THE PRECESSION OF AN ORBITING GYROSCOPE IN THE GRAVITATIONAL FIELD OF THE EARTH

by

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ABSTRACT

The motion of a gyroscope in a satellite orbiting the earth is considered. The axis of rotation of the gyroscope is assumed to be parallely propogated along its world line. Taking the satellite's path to be an ellipse, and using the true gravitational potential of the earth, including higher harmonics, one calculates the precession of the axis of rotation during one orbit of the earth with respect to the coordinate frames.

TABLE OF CONTENTS

		Page
• •	Introduction	l
I	Motion of the Satellite	5
II	Definition of, and Relations Between the Various	
	Coordinate Frames Used	10
III	Definition of Angular Velocity of Precession	15
IV	Precession in One Orbit	17
•••••	Summary	23
	Appendices:	
I	Justification of an Approximation Made in the	
	Orbit Calculation	24
II	A Comparison with some Previous Results for the	
	Special Case of A Circular Orbit in a Spherically	
	Symmetric Field.	27

References

. 30

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Introduction

The motion of a gyroscope in an orbiting satellite is an interesting problem. In the following we shall consider a spherically symmetric gyroscope. From a Newtonian point of view, and assuming a spherically symmetric gravitational field, the satellite's path is an ellipse, and after the completion of one orbit the gyroscope will have the same orientation as it did at the beginning of that orbit. Even if the gravitational field is not spherically symmetric the direction of the axis of rotation does not change in the Newtonian approximation. If, however, we treat the problem relativistically we find that the axis of rotation undergoes a slight precession. Thus the relativistic effects can be separated from the Newtonian ones and this experiment can be used to test theories of relativity and gravitation. (For a further discussion see Reference 1).

The gravitational potential $\Phi - \Phi_o$ of the earth is not spherically symmetric. One may expand Φ as an harmonic series;

$$\Phi - \Phi_{\circ} = \frac{K}{r} \left[1 + \sum_{n=2}^{\infty} \left(\frac{a_e}{r} \right)^n \sum_{m=0}^{n} \left(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \right) \right]$$

$$P_{nm}(\sin \phi)$$

where r is the distance from the earth's centre of gravity.

 ϕ is the latitude

 λ is the longitude from the Greenwich meridian

 a_e is the equatorial radius of the earth The first coefficients in this series have been evaluated by Izsak² and Guier³.

-1-

The relativistic effect of the first few harmonic terms on an orbiting gyroscope is considered in this thesis. The precession thus caused is compared with that predicted for a gyroscope orbiting in a spherically symmetric gravitational field.

There are two methods of looking at the motion of a gyroscope in a gravitational field. The first may be called the dynamical method, and is due to de Sitter, Papapetrou, Corindaldesi, and Pirani. De Sitter⁴ calculates corrections to the Newtonian theory in the case of the moon. Papapetrou⁵ takes as his model of the gyroscope a "pole-dipole particle" defined by the properties of the energy-momentum tensor in a small world tube. He is then able to define a spin tensor and to derive covariant equations for it's change along the world line of the particle. To solve these equations supplementary conditions are required. Corinaldesi and Papapetrou⁶ and $Pirani^7$ solve these equations using different supplementary Schiff⁸ extends the work of Papapetrou by solving conditions. his equations with the inclusion of effects of the earth's rotation (he adds the Lense-Thirring components to the Schwarzschild metric) and of non-gravitational constraining forces.

The second method of looking at the motion of a gyroscope in a gravitational field may be called the geometrical method. The following argument is due to Fokker⁹. To describe the motions taking place near the centre of the gyroscope we define axes by an orthonormal tetrad of vectors such that the time axis is always directed along the worldline and the origin

-2-

of the axes falls with the gyroscope. Also we demand that the tetrad of orthonormal vectors defining the axes remain parallel to themselves in the geodesic sense. With such a tetrad (which corresponds to a non-rotating frame¹⁰) we may expect that near the origin, free particles will move in straight lines under no force and that a top spinning around. its axis of symmetry will keep its axis of rotation in a fixed direction relative to the axes of reference. As the latter are carried along the worldline parallel to themselves, the same is true for the axis of rotation. Because of the curvature of space produced by the earth, we must expect the axis of rotation of the gyroscope to precess in the course of an orbit. Assuming a circular orbit and a Schwarzschild field Fokker 9 is able to derive an expression for the orientation of the gyroscopic axis of rotation after one revolution. Pirani⁷ shows that the methods of Papapetrou and Fokker are equivalent if one adds to Papapetrou's equation the supplementary condition that angular momentum is conserved.

Fokker's calculation assumes a circular orbit in a Schwarzschild gravitational field. Since the relativisitic effects predicted are rather small it is not clear whether or not the inclusion of the first few harmonic terms in the geopotential and the allowance for a non-circular orbit would seriously alter these predictions.

In this thesis we shall neglect the effects of constraining forces and the earth's rotation (Lense-Thirring of effect). We can then compare the precession of the gyroscope

-3-

axis in the actual field with the precession in a spherically symmetric field. It turns out to be a good approximation to replace the geodesics in both fields by ellipses characterized by eccentricity ϵ . We can then calculate the above precessions for several orbits using ϵ as a parameter. Taking the calculations to third order in $\frac{1}{r}$ (including the n=2 terms in the geopotential) we can compare the relative sizes of these effects.

In order to do the calculations we shall use a metric with components: $g_{km} = \delta_{km} e^{-f}$

$$g_{4k} = g_{k4} = 0$$
$$g_{44} = e^{f}$$

where $f = \frac{2}{c^2} (\Phi - \Phi_{\circ})$ and $\Phi - \Phi_{\circ}$ is the Newtonian potential. This metric¹¹ will give the Schwarzschild field to a sufficient approximation and will give the effects of higher harmonics to the Newtonian approximation. This metric has the advantage that it is directly determined from a knowledge of the gravitational potential. We shall assume that the path of the gyroscope about the earth is a geodesic of this field, and, after Fokker⁹ that the gyroscope's axis of rotation maintains a constant orientation with respect to the spatial triad of an orthonormal tetrad of vectors which is parallely propogated along the world line of the gyroscope. In measuring the rotation of this tetrad we use a co-moving observer to eliminate the effects of aberration due to differing velocities.

Chapter 1. Motion of the Satellite

In this chapter we shall find the equation of the orbit, some conservation laws, and some identities which will be needed in subsequent chapters.

