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FACULTY OF GRADUATE STUDIES

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FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

of

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B.Tech., Indian Institute of Technology, 1958

M.E., Indian Institute of Science, 1959

M.Sc., University of New Brunswick, 1963

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NUMERICAL AND ALGEBRAIC METHODS FOR COMPUTER-AIDED DESIGN OF LINEAR AND PIECE-WISE LINEAR SYSTEMS

ABSTRACT

A method is presented for linear control system design using functional relations between system parameters and system response. The functional relations are obtained by frequency domain evaluation of an integral performance criterion. The performance criterion is defined as a correlation measure between the response of a known reference system and the system to be designed.

A method is also presented for obtaining algebraic expressions relating the time-domain response of linear and piecewise linear systems with system parameters. By means of a rational fraction approximation to the exponential e^{st} and through use of a known technique for evaluating time-domain convolution integrals, it becomes possible to obtain the time-domain response without the necessity of first having to determine the poles of the system. The time-domain response is obtained as a ratio of polynomials in t with the coefficients as algebraic functions of the system parameters.

The extension of the linear design theory to cover nonlinear and multivariable systems is given. Several examples are given to illustrate the usefulness of the proposed technique.

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NUMERICAL AND ALGEBRAIC METHODS FOR COMPUTER-AIDED DESIGN OF LINEAR AND PIECE-WISE LINEAR SYSTEMS

by

BASANTA SARKAR

B. Tech., Indian Institute of Technology, Kharagpur, 1958

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department of
Electrical Engineering

We accept this thesis as conforming to the required standard

Research Supervisor ....

Members of the Committee ....

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Head of the Department ....

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THE UNIVERSITY OF BRITISH COLUMBIA

MAY, 1967

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1. INTRODUCTION

1.1 The Control Problem

The modern scientific approach to engineering in large measure consists of the formulation of problems so that methods of mathematical analysis may be applied. Engineering design of physical systems involving mathematical techniques have been rapidly developed and extended during the past three decades. One of the most important changes has been the broadening of interest from the frequency characteristics to the performance characteristics with the system excited by transient inputs or by actual typical inputs described statistically.

1.2 Mathematical Models

The analytical complexity that would result from a more or less exact description of a control system is avoided by simplified descriptions, called mathematical models, for the physical devices making up the system. Feedback control systems are conveniently classified in terms of the mathematical models that are employed as linear systems and nonlinear systems. It is in the field of linear systems that the greatest advances in design technique have taken place. However, in spite of the advanced state of linear system design there appears to be room for further development.

1.3 Trial-and-error Versus Analytical Design

More recently, control engineers have been exploring areas of performance analysis and design beyond the trial-and-error design of linear systems. Exploratory work is being done in the fields of analytical design techniques to supplement trial-and-error methods for the design of linear systems.

Analytical design techniques are in sharp contrast to trial-and-error design methods since they proceed directly from the problem specifications to the design without the need for human intuition. The trial-and-error design procedure provides no criterion for terminating the sequence of trials when difficulty is encountered in meeting the specifications. There is no way of knowing if the performance demanded in the specifications can be obtained or not. The ability to detect inconsistent specifications is a great advantage for the analytical design method. If the performance obtained by analytical procedure is not satisfactory, the designer is certain that either the performance specifications must be relaxed or some of the other specifications must be altered.

The design of control systems by application of the methods of mathematical analysis to idealized models which represent physical systems employs a more or less elaborate performance index as the basis on which the system performance is judged. The objective of the performance index is to encompass in a single number, a measure for the performance of the system.

The specifications that form the starting point of the analytical design procedure, in addition to the statement of the performance index to be used, must include a statement of the required property or value that the index must have for the system to be considered satisfactory. The analytical design procedure requires no explicit statement concerning the degree of stability of the over-all control system. All solutions include the twin requirements that the over-all system be stable and that it be physically realizable.

Ever since the suitability of functionals in engineering dynamical investigations was recognized, many authors have proposed various functionals as a quality measure of the performance of a control system. The control system error, e(t), defined as the difference between the actual and the desired value of a controlled quantity or defined as the difference between the input signal and the feedback signal of a feedback control system was used to form various functionals of the general form

$$F_1 = \int_0^{\infty} f[e(t)]\,dt \tag{1.1}$$

where $f[e(t)]$ is a function of $e(t)$

and $e(t)$ is a function of time, t.

