 1)T\right] \qquad (8.11a)$$

$$\dot{x}\left[(n + h)T\right](-1)^n > o > \dot{x}\left[(n + 1)T\right](-1)^n \qquad (8.11b)$$

$$(n = o, \pm 1, \pm 2, \ldots)$$

in the case of saturation without hysteresis or dead zone, and are

$$x\left[(n + h)T\right] = (-1)^n x_2,\ x\left[(n + 1)T\right] = (-1)^n(-x_1) \qquad (8.12a)$$

$$\dot{x}\left[(n + h)T\right](-1)^n > o > \dot{x}\left[(n + 1)T\right](-1)^n \qquad (8.12b)$$

$$(n = o, \pm 1, \pm 2, \ldots)$$

in the case of saturation with hysteresis and without dead zone.

In order to determine the possible half-periods of oscillation, we introduce the concept of the Tsypkin loci. These are defined by

$$\left.\begin{aligned} \mathcal{J}(T) &= \frac{T}{\pi}\dot{x}(T) + jx(T) \\ \text{and} \qquad \mathcal{J}(hT) &= \frac{T}{\pi}\dot{x}(hT) + jx(hT) \end{aligned}\right\} \qquad (8.13)$$

Since x(t) is determined by x(t) = f(t) - v(t), as shown in Figure 8.2, and v(t) is a function of h and T, as given by Eqs. (8.9) and (8.10), it follows that the Tsypkin loci $\mathcal{J}(T)$ and $\mathcal{J}(hT)$ are each functions of h and T.

Two Tsypkin loci are required because the system in Figure 8.2 has two switching instants within the half-period T. The imaginary parts of the Tsypkin loci determine the switching instants, and the real parts determine the switching directions. The proper switching instants occur at the intersections of the Tsypkin loci with the line jx_c (in the case of saturation without hysteresis and dead zone); also, from Figure 6.6 (a), the proper switching directions must be in the left-half plane for the $\mathcal{J}(T)$ loci, and the right-half plane for the $\mathcal{J}(hT)$ loci.

<u>Self oscillations in the case of the saturation characteristic.</u>

The Tsypkin loci are plotted with the help of Eqs. (8.10).

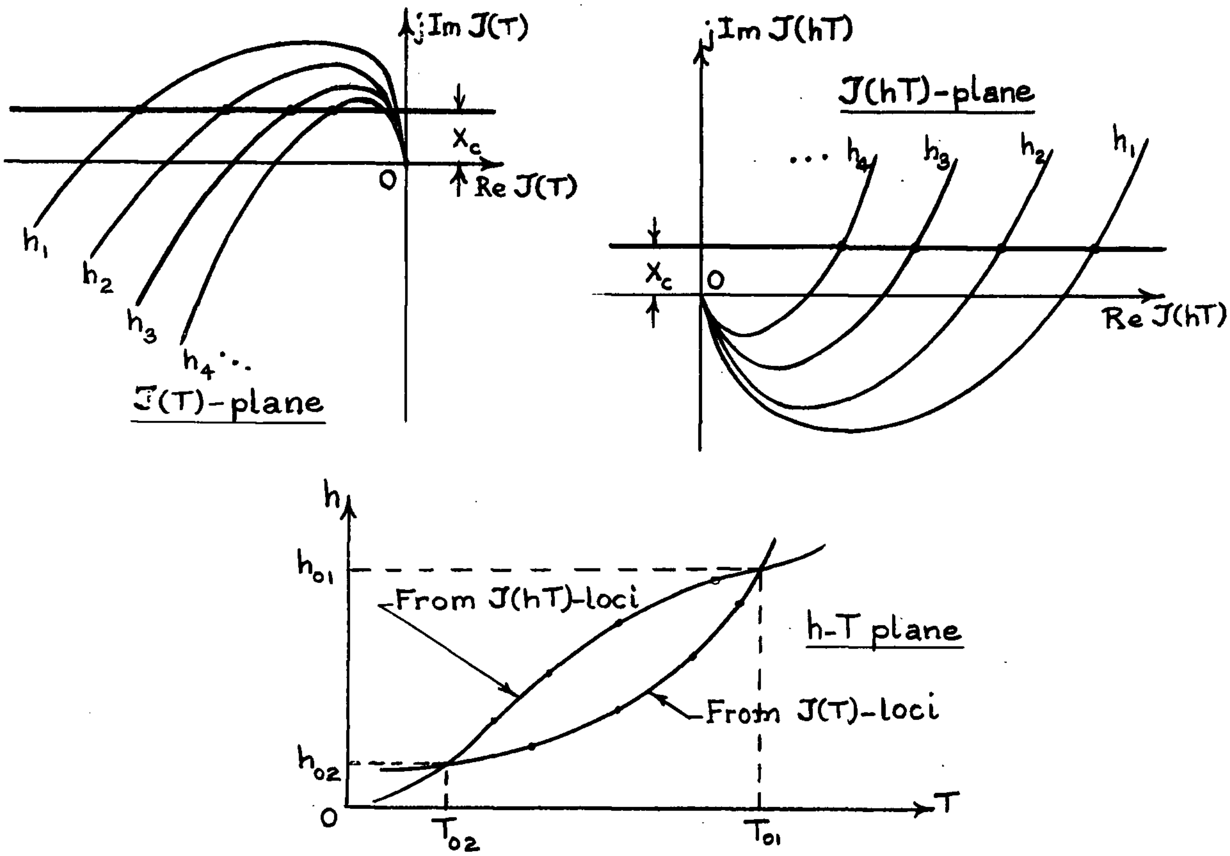


Figure 8.8. Construction for the determination of the possible half-periods of self oscillation.

Using h as the parameter and T as the variable. The straight lines jx_c are next inserted on the $\mathcal{J}(hT)$ and $\mathcal{J}(T)$ planes.

The values of h and T corresponding to the points of intersection of these loci with jx_c are then plotted on the h-T plane. The construction is shown in Figure 8.8. Any pair of values, such as (h_{o1}, T_{o1}) and (h_{o2}, T_{o2}), occurring at the intersection of the resulting curves in the h-T-plane may give rise to self oscillations.

Self oscillations in the case of saturation with hysteresis.

The construction in this case proceeds in precisely the same way as the above, except that instead of the straight lines jx_c we introduce the straight lines $-jx_1$ and jx_2 on the $\mathcal{J}(T)$ and $\mathcal{J}(hT)$ planes respectively, as shown in Figure 8.9.

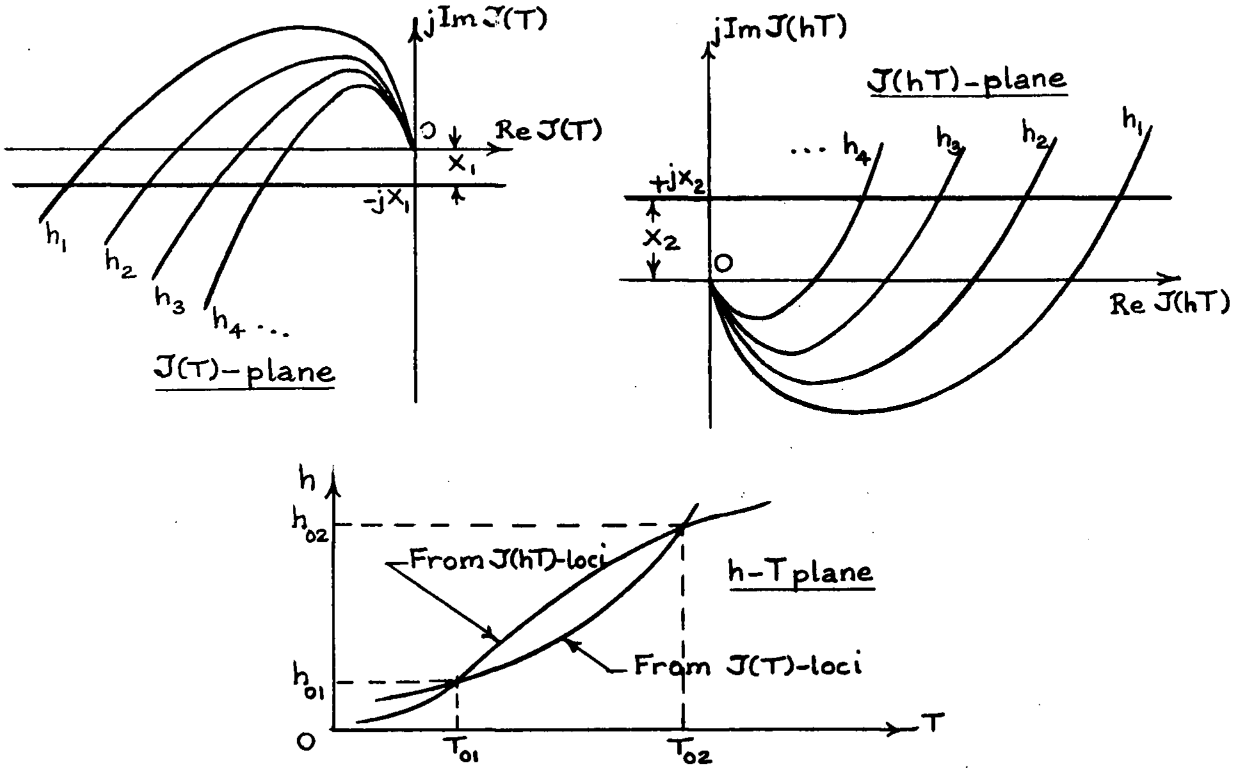


Figure 8.9. Construction for the determination of the possible half-periods of self oscillation in the case of saturation with hysteresis.

Forced oscillations in case of saturation.

In the case of self oscillations $x(t) = -v(t)$ and the unknown quantities are h and T. But in the case of forced oscillations $x(t) = f(t) - v(t)$, T_o the half-period of oscillation is known, and the sought-for quantities are now h and the phase shift τ of f(t) relative to v(t). As in Eq. (5.1), we let

$$f(t) = A_o f_o(t - \tau)$$

where $$A_o = \max \left| f(t) \right|, \text{ and } \max \left| f(t) \right| = 1.$$

The procedure for determining h and τ is as follows. As mentioned earlier, the imaginary parts of the Tsypkin loci determine the switching instants of x(t) and the real parts the switching directions $\dot{x}(t)$. We now have two contributions to x(t) and $\dot{x}(t)$, because x(t) consists of two parts, -v(t) and f(t), where v(t) is determined by Eq. (8.10). The h parameter, $o \leq h < 1$, is varied by choosing a sequence of values, $o < h_1 < h_2 \ldots < h_n = 1$.