The Newtonian potential $\Phi - \Phi_o$ is defined so that the force \vec{F} on a particle of mass mais $\vec{F} = -m \nabla (\Phi - \Phi_o)$. Our metric is then written:

$$g_{\kappa m} = \delta_{\kappa m} e^{-f}, \quad g_{\kappa 4} = g_{4\kappa} = 0, \quad g_{44} = e^{-f}$$
 (1.1)

where $f = \frac{2}{c^2}(\bar{x} - \bar{x}_o)$ and \bar{x}_o is a constant chosen so that $\bar{x} \rightarrow \bar{x}_o$ at spatial infinity. Coordinates with respect to an earth fixed frame are denoted by \bar{z}'' . (Greek letters have the range 1, 2, 3, 4, while latin letters have the range 1, 2, 3). It is useful to have a list of the Christoffel symbols for this metric:

$$\Gamma_{m \ \kappa}^{\kappa}(z) = \frac{1}{2} \left(\delta_{m n} \partial_{\kappa} f - \delta_{m \kappa} \partial_{n} f - \delta_{n \kappa} \partial_{m} f \right) \Big|_{z}$$

$$\Gamma_{4 \ \kappa}^{4}(z) = \frac{1}{2} \partial_{\kappa} f \Big|_{z}$$

$$\Gamma_{4 \ \kappa}^{\kappa}(z) = -\frac{1}{2} e^{2f} \partial_{\kappa} f \Big|_{z}$$

$$\Gamma_{4 \ \kappa}^{m}(z) = \Gamma_{m \ \kappa}^{4}(z) = \Gamma_{4 \ 4}^{4}(z) = 0$$

$$(1.2)$$

In the neighbourhood of the earth we have

$$f = -\frac{b}{r} \left[1 + \sum_{n=2}^{\infty} \left(\frac{ae}{r} \right)^n \sum_{m=0}^{\infty} \left(C_{nm} \cos m \lambda + S_{nm} \sin m \lambda \right) \cdot P_{nm} \left(\cos \theta \right) \right] \quad (1.3)$$

-5-

where	Ь	.=	$2\frac{GMB}{C^2}$					0	
	G	=	the gravitational	constant	Ш	6.670	x	-0 10 c	egs
	Me	=	mass of the earth		=	5.983	x	10 ²⁷	gm
	ae	=	equatorial radius	of earth	=	6.378	x	10 ⁸ 0	em
	С	=	speed of light		=	2.998	x	10 ¹⁰	cm/sec
	Θ	=	co-latitude (0+4=	whe:	re	, ¢ is th	ne	latit	ude)

and the first few coefficients are:²

n	m	Cinmila	S _{nm}
2	0	-1.08 x 10 ⁻³	0
2	l	0 _	0
2	2	9.68 x 10 ⁻⁷	-4.00×10^{-7}

The coordinates of the satellite are given in terms of a parameter s, the arc length along the world line.

 $z^{\mu}(s)$ are the satellite coordinates. we denote $\frac{d}{ds}$ by . (eg. $z^{\mu}(s) = \frac{d}{ds} z^{\mu}(s)$). <u>Lagrange's Method to Determine the Orbit and Some Identities</u>¹² We set $F(z(s), \dot{z}(s)) = \frac{1}{2}g_{\mu\rho}[z(s)]\dot{z}^{\mu}(s)\dot{z}^{\rho}(s)$ (1.4) Now the path of the Satellite is a geodesic of the field. Therefore: $\frac{S}{ss}\dot{z}^{\mu}(s) = 0$ (1.5)

which implies

$$\frac{d}{ds}\left(\frac{\partial F}{\partial z^{\mu}}\right) - \frac{\partial F}{\partial z^{\mu}} = 0$$

$$\frac{d}{ds}\left(F[z(s), \dot{z}(s)]\right) = 0 \qquad (1.6)$$

Therefore F is a constant. For a timelike geodesic (the path of a material particle) $F[z(s), \dot{z}(s)] = -\frac{1}{2}$ (1.7)

We assume that the gravitational field is time independent $\frac{\partial}{\partial z^4} f = \frac{\partial}{\partial z^4} F = O$

Then from (1.6) $\frac{d}{ds} \left(\frac{\partial F}{\partial \dot{z}^4} \right) = 0$ or $\frac{\partial F}{\partial \dot{z}^4} = K \text{ a constant.}$ Evaluating $\frac{\partial F}{\partial \dot{z}^4}$ from (1.4), (1.1) we have $\dot{z}^4(s) = K e^{-f C z(s) J}$ (1.8)

Then from (1.7)
$$\sum_{m=1}^{3} (\hat{z}^m)^2 = -\overline{I} = -(\kappa^2 + e^{\frac{f}{2}}).$$
 (1.9)

and from (1.5)
$$\vec{z}^{k}(s) = \vec{z}^{k}(s) f(z(s)) + \frac{1}{2}(z\kappa^{2} + e^{f})\partial_{\kappa}f$$
 (1.10)

Measurement of the Velocity of the Gyroscope

The velocity that an observer in an earth fixed frame would record cannot be equated with $: c \frac{dz^{\kappa}}{dz^{4}}$ because the metric is not Minkowskian. We define some new vectors by the equations:

$$\omega^{k} = \sqrt{g_{11}} dz^{k}$$
, $\omega^{4} = \sqrt{g_{44}} dz^{4}$ (1.11)

Then F is written $F = \frac{1}{2} \left((\omega^{1})^{2} + (\omega^{2})^{2} + (\omega^{3})^{2} + (\omega^{4})^{2} \right)$ which is the Minkowskian form. Therefore $\sqrt{k} [z(s)] = i c \frac{\omega^{k}}{\omega^{4}} = \frac{i c}{k} \frac{z^{k}(s)}{z^{k}(s)}$ (1.12)

as usual we define

$$\begin{aligned} \gamma[z(s)] &= \left[1 - \frac{1}{c^2} \sum_{k=1}^{3} \left(\sqrt{k} [z(s)] \right)^2 \right]^{-\frac{1}{2}} \\ &= -i \, k \, e^{-\frac{1}{2} f [z(s)]}$$
(1.13)

Determination of the Orbit

Here we assume that in the orbit calculation we can neglect terms of the order $\left(\frac{1}{r}\right)^3$ but not of order $\left(\frac{1}{r}\right)^2$. This will be shown valid by an estimate of error in appendix 1. We define new coordinates by $z^1 = r\sin\theta\cos\lambda$, $z^2 = r\sin\theta\sin\lambda$, $z^3 = r\cos\theta$

Then
$$F[z(s), \dot{z}(s)] = \frac{1}{2}e^{f[z(s')]} ((\dot{r})^2 + r^2(\dot{\theta})^2 + r^2sm^2\theta(\dot{\lambda})^2 + \kappa^2) = -\frac{1}{2}$$
 (1.14)
and $\partial F = \frac{1}{2}\frac{\partial f}{\partial A} = -\frac{2Gme}{c^2}\frac{1}{2}\sum_{n=2}^{\infty} (\frac{a_e}{r})^n \sum_{m=0}^n m \operatorname{Pnm}(c_{\theta}\theta) [-C_nmsmmn + S_nmc_{\theta}m_A]$

= 0 to this order

Then by (1.6) $\partial F = r^2 \sin^2 \Theta \lambda e^{f} = \frac{1}{p} = \text{constant}$ (1.15) For our initial conditions we take $\Theta = \frac{T}{2}$, $\Theta = O$. We now write the Θ equation for (1.6), noting that to this order $\begin{pmatrix} \partial f \\ \partial \Theta \end{pmatrix} = O$ $\frac{d}{ds} (e^{f} r^2 \Theta) - e^{-f} r^2 (\lambda)^2 Sm \Theta CO\Theta = O$

Therefore with the above initial condition we have

$$\Theta = \frac{I}{2} \quad \dot{\Theta} = 0 \qquad \text{for all s} \qquad (1.16)$$