The minimization of such an integral criterion was proposed as the basis of a procedure for the optimum design of a control system.

Analog methods for the optimization of Eq.(1.1) have been proposed by Bingulac and Kokotovic [1]. The sensitivity coefficients, that is, the derivatives of F_1 with respect to the system parameters are obtained through the use of a parameter influence analyser. The parameters of the system are obtained by means of a best match with a second order reference model. This thesis presents a method which can be used to obtain the sensitivity coefficients as algebraic functions of the system parameters.

The determination of the functional

$$F_2 = \int_0^{\infty} e^2(t)dt \tag{1.2}$$

for the case where e(t) has a known Laplace transform which is a ratio of polynomials was made by Phillips [2]; he used Parseval's theorem to replace the integral of Eq. (1.2) by a contour integral and gave tables showing the value of functional F_2 in terms of the transform coefficients. Analytical design theory has since been formulated to implement integral-square-error performance index for transient signals and mean-square-error performance index for stochastic signals. Westcott [3] used a similar technique and gave tables showing the value of the functional

$$F_3 = \int_0^{\infty} te^2(t)dt \tag{1.3}$$

in terms of the transform coefficients. Talbot [4] gave a method of computing functionals of the forms

$$F_4 = \int_0^{\infty} t^n x^2(t)dt \ ; \qquad n=0,\ 1,\ 2,\ \ldots. \tag{1.4}$$

and $$F_5 = \int_0^{\infty} t^n x(t)y(t)dt \ ; \qquad n=0,\ 1,\ 2,\ \ldots. \tag{1.5}$$

where the functions x(t) and y(t) have known rational Laplace transforms, showing how to determine the value of these functionals in terms of the transform coefficients. He gave the solutions in determinant forms.

A summary of the history of the performance index measures as functions of error is given in Table 1.1.

1.4 Possible Design Methods

Instead of establishing some rigid criterion of performance and applying it to the evaluation and design of all systems, a more flexible criterion may be used which can be adjusted to fit the particular application of each system. The system error suitably weighted can be used to obtain such a flexible criterion. The weighted error can be defined as some function of the actual system error, the specific form of the functional relationship depending upon the application of the system.

A more flexible performance criterion can be established by using the correlation function formed by the responses of two systems, the characteristics of one of the systems being known and taken as a reference.

Based on the method of computing functionals of the form F_5, given by Eq. (1.5), it is possible to make a transition from the frequency domain to the time domain and obtain the time response of the system in terms of system parameters and time.

1.5 Statement of the Problem

This thesis deals with the development of analytical

Table 1.1 A Summary of the Performance Index Measures as Functions of Error.

Performance Index	Year Proposed	Author
$\int_0^\infty e(t)dt$	1942	Obradovic [5]
	1948	Oldenbourg and Sartorius [6]
	1949	Mack [7]
	1950	Stout [8]
$\int_0^\infty e^2(t)dt$	1943	Hall [9]
	1943	Phillips [2]
	1949	Mack [7]
	1955	Rosenbrock [10]
$\int_0^\infty t^2e^2(t)dt$	1949	Mack [7]
	1952	Fickeisen and Stout [11]
	1953	Graham and Lathrop [12]
	1957	Crow [13]
$\int_0^\infty e^2(t,\tau)dt$	1949	Aigrain and Williams [14]
	1956	Spooner and Rideout [15]
	1957	Schultz and Rideout [16]
$\int_0^\infty te(t)dt$	1951	Nims [17]
	1952	Fickeisen and Stout [11]
$\int_0^\infty te^2(t)dt$	1952	Fickeisen and Stout [11]
	1953	Graham and Lathrop [12]
	1954	Westcott [3]
$\int_0^\infty \lvert e(t)\rvert dt$	1952	Fickeisen and Stout [11]
	1953	Graham and Lathrop [12]
	1953	Caldwell and Rideout [18]