The contribution of -v(t) to x(t) for a fixed half-period T_o and for $h = h_i$ appears as the points

$$0_{T,i} = - \frac{T_o}{\pi} \dot{v}(T_o) - jv(T_o) \quad \text{in the } \mathcal{J}(T)\text{-plane,}$$

and the points

$$0_{hT,i} = - \frac{T_o}{\pi} \dot{v}(hT_o) - jv(hT_o) \text{ in the } \mathcal{J}(hT)\text{-plane,}$$

for i = 1, 2,...,n. Using the points $0_{T,i}$ and $0_{hT,i}$ as origins, we next add the contribution due to $f(t) = A_o f_o(t - \tau)$; these contributions, denoted by

$$\mathcal{F}(T_0, \tau) = A_0 \left[\frac{T_0}{\pi} \dot{f}_0 (T_0 - \tau) + j f_0 (T_0 - \tau) \right],$$

appear as closed curves, as τ varies over the range $0 \leq \tau < 2T_0$. The pairs of values (h,τ) at the intersection of the $\mathcal{F}$-curves with the straight lines jx_c', (such pairs must be in the left-half $\mathcal{J}(T)$-plane and in the right-half $\mathcal{J}(hT)$-plane to satisfy the proper switching instants and switching directions) may give rise to forced oscillations. The (h,τ) values are plotted in the $h-\tau$ plane, as shown in Figure 8.10, to give two curves corresponding to each of the $\mathcal{J}$-planes. The points of intersection of the $h-\tau$ curves yield pairs of values (h,τ) for which forced oscillations may occur.

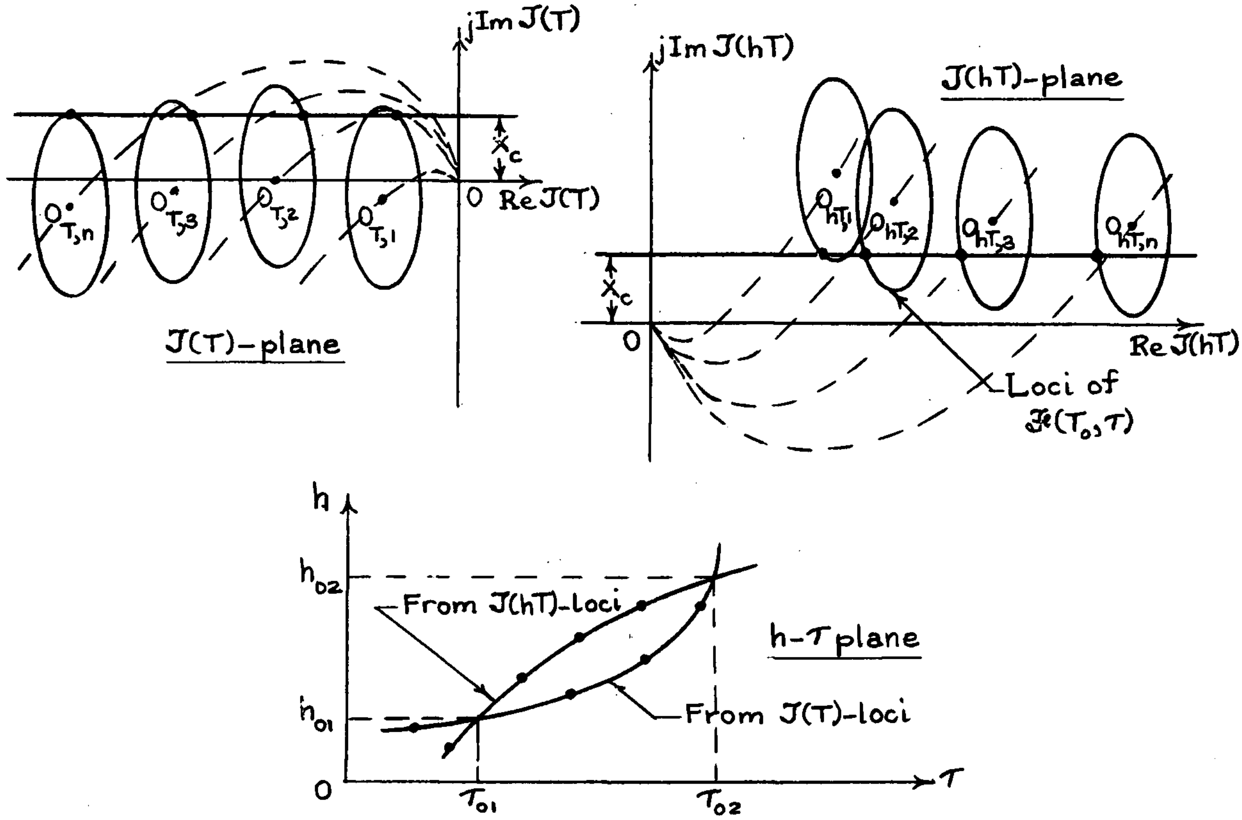


Figure 8.10. Construction to determine values of h and τ that may give rise to forced oscillation.

We observe that we may get more than one h- τ curve from each $\mathcal{J}$-plane, depending upon the complexity of f(t).

An analogous procedure can be used to determine pairs of values (h, τ) that may give rise to forced oscillations in the case of the saturation characteristic with hysteresis.

2. The Integral Equation Approach.

Referring to the exact output y(t) of the nonlinear element, let

$$y_2(t) = A\ x(t)\left[u(t) - u(t - hT)\right]$$

Then the Laplace transform of the output of the linear part of the system, V(s), has, by an argument analogous to that used in deriving Equation (8.10), the form

$$V(s) = \left\{ \begin{array}{ll} \left[\dfrac{Y_2(s) - Y_1(s)\ e^{sT}}{1 + e^{sT}}\right] H(s) \ , & \text{for } o \leq t < hT \\ \left[\dfrac{Y_2(s) + Y_1(s)}{1 + e^{sT}}\right] H(s) \ , & \text{for } hT \leq t < T \end{array} \right\} \quad (8.14)$$

where

$$Y_2(s) = A\ \mathcal{L}\left(x(t)\ \left[u(t) - u(t - hT)\right]\right)$$

and

$$Y_1(s) = \frac{M}{s}\ (\ e^{-shT} - e^{-sT}).$$

Let

$$v_1(t) = \frac{1}{2\pi j} \oint_{C_1 or C_2} \frac{H(s)\ Y_1(s)}{1 + e^{sT}}\ e^{st}\ ds,$$

where C_1 is a contour which encloses only the poles of $H(s)Y_1(s)$ and C_2 encloses only the poles of $1/(1 + e^{sT})$. This expression for $v_1(t)$ can be evaluated by the methods described in Chapter 3. Furthermore, let

$$\varpi(t) = \frac{1}{2\pi j} \oint_{C_1 \text{ or } C_2} \frac{H(s)}{1+e^{sT}} e^{st} \, ds$$

where C_1 encloses only the poles of H(s) and C_2 encloses only the poles of $1/(1 + e^{sT})$. Recall that

$$v(t) = f(t) - v(t).$$

By using the real convolution integral, and the expressions for $\varpi(t)$, v(t) and $Y_2(s)$ above, the inverse Laplace transform of Eq. (8.14) yields, upon rearrangement of its terms,

$$\begin{aligned} x(t) &= f(t) + v_1(t+T) - A\int_0^t x(\tau)\left[u(t) - u(t-hT)\right]\varpi(t-\tau)d\tau \\ &\qquad \text{for } o \leq t < hT, \\ x(t) &= f(t) - v_1(t) - A\int_{hT}^t x(\tau)\left[u(t) - u(t-hT)\right]\varpi(t-\tau)d\tau \\ &\qquad \text{for } hT \leq t < T. \end{aligned} \tag{8.15}$$

These equations are linear Volterra integral equations of the second kind with x(t) as the only unknown.[14] Such equations are readily solved by Picard's process of successive approximations. Practical solutions of such equations may be found by means of a repetitive differential analyzer.[15]

PART IV

THE STABILITY PROBLEM

9. STABILITY OF PERIODIC STATES IN ON-OFF SYSTEMS WITH OR WITHOUT A PROPORTIONAL BAND

The investigation of the possible periods of the periodic states, including both self and forced oscillations, was considered in the preceding chapters. Now the question of the stability of these periodic states acquires considerable importance. Only when stable can these periodic states be observed in systems physically. Before investigating the stability problem, let us first review the concept of stability that will be used.

9.1 THE CONCEPT OF STABILITY OF PERIODIC STATES

In this study we will consider the concept of stability in the sense of Lyapunov,[16] and in particular asymptotic stability in the small, or, as it is sometimes called, local stability.

Let $\tilde{x}(t)$ define a periodic state, the stability of which is to be investigated. According to A. M. Lyapunov, we determine the stability of the periodic state by studying the behaviour of the neighbouring non-periodic states. The non-periodic states close to the periodic one are excited by the introduction of a sufficiently small disturbance; such a non-periodic state may be represented by

$$x(t) = \tilde{x}(t) + \xi(t), \tag{9.1}$$

where $\xi(t)$ is the deviation from the periodic state.

Definition 1. If the deviation $\xi(t)$, after the removal of the sufficiently small disturbance, approaches zero asymptotically as time increases, that is

$$\lim_{t\to\infty} \xi(t) = o, \tag{9.2}$$

then the periodic state investigated is said to be asymptotically stable in the small or in the sense of Lyapunov. This means that as time increases all sufficiently close non-periodic states approach the periodic state asymptotically.

If, however, under the above-mentioned conditions $|\xi(t)|$ increases indefinitely as time becomes indefinitely large, then the periodic state under consideration is said to be unstable.

Definition 2. In this case we consider any non-periodic state; all states other than the periodic state investigated are referred to as non-periodic states. The quantity $\xi(t)$ is now the deviation (from the periodic state) caused by any disturbance, regardless of size. If $|\xi(t)|$ approaches zero as time increases, no matter what the disturbance may be, then the periodic state investigated is said to be asymptotically stable in the large or globally stable.

In this thesis we will be concerned with only the problem of asymptotic stability in the small. For simplicity, whenever we speak of stability in the remainder of this chapter we shall always mean asymptotic stability in the small.

To investigate the asymptotic stability in the small of the on-off systems considered, we will use one of the classical methods of Lyapunov. In this method we form the equation of motion with respect to the deviation $\xi(t)$ by replacing, in the general equations governing the behaviour of the system, the periodic solution $\tilde{x}(t)$ by $x(t) = \tilde{x}(t) + \xi(t)$ and rejecting in these equations all terms containing powers of $\xi(t)$ exceeding the first. Consequently, a linear equation in $\xi(t)$ is obtained; this equation is referred to as the equation of the first approximation or the variational equation. Moreover, in the case under consideration this equation has periodic coefficients.

According to a theorem of A. M. Lyapunov, if the solution $\xi(t)$ of the variational equation approaches zero as time approaches infinity, then the periodic state investigated is asymptotically stable, regardless of the nonlinear terms neglected in the initial equation. In the case of an unbounded increase of $|\xi(t)|$ the periodic state is said to be unstable.

It may happen that the solution $\xi(t)$ of the variational equation neither approaches zero nor approaches infinity in absolute value as time increases indefinitely, but merely remains bounded in absolute value. In such cases it is impossible, in general, to ascertain the stability or instability of the system by means of the variational equation. But in the systems under consideration, a theorem of I. G. Malkin[17, 18] shows that in this critical case the variational equation still gives an answer to the stability problem.

Lyapunov's method applies to equations containing continuous nonlinear and linear functions. On-off systems, however, are usually described in terms of discontinuous functions. Hence, a rigorous investigation in such cases requires that all arguments be conducted with continuous functions which approximate the discontinuous functions with any degree of accuracy, and uses the limiting process to obtain the behaviour of the system described by discontinuous functions.

Without claiming mathematical rigor, we will use a method which makes use of the unit step and delta functions for the systems under consideration. This method, besides leading to the very same results as the rigorous but cumbersome approach, possesses the advantage that, from the physical point of view, it is very graphic.