Then (1.15) becomes $r^2 \dot{\lambda} e^{f} = \frac{1}{\mu}$ (1.17) and (1.14) and (1.16) imply $\left(\frac{dr}{dS}\right)^2 + r^2 \left(\frac{d\lambda}{dS}\right)^2 = -(\kappa^2 + e^f)$ We now divide by $r^4 (\dot{\lambda})^2$ using (1.17) $\left[\frac{d}{d\lambda} \left(\frac{1}{r}\right)\right]^2 + \left(\frac{1}{r}\right)^2 = A + B\left(\frac{1}{r}\right) + C\left(\frac{1}{r^2}\right)$ + terms of order $\left(\frac{1}{r}\right)^3$ (1.18)

where
$$A = -\mu^{2}(1+K^{2})$$
 $B = -\mu^{2}b(1+2K^{2})$ $(=-\mu^{1}b^{1}(\frac{1}{2}+2K^{2}))$
We neglect terms in $(\frac{1}{r})^{3}$ and differentiate $w.r.t.$ Λ
 $\frac{d^{2}}{d\Lambda^{1}}(\frac{1}{r})+(1-c)(\frac{1}{r})=\frac{B}{2}$ which has the solution
 $h = \frac{\alpha}{1+\epsilon\cos[\sqrt{1-c^{2}}\Lambda-\Lambda_{0}]}$ (1.19)

Order of Magnitude Estimate for C

$$b = \frac{2Gm_e}{c^2} = 8.9 \times 10^{-1} \sim 1$$

To estimate K we note that speed of the satellite is small compared to C ie. $\Im \sim I$ (1.13) then implies $K \sim i$ To estimate μ we take $r \sim a_e$ and for a typical satellite we have a period of 100 minutes. Also $f \sim -\frac{b}{r} \sim 10^{-8}$ so $e^{f} \sim I$ Then $\Lambda = \frac{d}{ds}\Lambda = \frac{z^{\mu}}{dz^{\mu}} = \frac{Ke^{-f}}{ic}\frac{d\Lambda}{dt} \sim \frac{1}{c} \times \frac{2\Pi}{6\times 10^{3}} \sim 3.5 \times 10^{-4}$ cgs. (1.17) gives $\mu = \frac{e^{f}}{r^{2}\Lambda} = (a_{e})^{*}\Lambda \sim 10^{-4}$ Therefore $C \sim -\mu^{*}b^{*}(\frac{1}{2}+2r^{*}) \approx \frac{3}{2} \times 10^{-8}$ which is much smaller than 1. Thus we can approximate the orbit by an ellipse. $r = \frac{a}{1+\epsilon \cos(\Lambda - \Lambda_{0})}$ (1.20)

(1.20)

Chapter 2. Definition of, and Relations Between the Various Observers and Their Coordinate Systems

In this problem there are three observers to consider. There is the observer who is fixed on the earth (which is assumed to be non-rotating), the observer who is fixed in the frame of the gyroscope, and the observer who is in the co-moving frame. In this chapter we shall derive relations between there orthonormal basis tetrads.

The first coordinate system to consider is the earthfixed frame S_e . This will be the reference frame used by an earth-bound observer. (Note, nowe do not consider the effects of the earth's rotation). In this frame the coordinates are $z^{P'(s)}$ and we are given the equations of motion of the satellite in terms of the parameter $s \cdot \{z^{P'(s)}\}$ We have a natural basis for S_e , the tetrad of vectors $\frac{\partial}{\partial z^{P'}|_{z}}$ However this is not orthonomal. Indeed, $g(\frac{\partial}{\partial z^{P'}|_{z}}, \frac{\partial}{\partial z^{C'}}|_{z}) = g_{P''(z)}$ (2.1)

Insterms of this basis we may define an orthonomal tetrad

$$e_{(\tau)}(z) = \sum_{\pi=1}^{4} e_{(\tau)}^{\pi}(z) \frac{\partial}{\partial z^{\pi}} \Big|_{z} \quad (\tau = 1, 2, 3, 4) \quad (2.2)$$

where

$$e_{(\tau)}^{\Pi}(z) = \delta_{\nabla \Pi} \frac{1}{\sqrt{g_{\tau \sigma}(z)}}$$
(2.3)

which will also serve as a basis for
$$\sum_{e}$$

We then have $g[e_{(r)}(z), e_{(x)}(z)] = \delta_{rx}$ (2.4)

-10-

Now we are treating the motion of the gyroscope by the geometric method. That is to say, we assume that the gyroscope's axis of rotation maintains a constant orientation with respect to the "spatial components" of a tetrad of vectors which is parallely propogated along the world line of the gyroscope. We therefore define an orthonormal tetrad of vectors along the world line of the gyroscope and demand that they be parallely propogated. They will define a reference frame S_g . We express these vectors in terms of the basis of S_e $\lambda_{(\mu)}[z(s)] = \sum_{n=1}^{4} \Lambda_{(\mu)(n)}[z(s)] e_{(n)}[z(s)]$ (2.5)

Orthonormality implies that

$$g[\Lambda_{(\mu)}[z(s)],\Lambda_{(\alpha)}[z(s)]] = \delta_{\mu\alpha}$$

from which $\Lambda[z(s)] \Lambda^{T}[z(s)] = I$ (2.6)

 $\Lambda[z(s)]$ defines a Lorentz transformation. To further determine the matrix $\Lambda[z(s)]$ we use the fact that the tetrad is parallely propogated. The covariant derivative,

$$\frac{D}{ds}\lambda_{(\mu)}[z(s)] = \sum_{\pi=1}^{4} \left(\frac{D}{ds}\lambda_{(\mu)}[z(s)]\right)^{\pi} \frac{\partial}{\partial z^{\pi}}\Big|_{z(s)} = 0 \qquad (2.7)$$

which implies that $\begin{pmatrix} p \\ ds \end{pmatrix}^{\prime} \langle \psi \rangle [z(s)] \end{pmatrix}^{\prime\prime} = 0$ To calculate this term we must expand $\lambda_{\psi} \langle z(s) \rangle$ in terms of the basis $\frac{\partial}{\partial z} \psi$ of the tangent space.

where
$$\begin{split} \lambda_{(\mu)}[z(s)] &= \sum_{\pi=1}^{4} \lambda_{(\mu)}[z(s)] \frac{\partial}{\partial z^{\pi}} \Big|_{z(s)} \\ &= \sum_{\alpha'=1}^{4} \lambda_{(\mu)(\alpha)}[z(s)] e_{(\alpha')}^{\pi}[z(s)] \end{split}$$
(2.8)

G

Then $\begin{pmatrix} p & \lambda_{(\mu)}[z_{1}(s_{1})] \end{pmatrix}^{T} = \begin{pmatrix} d & \lambda_{(\mu)}^{T}[z_{1}(s_{1})] + \sum_{\beta,\alpha=1}^{4} \int_{\beta}^{T} [z_{1}(s_{1})] \dot{z}^{\alpha}(s) \lambda_{(\mu)}^{\beta}[z_{2}(s_{1})] = 0$ Substituting in the values of $\lambda_{(\mu)}^{T}[z_{1}(s_{1})]$ (2.8) and of $e_{(\alpha)}^{T}[z_{1}(s_{1})]$ (2.2) we find the result that $\Lambda[z_{1}(s_{1})] = \begin{pmatrix} d & \Lambda[z_{1}(s_{1})] = \Lambda[z_{1}(s_{1})] \\ d & \Lambda[z_{1}(s_{1})] = \frac{1}{2} \int_{\beta\alpha} \left(\begin{pmatrix} d & \log g_{\alpha\alpha}[z_{1}(s_{1})] \end{pmatrix} - \sqrt{\frac{g_{\alpha\alpha}[z_{1}(s_{1})]}{g_{\beta\beta}}} \sum_{\pi=1}^{4} \int_{\beta}^{\alpha} [z_{1}(s_{1})] \dot{z}^{\alpha}(s_{1})$ Where $M_{(\beta)(\alpha)}[z_{1}(s_{1})] = \frac{1}{2} \int_{\beta\alpha} \left(\begin{pmatrix} d & \log g_{\alpha\alpha}[z_{1}(s_{1})] \end{pmatrix} - \sqrt{\frac{g_{\alpha\alpha}[z_{1}(s_{1})]}{g_{\beta\beta}}} \sum_{\pi=1}^{4} \int_{\beta}^{\alpha} [z_{1}(s_{1})] \dot{z}^{\alpha}(s_{1})$ We have not yet specified the initial orientation of the tetrad $\lambda_{(\mu)}[z_{1}(s_{1})]$. However, apart from this the tetrad is uniquely determined by the above differential equation.

