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COMMITTEE IN CHARGE
Chairman: B. N. Moyls
E. V. Bohn A. C. Soudack ..... Y. N. Yu
R. W. Donaldson
V. J. Modi ..... M. Davi'es
External Examiner: Dr. I. H. Mofti Analysis Laboratory National Research Council, Ottawa, Ontario.
Research Supervisor: ..... E. V. Bohn

# THE SYNTHESIS OF OPTIMAL CONTROLLERS FOR A 

CLASS OF AERODYNAMICAL SYSTEMS, AND THE
NUMERICAL SOLUTION OF NONLINEAR
OPTIMAL CONTROL PROBLEMS

## ABSTRACT

This dissertation is divided into two parts. In Part I, a method is developed for determining the optimal control laws for a class of aerodynamical systems whose dynamics are linear in the thrust and nonlinear in the lift and thrust angle. Conditions under which the adjoint variables can be eliminated from the control equations are derived, and expressions for the thrust and rate of change of lift and thrust angle are obtained which depend only on state variables and a small number of time invariant parameters. The optimal values of the unknown parameters are determined by a direct search in parameter space. It is shown that the proposed technique is considerably simpler than standard gradient techniques which require a separate search in function space for each component of the control vector. Furthermore, since the controls are generated by the direct solution of differential equations, the method appears suitable for use with in-flight guidance computers.

In Part II, a three stage numerical algorithm is developed for a general class of optimal control problems. The first two stages of the algorithm are based on a gradient search in the parameter space of initial Lagrange multipliers. The first stage attempts to satisfy the given end constraints without regard to system performance, and the second stage attempts to improve the system performance while simultaneously maintaining the end constraints set by the first stage. The final stage of the algorithm is based on either a modified method of matching end points, or a method of determining the optimal step size for the gradient method of the second stage. Either combination results in a three stage algorithm which has good initial convergence, good final convergence, and which requires storage at terminal points only.

## GRADUATE STUDIES

Field of Study: Electrical Engineering

| Analogue Computer's | $\cdots$ | E.V. Bohn |
| :--- | :--- | :--- |
| Nonlinear System Theory |  | A.C. Soudack |
| Network Theory | $\ddots$ | L. Mason |
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| Applied Electromagnetic Theory | M.M. Kharadly |  |
| Communications Theory. | $\therefore$ | A.D. Moore |
| Digital Computers |  | H.P. Zeiger |

Related Studies:

The Calculus of Variations
Space Dynamics
J.F. Scott-Thomas
V.J. Modi

## PUBLICATIONS

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by

JAMES WII工IAM SUTHERLAND

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We accept this thesis as conforming to the required standard

Research Supervisor
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## ABSTRACT

In Part I, a method is developed for determining the optimal control laws for a class of aerodynamical systems whose dynamics are linear in the thrust and nonlinear in the lift and thrust angle. Due to the presence of the linear thrust control, a singular subarc exists along which it is often possible to eliminate the Lagrange multipliers from the control equations. Conditions under which this elimination is possible are derived, and expressions for thrust and the rate of change of lift and thrust angle are obtained that depend only on state variables and a small number of time-invariant parameters. The optimal values of the unknown parameters are determined by a direct search in parameter space for that set which minimizes the system performance function. As a result, the proposed method is considerably simpler than standard numerical techniques that require a separate search in function space for each component of the control vector. Furthermore, since the control vector is generated by the direct solution of differential equations, the method appears suitable for use with in-flight guidance computers. Several numerical examples are presented consisting of one, two, and three dimensional control. In each case, it is shown that the search in multi-dimensional function space can be replaced by an equivalent search in the parameter space of initial conditions.

In Part II, a three stage numerical algorithm is developed for a general class of optimal control problems. The technique is essentially a combination of the direct and
indirect approaches. Like the indirect approach, the control law equations are used to eliminate the control vector from the system and adjoint equations. However, instead of trying to solve the two point boundary-value problem directly, the augmented performance function is first considered to be a function of the unknown initial conditions and is minimized by a gradient search in the initial condition space. It is shown that it is sufficient to search over the surface of any sphere for the intersection of the line $\mu \lambda_{0}^{*}$, where $\lambda_{0}^{*}$ is the classical solution of initial values. As a result, this first approach is not dependent on a good initial estimate of the optimal trajectory, and is therefore used in the first two stages of the proposed algorithm to provide the property of rapid initial convergence. The property of rapid final convergence is obtained by employing either a modified method of matching end points, or a method of determining the optimal step size for the gradient method of the first two stages. Either combination results in a three stage numerical algorithm that has good initial convergence, good final convergence, and which requires storage at terminal points only. Several examples are presented consisting of both bounded and unbounded control.
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## I. INTRODUCTION

### 1.1 The Indirect Approach

The classical theory of the calculus of variations was developed nearly two-hundred years ago by Euler and Lagrange. Recently, a more complete and mathematically rigorous treatment of optimal control theory was presented by Pontryagin in the form of the maximum principle [1]. Both techniques are essentially equivalent and form the basis of the indirect approach to the solution of optimal control problems.

The indirect approach is based on establishing a set of conditions which are necessary but not sufficient for an extremum. The solution to the optimal control problem is then taken as any trajectory along which these necessary conditions are satisfied. As a result, the second variation may have to be computed to insure a true extremum of the solution and not merely a stationary point. Although this approach often allows some analytical information to be obtained about the optimal trajectory, a complete analytical solutions is usually not possible and; therefore, numerical techniques are employed in association with the indirect approach. However, these numerical techniques usually require a high-capacity digital computer and usually need a good initial estimate of the optimal trajectory before the solution will converge. On the other hand, once a trajectory is found which is in the neighbourhood of the extremum, very rapid final convergence is realized.

Typical examples of numerical techniques based on the indirect approach are the two-point boundary-value problem $[2,3]$, the successive sweep method $[4]$, and the min-H strategy

### 1.2 The Direct Approach

A further computational scheme available to solve optimal control problems is the method of steepest descent (or ascent) which is based on a standard hill-climbing approach. In contrast to the indirect approach, which satisfies the necessary conditions for an extremum, this hill-climbing technique seeks directly that trajectory along which the system performance is an extremum. As a result, the method of steepest descent is known as a direct approach to the solution of optimal control problems. However, unlike most hill-climbing techniques, which are based on a gradient search in parameter space, this approach is based on an iterative scheme for improving the control function by means of a gradient search in function space. Consequently, as these control functions must be stored at many points along the trajectory, the resulting memory requirements of the computer may become excessively large. Furthermore, since the technique is predicated on a gradient method, the convergence slows down as the extremum is approached and it is not known when the search should be terminated. On the other hand, a main advantage of the direct approach is that the solution does not depend upon a good initial estimate of the optimal trajectory. In fact, once a nominal solution is obtained, initial convergence is guaranteed.

Typical examples of numerical techniques based on the direct method are the method of gradients by Kelly [6] and the method of steepest descent by Bryson and Denham [7].

### 1.3 The Dynamic Programming Approach

The third approach to the solution of optimal control problems is the dynamic programming approach recently developed by Bellman [8]. This approach is based on the principle of optimality which states that "an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy from the state resulting from the first decision". Employing this principle, the dynamic programming technique works backward from the desired final conditions and evaluates the optimal controls at discrete points in the entire space of permissible states. This flooding of the solution throughout the state space has, in principle, three main advantages. First, the technique is capable of handling a very general control problem including problems with bounded state and/or bounded control variables. Second, the solution is good for any set of allowable initial conditions, and; third, as the optimal controls are known at all points in state space, the solution is useful for realtime optimal control. However, for all but the simplest cases, the computer memory that is required to store the complete solution is prohibitively large [9]. This severe restriction is what Bellman calls "the curse of dimensionality". Some techniques have been recently developed that significantly reduce this problem of dimensionality; however, the dynamic programming approach is still limited to relatively simple problems [10]. Due to this limitation, the emphasis in this thesis will be placed on the use of the direct and the indirect
approaches with an aim of developing techniques that reduce some of the difficulties currently experienced with these methods. 1.4 The Proposed Techniques

The optimization techniques developed in this thesis are primarily a combination of the direct and indirect approaches. Two classes of problems are studied, and since the results are essentially unique, the material is presented in two parts.

In Part I, a method is developed for determining the optimal control laws for a class of aerodynamical systems of which the dynamics are linear in the thrust and nonlinear in the lift and thrust angle. Due to the presence of the linear control, a variable thrust or singular subarc exists along which the maximum principle cannot be applied. To overcome the difficulty associated with the singular control, a method is developed to eliminate the unknown Lagrange multipliers along this variable thrust subarc. Conditions under which this elimination is possible are derived and expressions for thrust and rate of change of lift and thrust angle are obtained which depend only on state variables and a small number of scalar time-invariant parameters. The unknown parameters are then determined by a" search in parameter space, based on a direct approach, for the set which minimizes the performance function. The proposed method is considerably simpler than the standard numerical techniques which require a separate search in function space for each component of the control vector. Furthermore, since the control vector is generated by the direct solution of differential equations, the method appears suitable for in-flight
guidance computers; that is, either the control law is obtained as a feedback law involving state variables only, or it can be generated from state variables and a small number of scalar parameters. Several numerical examples are given illustrating this technique.

In Part II, a numerical algorithm is developed to solve optimal control problems which do not contain singular control. The technique is based on replacing a gradient search in function space by an equivalent search in parameter space through the use of the necessary conditions of the indirect approach. To accomplish this, the control law equations of the calculus of variations are used to eliminate the control vector from the system and the adjoint equations. However, instead of trying to solve the resulting two point boundary-value problem, the augaugmented performance function is considered to be a function of all unknown initial conditions and is minimized by a gradient search in the parameter space of these initial conditions. It is shown that it is sufficient to search over any sphere for the intersection of the line $\mu \lambda_{0}^{*}$, where $\lambda_{0}^{*}$ is the classical solution of initial values. As a result, the proposed method is not dependent on a good initial estimate of the optimal trajectory. However, since the technique is based on a gradient search, the final convergence slows down as the optimum is approached. To provide improved final convergence, three techniques are developed. The first is based on the method of matching end points which uses an optimal scale factor for the Lagrange multipliers such that the error in transversality is a minimum at each step
in the iteration. The resulting algorithm is independent of the initial scale factor for the Lagrange multipliers and, hence, can be conveniently used with the proposed gradient technique. The other two method are based on determining the optimal step size for the gradient technique once in the vicinity of the optimum. One approach uses a second variation of the augmented performance function, and the other uses a method of curve fitting. It is shown that by combining these techniques, a three stage algorithm can be developed which has good initial convergence, good final convergence, and which requires storage at terminal points only. It is also shown that a similar approach can be used to improve the final convergence properties of the gradient search in function space. Several examples are presented, consisting of both bounded and unbounded control.

## PART I

OPTIMAL CONTROL LAWS FOR A CLASS OF
AERODYNAMICAL SYSTEMS

### 2.1 Optimal Control of Aerodynamical Systems

Miele has given a general variational theory for optimal flight paths of aerodynamical systems and derived general expressions for the Euler-Lagrange equations [II]. In special cases, these equations are useful in deriving analytical results concerning the nature of optimal flight paths [12]. However, with the exception of these few particular cases, numerical techniques are required to solve optimization problems and, as mentioned earlier, these techniques usually require the storage capability of a large-size general purpose, digital computer [13-14]. However, it has been shown that in the special case of the sounding rocket, the Euler-Lagrange equations can be used to solve the synthesis problem and the optimal thrust is expressed in closed form as a function of state variables only [12]. A more complex and more interesting problem is the general case of a missile moving within the earth.'s atmosphere under the control of thrust, thrust angle, and lift such that a specified performance function is a minimum. This is an example of multi-dimensional control, and the standard numerical techniques are time consuming to apply since a separate search in function space must be performed for each component of the control vector. It is the purpose of Part I of this thesis to extend the techniques, used to solve the sounding rocket problem, to the case of multi-dimensional control.
2.2 Problem Statement

The class of flight systems to be discussed are those
which can be represented by the following state vector differential equation:

$$
\begin{equation*}
\dot{x}=G_{0}(x, L)+u G_{1}(x, \beta) \tag{2.1}
\end{equation*}
$$

where

$$
\begin{aligned}
& x=\left(\begin{array}{c}
x_{n} \\
\vdots \\
x_{n}
\end{array}\right)=n \text { - vector of state variables } \\
& \dot{x}=\frac{d x}{d t} \text { denotes the time derivative of } x \\
& G_{O}(x, I)=n \text { - vector of functions } g_{0 k} \text { of } x \text { and } L \\
& G_{1}(x, \beta)=n-\text { vector of functions } g_{I k} \text { of } x \text { and } \beta
\end{aligned}
$$

and where the dynamics are linear in the thrust $u$ and nonlinear in the lift $I$ and thrust angle $\beta$. (See Figure 2.1.).


Figure 2.1 The Motion of a Rocket Within the Earth's Atmosphere

The derivation of (2.1) for the case of a missile moving in the earth's atmosphere is given in Appendix A. Consider now the problem of determining the set of controls (u, $L, \beta$ ) which takes the system (2.1) from some initial manifold defined by the k-vector constraint

$$
\begin{equation*}
H\left(x\left(t_{0}\right), t_{0}\right)=0 \tag{2.2}
\end{equation*}
$$

to some final manifold defined by the p-vector constraint

$$
\begin{equation*}
G\left(x\left(t_{f}\right), t_{f}\right)=0 \tag{2.3}
\end{equation*}
$$

where $u$ is subject to the constraint

$$
\begin{equation*}
u\left(u_{\max }-u\right)-\alpha^{2}=0 \tag{2.4}
\end{equation*}
$$

and where the performance function

$$
\begin{equation*}
P=P\left(x\left(t_{f}\right), t_{f}\right) \tag{2.5}
\end{equation*}
$$

is to be minimized. The constraints (2.2) and (2.3) are vector equations whose dimensions are equal to or less than $n$. The statea problem is of the Mayer type and can be solved by introducing the augmented function [1]

$$
\begin{equation*}
F \triangleq \lambda^{T}\left[\dot{x}-G_{0}-u G_{1}\right]+\lambda_{n+1}\left[u\left(u_{\max }-u\right)-\alpha^{2}\right] \tag{2.6}
\end{equation*}
$$

Here $\lambda$ is an n-vector of Lagrange multipliers, $\lambda^{T}$ is the transpose of $\lambda$ and $\lambda_{n+1}$ is the $(n+1)$ - th Lagrange multiplier. The Euler-Lagrange equations obtained from (2.6) are

$$
\begin{align*}
& \dot{\lambda}=-\left[\left(\frac{\partial G_{0}}{\partial x}\right)^{T}+u\left(\frac{\partial G_{1}}{\partial x}\right)^{T}\right] \lambda  \tag{2.7}\\
& 0=k_{L}  \tag{2.8}\\
& 0=u k_{\beta}  \tag{2.9}\\
& 0=k_{u}+\lambda_{n+1}\left(2 u-u_{\max }\right) \tag{2.10}
\end{align*}
$$

$$
\begin{equation*}
0=\alpha \lambda_{n+1} \tag{2.11}
\end{equation*}
$$

where

$$
\begin{align*}
& k_{L} \triangleq\left(\frac{\partial G_{O}}{\partial I}\right)^{T} \lambda  \tag{2.12}\\
& k_{\beta} \triangleq\left(\frac{\partial G_{1}}{\partial \beta}\right)^{T} \lambda  \tag{2.13}\\
& k_{u} \triangleq G_{I} T_{\lambda} \tag{2.14}
\end{align*}
$$

and where the following abbreviated notation is used:

$$
\begin{align*}
& \frac{\partial G_{0}}{\partial x} \triangleq\left[\begin{array}{cccc}
\frac{\partial g_{01}}{\partial x_{1}} & \cdot & \cdot & \frac{\partial g_{01}}{\partial x_{n}} \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\frac{\partial g_{0 n}}{\partial x_{1}} & \cdot & \cdot & \frac{\partial g_{0 n}}{\partial x_{n}}
\end{array}\right]  \tag{2.15}\\
& \frac{\partial G_{0}}{\partial I} \triangleq\left[\begin{array}{c}
\frac{\partial g_{01}}{\partial I} \\
\vdots \\
\frac{\partial g_{0 n}}{\partial I}
\end{array}\right] \tag{2.16}
\end{align*}
$$

Since $G_{0}$ and $G_{1}$ are formally independent of $t$, $a$ first integral

$$
\begin{equation*}
(\dot{x})^{T} \lambda=c \tag{2.17}
\end{equation*}
$$

exists, where $c$ is a constant of integration. Substituting (2.1) into (2.17) yields

$$
\begin{equation*}
G_{0}{ }^{T} \lambda+u G_{1}{ }^{T} \lambda=c \tag{2.18}
\end{equation*}
$$

The transversality condition for the stated problem is

$$
\begin{equation*}
\left[d P+\nu^{T} d G+(d x)^{T} \lambda-c d t\right]_{t_{0}}^{t_{f}}=0 \tag{2.19}
\end{equation*}
$$

where $d x$ and de must be consistent with the terminal constraints (2.2) and (2.3) and where $\nu$ is a p-vector of constant Lagrange multipliers which are introduced to account for (2.3).

### 2.3 Discussion of the Necessary Conditions for an Optimal Trajectory

Because the system (2.1) is linear in $u$, and because of the constraint (2.4), the optimal trajectory will in general consist of maximum thrust subarcs where $u=u_{\max }$, variable thrust subareas where $0<u<u_{\max }$, and minimum thrust or coasting subarcs where $u=0$. The variable thrust subarcs are also known as singular subaros since the Hamiltonian is then independent of $u$ and, consequently, the maximum principle cannot be applied to determine $u$. It follows from (2.4) that $\alpha=0$ along maximum and minimum thrust subareas and hance (2.11) Is satisfied. Along the variable thrust aubaro it follows from (2.11) that $\lambda_{n+1}=0$ and hence (2.10) yields

$$
\begin{equation*}
k_{u} \equiv 0 \tag{2.20}
\end{equation*}
$$

The method to be discussed requires a knowledge of the sequence of subaros. This can be determined from the LegendreOlebsch condition and the Erdman-Weierstrase corner conditions. The Legendre-Clebsch condition applied to (2.6) requires that

$$
\begin{equation*}
\frac{\partial^{2} F}{\partial L}(\delta I)^{2}+\frac{\partial^{2} F}{\partial \beta}\left(\delta_{\beta}\right)^{2}+\frac{\partial^{2} F}{\partial u} \cdot\left(\delta_{u}\right)^{2}+\frac{\partial^{2} F}{\partial \alpha^{2}}(\delta \alpha)^{2} \geq 0 \tag{2.21}
\end{equation*}
$$

where $\delta_{u}$ and $\delta_{\alpha}$ are related by (2.4). Substituting (2.6) and (2.10) into (2.21) yields

$$
\begin{equation*}
G_{O L L^{\lambda}}^{T}(\delta I)^{2}+G_{I \beta \beta}^{T}(\delta \beta)^{2}+\frac{k_{u}}{2 u-u_{\max }}\left[(\delta u)^{2}+(\delta \alpha)^{2}\right] \geqslant 0 \tag{2.22}
\end{equation*}
$$

where the following abbreviated notation is used:

$$
G_{O L L} \triangleq\left[\begin{array}{c}
\frac{\partial^{2} g_{O I}}{\partial I^{2}}  \tag{2.23}\\
\vdots \\
\frac{\partial^{2} g_{O n}}{2}
\end{array}\right]
$$

as $\delta_{I}, \delta_{\beta}$ and $\delta_{u}$ are independent, it follows from (2.22) that

$$
\begin{align*}
& G_{O L I^{\prime}}^{T} \supseteq 0  \tag{2.24}\\
& G_{I \beta \beta}^{T} \lambda \geq 0 \tag{2.25}
\end{align*}
$$

everywhere along the optimal trajectory and that

$$
\begin{array}{ll}
k_{u}>0, & \left(u=u_{\max }\right) \\
k_{u}=0, & \left(0<u<u_{\max }\right) \\
k_{u}<0, & (u=0)
\end{array}
$$

Due to the Erdmann-Weierstrass corner conditions, the Lagrange multipliers and the first integral (2.18) are continuous along the optimal trajectory. Substituting (2.14) into (2.18) yields

$$
\begin{equation*}
G_{O}^{T} \lambda+u k_{u}=c \tag{2.27}
\end{equation*}
$$

It follows from (2.27) and the Erdmann-Weierstrass corner conditions that a discontinuity in $u$ is possible provided that

$$
\begin{equation*}
\left(k_{u}\right)_{-}=\left(k_{u^{\prime}}\right)_{+}=0 \tag{2.28}
\end{equation*}
$$

where the minus and plus subscripts denote evaluation of the
brackets just before and just after each switching point respectively. Hence, $k_{u}$ is a continuous function which vanishes at each switching point (see (2.26)). For this reason, $k_{u}$ will be defined as the switching function. The switching function is a fundamental importance in determining the allowable sequence of subarcs. To investigate the properties of $k_{u}$, (2.14) is differentiated with respect to time. Using (2.1) and (2.7) to eliminate $\dot{x}$ and $\dot{\lambda}$ yields

$$
\begin{align*}
\dot{k}_{u}= & {\left[G_{0}^{T} G_{l x}^{T}-G_{1}^{T} G_{0 x}^{T}\right] \lambda+u\left[G_{1}^{T} G_{l x}^{T}-G_{l}^{T} G_{l x}^{T}\right] \lambda } \\
& +\dot{\beta}_{\beta} \tag{2.29}
\end{align*}
$$

It is seen that the coefficient of $u$ in (2.29) is identically zero. Furthermore, the term $\dot{\beta} k_{\beta}$ is zero, since either $u \neq 0$ and $k_{\beta}=0$ (see (2.9)) or if $u=0$, the thrust angle can be defined to be constant so that $\dot{\beta}=0$. Hence (2.29) yields

$$
\begin{equation*}
\dot{\mathrm{k}}_{\mathrm{u}}=\mathrm{K}^{\mathrm{T}} \lambda \tag{2.30}
\end{equation*}
$$

where

$$
\begin{equation*}
K \triangleq G_{I x} G_{0}-G_{0 x} G_{I} \tag{2.31}
\end{equation*}
$$

It follows from (2.30) and the Erdmann-Weierstrass corner conditions that at the corners of the optimal trajectory

$$
\begin{equation*}
\left(\dot{k}_{u}\right)_{-}=\left(\dot{k}_{u}\right)_{+} \tag{2.32}
\end{equation*}
$$

The condition (2.32) can be used to determine. the possible sequences of subarcs. Expanding $k_{u}$ in a Taylor series in $t$ at the switching instant $t_{s}$ yields

$$
\begin{equation*}
k_{u}\left(t_{s}+\Delta t\right)=k_{u}\left(t_{S}\right)+\dot{k}_{u}\left(t_{S}\right) \Delta t+\ldots \tag{2.33}
\end{equation*}
$$

Substituting (2.28) and (2.33) into the condition (2.26), and choosing $\Delta t$ so that second and higher order terms are negligible, yields the conditions

$$
\begin{equation*}
\left[K^{t}\right]_{t_{S}} \Delta t \geqslant 0 \tag{2.34}
\end{equation*}
$$

which apply for maximum thrust, variable thrust, and zero thrust respectively.

Table 2.1

| $u\left(t_{s}-\Delta t\right)$ | $\dot{k}_{u}\left(t_{s}\right)=K^{T} \lambda$ | $u\left(t_{s}+\Delta t\right)$ |
| :---: | :---: | :---: |
| (1) $\begin{aligned} & u=0 \\ & k_{u}<0 \end{aligned}$ <br> (2) $\begin{aligned} & u=u_{\max } . \\ & k_{u}>0 . \end{aligned}$ <br> (3) $\begin{aligned} & u=u(t) \\ & k_{u}=0 \end{aligned}$ | (la) $K^{T} \lambda>0$ <br> (Ib) $K^{T} \lambda=0$ <br> (2a) $K^{T} \lambda<0$ <br> (2b) $K^{T} \lambda=0$ <br> (3a) $K^{T} \lambda=0$ <br> (3b) $K^{\mathbb{T}^{\mathrm{I}}} \lambda=0$ | (la) $u=u_{\max .}, k_{u}>0$ <br> (lb) $u=u(t), k_{u}=0$ <br> (2a) $u=0, k_{u}<0$ <br> (2b) $u=u(t), k_{u}=0$ <br> (3a) $\quad u_{k_{u}}>0,0$ if $u\left(t_{s}\right)=0$ <br> (3b) $\left.\begin{array}{rl} u=u_{\max } \cdot \\ k_{u}<0 \end{array}\right\} \begin{aligned} & \text { if } \\ & u\left(t_{s}\right) \\ & \\ & =u_{\max } \end{aligned}$ |

Table 2.1 illustrates the possible sequences of subarcs which satisfy the Legendre-Clebsch condition. The sign of $k_{u}$ defines the state of $u($ see (2.26)). The instants of time where $k_{u}$ vanishes defines the switching points $t_{s}$. The sign of $K^{T} \lambda$ at $t=t_{s}$ determines the state of $u$ after switching. The symbolism (la), ...., (3b) used to denote these states will be made use of later in obtaining a sequence diagram. It will be shown by means of examples that an analytical study of the system and its constraints can often provide the necessary
information to determine a unique sequence of subarcs by use of (2.28) and Table 2.1.

### 2.4 The Control Equations for Thrust, Thrust Angle and Lift

To determine suitable control equations, it is desirable to eliminate the Lagrange multipliers and obtain equations relating the state variables and $u, \beta$ and $I$ only. If these equations do not involve time derivatives of $u, \beta$ and $L$, a feedback control law is obtained. However, this may not always be possible. It is the purpose of this section to determine conditions under which a feedback law can be directly obtained and to develop suitable alternatives when these conditions are not satisfied. The elimination of the Lagrange multipliers is advantageous since the magnitudes of the control variables are usually known approximately while the magnitude of the Lagrange multipliers are unknown.
To eliminate the Lagrange multipliers, (2.8) and (2.9)
are first differentiated with respect to time. Eliminating $\dot{x}$ and $\dot{\lambda}$ by means of (2.1) and (2.7) yields

$$
\begin{equation*}
\dot{L}=L_{0}+u L_{1} \tag{2.35}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{\beta}=\beta_{0}+u \beta_{1} \tag{2.36}
\end{equation*}
$$

where

$$
\begin{align*}
& L_{0} \triangleq \frac{\left(G_{O L}^{T} G_{O x}^{T}-G_{0}^{T} G_{O L X}^{T}\right) \lambda}{G_{O L L}^{T} \lambda}  \tag{2.37}\\
& I_{1} \triangleq \frac{\left(G_{O L}^{T} G_{I X}^{T}-G_{1}^{T} G_{O L X}^{T}\right) \lambda}{G_{O L L}^{T} \lambda} \tag{2.38}
\end{align*}
$$

$$
\begin{align*}
& \beta_{0} \triangleq \frac{\left(G_{1 \beta}^{T} G_{0 x}^{T}-G_{0}^{T} G_{1 \beta x}^{T}\right) \lambda}{G_{1}^{T} \beta_{\beta}^{T} \lambda}  \tag{2.39}\\
& \beta_{1} \triangleq \frac{\left(G_{1 \beta}^{T} G_{1 X}^{T}-G_{1}^{T} G_{1 \beta x}^{T}\right) \lambda}{G_{1 \beta \beta}^{T} \lambda} \tag{2.40}
\end{align*}
$$

and where the following abbreviated notation is used:

$$
G_{O L x} \triangleq\left[\begin{array}{ccc}
\frac{\partial^{2} g_{0 I}}{\partial I \partial x_{1}} & \cdots & \cdots
\end{array} \frac{\partial^{2} g_{0 I}}{\partial I \partial x_{n}}\left[\begin{array}{ccc}
\vdots & & \vdots  \tag{2.41}\\
\vdots & & \vdots \\
\frac{\partial^{2} g_{0 n}}{\partial I \partial x_{1}} & \cdots & \cdots
\end{array}\right] \frac{\partial^{2} g_{0 n}}{\partial I \partial x_{n}}\right]
$$

If the inequality sign holds in (2.21), it follows from (2.24) and (2.25) that the division by $G_{O L I}^{T} \lambda$ and $G_{I \beta \beta}^{T} \lambda$ is permissible.

Along the variable thrust subarc $k_{u}$ is identically zero (see 2.20). Thus the time derivatives of $k_{u}$ are also zero. It follows from (2.30) that during variable thrust

$$
\begin{align*}
& \dot{k}_{u}=K^{T} \lambda=0  \tag{2.42}\\
& \ddot{k}_{u}=\dot{k}^{T} \lambda+K^{T} \dot{\lambda}=0 \tag{2.43}
\end{align*}
$$

Using (2.31) to evaluate $\dot{K}$ and eliminating $\dot{x}, \dot{\lambda}$, $\dot{L}$ and $\dot{\beta}$ by means of (2.1), (2.7), (2.35) and (2.36) yields
$u=\frac{\left[K^{T} G_{0 x}^{T}-G_{0}{ }^{T} K_{X}^{T}-\beta_{0}^{T} K_{\beta}^{T}-L_{0}^{T} K_{L}^{T}\right] \lambda}{\left[G_{1}{ }^{T} K_{x}{ }^{T}-K^{T} G_{l x}{ }^{T}+\beta_{1}{ }^{T} K_{\beta}^{T}+L_{1}{ }^{T} K_{L}^{T}\right] \lambda}$
If $\lambda$ can be eliminated from (2.35), (2.36) and (2.44), equations are obtained for $u, \dot{L}$ and $\dot{\beta}$ in terms of state variables
only. These equations can then be used to investigate the possibility of optimal and sub-optimal feedback laws which hold during the variable thrust subarc. The elimination of $\lambda$ is possible if a sufficient number of linear independent equations between the components of $\lambda$ can be found. There are five such linear equations given by (2.8), (2.9), (2.18), (2.20) and (2.42). It may be possible to augment these five equations by further linear equations obtained by a direct integration of the Euler-Lagrange equations. Let $m$ be the total number of such linear equations. These equations can be written in the matrix form

$$
\begin{equation*}
\mathrm{A} \lambda=\mathrm{b} \tag{2.45}
\end{equation*}
$$

where $A$ is a mxn matrix, where

$$
b \triangleq\left[\begin{array}{l}
c_{1}  \tag{2.46}\\
\cdot \\
\cdot \\
\cdot \\
c_{m}
\end{array}\right]
$$

and where the $c_{k}$ are integration constants. If $b \neq 0$, (2.45) can be used to determine $\lambda$ uniquely; in terms of the $c_{k}$ provided that

$$
\begin{equation*}
\operatorname{rank}(A)=n \leq m \tag{2.47}
\end{equation*}
$$

(It should be noted that for the case $m>n$, the integration constants $c_{k}$ cannot be independently specified.)

$$
\text { If } b=0,(2.45) \text { is a set of linear homogeneous }
$$

equations in $\lambda_{1}, \ldots, \lambda_{n}$ and if

$$
\begin{equation*}
\operatorname{rank}(A)=n-1 \leq m \tag{2.48}
\end{equation*}
$$

it is possible to obtain a unique non-trivial solution of (2.45) in the form

$$
\begin{equation*}
\lambda_{i}=Q_{i} \lambda_{k}, \quad(i \neq k) \tag{2.4}
\end{equation*}
$$

so that all $\lambda_{i}(i \neq k)$ are expressed in terms of one Lagrange multiplier $\lambda_{k}$. Introducing (2.49) into (2.37), (2.38), (2.39), (2.40) and (2.44) then yields the control equations of $u, \dot{L}$ and $\dot{\beta}$ in terms of state variables alone. Conditions (2.47) and (2.48) therefore serve as a test to see if it is possible to eliminate the Lagrange multipliers during a variable thrust subarc. Several examples will now be given to illustrate various possibilities which can occur.
3. THE SOUNDING ROCKET PROBLEM

### 3.1 Derivation of Optimal Control Law

The derivation of the feedback control law for the variable thrust subarc has been given in [12]. However, the use of the theory outlined in Chapter 2 can be used to prove the existence of feedback control laws for more general cases and, also, to provide a more systematic means for obtaining analytic expressions for control laws which are a function of the state variables only. The terminal conditions for the sounding rocket problem are

$$
\begin{align*}
t_{0} & =0 \\
y(0) & =0 \\
v(0) & =v_{0}  \tag{3.1}\\
m(0) & =m_{0} \\
v\left(t_{f}\right) & =0 \\
m\left(t_{f}\right) & =m_{f}
\end{align*}
$$

and the final altitude $y_{f}$ is to be maximized for a given amount of fuel. The performance function $P=-y_{f}$ is to be minimized. Appendix $B$ gives the analysis associated with this problem. The transversality condition (B-ll) yields

$$
\begin{equation*}
c=0, \quad \lambda_{2 f}=1 \tag{3.2}
\end{equation*}
$$

It follows that $b=0$ (see (2.46) and (B-13)), and since $n=m$ $=3$, condition (2.48) requires that rank (A) = 2 if a feedback control law is to exist. To determine rank (A), (B-12) can
be triangularized yielding the matrix

$$
B=\left[\begin{array}{ccc}
1 & -\frac{g}{v}-\frac{D}{m v} & 0  \tag{3.3}\\
0 & 1 & -\frac{m}{v_{e}} \\
0 & 0 & f_{s}
\end{array}\right]
$$

where

$$
\begin{equation*}
f_{s} \triangleq m g-\left(D+\frac{v}{v_{e}} D\right) \tag{3.4}
\end{equation*}
$$

and where rank $(A)=r a n k(B)$.
It is seen from (3.3) that the condition (2.48) is satisfied along the variable thrust subarc if

$$
\begin{equation*}
f_{s}=0 \tag{3.5}
\end{equation*}
$$

Substituting the first integral ( $B-10$ ) into ( $B-9$ ) yields the time-derivative of the switching function.

$$
\begin{equation*}
\dot{k}_{u}=\frac{v_{e}}{m v}\left[u k_{u}-\frac{\lambda_{3}}{m} f_{s}\right] \tag{3.6}
\end{equation*}
$$

During a variable thrust subarc, it is seen from (3.6), (3.5) and (2.26) that $\dot{k}_{u}=0$ as required. The function $f_{s}$, which is a function of state variables only, can be used to determine the optimal feedback control law. The time derivative of $f_{S}$ is

$$
\begin{equation*}
\dot{f}_{s}=-u \mathbb{M}+\mathbb{N} \tag{3.7}
\end{equation*}
$$

where

$$
\begin{align*}
& M \triangleq g+\frac{D}{m}\left(3+\frac{2 v_{e}}{v}\right)  \tag{3.8}\\
& N \triangleq\left(g+\frac{D}{m}\right)\left(\frac{2 D}{v}+\frac{3 D}{v_{e}}\right)+a D v\left(1+\frac{v}{v_{e}}\right) \tag{3.9}
\end{align*}
$$

(For the definition of a see ( $B-4$ ).)
It follows from (3.5) that $\dot{f}_{s}=0$ along a variable thrust subarc. Equation (3.7) can then be solved for the optimal control law.

$$
\begin{equation*}
u=\frac{N}{M} \tag{3.10}
\end{equation*}
$$

To determine the control law for the complete trajectory requires additional information about the sequence of subarcs. Consider first the case where the maximum thrust subarc is one of impulsive boosting ( $u_{\max }=\infty$ ). Since $M>0$, it follows from (3.7) that $\dot{f}_{s}<0$ along the maximum thrust subarc. The sequence of possible subarcs can be obtained with the aid of the sequence diagram illustrated in Figure 3.1 which is a graphical representation of the five possible states associated with the signs of $k_{u}, \dot{k}_{u}$ and condition $f_{s}=0$. Eight gates, (la), (Ib) to (4a), (4b) are provided to indicate the allowable change in state. The position of gates (la), (lb) to (3a), (3b) for all problems of the type discussed in Section 2.1 are determined by the use of Table 2.1. From (2.26) and (2.42), it is seen that regions 1 and II in the sequence diagram are regions of maximum thrust, regions III and IV are regions of zero thrúst, and region $V$ is a region of variable thrust where


Figure 3.1 Sequence Diagram for the Sounding Rocket with
Impulsive Boosting
$u$ is given by (3.10). The positions of gates (4a) and (4b) are not given by Table 2.1 and must be determined by further analysis of each particular problem. Consider, for example, the case of the sounding rocket. To determine the position of gate (4b) the sign of $\ddot{k}_{u}$ must be determined at the instant where $\dot{k}_{u}=0$. Differentiating (3.6) with respect to time and evaluating $\dddot{k}_{u}$ when $\dot{k}_{u}=0$ and $u=0$ yields

$$
\begin{equation*}
\ddot{\mathrm{k}}_{\mathrm{u}}=-\frac{\lambda_{3} \mathrm{v} e}{\mathrm{~m}^{2} \mathrm{v}} \dot{f}_{s} \tag{3.11}
\end{equation*}
$$

From (3.7) it is seen that during coasting ( $u=0$ )

$$
\begin{equation*}
\dot{f}_{S}=N>0 \tag{3.12}
\end{equation*}
$$

In Appendix $B$ it is shown that $\lambda_{3}>0$ along the interior of the optimal trajectory. It then follows from (3.12) and (3.11) that $\ddot{k}_{u}<0$ when $\dot{k}_{u}=0$ and gate ( $4 b$ ) must therefore open down as shown in Figure 3.1. The determination of the position of gate (4a) proceeds in a similar manner. Differentiating (3.6) with respect to time and using $u=u_{\max }$. yields

$$
\begin{equation*}
\ddot{k}_{u}=\frac{\lambda_{3} v_{e}}{m^{2}}\left[u_{\max } \cdot \frac{D}{m}\left(4+\frac{2 v_{e}}{v}+\frac{v}{v_{e}}\right)-N+\frac{f_{s} D}{m v_{e}}\right] \tag{3.13}
\end{equation*}
$$

It follows from (3.13) and $\lambda_{3}>0$ that during maximum thrust, the condition for $\ddot{k}_{u}>0$ is

$$
\begin{equation*}
u_{\max }>\frac{\mathbb{N}-\frac{f_{s} D}{m v_{e}}}{\frac{D}{m}\left(4+\frac{2 v_{e}}{v}+\frac{v}{v_{e}}\right)} \tag{3.14}
\end{equation*}
$$

In this example, it is assumed that $u_{\text {max. }} \rightarrow \infty$, and hence (3.14)
is satisfied. Gate (4a) must therefore be directed up as shown. The sequence diagram for the sounding rocket is now complete and the optimal sequence for any set of terminal constraints can be determined. It is proven in Appendix $B$ that for the end constraints (3.1) the final subarc must be a coasting subarc. Also, from (3.7), it is seen that

$$
\begin{equation*}
\dot{f}_{s} \geqslant 0 \tag{3.15}
\end{equation*}
$$

for the cases of a coasting, a variable thrust, and a maximum thrust subarc respectively. Combining this information with the sequence diagram permits the evaluation of an acceptable sequence of subarcs. Several possibilities are illustrated in Figure 3.1. The type of initial subarc is determined from the initial value of $f_{s}$ as shown. This initial subarc is maintained until either $f_{s}$ vanishes or until all fuel is consumed. The vanishing of $f_{s}$ indicates a switch to the variable-thrust subarc where $u$ is programmed according to (3.10), and the instant of burn-out indicates a switch to the final coasting subarc. It is seen, therefore, that (3.4) and (3.10) completely define the optimal control as a function of state. Also, for any system satisfying condition (3.14), there can be at most three subarcs in the optimal trajectory.