9.2 VARIATIONAL EQUATION FOR SINGLE-LOOP SYSTEM CONTAINING AN ELEMENT WITH A SATURATION CHARACTERISTIC

For the purpose of investigating the stability of a given periodic state in a single-loop system containing an on-off element with a proportional band, let us first form the variational equation. Without loss of generality, we assume that the nonlinear characteristic ($y = \Phi(x)$) is an odd function.

Let us suppose that

$$\tilde{x}(t) = \tilde{f}(t) - \tilde{v}(t) \tag{9.3}$$

corresponds to the periodic state of frequency ω_o. The quantity $\tilde{x}(t)$, defining the periodic control signal to the nonlinear element, satisfies the equation

$$\mathcal{L}\left(\tilde{x}(t)\right) = \mathcal{L}\left(\tilde{f}(t)\right) - H(s)\,\mathcal{L}\left(\Phi(\tilde{x}(t))\right) \tag{9.4}$$

Suppose that somewhere in the system at time $t = o$, there arises a sufficiently small disturbance (for example, a change in initial conditions, or the application of some external action), which breaks the periodic state $\tilde{x}(t)$ and excites the neighbouring non-periodic state $x(t) = \tilde{x}(t) + \xi(t)$. The small

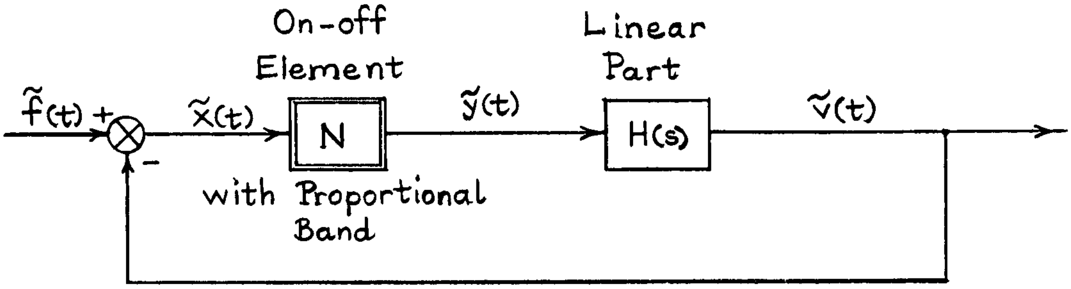


Figure 9.1. A single-loop system containing one on-off element.

disturbance can be transferred to the input of the system, where it will be designated by $f_d(t)$. Equation (9.4) now becomes

$$\mathcal{L}\left(\tilde{x}(t) + \xi(t)\right) = \mathcal{L}\left(\tilde{f}(t) + f_d(t)\right) - H(s)\mathcal{L}\left(\Phi\left[\tilde{x}(t) + \xi(t)\right]\right) . \tag{9.5}$$

The difference between Equations (9.5) and (9.4) gives the equation for the deviation $\xi(t)$ from the periodic state:

$$\mathcal{L}\left(\xi(t)\right) = \mathcal{L}\left(f_d(t)\right) - H(s)\mathcal{L}\left(\Phi\left[\tilde{x}(t) + \xi(t)\right] - \Phi\left(\tilde{x}(t)\right)\right).$$

This equation is nonlinear in $\mathcal{L}\left(\xi(t)\right)$. Assume that $\xi(t)$ is sufficiently small; then

$$\Phi\left[\tilde{x}(t) + \xi(t)\right] - \Phi\left(\tilde{x}(t)\right) \doteq \frac{\Phi\left[\tilde{x}(t) + \xi(t)\right] - \Phi\left(\tilde{x}(t)\right)}{\xi(t)}\,\xi(t)$$

$$= \Phi'\left[\tilde{x}(t)\right]\,\xi(t) + \text{higher order terms},$$

where Φ' denotes the derivative with respect to its argument. Disregarding terms in $\xi(t)$ of degree higher than the first, we obtain the variational equation for the system under consideration:

$$\mathcal{L}\left(\xi(t)\right) = \mathcal{L}\left(f_d(t)\right) - H(s)\mathcal{L}\left(\Phi'\left[\tilde{x}(t)\right]\xi(t)\right) \tag{9.6}$$

This equation is linear in $\xi(t)$ and has periodic coefficients by virtue of the presence of $\Phi'\left[\tilde{x}(t)\right]$. As indicated earlier, the behaviour of the solution of this equation determines the asymptotic stability of the periodic state $\tilde{x}(t)$.

In the general case of an arbitrary $\Phi(x)$ the investigation of the exact solutions of this variational equation meets with insurmountable difficulties. By virtue of the specific characteristics $\Phi(x)$ under consideration, it is possible to carry out the investigation of the stability of the periodic states by comparatively simple and well-known methods.

Let us first consider the case where $\Phi(x)$ is the saturation characteristic, as shown in Figure 9.2 (a). The derivative of this characteristic is

$$\Phi'(x) = A\left[u(x+x_c) - u(x-x_c)\right] \tag{9.7}$$

so that

$$\Phi'\left[\tilde{x}(t)\right] = A\left[u(\tilde{x}+x_c)-u(\tilde{x}-x_c)\right]$$

where $\tilde{x} \equiv \tilde{x}(t)$ is a periodic solution of frequency ω_0.

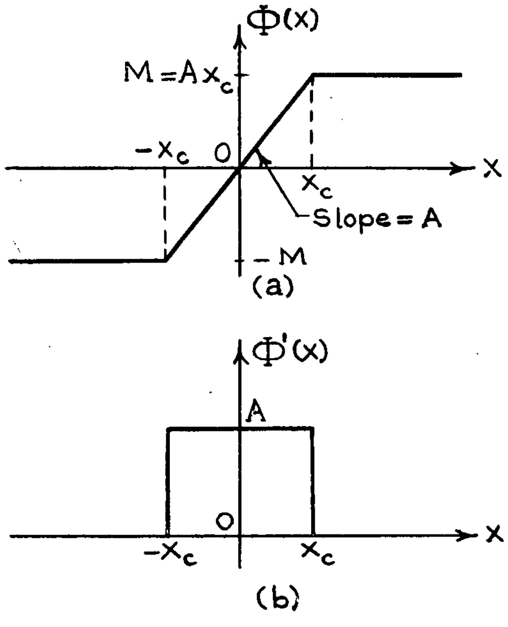


Figure 9.2. (a) Saturation characteristic,
(b) Its derivative.

The expression for $\Phi'\left[\tilde{x}(t)\right]$ is easily and graphically determined by means of the transfer diagram with the help of $\Phi'\left[x\right]$ as shown in Figure 9.3. Furthermore, let us assume that $\tilde{x}(t)$ is a simple symmetric periodic state of half-period T. With no loss in generality, we can choose the time axis t such that $\tilde{x}(o) = -x_c$ and $\tilde{x}'(o) > o$. Let $\tilde{x}(t)$ be equal to x_c at $t = h < T$. Then

$$\Phi'\left[\tilde{x}(t)\right] = A \sum_{k=o}^{\infty} \left[u(t - kT) - u(t - kT - h)\right] \tag{9.8}$$

where u(t) is the unit step function.

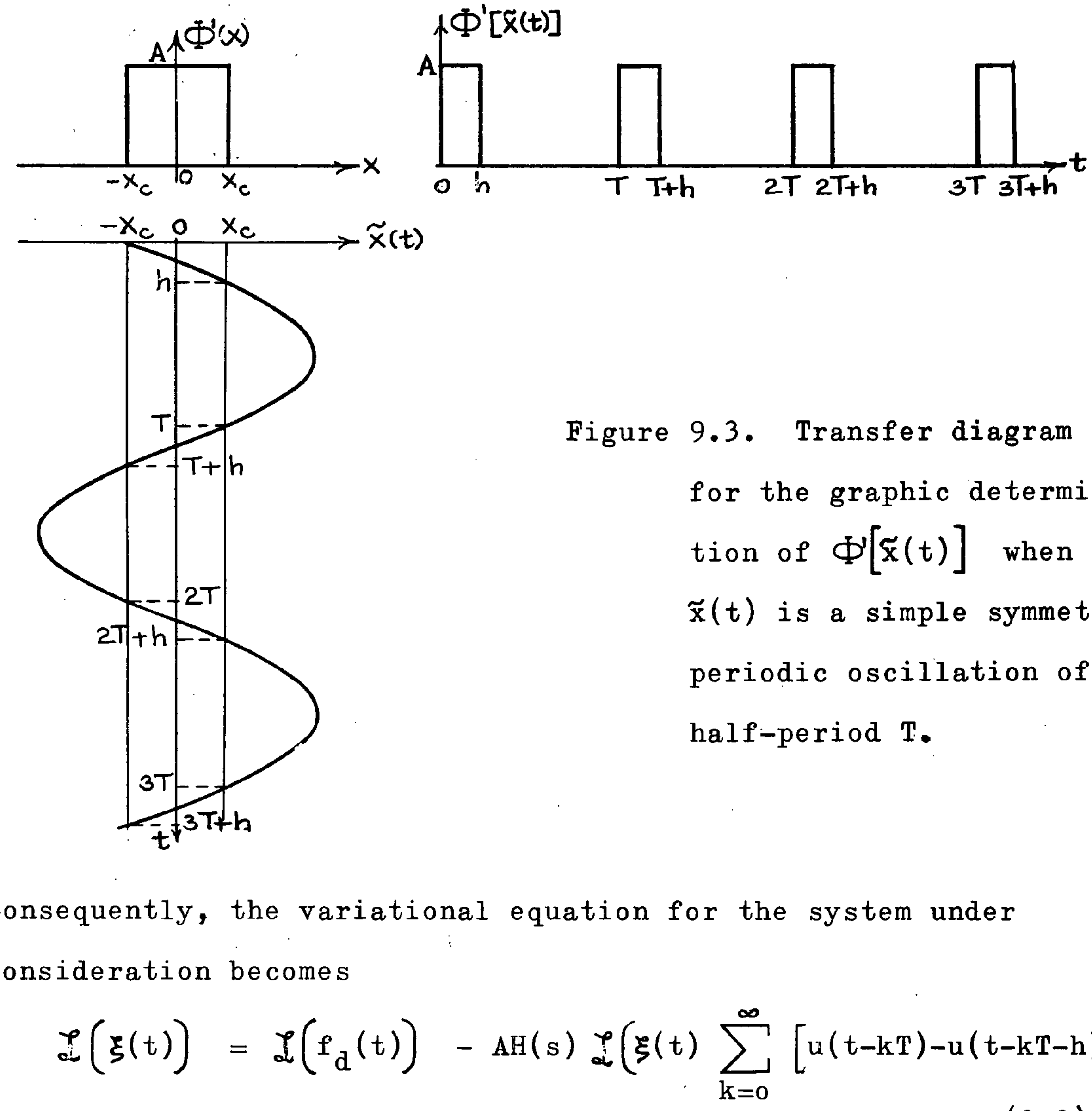


Figure 9.3. Transfer diagram for the graphic determination of $\Phi'\left[\tilde{x}(t)\right]$ when $\tilde{x}(t)$ is a simple symmetric periodic oscillation of half-period T.