We now consider the problem of observing the orientation If we look at it from the point of view of of the gyroscope. an observer in the ground frame \mathcal{F}_{e} , as well as seeing real effects due to a change in orientation we will also observe special-relativistic aberration effects due to the difference in velocities of the observer and the gyroscope. We therefore have a co-moving observer measure the orientation of the axis We denote the reference frame of this observer of rotation. by S_c , and his orthonomal basis tetrad by $\chi_{(\mu)}[z(s)]$. We want the frame S_c to have a known orientation with respect to S_e . One way to partially define this orientation is to demand that if an observer in S_e measures the velocity of S_c as \vec{v} , then an observer in S_c measures the velocity of S_e as $-\vec{v}$. We then have:

$$\chi^{(\mu)}[\Xi(s)] = \sum_{\alpha=1}^{\infty} H_{(\mu)(\alpha)}[\Xi(s)] e_{(\alpha)}[\Xi(s)]$$
(2.10)

-12-

where¹³

$$H_{(\kappa)(m)}[z(s)] = \delta_{\kappa m} + \frac{\partial [z(s)] - i}{(V [z(s)])^{2}} \sqrt{\kappa} [z(s)] \sqrt{m} [z(s)]$$

$$H_{(\kappa)(4)}[z(s)] = -H_{(4)(\kappa)}[z(s)] = \frac{i \partial [z(s)] \sqrt{\kappa} [z(s)]}{c} \qquad (2.11)$$

$$H_{(4)(4)}[z(s)] = \Im[z(s)]$$

Then H[z(s)] is also a Lorentz transformation.

ie.
$$H[z(s)] H[z(s)] = \overline{I}$$
 (2.12)

we may now find a relation between the two reference frames S_{g} and S_{c} , "attached" to the gyroscope.

$$\lambda_{(\mu)}[z(s)] = \sum_{\beta=1}^{4} R_{(\mu)(\beta)}[z(s)] \lambda_{(\beta)}[z(s)]$$
(2.13)

where, from (2.10) and (2.5)

 $\Lambda[z(s)] = R[z(s)] H[z(s)] \qquad (2.14)$

also from (2.6), (2.12) $\mathcal{R}[z(s)] \mathcal{R}^{T}[z(s)] = I$ (2.15) Now we can uniquely determine the tetrad $\lambda_{(\mu)}[z(s)]$ by demanding that at some point $\lambda = \lambda$, it coincide with the tetrad $\mathcal{X}_{(\mathbf{x})}[z(s)]$: Using the definition (2.11) one can show that $\mathcal{X}_{(\mathbf{y})}[z(s)] = -it[z(s)]$ for all S. [t is the unit tangent to the world line]. Therefore $\mathcal{X}_{(\mathbf{y})}$ is parallely propogated and consequently $\mathcal{X}_{(\mathbf{y})}[z(s)] = \lambda_{(\mathbf{y})}[z(s)]$ for all S. (2.16) Then from (2.13) and (2.15) we conclude that

$$K(u)(t)[z(s)] = K(t)(4)[z(s)] = 0$$
 for all s

(2.17)

$$R(4)(4)[z(5)] = i \qquad \text{for all } S$$

Thus $\mathcal{R}[\epsilon(s)]$ is a matrix corresponding to the rotation of the "spatial" vectors of the two tetrads. We may now find a differential equation for $\mathcal{R}[\epsilon(s)]$. (We leave out the arguments here, all understood to be $\epsilon(s)$). from $\dot{\Lambda}=\Lambda M$ (2.9) and $\Lambda=\mathcal{R}H$ (2.14)

$$\Lambda = RHM = (RH) = RH + RH$$
 which implies that

$$R = R(HM - \dot{H})H^{T} = RB$$

$$B = (HM - \dot{H})H^{T}$$
(2.18)

where

Properties of the matrix B from (2.17) $\dot{R}_{(4)(4)} = 0 = \sum_{\beta=1}^{4} R_{(4)(\beta)} B_{(\beta)(4)} = B_{(4)(4)} = 0$ $B + B^{T} = (R^{T}\dot{R}) + (R^{T}\dot{R})^{T} = R^{T}\dot{R} + \dot{R}^{T}R = \frac{d}{ds}(R^{T}R) = 0$ (2.19) ie. is antisymmetric and $B_{(4)(\alpha)} = B_{(\alpha)(4)} = 0$ (2.20) Chapter 3. Definition of Angular Velocity of Precession

We now have three reference frames defined. Since the axis of rotation vector is assumed to have constant components along the spatial basis vectors of S_g , and since the basis tetrad of S_g is related to the basis tetrad of S_c by a spatial rotation, one can find the rotation of the axis of rotation vector with respect to a co-moving observer by finding the rotation of the "spatial triad" $\lambda_{(\kappa)} [z(s)]$ with respect to the "spatial triad" $\lambda_{(\kappa)} [z(s)]$ with respect to the matrix $\mathcal{R}[z(s)]$ and hence by $\mathcal{B}[z(s)]$

Angular Velocity

We wish to calculate $\widehat{\mathcal{A}}[\overline{z}(s)]$, the angular velocity (with respect to time) of rotation of the triad $\mathcal{A}_{(\mathcal{K})}[\overline{z}(s)]$ with respect to $\mathcal{A}_{(\mathcal{K})}[\overline{z}(s)]$. Instead of calculating $\widehat{\mathcal{A}}[\overline{z}(s)]$ directly we calculate $\widehat{\omega}[\overline{z}(s)]$, the angular velocity (with respect to arc length) of rotation of these triads. We then have $\widehat{\mathcal{A}}[\overline{z}(s)] = \frac{ic}{i\frac{\pi}{2}} \widehat{\omega}[\overline{z}(s)]$ (3.1)

$$\lambda_{(\kappa)}[z(s)] = \sum_{m=1}^{3} R_{(\kappa)(m)}[z(s)] \, \chi_{(m)}[z(s)] \qquad (3.2)^{\frac{1}{2}}$$

Using (2.18)
$$R_{(\kappa)(m)}[z(s+h)] = \sum_{p=1}^{3} R_{(\kappa)(p)}[z(s)] V_{p(m)}[s,h]$$
 (3.3)

where (to first order in h)
$$V_{p,(m)}[s,h] = \delta_{pm} + h B_{pm}[z(s)]$$
 (3.4)