The case of $u_{\max }<\infty$ is of particular interest. If (3.14) is not satisfied at the instant when $\dot{k}_{u}$ vanishes, then it follows from (3.13) that gate (4a) must be changed to the down position as illustrated in the sequence diagram shown in Figure 3.2. Assuming that this be the case, it is seen that a closed loop in the sequence diagram can exist whose


Figure 3.2 Sequence Diagram for the Sounding Rocket with
sequence is ....., $u=u_{\max .}, 0<u<u_{\max .}, u=u_{\max }, \ldots \ldots$ (case Ia). This closed loop presents no analytical difficulty as switching to a maximum thrust subarc is defined when $u$ given by (3.10) equals $u_{\max }$, and switching back to a variable thrust subarc occurs when $f_{s}$ vanishes. However, difficulty does arise if the initial subarc is a coasting subarc. For this case, the first switching may be to a maximum thrust subarc (case IIIb) or to a variable thrust subarc (case IIIa), depending on the value of $k_{u}$ when $k_{u}=0$. To avoid the use of $k_{u}$, which contains unknown lagrange multipliers, the first switching instant $t_{s}$, where $k_{u}$ vanishes, can be introduced as an unknown parameter. Let $\tau_{s}$ be the instant where $f_{s}$ vanishes which determines the switching instant to a variable thrust subarc. If $t_{s}<\tau_{s}$ the control switches to a maximum thrust subarc and if $t_{s}=\tau_{s}$ the control switches to a variable thrust subarc. To determine $t_{s}$, the performance function $P$ can be considered to be a function of $t_{s}$ and a search over the interval $0 \leq t_{s} \leq \tau_{s}$ can be performed to determine the minimum of $P$. A numerical example of this type of search is given in [12].

## 4. THE MAXIMUM RANGE PROBLEM

4.1 Derivation of Optimal Control Laws for the Case $I=0$, $\hat{B}=0$

Consider the problem of maximizing the range for a given amount of fuel when the terminal constraints are

$$
\begin{array}{rlrl}
t_{0} & =0 & \theta(0) & =\theta_{0} \\
\mathrm{x}(0) & =0 & \mathrm{~m}(0) & =\mathrm{m}_{0} \\
\mathrm{y}(0) & =0 & \mathrm{y}\left(\mathrm{t}_{\mathrm{f}}\right) & =0 \\
\mathrm{v}(0) & =\mathrm{v}_{0} & \mathrm{~m}\left(\mathrm{t}_{\mathrm{f}}\right) & =\mathrm{m}_{\mathrm{f}} \tag{4.1}
\end{array}
$$

where the control constraints $\beta=0, L=0$ are applied, and where the performance function

$$
\begin{equation*}
P=-x_{f} \tag{4.2}
\end{equation*}
$$

is to be minimized. Appendix C gives the analysis associated with this problem. It is assumed that the initial conditions for the rocket are obtained by means of a launching platform. Also, it is assumed that $u_{\text {max. }}=\infty$ is a good approximation to the maximum thrust subarc. These assumptions do not impose any restrictions on the control laws derived for the variable thrust subarc. However, they do simplify the numerical computation and attention can thus be focused on the derivation of the control laws and their application in determining optimal trajectories.

The first step, in obtaining these control laws, is to evaluate the transversality condition ( $C-14$ ) according to (4.1) and (4.2) which yields

$$
\begin{array}{cc}
\lambda_{3 f}=0 & \lambda_{I f}=c_{1}=1  \tag{4.3}\\
\lambda_{4 f}=0 & c=0
\end{array}
$$

Taking $c=0$ in the first integral (C-13) and substiluting into (C-12) yields

$$
\begin{equation*}
\dot{k}_{u}=\frac{v_{e}}{m v}\left(-\frac{\lambda_{3} f_{s}}{m}-\frac{2 \lambda_{4}}{v} g \cos \theta+u k_{u}\right) \tag{4.4}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{S} \triangleq m g \sin \theta-D\left(1+v / v_{e}\right) \tag{4.5}
\end{equation*}
$$

Substituting (4.3) into ( $\mathrm{C}-16$ ) it is seen that $b \neq 0$. As (C-15) is a $4 \times 5$ matrix where $m=4, n=5$, the condition (2.47) is violated. Hence, a feedback control law, in terms of state variables only, cannot be obtained directly and a modified approach must be adopted. During a variable thrust subarc, equation (2.26) must be satisfied. Substituting ( ( $1-11$ ) into (2.26) yields

$$
\begin{equation*}
\mathrm{k}_{\mathrm{u}}=\frac{\lambda_{3}{ }^{\mathrm{v}} \mathrm{e}}{\mathrm{~m}}-\lambda_{5}=0 \tag{4.6}
\end{equation*}
$$

Substituting (4.6) into (C-33) and equating $\ddot{\mathrm{k}}_{u}$ to zero gives

$$
\begin{equation*}
u=\frac{2 \lambda_{1} g \cot \theta+\lambda_{3} \mathbb{N}}{\lambda_{3} M} \tag{4.7}
\end{equation*}
$$

where $M$ and $\mathbb{N}$ are given by $(C-34)$ and ( $C-35$ ) respectively. As $\lambda_{3} \neq 0$ along a variable thrust subarc (see Appendix $C$ ), division by $\lambda_{3}$ is permissible. With the aid of (4.6), therefore, it is possible to write (4.7) in the form

$$
\begin{equation*}
u=\frac{2 \gamma_{v_{e}} g \cot \theta / m+N}{M} \tag{4.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma \triangleq \lambda_{1} / \lambda_{5} \tag{4.9}
\end{equation*}
$$

Differentiating (4.9) with respect to time and using (4.8) and
(C-10) yields

$$
\begin{equation*}
\dot{\gamma}=\frac{\gamma}{m}\left(\frac{D}{v_{e}}-u\right) \tag{4.10}
\end{equation*}
$$

which is valid for all values of $\lambda_{1}$. Equations (4.8) and (4.10) give the optimal control law for this problem. As a result of (4.10), an unknown parameter $\gamma_{S}$ is introduced which is the initial value of $\gamma$ at the beginning of the variable thrust subarc. One other unknown parameter exists, however, and that is the instant of switching $\tau_{s}$ to the variable thrust subarc. This instant is characterized by the vanishing of the switching function $k_{u}$. However, using $k_{u}$ to define $\tau_{s}$ involves the use of the unknown Lagrange multipliers. To avoid determining the Lagrange multipliers, an alternate approach is adopted. The unknown switching instant $\tau_{s}$ is replaced by an unknown switching velocity $\mathrm{v}_{\mathrm{s}}$ 。 Consequently, the unknown parameters for this problem are $\gamma_{s}$ and $v_{s}$. The proposed technique, for obtaining the value of any unknown parameter $\alpha_{k}$, is to consider the performance function as a function of the parameters and then solve the minimization problem

$$
\begin{equation*}
\operatorname{Min}_{\alpha_{k}}[P] \tag{4.11}
\end{equation*}
$$

by a direct search in parameter space.
Using ( $0-3$ ) and a sequence diagram, it can be shown that acceptable sequences for this problem are given by
(1) $\dot{v}_{0}^{\prime}<v_{s}, u=u_{\max }, \quad 0 \leqslant u \leqslant u_{\max }, \quad u=0$
(2) $v_{0}>v_{s}, u=0,0 \leqslant u \leqslant u_{\max }, \quad u=0$
(3) $v_{0}=v_{s}, \quad 0 \leqslant u \leqslant u_{\max }, \quad u=0$

In case ( 1 ), $\mathrm{v}_{0}<\mathrm{v}_{\mathrm{S}}$ and hence the initial subarc must be a maximum thrust subarc along which $v_{0}$ increases to $\mathrm{v}_{\mathrm{S}}$. In case (2) $\mathrm{v}_{0}>\mathrm{v}_{\mathrm{S}}$ and hence the initial subarc must be a coasting subarc along which $\mathrm{v}_{0}$ decreases to $\mathrm{v}_{\mathrm{s}}$. When $\mathrm{v}_{0}=\mathrm{v}_{\mathrm{s}}$, the thrust switches to variable thrust until all the fuel is consumed.

As an illustrative numerical example, consider the following data

$$
\begin{align*}
& \mathrm{m}_{0}=35.0 \mathrm{slugs} \\
& \mathrm{~m}_{\mathrm{f}}=10.0 \mathrm{slugs} \\
& \theta_{0}=45^{\circ} \\
& \mathrm{v}_{0}=1000 \mathrm{ft} / \mathrm{sec}  \tag{4.12}\\
& \mathrm{v}_{\mathrm{e}}=5500 \mathrm{ft} / \mathrm{sec} \\
& \mathrm{a}=(22000 \mathrm{ft})^{-1} \\
& \mathrm{~K}_{a}=10^{-4} \mathrm{slug} / \mathrm{ft} \quad(\text { see }(\mathrm{B}-4))
\end{align*}
$$

In this example, $\mathrm{v}_{0}$ is a relatively small initial velocity and hence the sequence associated with case (1) results. As $u_{\text {max. }}=\infty$ is assumed, (c-1) to (c-5) can be integrated over the maximum thrust subarc using (4.1) from $v=v_{0}$ to $v=v_{s}$ to yield

$$
\begin{align*}
& x_{s}=0 \\
& y_{s}=0 \\
& \theta_{s}=\theta_{0}  \tag{4.13}\\
& m_{s}=m_{0} \exp \left(\left(v_{0}-v_{s}\right) / v_{e}\right)
\end{align*}
$$

The integration along the variable thrust subarc is now performed using (4.13) as initial values. The unknowns are $v_{s}$ and
$\gamma_{s}$ as $m_{s}$ can be obtained from $v_{s}$ through (4.13). The performance function to be minimized is $P=-x_{f}\left(v_{S}, \gamma_{s}\right)$. The computation was performed on an IBM 7040 digital computer using the following algorithm:

1. Select a set of values $\left(v_{s}, \gamma_{s}\right)$.
2. Solve (4.13) to obtain $m_{s}$.
3. With $u$ and $\gamma$ given by (4.8) and (4.10), integrate the system equations ( $\mathrm{C}-1$ ) to ( $C-5$ ) and determine $x_{f}\left(v_{s}, \gamma_{s}\right)$ 。
4. Repeat 1, 2, 3 and carry out a direct search in parameter space ( $v_{s}, \gamma_{s}$ ) for the maximum $X_{f}$.

The values for the state variables at the switching points along the optimal trajectory are shown in Table 4.l.

Table 4.1

|  | Initial Point | End of <br> Max. Thrust | Burnout | Final Point |
| :--- | :---: | :---: | :---: | :---: |
| $t(s e c)$. | 0 | $0+$ | 26.53 | 288.1 |
| $x\left(f t_{0}\right)$ | 0 | 0 | 90560 | 1470000 |
| $\mathrm{y}\left(f t_{0}\right)$ | 0 | 0 | 74870 | 0 |
| $\mathrm{v}(\mathrm{ft} / \mathrm{sec})$. | 1000 | 3045 | 6670 | 4970 |
| (deg.) | 45 | 45 | 36.6 | -41.0 |
| $m$ (slug) | 35 | 24.13 | 10.0 | 10.0 |

Figure 4.l illustrates $x_{f}$ as a function of $v_{s}$ for the optimal value of $\gamma_{s}$, and $x_{f}$ as a function of $\gamma_{s}$ for the optimal value of $v_{s}$. The graphs of $u(t)$ and $\gamma(t)$ for the optimal trajectory are shown in Figure 4.2.


Figure 4.1 The Final Range as a Function of $V_{S}$ and $\dddot{O}_{s}$



Figure $4.2 u(t)$ and $\gamma(t)$ for the Optimal Trajectory

The zero lift case where $u$ and $\beta$ are the control variables is a two dimensional control problem. Appendix D gives the analysis associated with this case. If $P$ is ingependent of time, the transversality condition (D-15) yields $c=0$. The class of problems for which $c=0, c_{1} \neq 0$, will be discussed in this section as the maximum range problem is a particular example. For this class of problems, it follows that $b \neq 0$ (see $D-18$ ). Furthermore, if the first row of matrix $A$ given by (D-16) is interchanged successively with the second and then with the third row of $A$, the resulting matrix can be triangularized yielding the matrix
$B=\left[\begin{array}{ccccc}1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{g \sin \theta+D / m}{v \sin \theta} & -\frac{g \cot \theta}{v^{2}} & 0 \\ 0 & 0 & 1 & -\frac{\cot \beta}{v} & 0 \\ 0 & 0 & 0 & 1 & -\frac{m v \sin \beta}{v_{e}} \\ 0 & 0 & 0 & 0 & f_{\beta}\end{array}\right]$
which has the same rank as $A$ and where

$$
\begin{align*}
f_{\beta} \triangleq & m g \sin ^{2}(\theta+\beta)-D \sin \theta\left(1+\frac{v \cos \beta}{v_{e}}\right)+D \cos \theta \\
& \cdot \sin \beta \cos \beta \tag{4.15}
\end{align*}
$$

For the case where $f_{\beta} \neq 0$, (2.47) is satisfied (rank ( $A$ ) = rank $(B)=5=m$ ). It then follows that the Lagrange multipliers can be eliminated from (2.39), (2.40), and (2.44), yielding equations for $u$ and $\dot{\beta}$ in terms of state variables only. Substituting ( $D-16$ ) and (D-18) into (2.45) yields a set of
linear nonhomogeneous equations in $\lambda$. This set of equations can be used to express all Lagrange multipliers in terms of $c_{1}$ for the case that $c_{1} \neq 0$. However, to obtain a control law which is more generally applicable, and which includes the special case $c_{1}=0$ for which $b=0$, it is more convenient to express all Lagrange multipliers in terms of $\lambda_{3}$. Consequently, a set of equations of the form (2.49) is obtained. The triangularized form given by (4.14) is useful in obtaining these relationships. Substituting (D-19), (D-25) and the relations (2.49) into (2.36), (2.39), and (2.40) yields

$$
\begin{align*}
\dot{\beta}= & \frac{1}{m v}\left[2 m g \cos \theta-\cot \beta\left\langle D\left(1+\frac{v}{v_{e}} \cos \beta\right)-m g \sin \theta\right\rangle\right. \\
& \left.-v_{e} u \sin \beta\right] \tag{4.16}
\end{align*}
$$

Comparing (2.30) and (D-12) yields

$$
K=\frac{v_{e}}{m}\left[\begin{array}{l}
\sin \theta \sin \beta-\cos \theta \cos \beta  \tag{4.17}\\
-\cos \beta \sin \theta-\sin \beta \cos \theta \\
\frac{2 D}{m v} \cos \beta+\frac{D}{m v_{e}}+\frac{g}{v} \sin \beta \cos \theta \\
-\frac{g}{\mathrm{v}^{2}} \cos \theta \cos \beta+\frac{D}{m v^{2}} \sin \beta \\
0
\end{array}\right]
$$

The matrices
and

$$
\mathrm{K}_{\beta}=\left[\begin{array}{c}
\frac{\partial \mathrm{k}_{1}}{\partial \beta}  \tag{4.19}\\
\vdots \\
\vdots \\
\frac{\partial \mathrm{k}_{5}}{\partial \beta}
\end{array}\right]
$$

are now evaluated where $k_{1}, \ldots ., k_{5}$ are the elements of $k$. Substituting (4.17), (4.18), (4.19), and (D-19) into (2.44), eliminating the Lagrange multipliers by use of (2.49), and performing a series of algebraic manipulations yields

$$
\begin{equation*}
u=\mathbb{N} / \mathbb{M} \tag{4.20}
\end{equation*}
$$

where

$$
\begin{align*}
M \triangleq D & {\left[2 v\left(\cos \beta+\frac{1}{\cos \beta}\right)+v_{e}\left(2+\sin ^{2} \beta\right)+\frac{v^{2}}{v_{e}}\right] \quad(4.21) }  \tag{4.21}\\
N \triangleq & -\cos \beta\left[\frac{D\left(1+\frac{v \cos \beta}{v_{e}}\right)-m g \sin \theta}{\sin \beta}-\frac{m g \cos \theta}{\cos \beta}\right]^{2} \\
& +m D\left(1+\frac{v}{v_{e}} \cos \beta\right)[g(2 \cos \theta \sin \beta+3 \sin \theta) \\
& \left.+\frac{a v^{2} \sin \theta}{\cos \beta}+\frac{D}{m \cos \beta}\right]+\frac{D}{m v}+\left(g-a v^{2}\right) \cos \theta \sin \beta \\
& +\frac{2 g \sin ^{2} \beta}{\cos \beta}\left(\frac{v}{v_{e}} \cos \theta \sin \beta-\sin \theta\right) \tag{4.22}
\end{align*}
$$

Equations (4.16) and (4.20) are the desired control laws which are valid for all problems where $c=0$.

An illustrative numerical example of ( $u, \beta$ ) control will now be presented. To allow a comparison with the case
where $\beta$ was constrained to be zero, the maximum range problem of Section 4.1 will be solved, with the addition of $\beta$ control. For the maximum thrust subarc, equations (D-34), (D-37), and ( $D-40$ ) apply for $v_{0} \leqslant v^{*} \leqslant v_{S}$. (The subscript s denotes the values of the variables at the instant $\tau_{s}$ which terminates the maximum thrust subarc.) The unknown parameters in this problem are $\beta_{0}$ and $\nabla_{S}$. Note that although there has been an increase in control dimension, there has been no increase in the number of unknown parameters. The computation was carried out on an IBM 7040 digital computer using an algorithm similar to that in Section 4.1.

The values for the state variables at the switching points along the optimal trajectory are given in Table 4.2.

$$
\text { Table } 4.2
$$

|  | Initial Point | End of Max. Thrust | Burnout | Final Point |
| :---: | :---: | :---: | :---: | :---: |
| $t$ (sec.) | 0 | $0^{+}$ | 30.32 | 331.9 |
| $x(f t$. | 0 | 0 | 80533 | 1545400 |
| y (ft.) | 0 | 0 | 83440 | 0 |
| v (ft./sec.) | 1000 | 2357.4 | 6727.0 | 5252 |
| $\theta$ (deg.) | 45 | 45 | 43.1 | -46.95 |
| m(slug) | 35 | 27.29 | 10.0 | 10.0 |

Figure 4.3 illustrates $x_{f}$ as a function of $v_{s}$ for the optimal value of $\beta_{0}$, and $x_{f}$ as a function of $\beta_{0}$ for the optimal value of $\nabla_{s}$. The optimal controls resulting from these parameters are shown in Figure 4.4. For purposes of comparison, a ballistic trajectory from the same initial conditions is presented for which all the fuel is consumed during the boosting subarc.
FINAL RANGE (Ft)

$V_{S}(F E E T / S E C)$


Figure 4.3 The Final Range as a Function of $v_{s}$ and $\beta_{o}$


Figure 4.4 The Optimal Controls $u$ and $\beta$ for the Maximum Range Problem

The graphs of the optimal trajectories for the $u$ control, $(u, \beta)$ control, and the ballistic trajectory are shown in Figure 4.5. The final ranges for these three cases are:

$$
\begin{align*}
& \text { 1. } u=0, \beta=0 \text { (ballistic), } x_{f}=1,050,000 \mathrm{ft} \text {. } \\
& \text { 2. } \beta=0 \\
& \text {, } x_{f}=1,470,000 \mathrm{ft} . \\
& \text { 3. } \beta \neq 0 \\
& \text {, } x_{f}=1,545,400 \mathrm{ft} \text {. } \tag{4.23}
\end{align*}
$$

The sequence boundary, shown in Figure 4.3 and subsequent figures, represents the locus of all points at which $u=0$ at burnout. For points on the other side of this boundary, the variable thrust goes to zero before all the fuel is consumed, and a sequence is required of the form $u=u_{\text {max. }}$, $0 \leqslant u \leqslant u_{\max .}, u=0, u=u_{\max }, 0 \leqslant u \leqslant u_{\max }, u=0, \ldots$. . However, as a true extremum was obtained for the acceptable sequence $u=u_{\max }, 0 \leqslant u \leqslant u_{\text {max. }}, u=0$, other possible sequences were not investigated.

### 4.3 The Maximum Range Problem with Control of Firing Angle

Consider the maximum range problem of Section (4.1)
when the terminal constraints are

$$
\begin{array}{rlrl}
t_{0} & =0 & m(0)=50.0 \text { slug } \\
x(0) & =0 & y\left(t_{f}\right)=0 \\
y(0) & =0 & m\left(t_{f}\right)=10.0 \text { slug } \\
v(0) & =\varepsilon>0 &
\end{array}
$$

In this problem, the firing angle, $\theta(0)=\theta_{0}$, is free and hence, the transversality condition ( $\mathrm{C}-14$ ) yields

$$
\begin{equation*}
\lambda_{4}(0)=\lambda_{40}=0 \tag{4.25}
\end{equation*}
$$



Figure 4.5 The Optimal Trajectories for the $u$ Control, the ( $u, \beta$ ) Control, and the Ballistic Trajectory

The initial velocity is assumed to be zero. However, to keep $\dot{\theta}$ finite during the boosting stage, the initial velocity is taken equal to a small non-zero value $\varepsilon$. For this case, as $\varepsilon \rightarrow 0$, the first subarc must be a maximum thrust subarc. It cannot be a coasting subarc for the system would remain at rest for $v(0)=0$. For a variable thrust subarc, $k_{u}$ (see C-ll) and $\dot{k}_{u}($ see (4.4)) are both zero. Therefore, if the conditions (4.24), (4.4), and (4.5) are evaluated at $t=0$, and if condition (4.25) is used, it is seen that $\dot{k}_{u}(0)=0$ only if $\lambda_{3}(0)$ $=0$. However, it is seen in Appendix $C$. that $\lambda_{3} \neq 0$ along a variable thrust subarc. Consequently, the only remaining possibility is that the first subarc be a maximum thrust subarc. It then follows from (4.25) and (C-9) that at the end of the maximum thrust subarc for $u_{\max } \rightarrow \infty$

$$
\begin{equation*}
\lambda_{4 \mathrm{~s}} \cong \lambda_{40}=0 \tag{4.26}
\end{equation*}
$$

Substituting (4.26) into (4.4) and evaluating $\dot{k}_{u}$ at the start of the variable thrust subarc yields

$$
\begin{equation*}
\dot{k}_{u}\left(\tau_{s}\right)=-\frac{v_{e} \lambda_{3 s}{ }^{f} s}{m_{s}{ }^{2} v_{s}}=0 \tag{4.27}
\end{equation*}
$$

Hence, as $\lambda_{3} \neq 0$ along a variable thrust subarc (see Appendix C), $\tau_{s}$ is determined as the instant $f_{s}$ vanishes, which from (4.5) yields

$$
\begin{equation*}
m_{S} g \sin \theta_{S}-D_{S}\left(1+v_{S} / v_{e}\right)=0 \tag{4.28}
\end{equation*}
$$

By use of a sequence diagram, an acceptable sequence is found to be

$$
\begin{array}{ll}
u=u_{\max \cdot}, & 0 \leqslant t<\tau_{s} \\
0<u<u_{\max }, & \tau_{s} \leqslant t \leqslant \tau_{b} \\
u=0, & \tau_{b} \leqslant t \leqslant t_{f}
\end{array}
$$

where $\tau_{s}$ is defined by $(4.28)$ and $\tau_{b}$ is the instant of burnout. The control laws for the problem are still given by (4.8) and (4.10). Furthermore, it is interesting to note that the free firing angle does not result in an increase in the number of unknown parameters as $\gamma_{s}$ and $\theta_{s}$ can be taken as the two unknown parameters with $\theta_{0}, v_{s}$, and $m_{s}$ being obtained from (4.28) and (4.13). The performance function to be minimized is then $P=-x_{f}\left(\gamma_{S}, \theta_{S}\right)$ and an algorithm similar to Section (4.1) is used.

The resulting values for the state variables along the optimal trajectory are shown in Table 4.3.

Table 4.3

|  | Initial Point | End of <br> Max. Thrust | Burnout | Final Point |
| :--- | :---: | :---: | :---: | :---: |
| $t($ sec. $)$ | 0 | $0^{+}$ | 30.035 | 378.8 |
| $x\left(f t_{0}\right)$ | 0 | 0 | 86000.0 | 2014212.0 |
| $y\left(f t_{0}\right)$ | 0 | 92000.0. | 0 |  |
| $v\left(f t_{0} /\right.$ sec. $)$ | $\varepsilon \cong 0$ | 2410 | 7731.0 | 5995.6 |
| $\theta($ deg. $)$ | 53.2 | 53.2 | 43.9 | -47.1 |
| $m(s l u g)$ | 50.0 | 32.26 | 10.0 | 10.0 |

The variation of $x_{f}$ in the neighbourhood of the optimum is shown in Figure 4.6 for $x_{f}$ as a function of $\theta_{S}$ with $\gamma_{S}$ optimum ard in Figure 4.7 for $x_{f}$ as a function of $\gamma_{S}$ with $\theta_{S}$ optimum. The resulting optimal control, $u(t)$, is shown in Figure 4.8.

$\theta_{s}-$ RADLANS (DEGREES)
Figure 4.6 The Final Range as a Function of $\theta_{S}$


Figure 4.7 The Final Range as a Function of $X_{S}$


Figure 4.8 The Optimal Control u for the Maximum Range

It is interesting to note that the optimal firing angle for this case is about $53^{\circ}$; whereas, the optimal firing angle of a ballistic missile in a vacuum is $45^{\circ}$.

This same problem was also studied with the addition of thrust angle control similar to the problem in Section 4.2. However, it was found that with the freedom to select an optimal firing angle, the $\beta$ control was extremely small and that the resulting increase in range was insignificant.
(1) $\beta=0, x_{f}=2,014,272 \mathrm{ft}$ 。
(2) $\beta \neq 0, \quad x_{f}=2,014,425 \mathrm{ft}$.

The insignificant increase in range indicates that, if the firing angle is free, it is impractical to employ $\beta$ control to maximize range. However, if the firing angle is fixed, a significant improvement in range can result through $\beta$ control. Furthermore, the use of thrust angle control could be extremely important for optimal trajectories requiring some maneuverability of the rocket during flight.

### 4.4 The Fixed End Point Problem

An interesting variation of the type of problem handled in Section 4.3 is that of minimizing the fuel to deliver a rocket between two fixed points in space. For this example it is assumed that the initial mass is given and that the final mass is to be maximized. As a result, there exists an additional unknown parameter $\tau_{b}$ which is the instant of burnout. However, the addition of $\tau_{b}$ does not result in a three dimensional search for the extremum of the performance function as one of the unknown parameters must be used to insure that the desired
final conditions are met. To illustrate this example, consider the following data:

$$
\begin{array}{rlrl}
t_{0} & =0 & m(0)=50 \text { slugs } \\
x(0) & =0 & x\left(t_{f}\right)=2,000,000 \mathrm{ft} . \\
y(0) & =0 & y\left(t_{f}\right)=0 \\
v(0) & =\varepsilon>0 & \tag{4.29}
\end{array}
$$

The performance function to be maximized is $P=m_{f}\left(\gamma_{S}, \theta_{S}\right)$ and the parameter $\tau_{b}$ is used to insure that (4.29) is satisfied for each set of parameters $\left(\gamma_{s}, \theta_{s}\right)$. The computational procedure is otherwise the same as Section 4.3.

The resulting state variables at the switching points along the optimal trajectory are given in Table 4.4. The variation of $m_{f}$ in the neighbourhood of the optimum is shown in

Table 4.4

| $\ldots$ | Initial <br> Point | End of <br> Max. Thrust | Burnout | Final Point |
| :--- | :---: | :---: | :---: | :---: |
| $t($ sec. $)$ | 0 | $0^{+}$ | 29.99 | 376.3 |
| x (ft.) | 0 | 0 | 85889 | 2000000 |
| $\mathrm{y}\left(\mathrm{ft}_{0}\right)$ | 0 | 0 | 91630 | 0 |
| $\mathrm{v}\left(f t_{0} /\right.$ sec. $)$ | $\varepsilon \cong 0$ | 2408.0 | 7703.0 | 5977.0 |
| $\theta($ deg. $)$ | 53.3 | 53.3 | 43.3 | -46.8 |
| $m$ (slug) | 50.0 | 32.27 | 10.05 | 10.05 |

Figure 4.9 for $m_{f}$ as a function of $\theta_{s}$ with $\gamma_{s}$ optimum, and for $m_{f}$ as a function of $\gamma_{s}$ with $\theta_{S}$ optimum. The optimal trajectory is similar in form to that of Section 4.3 .
4.5 The Direct Search by the Modified Relaxation Method

The direct search employed in these examples deals

with the problem of finding the extremum of a function

$$
\begin{equation*}
P=P\left(\alpha_{k g}\right) \tag{4.30}
\end{equation*}
$$

over a set of parameters $\alpha_{k}$ for which the functional relationship (4.30) is not known explicitly. In this part of the thesis, the search is accomplished through the use of a modified relaxation method.

In the standard relaxation technique, one parameter at a time is varied while the remaining parameters are held fixed. The optimal value of the parameter being varied is determined by a one dimensional search procedure as that value which maximizes $P$. The parameter then maintains this optimal value while the next parameter in the sequence is varied and so on. The technique is iterative and the cycle is repeated until all the parameters converge to an optimal value. The one dimensional search procedure used at each step in the iteration is based on finding a parabolic approximation to the curve $P\left(\alpha_{k}\right)$ in the vicinity of the optimum (see Figure 4.10). From an initial guess $\alpha_{k}^{1}$, the parameter $\alpha_{k}$ is varied in steps of $\Delta \alpha_{k}$ in the direction of increasing $P$ until a value of $\alpha_{k}$ is found which yields a larger value of $P$ than for points on either side of $\alpha_{k}$. Assume that the following inequalities hold

$$
\begin{equation*}
P\left(\alpha_{k}^{n-1}\right)<P\left(\alpha_{k}^{n}\right)>P\left(\alpha_{k}^{n+1}\right) \tag{4.31}
\end{equation*}
$$

As shown in Appendix E, the three points with coordinates $\left(P_{n-1}, \alpha_{k}^{n-1}\right),\left(P_{n}, \alpha_{k}^{n}\right)$, and $\left(P_{n+1}, \alpha_{k}^{n+1}\right)$ can be used to determine a parabolic approximation to the curve of the form

$$
\begin{equation*}
P=a+b \alpha_{k}+c \alpha_{k}^{2} \tag{4.32}
\end{equation*}
$$



Figure 4.10 One Dimensional Search Employing a Parabolic Approximation

The optimal value of $\alpha_{k}$ is taken to be the value of $\alpha_{k}$ that maximizes (4.32). It is shown in Appendix $E$ that this value is

$$
\begin{equation*}
\alpha_{k}(o p t)=\alpha_{k}+\frac{\Delta \alpha_{k}}{2} \frac{\left(P_{n-1}-P_{n+1}\right)}{\left(P_{n-1}-2 P_{n}+P_{n+1}\right)} \tag{4.33}
\end{equation*}
$$

For most problems, this standard relaxation approach is a convenient means of accomplishing the direct search in parameter space. However, the speed of convergence is dependent on the nature of the surface $P\left(\alpha_{k}\right)$ and often the convergence slows down long before the true optimum is reached. A good example of this difficulty is illustrated by the search over the $\left(\beta_{0}, v_{s}\right)$ plane for the maximum range problem of

Section 4.2. It is seen in Figure 4.11 that from an initial guess $\left(\beta_{o}, v_{s}\right)=(0.3,2400)$, the standard relaxation technique appears to converge to a solution in the vicinity of ( 0.24 , 2400). However, the true extremum is at (0.1908, 2357.4). The cause for this "apparent" convergence is that the contours of $P\left(\beta_{o}, v_{S}\right)$ in the $\left(\beta_{o}, v_{S}\right)$ plane are very nearly ellipses with a common major axis tilted at approximately $45^{\circ}$ to the coordinate axis. As the extremum is approached, the eccentricity of the ellipses becomes increasingly smaller which causes the solution obtained by the relaxation method to oscillate very rapidly, thus giving the impression of convergence. However, it can be observed that if the major axis were parallel to one of the coordinate axes, the search technique would be fairly independent of the eccentricity. Indeed, if the contours were true ellipses with a major axis parallel to one of the coordinate axes, then one step convergence would result. To benefit from this property, a modified relaxation method was developed. The technique is based on rotating the coordinate axes such that the search is along a new coordinate which is parallel to the direction of the major axis at that point. As this "major axis" is the projection of a ridge on the $P\left(\alpha_{k}\right)$ surface onto the $\alpha_{k}$ hyperplane, it is generally not a straight line and hence the direction of the new coordinate axis has to be recomputed several times during the search procedure. In essence, the procedure is as follows:
(1) Carry out one complete cycle of the standard relaxation technique to establish a point $\left\langle\alpha_{k}\right\rangle^{I}$ on the ridge of the $P\left(\alpha_{k}\right)$ surface.


Figure 4.11 A Standard Relaxation Search in the ( $\beta_{0}, v_{S}$ ) Plane
(2) Starting from $\left\langle\alpha_{k}\right\rangle^{1}$, carry out one more complete cycle of the standard relaxation method to establish another point $\left\langle\alpha_{k}\right\rangle^{2}$ on this ridge.
(3) Using $\left\langle\alpha_{k}\right\rangle^{1}$ and $\left\langle\alpha_{k}\right\rangle^{2}$ determine the direction of this ridge. Carry out a one dimensional search along this direction until an extremum of $P$ is found at $\left\langle\alpha_{k}\right\rangle$ opt ${ }^{\text {. }}$
(4) Repeat steps (1), (2), and (3) from $\left\langle\alpha_{k}\right\rangle_{o p t}$ until the solution converges.
(5) Investigate the surface $P\left(\alpha_{k}\right)$ in the neighbourhood of the solution found by the above procedure to prove that a true extremum has been obtained. The above technique is called the modified relaxation method and the resulting improved convergence is illustrated in Figure 4.12. It can be observed that since the search is carried out in a finite dimensional parameter space, it is possible to prove whether or not the solution is a true extremum (see Figure 4.1l and 4.12). This fact is further illustrated by Figure 4.13 which is the contour map resulting from step (5) of the above method. Note that such a contour map cannot be obtained when the search is carried out in function space since a function can only be exactly represented in a space of infinite dimensions.


5. PROBLEMS FOR WHICH $c_{1}=0$

### 5.1 Introduction

In Chapter 4, the problem of maximum range was studied using combinations of thrust and thrust-angle controls. In this chapter the three dimensional control problem is introduced which consists of thrust, thrust-angle, and lift control. The analysis associated with this case is given in Appendix A. Problems consisting of one, two, and three dimensional control are studied for which the terminal conditions are independent of range and, hence, the transversality conditions yield $\lambda_{I}=c_{1}=0$. The maximum altitude problem, for the two dimensional control of thrust and thrust-angle, is studied in detail to show that the assumption $u_{\max }=\infty$ is valid for most cases, and to illustrate that optimal solutions obtained by the proposed techniques do indeed satisfy all the necessary conditions of the calculus of variations. Subsequently, a problem of maximizing a performance function at burnout is investigated using one, two, and three dimensional control. It is shown that this problem is equivalent to the maximum altitude problem if burn-out occurs outside the earth:'s atmosphere and that, in all cases, the search in multi-dimensional function space is reduced to a search over one time-invariant parameter.