Consequently, the variational equation for the system under consideration becomes

$$\mathcal{L}\Big(\xi(t)\Big) = \mathcal{L}\Big(f_d(t)\Big) - AH(s)\,\mathcal{L}\Big(\xi(t) \sum_{k=o}^{\infty} \left[u(t-kT)-u(t-kT-h)\right]\Big) \tag{9.9}$$

Using the notation

$$\mathcal{L}\left(\xi(t)\right) = \Xi(s) \ , \ \mathcal{L}\left(f_d(t)\right) = F_d(s) \ ,$$

$$\mathcal{L}\left(\xi(t)\sum_{k=o}^{\infty}\left[u(t-kT) - u(t-kT-h)\right]\right) = P_{h,T}\left[\Xi(s)\right] \quad ,$$

where the symbol $P_{h,T}\left[\quad\right]$ represents the p-transform notation used by Farmanfarma and Jury,[13, 19] Eq.(9.9) takes the form

$$\Xi(s) = F_d(s) - P_{h,T}\left[\Xi(s)\right] \ AH(s). \tag{9.10}$$

We now make the observation that equation (9.9) or (9.10) corresponds to the linear feedback finite pulse width sampling system, as shown in Figure 9.4, in which $\xi(t)$ is sampled periodically with period T for finite durations of length h and then fed to the linear transfer function AH(s). Hence the asymptotic stability of the periodic state $\tilde{x}(t)$ can be deduced

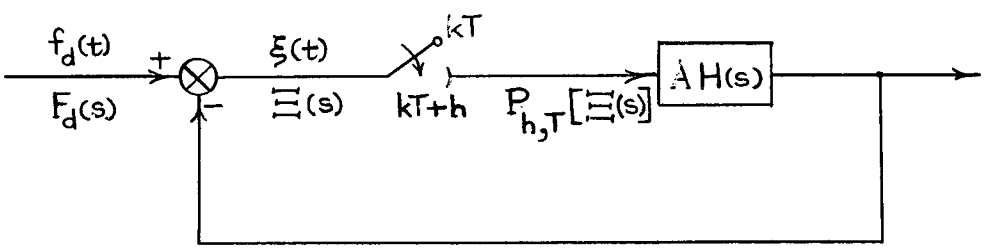


Figure 9.4. Linear system equivalent to Equation (9.9) or (9.10).

from an investigation of the stability of the equivalent finite pulse width sampled-data system depicted in Figure 9.4. The stability of the latter system is well-known, and an excellent discussion of this topic can be found in Farmanfarma[19] and in Jury.[13]

The above solution of the (asymptotic) stability problem is a generalization of that given by Tsypkin. It is of interest to consider the limiting cases of the above system:

1. h = T. In this case $u_p(t) \stackrel{\Delta}{=} \sum_{k=o}^{\infty} \left[u(t-kT) - u(t-kT-h)\right]$ becomes the unit step function u(t). This means that operation is confined to the linear portion of the characteristic, and the problem is reduced to a consideration of the stability of a simple linear feedback system.

2. h = o and $u_p(t)$ has any finite amplitude.

In this case $\mathcal{L}\left(u_p(t)\right) = o$, so that the sampler output is zero, and the system remains at rest. This case would be possible if $\tilde{x}(t)$ were a square wave of amplitude $>x_c$ with half-period T.

3. h = o but $u_p(t)$ becomes $\delta_T(t)$, a sequence of unit impulses.

Under these conditions, $x_c = o$, and the nonlinear characteristic $\Phi(x)$ becomes the ideal on-off element without a proportional band. This is the case considered by Tsypkin.[6] We now obtain

$$\Phi' \left[\tilde{x}(t)\right] = 2M\delta\left[\tilde{x}(t)\right].$$

Since

$$\dot{u}\left[\tilde{x}(t)\right] = \sum_{k=o}^{\infty} (-1)^k \delta(t - kT),$$

and

$$\dot{u}\left[\tilde{x}(t)\right] = \delta\left[\tilde{x}(t)\right] \dot{\tilde{x}}(t),$$

it follows that the delta function of a periodic argument can be expressed as

$$\delta\left[\tilde{x}(t)\right] = \sum_{k=o}^{\infty} \frac{\delta(t - kT)}{|\dot{\tilde{x}}(kT)|} ,$$

where kT $(k = o, 1, \ldots)$ are the roots of the equation $\tilde{x}(t) = o$, assuming, of course, that $\tilde{x}(o) = o$. Because of the periodicity of $\tilde{x}(t)$ we have

$$\delta\left[\tilde{x}(t)\right] = \frac{1}{|\dot{\tilde{x}}(T)|} \sum_{k=o}^{\infty} \delta(t - kT)$$

$$= \frac{1}{|\dot{\tilde{x}}(T)|} \quad \delta_T(t) .$$

Consequently, Eq. (9.10) reduces to

$$\Xi(s) = F_d(s) - \frac{H(s)}{|\dot{\tilde{x}}(T)|} \; \Xi^*(s) \qquad (9.11)$$

where

$$\Xi^*(s) = \mathcal{L}\left(\xi(t) \quad \delta_T(t)\right) .$$

Hence, the problem of the asymptotic stability in the case of the simple on-off characteristic is reduced to a consideration of a simple linear feedback sampled-data system corresponding to the system in Figure 9.4, but in which A is now replaced by $1/|\dot{\tilde{x}}(t)|$.

4. <u>h is small compared to the time constants of the system.</u>

This situation arises if $T >> h$, i.e. the magnitude and periodicity of $\tilde{x}(t)$ are such that, effectively, the nonlinear characteristic possesses an exceedingly narrow proportional band. The output of the nonlinearity, due to the input over this duration, can be approximated by replacing the finite pulses by impulses of equivalent area. Let us remark that if H(s) has a discontinuous impulse response the modified z-transform, and not the z-transform, may be used to give a true approximation of the component of the response for the time duration $nT + h \leq t \leq (n + 1)T$ arising from the input component $\xi(t)\left[u(t - nT) - u(t - nT - h)\right]$; whereas if H(s) has a continuous impulse response, we may use either the z-transform or the modified z-transform for this purpose. But the true approximation of the response during the interval $nT < t < nT + h$ cannot be estimated. On the other hand this effect will be negligible when h is sufficiently small and H(s) has a continuous impulse response. The exact behaviour, however, can be evaluated by means of p-transform methods.

So far we have considered only the case of the saturation characteristic shown in Figure 9.2 (a). Let us now consider the (asymptotic) stability problem for various types of saturation characteristics. The other types of characteristics considered and their derivatives are shown in Figure 9.5.

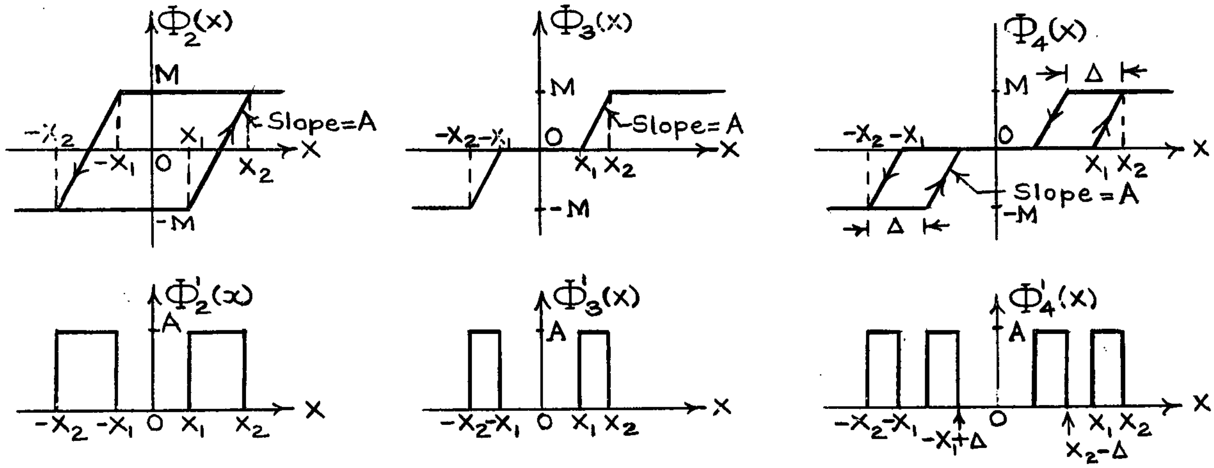


Figure 9.5. Form of derivatives $\Phi'(x)$ for various types of saturation characteristics.

Case of $\Phi_2(x)$

For the saturation characteristic with hysteresis, illustrated in Figure 9.5 (a), we have

$$\Phi_2'(x) = \begin{cases} A\left[u(x - x_1) - u(x - x_2)\right], & \text{for } \dot{x} > 0 \\ A\left[u(x + x_2) - u(x + x_1)\right], & \text{for } \dot{x} < 0 \end{cases} \tag{9.12}$$

The transfer diagram for the determination of $\Phi_2'\left[\tilde{x}(t)\right]$

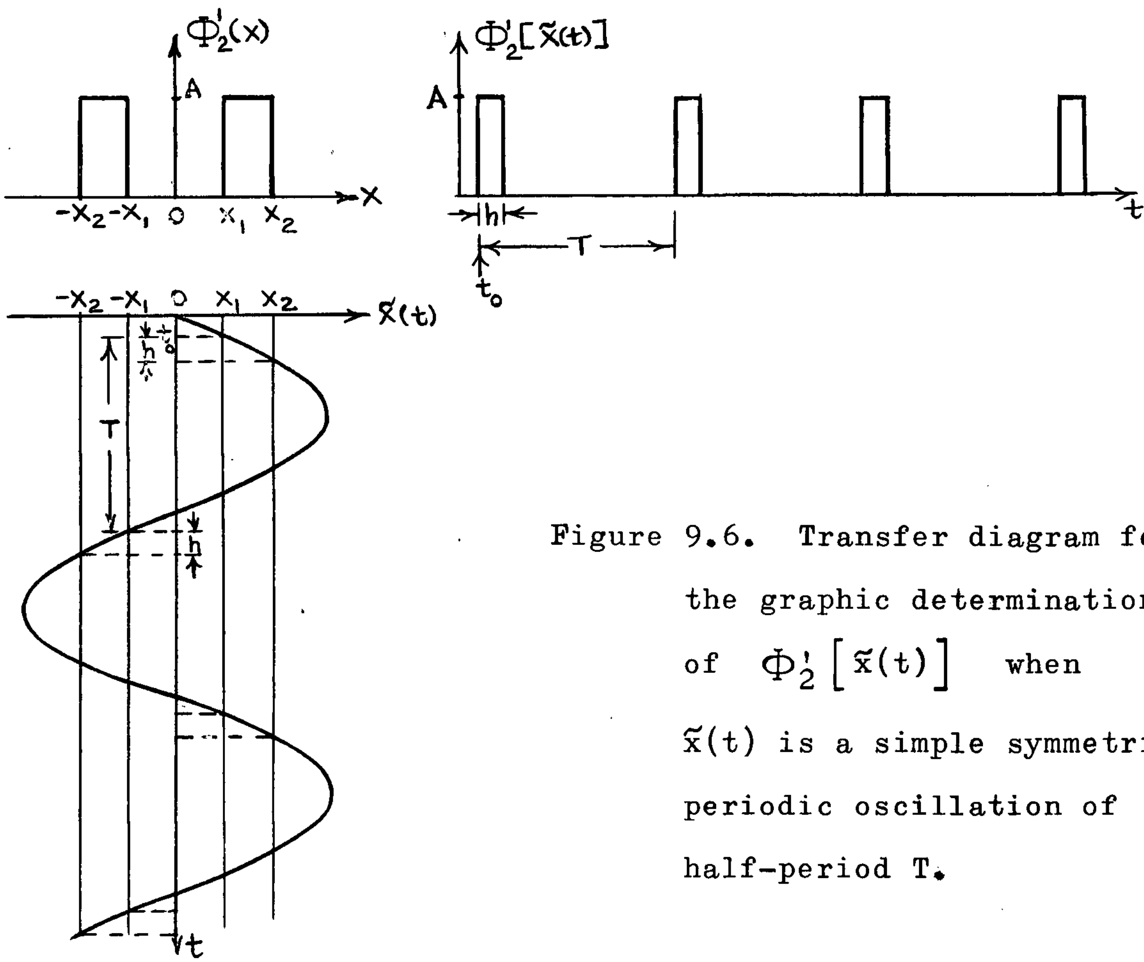


Figure 9.6. Transfer diagram for the graphic determination of $\Phi_2'\left[\tilde{x}(t)\right]$ when $\tilde{x}(t)$ is a simple symmetric periodic oscillation of half-period T.