One can show in flat space for the rotation $\widehat{\omega}[z(s)]$ that $\frac{3}{R(\kappa)(m)}\left[z(s+h)\right] = \sum_{\substack{p=1\\p=1}}^{3} R(\kappa)(p)[z(s)]\left(\delta_{pm} + h\sum_{\substack{p=1\\p=1}}^{3} \omega_{p}[z(s)]e_{rpm}\right) \quad (3.5)$ Therefore we interpret $\mathcal{B}_{PIm}[z(4)] = \sum_{r} \omega_{r}[z(6)] \in rpm$ (3.6) From (3.1) therefore

$$\widehat{\int} \left[\overline{z}(s) \right] = \frac{ic}{\dot{z}^{4}(s)} \left[B_{23} \left[\overline{z}(s) \right], B_{3} \left[\overline{z}(s) \right], B_{12} \left[\overline{z}(s) \right] \right]$$
(3.7)
Evaluation of the Matrix $B[\overline{z}(s)]$

$$B_{(4)(\alpha)}[z(s)] = B_{(\alpha)(4)}[z(s)] = 0 \qquad (2.20)$$

$$B_{(\kappa)(m)}[z(s)] = [(HM - \dot{H})H^{T}](z(s))_{\kappa m} \qquad (2.18)$$
Then using the values of H from (2.11) and those for M from

(2.9) we have:

$$B_{(\kappa)(m)}[z(s)] = A[z(s)](\dot{z}^{\kappa}(s)\partial_{m}f|_{z(s)} - \dot{z}^{m}(s)\partial_{\kappa}f|_{z(s)})$$
(3.8)

where
$$A[z(s)] = \frac{-1}{2(\kappa^2 + e^f)} [2\kappa^2 + e^f - i\kappa e^f]_{z(s)}$$
 (3.9)

Using the approximation that f is small and that $\partial \cong I$ we have $K \sim i$ and $A [\Xi(S)] = -\frac{3}{4}$ (3.10)

Therefore to a good approximation we have

$$B_{(k)(m)} [z(s)] = -\frac{3}{4} [z^{k}(s)\partial_{m}f]_{z(s)} - z^{m}(s)\partial_{k}f|_{z(s)}] \qquad (3.11)$$

(3.12)

As a check of the above method we calculate Ω for a Schwarzschild field. here $f = -\frac{b}{h}$ $h = \sqrt{\sum_{n=1}^{3} [z^n(s)]^2}$

$$b = \frac{26me}{c^2}$$

$$B_{(m(m))} = -\frac{3}{4} \frac{b}{r^{3}} \frac{k}{c} \left[\sqrt{\frac{r_{z}}{z}} - \sqrt{\frac{m_{z}}{z}} \right]_{z(s)}$$
(3.13)

 $\vec{\Omega} = \underbrace{3}_{2} \underbrace{G_{2}}_{C^{2}} \underbrace{F_{2}}_{F^{2}} \overrightarrow{F} \times \overrightarrow{V}^{*}.$ This agrees with the value calculated by Schiff⁸.

Chapter 4. Precession In One Orbit

We have equations of motion for a gyroscope in a special orbit about the earth $(\Theta=0, \Theta=\frac{T}{2})$. We shall now calculate the rotation of the spin axis in the course of one revolution about the earth for this orbit. We calculate this rotation with respect to the frame S_c (with basis tetrad $X_{(\alpha)}[z(s)]$). It is convenient to change parameters from the interval S to the longitude λ . We will, for ease, denote $z[\lambda(s)] = z(\lambda)$ while will still denote $\frac{d}{ds}$ ie. $\dot{z}(\lambda) = \frac{d}{ds} z[\lambda(s)]$.

After computing the calculations for this orbit if we wish to find the result for any other orbit we may rotate coordinate system until in terms of the new coordinates (\cdot , \dot{o} , \dot{n}) the orbit has $\Theta' = \frac{\pi}{2} \dot{\phi}' = 0$. Knowing how spherical harmonics transform under a rotation, one can expand f in terms of them.

Now
$$\lambda(\kappa) [2(\pi)] = \sum_{m=1}^{3} R(\kappa)(m) [2(\pi)] \chi(m) [2(\pi)]$$
 (4.2)
from (2.13), (2.17)

The observer in S_c will see the triad $A_{(\kappa)}[\Xi(n)]$ rotating and he can compare it before and after one orbit.

$$\lambda_{(\kappa)}[z(\lambda_{1}+z\pi)] = \sum_{p=1}^{3} G[z(\lambda_{1})_{2}z\pi]_{(\kappa)(p)} \lambda_{(p)}[z(\lambda_{1})]$$
(4.3)

As mentioned in Chapter 2 we choose the two tetrads to coincide at $\lambda = \lambda_1$

Therefore

$$G[\mathcal{Z}[\lambda_{i}), \mathcal{Z}\Pi]_{(\kappa)(n)} = \delta_{\kappa n} + \int_{S(\lambda_{i})} \frac{d}{ds} R_{\kappa n}[\mathcal{Z}(s)] ds \qquad (4.4)$$

$$= \delta_{\kappa n} + \int_{\lambda_{i}}^{\lambda_{i}+\mathcal{Z}\Pi_{3}} R_{(\kappa)(p)}[\mathcal{Z}(s)] B_{(p)(n)}[\mathcal{Z}(s)] \frac{d\lambda}{\lambda}$$

Since we have only one orbit it is a good approximation to take $\mathcal{R}[\mathcal{Z}(\Lambda)] = I$ Thus $\mathcal{G}[\mathcal{Z}(\Lambda), 2\pi]_{(\kappa)(\Lambda)} = \delta_{\kappa\Lambda} + \int_{\Lambda_{I}}^{\Lambda_{I}+2\pi} \mathcal{B}_{(\kappa)(\Lambda)}[\mathcal{Z}(\Lambda)]d\Lambda$ (4.6) We use the result (3.11) to do the integral. We therefore must calculate $\dot{\mathcal{Z}}^{\kappa}(\Lambda)$ and $\partial_{\kappa}f$ The orbit is given by $\kappa(\Lambda) = \frac{\alpha}{I+\epsilon c_{\mathcal{D}}(\Lambda-\Lambda_{0})}$ Thus $\mathcal{Z}'(\Lambda) = \kappa(\Lambda) c_{\mathcal{D}}\Lambda = \frac{\alpha c_{\mathcal{D}}\Lambda}{I+\epsilon c_{\mathcal{D}}(\Lambda-\Lambda_{0})}$ $\mathcal{Z}^{2}(\Lambda) = \kappa(\Lambda) Sm\Lambda = \frac{\alpha Sm\Lambda}{I+\epsilon c_{\mathcal{D}}(\Lambda-\Lambda_{0})}$ (4.7)

Therefore we have

$$\vec{z}'(\lambda) = -\frac{e^{f(z(\lambda))}}{\mu \alpha} \left[Jn \lambda + \epsilon Sn \lambda_0 \right]$$

$$\vec{z}'(\lambda) = \frac{e^{f(z(\lambda))}}{\mu \alpha} \left[cn \lambda + \epsilon cn \lambda_0 \right]$$

$$\vec{z}^3(\lambda) = 0$$
(4.8)