### 5.2 The Three Dimensional Control Problem

The development of the optimal control laws for the three dimensional control problem proceeds in a similar manner to that of the two dimensional control case in Section 4.2. Consider the case where the final time is unspecified. The
transversality condition (A.21) yields

$$
\begin{equation*}
c=0 \tag{5.1}
\end{equation*}
$$

Using (A.16) and (A.17) in (A.22), it is seen that for a nontrivial solution for $\lambda_{3}$ and $\lambda_{4}$

$$
\begin{equation*}
\tan \beta=D_{L} \tag{5.2}
\end{equation*}
$$

Due to (5.2), the first two rows of matrix A are not independent and hence the first row of $A$ may be eliminated. Rearranging the remaining rows of (A.23), the matrix $A$ can be triangularized to yield the matrix

$$
B=\left[\begin{array}{ccccc}
I & 0 & 0 & 0 & 0  \tag{5.3}\\
0 & I & \left(-\frac{g}{v}-\frac{D}{m v \sin \theta}\right) & \left(\frac{L}{m v}-\frac{g \cos \theta}{\mathrm{~V}}\right) / v \sin \theta & 0 \\
0 & 0 & I & -\cot \beta / v & 0 \\
0 & 0 & 0 & I & -\frac{m v \sin ^{v} \beta}{v_{e}} \\
0 & 0 & 0 & 0 & f_{L}
\end{array}\right]
$$

where B has the same rank as $A$ and where

$$
\begin{align*}
f_{L}= & m g \sin ^{2}(\theta+\beta)-D \sin \theta\left(\sin ^{2} \beta-\cos ^{2} \beta+v \cos \beta / v_{e}\right) \\
& +D \cos \theta \sin \beta \cos \beta-v D_{v} \sin \theta \cos ^{2} \beta-I \sin \beta \\
& \cdot\left(2 \sin \theta \cos \beta+\sin \beta \cos \theta-v \sin \theta / v_{e}\right) \tag{5.4}
\end{align*}
$$

For the case $f_{I} \neq 0$ and $c_{I} \neq 0$, (A.24) provides $b \neq 0$. It then follows from (5.3) that (2.47) is satisfied (rank $(A)=\operatorname{rank}(B)=5=m$ ) and hence, the Lagrange multipliers can be eliminated from (2.37), (2.38), and (2.44) yielding equations for $u$ and $\dot{I}$ in terms of state variables only. Equation (5.2) is used to define $\beta$. Substituting (A.23) and (A.24) into
(2.45) yields a set of linear, non-homogeneous equations in $\lambda$. This set of equations can be used to express all the Lagrange multipliers in terms of $c_{1}$ for the case $c_{1} \neq 0$. However, to obtain a solution which is valid for $c_{1}=0$, the Lagrange multipliers are expressed in terms of $\lambda_{3}$ to obtain a set of equations of the form (2.49). Consequently, the optimal control laws derived from this set of equations will be valid for all $c_{1}$. Following the procedure of Section (4.2), the desired optimal control laws obtained are

$$
\begin{align*}
& \dot{L}=X+Y / \sin \beta-\frac{v^{u}}{m D_{L L}}\left[\frac{\sin \beta}{v \cos ^{2} \beta}+\frac{D_{v L}}{D_{L L}} \cos \beta\right]  \tag{5.5}\\
& \beta=\tan ^{-1}\left(D_{L}\right)  \tag{5.6}\\
& u=N / M \tag{5.7}
\end{align*}
$$

where

$$
\begin{aligned}
& X=\frac{I}{D_{\mathrm{IL}}}\left[\frac{\mathrm{~g}}{\mathrm{~V}}\left(\frac{2 \cos \theta}{\cos ^{2} \beta}+\sin \theta \tan \beta\right)+\frac{D \tan \beta}{\mathrm{mv}}\right. \\
& \left.-D_{v} \frac{\tan \beta}{m}\right]+\frac{I}{D_{L L}}\left[\frac{I}{m v}\left(\frac{v}{v_{e} \cos \beta}-\frac{2}{\cos ^{2} \beta}\right)\right. \\
& \left.-D_{L y} v \sin \theta+D_{I v}\left(g \sin \theta-\frac{D}{m}\right)\right] \\
& Y=\frac{\cos \beta}{D_{L L}}\left[\frac{g}{v} \sin \theta+\frac{D}{m v}-\frac{D}{m v_{e} \cos \beta}-\frac{D_{v}}{m}\right] \\
& N=\frac{I}{\sin ^{2} \beta}\left[D_{L L} \cos ^{2} \beta Y P\right]+V \sin \theta\left[V D_{V y} \cos \beta\right. \\
& \left.-D_{y}\left(\cos \beta-\frac{\sin ^{2} \beta}{\cos \beta}-v / v_{e}\right)\right]+\frac{1}{\sin \beta}\left[v Y D_{V L^{\prime}}\right. \\
& \left.\cdot \cos \beta+P\left(\frac{2 g}{v} \cdot \cos \theta+\frac{L}{m}\left(\frac{\cos \beta}{v_{e}}-\frac{2}{v}\right)\right)\right]+P\left(g_{v}^{g} \sin \theta .\right. \\
& \left.\cdot \cos \beta+\frac{D \cos \beta}{m v}-\frac{D_{v} \cos \beta}{m}\right)+\frac{E g}{v}(\sin \theta+2 \cos \theta \cdot
\end{aligned}
$$

$$
\begin{aligned}
& \cdot \tan \beta)+\frac{E g}{v}\left(\frac{D}{m v}-\frac{D}{m}-2 L \frac{\tan \beta}{m v}\right)-\left(g \sin \theta+\frac{D}{m}\right) . \\
& \cdot\left[\frac{E}{v}+\frac{D}{v_{e}}-\frac{L \sin \beta}{v_{e} \cos \beta}+D_{v}\left(\frac{\sin ^{2} \beta}{\cos \beta}+\frac{v}{v_{e}}\right)+v D_{v v} \cos \beta\right] \\
& +\left(\frac{L}{m v}-\frac{g \cos \theta}{v}\right)\left[\operatorname { m g } \left(2 \sin \theta \sin \beta+\cos \theta\left(\frac{\sin ^{2} \beta}{\cos \beta}\right.\right.\right. \\
& \left.-\cos \beta))+D \sin \beta-\frac{L \sin ^{2} \beta}{\cos \beta}\right]+X\left(\frac{\sin \beta}{\cos ^{2} \beta}+v D_{v I} .\right. \\
& \cdot \cos \beta)+Y / \cos ^{2} \beta-D_{y} v \sin \beta \cos \theta \\
& M=\frac{D}{m}\left[-\frac{2 \sin ^{2} \beta}{\cos \beta}+\frac{3 v_{e} \sin ^{2} \beta}{v}-\frac{v}{v_{e}}\right]-\frac{v D_{v}}{m} . \\
& \cdot\left[2 \cos \beta+\frac{v_{e}}{v}\left(3 \sin ^{2} \beta\right)\right]+\frac{L}{m}\left[\frac{v}{v} \sin \beta \cdot \frac{1}{v_{e} \cos \beta}\right. \\
& \left.-2 \sin \beta+\frac{v e}{v}\left(2 \sin \beta \cos \beta-\sin ^{3} \beta / \cos \beta\right)\right] \\
& -v v_{e} \frac{D_{v v}}{m} \cos ^{2} \beta+\frac{v_{e}}{m D_{L L}}\left(\frac{\sin \beta}{v \cos ^{2} \beta}+D_{L v} \cos \beta\right) . \\
& \cdot\left(\frac{\sin \beta}{\cos ^{2} \beta}+\mathrm{v} \mathrm{D}_{\mathrm{vL}} \cos \beta\right) \\
& E=m g \sin (\theta+\beta)+D\left(\cos \beta-\frac{\sin ^{2} \beta}{\cos \beta}-v / v_{e}\right) \\
& -v D_{v} \cos \beta+I\left(\frac{V}{v_{e}} \frac{\sin \beta}{\cos \beta}-2 \sin \beta\right) \\
& P=m g \sin \theta+D / \cos ^{2} \beta-D_{v} \cos \beta / \dddot{H}_{e}-v D_{v}-\frac{L v}{v_{e}} \\
& \text { - } \frac{\sin ^{3} \beta}{\cos ^{2} \beta}
\end{aligned}
$$

5.3 The Class of Problems for which $c_{1}=0$
is seen that the final transversality condition yields $\lambda_{l}=$ $c_{1}=0$ whenever the final range is unspecified. Two examples of this type of problem are considered in this chapter. The first is that of maximizing the final altitude for a given amount of fuel. The second, which is an approximation to the first and which eliminates the final coasting subarc, is that of maximizing the function

$$
\begin{equation*}
G_{b}=\left(y+\frac{(v \sin \theta)^{2}}{2 g}\right)_{t}=\tau_{b} \tag{5.8}
\end{equation*}
$$

at burnout. This function is derived from the energy equation

$$
\begin{equation*}
\mathrm{E}_{\mathrm{yb}}=\mathrm{m}_{\mathrm{b}} g \mathrm{y}_{\mathrm{b}}+\frac{1}{2} \mathrm{~m}_{\mathrm{b}}\left(\mathrm{v}_{\mathrm{b}} \sin \theta_{\mathrm{b}}\right)^{2} \tag{5.9}
\end{equation*}
$$

by dividing by the constant $m_{b} g$. (The subscript $b$ denotes evaluation at the burnout instant $\tau_{b}$.) The kinetic energy term in (5.9) is associated with the $y$ component of velocity. Assuming that burnout occurs outside the earth's atmosphere, all aerodynamical forces are zero and the conservation of energy must apply. Consequently, (5.8) is constant during the coasting subarc and maximizing (5.8) at burnout is equivalent to maximizing the final altitude at $(v \sin \theta)_{t}=t_{f}=0$. Therefore, whenever burnout occurs at a relatively high altitude (say above $75,000 \mathrm{ft}$ for this case), the problem of maximizing $G_{b}$ at burnout is a good approximation to the maximum altitude problem. . Both of these problems have $c_{1}=0$ and the resulting analysis for the various cases is given in the following sections. 5.3.1 The Case of Thrust Control Only

For the thrust control case of Section (4.1)

$$
\begin{equation*}
X=\lambda_{1} / \lambda_{5} \tag{5.10}
\end{equation*}
$$

However, as $\lambda_{1}=c_{1}=0,(5.10)$ yields $\gamma=0$ and hence by equation (4.8) the optimal control law reduces to

$$
\begin{equation*}
u=N / M \tag{5.11}
\end{equation*}
$$

where $M$ and $N$ are given by ( $C-34$ ) and ( $C-35$ ) respectively. It is seen, therefore, that for $c_{1}=0$ the optimal control law (5.11) is a function of state variables only. It can also be shown that for the special case $\theta(0)=90^{\circ}$, (5.11) reduces to the optimal control law for the sounding rocket [12]. 5.3.2 The Case of Thrust and Thrust Angle Control

The optimal control laws developed in Section (4.2) are valid for $c=0$ and all values of $c_{1}$. However, note that ( $D-18$ ) demands that $b=0$ when $c=c_{1}=0$. For this case, condition (2.48) requires that rank $(A)=r a n k(B)=4$ and hence from (4.14) it is seen that

$$
\begin{equation*}
f_{\beta}=0 \tag{5.12}
\end{equation*}
$$

is required everywhere along the variable thrust subarc where $f_{\beta}$ is defined in (4.15). This function $f_{\beta}$ is analogous to the switching function $f_{s}$ for the sounding rocket case, (see (3.4)). Also note, that for the special case $\theta(0)=90^{\circ}, f_{\beta}$ reduces to $f_{s}$.
5.3.3 The Case of Thrust, Thrust-Angle, and Lift Controls

The optimal control laws for this problem are given by (5.5), (5.6) and (5.7). However, for the case $c=c_{1}=0$, (A.24) gives $b=0$ and condition (2.48) demands that rank $(A)=$ rank $(B)=4$. Trom the definition of matrix $B$ in equation (5.3), it is seen that condition (2.48) is satisfied if everywhere along the variable thrust subarc

$$
\begin{equation*}
f_{L}=0 \tag{5.13}
\end{equation*}
$$

where $f_{I}$ is defined in (5.4). The function $f_{I}$ is the switching function for the three dimensional control problem. Note that for the case $L=0, f_{I}$, (5.4), reduces to $f_{\beta}$, (4.15), and for the case $L=0$ and $\theta(0)=90^{\circ}$, $f_{I}$ reduces to $f_{S}$, (3.4). The function $f_{L}$ is, therefore, the general switching function from which the other switching functions can be derived.

### 5.4 The Maximum Altitude Problem for the Case of Thrust and Thrust-Angle Control

Consider the two dimensional control problem. with terminal constraints

$$
\begin{align*}
t_{0} & =0 \\
\mathrm{x}(0) & =0 \\
\mathrm{y}(0) & =0 \\
\mathrm{v}(0) & =1000 \mathrm{ft} . / \mathrm{sec} . \\
\theta(0) & =70^{\circ} \\
\mathrm{m}(0) & =41.69 \text { slugs } \tag{5.14}
\end{align*} \quad \theta\left(\mathrm{t}_{f}\right)=0 .
$$

The performance function to be minimized is

$$
\begin{equation*}
P=-y\left(t_{f}\right) \tag{5.15}
\end{equation*}
$$

Substituting (5.14) and (5.15) into the transversality equation (D-15) yields

$$
\begin{align*}
\lambda_{1} & =c_{1}=0 \\
c & =0 \\
\lambda_{2}\left(t_{f}\right) & =1  \tag{5.16}\\
\lambda_{3}\left(t_{f}\right) & =0
\end{align*}
$$

As $c=c_{1}=0$, the optimal control laws (4.16) and (4.20) apply, and condition (5.12) must be satisfied everywhere along the singular subarc. Using the assumption that $u_{\max } .=\infty$, equations ( $D-34$ ), ( $D-37$ ), and ( $D-40$ ) completely define the system during the maximum thrust subarc. Hence the problem is solved up to the unknown parameter $\beta(0)=\beta_{0}$. It is desired, however, to test the accuracy of the $\beta$ control generated by (D-37) against the exact solution for the case of $u_{\max }$ finite. To obtain the exact solution it is necessary to generate the Lagrange multipliers during the maximum thrust subarc and to programme $\beta$ according to (D-3l) which is

$$
\begin{equation*}
\beta=\tan ^{-1}\left(\frac{\lambda_{4}}{v \lambda_{3}}\right) \tag{5.17}
\end{equation*}
$$

Let $\tau_{s}$ be the instant of switching to the variable thrust subarc and let $x_{s}=x\left(\tau_{s}\right)$ where $x$ is any variable which depends on time. Then using the matrix $B$ in (4.14) and the fact that $b=f_{\beta}=0$, it is seen that at $\tau_{s}$

$$
\begin{equation*}
\lambda_{2 s}=\left[\frac{\lambda_{3}}{v}(g+D / m \sin \theta+g \cot \theta \tan \beta)\right]_{t}=\tau_{s} \tag{5.18}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{5 s}=\left[\frac{\lambda_{3} \mathrm{v}_{e}}{\mathrm{~m} \cos \beta}\right]_{t=\tau_{s}} \tag{5.19}
\end{equation*}
$$

For this problem it is convenient to scale the Lagrange multipliers such that $\lambda_{3}(0) \triangleq \lambda_{30}=1$. After the optimal solution is obtained, the Lagrange multipliers can be rescaled to yield the classical solution $\lambda_{2}\left(t_{f}\right)=1$. The computing algorithms for the
5.4.1 Approximate Solution for $u_{\max }$. Finite
(1) For a given value $u=u_{\max }$. and the initial conditions (5.14), select a value for $\beta(0)=\beta_{0}$ and using ( $D-37$ ) to define $\beta$, integrate ( $D-1$ ) to ( $D-5$ ) from $t=0$ until $f_{\beta}$ vanishes which defines $\tau_{S}$.
(2) Use (4.16) and (4.20) to define $\beta$ and u respectively, and continue integrating until $m\left(\tau_{b}\right)=m_{f}$.
(3) Let $u=\beta=0$, and continue integrating until $\theta\left(t_{f}\right)=0$ which defines $t_{f}$. Record $y\left(t_{f}\right)$.
(4) Return to (1) and perform a one dimensional search over $\beta_{\rho}$ for the maximum $y\left(t_{f}\right)$.
5.4.2 The Exact Solution for $u_{\max }$. Finite
(1) For each value of $u_{\max }$, use the value of $\beta_{o}$ found by the approximate technique as the initial estimate. With $\lambda_{30}=1$, select values for $\lambda_{20}$ and $\lambda_{50}$. Using these values, solve $(5.17)$ for $\lambda_{40^{\circ}}$. Integrate (D-1) to ( $D-10$ ) from (5.14) until $f_{\beta}$ vanishes which defines $\tau_{s}$ 。
(2) In general, $\lambda_{2}\left(\tau_{s}\right)$ and $\lambda_{5}\left(\tau_{s}\right)$ will not satisfy (5.18) and (5.19) respectively. Return to (1) and select as an improved set of values

$$
\begin{aligned}
& \lambda_{20}(\text { new })=\lambda_{20}(\text { old })+\lambda_{2 s}-\lambda_{2}\left(\tau_{s}\right) \\
& \lambda_{50}(\text { new })=\lambda_{50}(\text { old })+\lambda_{5 s}-\lambda_{5}\left(\tau_{s}\right)
\end{aligned}
$$

66. 

where $\lambda_{2 s}$ and $\lambda_{5 s}$ are found from (5.18) and (5.19), and $\lambda_{2}\left(\tau_{\mathrm{s}}\right)$ and $\lambda_{5}\left(\tau_{\mathrm{s}}\right)$ are the actual values obtained. Repeat (1) and (2) until (5.18) and (5.19) are satisfied.
(3) Use (4.6) and (4.20) to generate $\dot{\beta}$ and $u$ respectively, and continue integrating until $m\left(\tau_{b}\right)=m_{f}$.
(4) Set $u=\beta=0$ and continue until $\theta\left(t_{f}\right)=0$ which defines $t_{f}$.
(5) Return to (1) and perform a one dimensional search over $\beta_{o}$ for the maximum $y\left(t_{f}\right)$ starting with the values of $\lambda_{20}$ and $\lambda_{50}$ found by the previous iteration.
(6) Rescale the Lagrange multipliers such that $\lambda_{2}\left(t_{f}\right)=1$.

### 5.4.3 Comparison of Exact and Approximate Solutions

Table 5-l illustrates the exact and approximate values of the variables at the end of the maximum thrust subarc for $u_{\text {max. }}$ ranging from 2.0 slugs $/ \mathrm{sec}$. to infinity. It is seen that the approximate solution for $\beta$ given by ( $D-37$ ) is accurate to within one percent and that the difference in final range is insignificant for all values of $u_{\text {max }}$. greater than 10 slugs $/ \mathrm{sec}$. It can be concluded, therefore, that the use of (D-37) to programme $\beta$ during the maximum thrust subarc is justified for all medium and high thrust rocket engines.

### 5.5 Testing the Necessary Conditions of the Calculus of Variations

It was previøusly shown that the calculus of variations could be used to obtain: (a) analytical forms for the

Table 5.1

| $\begin{gathered} \mathrm{U}_{\max } \\ (\mathrm{Slug} / \mathrm{Sec}) \end{gathered}$ | Exact Approx. \% Error | $\begin{aligned} & \tau_{S} \\ & (\mathrm{Sec}) \end{aligned}$ | $\begin{gathered} \mathrm{X}_{\mathrm{S}} \\ (\mathrm{feet}) \end{gathered}$ | $\begin{gathered} \mathrm{y}_{\mathrm{S}} \\ (\text { feet }) \end{gathered}$ | $\begin{gathered} \mathrm{v}_{\mathrm{s}} \\ (\mathrm{ft} / \mathrm{sec}) \end{gathered}$ | $\begin{gathered} \theta_{\mathrm{S}} \\ (\mathrm{rad} .) \end{gathered}$ | $\begin{gathered} \mathrm{m}_{\mathrm{s}} \\ (\mathrm{slug}) \\ \hline \end{gathered}$ | $\begin{gathered} \beta_{\mathrm{S}} \\ (\mathrm{rad} .) \end{gathered}$ | $\begin{gathered} \mathrm{y}_{\mathrm{f}} \\ (\text { feet }) \\ \hline \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\infty$ | E $A$ $\% e$ | $0$ | 0 | 0 | 2621,32 | 1.4453 | 30.82 | 0.1339 | 1026330 |
| $10^{6}$ | $\begin{aligned} & \tilde{E} \\ & A \\ & \% e \end{aligned}$ | $\begin{aligned} & .11 \times 10^{-4} \\ & .11 \mathrm{xlO} \\ & 0 \end{aligned}$ | $\begin{gathered} 3.6 \times 10^{-3} \\ 3.6 \times 10^{-3} \\ 0 \end{gathered}$ | $\begin{aligned} & 0.0188 \\ & 0.0188 \\ & 0 \end{aligned}$ | $\begin{gathered} 2621.28 \\ 2621.28 \\ 0 \end{gathered}$ | $\begin{aligned} & 1.4453 \\ & 1.4453 \\ & 0.4 \end{aligned}$ | $\begin{gathered} 30.82 \\ 30.82 \\ 0 \end{gathered}$ | $\begin{aligned} & 0.1339 \\ & 0.1339 \\ & 0 \end{aligned}$ | $\begin{array}{r} 1026330 \\ 1026330 \\ 0 \end{array}$ |
| $10^{5}$ | $\begin{gathered} \mathrm{E} \\ \mathrm{~A} \\ \% \mathrm{e} \end{gathered}$ | $\begin{aligned} & .11 \times 10^{-3} \\ & .11 \times 10^{-3} \\ & 0 \end{aligned}$ | $\begin{gathered} 3.6 \times 10^{-2} \\ 3.6 \times 10^{-2} \\ 0 \end{gathered}$ | $\begin{aligned} & 0.188 \\ & 0.188 \\ & 0 \end{aligned}$ | $\begin{gathered} 2621.28 \\ 2621.28 \\ 0 \end{gathered}$ | $\begin{aligned} & 1.4453 \\ & \hline 1.4453 \\ & 0 . \end{aligned}$ | $\begin{gathered} 30.82 \\ 30.82 \\ 0 \end{gathered}$ | $\begin{aligned} & 0.1339 \\ & 0.1339 \\ & 0 \end{aligned}$ | $\begin{array}{r} 1026330 \\ 1026330 \\ 0 \end{array}$ |
| $10^{4}$ | $\begin{gathered} E \\ A \\ \% \end{gathered}$ | $\begin{aligned} & .11 \times 10^{-2} \\ & .11 \times 10^{-2} \\ & 0 \end{aligned}$ | .36 0.36 | $\begin{aligned} & 1.88 \\ & 1.88 \\ & 0 \end{aligned}$ | $\begin{gathered} 2621.38 \\ 2621.38 \\ 0 \end{gathered}$ | $\begin{aligned} & 1.4453 \\ & 1.4453 \\ & 0 \end{aligned}$ | $\begin{gathered} 30.82 \\ 30.82 \\ 0 \end{gathered}$ | $\left\|\begin{array}{l} 0.13389 \\ 0.13389 \\ 0 \end{array}\right\|$ | $\begin{array}{r} 1026330 \\ 1026330 \\ 0 \end{array}$ |
| $10^{3}$ | $\begin{gathered} \vec{E} \\ \frac{A}{\%} \end{gathered}$ | $\begin{aligned} & .11 \times 10^{-1} \\ & .11 \times 10^{-1} \\ & 0 \end{aligned}$ | $\begin{aligned} & 3.6 \\ & 3.6 \\ & 0 \end{aligned}$ | $\begin{gathered} 18.81 \\ 18.81 \\ 0 \end{gathered}$ | $\begin{gathered} 2622.0 \\ 2622.0 \\ 0 \end{gathered}$ | $\begin{aligned} & 1.4454 \\ & 7.4454 \\ & 0 \end{aligned}$ | $\left\|\begin{array}{c} 30.815 \\ 30.815 \\ 0 \end{array}\right\|$ | $\left\lvert\, \begin{aligned} & 0.13385 \\ & 0.13385 \\ & 0 \end{aligned}\right.$ | $\begin{array}{r} 1026330 \\ 1026330 \\ 0 \end{array}$ |
| $10^{2}$ | $\vec{E}$ $A$ $\%$ | $\left.\right\|_{0} ^{.21}$ | $\begin{gathered} 36.68 \\ 36.67 \\ 0.027 \end{gathered}$ | $\begin{aligned} & 189.5 \\ & 189.3 \\ & 0.100 \end{aligned}$ | $\begin{array}{r} 2628.44 \\ 2628.50 \\ 0.006 \end{array}$ | $\begin{aligned} & 1+4458 \\ & 1.4460 \\ & =0.004 \end{aligned}$ | $\begin{array}{r} 30.756 \\ 30.775 \\ 0.003 \end{array}$ | $\begin{aligned} & 0.13338 \\ & 0.13355 \\ & 0.13 \end{aligned}$ | $\begin{array}{r} 1026328 \\ 1026328 \\ 0 \end{array}$ |
| 50 | $\begin{aligned} & \mathrm{E} \\ & \mathrm{~A} \\ & \% \mathrm{e} \end{aligned}$ | $\begin{aligned} & .22 \\ & 0^{.22} \end{aligned}$ | $\begin{array}{r} 73.74 \\ 73.71 \\ 0.04 \end{array}$ | $\begin{array}{r} 381.7 \\ 381.7 \\ 0 \end{array}$ | $\begin{array}{r} 2635.78 \\ 2636.00 \\ 0.01 \end{array}$ | $\begin{aligned} & 1.4462 \\ & 1.4460 \\ & 0.012 \end{aligned}$ | $\begin{gathered} 30.69 \\ 30.69 \\ 0 \end{gathered}$ | $\left\lvert\, \begin{aligned} & 0.13287 \\ & 0.13320 \\ & 0.28 \end{aligned}\right.$ | $\begin{array}{\|} 1026322 \\ 1026320 \\ 0.0002 \end{array}$ |
| 10 |  | $\begin{aligned} & E: 157 \\ & 1: 157 \\ & 0 \end{aligned}$ | $\begin{array}{r} 386.8 \\ 386.1 \\ 0.12 \end{array}$ | $\begin{gathered} 2040 . \\ 20.40 \\ 0 \end{gathered}$ | $\begin{array}{r} 2699.09 \\ 2699.20 \\ 0.004 \end{array}$ | $\begin{aligned} & 17.4498 \\ & 10.451 \\ & 40.06 \end{aligned}$ | $\begin{gathered} 30.12 \\ 30.12 \\ 0 \end{gathered}$ | $\begin{aligned} & 0.12857 \\ & 0.12999 \\ & 1.09 \end{aligned}$ | $\begin{array}{\|} 1026320 \\ 1026319 \\ 0.0001 \end{array}$ |
| 2 | $\begin{gathered} \tilde{\mathrm{E}}-\times \\ \mathrm{A} \\ \% \text { \% } \end{gathered}$ | $\begin{aligned} & 8.05 \\ & 8.05 \\ & 0 \end{aligned}$ | $\begin{array}{r} 2647.0 \\ 2599.0 \\ 1.8 \end{array}$ | $\begin{array}{\|} 16230 . \\ 16250 . \\ 0.125 \end{array}$ | $\begin{gathered} 3301.74 \\ 3301.0 \\ 0.022 \end{gathered}$ | $\begin{array}{r} 1.475 \\ 2.481 \\ 20.38 \end{array}$ | $\begin{gathered} 25.585 \\ 25.580 \\ 0.02 \end{gathered}$ | $\begin{aligned} & 0.09886 \\ & 0.1602 \\ & 7.3 \end{aligned}$ | $\begin{array}{r} 1026317 \\ 1026315 \\ 0.0002 \end{array}$ |

optimal control laws, (b) the correct sequences of subarcs, (c) switching functions of state variables only, and (d) information about the initial values of the state variables. In most cases, once this information was obtained, the adjoint system could be completely disregarded and the optimal solution could be found by a search in the parameter space of initial conditions. It is now desired to show that the solution obtained in this manner is a true extremal of the calculus of variations as manifested by the fact that along this trajectory all the necessary conditions are satisfied. Using the two dimensional control problem of the previous sections as an example, these necessary conditions are
(1) from (2.9) and (D-13)

$$
K_{\beta}=\frac{v_{e}}{m}\left(\lambda_{3} \sin \beta-\frac{\lambda_{4} \cos \beta}{v}\right)=0 \text {, for } u \neq 0
$$

(2) from the Legendre-Clebsch conditions (2.26) and (D-21)

$$
\left.\mathrm{K}_{\mathrm{u}}=\frac{\mathrm{v}}{\mathrm{e}}\left(\lambda_{3} \cos \beta\right\}+\frac{\lambda_{4} \sin \beta}{\mathrm{v}}\right) \geqslant 0
$$

for $u=u_{\max }, 0<u<u_{\max }, u=0$ respectively.
(3) from the transversality condition (5.16)

$$
\begin{array}{r}
\lambda_{1}=c_{1}=0 \\
\lambda_{2}\left(t_{f}\right)=1 \\
\lambda_{3}\left(t_{f}\right)=0 \\
c=0
\end{array}
$$

(4) from (5.16) and the first integral (D-14)

$$
\lambda_{4}\left(t_{f}\right)=0
$$

(5) from equations (5.12) and (5.4.)

$$
f_{\beta}=0, \quad \text { for } 0<u<u_{\max }
$$

(6) and, from the Erdmann-Weierstrass corner conditions, all Lagrange multipliers, state variables and the constant of integration are continuous functions of time.

To test these conditions, an approximate solution was first obtained for $u_{\max }=50 \mathrm{slugs} / \mathrm{sec}$, (see Section 5.4.1). Using this trajectory, equations ( $D-6$ ) to ( $D-10$ ) were integrated and the correct initial conditions for the lagrange multipliers were determined as in Section 5.4.2. The resulting search over $\beta_{0}$ is shown in Figure 5.1 and the associated optimal controls for $u$ and $\beta$ are illustrated in Figure 5.2. It is seen from Figure 5.3 and Figure 5.4 that all the necessary conditions (i) to (6) are satisfied along this trajectory and, hence, it can be concluded'that the solutions obtained by the proposed technique are true extremals of the calculus of variations. 5.6 The Problem of Maximizing the Function $G_{b}$ at Burnout

In this section, the problem of maximizing

$$
G_{b}=\left(y+\frac{(v \sin \theta)^{2}}{2 g}\right)_{t=\tau_{b}}
$$

is investigated using one, two, and three dimensional control. For this problem the final time is the instant of burnout $\tau_{b}$. The terminal conditions are

$$
\begin{array}{rlr}
t_{0}=0 & \theta(0)=70^{\circ} \\
x(0)=0 & m(0)=41.69 \text { slugs }
\end{array}
$$



Figure 5.1 The Final Altitude as a Function of $\beta_{0}$


Figure 5.2 Optimal Controls $u$ and $\beta$, (Max. alt.)



Figure 5.4 The Control Law Equations for the Maximum
Altitude Problem

$$
\begin{array}{ll}
y(0)=0 & m\left(t_{f}\right)=m_{f} \\
v(0)=1000 \mathrm{ft} / \mathrm{sec} . &
\end{array}
$$

and the performance function to be minimized is

$$
\begin{equation*}
P=-\left[y+\frac{(v \sin \theta)^{2}}{2 g}\right]_{t=t_{f}} \tag{5.21}
\end{equation*}
$$

A value of $u_{\max } .=50 \mathrm{slugs} / \mathrm{sec}$. is used during the maximum thrust subarc. The computing algorithms for the three cases studied are given in the following sections and the acceptable sequence $u=u_{\max .}, 0 \leqslant u \leqslant u_{\max .}, u=0$ is assumed.

### 5.6.1 Computing Algorithm for Thrust Control Only

(1) Select a value for $\nabla_{S}$ which represents the velocity at the switching instant $\tau_{s}$ between maximum thrust and variable thrust. Starting from (5.21) integrate ( $0-1$ ) to ( $0-5$ ) with $u=u_{\max }$. until the velocity increases to $v_{s}$ which defines $\tau_{S}$.
(2) Programme $u$ according to (5.11) and continue integrating until $m=m_{f}$. Record $G_{b}$.
(3) Return to (1) and perform a one dimensional search over $\mathrm{v}_{\mathrm{s}}$ for the maximum $\mathrm{G}_{\mathrm{b}}$.

Figure 5.5 illustrates the search over $v_{s}$ for the maximum $G_{b}$ and Figure 5.6 illustrates the associated optimal thrust.
5.6.2 Computing Algorithm for Thrust and Thrust-Angle Control
(1) Using $u=u_{\max }$. and defining $\beta$ by ( $D-37$ ), select
a value for $\beta_{0}$ and integrate ( $D-1$ ) to ( $D-5$ ) from
(5.21) until $f_{\beta}$ vanishes which defines $\tau_{s}$.
 Figure 5.5 The Performance $G_{b}$ as a Function of $v_{s}$

(2) Programme $\beta$ and $u$ according to (4.16) and (4.20) and continue integrating until $m=m_{f}$. Record $G_{b}$.
(3) Return to (1) and perform a one dimensional search over $\beta_{o}$ for the maximum $G_{b}$.

Figure 5.7 illustrates the search over $\beta_{0}$ for the maximum $G_{b}$ and Figure 5.8 illustrates the resulting optimal controls for $u$ and $\beta$.
5.6.3 Computing Algorithm for Thrust, Thrust-Angle, and Lift Control

It is assumed that for the case where lift is nonzero, the drag force $D(y, v, L)$ is of the form $[17]$

$$
\begin{equation*}
D=K_{a} v^{2} e^{-a y}+K_{L} e^{a y_{L}{ }^{2} v^{-2}} \tag{5.22}
\end{equation*}
$$

where $K_{a}=10^{-4}$

$$
\mathrm{K}_{\mathrm{L}}=500
$$

and

$$
a=(22,000 \mathrm{ft})^{-1}
$$

During the maximum thrust subarc $u_{\max } .=\infty$ is assumed. The equation for lift is found by substituting (5.22) into (5.2)

$$
\begin{equation*}
L=\frac{1}{2 K_{L}}\left(v^{2} e^{-a y} \tan \beta\right) \tag{5.23}
\end{equation*}
$$

Using (A-1) to (A-5), (A-10) to (A-14), and (5.23), it is seen that

$$
\begin{equation*}
\beta=\sin ^{-1}\left(v_{0} \sin \beta_{0} / v\right) \tag{5.24}
\end{equation*}
$$

along the maximum thrust subarc. Under these assumptions, the resulting computing algorithm is
(1) For $u=u_{\max }$, select a value for $\beta_{0^{\circ}}$ Define $L$ and $\beta$ by (5.23) and (5.24) respectively, and integrate


Figure 5.7 The Performance $G_{b}$ as a Function of $\beta_{o}$


Figure 5.8 The Optimal Controls $u$ and $\beta$, (max. energy)
(A-1) to (A-5) from (5.14) until $f_{I}$ vanishes which defines $\tau_{s}$.
(2) Programme $L, \beta$, and $u$ according to (5.5), (5.6), and (5.7) respectively, and continue integrating until $m=m_{f}$. Record $G_{b}$.
(3) Return to (1) and perform a one dimensional search over $\beta_{0}$ for the maximum $G_{b}$.

Figure 5.9 illustrates the one dimensional search over $\beta_{o}$ for the maximum of $G_{b}$, and the optimal controls are shown in Figure 5.10.

### 5.6.4 A Comparison of the Three Cases

Comparing these three cases with the case of a ballistic trajectory for which all the fuel is consumed during boosting, it is found that
(1) Ballistic $, G_{b}=681,000 \mathrm{ft}$.
(2) $u$ control $, G_{b}=895,000 \mathrm{ft}$.
(3) $(u, \beta)$ control $, G_{b}=1,033,400 \mathrm{ft}$.
(4) (u, $\beta, L$ ) control , $G_{b}=1,036,400 \mathrm{ft}$.

As predicted by theory, the value of $G_{b}$ increases with an increase in the control vector. In particular, significant increases are realized between cases (1) and (2), and between cases (2) and (3). The increase between cases (3) and (4) is relatively small; however, it can be observed from Figure 5.8 and Figure 5.10 that the demand on $\beta$ control is reduced when lift control is added. This may be an important feature since, in all practical cases, there will be an upper and lower bound


Pigure 5.9 The Performance $G_{b}$ as a Function of $\beta_{0}$


Figure 5.10 The Optimal Controls $u, \beta$, and L. (max. energy)
on $\beta$ which could be violated by case (3) and not by case (4). Furthermore, it may be economically advantageous to reduce $\beta$ control at the expense of lift control. The answers to such problems of course will depend on the particular system under study and the type of trajectory desired. Certainly those trajectories which require much maneuverability would favour the use of both $\beta$ and lift controls.

### 6.1 Introduction

It is sometimes economically advantageous to trade off
a loss in system performance for a simplification in the design of the optimal controller. This type of consideration leads to the area of suboptimal control. By definition, a suboptimal control is any control, other than the optimal, which takes the system from a given initial manifold to a given final manifold. This type of control always experiences a loss in system performance. However, as shown in this chapter, under certain conditions this loss may be insignificant. For an example, the maximum range problem is studied and two means of generating suboptimal controls which are functions of state variables only are presented.

### 6.2 Eliminating $\gamma$ from the Control Equations

Consider the optimal control problem of Section 4.3. For this problem, a value $\gamma_{s}=0.8 \times 10^{-5}$ is found to be the optimum value of $\gamma$ at $t=\tau_{s}$. During the remainder of the trajectory $\gamma$ is generated from the differential equation (4.10) using $\gamma_{s}$ as the initial value. However, this solution is valid only if no disturbances occur during flight. Should a disturbance occur at $t=\tau_{1}$, the optimal trajectory for $\tau_{1} \leqslant t \leqslant t_{f}$ requires that a new optimal value of $\gamma_{\tau_{1}} \triangleq \gamma\left(\tau_{1}\right)$ be found. To illustrate this property, disturbances to the velocity and path inclination are provided at various times during the optimal trajectory found in Section 4.3. Three cases are studied. First, the new optimal value of $\gamma$ is found; second, no change in the value of $\gamma$ is made; and third, $\gamma$ is
kept equal to zero throughout the trajectory. The results are shown in Table 6.1 and it can be observed that for perturbations in state variables up to $10 \%$, the resulting difference in final range for the last two cases is less than $1 \%$ of the optimum. Furthermore, since $\gamma$ is localized to zero for all disturbances, the case for which $\gamma$ is kept equal to zero appears to be a good choice for the thrust control equation. Using $\gamma \equiv 0$ in ( 54.8 , , a suboptimal control of the type

$$
\begin{equation*}
u_{s o l}=N / \mathbb{M} \tag{6.1}
\end{equation*}
$$

is developed where $N$ and $M$ are given by (C-34) and (C-35) respectively. Note that the suboptimal control (6.1) is a function of state variables only.
6.3 Using $\beta=0$ in the Two Dimensional Control Case

In Section 4.2, the optimal controls for the two dimensional control problem were found to be (see (4.16) and (4.20))

$$
\begin{align*}
\dot{\beta}= & \frac{I}{m v}\left[2 m g \cos \theta-\frac{\cos \beta}{\sin \beta}\left\langle D\left(1+\frac{v}{v_{e}} \cos \beta\right)-m g \sin \theta\right\rangle\right. \\
& \left.-v_{e} u \sin \beta\right] \tag{6.2}
\end{align*}
$$

and $\quad u=N / M$
where $N$ and $M$ are functions of state variables only as given by (4.21) and (4.22) respectively. The object here is to reduce this two dimensional control problem to a one dimensional control problem by forcing $\beta$ to be identically zero. This requires that (6.2) vanish. Defining

$$
\begin{equation*}
R \triangleq\left[D\left(I+\frac{v}{v_{e}} \cos \beta\right)-m g \sin \theta\right] / \sin \beta \tag{6.4}
\end{equation*}
$$

and evaluating (6.2) for $\beta=0$, it is seen that $\dot{\beta}$ is zero only if

SCALE: $\quad \mathrm{v}(\mathrm{ft} / \mathrm{sec}), \quad \theta(\mathrm{rad}), \mathrm{y}(\mathrm{ft})$, and $\gamma_{\mathrm{o}} \triangleq \gamma_{\mathrm{opt}}$.

| $\stackrel{r i m e}{\tau}_{\tau_{1}}$ | Values Before |  | Values After |  | $\begin{aligned} & \text { Percent } \\ & \text { Change } \\ & \delta_{v}, \delta_{\theta} \end{aligned}$ | $\left\|\begin{array}{r} \gamma_{0}\left(\tau_{1}^{-}\right) \\ x=5 \end{array}\right\|$ | $\gamma_{0}\left(\tau_{1}^{+}\right)$ | $\begin{aligned} & y\left(t_{f}\right) \\ & W_{i} t_{h} \\ & \gamma_{0}\left(\tau_{1}^{+}\right) \end{aligned}$ |  | $\begin{aligned} & y\left(t_{f}\right) \\ & \text { With } \\ & \gamma \equiv 0 \end{aligned}$ | $\begin{aligned} & \% \text { Error } \\ & \text { With } \\ & \gamma_{0}\left(\tau_{1}^{-}\right) \\ & \hline \end{aligned}$ | $\begin{gathered} \% \text { Error } \\ \text { With } \\ \gamma \equiv 0 \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\left.\nabla^{\prime} \tau_{工}^{-}\right)$ | $\theta\left(\tau_{I}^{-}\right)$ | $\stackrel{\mathrm{v}}{ }\left(\tau_{1}^{+}\right)$ | $\theta\left(\tau_{1}^{+}\right)$ |  |  |  |  |  |  |  |  |
| 0 | 2410. | . 8960 |  | -- | -- | . 8 | -- | 2014212 | -- | 2014027 | . -- | . 009 |
| 5 | 2752.4 | -- | 3000 | -- | +9.1, - | . 7345 | . $1 \times 10^{-4}$ | 2139186 | 2140092 | 2139638 | . 0425 | . 0212 |
| 5 | 2752.4 | -- | 2500 | -- | -9.1, - | . 734 | . $5 \times 10^{-4}$ | 1886340 | 1888260 | 1885098 | . 1020 | . 168 |
| 5 | -- | . 8960 | -- | . 92 | -, 3.4 | . 734 | -. $1 \times 10^{-4}$ | 2009796 | 2010262 | 2010239 | . 0425 | . 0204 |
| 5 | -- | . 8960 | -- | . 86 | -, -3.4 | . 734 | . $5 \times 10^{-4}$ | 2006504 | 2009293 | 2005132 | . 139 | . 207 |
| 10 | 3200. | -- | 3500 | -- | 10, 0. | . 662 | . $1 \times 10^{-10}$ | 2164927 | 2165166 | 2165166 | . 011 | . 00 |
| 10 | 3200.6 | -- | 3000 | -- | -6.66, - | . 662 | . $5 \times 10^{-4}$ | 1915000 | 1915257 | 1914524 | . 010 | . 0250 |
| 10 | -_ | . 8613 | -- | . 93 | -, 8.1 | . 662 | -. $1 \times 10$ | 1986387 | 1987732 | 1986931 | . 068 | . 0405 |
| 10 | -- | . 8613 | -- | . 85 | -, -1.17 | . 662 | . $1 \times 10^{-4}$ | 2013959 | 2014105 | 2013721 | . 0073 | . 0241 |
| 10 | 3200.6 | . 8113 | 3000 | . 9 | -6.7,5.0 | . 662 | $.1 \times 10^{-10}$ | 1908300 | 1908576 | 1908576 | . 010 | . 000 |
| 20 | 4646.8 | -_ | 5100 | -_ | 10.08 | . 487 | -. $1 \times 10^{-4}$ | 2240427 | 2240541 | 2240541 | . 005 | . 000 |
| 20 | 4646.8 | -- | 4200 | -- | -9.8, | . 487 | . $5 \times 10^{-4}$ | 1802958 | 1803487 | 1802773 | . 029 | . 038 |
| 20 | -_ | . 8043 | -_ | . 88 | -, 10.0 | . 487 | -. $5 \times 10^{-4}$ | 1987371 | 1988354 | 1987438 | . 0495 | . 0461 |
| 20 | -- | . 8043 | -- | . 72 | -, -10.0 | . 487 | $.1 \times 10^{-3}$ | 1988129 | 1990283 | 1987753 | . 108 | . 127 |

Table 6.1

$$
\begin{equation*}
2 m g \cos \theta=R \tag{6.5}
\end{equation*}
$$

Substituting (6.5) and $\beta=0$ in (6.3) yields

$$
\begin{equation*}
u_{\text {so2 }}=\mathbb{N}_{0} / \mathbb{M}_{0} \tag{6.6}
\end{equation*}
$$

where

$$
N_{0}=-m^{2} g^{2} \cos ^{2} \theta+\frac{D}{m v_{e}}+m D\left[( 1 + \frac { v } { v _ { e } } ) \left(3 g \sin \theta+a v^{2}\right.\right.
$$

$$
\begin{equation*}
\left.\left.\cdot \sin \theta+\frac{D}{m}\right)\right] \tag{6.7}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{0}=D\left[4 v+2 v_{e}+v^{2} / v_{e}\right] \tag{6.8}
\end{equation*}
$$

Equation (6.6) is the desired suboptimal control. However, for $\beta=0$, (6.4) yields a finite $R$ if and only if

$$
\begin{equation*}
D\left(1+\frac{v}{v_{e}}\right)-m g \sin \theta \triangleq f_{s o}=0 \tag{6.9}
\end{equation*}
$$

Differentiating (6.9) with respect to time and using (D-1) to (D-5), it can be shown that (6.9) is satisfied if

$$
\begin{equation*}
u=N_{0} / M_{0}=u_{s o 2} \tag{6.10}
\end{equation*}
$$

which is consistent with (6.6). 6.4 Comparison of Suboptimal and Optimal Controls

The maximum range problem of Section 4.3 was solved using the suboptimal controls (6.1) and (6.6). Comparing with the optimal control, the results are:
(1) $u$ optimal $, \quad x_{f}=2,014,425 \mathrm{ft}$.
(2) $\mathrm{u}_{\mathrm{so2}}$, $\mathrm{x}_{\mathrm{f}}=2,014,212 \mathrm{ft}$.
(3) $u_{\text {sol }}, x_{f}=2,014,156 \mathrm{ft}$.