gives
$$\Phi_2'\left[\tilde{x}(t)\right] = A\sum_{k=0}^{\infty}\left[u(t-t_o-kT) - u(t-t_o-kT-h)\right] \tag{9.13}$$

Since the choice of the initial time instant is arbitrary, then the displacement t_o does not influence the form of the variational equation, which is thus given by

$$\Xi(s) = F_d(s) - AH(s)\, P_{h,T}\left[\Xi(s)\right] \quad . \tag{9.14}$$

Equations (9.10) and (9.14) are the same, except that the values of h are, in general, different. Hence, the stability of the system containing a characteristic with saturation and hysteresis can again be deduced from the behaviour of the simple feedback sampled-data system with finite pulse width.

Cases of $\Phi_3(x)$ and $\Phi_4(x)$

The cases of characteristics with dead zone and with or without hysteresis will yield variational equations of the same form - just as the cases of characteristics without dead zone and with or without hysteresis. Consequently, it is sufficient to consider the case of $\Phi_4(x)$.

Clearly, from Figure 9.5 (c),

$$\Phi_4'(x) = \begin{cases} A\left[u(x - x_1) - u(x - x_2) \right. \\ \quad \left. + u(x + x_2 - \Delta) - u(x + x_1 - \Delta)\right] & \text{for } \dot{x} > o \\ A\left[u(x + x_2) - u(x + x_1) \right. \\ \quad \left. + u(x - x_1 + \Delta) - u(x - x_2 + \Delta)\right] & \text{for } \dot{x} < o \end{cases} \tag{9.15}$$

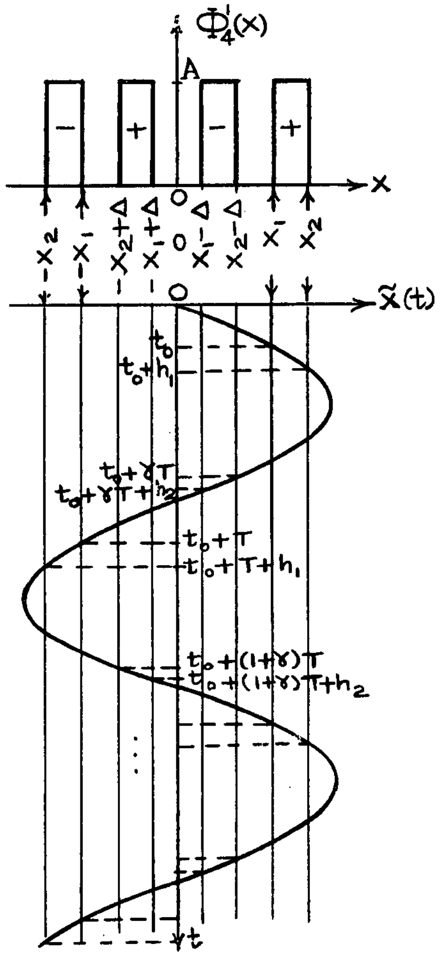
By substituting $\Delta = o$ in Eq. (9.15) we get $\Phi_3'(x)$.

In this case

$$\Phi_4'\left[\tilde{x}(t)\right] = A\sum_{k=o}^{\infty}\left[u(t-t_o-kT) - u(t-t_o-kT-h_1)\right. \tag{9.16}$$

$$\left. + u(t-t_o-kT-\gamma T) - u(t-t_o-kT-\gamma T-h_2)\right]$$

i.e. $\Phi_4'\left[\tilde{x}(t)\right]$ corresponds to the sum of two sequences of pulse functions. The periodicity of each sequence is the same and is equal to T the half-period of the periodic state $\tilde{x}(t)$ (we are assuming simple symmetric oscillations for $\tilde{x}(t)$). The second is displaced relative to the first by a fixed time interval γT. The geometric transformation into the indicated sequences of pulse functions is shown in Figure 9.7 with the help of the derivative of the characteristic $\Phi_4'(x)$. By an appropriate choice of the initial time instant (set $t_o = o$),



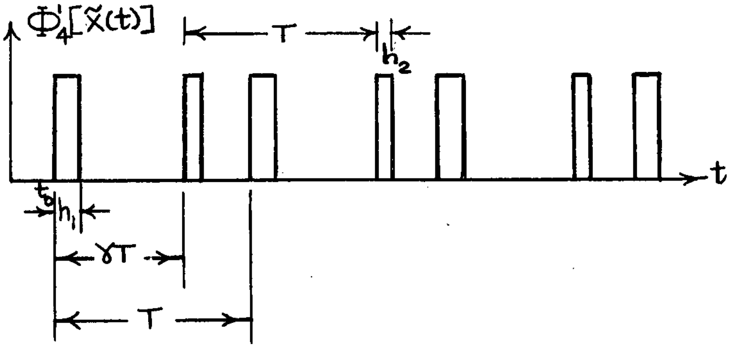


Figure 9.7. Transfer diagram for for the determination of $\Phi_4'\left[\tilde{x}(t)\right]$ when $\tilde{x}(t)$ is a simple symmetric periodic oscillation of half-period T.

the variational equation for this particular characteristic $\Phi_4'(x)$ has the form

$$\Xi(s) = F_d(s) - AH(s) \quad \mathcal{L}\Big(\xi(t) \sum_{k=0}^{\infty} \Big[u(t-kT) - u(t-kT-h_1)$$
$$+ u(t-kT-\gamma T) - u(t-kT-\gamma T-h_2)\Big]\Big) \tag{9.17}$$

Using the p-notation

$$P_{h_i,T}\left[\Xi(s)\right] = \mathcal{L}\Big(\xi(t) \sum_{k=0}^{\infty} \Big[u(t-kT) - u(t-kT-h_i)\Big]\Big),$$

Eq. (9.17) can be rewritten as

$$\Xi(s) = F_d(s) - AH(s)\Big[P_{h_1,T}\left[\Xi(s)\right] + e^{-s\gamma T}P_{h_2,T}\left[\Xi(s)e^{s\gamma T}\right]\Big] \tag{9.18}$$

Equation (9.17) or (9.18) corresponds to the linear feedback finite pulse width sampled-data system in Figure 9.8. It consists of two samplers in parallel and a feedback link containing a linear transfer function AH(s). The samplers close synchronously and their outputs have uniform pulse widths h_1 and h_2. However, the second sampler operates with a delay γT with respect to the first. Even though this system contains an

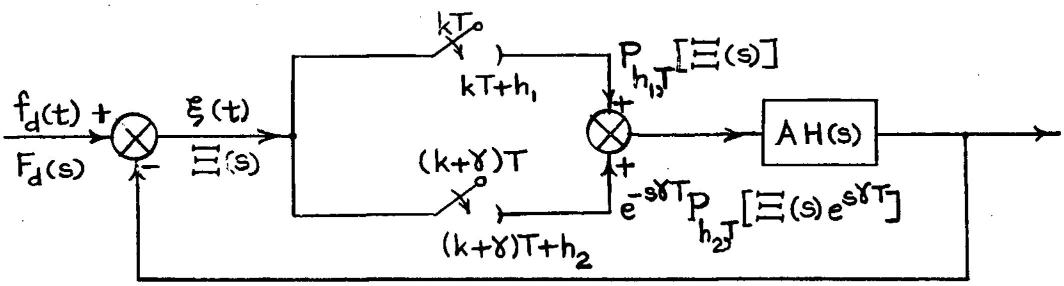


Figure 9.8. Linear system equivalent to Equation (9.17) or (9.18).

additional sampler, as compared to that for the case without dead zone, the analysis of the behaviour of the former is no more difficult than that of the latter, because of the fact that the samplers operate synchronously.

The Case of More Complicated Forms of Periodic Oscillations.

The method described above can be extended easily to the study of the stability of any given complicated form of periodic oscillation. As a example, let us consider the case of the simple saturation characteristic. Without deducing the variational equation in $\xi(t)$, we make use of the transfer diagram shown in Figure 9.9. The derivative of the periodic function $\bar{x}(t)$ of period 2T now consists of n sequences of pulses. The duration of the pulses in the successive sequences, initiated at times o, $\gamma_1 2T$, $\gamma_2 2T, \ldots, \gamma_{n-1} 2T$ with respect to the first, are in general different, and are denoted by h_o, h_1, $h_2, \ldots, h_{n-1}$ respectively.

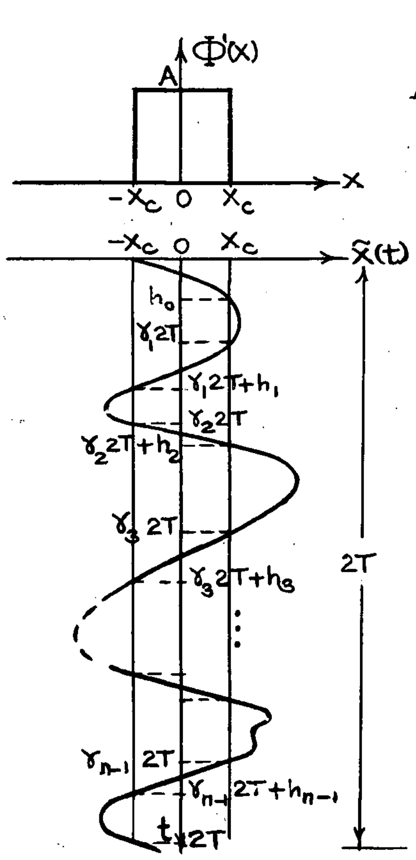


Figure 9.9. Transfer diagram for the determination of $\Phi'\left[\bar{x}(t)\right]$ where $\Phi(x)$ is the simple saturation characteristic, and $\bar{x}(t)$ is a complicated periodic waveform of period 2T.

Clearly, the linear system corresponding to the variational equation in this case will consist of n samplers in parallel of uniform pulse widths h_o, h_1, $h_2, \ldots$, h_{n-1} and a feedback link containing the linear transfer function AH(s). The samplers close synchronously with periodicity 2T, but are not in phase. This system is shown in Figure 9.10.