Now we need to calculate the terms $\frac{\partial}{\partial z} mfEz(\lambda)J$. We have $f = -\frac{b}{h} \left[1 + \sum_{n=2}^{\infty} \left(\frac{ae}{h} \right)^n \sum_{m=0}^{n} P_{nm}(cn0) \left[(nm com\lambda + S_{nm}Smm m\lambda) \right]$ It is convenient to expand f in terms of spherical harmonics. $C_{nm} cosm\lambda + S_{nm}Smm\lambda = A_{nm}e^{im\lambda} + B_{nm}e^{-im\lambda}$

where $Anm = \frac{1}{2} \begin{bmatrix} C_{nm} - iS_{nm} \end{bmatrix}$ (4.9) $Bnm = \frac{1}{2} \begin{bmatrix} C_{nm} + iS_{nm} \end{bmatrix}$

$$P_{nm}(coo)e^{im\eta} = D_{nm}Y_{nm}(\partial,\lambda)$$

$$P_{nm}(\omega, \theta) \in [m\lambda] = (-1) D_{nm} Y_{h,-m}(\theta, \lambda)$$
where $D_{nm} = \sqrt{\frac{4\pi (n+m)!}{(2n+i)(n-m)!}}$

$$(4.10)$$

Therefore we set
$$H_{nm} = A_{nm} D_{nm}$$

 $L_{nm} = (-1)^{m} B_{nm} D_{nm}$
(4.11)
and we have $f = -\frac{b}{r} \left[1 + \sum_{n=2}^{\infty} \left(\frac{ae}{r} \right)_{m=0}^{2} \left[H_{nm} Y_{nm}(\theta, \lambda) + L_{nm} Y_{n,-m}(\theta, \lambda) \right] \right]$
Now $\frac{\partial f}{\partial zm} = \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial zm} + \frac{\partial f}{\partial \lambda} \frac{\partial \lambda}{\partial zm} + \frac{\partial f}{\partial r} \frac{\partial r}{\partial zm}$
(4.12)
For the orbit $\Theta = \frac{\pi}{2}$ (from equation 4.7)
 $\frac{\partial f}{\partial z_{1}} = \cos \lambda$
 $\frac{\partial \theta}{\partial z_{2}} = 0$
 $\frac{\partial A}{\partial z_{2}} = -\frac{i}{r} \sin \lambda$
 $\frac{\partial A}{\partial z_{2}} = -\frac{i}{r} \cos \lambda$
 $\frac{\partial F}{\partial z_{3}} = 0$
(4.13)
Setting $\Gamma_{nm} = H_{nm} Y_{nm}(\pi_{z}'\lambda) + L_{nm} Y_{n-m}(\pi_{z}'\lambda)$
 $\pi_{nm} = H_{nm} Y_{nm}(\pi_{z}'\lambda) - L_{nm} Y_{n-m}(\pi_{z}'\lambda)$
(4.14)
and $\sqrt{nm} = H_{nm} \frac{\partial}{\partial \theta} Y_{nm}(\theta, \lambda) \Big|_{\theta = \pi/2} + L_{nm} \frac{\partial}{\partial \theta} Y_{n-m}(\theta, \lambda) \Big|_{\theta = \pi/2}$

write

$$f = -\frac{b}{r} \left[1 + \sum_{n=2}^{\infty} {\binom{ae}{r}}^{n} \sum_{m=0}^{n} \Gamma_{nm} \right]$$

$$\frac{\partial f}{\partial \lambda} = -\frac{b}{r} \sum_{n=2}^{\infty} {\binom{ae}{r}}^{n} \sum_{m=0}^{n} im \Gamma_{nm}$$

$$\frac{\partial f}{\partial r} = \frac{b}{r^{2}} \left[1 + \sum_{n=2}^{\infty} {\binom{ae}{r}}^{n} \sum_{m=0}^{n} {\binom{n+1}{r}} \Gamma_{nm} \right]$$

$$\frac{\partial f}{\partial \theta} = -\frac{b}{r} \sum_{n=2}^{\infty} {\binom{ae}{r}}^{n} \sum_{m=0}^{m} V_{nm}$$
(4.15)

Then from (4.12) and (3.13) we can find:

$$\frac{h^{2}\mu}{e^{r}}B_{12}[z(\lambda)] = -\frac{34}{4\alpha}\left[\epsilon\sin(\lambda_{0}-\lambda)\sum_{n=2}^{\infty}\left(\frac{ae}{r}\right)_{m=0}^{n}im\overline{n}m - \frac{ab}{r}\left(1+\sum_{n=2}^{\infty}\left(\frac{ae}{r}\right)_{m=0}^{n}i(n+1)\overline{n}m\right)\right]\right]$$

$$\frac{h^{2}\mu}{e^{r}}B_{23}[z(\lambda)] = -\frac{3b}{4\alpha}\left(co\lambda + \epsilon\cos\lambda_{0}\right)\sum_{n=2}^{\infty}\left(\frac{ae}{r}\right)_{m=0}^{n}Vnm$$
(4.16)

$$\frac{h^{2}\mu}{e^{r}}B_{31}[z(\lambda)] = -\frac{3b}{4\alpha}(sm\lambda + \epsilon\sin\lambda_{0})\sum_{n=2}^{\infty}\left(\frac{ae}{r}\right)_{m=0}^{n}Vnm$$
We are interested in the effects after one orbit. We therefore
set $\lambda_{2} = \lambda_{1} + 2\pi$. To do the integration (4.6) we carry terms to
order $\left(\frac{1}{r}\right)^{2}$ but neglect those of order $\left(\frac{1}{r}\right)^{3}$.

-19-

5

Doing the integration (4.6) to this order we find that

$$G[z(\lambda_{i}), 2\pi]_{23} = -\frac{3b}{4a^{3}}\sqrt{\frac{45}{8\pi}} a^{2} Intz3$$

$$G[z(\lambda_{i}), 2\pi]_{31} = -\frac{3b}{4a^{3}}\sqrt{\frac{45}{8\pi}} a^{2} Int_{31}$$

$$G[z(\lambda_{i}), 2\pi]_{12} = \frac{3b\pi}{2a} + i\epsilon \frac{3b}{8a^{3}}\sqrt{\frac{45}{2\pi}} a^{2} Int_{31}$$

$$(4.17)$$

$$+\frac{9b}{4a^{3}} [\int_{20}^{2} Int_{2} + \frac{1}{4}\sqrt{\frac{45}{4\pi}} Int_{3}]a^{2}_{e}$$

where:

$$Int_{23} = \pi \left[(I + \frac{e^{2}}{4})(H_{2I} - L_{2I}) + \frac{s}{2} e^{2} cod \lambda_{0} (H_{2I} e^{i\lambda_{0}} - L_{2I} e^{-i\lambda_{0}}) \right]$$

$$Int_{3I} = \pi \left[i(I + \frac{e^{2}}{4})(H_{2I} + L_{2I}) + \frac{s}{2} e^{2} j\lambda_{0} \lambda_{0} (H_{2I} e^{i\lambda_{0}} - L_{2I} e^{-i\lambda_{0}}) \right]$$

$$Int_{3I} = i \epsilon \pi \left[H_{22} e^{2i\lambda_{0}} + L_{22} e^{-2i\lambda_{0}} \right]$$

$$Int_{2} = 2\pi + 3e^{2}\pi$$

$$Int_{3} = 3e^{2} \frac{\pi}{2} \left[H_{22} e^{2i\lambda_{0}} + L_{22} e^{-2i\lambda_{0}} \right]$$
Now for this orbit we are given the values of H_{nm} , L_{nm}
See (4.9), (4.10), and (4.11)
Substituting in (4.17), (4.18) we find:

$$G \left[\frac{2(\lambda_{1})}{2\pi} \frac{2\pi}{23} = -\frac{96\pi}{4a^{3}} \left[(I + \frac{e^{2}}{4})C_{2I} + \frac{s}{2}e^{2}cod\lambda_{0} (C_{2I} co \lambda_{0} + S_{2I} sm\lambda_{0}) \right] a_{e}^{2}$$