It is seen that the suboptimal controls are excellent approximations to the optimal. Also, the advantage of the suboptimal controls, $u_{\text {sol }}$ and $u_{\text {so2 }}$, is that they are functions of state
variables only and hence can be generated by standard feedback techniques, $u_{s o 2}$ has the added advantage that the function $f_{\text {so }}$ in (6.9) must be identically zero during variable thrust and hence the thrust can be generated by (6.9) and a high gain amplifier [9]. It can be concluded, therefore, that since the suboptimal controls are simpler to implement, and since the loss in system performance is negligible, the use of (6.1) or (6.6) to generate the thrust is justified in this case. In a similar manner, suboptimal controls for other cases could be generated and tested. As the true optimum for each case is known, a performance measure can be assigned to any candidate for suboptimal control.
7. OPTIMAL CONTROLLERS DURING SINGULAR SUBARC

### 7.1 Introduction

For the purpose of synthesizing optimal controllers, the various optimal control laws developed in the previous chapters can be separated into three cases: (1) the case for which the control is a function of state variables only, (2) the case for which one control is integrated from a differential equation, and (3) the case for which a function of state and control variables exists which is to be zero throughout the variable thrust subarc. The form of the optimal controller for each of the three cases is different as illustrated in the following sections.

### 7.2 Direct Feedback Control

For the optimal control law in (5.11), and for the suboptimal controls (6.1) and (6.6), the controller is obtained by standard feedback techniques as illustrated in Figure 7.l.

### 7.3 Hybrid Optimal Controllers

For the second type of controller, one parameter exists whose value must be up-dated along the trajectory to account for disturbances to the system during flight. The class of problems which require this type of controller are those for which $c_{1}$ is not equal to zero and one of the controls is generated from a differential equation, (see Chapter 4). The resulting controller is of the hybrid computer variety which uses an analog simulator to perform high-speed trajectory computations and a hill-climbing digital computer to carry out the one dimensions search [9]. Figure $7.2(\mathrm{a})$ shows the controller


Figure 7.1 The Direct Feedback Optimal Controller


Figure 7.2a The Hybrid Optimal Controller - Maximum Range with Thrust Control


Figure 7.2b The Hybrid Optimal Controller - Maximum Range With Thrust and Thrust Angle Control
for the case of thrust only, and Figure 7.2(b): shows the controller for the two dimensional control of $u$ and $\beta$.

### 7.4 Implicit Function Generation

The third type of controller deals with those cases for which a function of state variables and at most one control variable exists which must be zero throughout the singular subarc. In general, the control cannot be solved explicitly from this function and an implicit solution is required. Figure 7.3 shows the case of the sounding rocket and the suboptimal control of Section 6.3. Figure 7.4(a) shows the case of the two dimensional control of Section 5.3.2, and Figure 7.4(b) shows the case of the three dimensional control of Section 5.3.3.

### 7.5 Conclusions

It has been shown that for systems whose dynamics are linear in control $u$, it is possible to derive control equations for $u, \dot{\beta}$, and $\dot{I}$ which are functions of state variables only for a variety of optimization problems. Furthermore, these control equations are convenient for the study of optimal and suboptimal feedback control laws which can be implemented by direct feedback, up-dating the parameters through hybrid computation, or by implicit solution of a switching function to obtain the desired control. All unknown parameters $\alpha_{k}$ which enter into the problem are found by a direct search in a parameter space for the minimum of the performance functionom, It was shown that a modified relaxation method is a suitable technique for accomplishing this search and that, since the search is carried out in a


Figure 7.3 Implicit Solution Using a High Gain Amplifier


Figure 7.4a Implicit Solution Controller - Two Dimensional Control With $c_{1}=0$


Figure 7.4k Implicit Solution Controller - Three Dimensional Control with $c_{1}=0$
parameter space of finite dimensions, the solution can be easily tested to insure that it is a true extremum and not merely a stationary point.: The control components $u, \beta$, and $L$ can then be generated from the state variables and the optimal parameters.

For the class of systems given by (2.1), the proposed technique is considerably more convenient than standard numerical procedures which require not only a search in multi-dimensional function space but are also unsuitable for real-time control by in-flight guidance computers.

## PART II

NUMERICAL ALGORITHMS

### 8.1 Introduction

In Part I of this thesis, the optimal control laws for a class of aerodynamical systems were obtained as a function of state variables and, at most, one time-invariant parameter. These control laws provided an efficient means of generating the optimal trajectories and allowed the possibility of implementing real-time control. In general, however, such analytical forms for the optimal controls cannot be obtained and it becomes necessary to employ numerical techniques for the solution of the optimization problem. As mentioned previously, these numerical methods are basically iterative schemes which require the use of large size digital or hybrid computers. Although some success has been realized with these techniques, problems still exist in the areas of initial and final convergence, computer storage requirements, and computational algorithms. In this part of the thesis, numerical algorithms are discussed which are essentially a combination of the direct and indirect approaches and which alleviate some of these present difficulties. It is shown that the concepts used to develop these new algorithms can also be used to improve the properties of existing techniques. Essentially, there are three basic concepts used:
(I) a first variation approach applied to the augmented performance function which results in a gradient search in the parameter space of initial Lagrange multipliers,
(2) a second variation approach applied to the augmented performance function which determines the optimal step size for the gradient approach in (1), and (3) an approach which determines the optimal scale factor for the Lagrange multipliers such that the error in final transversality is a minimum at each step in the iteration.

It is shown that for algorithms based on (1), the scale factor for the Lagrange multipliers is arbitrary, and instead of searching over the entire $\lambda_{0}$-space, it is sufficient to determine the intersection of a line with any sphere $\lambda_{0} T_{\lambda_{0}}=$ constant. Consequently, the initial convergence does not depend on a good estimate of the optimal trajectory. Furthermore, as the gradient search in (1) is performed in parameter space, computer storage is required at the terminal points only. A disadvantage to (l) is that, since it is a gradient technique, the convergence slows dewn as the optimum is approached and it is not known when the search should be terminated. To overcome this difficulty, (2) and (3) are used to determine the optimal step size for the gradient technique in the vicinity of the extremum. Concepts (2) and (3) are also applied to the method of steepest descent and the indirect methods based on matching end points. It is shown that some of the undesirable properties previously associated with these techniques can be significantly reduced.

In this chapter, the fundamental concepts of numerical methods will be discussed and some of the existing numerical
techniques will be presented.

### 8.1.1 The Direct and Indirect Approaches

The direct and indirect approaches are iterative schemes which start from some initial estimate of the optimal trajectory and generate a series of trajectories that eventually converges to the optimum. Each trajectory in the series is obtained by a search in the neighbourhood of the previous trajectory, called the nominal trajectory, for that trajectory which best satisfies the search criterion. As a result, the new trajectory in the series is, in some sense, "better" than its predecessor and is therefore closer to the optimum. This procedure is repeated until the search criterion is satisfied. Thus, it can be observed that there are three basic features of these numerical techniques. These features are based on the manner of defining (1) the nominal trajectory about which the search is conducted, (2) the space in which the search is carried out, and (3) the criterion upon which the search is based. The method of generating the neighbouring trajectories is common to all techniques and is based on a technique of linearization about a nominal trajectory. This technique will be discussed in the following sections beginning with a review of linear system theory.
8.2 Linear Time-Varying Differential Systems [16]

To begin, consider the zero input response and the forced response of systems described by

$$
\begin{equation*}
\dot{x}(t)=A(t) x(t)+B(t) u(t) \tag{8.1}
\end{equation*}
$$

where $A(t)$ is an $n \times n$ matrix of scalar functions assumed to
be continuous for all $t$; similarly, $B(t)$ is assumed to be an $n \mathrm{x} m$ continuous matrix, $\mathrm{x}(\mathrm{t})$ is the state vector, and $u(t)$ the input.

### 8.2.1 The Zero Input Response

Theorem 1: Let $\Phi\left(t, t_{0}\right)$ be an $n \times n$ matrix which is
the solution of the matrix equation
$\frac{d \Phi\left(t, t_{0}\right)}{d t}=A(t) \Phi\left(t, t_{0}\right)$
where $\Phi\left(t_{0}, t_{0}\right)=I$
Then the zero-input response of (8.1)

$$
\begin{equation*}
\dot{x}(t)=A(t) x(t), \quad x\left(t_{0}\right)=x_{0} \tag{8.3}
\end{equation*}
$$

is given by

$$
\begin{equation*}
x(t)=\Phi\left(t, t_{0}\right) x\left(t_{0}\right) \quad \forall t, \forall x_{0} \tag{8.4}
\end{equation*}
$$

Proof: By the definition of $\Phi\left(t, t_{0}\right)$, observe that (8.4) reduces the $x_{0}$ at $t=t_{0}$. Finally, (8.3) is satisfied by differentiating (8.4). The matrix $\Phi\left(t, t_{0}\right)$ is called the state transition matrix for the system (8.3).

### 8.2.2 The Forced Response

Theorem 2: Let $\Phi\left(t, t_{0}\right)$ be defined by (8.2). Then the forced response of (8.1) which goes through $x_{0}$ at $t_{0}$ is given by

$$
\begin{equation*}
x(t)=\Phi\left(t, t_{0}\right) x_{0}+\int_{t_{0}}^{t} \Phi(t, \alpha) B(\alpha) u(\alpha) d a \tag{8.5}
\end{equation*}
$$

$$
\forall t, \forall x_{0}
$$

Proof: The proof is immediate by direct verification of initial conditions and direct substitution of (8.5) into (8.1). To effect this substitution note that $\frac{d}{d t} \int_{t_{0}}^{t} \Phi(t, a) B(a) u(a) d a=B(t) u(t)+A(t) \int_{t_{0}}^{t} \Phi(t, a)$.

$$
\begin{equation*}
\cdot B(a) u(a) d a \tag{8.6}
\end{equation*}
$$

In practice, the particular solution which is given by the integral in (8.5) is obtained by solving (8.1) with $x_{o}$ identically zero. Also, the $i^{\text {th }}$ column of $\Phi\left(t, t_{0}\right)$ is obtained by finding the zero input solution to (8.1) with $x_{i}\left(t_{0}\right)$ equal to unity and the remaining initial conditions equal to zero.

### 8.2.3 The Adioint System

For the input response of (8.1)

$$
\begin{equation*}
\dot{x}(t)=A(t) x(t) \tag{8.7}
\end{equation*}
$$

the system defined by

$$
\begin{equation*}
\dot{Z}(t)=-A^{T}(t) Z(t) \tag{8.8}
\end{equation*}
$$

is called the adjoint system. According to Theorem l, a state transition matrix $\Psi\left(t, t_{0}\right)$ exists for the adjoint system such that

$$
\begin{equation*}
\frac{d}{d t} \Psi\left(t, t_{0}\right)=-A^{T}(t) \Psi\left(t, t_{0}\right) \tag{8.9}
\end{equation*}
$$

where

$$
\Psi\left(t_{0}, t_{0}\right)=I
$$

As a result, the solution of (8.8) which passes through $Z_{o}$ at $t_{0}$ is given by

$$
\begin{equation*}
z(t)=\Psi\left(t, t_{0}\right) z_{0} \quad \forall t, \forall z_{0} \tag{8.10}
\end{equation*}
$$

The following lemmas, which relate the system and adjoint system, will be stated without proof: (see [16])

Lemma 1: $\Psi\left(t_{1}, t_{0}\right)=\Psi^{-1}\left(t_{0}, t_{1}\right)$
Lemma 2: $\Phi^{T}\left(t_{1}, t_{0}\right) \Psi\left(t_{1}, t_{0}\right)=I$
Lemma 3: $\Psi\left(t_{0}, t_{1}\right)=\Phi^{T}\left(t_{1}, t_{0}\right)$
Lemma 4: $x^{T}(t) Z(t)=$ constant

### 8.3 Linearization About a Nominal Trajectory

In optimization problems, the system to be controlled is generally described by a set of nonlinear differential equations of the form

$$
\begin{equation*}
\dot{x}(t)=f(x, u) \tag{8.15}
\end{equation*}
$$

where x is the state vector of n components, $u$ is a control vector of $m$ components, and $f(x, u)$ is an $n x$ vector whose components are continuous functions of $x$ and $u$.

Ieten nominal trajectory $\hat{\mathrm{x}}(\mathrm{t})$ of (8.15) be defined by some control $\hat{u}(t)$ and a set of initial conditions $\hat{x}\left(t_{0}\right)$. It is desired to examine the effect of perturbing the initial state by $\delta x_{0}$ and perturbing the control by $\delta u(t)$. Equation (8.15) can be expanded in a Taylor series about the nominal trajectory to yield

$$
\begin{equation*}
\delta \dot{x}=A(t) \delta_{x}+B(t) \delta u \tag{8.16}
\end{equation*}
$$

where
and

$$
A(t) \triangleq \frac{\partial \hat{f}}{\partial x}, \quad B(t) \triangleq \frac{\partial \hat{f}}{\partial u}
$$

$$
\delta x\left(t_{0}\right)=\delta x_{0}
$$

and where the partial derivatives are evaluated along the
nominal trajectory. Applying Theorem 2 to the linear timevarying system (8.16) yields

$$
\begin{equation*}
\delta_{x}(t)=\Phi\left(t, t_{0}\right) \delta_{x_{0}}+\int_{t_{0}}^{t} \Phi(t, a) B(a) \delta_{u}(a) d a \tag{8.17}
\end{equation*}
$$

where $\Phi\left(t, t_{o}\right)$ is the state transition matrix of the zero input response of (8.16). The trajectories given by

$$
\begin{equation*}
x(t)=\hat{x}(t)+\delta x(t) \tag{8.18}
\end{equation*}
$$

are the trajectories in the neighbourhood of the nominal trajectory and they are functions of $\delta_{x_{0}}$ and $\delta_{u}(t)$ as given by (8.17). 8.4 The Optimal Control Problem

To illustrate the basic principles of the various numerical techniques and because there is no loss in generality, a somewhat simplified optimal control problem will be used as an example. Extension of the techniques to problems involving free final time, additional end constraints, bounded control, etc., may be obtained by consulting the references given in each section. The problem will be to find that set of controls $u(t)$ which will minimize the system performance function

$$
\begin{equation*}
J=\emptyset(x(T)) \tag{8.19}
\end{equation*}
$$

subject to the constraints

$$
\begin{align*}
\dot{x} & =f(x, u)  \tag{8.20}\\
x(0) & =x_{o} \tag{8.21}
\end{align*}
$$

over the fixed time interval

$$
\begin{equation*}
0 \leq t \leq T \tag{8.22}
\end{equation*}
$$

8.4.1 The Necessary fonditions for a Local Extremum [27] : The constroirts $(8.20)$ are adjoined to the perfor-
mance function (8.19) by means of an $n \times 1$ vector of Lagrange multipliers $\lambda$ to yield the augmented performance function

$$
\begin{equation*}
J_{a}=\phi(x(\mathbb{T}))+\int_{0}^{T}\left(\lambda^{T} x-H\right) d t \tag{8.23}
\end{equation*}
$$

where $H \triangleq \lambda^{T}{ }_{f}$
is the variational Hamiltonian. The problem of minimizing $J$ is thus transformed to a problem of minimizing $J_{a}$. Taking the first variation of (8.23) yields

$$
\begin{equation*}
\delta_{J_{a}}=\delta \phi(x(T))+\int_{0}^{T}\left(\delta \lambda T_{x}^{\cdot}+\lambda^{T} \delta_{x}^{\cdot}-\delta H\right) d t \tag{8.25}
\end{equation*}
$$

$$
\begin{equation*}
\delta H=\delta_{u} T_{u}+\delta x^{T} H_{x}+\delta \lambda^{T_{H}} \tag{8.26}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{T} \lambda^{T} \dot{x} d t=\delta x^{T} \lambda_{0}^{T}-\int_{0}^{T} \delta x^{T} \dot{\lambda} d t \tag{8.28}
\end{equation*}
$$

Substituting (8.21), (8.26), (8.27) and (8.28) into (8.25) yields

$$
\begin{align*}
\delta J_{a}= & \delta x_{f}^{T}\left[\emptyset_{x f}+\lambda_{f}\right]-\int_{0}^{T}\left\{\delta u^{T} H_{u}+\delta x^{T}\left(H_{x}+\dot{\lambda}\right)\right. \\
& \left.+\delta \lambda^{T}\left(H_{\lambda}-\dot{x}\right)\right\} d t \tag{8.29}
\end{align*}
$$

where the subscript $f$ denotes evaluation at the final time $\mathbb{T}$. Hence, for $J_{a}$ to be a minimum, it is necessary that $\delta J_{a}$ be zero. Equating (8.29) to zero, for independent variations in $\delta_{u}, \delta_{x}$, and $\delta \lambda$, provides the following set of necessary con-
ditions:
(1) the system equations,

$$
\begin{equation*}
\dot{x}=H_{\lambda}=f(x, u) \tag{8.30}
\end{equation*}
$$

(2) the Euler-Lagrange equations,

$$
\begin{equation*}
\dot{\lambda}=-H_{x}=-f_{x} T_{\lambda} \tag{8.31}
\end{equation*}
$$

(3) the gradient condition (control equation),

$$
\begin{equation*}
0=H_{u}=f_{u}{ }^{T} \lambda \tag{8.32}
\end{equation*}
$$

(4) the initial conditions,

$$
\begin{equation*}
x(0)=x_{0} \tag{8.33}
\end{equation*}
$$

(5) and, the final conditions (transversality),

$$
\begin{equation*}
\lambda_{f}=-\phi_{\mathrm{xf}} \tag{8.34}
\end{equation*}
$$

where the shorthand notation

$$
f_{x} \triangleq \frac{\partial f}{\partial \mathrm{x}}=\left[\begin{array}{ccccc}
\mathrm{f}_{\mathrm{lx}_{1}} & \cdots & \cdots & \cdots & \cdot  \tag{8.35}\\
\vdots & & & & f_{l x_{n}} \\
\vdots & & & \cdots & \\
\vdots & & & & \\
\vdots & & & & \vdots \\
f_{n x_{1}} & \cdots & \cdots & \cdots & \cdots \\
f_{n x_{n}}
\end{array}\right]
$$

and

$$
\phi_{\mathrm{x}_{\mathrm{f}}} \triangleq \frac{\partial \phi(\mathbb{T})}{\partial \mathrm{x}} \quad\left[\begin{array}{c}
\phi_{\mathrm{x}_{\mathrm{l}}}(\mathbb{T})  \tag{8.36}\\
\vdots \\
\vdots \\
\phi_{\mathrm{x}_{\mathrm{n}}}(\mathbb{T})
\end{array}\right]
$$

is used to denote partial derivatives of vectors and scalars respectively.

A solution which satisfies the above necessary conditions is called an extremal solution. Such a solution is a candidate for optimality but is not necessarily the global optimum since the conditions (8.30) to (8.34) are (1) local in nature and (2) generally not sufficient. An additional test, such as a second variation test, is required to separate the local optima from the extremals, and, subsequently, a search over all local optima is needed to determine the global optimum. However, for the present argument, it is tacitly assumed that the local optimum and the extremal are unique so that only the conditions (8.30) to (8.34) need be considered. Based on these assumptions, the general approach to obtain a numerical solution is as follows:
(I) select a nominal trajectory which satisfies as many of the necessary conditions as possible,
(2) determine the space over which the search is to be conducted by selecting those parameters and/or functions which will be perturbed to generate the neighbouring trajectories, and,
(3) select as a search criterion a direct approach (most improvement in systems performance) or an indirect approach (most improvement in meeting the necessary conditions not satisfied in (1)).

To illustrate this approach, several of the more common numericai techniques will be presented in the following sections.

### 8.5 The Method of Steepest Descent $[6,7]$

For the method of steepest descent, the search is carried out in the function space of the control vector $u(t)$. An initial control $\hat{u}(t)$ is selected to define the nominal trajectory. The system equations (8.30) are integrated forward from (8.33) at $t=t_{o}$ until $t=T$. The Euler-Lagrange equations (8.31) are integrated backward from the final conditions (8.34) using the values of $\hat{X}(t)$ obtained in the forward integration. As a result, only condition (8.32) is not satisfied and hence equation (8.29) reduces to

$$
\begin{equation*}
\delta_{J_{a}}=-\int_{0}^{T}\left(\delta_{u} \hat{H}_{u}^{n}\right) d t \tag{8.37}
\end{equation*}
$$

The method of steepest descent is based on the direct approach and a neighbouring trajectory is sought which results in a minimum of (8.37). Using Schwarz's inequality, $\delta_{J_{a}}$ is a minimum when

$$
\begin{equation*}
\delta u(t)=k \hat{H}_{u} \tag{8.38}
\end{equation*}
$$

where $k$ is a positive constant and $\hat{H}_{u}$ is evaluated along the nominal trajectory. To insure that the linearity requirements are not violated, a constraint on $\delta u(t)$ is imposed such that

$$
\begin{equation*}
\int_{0}^{T} \delta_{u}{ }^{T}(t) \delta u(t) d t=\delta l^{2} \tag{8.39}
\end{equation*}
$$

where $\delta_{1}{ }^{2}$ is chosen arbitrarily small. The new nominal trajectory is

$$
\begin{equation*}
\hat{u}(t)=\hat{u}+\delta u(t) \tag{8.40}
\end{equation*}
$$

where $u(t)$ is defined by (8.38) and is subject to the constraint (8.39). This process is repeated until $\delta u(t)$ goes to zero. It is seen from (8.38) that this condition will occur when $H_{u}$ is identically zero and hence the remaining necessary condition (8.32) will be satisfied.

The main advantage of this method is that initial convergence does not depend on a good initial estimate of the optimal control $u^{*}(t)$. The disadvantages are that computer storage is required at many points along the trajectory and that the convergence slows down as the optimum is approached. In Section (10.4), a second variation technique is developed which determines the optimal value or the parameter $k$ in the vicinity of the extremum. This modification to the steepest descent technique provides a means of improving final convergence without significantly increasing the computational requirements.

### 8.6 The Min-H Strategy $[5]^{*}$

The min-H strategy is similar to the method of steepest descent except for the criterion upon which $\delta u(t)$ is selected. In this process, $a \delta_{u}(t)$ is found which drives $H_{u}$ closer to zero. Consequently, as the technique is based on satisfying the remaining necessary condition (8.32), the min-H strategy is an indirect approach. The name "min-H" is derived from the fact that $H_{u}$ is zero when $H$ is a mimimum (condition (8.32)). For the nominal trajectory, $\hat{H}_{u}$ will in general not be zero' Expanding $H_{u}$ in a Taylor series yields

$$
\begin{equation*}
\dot{H}_{u}=\hat{H}_{u}+\hat{H}_{u u} \delta u+\hat{H}_{u \lambda} \delta \lambda+\hat{H}_{u x} \delta x \tag{8.41}
\end{equation*}
$$

*Correctly, the choise of sign in (8.34) makes this "Max-H".
where the partial derivatives are evaluated along the nominal trajectory. Equating (8.41) to zero such that condition (8.32) is satisfied, the desired variation is $\delta_{u}(t)$ becomes

$$
\begin{equation*}
\delta_{u}(t)=-\hat{H}_{u u}-1\left[\hat{H}_{u x} \delta_{x}+\hat{H}_{u \lambda} \delta \lambda+\hat{H}_{u}\right] \tag{8.42}
\end{equation*}
$$

Expanding (8.30) and (8.31) in a Taylor series and replacing $\delta_{u}(t)$ by (8.42) yields
where

$$
\begin{align*}
& C_{o}=\left[\begin{array}{ll}
\hat{f}_{x}-\hat{f}_{u} \hat{H}_{u u}^{-1} \hat{H}_{u x} & -\hat{f}_{u}^{T} \hat{H}_{u u}{ }^{-1} \hat{f}_{u} \\
-\hat{H}_{x x}+\hat{H}_{u x} T_{u u} \hat{H}_{u x}^{-1} \hat{H}_{u x} & -\hat{f}_{x}^{T}+\hat{H}_{u x}^{T} \hat{H}_{u u p}^{-1} \hat{f}_{u}^{T}
\end{array}\right]  \tag{8.44}\\
& d(t)=-\hat{f}_{u} \hat{H}_{u u}{ }^{-1} \hat{H}_{u} \tag{8.45}
\end{align*}
$$

and

$$
\begin{equation*}
e(t)=+\hat{H}_{x u} \hat{H}_{u u}-1 \hat{H}_{u} \tag{8.46}
\end{equation*}
$$

and where all partial derivatives are evaluated along the nominal trajectory. $A s \delta_{x_{0}}=0$, a $\delta \lambda_{0}$ must be determined that will preserve the desired final condition (8.34). Expanding (8.34) about the nominal trajectory yields the desired change in $\delta \lambda_{f}$

$$
\begin{equation*}
\delta \lambda_{f}=-\hat{\phi}_{\mathrm{xxf}} \delta \mathrm{x}_{\mathrm{f}} \tag{8.47}
\end{equation*}
$$

This desired change in final value can be transferred (or "swept") back to the initial point by means of the Riccati
transformation.

$$
\begin{equation*}
\delta \lambda(t)=P(t) \delta x(t)+W(t) \tag{8.48}
\end{equation*}
$$

Substituting (8.48) into (8.43) yields

$$
\begin{align*}
& \delta \dot{x}=(A+B P) \delta x+B W+d  \tag{8.49}\\
& \delta \dot{\lambda}=\left(C-A^{T} P\right) \delta x-A^{T} W+e \tag{8.50}
\end{align*}
$$

where

$$
\begin{equation*}
A=\hat{f}_{x}-\hat{f}_{u}-\hat{f}_{u} \hat{H}_{u u}^{-l} \cdot \hat{H}_{u x} \tag{8.51}
\end{equation*}
$$

$$
\begin{equation*}
B=-\hat{f}_{u}^{T} \hat{H}_{u u}{ }^{-1} \hat{f}_{u} \tag{8.52}
\end{equation*}
$$

and

$$
\begin{equation*}
C=-\hat{H}_{x x}+\hat{H}_{u x}{ }^{T} \hat{H}_{u u}{ }^{-l} \hat{H}_{u x} \tag{8.53}
\end{equation*}
$$

Differentiating (8.48) with respect to time and using (8.49) gives

$$
\begin{equation*}
\delta \dot{\lambda}=(\dot{P}+P A+P B A) \delta x+P B W+P \dot{X}+\dot{W} \tag{8.54}
\end{equation*}
$$

Equating (8.54) to (8.50) for $\delta x$ arbitrary yields

$$
\begin{align*}
& \dot{P}=-P A-P B P-A^{T} P+C, P(\mathbb{T})=-\emptyset_{x x}(\mathbb{T})  \tag{8.55}\\
& \dot{W}=-P B W-P d-A^{T} W+e, \quad W(T)=0 \tag{8.56}
\end{align*}
$$

Substituting (8.48) into.(8.42) gives

$$
\begin{equation*}
\delta u(t)=-\hat{H}_{u u}-1\left[\left(\hat{H}_{u x}+\hat{H}_{u \lambda} \hat{P}\right) \delta x+\hat{H}_{u \lambda} \hat{W}+\hat{H}_{u}\right] \tag{8.57}
\end{equation*}
$$

where $\delta_{x}$ is determined from

$$
\begin{equation*}
\delta x=x(t)=\hat{x}(t) \tag{8.58}
\end{equation*}
$$

The procedure is repeated until $H_{u}$ is driven to zero and hence all the necessary conditions are satisfied. The initial convergence for this technique is not as good as the method of steepest descent. However, this approach offers good final convergence; in fact, the speed of convergence increases as the extremum is approached. The computational algorithm, however, is much more
complex and computer storage is required at many points along the trajectory.

### 8.7 The Newton-Raphson Technique [2]

For the Newton-Raphson technique, the search is carried out in the function space of the state variables $x(t)$ and the adjoint variables $\lambda(t)$. An initial guess at the time functions $x(t)$ and $\lambda(t)$ is made such that the boundary conditions (8.33) and (8.34) are satisfied. The control $u(t)$ is determined by solving the control equation (8.34). It is assumed here that (8.32) can be solved explicitly for $u$ in the form

$$
\begin{equation*}
u=u(x, \lambda) \tag{8.60}
\end{equation*}
$$

This assumption however, is not a restriction on the numerical techniques and the case where (8.60) cannot be found explicitly is covered in Section (9.7). As a result of (8.60), the only necessary conditions which are not satisfied are the state equations (8.30) and the Euler-Lagrange equations (8.31). The search criterion is to find the neighbouring trajectory for which (8.30) and (8.31) are more closely satisfied. Substituting (8.60) into (8.30) and (8.31) yields two functions of the form

$$
\begin{align*}
& h(t)=\dot{x}-f(x, u(x, \lambda) \triangleq \dot{x}-r(x, \lambda)  \tag{8.61}\\
& p(t)=\dot{\lambda}+f_{x}{ }^{T}(x, u(x, \lambda)) \lambda \triangleq \dot{\lambda}-s(x, \lambda) \tag{8.62}
\end{align*}
$$

and hence the necessary conditions (8.30) and (8.31) can be replaced by the conditions

$$
\begin{align*}
& h(t)=0  \tag{8.63}\\
& p(t)=0 \tag{8.64}
\end{align*}
$$

and
everywhere along the extremal trajectory. Expanding (8.61) and (8.62) in a Taylor series about the nominal trajectory yiè ${ }^{2} s$.

$$
\begin{align*}
& h(t)=\hat{h}+\delta_{x}-\hat{r}_{x} \delta_{x}-\hat{r}_{\lambda} \delta_{\lambda}  \tag{8.65}\\
& p(t)=p+\delta_{\lambda}-s_{x} \delta_{x}-s_{\lambda} \delta_{\lambda} \tag{8.66}
\end{align*}
$$

where $h(t)$ and $p(t)$ are the values of $h$ and $p$ along the neighbouring trajectories. Substituting (8.63) and (8.64) into (8.65) and ( 8.66 ) yields

$$
\begin{align*}
& \delta_{\dot{x}}=A(t) \delta x+B(t) \delta \lambda-\hat{h}  \tag{8.67}\\
& \delta_{\dot{\lambda}}=C(t) \delta x-A^{T}(t) \delta \lambda-\hat{p}
\end{align*}
$$

where $A, B$, and $C$ are defined in (8.51), (8.52) and (8.53)
respectively. In an identical manner to that used in the Min-H strategy, the desired changes in final values are swept back to the initial point by means of the Riccati transformation (8.48). The final result is

$$
\begin{align*}
& \delta \dot{x}=(A+B P) \delta x+B W-\hat{h}, \quad \delta x_{0}=0  \tag{8.69}\\
& \delta \dot{\lambda}=P \delta x+W  \tag{8.70}\\
& \dot{P}=-P A-P B P-A^{T} P+C, P(T)=-\varnothing_{x X}(T)  \tag{8.71}\\
& \dot{W}=-P B W+P \hat{C}-A^{T} W-\hat{p} ; W(T)=0 \tag{8.72}
\end{align*}
$$

where

Hence, the desired neighbouring trajectory is

$$
\begin{align*}
& x(t)=\hat{x}(t)+\delta_{x}(t)  \tag{8.73}\\
& \lambda(t)=\hat{\lambda}(t)+\delta \lambda(t) \tag{8.74}
\end{align*}
$$

where $\delta_{x}$ and $\delta_{\lambda}$ are defined by (8.69) and (8.70) respectively. The process is continued until (8.63) and (8.64) are satisfied.

The characteristics of this method are very similar to those of the Min-H Strategy in that the initial convergence is fair and the final convergence is very good (quadratic).

Also, the computing algorithm is fairly complex and storage is required at many points along the trajectory. 8.8 The Method of Matching End Points $\left[\begin{array}{ll}3 & 13\end{array}\right]$

In the previous techniques, the iterative search procedure was performed in function space and, as a result, the desired changes in these functions had to be computed and stored at many points along the trajectory. The present method is an example of a technique for which the iterative search procedure is performed in a parameter space and storage is required at the terminal points only. For this technique, the initial trajectory is determined by selecting a set of initial Lagrange multipliers $\hat{\lambda}(0)$. The control equation (8.32) is used to obtain $u$ in the form of (8.60). The system and EulerLagrange equations (8.30) and (8.31) are integrated forward from $t=0$ with $x(0)$ satisfying (8.33) and the assumed values for $\lambda_{0} \cdot$ Consequently, the only necessary condition which is not satisfied along this nominal trajectory is the final condition (8.34). Expanding (8.30) and (8.31) in a Taylor series about the nominal trajectory yields

$$
\left[\begin{array}{l}
\delta_{x}  \tag{8.75}\\
\delta_{i}
\end{array}\right]=c_{0}(t)\left[\begin{array}{l}
\delta_{x} \\
\delta_{\lambda}
\end{array}\right]
$$

where $C_{0}$ is defined in (8.44). Equation (8.75) can be solved by means of a state transition matrix $\Phi$ such that (see Section (8.2.1))

$$
\left[\begin{array}{l}
\delta_{x}(t)  \tag{8.76}\\
\delta \lambda(t)
\end{array}\right]=\Phi(t, 0)\left[\begin{array}{l}
\delta x_{0} \\
\delta \lambda_{0}
\end{array}\right]
$$

Evaluating (8.76) at $t=T$ and $\delta x_{0}=0$ yields

$$
\begin{equation*}
\delta x_{f}=\Phi_{12}(T, 0) \delta \lambda_{0} \tag{8.77}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta \lambda_{f}=\Phi_{22}(T, 0) \delta \lambda_{0} \tag{8.78}
\end{equation*}
$$

where

$$
\Phi=\left[\begin{array}{ll}
\Phi_{11} & \Phi_{12}  \tag{8.79}\\
\Phi_{21} & \Phi_{22}
\end{array}\right]
$$

and where the subscript $f$ denotes evaluation of time $t=T$. The necessary condition (8.34) can be expressed in the form

$$
\begin{equation*}
E_{f}=\lambda_{f}+\varnothing_{x f} \tag{8.80}
\end{equation*}
$$

such that

$$
\begin{equation*}
E_{f}=0 \tag{8.81}
\end{equation*}
$$

for the optimal trajectory. Expanding (8.80) in a Taylor series about the nominal trajectory and using (8.81) yields

$$
\begin{equation*}
E_{f}=\hat{E}_{f}+\delta \lambda_{f}+\hat{\phi}_{x x f} \delta x_{f}=0 \tag{8.82}
\end{equation*}
$$

Substituting (8.77) and (8.78) into (8.83) and solving for $\delta \lambda_{0}$ yields

$$
\begin{equation*}
\delta \lambda_{0}=-\left[\hat{\Phi}_{22 f}+\hat{\phi}_{x x f} \hat{\Phi}_{l 2 f}\right]^{-1}\left[\hat{\lambda}_{f}+\hat{\phi}_{x f}\right] \tag{8.83}
\end{equation*}
$$

As a result, the desired neighbouring trajectory becomes

$$
\begin{equation*}
\lambda(0)=\hat{\lambda}_{0}+\delta \lambda_{0} \tag{8.84}
\end{equation*}
$$

where $\delta \lambda_{0}$ is obtained from (8.83). Using (8.84) to define the new nominal trajectory, the procedure is repeated until $\delta \lambda_{\circ}$ goes to zero. This condition occurs when ( $\lambda_{f}+\emptyset_{x f}$ ) in (8.83) vanishes, and hence condition (8.34) is satisfied.