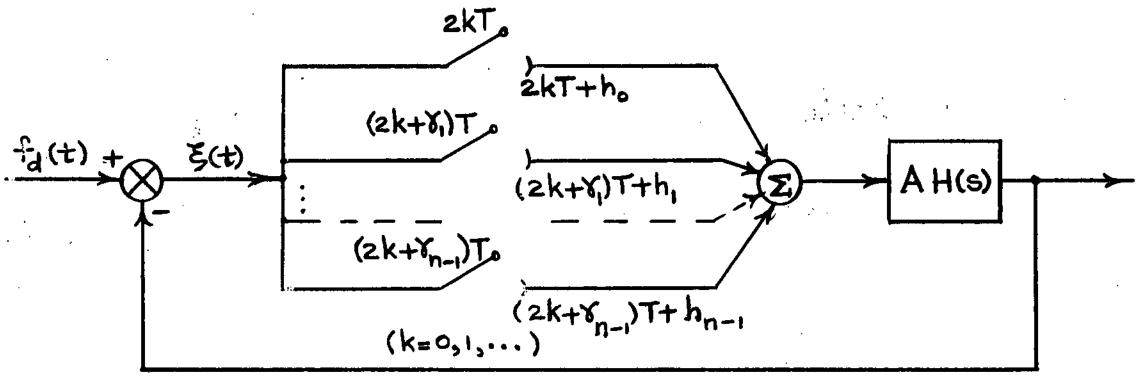


Figure 9.10. Linear system determining the stability of a complicated periodic state $\tilde{x}(t)$ for the saturation characteristic $\Phi(x)$.

9.3 AN APPROXIMATE SOLUTION TO THE ASYMPTOTIC STABILITY OF PERIODIC SOLUTIONS

In the preceding section we formulated an exact method, which reduces to well-known solved problems in sampled-data systems, for the determination of the asymptotic stability of periodic states. We now present an approximate solution to the above problem but without resorting to the sampled-data approach.

Let us assume that the linear transfer function H(s) is a fractional rational function, which may be written as

$$H(s) = \frac{P(s)}{Q(s)},$$

and that the degree of $P(s)$ is less than that of $Q(s)$. Then the variational equation (9.6) can be expressed in differential equation form thus:

$$Q(p)\,\xi(t) + P(p)\,\Phi'\left[\tilde{x}(t)\right]\xi(t) = Q(p)\,f_d(t)\,, \tag{9.19}$$

where $p = \frac{d}{dt}$, and $P(p)$ and $Q(p)$ are differential operators.

Since the derivative of the characteristic $\Phi'\left[\tilde{x}(t)\right]$ is periodic with period T, we can write it as an exponential Fourier series thus:

$$\left.\begin{aligned} \Phi'\left[\tilde{x}(t)\right] &= \sum_{\ell=-\infty}^{\infty} C_\ell\, e^{j\ell\omega t}\,, \\ \text{where} \quad & \\ C_\ell &= \frac{1}{T}\int_c^{c+T} \Phi'\left[\tilde{x}(t)\right] e^{-j\ell\omega t} dt \quad (c = \text{constant}) \\ \text{and} \quad & \\ \omega &= 2\pi/T \end{aligned}\right\} \tag{9.20}$$

We now seek a general solution of the homogeneous equation

$$Q(p)\,\xi(t) + P(p)\,\Phi'\left[\tilde{x}(t)\right]\xi(t) = o \tag{9.21}$$

of the form

$$\xi(t) = \sum_{k=-\infty}^{\infty} B_k e^{(\alpha + jk\omega)t}\,, \tag{9.22}$$

where the B's are the complex amplitudes and α is the so-called characteristic exponent which is to be determined. Clearly, if the real parts of the values of α are found to be negative, then the system is asymptotically stable.

Substituting (9.20) and (9.22) into (9.21), we obtain

$$\left[Q(p) + C_o\,P(p)\right]\sum_{k=-\infty}^{\infty} B_k e^{(\alpha + jk\omega)t}$$
$$+ P(p)\sum_{\ell=1}^{\infty}\sum_{k=-\infty}^{\infty}\left[C_\ell B_k e^{[\alpha+j(k+\ell)\omega]t} + C_{-\ell} B_k e^{[\alpha+j(k-\ell)\omega]t}\right] = o.$$

This last equation can be rewritten as

$$\left[Q(p) + C_o P(p)\right] \sum_{k=-\infty}^{\infty} B_k e^{(\alpha+jk\omega)t}$$

$$+ P(p) \sum_{\ell=1}^{\infty} \sum_{k=-\infty}^{\infty} \left[C_\ell B_{k-\ell} + C_{-\ell} B_{k+\ell}\right] e^{(\alpha+jk\omega)t} = o \tag{9.23}$$

By using the relation

$$F(p)\, e^{\zeta t} = e^{\zeta t}\, F(\zeta) \ ,$$

and equating the coefficients of like frequency components, we obtain

$$B_k \left[Q(\zeta_k) + C_o P(\zeta_k)\right] + \sum_{\ell=1}^{\infty} \left[C_\ell B_{k-\ell} + C_{-\ell} B_{k+\ell}\right] P(\zeta_k) = o \tag{9.24}$$

$$(k = o, \pm 1, \pm 2, \ldots)$$

where

$$\zeta_k = \alpha + jk\omega$$

Equation (9.24) is an infinite system of equations, each of which contains an infinite number of terms in B_k $(k = o, \pm 1, \pm 2, \ldots)$. The characteristic equation of the system is obtained by equating the determinant of Eq. (9.24) to zero. As it stands, this characteristic equation is of infinite degree in α.

Let the roots of the characteristic equation be $\alpha_i (i = 1, 2, \ldots)$. Then a necessary and sufficient condition that the system be stable is that the real parts of α_i lie in the left-half s-plane.

A Practical Approximation.

In practice, the linear parts of the systems considered are such that the frequency components lying outside certain finite bandwidths can be regarded as negligible. This can always be achieved by choosing the pertinent bandwidths sufficiently large. Let us assume that all frequency components larger than ω_c are negligible. Then all complex amplitudes for which

$$+\omega_c < \operatorname{Im} \zeta_i < -\omega_c \qquad (9.25)$$

may be neglected. Unfortunately, the values of ζ_i are unknown. However, by choosing sufficiently large values of k in $\zeta = \alpha + jk\omega$, say $|k| > M$, condition (9.25) can usually be fulfilled. Thus all complex amplitudes for $|k| > M$ may be neglected. Consequently, in place of the infinite system of equations (9.24), each containing an infinite number of terms, we now restrict our attention to the following finite system of equations, each containing a finite number of terms:

$$\sum_{k=-M}^{M} a_{ik} B_k = o \qquad (i = o, \pm 1, \ldots, \pm M)$$

where

$$a_{ik} = \begin{cases} Q(\zeta_i) + C_o P(\zeta_i), & \text{for } i = k \\ C_{k-i}\, P(\zeta_i), & \text{for } i \neq k \end{cases} \qquad (9.26)$$

The characteristic equation is now given by the determinant of the system (9.26), i.e.

$$|a_{ik}| = o,$$

which is polynomial of degree 2M + 1 in α. If all the roots α_i $(i = o, \pm 1, \ldots, \pm M)$ of this polynomial lie in the left-half s-plane, i.e. they all have negative real parts, then the periodic state under consideration is stable.

In the case of the saturation characteristic, with or without hysteresis and without dead zone, $\Phi'\left[\tilde{x}(t)\right]$ has the form

$$\Phi'\left[\tilde{x}(t)\right] = A \sum_{k=0}^{\infty} \left[u(t-t_0-kT) - u(t-t_0-kT-h)\right]$$

when $\tilde{x}(t)$ is a simple symmetric periodic oscillation of half period T. The Fourier series for this sequence of rectangular pulses is

$$\Phi'\left[\tilde{x}(t)\right] = A\left[\frac{\omega h}{2\pi} + \frac{2}{\pi}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\sin\frac{\ell\omega h}{2}\cos\ell\omega\left(t-t_0-\frac{h}{2}\right)\right]$$

where $\omega = 2\pi/T$. By choosing $t_0 - \frac{h}{2} = 0$, the exponential form for this series is

$$\Phi'\left[\tilde{x}(t)\right] = \frac{A}{\pi}\sum_{\ell=-\infty}^{\infty}\frac{1}{\ell}\left(\sin\frac{\ell\omega h}{2}\right) e^{j\ell\omega t} .$$

Similar expressions for the saturation characteristic with dead zone can be found.

When the characteristic of the nonlinear element $\Phi(x)$ ceases to be of the on-off or saturation type, the question of the stability of the periodic states cannot, in general, be reduced to a consideration of the stability of sampled-data systems. Under these conditions the present approximate method can still yield an answer to the stability problem in most cases of practical interest.

9.4 A DIRECT APPROACH TO THE STABILITY PROBLEM

The method to be presented below will be called the direct approach, in contrast to the sampled-data approach, because it is directly related to the physical definition of stability: that is, a disturbance is applied, and the deviation from the state of equilibrium is studied. If the deviation dies out the system is said to be stable; otherwise, it is unstable. This approach will be applied both to forced and self oscillations in the system shown in Figure 9.11.

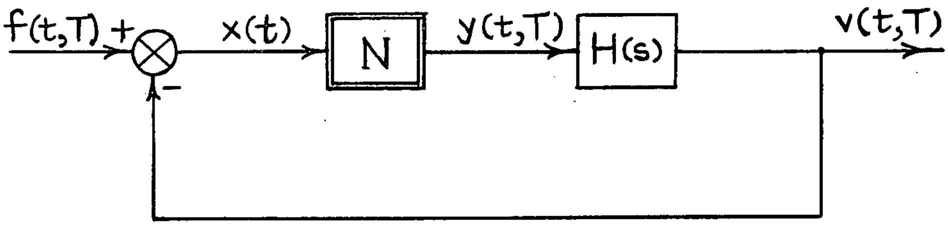


Figure 9.11. A single-loop system containing one on-off element.

Let f(t,T) be the periodic input with half-period equal to T, in the case of forced oscillations. Let y(t,T) and v(t,T) be the corresponding outputs of N and H(s), respectively. The input to N is denoted by x(t).

Stability of Forced Oscillations

The system in Figure 9.11 is assumed to be in a state of forced oscillations with half-period equal to T. Let a random disturbance $\Delta\tau_o$ occur in the zero-crossover at t = o as shown in Figure 9.12, so that the response v(t,T) for $t>o$ is modified to $v_m(t)$. We take $|\Delta\tau_o| \ll T$, and neglect higher order terms in $\Delta\tau_i$.