$$G \left[\frac{2(\lambda_{1})}{2\pi} \frac{2\pi}{24} = \frac{36\pi}{2a} - \frac{96\pi}{8a^{3}} C_{20} (2 + 3e^{2})a_{e}^{2} \right]$$

$$(4.19)$$

One can also determine the angle and the axis of rotation for this orbit. The method of doing this is straight foreward knowing G. If we denote the angle of rotation by ϕ and the axis of rotation by \vec{X} we find that $\vec{X} = H[G_{23}, G_{3'}, G_{12}]$ where $Sm \phi = \frac{1}{H} = \sqrt{(G_{12})^2 + (G_{3'})^2 + (G_{22})^2}$ Now to this order we have $C_{2,1} = S_{2,1} = 0$. If we use the values in Guier's paper³ they are of order 10^{-8} . Thus the G_{23} and $G_{3,1}$ terms are very small and may be set = 0. In the $G_{1/2}$ term we have three contributions $\frac{367}{2a}$ which is largest, $-\frac{96 \, a_e^2}{8a^3} \pi C_{20}(z+3\epsilon^2)$ which is smaller by 10^3 and the last term (see 4.19) which is smaller by 10^6 and can be set to 0. To this approximation

$$Sm \phi = \frac{1}{H} = |G_{12}| = \frac{36\pi}{2a} - \frac{96}{8a} \left(\frac{a_e}{a}\right)^2 \pi C_{20}(2+3\epsilon^2)$$

$$\vec{X} = (0, 0, HG_{12}) = (0, 0, 1)$$

The first term in the expression for $S_A \phi$ is the same as is given by Fokker⁹ (see appendix 2). The rotation occurs about the axis perpendicular to the plane of the orbit.

One may also do a similar calculation for other orbits. An easy way to do this is to rotate coordinates. We effect a rotation by Euler angles $\{\varkappa, \beta, \gamma\}$. (First rotate about the \mathcal{Z} axis by angle \mathcal{J} , then rotate about the new γ axis by angle β , then rotate about the new \mathcal{Z} axis by angle \prec) to arrive at a reference frame with spherical polar coordinates $r'_{,0}, \lambda'$ and such that the orbit is described by $\mathfrak{G}' = \overline{n'_{2}}$ We may then expand

 $Y_{nm}(\Theta,\lambda) = \sum_{j=-n}^{n} D^{(n)} [\{\alpha,\beta,\delta\}]_{jm} Y_{nj}(\Theta',\lambda')$ where $D^{(n)} [\{\alpha,\beta,\delta\}]_{jm}$ is a coefficient evaluated by Wigner¹⁵.

Since we have a rotation $r'_{=}r$ and in our new coordinate $f = -\frac{b}{F} \left[1 + \sum_{n=2}^{\infty} \left(\frac{ae}{r} \right)^{n} \sum_{m=0}^{n} \left[H'_{nm} Y_{nm}(\Theta', \Lambda') + L'_{nm} Y_{n,-m}(\Theta', \Lambda') \right] \right]$ To find H'_{nm} and L'_{nm} it is convenient to set

 $K_{no} = H_{no} + L_{no}$ $K'_{no} = H'_{no} + L'_{no}$ $K_{nm} = H_{nm}$ f_{m21} $K_{n,-m} = L_{nm}$ $K'_{n+m} = L'_{nm}$

-21-

$$f = -\frac{b}{r} \left[1 + \sum_{n=2}^{\infty} {\binom{ae}{r}}^n \sum_{m=-n}^n K_{nm} Y_{nm}(\Theta, \lambda) \right]$$
$$= -\frac{b}{r} \left[1 + \sum_{n=2}^{\infty} {\binom{ae}{r}}^n \sum_{m=-n}^n K_{nm} Y_{nm}(\Theta, \lambda') \right]$$

 $K'_{nj} = \sum_{m=-n}^{n} D^{(n)} [\{\alpha, \beta, \beta\}]_{jm} K_{nm}$

We then can solve for H'_{nm} and L'_{nm} and use equations (4.17), (4.18) to calculate precession for this orbit.

Summary

Neglecting the effect of the earth's rotation, we have been able to calculate the precession of the axis of rotation of a small spherically symmetric gyroscope in the course of one orbit in the earth's gravitational field. This precession is almost entirely (to 1 part in 10^6) about an axis normal to the plane of the orbit. The expression giving the angle of precession about this axis contains two main terms. The largest is analogous to the factor derived by Fokker⁹ for a circular orbit in a spherically symmetric field. The other main term (10^3 smaller than the first) contains the effect of the eccentricity and of one of the harmonics (C_{20}) of the gravitational potential of the earth.

In doing the calculation we have assumed that the orbit was an ellipse. This will affect the result by at most 2%. If, however, we wish to include the effect of the earth's rotation (itself one or two percent of the main precession term, see Schiff⁸) this approximation is no longer valid. Taking the above error into account we see that the effects of both eccentricity of the orbit and of the inclusion of harmonic terms in the geopotential do not seriously affect the inclusion calculated by Fokker for a circular orbit in a spherically symmetric gravitational field. Appendix 1. Justification of an Approximation Made in the

Orbit Calculation

Classically we have $\frac{d^2}{dt^2}\vec{r} = -\nabla(\vec{p}-\vec{p}_{\cdot})$. We can write $\vec{F} = \vec{r_o} - 4\vec{F}$ where $\vec{r_o}$ is the elliptical part and \vec{ar} is the remainder. For the elliptical part we have $\frac{d^2}{dt^2}\vec{F_o} = \nabla \frac{Gm_e}{r}$ Thus $\frac{d^2}{dt^2}\vec{\Delta r} = -\frac{c^2}{k^2}e^{it}f^{it}\frac{d^2}{dt^2}\vec{\Delta r} = \nabla \frac{Gm_e}{r}\sum_{n=2}^{\infty} \left(\frac{d_e}{r}\right)^{h}\sum_{m=0}^{n} P_{nm}(cn\theta)[c_{nm}(c\sigma m\Lambda) + S_{nm}Simm\Lambda]$ We now consider the N=2 term to see the effect of neglecting it in the calculation $\frac{d^2}{d\Lambda^{L}}\vec{\Delta r} = -\prod_{m=0}^{2} \left[\frac{-3f_{2m}(cn\theta)[c_{2m}c\sigma m\Lambda + S_{2m}Simm\Lambda]\vec{4}_{r}}{d\theta} \int_{2m}(cn\theta)[c_{2m}c\sigma m\Lambda + S_{2m}Simm\Lambda]\vec{4}_{r}} \right]$ where $\vec{4}_{r}, \vec{4}_{\theta}$ and $\vec{4}_{\Lambda}$ are unit vectors in direction of increasing coordinate and $\vec{T} = -\frac{k^2 e^{-\frac{i}{T}}\mu^2(a_e)Gm_e}{c^2} \sim \frac{\mu^2(a_e)^2Gm_e}{c^2}$ Using the same approximations as are used in Chapter 1, we

note that for this orbit $\Theta = \frac{\pi}{2}$ and that the only non-zero

coefficients are C_{20} , C_{22} , and S_{22} . Writing $\bar{I}_{r} = \hat{i} c_{P} \lambda + \hat{j} s_{M} \lambda$ $\bar{I}_{\lambda} = -\hat{i} s_{M} \lambda + \hat{j} c_{P} \lambda$ (for $\theta = \frac{\pi}{2}$ only)

and integrating twice we find

$$\Delta r_{2\pi} = \Delta r_{0} + \frac{3}{2}T \cdot 2\pi \cdot (r_{0}\hat{j} + T[-8\pi S_{22}\hat{c} - 2\pi C_{22}\hat{j}]$$