In the equation (8.83), $\Phi_{12 f}$ and $\Phi_{22 f}$ may be obtained by $n$ forward integrations of the linearized system of equations (8.75) (see Section (8.2.2)). Alternatively, n systems of (8.75) may be used in parallel, from appropriate sets of initial conditions, to provide $\Phi_{12 f}$ and $\Phi_{22 f}$ after one forward integration. In either case, computer storage is required at terminal points only.

The final convergence properties of this approach are exceptionally good in the vicinity of the optimum; "however, the initial convergence depends on a good estimate of the optimal trajectory. The computing algorithm associated with this technique is relatively complex and can involve much matrix inversion.

A modification to this method, proposed by Knapp and Frost [3], suggests placing the desired end constraints in a penalty function of the form

$$
\begin{equation*}
P=\sum_{i=1}^{n} K_{i}^{2}\left[\lambda_{i f}+\phi_{x_{i f}}\right]^{2} \tag{8.85}
\end{equation*}
$$

and using a direct approach to find the minimum of $P$. Comparing (8.34) and (8.85), it is seen that, when $P$ attains its minimum value of zero, the desired final conditions of the Lagrange multipliers are satisfied. The technique used to minimize $P$ is based on a gradient search in the parameter space of initial Lagrange multipliers. As a result, the low memory requirement of the computer is preserved and the computational algorithm is relatively simple. However, as the technique is
based on a gradient method, the final convergence slows down as the optimum is approached; and, as the technique is based on matching the final values of Lagrange multipliers, the initial convergence depends on a good estimate of the optimal trajectory. It is shown in Section (10.1) that the initial convergence properties for these techniques can be improved by selecting an optimal scale factor for the Lagrange multipliers such that the error in final transversality is kept at a minimum.

## 9. AN ALGORITHM BASED ON A FIRST VARIATION

### 9.1 Introduction

The method to be discussed is essentially a combination of the direct and indirect approaches. The state, cotate, and control variables are generated for each trajectory from the necessary conditions developed for the indirect approach. However, instead of attempting to match end conditions, the augmented performance function $J_{a}$ is considered to be a function of the unknown initial values of the Lagrange multipliers, and a direct search for the minimum of $J_{a}$ is carried out in the initial conditions space. The result is a technique which has good initial convergence and which is suitable for a digital or hybrid computer of limited memory. The method also brings out an interesting point concerning the arbitrary scaling of the Lagrange multipliers and helps to explain the difficulties, often encountered in applying the indirect method, if the initial estimate for the optimal trajectory is not a good one. The analytical relations necessary for formulating a computational algorithm are derived in the next section. To avoid unnecessary difficulties the control problem in Section (8.4) is used. In subsequent sections, it is shown that the technique can be extended to problems with bounded control, fixed terminal constraints, and free final time. 9.2 The Proposed Algorithm [18]

From (8.29) the first variation of the augmented performance function is

$$
\begin{align*}
\delta J_{a}= & \delta x_{f}^{T} \phi_{x f}+\delta x_{f} T_{\lambda_{f}}-\int_{0}^{T}\left\{\delta u^{T} H_{u}+\delta x^{T}\left(H_{x}+\dot{\lambda}\right)\right. \\
& \left.+\delta \lambda^{T}\left(H_{\lambda}-\dot{x}\right)\right\} d t \tag{9.1}
\end{align*}
$$

The conditions necessary for an extremum are given by equations (8.30) to (8.34). In this technique, the state equations (8.30), and the Euler-Lagrange equations (8.31), are integrated from the initial point given by (8.33) and an assumed set of values $\hat{\lambda}(0)$. During this integration, the controls are generated according to (8.60) which satisfies the necessary condition (8.32). The neighbouring trajectories are then generated by perturbing the initial Lagrange multipliers by $\delta \lambda_{0}$. The resulting linearized equation for (8.30) and (8.31) are given by

$$
\left[\begin{array}{l}
\delta_{\dot{x}}  \tag{9.2}\\
\delta_{i}
\end{array}\right]=c_{0}(t)\left[\begin{array}{l}
\delta_{x} \\
\delta_{\lambda}
\end{array}\right]
$$

where $C_{o}$ is defined by (8.44). The solution of (9.2) with $\delta_{x_{0}}=0$ is given in (8.76) to be

$$
\begin{align*}
& \delta x(t)=\Phi_{12}(t, 0) \delta \lambda_{0}  \tag{9.3}\\
& \delta \lambda(t)=\Phi_{22}(t, 0) \delta \lambda_{0} \tag{9.4}
\end{align*}
$$

where $\Phi_{12}$ and $\Phi_{22}$ are defined in (8.79). The linearized form of (8.30) can also be written as

$$
\begin{equation*}
\delta_{\dot{x}}=f_{x} \delta_{\dot{x}}+f_{u} \delta u \tag{9.5}
\end{equation*}
$$

Taking the transpose of (8.31)

$$
\begin{equation*}
-\lambda^{T}=\lambda^{T} f_{x} \tag{9.6}
\end{equation*}
$$

it follows from (9.5) and (9.6) that

$$
\begin{equation*}
\frac{d}{d t}\left(\lambda^{T} \delta x\right)=\lambda^{T} f_{u} \delta u=H_{u}^{T} \delta_{u} \tag{9.7}
\end{equation*}
$$

However, as (8.32) is satisfied along the nominal trajectory, then $\lambda^{T_{f_{u}}}=0$ and (9.7) reduces to

$$
\begin{equation*}
\frac{\left(\lambda^{T} \delta_{x}\right)}{d t}=0 \tag{9.8}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\lambda^{T} \delta x=\text { constant }=0 \tag{9.9}
\end{equation*}
$$

where the constant in (9.9) is zero since $\delta_{x_{0}}=0$ by (8.33). Substituting (8.30), (8.31), (8.32), (9.3) and (9.9) into (9.1) yields

$$
\begin{equation*}
\delta J_{a}=\delta \lambda_{0}{ }^{T} \hat{\Phi}_{12}^{T}(T, 0) \hat{\phi}_{\mathbf{x}}(\mathbb{T}) \tag{9.10}
\end{equation*}
$$

along the neighbouring trajectories generated by this approach. Equation (9.10) is the desired expression relating the incremental change in system performance to the incremental change in initial Lagrange multipliers. As mentioned previously, it is desired to find that $\delta \lambda_{0}$ which yields the greatest decrease in $J_{a}$; that is, which makes $\delta J_{a}$ a minimum. Using Schwarz' inequality

$$
\left(x^{T} y\right)^{2} \leq\left(x^{T} x\right)\left(y^{T} y\right)
$$

and noting that equality holds only if y is proportional to x , it is seen that $\delta J_{a}$ is a minimum when

$$
\begin{equation*}
\delta \lambda_{0}=-k \hat{\Phi}_{I 2}^{T}(T, 0) \hat{\phi}_{x}(T) \tag{9.11}
\end{equation*}
$$

where $\mathrm{k}>0$ is a constant which determines the step size. Equation (9.11) gives the incremental change in $\lambda_{0}$ which produces the largest decrease in $J_{a}$. The matrix $\Phi_{12}{ }^{T}(T, 0)$
114.
can be obtained by $n$ forward integrations of (9.2) (see Section (8.2)) or it can be obtained by one backward integration of the adjoint system to (9.2)

$$
\begin{equation*}
\dot{Z}=-C_{0}^{T}(t) Z \tag{9.12}
\end{equation*}
$$

Solving (9.12) backwards in time for the state transition matrix $\Psi$ defined by

$$
\begin{equation*}
Z(0)=\Psi(0, T) Z(T) \tag{9.13}
\end{equation*}
$$

and partitioning $\Psi$ :

$$
\Psi=\left[\begin{array}{ll}
\Psi_{11} & \Psi_{12}  \tag{9.14}\\
\Psi_{21} & \Psi_{22}
\end{array}\right]
$$

the relationship $\Psi_{21}(0, T)=\hat{\Phi}_{12}{ }^{T}(T, 0)$ is obtained by Lemma 3 , Section (8.2). Substituting this relationship into (9.11) yields

$$
\begin{equation*}
\delta \lambda_{0}=-k \Psi_{2 I}(0, T) \hat{\phi}_{x}(T) \tag{9.15}
\end{equation*}
$$

The vector $\Psi_{21}(O, T) \hat{\phi}_{X}(T)$ in (9.15) can be obtained from one backward integration of (9.12) with

$$
Z(T)=\left[\begin{array}{c}
\hat{\phi}_{X}(T)  \tag{9.16}\\
0
\end{array}\right]
$$

The values of $Z$ at $t=0$ are defined as

$$
Z(0)=\left[\begin{array}{l}
Y_{\emptyset}  \tag{9.17}\\
Z_{\phi}
\end{array}\right]
$$

where the last $n$ components of (9.17) are given by (9.13) to be

$$
\begin{equation*}
Z_{\emptyset}=\Psi_{21}(T, 0) \hat{\emptyset}_{x}(T) \tag{9.18}
\end{equation*}
$$

Substituting (9.18) into (9.15) yields the desired change in the initial Lagrange multipliers

$$
\begin{equation*}
\delta \lambda_{0}=-k Z_{\varnothing} \tag{9.19}
\end{equation*}
$$

To insure that the linearity requirements are not violated, a constraint on $\delta \lambda_{0}$ is imposed of the form

$$
\begin{equation*}
\delta \lambda_{0}{ }^{T} \delta \lambda_{0}=\delta I^{2} \tag{9.20}
\end{equation*}
$$

where $\delta_{1}{ }^{2}$ is chosen arbitrarily small. The desired neighbouring trajectory is

$$
\begin{equation*}
\lambda_{0}=\hat{\lambda}_{0}+\delta \lambda_{0} \tag{9.21}
\end{equation*}
$$

where $\delta \lambda_{0}$ is given by (9.19) subject to (9.20). Equation (9.21) becomes the new nominal trajectory and the procedure is repeated until $\delta \lambda_{0}$ goes to zero. A comparison of this technique with the techniques discussed in Chapter 8 is given in Table (9.1). 9.3 Extension of the Proposed Technique

In the previous section, the basic principle of the first variation approach was presented for a somewhat simplefied problem. This approach will now be extended to include problems with fixed terminal constraints and free final time. The assumptions are still made that the control is unbounded and that an explicit solution for the control can be obtained in the form of (8.60). However, it will be shown in subsequent examples that these assumptions are not a restriction on the applicability of the proposed method.

Consider the problem of determining a control vector. $u(t)$ in the free time interval $0 \leq t \leq t_{f}$, so that the performance function

Table 9.1

| Necessary Conditions of the Calculus of Variations | Steepest Descent | $\begin{aligned} & \text { Min - H } \\ & \text { Strategy } \end{aligned}$ | Newton Raphson | Matching End Points | Proposed Techniques |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | $\begin{array}{\|l\|} \hline \text { First } \\ \text { Variation } \end{array}$ | $\begin{aligned} & \text { Combined } \\ & \text { Algorithms } \end{aligned}$ |
| $\dot{\mathrm{x}}=\mathrm{f}(\mathrm{x}, \mathrm{u})$ | X | X |  | X | X | X |
| $\dot{\lambda}=-\frac{\partial r^{T}}{\partial x} \lambda=h(x, \lambda, u)$ | X | X |  | X | X | X |
| $\frac{\partial H}{\partial u}=\frac{\partial f^{T}}{\partial u} \cdot \lambda=0$ |  |  | X | X | X | X |
| $\lambda_{f}^{*}=-\left(\emptyset_{S_{f}}+g_{S_{f}}^{T} \nu\right)$ | X | X | X |  |  |  |
| $\delta_{x} \mathrm{~T}^{T}(0) \lambda(0)=0$ | X | X | X | X | X | X |
| $x(0)=x_{0}$ | X | X | X | X | X | X |
| $g\left(x\left(t_{f}\right), t_{f}\right)=0$ |  | $\cdots$ | X |  |  |  |
| $S\left(x\left(t_{f}\right), t_{f}\right)=0$ | X | X | X | X | X | X |
| Search Over | $u(t)$ | $u(t)$ | $x(t)$ $\lambda(t)$ | $\lambda(0)$ | $\lambda(0)$ | $\lambda(0)$ |
| Search Criterion | $\delta_{J_{a}<0}<0$ | $\begin{aligned} & \frac{\partial \mathrm{H}}{\partial \mathrm{u}} \rightarrow 0 \\ & \mathrm{~g} \rightarrow 0 \end{aligned}$ | $\begin{aligned} & (\dot{x}-f) \rightarrow 0 \\ & (\dot{\lambda}-h) \rightarrow 0 \end{aligned}$ | $\underset{\mathrm{g} \rightarrow 0}{\lambda_{f} \rightarrow \lambda_{f}^{*}}$ | $\delta J_{\mathrm{a}}<0$ | $\delta J_{\mathrm{g}}<0 \text { or } \lambda_{f} \rightarrow \lambda_{f}^{*}$ |
| Computer Storage | along trajectory | along trajectory | aiong trajectory | at end points | at end points | at end points |
| Initial Convergence | good | fair | fair | poor | good | good |
| Final Convergence | poor | good | good | good | poor | good |

$$
\begin{equation*}
J=\phi\left(x_{f}, t_{f}\right) \tag{9.22}
\end{equation*}
$$

is a minimum. The state vector x is subject to the constraints

$$
\begin{align*}
\dot{x} & =f(x, u)  \tag{9.23}\\
x(Q) & =x_{\dot{f}}  \tag{9.24}\\
g\left(x_{f}, t_{f}\right) & =0  \tag{9.25}\\
S\left(x_{f}, t_{f}\right) & =0 \tag{9.26}
\end{align*}
$$

where x is an $\mathrm{n} \times \mathrm{l}$ vector of state variables, f is an n x l vector of continuous functions, $g$ is a $p x l$ vector of terminal constraints, and $S$ is a scalar function of terminal values. The terminal constraints (9.25) and the dynamical constraints (9.23) are adjoined to the system performance function (9.22) to yield

$$
\begin{equation*}
J_{a}=\varnothing\left(x_{f}, t_{f}\right)+g^{T}\left(x_{f}, t_{f}\right) \nu+\int_{0}^{t_{f}}\left(\lambda^{T}{ }_{x}-H\right) d t \tag{9.27}
\end{equation*}
$$

where $\quad H=\lambda^{T} f(x, u)$
and $\nu$ is a $\mathrm{p} x .7$ vector of constant Lagrange multipliers. The function (9.26) is used as a stopping function to define $t_{f}$. Taking the first variation of (9.27) yields

$$
\begin{align*}
\delta J_{a}= & \delta \phi\left(x_{f}, t_{f}\right)+\delta\left(g^{T}\left(x_{f}, t_{1}\right) \nu\right)  \tag{9.28}\\
& +\int_{0}^{t_{f}}\left(-\delta H+\delta \lambda^{T} \dot{x}+\lambda^{T} \delta \dot{x}\right) d t+\left(\lambda^{T} \cdot{ }_{x}-H\right) \delta t_{f}
\end{align*}
$$

where

$$
\begin{equation*}
\delta \phi_{f}=d x_{f}^{T} \phi_{x}+\phi_{t} \delta t_{f} \tag{9.29}
\end{equation*}
$$

$$
\begin{align*}
& \delta\left(g^{T} \nu\right)=d x_{f}^{T} \delta g_{x}^{T} \nu+g_{t}^{T} \nu \delta t_{f_{i}}  \tag{9.30}\\
& d x_{f}=\delta x_{f}+\dot{x}_{f} \delta t_{f}  \tag{9.31}\\
& \delta H=\delta u^{T} H_{u}+\delta x_{H_{x}}+\delta \lambda^{T_{H}}{ }_{\lambda}  \tag{9.32}\\
& \left.\int_{0}^{t_{f}} \lambda^{T} \delta x d t=\delta x^{T} \lambda\right]_{0}^{t_{f}}-\int_{0}^{t_{f}} \delta_{x}{ }^{T} \lambda d t  \tag{9.33}\\
& \text { Using }(9.26) \text { to define } \delta t_{f} \text { yields } \\
& \delta S=d x_{f} T_{S_{x}}+S_{t} \delta t_{f}=0 \tag{9.34}
\end{align*}
$$

After substituting (9.29) to (9.34) in (9.28), the expression for $\delta J_{a}$ is

$$
\begin{align*}
\delta J_{a}= & \delta x_{f}^{T}\left(\phi_{s f}+g_{s f} T^{T} \nu\right)-\int_{0}^{t_{f}}\left(\delta u^{T_{H}}+\delta x^{T}\left(H_{x}+\dot{\lambda}\right)\right. \\
& \left.\left.+\delta \lambda^{T}\left(H_{\lambda}-\dot{x}\right)\right) d t+\delta x^{T} \lambda\right]_{0}^{t_{f}}+\lambda_{f}{ }^{T}(\dot{x}-f)_{f} \delta_{t_{f}} \tag{9.35}
\end{align*}
$$

where

$$
\begin{align*}
& \phi_{s f} \triangleq \phi_{x f}-S_{x f}\left(\frac{\dot{( } \dot{S}_{S}}{f}\right.  \tag{9.36}\\
& g_{S f} T \triangleq g_{x f}^{T}-S_{x f}\left(\dot{(\dot{G}}_{\dot{S}}^{\dot{S}}\right)_{f} \tag{9.37}
\end{align*}
$$

As in the previous case, the necessary conditions (8.30), (8.31), (8.32) and (8.33) are satisfied along the nominal trajectory and $\lambda^{7 \delta_{x}}=0$ along all neighbouring paths. As a result, using (9.3) to define $\delta_{x_{f}}$, equation (9.35) reduces to

$$
\begin{align*}
\delta J_{a} & =\delta \lambda_{0}^{T} \hat{\Phi}_{l 2 f}{ }^{T}\left(\hat{\phi}_{s f}+\hat{g}_{s f}^{T} \nu\right)  \tag{9.38}\\
& =\delta x_{f}^{T}\left(\hat{\phi}_{s f}+\hat{g}_{s f}{ }^{T} \nu\right)
\end{align*}
$$

for the neighbouring trajectories. In a manner identical to that of the previous section, it can be shown that the minimum of $\delta J_{a}$ occurs when.

$$
\begin{equation*}
\delta \lambda_{0}=-k\left(z_{\phi}+z_{g} \nu\right) \tag{9.39}
\end{equation*}
$$

where

$$
\begin{aligned}
& z_{\phi} \triangleq \hat{\Phi}_{12 f} \mathrm{~T} \hat{\phi}_{\mathrm{Sf}} \\
& \mathrm{z}_{\mathrm{g}} \triangleq \hat{\Phi}_{12 f} \mathrm{~T} \hat{\dot{g}}_{\mathrm{Sf}} \mathrm{~T}
\end{aligned}
$$

The vector $Z_{\phi}$ is found from the last $n$ components of Z(0) when

$$
Z(T)=\left[\begin{array}{c}
\hat{\phi}_{S f}  \tag{9.40}\\
0
\end{array}\right]
$$

and the $i^{\text {th }}$ column,$Z_{g i}$, of the matrix $Z_{g}$ is found from the last $n$ components of: $Z(0)$ when

$$
Z_{i}(t)=\left[\begin{array}{c}
\hat{\varepsilon}_{i s f} \\
0
\end{array}\right]
$$

where $g_{i}$ is the $i^{\text {th }}$ component of $g$.
In general, the boundary constraints (9.25) will not be satisfied along the nominal trajectory. If $g$ were small for the nominal trajectory, a $\delta \lambda_{0}$ could be chosen such that $\delta g=-g$. Thus, $\lambda_{0}+\delta \lambda_{0}$ would result in $g+\delta g=0$ as required. However, this choice could violate the linearity requirement and thus, in order to keep the error small,

$$
\begin{equation*}
g=-a \hat{g} \tag{9.41}
\end{equation*}
$$

is chosen where a is an arbitrary small positive quantity, $0 \leq a \leq 1$. Equation (9.41) imposes a constraint on $\delta \lambda_{0}$ which can be determined from the incremented equation

$$
\begin{equation*}
\delta g=\left(\hat{\mathrm{g}}_{\mathrm{x}}-(\underset{\mathrm{S}}{\dot{\mathrm{~g}}})_{\mathrm{S}}^{\mathrm{T}}\right)_{f} \delta \mathrm{x}_{f} \tag{9.42}
\end{equation*}
$$

Substituting (9.3), (9.41), and using the definition of $\mathrm{Z}_{\mathrm{g}}$ from (9.39), equation (9.42) becomes

$$
\begin{equation*}
\delta_{g}=z_{g}^{T I} \delta \lambda_{0}=-a \hat{g} \tag{9.43}
\end{equation*}
$$

To insure that the linearized equations are valid, a further constraint

$$
\begin{equation*}
\delta \lambda_{0}{ }^{T} \delta \lambda_{0}=\delta 1^{2} \tag{9.44}
\end{equation*}
$$

is imposed where $\delta 1^{2}$ can be chosen arbitrarily small. The evaluation of $\nu, k$ and $\delta \lambda_{0}$ is carried out in Appendix $G$ and yields

$$
\begin{align*}
& \delta \dot{\lambda}_{0}=-a Z_{g}\left(Z_{g} Z_{g}\right)^{-1} \hat{g}+\left(Z_{g}\left(Z_{g} T_{g}\right)^{-1} Z_{g} T_{\phi} Z_{\phi}-Z_{\phi}\right) \\
& \cdot \sqrt{\frac{\delta_{1^{2}}-a^{2} \hat{g}^{T}\left(Z_{g} Z_{g}\right)^{-1} \hat{g}}{Z_{\phi} Z_{\phi}-Z_{\phi} Z_{g}\left(Z_{g}{ }_{g} Z_{g}\right)^{-1} Z_{g}{ }_{g}^{T} Z_{\emptyset}}} \tag{9.45}
\end{align*}
$$

### 9.3.1 An Algorithm for Numerical Computation

An algorithm for numerical computation based on (9.45)
can now be formulated.
(1). Select an initial $\hat{\lambda}_{0}$ and integrate (8.30) and (8.31) from (8.33) at $t=0$ until $S=0$, which
defines $t_{f}$. During this integration (8.32) is used to define $u$.
(2) With the aid of the nominal trajectory determined in step (1), compute (9.40) and (9.41). Integrate (8.30) and (8.31) and (9.12) backwards $p+1$ times and find $Z_{\emptyset}$ and $Z_{g}$.
(3) Select a and $\delta_{1}^{2}$ and calculate $\delta_{1}^{2}-a^{2} \hat{g}^{T}\left(Z_{g}^{T} Z\right)^{-l} \hat{g}$. If this quantity is less than zero, adjust a to make it zero. If this quantity is greater than zero, no change need be made in a. (Note: $0 \leq a \leq 1$ )
(4) Compute $\delta \lambda_{0}$ using (9.45).
(5) Select a new trajectory using $\hat{\lambda}_{0}+\delta \lambda_{0}$ and repeat steps (1) to (5).

A considerable simplification in the computation can be achieved if the end constraints are considered in a penalty function of the form
$g=\sum_{i=1}^{P} K_{i} g_{i}{ }^{2}$
where the $K_{i}$ are assigned weighting factors. In this case, $Z_{g}$ is a vector and can be determined by one backward integration. In theory, the weighting factors $K_{i}$ approach infinite values as the $g_{i}$ go to zero. However, in numerical computations, it is not possible to have infinite values for the $K_{i}$ and, as a result, spurious extremals may be introduced. For most cases reported, however, this characteristic has not caused any restriction on the use of (9.46).

It can be noticed that this proposed algorithm requires a minimal amount of computer memory since only the initial and final values need be stored. The elements of
$C_{o}^{T}$ can be determined by function generation using the values of $x$ and $\lambda$ found from (8.30) and (8.31).

### 9.4 The Arbitrary Scaling of the Lagrange Multipliers

In the classical theory, the transversality condition for the problem of Section (9.2) is
and

$$
\begin{gather*}
\left.\left(\lambda^{*}+\emptyset_{x}+g_{x}^{T} \nu+s_{k} \nu_{o}\right)\right]=0  \tag{9.47}\\
\left(-f^{T} \lambda^{f}+\emptyset_{t}+g_{t}^{T} \nu+s_{t} \nu_{\underline{o}}\right)=0  \tag{9.48}\\
t=t_{f}
\end{gather*}
$$

for an unspecified $t_{f}$. However, if $S_{i}$ is used to define $t_{f}$, then the final transversailty conditions can be modified. Solving (9.17) and (9.48) for $V_{0}$ yields $\quad$.

$$
\begin{equation*}
\nu_{0}=-\frac{1}{\dot{S}}\left(\dot{\phi}+\dot{g}^{T} \nu\right)_{f} \tag{9.49}
\end{equation*}
$$

Substituting (9.49) into (9.47) yields

$$
\begin{equation*}
\lambda_{f}^{*}=-\left(\emptyset_{s f}+g_{s f}^{T} \nu\right) \tag{9.50}
\end{equation*}
$$

where

$$
\begin{aligned}
& \phi_{S f} \triangleq\left(\phi_{x}-S_{x}\left(\frac{\dot{\emptyset}_{S}^{\circ}}{\dot{S}}\right)_{f}\right. \\
& g_{S f}^{T} \triangleq\left(g_{x}^{T}-S_{x}\left(\frac{\circ^{T}}{\dot{S}}\right)\right)_{f}
\end{aligned}
$$

and where $\lambda^{*}$ is the Lagrange multipliers for the classical case. In the present approach, however, it is seen from (9.38) that a trajectory is sought for which any increment $\delta \lambda_{0}$ results in

$$
\begin{equation*}
\delta J_{a}=\delta x_{f}^{T}\left(\phi_{s f}+g_{s f}^{T} \nu\right)=0 \tag{9.51}
\end{equation*}
$$

Hence, the vector $\left(\emptyset_{S f}+g_{S f} T \nu\right)$ is normal to the hyperplane
formed by the $\delta x_{f}$. From (9.9)

$$
\begin{equation*}
\delta_{x_{f}}{ }^{T} \lambda_{f}=0 \tag{9.52}
\end{equation*}
$$

and hence $\lambda_{f}$ is also perpendicular to the variation $\delta x_{f}$. Since $\delta x_{f}$ is an arbitrary vector in the hyperplane, it can be concluded from (9.51) and (9.52) that $\lambda_{f}$ is colinear with $\left(\emptyset_{s f}+g_{S f} \stackrel{T}{\nu}\right)$ for the optimal trajectory. Hence at optimality

$$
\begin{equation*}
\lambda_{f}=\mu\left(\phi_{s f}+g_{s f}{ }^{T} \nu\right) \tag{9.53}
\end{equation*}
$$

Note that by $(9.50), \lambda_{f}=\lambda_{f}{ }^{*}$ only if $\mu=-1$. This choice for $u$ is actually an unnecessary restriction. Since (8.31), (8.32), (9.47), and (9.48) are linear in $\lambda$, it is seen that if $\lambda_{\odot}{ }^{*}$ results in a trajectory which extremizes $\varnothing$ with $g=S=0$, then, $\mu \lambda_{0}^{*}$ will result in a trajectory which extremizes $\mu \emptyset$ with $\mu g=\mu S=0$. However, the control variable $u$ is the same in both cases and it is $u$ which is desired. Thus, instead of searching for a point $\lambda_{0}^{*}$, as is done in the classical approach, it is sufficient to search for the line $\lambda_{0}=\mu \lambda_{0}^{*}$ (see Figure 9.1). It is shown in Appendix H, that a value of $J_{a}$ is associated with each radial line through the origin in the $\lambda_{0}$-space. If the initial estimate $\lambda_{0}{ }^{\prime}$ results in $J_{a}=J_{a}^{\prime}>J_{\min }$, a search can be carried out over the surface of a sphere

$$
\begin{equation*}
\lambda_{0}^{T} \lambda_{0}=\left(\lambda_{0}^{\prime}\right)^{T} \lambda_{0}^{\prime} \tag{9.54}
\end{equation*}
$$

for the minimum $J_{a}\left(J_{\text {amin. }}=\emptyset_{\text {min. }}\right)$ which occurs when the line $\mu \lambda_{0}^{*}$ intersects the sphere. As the search is conducted over a sphere, it is convenient to define $\delta 1^{2}$ in (9.44) to be


Figure 9.1 The Solution Line $\mu \lambda_{0}^{*}$

$$
\begin{equation*}
\delta I^{2}=\delta \alpha^{2} \lambda_{0}{ }^{T} \lambda_{0} \tag{9.55}
\end{equation*}
$$

where $\delta_{\alpha}$ is the angular incremental rotation of the $\lambda_{0}$ vector resulting from the incremented displacement $\delta \lambda_{0}$. For this choice of $\delta_{1}{ }^{2}$ it is seen in Appendix $H$ that for $\lambda_{0}^{\prime}=c \lambda_{0}$, where $c$ is any non zero scalar, then $\delta \lambda_{0}{ }^{\prime}=c \delta \lambda_{0}$ results from (9.45).

Hence the speed of convergence for this technique is independent of any initial scale factor. It is evident from this result and Figure 9.1, that the initial convergence is not dependent on a good first estimate of $\lambda_{0}{ }^{*}$.

### 9.5 Example I

Consider a system of the type illustrated by the state transition flow graph of Figure 9.2a. The reaction kinetics are

$$
\begin{aligned}
\dot{x}_{1}+k_{1} x_{1} & =0 \\
\dot{x}_{2}-k_{1} x_{1}+k_{2} x_{2} & =0
\end{aligned}
$$


(a)


Figure 9.2 (a) Transition flow graph for example 1.

$$
\begin{aligned}
& k_{1} \triangleq G_{1} e^{-E_{1} / R T} \\
& k_{2} \triangleq G_{2} e^{-E_{2} / R T}
\end{aligned}
$$

where the absolute temperature $T(t)$ is to be found such that $x_{2}\left(t_{f}\right)$ is a maximum, where $t_{f}$ is fixed and where

$$
\begin{aligned}
& t_{0}=0 \quad E_{1}=18,000 \mathrm{cal} / \mathrm{mole} \\
& t_{f}=8 \mathrm{~min} \quad E_{2}=30,000 \mathrm{cal} / \mathrm{mole} \\
& \mathrm{R}=2 \mathrm{cal} / \mathrm{mole} / 0 \quad G_{1}=0.535 \times 10^{11} / \mathrm{min} \\
& \\
& \\
& \\
& \\
& G_{2}=0.461 \times 10^{18} / \mathrm{min}
\end{aligned}
$$

For convenience, the control variable is taken as $u=k_{1}$ and is considered unbounded. The matrix $-C_{o}^{T}$ is (see (8.44) and (9.12))
$-C_{0}^{T}=u\left[\begin{array}{cccc}1+p & -\left(1+p \lambda_{1} / \lambda_{2}\right) & p \frac{\left(\lambda_{2}-\lambda_{1}\right)}{\bar{x}_{1}} & -p \frac{\left(\lambda_{2}-\lambda_{1}\right)}{x_{2}} \\ -p x_{1} / x_{2} & \frac{x_{1}}{\bar{n} x_{2}}\left(1+p \lambda_{1} / \lambda_{2}\right) & -p \frac{\left(\lambda_{2}-\lambda_{1}\right)}{x_{2}} & p \frac{\left(\lambda_{2}-\lambda_{1}\right)}{x_{2}} \frac{x_{1}}{x_{2}} \\ -p x_{1} /\left(\lambda_{2}-\lambda_{1}\right) & \frac{p x_{1}}{\left(\lambda_{2}-\lambda_{1}\right)} \lambda_{1} / \lambda_{2} & -(1+p) & p_{x_{2}}^{\prime} \\ p x_{1} \lambda_{1} / \lambda_{2}\left(\lambda_{2}-\lambda_{1}\right) & \left.\frac{-p x_{1}}{\left(\lambda_{1}-\lambda_{2}\right.}\right)\left(\frac{\lambda_{1}}{\lambda_{2}}\right)^{2} & 1+p \lambda_{1} / \lambda_{2} & \frac{-x_{1}}{n x_{2}}\left(1+\frac{p \lambda_{1}}{\lambda_{2}}\right)\end{array}\right]$
where $\mathrm{p}=\mathrm{E}_{1} /\left(\mathrm{E}_{2}-\mathrm{E}_{1}\right)$ and $\mathrm{n}=\mathrm{E}_{2} / \mathrm{E}_{1}$. An initial estimate $\lambda_{20}=1.0$ and $\lambda_{10}=0.1$ was arbitrarily selected. Table 9.2 illustrates the numerical results. Note that the classical theory requires that $\lambda_{1 f}{ }^{*}=0, \lambda_{2 f}{ }^{*}=1.0\left(\right.$ since $\left.\varnothing=-x_{2 f}\right)$. The final values for the initial trajectory are -8210 and 133, respectively, and are grossly in error. However, after five iterations using (9.45) and the proposed algorithm, it is

Table 9.2

|  | First Variation (9.45) |  |  | Combined Algorithm with Fl |  |  | Combined Algorithm with F2 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\lambda_{\text {If }}$ | $\lambda_{2 f}$ | $\mathrm{x}_{2 f}$ | $\lambda_{\text {If }}$ | $\lambda_{2 f}$ | $\overline{\mathrm{x}}_{2 f}$ | $\lambda_{\text {If }}$ | $\lambda_{2 f}$ | $\mathrm{x}_{2 f}$ |
| 0 | -8210. | 133.3 | 0.02005 | -8210. | 133.3 | 0.02005 | -8210. | 133.3 | 0.02005 |
| 1 | -7407. | 122.3 | 0.02144 | -1019.0 | 26.58 | 0.072622 | -1019.0 | 26.58 | 0.072622 |
| 2 | -1825. | 40.58 | 0.05160 | -64.92 | 4.882 | 0.274662 | -64.92 | 4.882 | 0.274662 |
| 3 | -317. | 12.10 | 0.1348 | -0.2057 | 1.103 | 0.679878 | -0.2057 | 1.103 | 0.679878 |
| 4 | -33. | 3.52 | 0.3472 | 0.01645 | 1.002 | 0.681691 | -0.003409 | 0.9253 | 0.681707 |
| 5 | -0.874 | 1.214 | 0.6626 | -0.0001526 | 1.000 | 0.681707 | -0.000005 | 1.0000 | 0.681707 |
| 6 | -0.3896 | 1.123 | 0.6761 | 0.0000004 | 1.000 | 0.681707 |  |  |  |
| 7 | -0.1665 | 1.075 | 0.6304 |  |  |  |  |  |  |
| 8 | -0.6952 | 1.053 | 0.6815 |  |  |  |  |  |  |
| 9 | -0.02866 | 1.043 | 0.6817 |  |  |  |  |  |  |
| 10 | -0.01174 | 1.039 | 0.6817 |  |  |  |  |  |  |
| 11 | -0.00930 | 1.037 | 0.6817 |  |  |  |  |  |  |
| 12 | -0.00196 | 1.037 | 0.6817 |  |  |  |  |  |  |
| 13 | -0.00079 | 1.036 | 0.6817 |  |  |  |  |  |  |

seen in Figure 9.3 that the Lagrange multipliers have converged very close to line $\mu \lambda_{0}{ }^{*}$. As the optimum is approached, the


## Figure 9.3 Iteration Path in the Initial Conditions Space of Lagrange Multipliers

rate of convergence of the gradient method slows down and it is not known when the search should be terminated. In the next chapter, methods of improving this final convergence are discussed. The first approach is based on the method of matching end points in which the Lagrange multipliers are continually re-scaled to maintain a minimum error in final
transversality. As a result, it is shown that this method of matching end points can be used in the final stages of the proposed technique to provide the property of rapid final convergence. Another technique discussed is a second variation method which determines the optimal step size for $\delta \lambda_{0}$ as the extremum is approached. The effect of using these two techniques is shown in Table 9.2. Note that for these last two methods, the solution converges to the classical solution, yet for the first variation approach,' the scale factor $\mu=$ 1.036 results. Figure 9.4 and Figure 9.5 illustrate the $\mathbb{T}$ and $\mathrm{x}_{2}$ profiles for various iterative cycles. The iteration path is illustrated in Figure 9.3, and Figure 9.6 illustrates the time variations of $x_{1}, x_{2}, \lambda_{1}, \lambda_{2}$, and $T$ for the optimal path. To plot these quantities on one graph, the following ordinate scales are used:

$$
\begin{array}{ll}
y=(T-326) / 2 & y=10\left(-\lambda_{1}+1\right) \\
y=10 x_{1} & y=10\left(-\lambda_{2}+1\right) \\
y=10 x_{2} &
\end{array}
$$

### 9.6 Example 2

To illustrate how the proposed method can be used if $u$ is bounded, consider the equation of constraint

$$
\begin{equation*}
\left(u-u_{\min .}\right)\left(u_{\max }-u\right)=\alpha^{2} \tag{9.57}
\end{equation*}
$$

Due to this constraint, (8.32) has the form

$$
\begin{equation*}
0=H_{u}+\lambda_{3}\left(u_{\max .}+u_{\min .}-2 u\right) \tag{9.38}
\end{equation*}
$$



Figure 9.4 Temperature for Various Iteration Cycles


Figure $9.5 \mathrm{x}_{2}$ Profile for Various Iteration Cycles


Figure 9.6 The time variations for the optimal path (更.1)


Figure 9.7 Temperature profiles for cases a to $\alpha$ (Ex.2)
where $\lambda_{3}$ must be introduced because of (9.57), and is determined by

$$
\begin{equation*}
0=\lambda_{3} \alpha \tag{9.59}
\end{equation*}
$$

Thus, when $u=u_{\min }$. or $u=u_{\max }, \alpha=0$ and (9.59) is satisfied. If $u_{\min }<u<u_{\max }, \lambda_{3}=0$ and (9.58) reduces once more to (8.32). If $\varnothing$ is to be a minimum, the Legandre Clebsch condition yields

$$
\begin{equation*}
\frac{2 H_{u}\left(\delta u^{2}+\delta \alpha^{2}\right)}{u_{\max }+u_{\min \cdot}-2 u} \geqslant 0 \tag{9.60}
\end{equation*}
$$

and hence $H_{u}<0$ when $u=u_{\max .}, H_{u}>0$ when $u=u_{\min }$, and $H_{u}=0$ if $u_{\min }<u<u_{\max }$.