Let $y_m(t)$ be the modified output of N, and let its deviation from $y(t,T)$ be denoted by $y_d(t)$: that is

$$y_d(t) = y_m(t) - y(t,T).$$

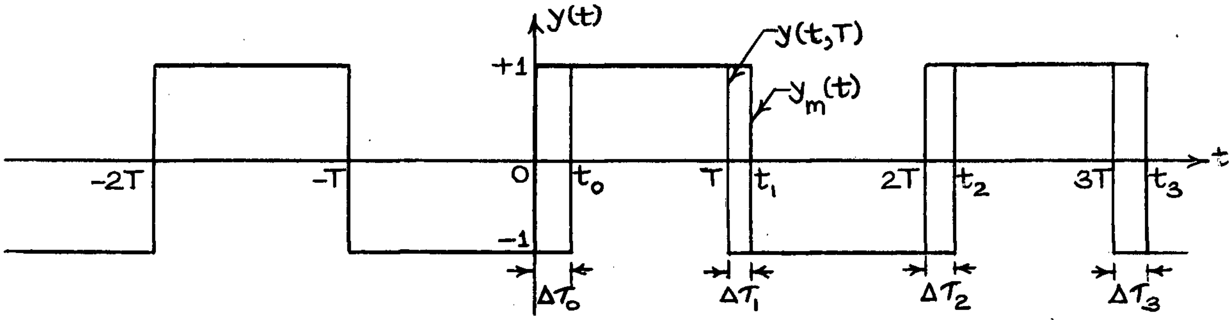


Figure 9.12. Periodic and modified outputs of N.

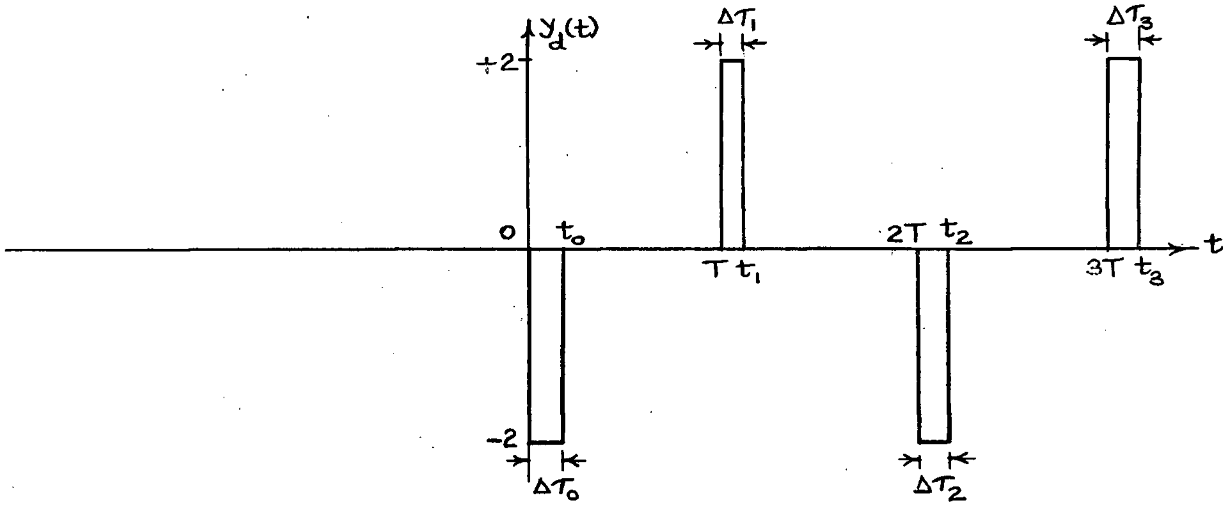


Figure 9.13. Deviation in the output of N.

The quantity $y_d(t)$ consists of a series of impulses as indicated in Figure 9.13. The deviation in the system response, $v_d(t) = v_m(t) - v(t,T)$, is the response of H(s) to $y_d(t)$.

Let $\quad y_m(t) = o \quad$ for $t = t_o, t_1, t_2, \ldots$

and $o < t < \infty$

$$\Delta\tau_n = t_n - nT, \; n = o, 1, 2, \ldots$$

The quantities $\Delta\tau_n (n = 1, 2, \ldots)$ are now determined in terms of $\Delta\tau_o$.

The change in the first crossover past the origin, $\Delta\tau_1$, can be found by solving

$$f(t_1,T) - v_m(t_1) = o \tag{9.27}$$

where

$$v_m(t) = v(t,T) - 2h(t)\ \Delta\tau_o \tag{9.28}$$

and $h(t)$ is the unit impulse response corresponding to the transfer function $H(s)$. Substitution of (9.28) into (9.27) gives

$$f(t_1,T) - v(t_1,T) = -2h(t_1)\ \Delta\tau_o \tag{9.29}$$

A Taylor series expansion of (9.29) about $t = T$ yields

$$f(T,T) - v(T,T) + \left[\dot{f}(T,T) - \dot{v}(T,T)\right] \Delta\tau_1 = -2h(T)\Delta\tau_o,$$

where

$$\dot{f}(T,T) \triangleq \left.\frac{\partial f(t,T)}{\partial t}\right]_{t=T} \quad \text{and } \dot{v}(T,T) \triangleq \left.\frac{\partial v(t,T)}{\partial t}\right]_{t=T}$$

But

$$f(T,T) - v(T,T) = o,$$

so that

$$\Delta\tau_1 = \eta\ h(T)\ \Delta\tau_o \tag{9.30}$$

where

$$\eta \triangleq 2\ (\ -\dot{f}(T,T) + \dot{v}(T,T)\ )^{-1} \tag{9.31}$$

The change in the next crossover $\Delta\tau_2$ is determined by

$$f(t_2,T) - v_m(t_2) = o \tag{9.32}$$

where $v_m(t)$ is now given by

$$v_m(t) = v(t,T) - 2h(t)\ \Delta\tau_o + 2h(t-T)\ \Delta\tau_1 \tag{9.33}$$

Substitution of (9.33) into (9.32), and expansion about $t = 2T$ yield

$$f(2T,T) - v(2T,T) + \left[\dot{f}(2T,T) - \dot{v}(2T,T)\right] \Delta\tau_2$$

$$= -2h(2T)\, \Delta\tau_o + 2h(T)\, \Delta\tau_1 \qquad (9.34)$$

Since

$$f(2T,T) - v(2T,T) = o$$

and

$$f(t,T) - v(t,T) = - f(t-T,T) + v(t-T,T) \ ,$$

equation (9.34) yields

$$\Delta\tau_2 = \eta \left[-h(2T)\, \Delta\tau_o + h(T)\, \Delta\tau_1\right] \qquad (9.35)$$

This equation for $\Delta\tau_2$ may be written in terms of $\Delta\tau_o$ using (9.30) but this is not necessary as will be shown later.

In general, the expressions for $\Delta\tau_n$ are given by

$$\begin{aligned}
\Delta\tau_1 &= \eta \left[h(T)\, \Delta\tau_o\right] \\
\Delta\tau_2 &= \eta \left[-h(2T)\, \Delta\tau_o + h(T)\, \Delta\tau_1\right] \\
\Delta\tau_3 &= \eta \left[h(3T)\, \Delta\tau_o - h(2T)\, \Delta\tau_1 + h(T)\, \Delta\tau_2\right] \qquad (9.36) \\
\Delta\tau_4 &= \eta \left[-h(4T)\, \Delta\tau_o + h(3T)\, \Delta\tau_1 - h(2T)\, \Delta\tau_2 + h(T)\, \Delta\tau_3\right]
\end{aligned}$$

etc.

The deviation in the response is

$$v_d(t) = -2h(t)\, \Delta\tau_o + 2h(t-T)\, \Delta\tau_1 - 2h(t-2T)\, \Delta\tau_2 + \cdots$$

or

$$\frac{V_d(s)}{-2H(s)\, \Delta\tau_o} = 1 - e^{-Ts} \frac{\Delta\tau_1}{\Delta\tau_o} + e^{-2Ts} \frac{\Delta\tau_2}{\Delta\tau_o} - e^{-3Ts} \frac{\Delta\tau_3}{\Delta\tau_o} + \cdots \qquad (9.37)$$

Substitution of (9.36) into (9.37) yields

$$\frac{V_d(s)}{-2H(s)\Delta\tau_o} = 1 - \eta e^{-Ts}\left[h(T)\right]$$
$$+ \eta e^{-2Ts}\left[-h(2T) + h(T)\frac{\Delta\tau_1}{\Delta\tau_o}\right]$$
$$- \eta e^{-3Ts}\left[h(3T) - h(2T)\frac{\Delta\tau_1}{\Delta\tau_o} + h(T)\frac{\Delta\tau_2}{\Delta\tau_o}\right]$$
$$+ \dots$$
$$= 1 - \eta\left[\sum_{n=1}^{\infty} h(nT)\; e^{-nTs}\right]\left(1 - e^{-Ts}\frac{\Delta\tau_1}{\Delta\tau_o} + e^{-2Ts}\frac{\Delta\tau_2}{\Delta\tau_o} - \dots\right) \tag{9.38}$$

From (9.37) and (9.38) there results

$$V_d(s) = \frac{-2H(s)\;\Delta\tau_o}{1 + \eta\sum_{n=1}^{\infty} h(nT)e^{-nTs}} \tag{9.39}$$

where η is given by Eq. (9.31).

Stability requires that all the poles of (9.39) lie in the left-half s-plane or that all the zeros of $1 + \eta \sum_{n=1}^{\infty} h(nT)d^{-nTs}$ lie in the left-half s-plane. Equivalently, if we substitute $z = e^{Ts}$, stability requires that all the roots of $1 + \eta \sum_{n=1}^{\infty} h(nT)z^{-n} = 0$ are inside the unit circle, with centre at the origin, in the z-plane.

Comparison with the sampled-data approach:

As mentioned by Tsypkin,[6] the study of the above stability problem is equivalent to the study of the stability of the linear sampled-data feedback system shown in Figure 9.14.

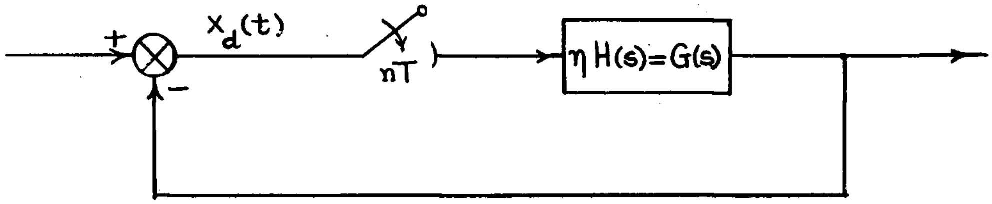


Figure 9.14. Equivalent sampled-data system for the stability problem.

The $\mathfrak{z}$-transform of $G(s) = \eta H(s)$ is

$$G(z) = \eta H(z) = \eta \sum_{n=o}^{\infty} h(nT)\ z^{-n}, \quad (z = e^{Ts})$$

The sampled-data feedback system is stable provided that all the roots of

$$1 + G(z) = 1 + \eta \sum_{n=o}^{\infty} h(nT)\ z^{-n} = o \qquad (9.40)$$

lie inside the unit circle in the z-plane. The results of the direct and sampled-data approaches differ: the term $\eta h(o)$ in (9.40) is absent in (9.39). The sampled-data result in (9.40) was derived on the assumptions that (1) $x_d(t)$ has small average amplitude as compared to $x(t,T)$ and (2) the time derivative of $x_d(t)$ does not take too large values. These assumptions imply that $h(t)$ must not be discontinuous at $t = o$, or, equivalently, that $h(o+) = o$. Consequently, the result derived by the sampled-data approach should be used only in cases where $h(o+) = o$; but it does not say what should be used when $h(o+) \neq o$. The result derived by the direct approach, Eq. (9.39), is valid both for $h(o+) \neq o$ and $h(o+) = o$.