We can take $\Delta \overline{r}_{0} = 0$. The large term is the one containing C_{20} and is of order $3T \pi (2_{0} \sim 130 \text{ km})$. We use the result (4.6) $G[E(\lambda_{1}), 2\pi]_{kn} = \delta_{kn} + \int \frac{\mu r^{2}}{e^{f}} B_{(k\lambda_{1})}[E(\lambda_{1})] d\lambda$ and compare the result where we take r for an ellipse as compared with r for the actual orbit. Using the approximation (3.13) and the results (1.8), (1.17) we have

$$G[z(\lambda_{i}), 2\pi]_{Kn} = \delta_{Kn} - \frac{3b}{4} \int_{4}^{2} \int_{4}^{2} e^{f} [\sqrt{k_{z}}^{m} - \sqrt{m_{z}}^{m} z^{k}] dt$$

where t_{i}, t_{2} are the times for the start and end of the orbit.
As an approximation we take $r \sim \alpha$, $e^{f} \sim 1$ so that

 $G[z(\lambda_{1}), 2\pi]_{Kn} = \delta_{Kn} - \frac{36}{49} \int_{4}^{t_{2}} (v^{k} z^{n} - v^{n} z^{k}) dt$ If we let \vec{A} be the vector area of the orbit we have $\frac{d\bar{A}}{dt} = \frac{1}{2}(\bar{F} \times \bar{V})$. Therefore if K, n, s are cyclic we have The difference between $G[z(\lambda_1), ZT]_{Kn} = \delta_{Kn} - \frac{36}{40^3} A_5$ $G_{\kappa n}$ for the real orbit and $G_{\kappa n}$ for the elliptical orbit is therefore proportional to their difference in area. We know the maximum difference in position is 130 Km . We let A be the average area and \triangle A be the difference between the area of the actual orbit and the elliptical orbit. Since both orbits are planar A_1 and A_2 are zero. Thus there is no error in G_{23} or G_{31} due to taking the path elliptical. The error in G_{12} is of the order $\frac{\Delta A}{A}$. To find an estimate of the error we take the case of two circles, the smaller of radius a, touching at one point with a difference in diameter of ar

Then
$$A = \frac{\pi(a+\frac{\Delta r}{2})^2 - \pi a^2}{\pi a^2} = \frac{\Delta r}{a} + \frac{1}{4} \left(\frac{\Delta r}{a}\right)^2 \sim 2 \times 10^{-2}$$

Thus we can expect an error of about 2% in the precession calculation. Were we also concerned with the effect due to the earth's rotation (Lense-Thirring effect) which contributes at most a few percent 8 to the precession of the axis of rotation this approximation would not be justified.

Appendix 2. A Comparison With Some Previous Results for the Special Case of a Circular Orbit in a Spherically Symmetric Field

In this appendix we shall show that our results agree with those calculated by Fokker and by Schiff for the special case of a circular orbit in a spherically symmetric gravitational field. Recall, for comparison that our calculation predicts a precession of angle $\phi = \frac{3\pi}{2a} \cdot 2 \frac{G_{me}}{c^2} rad$ with axis in the direction $\vec{F} \times \vec{V}$ (perpendicular to the orbital plane) after one revolution.

To show this agrees with Schiff's result we use the relevant term $\frac{3m_e}{2a^3}(\vec{r}\times\vec{v})$ in his expression for the rate of precession. (Equation 40, Page 879, Ref. 8). To calculate the angle of precession after one revolution we note that

$$\Delta \phi = 2\pi \frac{d\phi}{dt} / \frac{d\lambda}{dt}$$

 $= 2\pi \frac{3}{2} \left(\frac{M_{e}}{\Delta}\right) \text{ in one revolution. To convert to c.g.s.}$ unit we replace $\frac{M_{e}}{\Delta}$ by $\frac{GM_{e}}{C^{*}\Delta}$ to find that $\Delta \phi = \frac{3\pi}{2\Delta} \cdot 2 \cdot \frac{GM_{e}}{C^{*}}$ with axis in the direction $\vec{r} \cdot \vec{v}$ Agreement with the result given in Fokkers paper is not so immediate (and is the reason for this appendix). He uses spherical polar coordinates with basis vectors $\frac{2}{\Delta r}, \frac{2}{\Delta \theta}, \frac{2}{\Delta \theta}$ $\frac{2}{\Delta \theta} and \frac{2}{\Delta t}$. He then defines a tetrad of vectors $U_{(x)}$ along the worldline of the gyroscope. These are parallely propogated and begin (at $\lambda = 0$) as a known set $A_{(x)}$ which corresponds to an orthonormal tetrad. $(A_{(r)})$ in direction $\frac{2}{\Delta r}, A_{(z)}$ in direction $\frac{2}{\Delta \theta}$ and thus perpendicular to the orbital plane, and A_{i_3} in direction $\frac{\partial}{\partial \lambda}$). Using the condition that the set $U_{(\alpha)}$ is parallely propogated one can derive expression for $U_{(\alpha)}[\lambda]$ when the gyroscope has momed through angle λ . These are given below.

$$U_{(\alpha)}[\lambda] = \sum_{\beta=1}^{4} U_{(\alpha)}{}^{\beta}[\lambda] \frac{\partial}{\partial x^{\beta}}$$

We set $b = \frac{2Gm_e}{C^2}$. The initial components are

	8=0	8 = 1	8=2	8=3
$A_{(o)}^{\gamma}$	$\sqrt{\frac{2a}{2a-3b}}$	0	0	a 12a-36
A & (1)	O	$\sqrt{1-\frac{b}{a}}$	0	0
A (2)	0	0	11a	0
A8(3)	$\sqrt{\frac{ba}{(a-b)(za-b)}}$, 0	0	$\frac{1}{2}\sqrt{\frac{2(a-b)}{2a-3b}}$

Then at angle λ

One can see that the vectors $U_{i,i}(\Lambda)$, the unit tangent and $U_{(1)}(\Lambda)$ the vector perpendicular to the orbital plane do not change components with respect to the basis $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$, in the course of the motion. This would be expected. We also note that, relative to an observer in this frame $\begin{bmatrix} 2 & 2 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$, the spatial parts of the vectors $U_{(i)}$ and $U_{(3)}$

undergo a rotation of angle $\beta \Lambda$ (after the gyroscope has moved through angle Λ) in a direction $-\vec{F} \times \vec{V}$. Thus if an external observer (say with "Cartesian" basis $\left(\frac{\lambda}{\lambda \epsilon}, \frac{\lambda}{\delta \chi}, \frac{\lambda}{\delta \chi}, \frac{\lambda}{\delta \chi}\right)$) watches the process he will see the gyroscope precess: by angle $\Lambda - \beta \Lambda$ in direction $\vec{F} \times \vec{V}$ as the gyroscope moved through angle Λ . Thus the total precession after one revolution is $2\pi [I - b] \Lambda \alpha d$.

or $2\Pi \cdot \frac{3b}{4a} = \frac{3\pi}{2a} \cdot \frac{6m}{c^2}$ radians in a direction $\overline{r} \times \sqrt{2}$

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