Consider the case of Example 1 where $T$ has the following upper bounds:
(a) $T_{m}=345$
(b) $T_{m}=342$
(c) $T_{m}=340$
(d) $T_{m}=338$

In this problem it is known that $u=u_{\max }$ for $0 \leqslant t \leqslant t_{s}$, and $u<u_{\max }$ for $t_{s}<t \leqslant t_{f}$. The instant $t_{s}$ is determined when $H_{u}$ vanishes. After this instant, $u$ is computed as in the unbounded case. For $t \leqslant t_{S}$, the elements in $C_{o}$ (see (8.44)) change, since $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \lambda}=0$ when $u=u_{\max }$. However, this change is readily carried out in the backward sweep by storing the value of $t_{s}$ found during the forward sweep. With this slight modification, the computation is the same as in Example". The temperature profiles for cases (a) to (d) are illustrated in Figure 9.7. In Figure 9.8, the ordinate scales used are:


Figure 9.8 The Time Variations for the Optimal Path (Ex. 2)

Figure 9.9 The Time Variations for the Optimal Path (Ex. 3)

$$
\begin{array}{ll}
\mathrm{y}=\mathrm{T}-331 & \mathrm{y}=\frac{10 \lambda_{1}}{3}+10 \\
\mathrm{y}=\frac{100 \mathrm{H}_{\mathrm{u} \cdot}+10}{2} & \mathrm{y}=5\left(-\lambda_{2}+2.4\right) \\
\mathrm{y}=10 \mathrm{x}_{2} & \mathrm{y}=10 \mathrm{x}_{1}
\end{array}
$$

### 9.7 Example 3

It is not always possible to use (8.32) to obtain an explicit analytical expression of the form (8.60) for $u$ as a function of x and $\lambda$. This would complicate the computation since ( 8.32 ) would have to be used to determine $u$ implicitly. Note, however, that (8.32) yields the linearized equation

$$
\begin{equation*}
\delta u=-\hat{H}_{u u}{ }^{-1}\left(\hat{H}_{u x} \delta x+\hat{H}_{u \lambda} \delta \lambda\right) \tag{9.61}
\end{equation*}
$$

To avoid the implicit computation of $u$, note that (8.32) can be differentiated with respect to time to yield

$$
\begin{equation*}
\dot{u}=-\hat{H}_{u u}{ }^{-1}\left(\hat{H}_{u x} \dot{x}+\hat{H}_{u \lambda} \dot{\lambda}\right) \tag{9.62}
\end{equation*}
$$

where the right hand side is a function $x, \lambda$, and $u$. The initial value of $u_{0}$ can be found by an implicit solution of (8.32) at $t=0$. With $u_{0}$ known, (8.30), (8.31) and (9.62) can be used to compute u for the nominal trajectory during the forward and backward sweeps. The procedure is otherwise the same as before.

To illustrate this modification, consider the batch reactor problem of Example 1 where there is an extra unwanted by-product $x_{4}$ (see Figure 9.2b). The equations are

$$
\begin{align*}
& \dot{x}_{1}=-\left(k_{1}+k_{3}\right) x_{1} \\
& \dot{x}_{2}=k_{1} x_{1}-k_{2} x_{2} \tag{9.63}
\end{align*}
$$

Let $u=k_{1}$. In this case (8.32) has the form

$$
\begin{equation*}
H_{u}=\lambda_{1} x_{1}\left(1+n_{3} G_{3} u^{n_{3}+1}\right)^{\prime \prime}+\lambda_{2}\left(-x_{1}+n_{2} G_{2} x_{2} u^{n_{2}-1}\right) \tag{9.64}
\end{equation*}
$$

where $\quad n_{2} \triangleq E_{2} / E_{1}, n_{3} \triangleq E_{3} / E_{1}, G_{2}{ }^{\prime} \triangleq G_{2} / G_{1}{ }^{n_{2}}$, $G_{3}{ }^{\prime} \triangleq G_{3} / G_{1} n_{3}$. Hence

$$
\begin{equation*}
\dot{u}=N / \mathbb{M} \tag{9.65}
\end{equation*}
$$

where $N=-\left(\lambda_{1}\left(1+n_{3} G_{3} u^{n^{n}-1}\right)-\lambda_{2}\right) \dot{x}_{1}-\lambda_{2} n_{2} G_{2} u^{n_{2}-1} \dot{x}_{2}$

$$
-x_{1}\left(1+n_{3} G_{3} u^{n_{3}-1}\right) \dot{\lambda}_{1}-\left(-x_{1}+n_{2} G_{2} x_{2} u^{n_{2}-1}\right) \dot{\lambda}_{2}
$$

$$
M=\lambda_{1} x_{1} n_{3}\left(n_{3}-1\right) G_{3} u^{\prime} u^{-2}+\lambda_{2} n_{2}\left(n_{2}-1\right) G_{2} u^{n_{2}-2} x_{2}
$$

Figure 9.9 illustrates the computed results for the case where

$$
\begin{array}{ll}
\mathrm{E}_{1}=18,000 \mathrm{cal} / \mathrm{mole}, & G_{1}=0.53510^{11} / \mathrm{min} \\
\mathrm{E}_{2}=30,000 \mathrm{cal} / \mathrm{mole}, & G_{2}=0.40110^{18} / \mathrm{min} \\
\mathrm{E}_{3}=27,000 \mathrm{cal} / \mathrm{mole}, & G_{3}=0.50010^{16} / \mathrm{min}
\end{array}
$$

The ordinate scales used in Figure 9.9 are

$$
\begin{array}{ll}
\mathrm{y}=\mathrm{T}-330, & \mathrm{y}=10\left(1-\lambda_{1}\right) \\
\mathrm{y}=10 \mathrm{x}_{1}, & \mathrm{y}=10\left(1.1-\lambda_{2}\right) \\
\mathrm{y}=10 \mathrm{x}_{2}, &
\end{array}
$$

10. TECHNIqUES FOR IMPROVING FINAL CONVERGENCE
10.1 Matching End Points Using an Optimal Scale Factor [20] Consider the optimal control problem in section (8.4)
which is to determine the control $u(t)$ over the fixed time interval $0 \leqslant t \leqslant \mathbb{T}$ such that the system performance function

$$
\begin{equation*}
J=\varnothing(x(\mathbb{T})) \tag{10.1}
\end{equation*}
$$

is a minimum. The constraints on the state variables are

$$
\begin{align*}
\dot{x} & =f(x, u)  \tag{10.2}\\
x(0) & =x_{0} \tag{10.3}
\end{align*}
$$

As in section (8.4), the constraint (10.2) is adjoined to the system performance function (10.1) by an $n \times 1$ vector of Lagrange multipliers $\lambda$. However, since it was shown in Chapter 9 that the scale factor for the Lagrange multipliers is arbitrary, the augmented performance function is taken as

$$
\begin{equation*}
J_{a}=\varnothing(x(\mathbb{T}))+\mu \int_{0}^{T}\left(\lambda^{T} \dot{x}-H\right) d t \tag{10.4}
\end{equation*}
$$

where

$$
H \triangleq \lambda^{T}{ }_{f}
$$

and where $\mu$ is introduced as the arbitrary scale factor. For this case, the transversality condition (8.34) is

$$
\begin{equation*}
\mu \lambda_{f}+\emptyset_{x f}=0 \tag{10.5}
\end{equation*}
$$

Substituting (10.5) in (8.80) yields

$$
\begin{equation*}
\hat{\mathbb{E}}_{f}=\mu \hat{\lambda}_{f}+\hat{\phi}_{x f} \tag{10.6}
\end{equation*}
$$

where $E_{f^{\prime}}=0$ on the optimal trajectory. Expanding (10.6) in a Taylor series about the nominal trajectory yields

$$
\delta E_{f}=\mu \delta \lambda_{f}+\hat{\phi}_{x x f} \delta x_{f}
$$

Using $\delta E_{f}=-\hat{E}_{f}$, and substituting for $\delta x_{f}$ and $\delta \lambda_{f}$ from (9.3) and (9.4) yields

$$
\begin{equation*}
\delta \lambda_{0}=-\left[\mu \hat{\Phi}_{22 f}+\hat{\phi}_{x x f} \hat{\Phi}_{12 f}\right]^{-1}\left[\mu \hat{\lambda}_{f}+\hat{\phi}_{x f}\right] \tag{10.7}
\end{equation*}
$$

which differs from (8.33) by the scale factor $\mu$. $A s \delta \lambda_{o}$ is proportional to $\hat{\mathrm{E}}_{\mathrm{f}}, \mu$ is selected such that the square of the error in final transversality is a minimum and hence $\delta \lambda_{0}^{T} \delta \lambda_{0}$ is minimized for each iteration. From (10.6), the square of the error is

$$
\begin{equation*}
\left|\hat{E}_{f}\right|^{2}=\hat{E}_{f} \mathbb{T}_{\hat{E}_{f}}=\mu \hat{\lambda}_{f}^{T} \hat{\lambda}_{f}+2 \mu \hat{\lambda}_{f}^{T} \hat{\phi}_{x f}+\hat{\phi}_{x f} \mathbb{T} \hat{\phi}_{x f} \tag{10.8}
\end{equation*}
$$

Differentiating (10.8) with respect to $\mu$ and equating the derivative to zero yields

$$
\begin{equation*}
2 \mu \hat{\lambda}_{f}^{T} \hat{\lambda}_{f}+2 \hat{\lambda}_{f}^{T} \hat{\phi}_{x f}=0 \tag{10.9}
\end{equation*}
$$

Hence the value of $\mu$ which minimizes the final error in transversality is

$$
\begin{equation*}
\mu_{\text {opt }}=-\frac{\hat{\lambda}_{f}^{T} \hat{\phi}_{X f}}{\hat{\lambda}_{f}^{T} \hat{\lambda}_{f}} \tag{10.10}
\end{equation*}
$$

Substituting (10.10) into (10.7) yields the desired incremental change

$$
\begin{equation*}
\delta \lambda_{o}=-\left[\frac{\hat{\lambda}_{f}^{T} \hat{\phi}_{x f}}{\hat{\lambda}_{f}^{T} \hat{\lambda}_{f}} \hat{\Phi}_{22 f}+\hat{\phi}_{x x f} \hat{\Phi}_{I 2 f}\right]^{-1}\left[-\frac{\hat{\lambda}_{f}^{T} \hat{\phi}_{x f}}{\hat{\lambda}_{f}^{T} \hat{\lambda}_{f}} \hat{\lambda}_{f}+\hat{\phi}_{x f}\right] \tag{10.11}
\end{equation*}
$$

It is shown in Appendix $H$ that, using (10.11) to define $\delta \lambda_{0}$, the rate of convergence is independent of the initial scale factor, and that the procedure converges to the solution line
$\mu \lambda_{0}{ }^{*}$. This fact is illustrated in Figure 10.1, for the problem of Example 1 in Section (9.5). It is seen that a "cone of convergence ", region $R_{c}$, exists about the solution line $\mu \lambda_{o}{ }_{0}^{*}$ Within $R_{c}$, any initial estimate for $\lambda_{0}$ will converge to the solution line independent of the initial scale factor. The number of steps required for convergence is indicated on the radial lines. For an initial estimate which falls outside the cone of convergence, region $R_{n c}$, an unacceptable trajectory results. For this particular example, matrix $C_{o}{ }^{T}$ in (9.56) contains terms which are divided by $\left(\lambda_{2}-\lambda_{1}\right)$ and which become infinite if $\lambda_{2}=\lambda_{1}$. Therefore, should this situation exist, the trajectory is unacceptable. To find an initial estimate which lies inside the cone of convergence, a random search can be employed or a method of relaxing the final constraints, as done by Isaacs et al [15], can be used. If it is desired to have the procedure converge to the classical solution $\lambda_{0}^{*}$, then it is shown in Appendix $H$ that the neighbouring trajectory should be taken as

$$
\begin{equation*}
\lambda_{0}=\mu_{o p t}\left(\hat{\lambda}_{0}+\delta \lambda_{0}\right) \tag{10.12}
\end{equation*}
$$

where $\hat{\lambda}_{0}$ is defined by the nominal trajectory, $\mu_{o p t}$ is defined by (10.10) and $\delta \lambda_{0}$ by (10.11). The effect of using (10.12) for the solution of Example 1 of Section (9.5) is illustrated in Figure 10.2. For any initial estimate of $\lambda_{0}$ which lies inside the region $R_{c a}$, the first step in the iteration establishes the initial conditions on the solution curve $C_{a}$. Once on this solution curve, $\lambda_{0}$ moves along the curve until $\lambda_{0}{ }^{*}$ is reached. If.the initial erstimate lies in region $R_{c b}$,


Figure 10.1 The Solution Line $\mu \lambda_{0}^{*}$ and the Cone of Convergence $R_{c}$


Figure 10.2 Convergerce to $\lambda_{o}^{*}$ by the Modifie $\bar{\alpha}$ Methō of Matchirg Enū Points
the first step in the iteration establishes the initial conditions on the line $C_{b}$ which is in region $R_{c a}$. The procedure is then the same as before.

For the technique developed by Knapp and Frost, (Section (8.8)), a similar improvement in initial convergence can be obtained when the penalty function in (8.85) is replaced by

$$
\begin{equation*}
P=\sum_{i=1}^{n} K_{i}^{2}\left(\mu \lambda_{i f}+\phi_{x i f}\right)^{2} \tag{10.13}
\end{equation*}
$$

In this case $\mu$ is selected to minimize $P$ and hence the optimal value for $\mu$ is
$\mu_{\text {opt }}=-\frac{\sum\left(K_{i}{ }^{2} \lambda_{\text {if }} \emptyset_{\text {xif }}\right)}{\sum\left(K_{i}{ }^{2} \lambda_{i f}{ }^{2}\right)}$
Using (10.14) in (10.13), it can be shown that the technique is then independent of the initial scale factor, and hence the rate of initial convergence has been improved. 10.1.1 Extension of the Method of Matching End Points Consider the control problem in Section (9.3) and let the augmented functional be

$$
\begin{equation*}
J_{a}=\emptyset\left(x_{f}, t_{f}\right)+g^{T}\left(x_{f}, t_{f}\right) \nu+\mu \int_{0}^{t_{f}}\left(\lambda^{T}{ }_{x}-H\right) d t \tag{10.15}
\end{equation*}
$$

where $\mu$ is the scale factor for the Lagrange multipliers. As before, the final time is defined by

$$
\begin{equation*}
S\left(x_{f}, t_{f}\right)=0 \tag{10.16}
\end{equation*}
$$

and the variation in final time for the neighbouring trajectory is defined from (10.16) to be

$$
\begin{equation*}
\delta_{t_{f}}=-\delta x_{f}{ }^{T} \frac{S_{x f}}{S_{f}} \tag{10.17}
\end{equation*}
$$

Using (10.17) to define $\delta t_{f}$, the first variation of $J_{a}$ in (10.15) is

$$
\begin{aligned}
\delta J_{a}= & \delta x_{f}{ }^{T}\left(\phi_{s f}+g_{s f}{ }^{T} \nu\right)+\mu \delta x_{f}{ }^{T} \lambda_{f}+\left(\lambda^{T}{ }^{T}-H\right)_{f} \delta_{t_{f}} \\
& -\delta x_{o}{ }^{T} \lambda_{0}+\mu \int_{0}^{t_{f}}\left\{-\delta u^{T} H_{u}+\delta \lambda^{T}\left(\dot{x}-H_{\lambda}\right)+\delta x ?\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.\cdot\left(\dot{\lambda}+H_{\lambda}\right)\right\} d t \tag{10.18}
\end{equation*}
$$

where

$$
\phi_{s f} \triangleq \phi_{x f}-S_{x f}\left(\frac{\dot{\not}}{\dot{S}}\right)_{f}
$$

and

$$
g_{S f} T \triangleq g_{x f}^{T}-S_{x f}\left(\underset{S}{\stackrel{\cdot}{\dot{G}}}{ }_{S}^{T}\right)_{f}
$$

The transversality condition for this case is

$$
\begin{equation*}
\mu \lambda_{f}+\phi_{s f}+g_{s f}^{T} \nu=0 \tag{10.19}
\end{equation*}
$$

Therefore, for the nominal trajectory, the error in transversalify is defined as

$$
\begin{equation*}
\hat{E}_{f}=\mu \hat{\lambda}_{f}+\left(\hat{\phi}_{s f}+\hat{E}_{s f}^{T} \nu\right) \tag{10.20}
\end{equation*}
$$

Expanding $\mathrm{E}_{\mathrm{f}}$ in a Taylor series about the nominal trajectory yields

$$
\begin{equation*}
\delta E_{f}=\mu \delta \lambda_{f}+\hat{\phi}_{s s f} \delta x_{f}+\hat{g}_{s s f}{ }^{T} \nu \delta x_{f}+\hat{g}_{s f}^{T} \delta \nu \tag{10.21}
\end{equation*}
$$

where

$$
\begin{aligned}
& \phi_{S S f} \triangleq\left(\phi_{X X}-2 S \frac{\dot{\phi}_{x}^{T}}{\dot{S}_{S}^{T}}+\frac{S_{x} \Psi_{\phi} S_{x}^{T}}{\dot{S}^{2}}\right)_{f} \\
& g_{S S f} \nu \triangleq\left(g_{x x}^{T} \nu-\frac{2 S_{x} \nu^{T} g_{x}}{\dot{S}}+\frac{S_{x} \Psi_{g} S_{x}^{T}}{\dot{S}^{2}}\right)_{f} \\
& \Psi_{\emptyset} \triangleq\left(\dot{x}^{T} \phi_{\bar{x}} \dot{x} \dot{x}+2 \dot{x}^{T} \phi_{x t}+\phi_{t t}\right)_{f} \\
& \Psi_{g}=\left(\dot{x}^{T} g_{x \dot{x}} T \dot{x}+2 \dot{x}^{T} g_{x t} T_{V}+g_{t t} T^{T} V\right)_{f}
\end{aligned}
$$

Substituting for $\delta x_{f}$ and $\delta \lambda_{f}$ from (9.3) and (9.4), and using

$$
\delta E_{f}=-\hat{E}_{f} \text { yields }
$$

$$
\begin{equation*}
\Phi_{\lambda} \delta \lambda_{0}+\hat{g}_{S}^{T} \delta \nu=-\hat{E}_{f} \tag{10.22}
\end{equation*}
$$

where

$$
\Phi_{\lambda} \triangleq \hat{\Phi}_{22} \mu+\left(\hat{\emptyset}_{\mathrm{Ss}}+\hat{छ}_{\mathrm{SS}} T_{\nu}\right) \hat{\Phi}_{12}
$$

Solving (10.21) for $\delta \lambda_{0}$ yields

$$
\begin{equation*}
\delta_{\lambda_{0}}=-\Phi_{\lambda}^{-1}\left(\hat{g}_{S}^{T} \delta \nu+\hat{E}_{f}\right) \tag{10.23}
\end{equation*}
$$

From (9.25), it is required that $g$ be zero on the neighbouring trajectory. Expanding (9.25) in a Taylor series about the nominal trajectory and using (9.3) yields

$$
\begin{equation*}
\delta_{g}=\hat{g}_{S} \hat{\Phi}_{12} \delta \lambda_{\odot} \tag{10.24}
\end{equation*}
$$

Substituting (10.23) into (10.24) and using $\delta_{g}=-\hat{g}$ yields

$$
\begin{equation*}
-\hat{g}_{S} \hat{\Phi}_{12} \Phi_{\lambda}^{-1}\left(\hat{g}_{S}^{T} \delta \nu+\hat{E}_{f}\right)=-\hat{g} \tag{10.25}
\end{equation*}
$$

Equation (10.24) can be solved for $\delta \nu$ to provide

$$
\delta \nu=\left[\begin{array}{llll}
\hat{g}_{S} & \hat{\Phi}_{12} & \Phi_{\lambda}^{-1} & \hat{g}_{S}^{T}
\end{array}\right]^{-1}\left[\begin{array}{lll}
\hat{g}-\hat{g}_{S} \hat{\Phi}_{12} \Phi_{\lambda}^{-1} \hat{\mathrm{E}}_{f}
\end{array}\right]
$$

Replacing $\delta V$ in (10.23) by (10.26), the desired incremental change becomes

$$
\begin{equation*}
\delta \lambda_{0}=\delta \lambda_{00}+\delta \lambda_{01} \tag{10.27}
\end{equation*}
$$

where

$$
\begin{aligned}
& \delta \lambda_{00} \triangleq-\Phi_{\lambda}^{-1} \hat{\mathrm{~g}}_{\mathrm{Sf}} \mathrm{~T}\left[\hat{\mathrm{~g}}_{\mathrm{Sf}} \hat{\Phi}_{12 f} \Phi_{\lambda^{-1}} \hat{\mathrm{~g}}_{\mathrm{Sf}}^{\mathrm{T}}\right] \hat{\mathrm{g}}_{f} \\
& \delta \lambda_{o l} \triangleq\left[\Phi_{\lambda}^{-1} \hat{G}_{S f}^{T}\left(\hat{\mathrm{~g}}_{\mathrm{Sf}} \hat{\Phi}_{12 f} \Phi_{\lambda}^{-1} \hat{\mathrm{~g}}_{\mathrm{Sf}}{ }^{\mathrm{T}}\right)^{-1} \hat{\mathrm{~g}}_{\mathrm{Sf}} \hat{\Phi}_{12 f} \Phi_{\lambda}^{-1}\right. \\
& \left.-\Phi_{\lambda}^{-I}\right] \hat{E}_{f}
\end{aligned}
$$

Note that for $\delta \lambda_{0}=\delta \lambda_{00}$, equation (10.24) becomes

$$
\begin{align*}
\delta_{g} & =-\hat{g}_{S f} \hat{\Phi}_{l 2 f} \Phi_{\lambda}^{-1} \hat{g}_{S f}^{T}\left[\hat{E}_{S} \hat{\Phi}_{12} \Phi_{\lambda}^{-1} \hat{g}_{S}^{T}\right]_{f}^{-1} \hat{g}_{f} \\
& =-\hat{g}_{f} \tag{10.28}
\end{align*}
$$

and hence $\delta \lambda_{00}$ is that component of $\delta \lambda_{0}$ which attempts to satisfy the end constraints (9.25). Using $\delta \lambda_{0}=\delta \lambda_{0 l}$ in (10.24) yields

$$
\begin{align*}
\delta_{g}= & \hat{g}_{S f} \hat{\Phi}_{l 2 f} \Phi_{\lambda}^{-1} \hat{\underline{g}}_{S f} T\left[\hat{g}_{S} \hat{\Phi}_{I 2} \Phi_{\lambda}^{-1} \hat{g}_{S}^{T}\right]_{f}^{-1} \hat{g}_{S f} \hat{\Phi}_{I 2 f} \\
& \Phi_{\lambda}^{-1} \hat{\mathrm{E}}_{f}-\hat{\underline{g}}_{S f} \hat{\Phi}_{12 f} \Phi_{\lambda}^{-1} \hat{\mathrm{E}}_{f} \\
\ddots & 0 \tag{10.29}
\end{align*}
$$

and hence $\delta \lambda_{o l}$ is that component of $\delta \lambda_{0}$ which attempts to satisfy the transversaiity condition without affecting the end constraints set by $\delta \lambda_{00^{\circ}}$. Equation $(10.27)$, therefore, provides the desired incremental change in $\lambda_{0}$. However, to evaluate $\Phi_{\lambda}$ (equation (10.22)), values of $\mu$ and $V$ are required. As in the previous section, these values are selected to minimize the square of the error in transversality. By (10.20) this error is

$$
\begin{equation*}
\left|E_{f}\right|^{2}=E_{f}^{T} E_{f}=\phi_{s f}^{T} \phi_{s f}+2 \bar{\nu}^{T} \Psi_{\nu}^{T} \phi_{s f}+\bar{\nu}^{T} \Psi_{\nu}^{T} \Psi_{\nu} \bar{\nu} \tag{10.30}
\end{equation*}
$$

where $\quad \Psi_{\nu}^{T} \triangleq\left[g_{s f}^{T}, \lambda_{f}\right]$
and $\quad \because \quad \bar{\nu} \triangleq\left[\begin{array}{l}\nu \\ \mu\end{array}\right]$
Differentiating (10.30) with respect to $\bar{\nu}$, and equating the derivative to zero yields

$$
\begin{equation*}
2 \Psi_{\nu}^{T} \emptyset_{\text {Sf }}+2 \Psi_{\nu}^{T} \Psi_{\nu} \cdot \bar{\nu}=0 \tag{10.31}
\end{equation*}
$$

The optimal values for the augmented vector $\bar{\nu}$ is found from (10.31) to be.

$$
\begin{equation*}
\bar{\nu}_{\text {opt }}=-\left[\Psi_{\nu}^{\mathrm{T}} \Psi_{\nu}\right]^{-1}\left[\Psi_{\nu}^{\mathrm{T}} \emptyset_{s f}\right] \tag{10.32}
\end{equation*}
$$

As a result, the desired neighbouring trajectory becomes

$$
\begin{equation*}
\lambda_{0}=\ddot{\mu}_{o p t}\left(\hat{\lambda}_{o}+\delta \lambda_{0}\right) \tag{10.33}
\end{equation*}
$$

where $\delta \lambda_{0}$ is defined by (10.27), $\mu_{o p t}$ and $\nu_{\text {opt }}$ are defined by (10.32), and $\hat{\lambda}_{0}$ is defined by the nominal trajectory.
10.1.2 Computing Algorithm FI using Matching End Points
(1) From $\hat{\lambda}_{0}$, which defines the nominal trajectory, and (8.33), integrate (8.30), (8.31) and (9.32) from appropriate initial conditions until $S=0$, which defines $t_{f}$. (Note: $n$ systems of (9.32) are integrated in parallel from the initial conditions given in Section (8.2.2)).
(2) Test $\left|\delta J_{a}\right|<\varepsilon_{I}$ and/or $|g|<\varepsilon_{2}$ for exit, where $\varepsilon_{1}$ and $\varepsilon_{2}$ are chosen to provide the desired degree of accuracy.
(3) Compute $\dot{\mu}_{\mathrm{opt}}, \nu_{\mathrm{opt}}$, and $\delta \lambda_{\circ}$ by (10.33) and (10.27) respectively.
(4) Replace $\hat{\lambda}_{0}$ by (10.33) and repeat from (1).
10.2 Determining Optimal Step Size for the First Variation

## Approach

In Chapter 9, the incremental change in $\lambda_{0}$ based on a first variation approach was shown to be (see (9.45))

$$
\begin{equation*}
\delta \lambda_{0}=a \delta \lambda_{00}+k \delta \lambda_{0 l} \tag{10.34}
\end{equation*}
$$

where

$$
\begin{aligned}
\delta \lambda_{00} & =-z_{g}\left(z_{g}{ }^{T} z_{g}\right)^{-1} \hat{g} \\
\delta \lambda_{o l} & =z_{g}\left(z_{g}{ }^{T} z_{g}\right)^{-1} z_{g}^{T} z_{\phi}-z_{\phi} \\
k & =\sqrt{\frac{\delta_{1}^{2}-a^{2} \hat{g}^{T}\left(z_{g} Z_{g}\right)^{-1} \hat{g}}{z_{\phi}^{T} Z_{\phi}-z_{\phi} Z_{g}\left(z_{g}{ }^{T} Z_{g}\right)^{-1} z_{g}{ }^{T} Z_{\phi}}}
\end{aligned}
$$

and $0 \leqslant a \leqslant 1$
It was further shown that since the resulting search is carried out over the surface of a sphere, it is convenient to define $\delta_{I}{ }^{2}$ as $(\operatorname{see}(9.55))$

$$
\begin{equation*}
\delta_{1}{ }^{2}=\delta \hat{\alpha}^{2} \hat{\lambda}_{0}^{T} \hat{\lambda}_{0} \tag{10.35}
\end{equation*}
$$

where $\delta \hat{\alpha}$ is the angular incremental rotation of the $\lambda_{0}$ vector resulting from the incremental displacement $\delta \lambda_{0}$. The vector $\delta \lambda_{0}$ in (10.34) is a linear combination of the vectors $\delta \lambda_{00}$ and $\delta \lambda_{01}$. It is shown in Appendix $F$ that $\delta \lambda_{00}$ is the component of $\delta \lambda_{0}$ which attempts to satisfy the end conditions (9.25), and that $\delta \lambda_{o l}$ is that component of $\delta \lambda_{0}$ which attempts to minimize Ja without affecting the end conditions set by $\delta \lambda_{00}$. The relative emphasis placed on these two effects depends on
the values selected for the parameters a and k. From (10.34), it is seen that these values are subject to the constraints

$$
\begin{equation*}
k=\delta \hat{\alpha} \sqrt{\frac{\hat{\lambda}_{0}^{T} \hat{\lambda}_{0}\left(1-a^{2} / b^{2}\right)}{z_{\phi}^{T} z_{\phi}-z_{\phi}^{T} z_{g}\left(z_{g}^{T} z_{g}\right)^{-1} z_{g}^{T} z_{\phi}}} \tag{10.36}
\end{equation*}
$$

$0 \leqslant a \leqslant 1$
where

$$
\begin{equation*}
b \triangleq \sqrt{\frac{\delta \hat{\alpha}^{2} \hat{\lambda}_{0}^{T} \hat{\lambda}_{0}}{\hat{g}^{T}\left(z_{g}{ }^{T} z_{g}\right)^{-1} \hat{g}}} \tag{10.37}
\end{equation*}
$$

As a result of (10.36) and (10.37), two sets of values can be assigned to $a$ and $k$ depending on the value of $b . .$. The two cases are:
(1) for $b \leqslant l, a=b$ and $k=0$
(2) for $b>1, a=1$ and $k$ is defined in (10.36) For the first case, the error in the final end conditions is large and full emphasis is placed on minimizing this error. For the second case, the error in end constraint is small and, within the step $\delta \hat{\alpha}$, it is possible to reduce Ja while also satisfying the desired end conditions. To take advantage of the good.initial convergence properties of this gradient technique, a three stage algorithm can be developed that uses the gradient technique in the first two stages to rapidly locate the region of the optimum. However, as the final convergence of the gradient method is relatively poor, a third stage is used which has good final convergence properties and which can be used with the gradient method. An example of such a technique is the modified method of matching end points of Section 10.1. Aiso, in the next sections, two techniques are developed which
determine the optimal step size for the gradient method as the optimum is approached. The first is based on a second variation approach, and the second is based on a method of curve fitting. Combining these techniques, therefore, the three stage algorithm is as follows: (see Figure 10.3)
(a) First stage: if $b \leqslant 1$, $a$ and $k$ are defined by

Case (I) and hence

$$
\delta \lambda_{0}=b \delta \lambda_{00}
$$

This region is called the Rg region since full emphasis is placed on satisfying the end conditions,
$g=0$. The search is carried out over the surface of a sphere with a constant rotation $\delta \hat{\alpha}$ until $b>1$ which defines the second stage.
(b) Second stage: if $b>1, a$ and $k$ are defined in Case (2) and hence

$$
\delta \lambda_{0}=\delta \lambda_{00}+k \delta \lambda_{01}
$$

This region is called the RJa region since the emphasis is now on reducing Ja without affecting the end conditions set by $\delta \lambda_{00}$. The search continues on the surface of the sphere with a constant rotation $\delta \hat{\alpha}$ until Ja increases which defines the third stage. (c) Third stage: At the point before Ja increases, it can be concluded that the rotation $\delta \hat{\alpha}$ was too large and the region of the optimum was overstepped. This region, calied the $\mathrm{R} \alpha$ region, lies in the interior of a cone-shaped surface which has a maximum angular width $\delta \hat{\alpha}$ and which contains the solution line $\mu \lambda_{0}^{*}$.


Eigure 10.3 The Regions $R g, R J$, and $R a$ About the Solution Line $\mu \lambda_{0}^{*}$

Within this region one of the techniques with good final convergence properties is used to complete the search procedure.

### 10.2.1 The Second Variation Method of Determining the Optimal Step Size

Taking the variation of Ja defined in (10.15) and keeping all terms up to second order yields

$$
\left.+\mu \int_{0}^{t_{f}}\left(\delta \lambda^{T}\left(\dot{x}-H_{\lambda}\right)+\delta_{x} \mathbb{T}^{( } \dot{\lambda}_{\lambda} H_{x}\right)+\delta_{u} T_{H_{u}}\right) d t
$$

$$
+\left[\lambda^{T} \cdot x-H\right]_{f} \delta t_{f}+\mu \int_{0}^{t_{f}} \delta \lambda^{T}\left(\delta \dot{x}-H_{\lambda x} \delta x-H_{\lambda u} \delta u\right) d t
$$

$$
\begin{equation*}
\left.+\delta x^{T} \lambda\right]_{0}^{t_{f}} \tag{10.38}
\end{equation*}
$$

where $g_{s f}{ }^{T}, \phi_{s f}, g_{s s f}{ }^{T} \nu$ and $\phi_{\text {sEf }}$ are defined in (10.18) and (10.21) respectively. Using the first variation approach developed in Chapter 9, the following relations hold for the nominal trajectory:
(1) $\dot{x}-H_{\lambda}=0$, from (8.30)
(2) $\dot{\lambda}+H_{x}=0, \quad$ from (8.31)
(3) $H_{u}=0$, from (8.32)
(4) $H_{\lambda \lambda}=0 \quad, \quad$ from (8.24)

$$
\begin{aligned}
& \delta_{J_{a}}=\delta x_{f}{ }^{T}\left(\phi_{s f}+\delta_{s f} \mathcal{T}\right)+\frac{1}{2} \delta x_{f}^{T}\left(\phi_{\text {gef }}+\delta_{s s f}{ }^{T} \nu\right) \delta x_{f} \\
& -\frac{\mu}{2} \int_{0}^{t_{f}}\left(\delta u^{T_{H}}{ }_{u u} \delta_{u+}+\frac{1}{2} \delta_{u}{ }^{T} H_{H u} \delta x+\delta x^{T_{H}}{ }_{x x} \delta x+\delta \lambda{ }^{T}{ }_{H_{\lambda \lambda}} \delta_{\lambda}\right) d t
\end{aligned}
$$

(5) $\delta_{x}^{0}-H_{\lambda \lambda} \delta x-H_{\lambda u} \delta_{u}=0, \quad$ from (9.5)
(6) $x(0)=x_{0}$
from (8.33)
(7) $\delta x_{f}{ }^{T} \lambda_{f}=0$
from (9.9)

Substituting (1) to (7) into (10.38) yields

$$
\begin{align*}
\delta J_{a}= & \delta x_{f}{ }^{T}\left(\phi_{s f}+g_{s f}{ }^{T} \nu\right)-\frac{1}{2} \delta x_{f}{ }^{T}\left(\phi_{s s f}+g_{s f f} T_{\nu}\right) \delta x_{f} \\
& -\frac{\mu}{2} \int_{0}^{t_{f}}\left(\delta{ }_{u} T_{u u} \delta_{u} u+2 \delta{ }_{u} T_{u x} \delta x+\delta x_{x} T_{x x} \delta_{x}\right) d t \tag{10.39}
\end{align*}
$$

From (9.61) the variation in $u$ for the neighbouring trajectory is given by

$$
\begin{equation*}
\delta u=-\hat{H}_{u u}{ }^{-1}\left(\hat{H}_{u x} \delta_{x}+\hat{H}_{u \lambda} \delta \lambda\right) \tag{10.40}
\end{equation*}
$$

For $\delta x_{0}=0$, the value of $\delta_{x}(t)$ and $\delta_{\lambda}(t)$ can be obtained from (9.3) and (9.4) in terms of the submatrices $\Phi_{12}$ and $\Phi_{22}$ : Using (9.3), (9.4), and (10.40) in (10.39), $\delta_{J_{a}}$ becomes

$$
\begin{align*}
\delta J_{a}= & \delta \lambda_{0}^{T}\left(Z_{\phi}+Z_{g} \nu\right)+\frac{1}{2} \delta \lambda_{0}^{T}\left(\hat{\Phi}_{12 f}\left(\hat{\phi}_{S S f}+\hat{S}_{S S f} T_{\nu}\right) \hat{\Phi}_{12 f}\right) \delta \lambda_{0} \\
& -\frac{\mu}{2} \delta \lambda_{0}^{T}\left[\int_{0}^{t_{f}}\left(D^{T} \hat{H}_{u u} D+2 D \hat{H}_{u x} \hat{\Phi}_{12}+\hat{\Phi}_{12}^{T} \hat{H}_{x x} \hat{\Phi}_{12}\right) d t\right] \delta \lambda_{0} \tag{10.41}
\end{align*}
$$

where

$$
\begin{aligned}
& z_{\phi} \triangleq \hat{\Phi}_{l 2 f} T \hat{\phi}_{S f} \\
& z_{g} \triangleq \hat{\Phi}_{l 2 f}^{T} \hat{\mathrm{~g}}_{S f} T
\end{aligned}
$$

and

$$
D \triangleq-\hat{H}_{u u}{ }^{-1}\left(\hat{H}_{u x} \hat{\Phi}_{12}+\hat{H}_{u \lambda} \hat{\Phi}_{22}\right)
$$

By the first variation approach, equation (10.34), the value of $\delta \lambda_{\odot}$ that minimizes $J_{a}$ subject to the constraint $\delta g=-\hat{g}$
( $a=1$ ) is

$$
\begin{equation*}
\delta \lambda_{0}=\delta \lambda_{00}+k \delta \lambda_{01} \tag{10.42}
\end{equation*}
$$

By ( $G-2$ ), the value for $\nu$ was found to be

$$
\begin{equation*}
\nu=\frac{\nu_{0}}{k}+\nu_{1} \tag{10.43}
\end{equation*}
$$

where

$$
\begin{aligned}
& \nu_{0}=\left(\mathrm{Z}_{g} \mathrm{~T}_{\mathrm{Z}}\right)^{-1} \hat{g} \\
& \nu_{1}=-\left(\mathrm{Z}_{g} \mathrm{~T}_{\mathrm{g}}\right)^{-1} Z_{g} \mathrm{~T}_{\phi}
\end{aligned}
$$

For this problem, the error in final transversality is.

$$
\begin{equation*}
\hat{E}_{f}=\mu \hat{\lambda}_{f}+\left(\hat{\phi}_{S f}+\hat{g}_{s f}^{T}{ }^{T}\right) \tag{10.44}
\end{equation*}
$$

Substituting (10.43) into (10.44), and determining $\mu$ as that value which minimizes $E_{f}{ }^{T} E_{f}$ yields

$$
\begin{equation*}
\mu=\frac{\mu_{0}}{k}+\mu_{1} \tag{10.45}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mu_{0}=-\hat{\lambda}_{f}^{T} \hat{g}_{s f}^{T} \nu_{0} / \hat{\lambda}_{f}^{T} \hat{\lambda}_{f} \\
& \mu_{1}=-\hat{\lambda}_{f}^{T}\left(\hat{\phi}_{s f}+\hat{E}_{s f}^{T} \nu_{l}\right) / \hat{\lambda}_{f}^{T} \hat{\lambda}_{f}
\end{aligned}
$$

For this approach, it is desired to find the value for $k$ in (10.42) which causes $\delta_{J}$ to be a minimum. Substituting (10.42), (10.43), and (10.45) into (10.41) yields

$$
\begin{aligned}
& \delta J_{a}==\frac{1}{k} \delta \lambda_{00}{ }^{T} W_{4} \delta \lambda_{00}+\left(2 \delta \lambda_{00}{ }^{T} W_{4} \delta \lambda_{01}+\delta \lambda_{00}{ }^{T} \frac{W_{5}}{2} \delta \lambda_{00}\right) \\
&+k\left(\delta \lambda_{01}{ }^{T} W_{4} \delta \lambda_{01}+\delta \lambda_{O 1}{ }^{T} W_{5} \delta \lambda_{00}\right)+\frac{k^{2}}{2} \delta \lambda_{01}{ }^{T} W_{5} \delta \lambda_{01} \\
&(10.46)
\end{aligned}
$$

where

$$
\begin{aligned}
& W_{1} \triangleq \frac{1}{2} \hat{\Phi}_{12 f}^{T}\left(\hat{\phi}_{S S f}+\hat{G}_{S S f}{ }^{T} V_{1}\right) \hat{\Phi}_{12 f} \\
& W_{2} \triangleq \frac{1}{2} \hat{\Phi}_{12 f}{ }^{T}\left(\hat{\mathrm{E}}_{\mathrm{SSf}}{ }^{T} \nu_{0}\right) \hat{\Phi}_{\text {Inf }} \\
& w_{3} \triangleq-\frac{1}{2} \int_{0}^{t} \cdot\left(D^{T} \hat{H}_{u u} D+2 D^{T} \hat{H}_{u x} \hat{\bar{\Phi}}_{12}+\hat{\bar{\Phi}}_{12}{ }^{T} \hat{H}_{x x} \hat{\bar{\Phi}}_{12}\right) d t
\end{aligned}
$$

$$
\begin{aligned}
& W_{4} \triangleq-I+W_{2}+\mu_{0} W_{3} \\
& W_{5} \triangleq 2\left(W_{1}+\mu_{1} W_{3}\right)
\end{aligned}
$$

As (10.46) is a function of $k$ only, the minimum of $\delta_{J}$ occurs when the derivative of (10.46) with respect to $k$ is zero. This yields the cubic equation

$$
\begin{equation*}
\mathrm{k}^{3}+\mathrm{pk}{ }^{2}+r=0 \tag{10.47}
\end{equation*}
$$

where

$$
p \triangleq \frac{\left(\delta \lambda_{01}{ }^{T} W_{4} \delta \lambda_{01}+\delta \lambda_{01}^{T} W_{5} \delta \lambda_{00}\right)}{\left(\lambda_{01}{ }^{T} W_{5} \lambda_{01}\right)}
$$

and

$$
r \triangleq \frac{-\delta \lambda_{00}^{T} W_{4} \delta \lambda_{00}}{\left(\delta \lambda_{01}^{T} W_{5} \delta \lambda_{01}\right)}
$$

Using Cardan's cubic formula for the real root of (10.47) yields

$$
\begin{equation*}
k_{o p t}=-\frac{p}{3}\left(2+\left(1+\frac{2.7 r}{p^{3}}\right)^{1 / 3}\right) \tag{10.48}
\end{equation*}
$$

It is shown in Appendix $H$ that the rate of convergence for the approach based on (10.48) is independent of the initial scale factor. As a result, this approach can be used within the R $\boldsymbol{\alpha}$ region as a means of providing improved final convergence. The associated neighbouring trajectory is given by

$$
\begin{equation*}
\lambda_{0}=\mu_{\mathrm{opt}}\left(\hat{\lambda}_{0}+\delta \lambda_{0}\right) \tag{10.49}
\end{equation*}
$$

where $\hat{\lambda}_{0}$ is defined on the nominal trajectory, $\delta \lambda_{0}$ is defined by ( 10.42 ), k is defined by (10.48) and $u_{\text {opt }}$ by (10.45).
(1) From the nominal trajectory defined by $\hat{\lambda}_{0}$ and (8.33), integrate (8.30), (8.31) and (9.32) from appropriate initial conditions until $S=0$, which defines $t_{f}$. (Note: $n$ systems of (9.32) are integrated in parallel from the initial conditions given in Section (8.2.2)).
(2) Test $\left|J_{a}\right|<\varepsilon_{1}$ and/or $|g|<\varepsilon_{2}$ for exit, where $\varepsilon_{1}$ and $\varepsilon_{2}$ are selected to provide the desired degree of accuracy.
(3) Calculate $k_{\text {opt }}$ from (10.41) and use (10.42) to define $\delta \lambda_{0}$ 。
(4). With $\lambda_{0}$ defined by (10.49) return to (1) and repeat.