Stability of Self Oscillations

A slight modification of the previous arguments will give the desired result for the stability of self oscillations. Let the half-period of self oscillation be T_o. Let the system in Figure 9.11 be undergoing forced oscillations of half-period T, $T \cong T_o$, up to $t = o$, after which the input $f(t,T)$ is removed and the ensuing oscillation periods are compared to T_o.

The modified response is

$$v_m(t) = v(t,T) - 2h(t)\,\Delta\tau_o \qquad (o \leq t < t_1) \tag{9.41}$$

Since

$$v_m(t_1) = v_m(T_o + \Delta\tau_1) = o \text{ and } v(T_o,T_o) = o,$$

a Taylor series expansion of (9.41) about (T_o,T_o) yields

$$\Delta\tau_1 = - a\Delta T + \eta\, h(T_o)\,\Delta\tau_o \tag{9.42}$$

where

$$\Delta T = T - T_o, \quad \eta = 2\left[\dot{v}(T_o,T_o)\right]^{-1} \tag{9.43}$$

and

$$a \triangleq \frac{v_T(T_o,T_o)}{\dot{v}(T_o,T_o)},$$

and where

$$v_T(T_o,T_o) \triangleq \left.\frac{\partial v(t,T)}{\partial T}\right]_{t = T_o \text{ and } T = T_o}$$

For the next interval $t_1 \leq t \leq t_2$,

$$v_m(t) = v(t,T) - 2h(t)\,\Delta\tau_o + 2h(t - T_o)\,\Delta\tau_1 .$$

Since

$$v_m(t_2) = o = v_m(2T_o + \Delta\tau_2) \text{ and } v(2T_o,T_o) = o,$$

then

$$\Delta\tau_2 = -a\Delta T + \eta\left[-h(2T_o)\,\Delta\tau_o + h(T_o)\,\Delta\tau_1\right].$$

In general,

$$\Delta\tau_n = -a\Delta T + \eta\sum_{m=1}^{n} h(mT_o)\,\Delta\tau_{n-m}(-1)^{m+1} \tag{9.44}$$

$$n = 1, 2, 3, \ldots$$

The deviation in response is

$$v_d(t) = v_m(t) - v(t,T_o)$$

$$= v(t,T) - v(t,T_o) - 2h(t)\,\Delta\tau_o + 2h(t-T_o)\,\Delta\tau_1$$

$$- 2h(t-2T_o)\,\Delta\tau_2 + \ldots$$

$$\cong v_T(t,T_o)\,\Delta T - 2h(t)\,\Delta\tau_o + 2h(t-T_o)\,\Delta\tau_1$$

$$- 2h(t-2T_o)\,\Delta\tau_2 + \ldots \tag{9.45}$$

The first term on the right-hand side of (9.45) is periodic with an infinitesimal amplitude and therefore can be neglected. Substitution of (9.44) into the Laplace transform of (9.45) yields

$$V_d(s) = \frac{-2H(s)\left[\Delta\tau_o + a\Delta T(e^{-T_os} - e^{-2T_os} + e^{-3T_os} - \ldots)\right]}{1 + \eta\sum_{n=1}^{\infty} h(nT_o)\,e^{-nT_os}} \tag{9.46}$$

Consequently, the condition for stability is the same as that found in the case of forced oscillations except that η is given by (9.43).

The zeros of $1 + \eta\sum_{n=1}^{\infty} h(nT_o)u^n$, $u \triangleq e^{-T_os}$, will be discussed further. Let

$$F(u) \triangleq (1 + \eta \sum_{m=1}^{\infty} h(mT_o)\, u^m)/\eta \tag{9.47}$$

Now in the case where H(s) has n simple poles all distinct from zero

$$\frac{1}{\eta} = \frac{1}{2}\dot{v}(T_o,T_o) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} \frac{e^{s_k T_o}}{1+e^{s_k T_o}}$$

$$= \sum_{m=1}^{\infty} (-1)^{m+1} h(mT_o) \tag{9.48}$$

so that (9.47) can be written as

$$F(u) = \sum_{m=1}^{\infty} h(mT_o) \left[u^m + (-1)^{m+1} \right]$$

A zero of F(u) is at u = -1, so that

$$F(u) = (1 + u)\, G(u)\,. \tag{9.49}$$

The form of G(u) is derived as follows:

$$\sum_{m=1}^{\infty} h(mT_o)\, e^{-mT_o s} = \sum_{m=1}^{\infty} \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} e^{mT_o(s_k - s)}$$

$$= \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} \frac{e^{-T_o(s-s_k)}}{1-e^{-T_o(s-s_k)}}$$

Now

$$(1 + u)\, G(u) = F(u) = \frac{1}{\eta} + \sum_{m=1}^{\infty} h(mT_o)\, e^{-mT_o s}$$

$$= \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} \left[\frac{e^{T_o s_k}}{1+e^{T_o s_k}} + \frac{u\, e^{T_o s_k}}{1-u\, e^{T_o s_k}} \right]$$

so that

$$G(u) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} \frac{e^{T_o s_k}}{1+e^{T_o s_k}} \frac{1}{1-u\, e^{T_o s_k}} \tag{9.50}$$

Since

$$\frac{1}{2}\,\dot{v}(t,T_o) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} \; \frac{e^{s_k t}}{1+e^{T_o s_k}}$$

and since the $\mathfrak{z}$-transform of $e^{s_k t}$ is given by

$$\mathfrak{z}\left(e^{s_k t}\right) = \frac{z}{z-e^{T_o s_k}} = \frac{1}{1-u\; e^{T_o s_k}}$$

it follows that

$$G(u) = \mathfrak{z}\left(\frac{1}{2}\,\dot{v}(t,T_o)\right)_{t\,=\,T_o} \tag{9.51}$$

The following partial fraction expansion is valid:

$$\frac{1}{1+\eta \sum_{m=1}^{\infty} h(mT_o)\; u^m} = \frac{1}{\eta G(-1)}\;\frac{1}{1+u} + \frac{f(u)}{G(u)} \tag{9.52}$$

In the first term on the right hand side of (9.52), $u = -1$ corresponds to periodic oscillations. Hence, the stability depends on the zeros of $G(u) = G(z^{-1})$, and these zeros should be within the unit circle in the z-plane. The stability question may therefore be answered by a Nyquist plot. A necessary condition is that $G(-1) > o$.

Additional notes on the function $G(u)$ are as follows:

$$G(o) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)}\;\frac{e^{T_o s_k}}{1+e^{T_o s_k}} = \frac{1}{2}\,\dot{v}(T_o,T_o)$$

Thus $\eta G(o) = 1$, which is the value for $s \rightarrow \infty$.

From

$$G(-1) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)}\;\frac{e^{T_o s_k}}{(1+e^{T_o s_k})^2} \quad ,$$

$$\frac{1}{2}\, v_T(T_o,T_o) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} \; \frac{-e^{T_o s_k}}{(1+e^{T_o s_k})^2} ,$$

and

$$\frac{1}{2}\, \dot{v}(T_o,T_o) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} \; \frac{e^{T_o s_k}}{1+e^{T_o s_k}}$$

it follows that

$$G(-1) = \frac{1}{2}\left[v_T(T_o,T_o) + \dot{v}(T_o,T_o)\right] \qquad (9.53)$$

Thus

$$\eta\, G(-1) = 1 + a$$

where a is given by (9.43). Now

$$G(+1) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} \; \frac{e^{T_o s_k}}{(1+e^{T_o s_k})^2} \left[1 + 2e^{T_o s_k} + 2e^{2T_o s_k} + \dots\right]$$

$$= \frac{1}{2}\left[v_T(T_o,T_o) + \dot{v}(T_o,T_o)\right] - v_T(T_o,T_o) - v_T(2T_o,T_o) - \dots$$

Therefore

$$\eta\, G(+1) = 1 - a + b \qquad (9.54)$$

where

$$b = -\,2 \sum_{m=2}^{\infty} v_T(mT_o,T_o) / \, \dot{v}(T_o,T_o)$$

If b is small, then Eq. (9.54) indicates that the $\eta G(u)$-plot does not enclose the origin for $|a| < 1$. This condition is much stronger than the previous one where $G(-1) > o$.

Illustrative Example

Consider the simple case where $H(s) = 1/s$. In this case,

$$h(t) = 1, \; t \geq o+, \text{ and } h(o+) = 1.$$

The sampled-data equation (9.40) should not be used in this case because it is not valid when $h(o+) \neq o$.

The use of (9.39), however, yields

$$1 + \eta\left[H(z) - h(o+)\right] = 1 + \frac{\eta}{z - 1} = o \qquad (9.55)$$

Thus

$$z = 1 - \eta$$

and stability requires that

$$o < \eta < 2 \qquad (9.56)$$

In the case of forced oscillations,

$$\eta = 2\left[- \dot{f}(T,T) + \dot{v}(T,T)\right]^{-1}$$

Since $\dot{v}(T,T) = 1$, the condition for the stability of forced oscillations yields

$$o < -\dot{f}(T,T) < \infty \qquad (9.57)$$

For this example, the quantities appearing in Figure 9.11 have the following description: $y(t,T)$ is a square wave as shown in Figure 9.12; $v(t,T)$ is the integral of the square wave $y(t,T)$ and is therefore sawtooth in shape; the waveform $f(t,T)$ is such that

$$x(o) = x(T) = o$$

$$\dot{x}(o) > o, \ \dot{x}(T) < o$$

$$o < -\dot{f}(T,T) < \infty,$$

and, provided that there are no more switchovers in the interval $o < t < T$, the shape of $f(t,T)$ is otherwise arbitrary.

CONCLUSIONS

Techniques and concepts for studying periodic phenomena in on-off feedback systems have been developed.

Three methods for evaluating the periodic response of the linear part of the on-off element have been presented: the first method uses the impulse response of the linear part of the system; the second method is in terms of the residues at the poles of H(s)/s, where H(s) is the transfer function of the linear part; the third method is in terms of H(jω), the frequency response of the linear part.

Concepts pertaining to the steady-state response of on-off elements are then examined: generalizations of the concepts of the Hamel and Tsypkin loci and of the phase characteristic of Neimark have been introduced. These concepts have been found to be useful in the study of self and forced oscillations in on-off feedback systems: they have been used to determine the possible periods of self and forced oscillations in single-, double-, and multiloop systems containing, in general, an arbitrary number of on-off elements.

The behaviour of on-off elements possessing a proportional band has been considered. The response of a single-loop system containing one such element has been determined by means of equivalent sampled-data systems, in which the samplers have finite pulse widths. However, in the study of the periodic oscillations in such a system, an approximate method, called the trapezoidal approximation, has been used; in general, this approximation is more accurate than that of the describing function, and is valid when there is sufficient filtering action by the linear part. The concept of the generalized Tsypkin loci has also been found useful in the determination of the possible periods of self and forced oscillations of such systems.

The results found by Tsypkin on the asymptotic stability in the small of single-loop systems having one on-off element without a proportional band have been generalized to include the case where the on-off element contains a proportional band. The investigations of the stability of these systems have been reduced to a consideration of the stability of equivalent sampled-data systems in which the samplers have finite pulse width; multiple samplers in parallel that close synchronously, but not in phase, have been found to enter in the case of hysteresis, dead zone and complicated forms of periodic oscillations. Finally, a direct approach to the stability problem has been presented: the direct use of the physical definition of asymptotic stability in the small has given results that agree with those obtained by the sampled-data approach.

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