### 10.2.3 A Curve Fitting Technique to Determine the Optimal Step. Size

In the first variation procedure of Section 10.2, the $\lambda_{0}$ vector is swept through the initial condition space at a constant angular rotation $\delta \hat{\alpha}$ until $J_{a}$ increases. At this final step $((n+1)$-th $)$, two points on the curve $J_{a}=J_{a}\left(\delta_{\alpha}\right)$ have been established, where $\delta \alpha$ is defined as the angular rotation in the gradient direction from the n-th step. The two points are $\left(J_{a 0}, 0\right)$ and $\left(J_{a l}, \delta \hat{\alpha}\right)$, where $J_{a o}$ is the value of $J_{a}$ at the $n$-th step, and $J_{a l}$ is the value of $J_{a}$ at the $(n+l)$-th step. As the rotation from the $n$-th step is in the gradient direction, an optimal rotation $\delta_{\alpha_{o p t}}$ must exist which provides the maximum decrease in $J_{a}$ and whichis within the range

An approximation to $\delta_{\alpha_{o p t}}$ can be conveniently obtained by approximating the curve $J_{a}(\delta \alpha)$ by a parabola of the form

$$
\begin{equation*}
J_{a}=a+b \delta \alpha+c \delta \alpha^{2} \tag{10.51}
\end{equation*}
$$

To solve for the coefficients $a, b$, and $c$ in (10.51), only one extra point on the curve $J_{a}(\delta \alpha)$ is required. For convenience this point is selected as ( $J_{a \frac{1}{2}}, \delta \hat{\alpha} / 2$ ). (See Figure 10.4)


Figure 20.4 The Graph of $J_{a}(\delta \alpha)$ in the Neighbourhood of the $n$-th Step

Substituting these three points into (10.51) yields the matrix equation

$$
\left[\begin{array}{c}
J_{a 0}  \tag{10.52}\\
J_{a \frac{1}{2}} \\
J_{a \cdot 1}
\end{array}\right]=\left[\begin{array}{ccc}
1 & 0 & 0 \\
1 & \frac{1}{2} \delta \hat{\alpha} & \frac{1}{4} \delta \hat{\alpha}^{2} \\
1 & \delta \hat{\alpha} & \delta \hat{\alpha}^{2}
\end{array}\right]\left[\begin{array}{l}
a \\
i \\
b \\
c
\end{array}\right]
$$

The value for $\delta_{\alpha_{o p t}}$ is taken as that value of $\delta_{\alpha}$ which minimizes (10.51). Differentiating (10.51) with respect to $\delta_{\alpha}$ and equating the derivative to zero yields

$$
\begin{equation*}
\delta_{\alpha_{o p t}}=-b / 2 c \tag{10.53}
\end{equation*}
$$

Solving for b and c from (10.52) yields

$$
\begin{align*}
& b=-\delta \hat{\alpha}^{2} \frac{\left(J_{a l}-4 J_{a \frac{1}{2}}+3 J_{a 0}\right)}{\delta \hat{\alpha}^{3}}  \tag{10.54}\\
& c=-2 \delta \hat{\alpha} \frac{\left(J_{a l}-2 J_{a \frac{1}{2}}+J_{a 0}\right)}{\delta \hat{\alpha}^{3}} \tag{10.55}
\end{align*}
$$

Substituting (10.54) and (10.55) into (10.53) yields the desired value

$$
\begin{equation*}
\delta_{\alpha_{\rho p t}}=\frac{\delta_{\alpha}}{4} \frac{\left(J_{a l}-4 J_{a \frac{1}{2}}+3 J_{a 0}\right)}{\left(J_{a l}-2 J_{a \frac{1}{2}}+J_{a 0}\right)} \tag{10.56}
\end{equation*}
$$

The value for $\delta \hat{\alpha}$ is replaced by $\delta_{\alpha_{o p, t}}$ and the procedure is repeated until the desired degree of accuracy is obtained. 10.2.4 Computing Algorithm F3 Using a Curve Fitting Technique
(1) Let $J_{a o}$ be a large positive number.
(2) Use the first variation approach of Section 10.2 with $\delta_{\alpha}=\delta \hat{\alpha}$.
(3) If $\left|J_{a}\right|<\varepsilon_{1}$ and/or $|g|<\varepsilon_{2}$ exit, where $\varepsilon_{1}$ and $\varepsilon_{2}$ are chosen to provide the desired accuracy.
(4) If $J_{a}<J_{a 0}$, set $J_{a 0}=J_{a}$ and store the final values for this trajectory. Compute: $\delta \lambda_{\circ}$ by (10.34)
and return to (2).
(5) If $J_{a} \geqslant J_{a 0}$, replace $\delta \alpha$ by $\frac{1}{2} \delta \hat{\alpha}$ and, using the final values of the previous trajectory, compute $k_{\frac{1}{2}}$ from (10.36) and store (Note: $a=1$ in R $\alpha$ ).
(6) Replace $\hat{\lambda}_{0}$ by ( $\hat{\lambda}_{0}+\left(k_{\frac{1}{2}}-\hat{k}\right) \delta \hat{\lambda}_{\circ 1}$ ) and integrate ( 8.30 ) and $(8.31)$ forward to obtain $J_{a \frac{1}{2}}$. (7) Compute $\delta_{\alpha_{o p t}}$ from (10.53). Replace $\delta_{\alpha}$ by $\delta \alpha_{o p t}$ and, using the stored final values, compute $k_{\text {opt }}$ from (10.36).
(8) Replace $\hat{\lambda}_{\circ}$ by $\left(\hat{\lambda}_{o}+\left(k_{o p t}-k_{\frac{k_{2}}{}}\right) \delta \hat{\lambda}_{o l}\right)$ and return to (2).

### 10.3 The Combined Computing Algorithm

In this section a combined algorithm is presented which uses the first variation approach to initiate the search procedure and to locate the $R \alpha$ region. Once in the $R \alpha$ region, one of the computing algorithms F1, F2, or F3 is used to provide the final convergence. A flow graph of the combined. algorithm is shown in Figure 10.5.

### 10.3.1 Example 1

As a first example, the problem of Section 9.5 is solved using the combined algorithm with FI and F2 as the final stage. The results are shown in Table 9.2. It is seen that the combined algorithms provide the desired improvement in final convergence over the algorithm based on first variation alone. A further comparison between $F 1$ and $F 2$ is illustrated in Table 10.1 and Table 10.2 for which the search procedure is initiated from different points in the initial conditions space. It can be observed that, for this example, the computing


Figure 10.5 Flow Chart for the Combined Algorithm

Table 10.1

| STEP |  | F2 - Second Variation on $k$ |  |  | Fl - Matching End Points |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $\mathrm{X}_{2 f}$ | $\lambda_{1 f}$ | $\lambda_{2 f}$ | $\mathrm{X}_{2 f}$ | $\lambda_{\text {If }}$ | $\lambda_{2 f}$ |
| $0^{\infty}$ | $\begin{aligned} & 0 \\ & 1 \\ & 2 \end{aligned}$ | $\begin{aligned} & 0.166528 \\ & 0.56796 \\ & 0.659119 \end{aligned}$ | $\begin{aligned} & -197.1 \\ & -4.459 \\ & 0.4128 \end{aligned}$ | $\begin{aligned} & 8.869 \\ & 1.627 \\ & 0.8804 \end{aligned}$ | $\begin{aligned} & 0.166528 \\ & 0.56796 \\ & 0.659119 \end{aligned}$ | $\begin{aligned} & -197.1 \\ & -4.459 \\ & 0.4128 \end{aligned}$ | $\begin{aligned} & 8.869 \\ & 1.627 \\ & 0.8804 \end{aligned}$ |
| ¢ | 3 4 5 6 | $\begin{aligned} & 0.67 .6695 \\ & 0.681707 \\ & 0.681707 \end{aligned}$ | $\begin{gathered} -0.3273 \\ 0.00262 \\ -0.0000043 \end{gathered}$ | $\begin{aligned} & 1.006 \\ & 0.8379 \\ & 1.001 \end{aligned}$ | $\begin{aligned} & 0.67 .7784 \\ & 0.681636 \\ & 0.681707 \\ & 0.681707 \end{aligned}$ | $\left\|\begin{array}{\|} -0.2944 \\ -0.03482 \\ -0.0006664 \\ 0.0000003866 \end{array}\right\|$ | $\begin{aligned} & 1.039 \\ & 1.004 \\ & 1.000 \\ & 1.0000 \end{aligned}$ |

Table 10.2

| STEP |  | F2 - Second Variation on $k$ |  |  | Fl - Matching End points |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $\chi_{2 f}$ | $\lambda_{\text {If }}$ | $\lambda_{2 f}$ | $\mathrm{X}_{2 f}$ | $\lambda_{\text {lf }}$ | $\lambda_{2 f}$ |
| 吅 ${ }_{\text {® }}$ | $\begin{aligned} & 0 \\ & 1 \\ & 2 \\ & 3 \\ & 4 \\ & 5 \end{aligned}$ | $\begin{aligned} & 0.006379 \\ & 0.011924 \\ & 0.029196 \\ & 0.094367 \\ & 0.339618 \\ & 0.670167 \end{aligned}$ | $\begin{array}{r} -85020 \\ -36610 . \\ -9714 . \\ -134.8 \\ -76.88 \\ 0.7321 \end{array}$ | $\begin{aligned} & 1113 . \\ & 529 . \\ & 175.3 \\ & 50.61 \\ & 7.783 \\ & 2.036 \end{aligned}$ | $\begin{aligned} & 0.006379 \\ & 0.011924 \\ & 0.029196 \\ & 0.094367 \\ & 0.339618 \\ & 0.670167 \end{aligned}$ | $\left\lvert\, \begin{array}{r} -85020 \\ -36610 \\ -9714 . \\ -1348 . \\ -76.88 \\ 0.7321 \end{array}\right.$ | $\begin{array}{\|c} 1113 . \\ 529 . \\ 175.3 \\ 40.61 \\ 7.783 \\ 2.036 \end{array}$ |
| ¢ $\square^{\circ}$ | 6 7 8 9 | $\begin{aligned} & 0.680692 \\ & 0.681707 \\ & 0.681707 \end{aligned}$ | $\left\lvert\, \begin{aligned} & -0.1371 \\ & -0.002836 \\ & -0.00000391 \end{aligned}\right.$ | $\begin{aligned} & 1.007 \\ & 0.9 .509 \\ & 0.9993 \end{aligned}$ | $\begin{aligned} & 0.680782 \\ & 0.681703 \\ & 0.681707 \\ & 0.681707 \end{aligned}$ | $\left\lvert\, \begin{aligned} & -0.1318 \\ & -0.008396 \\ & -0.00003993 \\ & 0.00000057 \end{aligned}\right.$ | $\begin{aligned} & 1.017 \\ & 1.001 \\ & 1.000 \\ & 1.0000 \end{aligned}$ |

algorithm F2, which is based on a second variation approach, appears to provide the best final convergence. For both $F l$ and F2, however, the convergence is essentially quadratic.
10.3.2 Example 2

Consider the problem for which the control $u(t)$ is to be found over the time interval $0 \leqslant t \leqslant 4.2$, such that the system performance

$$
\begin{equation*}
J=\left(2 x_{1}^{2}+x_{3}\right)_{t=4.2} \tag{10.57}
\end{equation*}
$$

is a minimum. The equations of constraint are

$$
\begin{array}{ll}
\dot{x}_{1}=\left(1-x_{2}^{2}\right) x_{1}-x_{2}+u, & x_{1}(0)=0 \\
\dot{x}_{2}=x_{1} & ,  \tag{10.58}\\
x_{2}(0)=1.0 \\
\dot{x}_{3}=x_{1}^{2}+x_{2}^{2}+u^{2} & ,
\end{array} x_{3}(0)=0
$$

and

$$
\begin{equation*}
g=x_{2}(4.2)=0 \tag{10.59}
\end{equation*}
$$

The associated Euler-Lagrange equations are

$$
\begin{aligned}
& \dot{\lambda}_{1}=-\lambda_{1}\left(1-x_{2}^{2}\right)-\lambda_{2}-2 \lambda_{3} x_{1} \\
& \dot{\lambda}_{2}=\lambda_{1}\left(1+2 x_{1} x_{2}\right)-\lambda_{2}-2 \lambda_{3} x_{1} \\
& \dot{\lambda}_{3}=0 \\
& H_{u}=-\lambda_{1}-2 \lambda_{3} u=0
\end{aligned}
$$

For the classical approach, the final transversality conditions are

$$
\begin{align*}
& \lambda_{1}(4.2)=-4 x_{1}(4.2)  \tag{10.61}\\
& \lambda_{2}(4.2)=-V \\
& \lambda_{3}(4.2)=-1.0
\end{align*}
$$

where $V$ is defined by the augmented performance function

$$
\begin{equation*}
J_{a}=\left(2 x_{1}^{2}+x_{2} \nu+x_{3}\right)_{t=4.2} \tag{10.62}
\end{equation*}
$$

This problem is an example of a problem with final end constraints. The solution is obtained using the combined algorithm with F1, F2, and F3 and the final stage. The optimal trajectory and optimal control $u(t)$ are shown in Figure 10.6. The associated Lagrange multipliers are illustrated in Figure 10.7. The values $\lambda_{1}\left(0^{\circ}\right)=0.5, \lambda_{2}(0)=5.0$, and $\lambda_{3}(0)=1.0$ were arbitrarily selected as an initial estimate. It can be observed in Table 10.3 that this initial estimate is in the region Rg since $\mathrm{b}=0.3<1$. An incremental rotation $\delta \hat{\alpha}=0.005$ radians is used during the first variation approach. Table 10.3 illustrates viery clearly the region Rg (steps 0 to 7) in which $a=b \leqslant l$ and $k=0$, the region $R J_{a}$ (steps 8 to 15) in which $a=1$ and $k \neq 0$, and the region $R \alpha$ (final steps) in which $\delta_{\alpha_{o p t}}<\delta \hat{\alpha}$. Notice that in region Rg , full emphasis is placed on reducing $g$, and in region $R J_{a}$ the emphasis is transferred to reducing $J_{a}$ while maintaining g close to zero. The intersection of the R $\alpha$ region is manifested by the increase in $J_{a}$ at the 16 -th step. Once in R $\alpha, F 1, F 2$, and $F 3$ are used to provide the final convergence. It can be observed that, for this example, F1 and F3 provide more rapid convergence than F2. The convexity of $\mathrm{J}_{\mathrm{a}}$ in the neighbourhood of the 15-th step is shown in Figure 10.8. Note that the method of curve fitting, F3, estimates quite accurately the minimum of $J_{a}$, and that the corresponding estimate by the second variation approach, F2, is slightly in error. It is believed that this error is caused


TIME (SECONDS)
Figure 10.6 The Optimal Trajectory for Example 2


Figure 10.7 The Optimal Lagrange Multipliers for Example 2

Table 10.3



Figure 10.8 J a in the Neighbourhood of the $n$-th Step
by the uncertainty in the value for $\nu$ on the nominal trajectory of the F2 approach. By (10.43), it is seen that $\nu$ is a function of the future step size, and hence $J_{a}$ on the nominal trajectory can only be computed after the future step has been specified. This type of problem is not present with FI and F3 since for Fl, $\nu$ is precisely specified on the nominal trajectory as that value which minimizes the error in transversality (10.32), and for F3, the actual curve $J_{a}\left(\delta_{\alpha}\right)$ is used. $10.4 \frac{\text { The Second Variation Technique Used with the Method of }}{\text { Ste }}$ Steepest Descent

A second variation approach can also be used to determine the optimal parameter $k$ for the method of steepest descent in function space. Consider the optimal control problem in Section (8.4). Using (8.23), the variation on $J_{a}$ up to terms of second order is

$$
\begin{align*}
& \text { T } \\
& \left.\delta J_{a}=\delta x_{f}{ }^{T} \phi_{x f}+\frac{1}{2} \delta x_{f}{ }^{T} \phi_{x x f} \delta x_{f}+\delta x^{T}{ }^{T}\right]_{0}^{T}-\frac{1}{2} \int_{0}\left(\delta u^{T} H_{u u} \delta u\right) d t \\
& -\frac{1}{2} \int_{0}^{T}\left(2 \delta u^{T} H_{u x} \delta x+\delta x^{T} H_{x x} \delta x\right) d t-\frac{1}{2} \int_{0}^{T}\left(\delta \lambda^{T} H_{\lambda \lambda} \delta \lambda\right) d t \\
& -\int_{0}^{T}\left(\delta u^{T} H_{u}+\delta x^{T}\left(H_{x}+\dot{\lambda}\right)+\delta \lambda^{T}\left(H_{\lambda}-\dot{x}\right) d t-\frac{1}{2} \int_{0}^{T}\left(\delta \lambda^{T} \cdot\right.\right. \\
& \text { - }\left(\delta \dot{x}-H_{\lambda x} \delta x-H_{\lambda u} \delta u\right) d t \tag{10.63}
\end{align*}
$$

For the method of steepest descent developed in Section (8.5), the following relations hold for the nominal trajectory:
(1) $\dot{x}-H_{\lambda}=0$, from (8.30)
(2) $\dot{\lambda}+H_{x}=0, \quad$ from (8.31)
(3) $H_{\lambda \lambda}=0$, from (8.24)
(4) $x(0)=x_{0} \quad$, from (8.33)
(5) $\lambda_{f}+\emptyset_{x f}=0, \quad$ from (8.34)
(6) $\delta \dot{x}-H_{\lambda x} \delta_{x}-H_{\lambda u} \delta_{u}=0$, from (9.25)

Using (1) to (6) in (10.63) provides

$$
\begin{align*}
\delta_{J_{a}}= & -\int_{0}^{T} \delta_{u} T_{H_{u}} d t+\frac{1}{2} \delta_{x_{f}} T_{\phi_{x x f}} \delta_{x_{f}} \\
& -\frac{1}{2} \int_{0}^{T}\left(\delta_{u}{ }^{T} H_{u u} \delta_{u}+2 \delta_{u} T_{H}{ }_{u x} \delta x+\delta x_{x} T_{H_{x x}} \delta_{x}\right) d t \tag{10.64}
\end{align*}
$$

From (8.38) the desired value for $\delta_{u}$ is

$$
\begin{equation*}
\delta_{u}=k \hat{H}_{u} \tag{10.65}
\end{equation*}
$$

The incremental variation $\delta x(t)$ which results from this incremental change $\delta u$, is given by (9.5) to be

$$
\begin{equation*}
\delta \dot{x}=\hat{H}_{\lambda x} \delta x+\hat{H}_{\lambda u} \delta u \tag{10.66}
\end{equation*}
$$

Substituting (10.65) into (10.66) yields

$$
\begin{equation*}
\delta \dot{x}=\hat{H}_{\lambda x} \delta x+\hat{H}_{\lambda u}{ }^{k \hat{H}_{u}} \tag{10.67}
\end{equation*}
$$

Let

$$
\begin{equation*}
\delta x=k z \tag{10.68}
\end{equation*}
$$

and hence, using (10.68) in (10.67) results in

$$
\begin{equation*}
\dot{z}=\hat{H}_{\lambda x} z+\hat{H}_{\lambda u} \hat{H}_{u} \tag{10.69}
\end{equation*}
$$

which is independent of the parameter $k$. The solution to (10.69) for $\delta_{x_{0}}=Z(0)=0$ is given in Section (8.2.2) in terms of the state transition matrix $\Phi$ :

$$
\begin{equation*}
Z(t)=\int_{0}^{t} \hat{\Phi}(t, \alpha) \hat{H}_{\lambda u}(\alpha) \hat{H}_{u}(\alpha) d \alpha \tag{10.70}
\end{equation*}
$$

Using (10.65) and (10.68) in (10.64) provides

$$
\begin{align*}
\delta_{J}= & -k \int_{0}^{T} \hat{H}_{u}^{T} \hat{H}_{u} d t+\frac{k^{2}}{2} z_{f}^{T} \hat{\phi}_{x x f} Z_{f} \\
& -\frac{k^{2}}{2} \int_{0}^{T}\left(\hat{H}_{u}^{T} \hat{H}_{u u} \hat{H}_{u}+2 \hat{H}_{u}^{T} \hat{H}_{u x} Z+z^{T} \hat{H}_{x x} Z\right) d t \tag{10.71}
\end{align*}
$$

For $\delta J_{a}$ to be a minimum with respect to $k, \delta_{J_{a}}^{2}$ must vanish and hence

$$
\begin{equation*}
\delta^{2} J_{a}=-\int_{0}^{T} \hat{H}_{u}^{T} \hat{H}_{u}+k W_{0}=0 \tag{10.72}
\end{equation*}
$$

where

$$
W_{o} \triangleq z_{f}^{T} \hat{\emptyset}_{x x f} z_{f}-\int_{0}^{T}\left(\hat{H}_{u}{ }^{T} \hat{H}_{u u} \hat{H}_{u}+2 \hat{H}_{u}^{T} \hat{H}_{u x} Z+z^{T} \hat{H}_{x x} Z\right) d t
$$

Solving (10.72) for $k_{\text {opt }}$ yields

$$
k_{o p t}=\frac{\int_{0}^{T} \hat{H}_{u}^{T} \hat{H}_{u} d t}{W_{0}}
$$

and hence, from (10.65) and (10.73),

$$
\delta_{u}(t)=\frac{\left.\int_{0}^{T} \hat{H}_{u} T_{H_{u}} \alpha t\right)}{W_{0}} \hat{H}_{u}(t)
$$

is the desired variation for $\delta_{u}$ in the vicinity of the extremum. Using a preselected value for $\delta_{1}{ }^{2}$ in (8.39), the value of $k$ for the first variation approach is

$$
\begin{equation*}
k=\delta_{I} /\left(\int_{0}^{T} \hat{H}_{u} \mathbb{H}_{u}\right)^{\frac{1}{2}} \tag{10.75}
\end{equation*}
$$

The method of steepest descent is carried out as before with $k$ defined by (10.75) until the situation exists where $k_{o p t} \leqslant k$. When this occurs, the second variation approach of (10.74) is used for the remainder of the search to determine the variation $\delta_{u}(t)$. In a manner similar to that of the previous sections, this approach can be extended to cover problems with free final time and additional terminal constraints.
10.5 Conclusions

An algorithm for the numerical solution of optimal control problems has been developed which is based on a combination of the direct and indirect approaches. The method is similar to the indirect method in that trajectories are computed using differential equations and the known and computed iritial values. However, instead of matching end conditions as is done in the classical indirect approach, the augmented performance function $J_{a}$ is considered to be a function of the unknown initial values. The minimum of $J_{a}$ is found by gradient search in the initial condition space based on the first variation. Unlike the indirect approach, convergence does not
depend on a good initial estimate of $\lambda_{0}$. It has been shown that the normalization of the Lagrange multipliers $\lambda_{0}{ }^{*}$, as carried out in the classical approach, is not essential. Thus, instead of a search over the complete $\lambda_{0}$-space, it is sufficient to determine the intersection of the line $\mu \lambda_{0}^{*}$ with the sphere $\hat{\lambda}_{0}^{T} \hat{\lambda}_{0}=$ constant. Also, it has been shown by means of examples that the method is applicable to the case of bounded control, and that it can be applied without computational difficulty to the case where $u$ cannot be determined as an explicit function of $X$ and $x^{\prime}$ In the case of bounded control, some prior knowledge of the sequence of arcs is required. Information of this type can be determined with the aid of the Legendre-Clebsch condition.

A disadvantage of the gradient technique, based on the first variation only, is that the convergence slows down as the optimum is approached. It is desirable, therefore, to use the gradient technique to initiate the search, and then to use a technique with good final convergence properties to complete the search. In this respect, a three stage computing algorithm was developed which is based on a systematic search in the initial condition space of Lagrange multipliers. The first two stages are steepest descent techniques which result in a search over the surface of a sphere, at a constant sweep angle, until the region of the optimum is overstepped. At this point, the third stage comes into effect to provide a rapid final convergence. For this third stage, three algorithms were developed. The first approach is based on a method of matching end points in which the Lagrange multipliers are continuously re-scaled to
provide a minimum error in final transversality. The other two approaches are based on finding the optimal step size for the method of steepest descent used in the second stage. One approach uses a second variation of the augmented performance function and the other uses a curve fitting technique to estimate the optimal step size. It is easily observed that these techniques vary in computational difficulty and in the rate of convergence. For the method of matching end points, some matrix inversion is required which may introduce computational difficulties. For this technique, however, it was demonstrated by examples that the rate of convergence is essentially quadratic. The method of curve fitting, on the other hend, requires no extra equipment when used with the steepest descent approach. However, one extra forward integration is required for each iteration. For this approach the rate of convergence was shown to be very satisfactory. In the second variation approach, no matrix inversion and no extra forward integrations are required, and the rate of convergence appears to range between linear and quadratic. The particular approach to use, therefore, will depend on the problem under study and the size and type of computing facilities available. A common feature of all these techniques is that storage is required at the end pointis only and, as such, these techniques are suitable for use with digital or hybrid computers of limited memory.

## Appendix A

The equations of motion for a missile moving in the earth's atmosphere are (see Figure 2.1, for simplicity, a flat earth and motion in the $x y-p l a n e$ is assumed).

$$
\begin{align*}
& \dot{x}=v \cos \theta  \tag{A-1}\\
& \dot{y}=v \sin \theta  \tag{A-2}\\
& \dot{v}=-g \sin \theta-\frac{D}{m}+\frac{v_{e} u \cos \beta}{m}  \tag{A-3}\\
& \dot{\theta}=-\frac{g \cos \theta}{v}+\frac{L}{m v}+\frac{v_{e} u \sin \beta}{m v}  \tag{A-4}\\
& \dot{m}=-u . \tag{A-5}
\end{align*}
$$

Here $D \triangleq D(y, v, L)$ is the drag and it is assumed that the engine thrust is given by the ideal equation $T=v_{e}$.
Equations (A-1) to (A-5) can be written in the form (2.1) by choosing $\mathrm{x}_{1}=\mathrm{x}, \mathrm{x}_{2}=\mathrm{y}, \mathrm{x}_{3}=\mathrm{v}, \mathrm{x}_{4}=\theta, \mathrm{x}_{5}=\mathrm{m}$ as the state variables and by taking
$G_{0} \triangleq\left[\begin{array}{l}v \cos \theta \\ v \sin \theta \\ -g \sin \theta-\frac{D}{m} \\ \frac{g \cos \theta}{v}+\frac{L}{m v} \\ 0\end{array}\right], \quad G_{I} \triangleq\left[\begin{array}{c}0 \\ 0 \\ \frac{v_{e} \cos \beta}{m} \\ \frac{v_{e} \sin \beta}{m} \\ -1\end{array}\right]$
It follows from (A-6) that

$$
G_{0 x} \triangleq\left[\begin{array}{ccccc}
0 & 0 & \cos \theta & -v \sin \theta & 0 \\
0 & 0 & \sin \theta & v \cos \theta & 0 \\
0 & -\frac{D}{m} & -\frac{D_{v}}{m} & -g \cos \theta & D / m^{2} \\
0 & 0 & (m g \cos \theta-L) / \mathrm{mv}^{2} & g \sin \theta / v & -I / m^{2} v \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\begin{aligned}
& G_{I x} \triangleq\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -v_{e} \sin \beta / \mathrm{mv}^{2} & 0 \\
0 & 0 & 0 & -v_{e} \cos \beta / \mathrm{m}^{2} \\
-v_{e} \sin \beta / \mathrm{m}^{2} \mathrm{v}
\end{array}\right]_{(\mathrm{A}-8)} \\
& G_{O L} \triangleq\left[\begin{array}{c}
0 \\
0 \\
-D_{I} / \mathrm{m} \\
I / \mathrm{mv} \\
0
\end{array}\right], \quad G_{I \beta} \triangleq\left[\begin{array}{c}
0 \\
0 \\
-v_{e} \sin \beta / \mathrm{m} \\
v_{e} \cos \beta / \mathrm{mv} \\
0
\end{array}\right]
\end{aligned}
$$

The Euler-Lagrange equations are

$$
\begin{equation*}
\dot{\lambda}_{I}=0 \tag{A-10}
\end{equation*}
$$

$$
\dot{\lambda}_{2}=\frac{\lambda_{3} D_{y}}{m}
$$

$$
\dot{\lambda}_{3}=-\lambda_{1} \cos \theta-\lambda_{2} \sin \theta+\frac{\lambda_{3}}{m} D_{v}-\lambda_{4} \frac{g}{v^{2}} \cos \theta
$$

$$
+\frac{\lambda_{4}}{m^{2}}\left(L+v_{e} u \sin \beta\right)
$$

$$
\dot{\lambda}_{4}=\lambda_{1} v \sin \theta-\lambda_{2} v \cos \theta+\lambda_{3} g \cos \theta-\lambda_{4} \underset{(\mathrm{~A}-13)}{\stackrel{g}{v} \sin } \theta
$$

$$
\begin{equation*}
\dot{\lambda}_{5}=-\frac{\lambda_{3}}{m^{2}}\left(D-v_{e} u \cos \beta\right)+\frac{\lambda_{5}}{m^{2}}\left(I+v^{u} u \sin \beta\right) \tag{A-14}
\end{equation*}
$$

Equation (A-10) yields

$$
\lambda_{1}=c_{1}
$$

where $c_{1}$ is a constant of integration. Substituting (A-6) and (A-9) into (2.12), (2.13) and (2.14) yields

$$
\begin{align*}
& k_{L}=\frac{1}{m}\left(\lambda_{3} D_{L}-\frac{\lambda_{4}}{v}\right)  \tag{A-16}\\
& k_{\beta}=\frac{v_{e}}{m}\left(\lambda_{3} \sin \beta-\frac{\lambda_{4}}{v} \cos \beta\right)  \tag{A-17}\\
& k_{u}=\frac{v_{e}}{m}\left(\lambda_{3} \cos \beta+\frac{\lambda_{4}}{v} \sin \beta-\frac{m}{v_{e}} \lambda_{5}\right)
\end{align*}
$$

Substituting (A-6) into (2.18) yields

$$
\begin{align*}
& \lambda_{1} v \cos \theta+\lambda_{2} v \sin \theta-\lambda_{3}\left(g \sin \theta+\frac{D}{m}\right) \\
& +\lambda_{4}\left(\frac{I}{m v}-\frac{g}{v} \cos \theta\right)+u k_{u}=c \tag{A-19}
\end{align*}
$$

Evaluating (2.30) with the aid of (A-6), (A-7) and (A-8) yields $\dot{k}_{u}=\frac{v_{e}}{m}\left[(\sin \theta \sin \beta-\cos \theta \cos \beta) \lambda_{1}-(\cos \beta \sin \theta+\sin \beta \cdot\right.$
$\cdot \cos \theta) \lambda_{2}+\left\langle\left(D_{v} \cos \beta+\frac{D}{v_{e}}\right) \frac{1}{m}+\frac{g}{v} \sin \beta \cos \theta\right\rangle \lambda_{3}$
$\left.+\left\langle-\frac{g}{v^{2}} \cos \theta \cos \beta+\frac{D}{m^{2}} \sin \beta+\frac{L}{m^{2}}\left(\cos \beta-\frac{v}{v_{e}}\right)\right\rangle \lambda_{4}\right]$
(A-20)
The transversality condition is

$$
\begin{equation*}
\left[d P+\lambda_{1} d x+\lambda_{2} d y+\lambda_{3} d v+\lambda_{4} d \theta+\lambda_{5} d m-c d t\right]_{t_{0}}^{t_{f}}=0 \tag{A-21}
\end{equation*}
$$

During a variable thrust subarc (2.45) represents the following system of equations

$$
\begin{align*}
k_{L} & =0 \\
k_{\beta} & =0 \\
\lambda_{I} & =c_{I} \\
G_{O}{ }^{T} \lambda & =c \tag{A-22}
\end{align*}
$$

$$
\begin{aligned}
& k_{u}=0 \\
& \dot{k}_{u}=0
\end{aligned}
$$

It follows that
$A=\left[\begin{array}{ccccc}0 & 0 & \frac{D_{L}}{m} & -\frac{l}{m v} & 0 \\ 0 & 0 & \frac{v_{e}}{m} \sin \beta & \frac{-v e}{m v} \cos \beta & 0 \\ 1 & 0 & 0 & 0 & 0 \\ v \cos \theta & v \sin \theta & -g \sin \theta-\frac{D}{m} & \frac{L}{m v}-\frac{g}{v} \cos \theta & 0 \\ 0 & 0 & \frac{v_{1}}{m} \cos \beta & \frac{v_{e}}{m v} \sin \beta & -1 \\ a_{51} & a_{52} & a_{53} & a_{54} & 0\end{array}\right]$
and

$$
b=\left[\begin{array}{c}
0  \tag{A-24}\\
0 \\
c_{1} \\
c \\
0 \\
0
\end{array}\right]
$$

where

$$
\begin{aligned}
& a_{51} \triangleq \sin \theta \sin \beta-\cos \theta \cos \beta \\
& a_{52} \triangleq-(\cos \beta \sin \theta+\sin \beta \cos \theta) \\
& a_{53} \triangleq \frac{I}{m}\left(D_{v} \cos \beta+\frac{D}{\mathrm{~V} \cdot e}\right)+\frac{g}{v} \sin \beta \cos \theta \\
& a_{54} \triangleq-\frac{g}{v^{2}} \cos \theta \cos \beta+\frac{D}{\mathrm{mv}^{2}} \sin \beta+\frac{L}{m^{2}}\left(\cos \beta-\frac{\mathrm{V}}{\mathrm{~V}_{e}}\right) \\
& \text { Appendix B }
\end{aligned}
$$

In the case of vertical flight, the missile is con-
strained so that $\theta=\pi / 2, \beta=0, \mathrm{~L}=0$ and the system dynamics (A-1) to (A-5) simplify to the form

$$
\begin{align*}
& \dot{y}=v  \tag{B-1}\\
& \dot{\mathrm{v}}=-g-\frac{D}{m}+\frac{v e^{u}}{m}  \tag{B-2}\\
& \dot{m}=-u \tag{B-3}
\end{align*}
$$

Using a standard drag function of the form

$$
\begin{equation*}
D=K_{a}^{2} e^{-a y} \tag{B-4}
\end{equation*}
$$

where $K_{a}$ and a are constants, results in the following EulerLagrange equations

$$
\begin{align*}
& \dot{\lambda}_{2}=-\frac{a \lambda_{3} D}{m}  \tag{B-5}\\
& \dot{\lambda}_{3}=-\lambda_{2}+\frac{2 \lambda_{3} D}{m v}  \tag{B-6}\\
& \dot{\lambda}_{5}=-\frac{\lambda_{3}}{m^{2}}\left(D-v e^{u}\right) \tag{B-7}
\end{align*}
$$

The switching function and its time derivative are given by

$$
\begin{align*}
& k_{u}=\frac{\lambda_{3} v_{e}}{m}-\lambda_{5}  \tag{B-8}\\
& \dot{k}_{u}=-\frac{\lambda_{2} v_{e}}{m}+\frac{\lambda_{3}}{m^{2}} D\left(1+\frac{2 v_{e}}{v}\right) \tag{B-9}
\end{align*}
$$

The first integral is

$$
\lambda_{2} v-\lambda_{3}\left(g+\frac{D}{m}\right)+u k_{u}=c
$$

The transversality condition is

$$
\begin{equation*}
\left[d P+\lambda_{2} d y+\lambda_{3} d v+\lambda_{5} d m-c d t\right]_{t_{0}}^{t_{f}}=0 \tag{B-lI}
\end{equation*}
$$

The matrix $A$ and the vector $b$ are given by (see (2.45) and (2.46))

$$
\begin{align*}
& A=\left[\begin{array}{ccc}
v & -g-\frac{D}{m} & 0 \\
0 & \frac{v_{e}}{m} & -1 \\
\frac{-v e}{m} & \frac{D}{m^{2}}\left(1+\frac{2 v_{e}}{v}\right) & 0
\end{array}\right]  \tag{B-12}\\
& b=\left[\begin{array}{l}
c \\
0 \\
0
\end{array}\right] \tag{B-I3}
\end{align*}
$$

In Chapter 3, use was made of the fact that $\lambda_{3} \supseteq 0$ in deriving the sequence diagram. To prove this assertion, note that during a variable thrust subarc ( $k_{u}=0$ ) or during a coasting subarc $(u=0),(B-10)$ reduces to

$$
\begin{equation*}
\lambda_{2} v-\lambda_{3}\left(g+\frac{D}{m}\right)=0 \tag{B-14}
\end{equation*}
$$

after substituting the condition (3.2). It follows from (3.1) and (B-14) that at the final time

$$
\begin{equation*}
\lambda_{3 f}=0 \tag{B-15}
\end{equation*}
$$

If $\lambda_{3}=0$ at any instant during the variable thrust or coasting subarc it follows from (B-14) that $\lambda_{2}=0$, and from ( $B-5$ ) and (B-6) it can then be concluded that $\lambda_{3}=0, \lambda_{2}=0$ everywhere, violating the terminal condition for $\lambda_{2 f}$ (see 3.2). Consider now a maximum thrust subarc. Substituting $c=0$ and ( $B-10$ ) into (B-6) yields

$$
\begin{equation*}
\dot{\lambda}_{3}=u k_{u}+\frac{\lambda_{3}}{m v}(D-m g) \tag{B-16}
\end{equation*}
$$

If $\lambda_{3}=0$ at any instant $t_{\text {s }}$ during a maximum thrust subarc, it follows from $k_{u}>0$ and $(B-16)$ that $\dot{\lambda}_{3}>0$ and consequently $\lambda_{3}>0$ for $t_{s}<t \leq t_{f}$. From (B-5) it is then seen that

$$
\begin{equation*}
\dot{\lambda}_{2}<0 \tag{B-17}
\end{equation*}
$$

for $t_{s}<t \leqslant t_{f}$. Evaluating (B-10) at $t=t_{s}$ yields

$$
\begin{equation*}
\lambda_{2}=-\frac{u k_{u}}{v}<0 \tag{B-18}
\end{equation*}
$$

Conditions (B-17) and (B-18) require that $\lambda_{2}<0$ for $t_{s}<t_{f}$, which violates the terminal condition $\lambda_{2 f}=1$. Thus $\lambda_{3} \neq 0$ for $t_{0}<t<t_{f}$. To prove that $\lambda_{3}>0$ it is sufficient to note that if $(B-4)$ is used to evaluate $(B-6)$ at $t=t_{f}$, it yields

$$
\begin{equation*}
\dot{\lambda}_{3 f}=-1 \tag{B-19}
\end{equation*}
$$

and thus the final value of $\lambda_{3}$ is approached through positive values, since $\lambda_{3 f}=0$ (see $\left.B-15\right)$.

From (B-2) it follows that $\dot{v}>0$ during an impulsive thrust subarc. The final subarc, therefore, cannot be an impulsive thrust subarc. From (3.1), (3.2) and (3.6) it follows that

$$
\begin{equation*}
\dot{k}_{u}\left(t_{f}\right)=-\frac{v_{e}}{m_{f}}<0 \tag{B-20}
\end{equation*}
$$

which violates the condition (2.42) for a variable thrust subarc. Thus the final arc is a coasting subarc.

## Appendix C

The system dynamics for flight with the control constraints $I=0, \beta=0$ are

$$
\begin{align*}
& \dot{x}=v \sin \theta  \tag{C-1}\\
& \dot{y}=v \cos \theta  \tag{c-2}\\
& \dot{y}=-g \sin \theta-\frac{D}{m}+\frac{v_{e} u}{m} \tag{c-3}
\end{align*}
$$

$$
\begin{align*}
& \dot{\theta}=-\frac{g}{v} \cos \theta  \tag{c-4}\\
& \dot{m}=-u \tag{C-5}
\end{align*}
$$

The Euler-Lagrange equations are

$$
\begin{align*}
& \dot{\lambda}_{1}=0  \tag{c-6}\\
& \dot{\lambda}_{2}=-\frac{a \lambda_{3} D}{m}  \tag{C-7}\\
& \dot{\lambda}_{3}=-\lambda_{I} \cos \theta-\lambda_{2} \sin \theta+\frac{\lambda_{3} 2 D}{m v}-\lambda_{4} \frac{g_{v}}{2} \cos \theta  \tag{c-8}\\
& \dot{\lambda}_{4}=\lambda_{1} v \sin \theta-\lambda_{2} v \cos \theta+\lambda_{3} g \cos \theta-\lambda_{4} \frac{g}{v} \sin \theta  \tag{c-9}\\
& (c-9)  \tag{c-10}\\
& \dot{\lambda}_{5}=-\frac{\lambda_{3}}{m^{2}}\left(D-v e^{u}\right)
\end{align*}
$$

The switching function and its time derivatives are

$$
k_{u}=\frac{\lambda_{3} v_{e}}{m}-\lambda_{5}
$$

$$
{\dot{k_{u}}}_{u}=\frac{v_{e}}{m}\left[-\lambda_{1} \cos \theta-\lambda_{2} \sin \theta+\frac{\lambda_{3} D}{m v}\left(2+\frac{v}{v_{e}}\right)\right.
$$

$$
\begin{equation*}
\left.-\lambda_{4} \frac{g}{v^{2}} \cos \theta\right] \tag{C-12}
\end{equation*}
$$

The first integral is
$\lambda_{1} v \cos \theta+\lambda_{2} v \sin \theta-\lambda_{3}\left(g \sin \theta+\frac{D}{m}\right)-\frac{\lambda_{4}}{v} g \cos \theta+u k_{u}=c$

The transversality condition is

$$
\begin{equation*}
\left[d P+\lambda_{1} d x+\lambda_{2} d y+\lambda_{3} d v+\lambda_{4} d \theta+\lambda_{5} d m-c d t\right]_{t_{0}}^{t_{f}}=0 \tag{C-14}
\end{equation*}
$$

The matrix $A$ and the vector $b$ are given by (see (2.45) and (2.46))

$$
\begin{gather*}
A=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
v \cos \theta & v \sin \theta & -g \sin \theta-\frac{D}{m} & -\frac{g}{v} \cos \theta & 0 \\
0 & 0 & \frac{v_{e}}{m} & 0 & -1 \\
-\cos \theta & -\sin \theta & \frac{D}{m v}\left(2+\frac{v}{v_{e}}\right) & -\frac{g}{v^{2}} \cos \theta & 0
\end{array}\right]_{(c-15)}^{180 .}  \tag{c-15}\\
b=\left[\begin{array}{c}
c_{1} \\
c \\
0 \\
0
\end{array}\right] \tag{c-16}
\end{gather*}
$$

where $\lambda_{I}=c_{1}=$ constant follows from ( $c-6$ ).
A proof will now be given of the fact that $\lambda_{3} \neq 0$, on a variable thrust subarc. During variable thrust (4.6) is valid. Substituting (4.6) into (C-IO) yields

$$
\begin{equation*}
\dot{\lambda}_{5}=-\frac{\lambda_{5}}{m v_{e}}\left(D-v_{e}^{u)}\right. \tag{c-17}
\end{equation*}
$$

If $\lambda_{3}$ becomes zero at any instant $t_{1}$ on a variable thrust subarc, it follows from (4.6) and ( $(0-17)$ that $\lambda_{3}$ and $\lambda_{5}$ then remain zero for the remaining time interval $t_{1} \leq t \leq t_{2}$ of the variable thrust subarc. From ( $\mathrm{C}-8$ ) and ( $(\mathrm{C}-13)$ it follows that since $\lambda_{3}$ and $k_{u}$ are zero

$$
\begin{align*}
& \lambda_{1} v \cos \theta+\lambda_{2} v \sin \theta-\lambda_{4} \frac{g}{v} \cos \theta=0  \tag{c-18}\\
& \lambda_{1} \cos \theta+\lambda_{2} \sin \theta+\lambda_{4} \frac{g}{v^{2}} \cos \theta=0
\end{align*}
$$

for $t_{1} \leq t \leq t_{2}$. Hence $\lambda_{4}=0$ and

$$
\begin{equation*}
\lambda_{1} \cos \theta+\lambda_{2} \sin \theta=0 \tag{c-19}
\end{equation*}
$$

From (c-6) and (c-7) it is seen that $\lambda_{1}=$ const., $\lambda_{2}=$ const.

This implies that $\theta=$ const. for $t_{1} \leq t \leq t_{2}$ which contradicts (c-4). Hence $\lambda_{3} \neq 0$ for the variable thrust subarc.

To determine an expression for $k_{u}$, (4.4) is first
written in the form

$$
\begin{equation*}
\dot{k}_{u}=\frac{v_{e}}{m v} f \tag{C-20}
\end{equation*}
$$

where

$$
\begin{equation*}
f \triangleq u k_{u}-\frac{\lambda_{3} f_{s}}{m}-\lambda_{4} \frac{2 g}{v} \cos \theta \tag{C-21}
\end{equation*}
$$

Substituting ( $\mathrm{C}-13$ ) with $\mathrm{c}=0$ into ( $\mathrm{C}-8$ ) and using ( $\mathrm{C}-20$ ) yields

$$
\begin{equation*}
\dot{\lambda}_{3}=\frac{m}{v_{e}} \dot{k}_{u}-\frac{\lambda_{3} D}{m v_{e}} \tag{c-22}
\end{equation*}
$$

Substituting (C-13) with $c=0$ into (c-9) yields
$\dot{\lambda}_{4}=\frac{I}{\sin \theta}\left(\lambda_{I} v-\lambda_{3} \frac{D \cos \theta}{m}-\lambda_{4} \frac{g}{v}+u k_{u} \cos \theta\right)$
Differentiating (4.5) with respect to time and using the system equations (C-1) to (C-5) yields

$$
\begin{equation*}
\dot{f}_{s}=u N_{1}+N_{1} \tag{c-24}
\end{equation*}
$$

where

$$
\begin{align*}
& M_{l} \triangleq-g \sin \theta-\frac{v_{e} D}{m v}\left(2+\frac{3 v}{v_{e}}\right)  \tag{c-26}\\
& N_{l} \triangleq-\frac{m^{2}}{}{ }^{2} \cos ^{2} \theta  \tag{c-26}\\
& v
\end{align*} \frac{D}{v}\left(g \sin \theta+\frac{D}{m}\right)\left(2+\frac{3 v}{v_{e}}\right)
$$

Differentiating (C-2l) with respect to time yields

$$
\begin{align*}
\dot{f}= & \dot{u}_{u}+u \dot{k}_{u}-\dot{\lambda}_{3} \frac{f_{s}}{m}-\lambda_{3} \frac{\dot{f}_{s}}{m}-\dot{\lambda}_{4} \frac{2 g \cos \theta}{v}+\lambda_{4} \frac{2 g}{v} \sin \theta \dot{\dot{\theta}} \\
& +\lambda_{4} \frac{2 g \dot{v}}{v^{2}} \cos \theta \tag{C-27}
\end{align*}
$$

The term $\dot{u}_{u}$ in (c-27) is always zero since $\dot{u}=0$ when $u$ is at its bounds of $k_{u}=0$ when $u$ is variable. Eliminating the time derivatives in ( $\mathrm{C}-27$ ) with the aid of $(\mathrm{C}-3),(\mathrm{C}-4),(\mathrm{C}-22)$, (C-23) and (C-24) yields

$$
\begin{align*}
\dot{f}= & u\left[\dot{k}_{u}+\lambda_{3} M_{2}+\lambda_{4} \frac{2 g v_{e}}{m} \cos \theta-\frac{2 k u^{2}}{v} \frac{\cos ^{2} \theta}{\sin \theta^{1}}-\frac{\dot{k}_{u}}{v_{e}} f_{s}\right. \\
& \left.+\lambda_{3} N_{2}+\lambda_{4} N_{3}-\lambda_{1} 2 g \cot \theta\right] \tag{c-28}
\end{align*}
$$

where

$$
\begin{align*}
N_{2} \triangleq & \frac{D}{m^{2}}\left(4+\frac{2 v_{e}}{v}+\frac{v}{v_{e}}\right)  \tag{c-29}\\
N_{2} \triangleq & \frac{f_{s} D}{m^{2} v_{e}}+\frac{2 D g \cos ^{2} \theta}{m v \sin \theta}+\frac{g^{2} \cos ^{2} \theta}{v}-\frac{D}{m v}\left(g \sin \theta+\frac{D}{m}\right) \\
& \cdot\left(2+\frac{3 v}{v_{e}}\right)-\frac{a D v \sin \theta}{m}\left(1+\frac{v}{v_{e}}\right)  \tag{c-30}\\
N_{3} \triangleq & \frac{2 g \cos \theta}{v^{2}}\left[g \frac{\cos ^{2} \theta-\sin ^{2} \theta}{\sin \theta}-\frac{D}{m}\right]
\end{align*}
$$

Evaluating $\ddot{\mathrm{k}}_{\mathrm{u}}$ when $\dot{\mathrm{k}}_{\mathrm{u}}$ equals zero yields

$$
\begin{equation*}
\ddot{k}_{u}=\frac{v_{e}}{m v} \dot{f} \tag{c-32}
\end{equation*}
$$

Substituting ( $C-28$ ) into ( $C-32$ ), using (4.5) and taking $\dot{k}_{u}=0$ yields

$$
\begin{equation*}
\ddot{k}_{u}=\frac{v_{e}}{m v}\left[\frac{u^{2} k_{u} v_{e}}{m v}-u k_{u}\left(\frac{q}{v \sin \theta}+\frac{D}{m v}\right)+\lambda_{3} u M-\lambda_{3} N-\lambda_{1} 2 g \cot \theta\right] \tag{c-33}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbb{M} \triangleq \frac{D}{m^{2}}\left(5+\frac{3 v_{e}}{v}+\frac{v}{v_{e}}\right)-\frac{v_{e}}{m v} \sin \theta \tag{c-34}
\end{equation*}
$$

$$
\begin{align*}
N \triangleq & \Delta \frac{D \sin \theta}{m v}\left[2+\frac{3 v}{v_{e}}-\left(3+\frac{v}{v_{e}}\right) \cot ^{2} \theta+\frac{D^{2}}{m^{2}} \cdot\right. \\
& \left.\cdot\left(3+\frac{5 v}{v_{e}}+\frac{v^{2}}{v_{e}^{2}}\right)-\frac{g^{2} \sin ^{2} \theta}{v}+\frac{a D v \sin \theta}{m}\left(1+\frac{v}{v_{e}}\right)\right] \tag{C-35}
\end{align*}
$$

## Appendix D

The system dynamics for flight with the control constraint $I=0$ are

$$
\begin{align*}
& \dot{x}=v \sin \theta  \tag{D-1}\\
& \dot{y}=v \cos \theta  \tag{D-2}\\
& \dot{v}=-g \sin \theta-\frac{D}{m}+\frac{v_{e} u}{m} \cos \beta  \tag{D-3}\\
& \dot{\theta}=-\frac{g}{v} \cos \theta+\frac{v_{e} u}{m v} \sin \beta  \tag{D-4}\\
& \dot{m}=-u \tag{D-5}
\end{align*}
$$

The Euler-Lagrange equations are

$$
\begin{align*}
& \dot{\lambda}_{1}=0  \tag{D-6}\\
& \dot{\lambda}_{2}=-\lambda_{3} \frac{\mathrm{aD}}{\mathrm{~m}} \tag{D-7}
\end{align*}
$$

$\dot{\lambda}_{3}=-\lambda_{1} \cos \theta-\lambda_{2} \sin \theta+2 \lambda_{3} \frac{D}{m}-\frac{\lambda_{4}}{v^{2}}\left(g \cos \theta-\frac{v_{e} e^{u}}{m} \sin \beta\right)$
$\dot{\lambda}_{4}=\lambda_{1} v \sin \theta-\lambda_{2} v \cos \theta+\lambda_{3} g \cos \theta-\lambda_{4} \stackrel{g}{\mathrm{v}} \sin \theta$
$\dot{\lambda}_{5}=-\frac{\lambda_{3}}{m^{2}}\left(D-v_{e}^{u} \cos \beta\right)+\lambda_{4} \frac{v_{e} u}{m^{2} u} \sin \beta$
The equations for the switching function, its time derivative and $\mathrm{k}_{\beta}$ are

$$
\begin{align*}
k_{u}= & \frac{v_{e}}{m}\left(\lambda_{3} \cos \beta+\frac{\lambda_{4}}{v} \sin \beta-\frac{m}{v_{e}} \lambda_{5}\right) \\
\dot{k}_{u}= & \frac{v_{e}}{m}\left[(\sin \theta \sin \beta-\cos \theta \cos \beta) \lambda_{1}-(\cos \beta \sin \theta\right. \\
& +\sin \beta \cos \theta) \lambda_{2}+\left\langle\frac{D}{m v}\left(2 \cos \beta+\frac{v}{v_{e}}\right)+\frac{g}{v} \sin \beta .\right. \\
& \left.\cdot \cos \theta\rangle \lambda_{3}+\left\langle-\frac{g}{v^{2}} \cos \theta \cos \beta+\frac{D}{m^{2}} \sin \beta\right\rangle \lambda_{4}\right] \tag{D-12}
\end{align*}
$$

$$
\begin{equation*}
\mathrm{k}_{\beta}=\frac{\mathrm{v}_{\mathrm{e}}}{\mathrm{~m}}\left(\lambda_{3} \sin \beta-\frac{\lambda_{4}}{\mathrm{v}} \cos \beta\right) \tag{D-13}
\end{equation*}
$$

The first integral is
$\lambda_{1} v \sin \theta+\lambda_{2} v \cos \theta-\lambda_{3}\left(g \sin \theta+\frac{D}{m}\right)-\lambda_{4} \underset{\mathrm{~V}}{\mathrm{~g}} \cos \theta+u \mathrm{k}_{\mathrm{u}}=\mathrm{c}$
and the transversality condition is

$$
\begin{equation*}
\left[d P+\lambda_{1} d x+\lambda_{2} d y+\lambda_{3} d v+\lambda_{4} d \theta+\lambda_{5} d m-c d t\right]_{t_{0}}^{t_{f}}=0 \tag{D-15}
\end{equation*}
$$

The matrix $A$ and the vector $b$ are given by (see (2.45) and (2.46))
$A=\left[\begin{array}{cc}0 & 0 \\ 1 & 0 \\ v \cos \theta & v \sin \theta \\ 0 & 0 \\ a_{51} & a_{52}\end{array}\right.$
$\left.\begin{array}{ccc}\frac{v_{e}}{m} \sin \beta & -\frac{v_{e}}{m v} \cos \beta & 0 \\ 0 & 0 & 0 \\ -g \sin \theta-\frac{D}{m} & -\frac{g}{v} \cos \theta & 0 \\ \frac{v_{e}}{m} \cos \beta & \frac{v_{e}}{m v} \sin \beta & -1 \\ a_{53} & a_{54} & 0\end{array}\right]$
where

$$
a_{51} \triangleq \sin \theta \sin \beta-\cos \theta \cos \beta
$$

$$
\begin{align*}
& a_{52} \triangleq-(\cos \beta \sin \theta+\sin \beta \cos ) \\
& a_{53} \triangleq \frac{D}{m v}\left(2 \cos \beta+\frac{v}{v_{e}}\right)+\frac{g}{v_{e}} \sin \beta \cos \theta  \tag{D-17}\\
& a_{54} \triangleq-\frac{g}{v^{2}} \cos \theta \cos \beta+\frac{D}{m v^{2}} \sin \beta
\end{align*}
$$

and

$$
b=\left[\begin{array}{c}
0  \tag{D-18}\\
c_{1} \\
c \\
0 \\
0
\end{array}\right]
$$

where $\lambda_{1}=c_{1}=$ constant.
For zero-lift flight, (A-6) to (A-9) take the forms
$G_{0}=\left[\begin{array}{c}v \cos \theta \\ v \sin \theta \\ -g \sin \theta-\frac{D}{m} \\ -\frac{g}{v} \cos \theta \\ 0\end{array}\right] \quad, \quad G_{1}=\left[\begin{array}{c}0 \\ 0 \\ \frac{v_{e}}{m} \cos \beta \\ \frac{v_{e}}{m} \sin \beta \\ -1\end{array}\right]$
$G_{O x}=\left[\begin{array}{ccccc}0 & 0 & \cos \theta & -v \sin \theta & 0 \\ 0 & 0 & \sin \theta & -v \cos \theta & 0 \\ 0 & \frac{a D}{m} & -\frac{2 D}{m v} & -g \cos \theta & \frac{D}{m^{2}} \\ 0 & 0 & \frac{g}{v^{2}} \cos \theta & \frac{g}{v} \sin \theta & 0 \\ 0 & 0 & 0 & 0 & 0\end{array}\right]$
$G_{l x}=\left[\begin{array}{ccccc}0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{v_{e}}{m^{2}} \cos \beta \\ 0 & 0 & -\frac{v_{e}}{m v^{2}} \sin \beta & 0 & -\frac{\mathrm{v}_{e}}{m^{2}} \sin \beta \\ 0 & 0 & 0 & 0 & 0\end{array}\right]$

$$
\begin{equation*}
G_{O L}=0 \tag{D-22}
\end{equation*}
$$

Differentiating ( $D-21$ ) with respect to $\beta$ yields

$$
G_{1 \beta X}=\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{v_{e}}{m^{2}} \sin \beta \\
0 & 0 & -\frac{v_{e}}{m v^{2}} \cos \beta & 0 & -\frac{v_{e}}{m v^{2}} \cos \beta \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Differentiating ( $D-23$ ) with respect to $\beta$ yields

$$
G_{O \beta \beta}=\left[\begin{array}{c}
0 \\
0 \\
-\frac{v_{e}}{m} \cos \beta \\
-\frac{v_{e}}{m} \sin \beta \\
0
\end{array}\right]
$$

For the case where $u_{\text {max. }}=\infty$, it is possible to derive a control law for $\beta$ for the maximum (impulsive) thrust subarc. Equations ( $D-3$ ), ( $D-4$ ), ( $D-8$ ), ( $D-9$ ) and ( $D-10$ ) for
$u \rightarrow \infty$, take the form

$$
\begin{align*}
& \dot{\mathrm{v}} \cong-\frac{v_{e} \dot{m}^{m}}{m} \cos \beta  \tag{D-26}\\
& \dot{\theta} \cong-\frac{v_{e} \dot{m}^{m}}{m v} \sin \beta  \tag{D-27}\\
& \dot{\lambda}_{3} \cong-\frac{\lambda_{4} v_{e} \dot{m}^{2}}{m v^{2}} \sin \beta  \tag{D-28}\\
& \dot{\lambda}_{5} \cong-\frac{\lambda_{3} v_{e} u}{m^{2}} \cos \beta \tag{D-29}
\end{align*}
$$

The remaining state variables and Lagrange multipliers do not change during the infinitesimally small interval of time in which the impulsive thrust occurs. Thus

$$
\begin{equation*}
\lambda_{4}=\text { const. } \tag{D-30}
\end{equation*}
$$

during the impulsive thrust subarc. It follows from (2.9)
and (2.13) that $k_{\beta}=0$ and hence

$$
\begin{equation*}
\tan \beta=\frac{\lambda_{4}}{v \lambda_{3}} \tag{D-3I}
\end{equation*}
$$

Differentiating (D-3I) with respect to time and noting (D-28) and (D-30) yields

$$
\begin{equation*}
\dot{\beta} \cong \frac{\mathrm{v}^{\dot{m}} \sin \beta^{2}}{\mathrm{mv}} \tag{D-32}
\end{equation*}
$$

Adding ( $\mathrm{B}-27$ ) and (D-32) yields

$$
\begin{equation*}
\dot{\beta}+\dot{\theta} \cong 0 \tag{D-33}
\end{equation*}
$$

Integrating (D-33) yields

$$
\begin{equation*}
\beta+\theta \cong \beta_{0}+\theta_{0}=\text { const. } \tag{D-34}
\end{equation*}
$$

Substituting (D-32) into (D-26) yields

$$
\begin{align*}
& \dot{\mathrm{v}} \cong-\mathrm{v} \beta \frac{\cos \beta}{\sin \beta}  \tag{D-35}\\
& \text { Integrating }(\mathrm{D}-35) \text { yields } \\
& \mathrm{c}_{2} \triangleq \mathrm{v}_{0} \sin \beta_{0} \cong \mathrm{v} \sin \beta \tag{D-36}
\end{align*}
$$

where $c_{2}$ is an integration constant. Hence

$$
\begin{equation*}
\beta \cong \operatorname{arc} \sin \left[\frac{\mathrm{v}_{0}}{\mathrm{v}} \sin \beta_{0}\right] \tag{D-37}
\end{equation*}
$$

Equation ( $D-37$ ) gives the control law for $\beta$ during impulsive boosting。 Substituting (D-36) into (D-32) yields

$$
\begin{equation*}
\dot{\mathrm{v}}=-\frac{\mathrm{v}_{e} \dot{m}}{m}\left(1-\frac{c_{2}^{2}}{v^{2}}\right)^{\frac{1}{2}} \tag{D-38}
\end{equation*}
$$

Separating the variables in ( $D-38$ ) and integrating yields

$$
\begin{equation*}
\left(c_{2}^{2}-v^{2}\right)^{\frac{1}{2}}-\left(c_{2}^{2}-v_{0}^{2}\right)^{\frac{1}{2}}=-v_{e} \ln \frac{m}{m_{0}} \tag{D-39}
\end{equation*}
$$

Substituting (D-36) into (D-39) and solving for m yields

$$
\begin{equation*}
m=m_{0} \exp \left[\frac{\mathrm{v}_{0} \cos \beta_{0}-\mathrm{v} \cos \beta}{\mathrm{v}_{\mathrm{e}}}\right] \tag{D-40}
\end{equation*}
$$

Equations (D-37) and (D-40) hold during the impulsive boosting
subarc.

## Appendix E

After the $n+$ l-st step in the ordedmensional search procedure, assume a condition exists of the form

$$
\begin{equation*}
P\left(\alpha_{k}^{n-1}\right)<P\left(\alpha_{k}^{n}\right)>P\left(\alpha_{k}^{n+1}\right) \tag{E-1}
\end{equation*}
$$

and, hence, the optimal value of $\alpha_{k}$ is located in the region (see Figure 4.10)

$$
\begin{equation*}
\alpha_{k}^{n-1}<\alpha_{k o p t}<\alpha_{k}^{n+1} \tag{E-2}
\end{equation*}
$$

To determine $\alpha_{k o p t}$ approximately, the region in the vicinity of the optimum is represented by a parabola of the form

$$
\begin{equation*}
P=a+b \alpha_{k}+c \alpha_{k}^{2} \tag{E-3}
\end{equation*}
$$

Using the three coordinates $\left(P_{n-1}, \alpha_{k}{ }^{n-1}\right),\left(P_{n}, \alpha_{k}^{n}\right)$, and $\left(P_{n+1}, \alpha_{k}^{n+1}\right)$ in (E-3) yields

$$
\left[\begin{array}{l}
P_{n+1}  \tag{E-4}\\
P_{n} \\
P_{n-1}
\end{array}\right]=\left[\begin{array}{ccc}
1 & \alpha_{k}^{n+1} & \left(\alpha_{k}^{n+1}\right)^{2} \\
1 & \alpha_{k}^{n} & \left(\alpha_{k}^{n}\right)^{2} \\
1 & \alpha_{k}^{n-1} & \left(\alpha_{k}^{n-1}\right)^{2}
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]
$$

from which the values of $a, b$, and $c$ can be established. From ordinary calculus $\left(\frac{d P}{d \alpha_{k}}=0\right)$, the extremum of $(E-3)$ is located at

$$
\alpha_{k o p t}=-\frac{b}{2 c}
$$

- Substituting for $b$ and $c$ from ( $E-4$ ), and noting that

$$
\begin{align*}
& \alpha_{k}^{n-1}=\alpha_{k}^{n}-\Delta \alpha_{k} \\
& \alpha_{k}^{n+1}=\alpha_{k}^{n}+\Delta \alpha_{k} \tag{E-5}
\end{align*}
$$

yields

$$
\begin{equation*}
\alpha_{k o p t}=\alpha_{k}+\frac{\Delta \alpha_{k}}{2} \frac{\left(P_{n-1}-P_{n+1}\right)}{\left(P_{n+1}-P_{n}\right)+\left(P_{n-1}-P_{n}\right)} \tag{E-6}
\end{equation*}
$$

## Appendix F

The optimal value of $\delta \lambda_{0}$ is given in (10.34) to be

$$
\begin{equation*}
\delta \lambda_{0}=a \delta \lambda_{00}+k \delta \lambda_{01} \tag{F-1}
\end{equation*}
$$

where

$$
\begin{align*}
& \delta \lambda_{\circ \circ}=-z_{g}\left(z_{g}^{T} z_{g}\right)^{-1} \hat{g}  \tag{F-2}\\
& \delta \lambda_{\circ 1}=z_{g}\left(z_{g}{ }^{T} z_{g}\right)^{-1} z_{g}{ }^{T} z_{\phi}-z_{\emptyset} \tag{F-3}
\end{align*}
$$

From (9.43), the variation $\delta_{g}$ for the neighbouring trajectory is

$$
\begin{equation*}
\delta_{g}=z_{g}^{T} \delta_{0} \tag{F-4}
\end{equation*}
$$

It is desired in this appendix to determine the effect of $\delta \lambda_{00}$ and $\delta \lambda_{01}$ on $\delta g$. Using ( $\mathrm{F}-2$ ) in ( $\mathrm{F}-4$ ) yields

$$
\begin{equation*}
\delta_{g}=z_{g}^{T}\left(-z_{g}\left(z_{g}^{T} Z_{g}\right)^{-1} \hat{g}\right)=-\hat{g} \tag{F-5}
\end{equation*}
$$

Hence, by ( $\mathrm{F}-5$ ) it can be concluded that $\delta \lambda_{00}$ is the component of $\delta \lambda_{\mathrm{o}}$ concerned with satisfying the desired end conditions. Using ( $F-3$ ) in ( $F-4$ ) yields

$$
\begin{equation*}
\delta_{g}=z_{g}^{T}\left(z_{g}\left(z_{g}^{T} z_{g}\right)^{-1} z_{g}^{T} z_{\phi}-z_{\phi}\right)=0 \tag{F-6}
\end{equation*}
$$

Therefore, by ( $\mathrm{F}-6$ ), it can be concluded that $\delta \lambda_{0}$ ol attempts to minimize the system performance without affecting the end conditions set by $\delta \lambda_{00}$.

## Appendix G

To find $\nu$, substitute (9.39) into (9.43). This yields

$$
\begin{equation*}
a \hat{g}=k Z_{g}^{T}\left(Z_{\phi}+Z_{g} \nu\right) \tag{G-I}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\nu=\left(Z_{g}^{T} Z_{g}\right)^{-l}\left(\frac{a}{k} \hat{g}-Z_{g}^{T} Z_{\phi}\right) \tag{G-2}
\end{equation*}
$$

To find $k$, substitute (9.39) into (9.44). Noting that
$\left[\left(Z_{g}^{T} Z_{g}\right)^{-1}\right]^{T}=\left(Z_{g}^{T} Z_{g}\right)^{-1}$, and using (G-1) and (G-2) yields

$$
\begin{equation*}
k=\sqrt{\frac{\delta_{1}^{2}-a^{2} \hat{g}^{T}\left(z_{g}^{T} z_{g}\right)^{-1} \hat{g}^{T}}{z_{\phi}^{T} z_{\phi}-z_{\phi}^{T} z_{g}\left(z_{g}^{T} z_{g}\right)^{-1} z_{g}^{T} z_{\emptyset}}} \tag{G-3}
\end{equation*}
$$

The result (9.45) is obtained by substituting (G-3) and (G-2) into (9.39).

## Appendix H

For the first variation approach, the differential
equations used are

$$
\begin{align*}
& \dot{x}=f(x, u), \quad \text { from (8.30) }  \tag{H-1}\\
& \dot{\lambda}=f_{x}^{T} \lambda, \quad \text { from }(8.31)  \tag{H-2}\\
& {\left[\begin{array}{l}
\delta_{\dot{\circ}}^{\circ} \\
\delta_{\lambda}^{\circ}
\end{array}\right]=c_{0}(t)\left[\begin{array}{l}
\delta x \\
\delta \lambda
\end{array}\right]} \tag{H-3}
\end{align*}
$$

The solution of $(\mathrm{H}-3)$ is given by (9.3) and (9.4) to be

$$
\begin{align*}
& \delta x(t)=\Phi_{12}(t, 0) \delta \lambda_{0}  \tag{H-4}\\
& \delta \lambda(t)=\Phi_{22}(t, 0) \delta \lambda_{0} \tag{H-5}
\end{align*}
$$

It is desired in this appendix to determine the effect of multiplying all Lagrange multipliers by a nonzero scale factor c. Let the primed quantities represent the values obtained when $\lambda_{0}{ }^{\prime}=c \hat{\lambda}_{0}$, and let the unprimed quantities represent the values obtained when $\lambda_{0}=\hat{\lambda}_{0}$. From (H-1) and (H-2) it is seen that
and

$$
\begin{equation*}
x^{\prime}(t)=x(t) \tag{H-6}
\end{equation*}
$$

and $\quad \lambda(t)=c \lambda(t)$
From (H-6) and (H-7), and the definition $H=\lambda^{T}$, it is seen that

$$
\begin{align*}
\mathrm{H}^{\prime} & =\mathrm{cH} \\
\mathrm{H}_{u}^{\prime} & =\mathrm{cH}_{u} \\
\mathrm{H}_{\lambda}^{\prime} & =\mathrm{H}_{\lambda} \\
\mathrm{H}_{\mathrm{x}}^{\prime} & =\mathrm{cH}_{\mathrm{x}}  \tag{H-8}\\
\mathrm{H}_{\mathrm{uu}}^{\prime} & =\mathrm{cH}_{u u} \\
\mathrm{H}_{u x}^{\prime} & =\mathrm{cH}_{\mathrm{ux}} \\
\mathrm{H}_{\mathrm{xx}}^{\prime} & =\mathrm{cH}_{\mathrm{xx}} \\
\left(\mathrm{H}_{u u}^{-1}\right)^{\prime} & =\frac{1}{\mathrm{c}} \mathrm{H}_{\mathrm{uu}}^{-1}
\end{align*}
$$

From equations (9.2), (9.3), (9.4), and the relations (H-8), it can be determined that

$$
\begin{align*}
& \Phi_{12}^{\prime}(t, 0)=\frac{1}{c} \Phi_{12}(t, 0) \\
& \Phi_{22}^{\prime}(t, 0)=\Phi_{22}(t, 0)  \tag{H-9}\\
& \delta_{x}^{\prime}(t)=\delta x(t) \\
& \delta_{\lambda^{\prime}}^{\prime}(t)=c \delta \lambda(t) \\
& \text { and } \quad U_{\text {sing }}(H-9) \text { in }(9.39) \text { yields }
\end{align*}
$$

and

$$
\begin{equation*}
z_{\phi}^{\prime}=\frac{1}{c} z_{\phi} \tag{H-10}
\end{equation*}
$$

and

$$
z_{g}^{\prime}=\frac{1}{c} z_{g}
$$

From the definition of $\delta 1^{2}$ in (9.55), and (H-10), it is found from ( $G-3$ ) and ( $G-2$ ) that

$$
\begin{align*}
\left(\delta I^{2}\right)^{\prime} & =c^{2} \delta I^{2} \\
k^{\prime} & =c^{2} k \\
\nu^{\prime} & =\nu
\end{align*}
$$

Hence, from (H-11) and the definition of $J_{a}$ in (9.27) and $\delta \lambda_{0}$ in (9.45), it is seen that

$$
\begin{equation*}
J_{a}^{\prime}=J_{a} \tag{H-12}
\end{equation*}
$$

and $\quad \delta \lambda_{0}^{\prime}=\sigma \delta \lambda_{0}$
The result of (H-12) is that a value of $J_{a}$ is associated with each radial line $c \hat{\lambda}_{o}$ in the initial condition space of Lagrange multipliers, and the result of (H-13) is that the rate at which the first variation technique converges to the line $\mu \lambda_{0}^{*}$ is independent of the initial scale factor.

$$
\text { Using ( } \mathrm{H}-6 \text { ) to ( } \mathrm{H}-10 \text { ) in (10.47) of the second vari- }
$$ ation approach, it is found that

and

$$
r^{\prime}=c^{6} r
$$

from which it can be determined by (10.48) and (10.42) that

$$
k_{o p t}^{\prime}=c^{2} k_{o p t}
$$

and

$$
\begin{equation*}
\delta \lambda_{\circ}^{\prime}=o \delta \lambda_{\circ} \tag{H-15}
\end{equation*}
$$

Note that by ( $\mathrm{H}-14$ ) and (H-15), if the neighbouring trajectory

$$
\begin{equation*}
\bar{\lambda}_{0}=\mu^{\prime}\left(\lambda_{0}^{\prime}+\delta \lambda_{0}^{\prime}\right) ; \tag{H-16}
\end{equation*}
$$

then

$$
\bar{\lambda}_{0}^{\prime}=\bar{\lambda}_{0}
$$

Hence, instead of converging to the line $\mu \lambda_{0}$, the sequence will converge to the classical solution $\lambda_{o}^{*}$ since this point has the minimum error in transversality on the solution line. The use of (H-16), however, is merely a means of obtaining the classical solution and does not provide an improvement in convergence. Using (H-6), (H-7) and (H-11) in (10.27), it can be shown that by using the modified method of matching end points then

$$
\begin{equation*}
\delta \lambda_{0}^{\prime}=c \delta \lambda_{0} \tag{H-18}
\end{equation*}
$$

From (H-18) it is seen that the rate of convergence for the, modified method of Section 10.1 is independent of the initial scale factor for the Lagrange multipliers. As a result, this modified approach has improved initial convergence, and can be conveniently used with the combined algorithm of Section 10.3 to provide the property of rapid final convergence.

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