Table 1.1 Continued

Performance Index	Year Proposed	Author
$\int_0^\infty t\|e(t)\|\,dt$	1953	Graham and Lathrop [12]
$\int_0^\infty t^2\|e(t)\|\,dt$	1953	Graham and Lathrop [12]
$\int_0^\infty t^n e^2(t)\,dt,\ n=0,1,2,..$	1954 1959	Westcott [19] Talbot [4]
$\int_0^\infty \|e(t,\tau)\|\,dt$	1957	Schultz and Rideout [16]
$\int_0^\infty \left[\frac{de(t)}{dt}\right]^2 dt$	1957	Babister [20]
$\int_0^\infty [\|e(t)\|]^r dt,\ r=1,2,3,..$	1959	Fuller [21]

relations between system parameters and the time domain system response. Two methods are proposed to determine analytical relations suitable for design purposes. One method is based on the use of a correlation function as a generalized performance function. The system parameters are chosen to obtain a maximum correlation between its response and the response of a reference system. One of the distinguishing features of this approach compared with other techniques is that the reference system has a specified configuration but is otherwise arbitrary. As a consequence, the use of this criterion does not place undesirable constraints on the system pole-zero locations which may be difficult to satisfy.

Also proposed is a generalized time domain design method for linear and piece-wise linear control systems which allows an easy and rapid transition from the pole-zero locations or frequency domain to the time domain. The mathematical theory has been applied to the design of linear control systems.

The application of the proposed methods to the design of a certain class of nonlinear system and multivariable systems is given.

2. FUNCTIONAL RELATIONS BETWEEN TIME DOMAIN RESPONSE AND SYSTEM PARAMETERS

2.1 Outline

Figure 2.1 shows the block diagram of a feedback control system configuration. This is a rather general block diagram in the sense that more complex configurations can be manipulated into this form. As far as the control system designer is concerned, he seldom has a completely free choice for the system. Usually he is faced with a system that is partially specified.

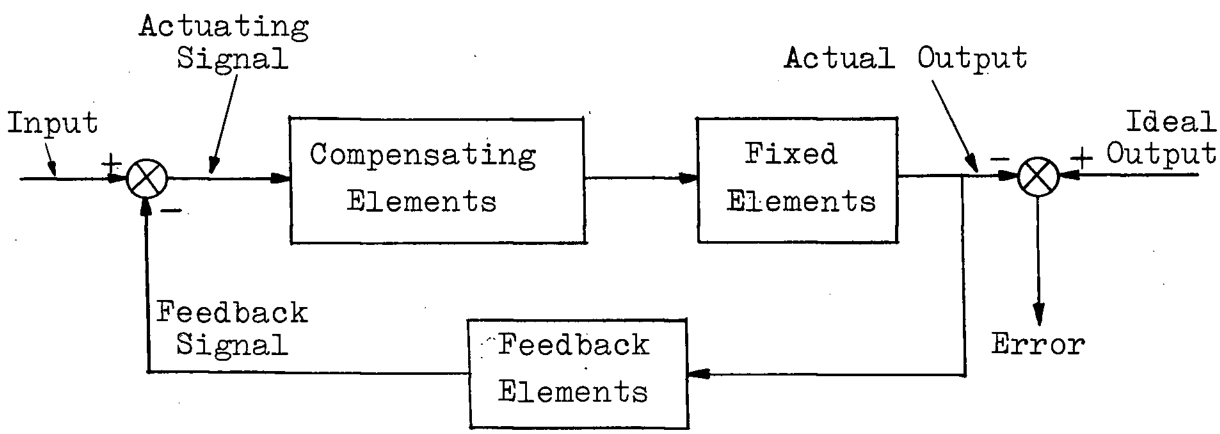


Figure 2.1 Block Diagram of a Feedback Control System.

It is a common practice in the design of practical systems to idealize it in one or more ways, to reduce excessively complicated mathematics, by a simpler model which retains some of the more important features of the original specifications.

In analytical design methods using performance indices based on system error, the error is defined as the difference between the actual output and ideal or desired output. The concept of actual output and ideal or desired output will be used here to define a generalized performance index for the analytical design of control systems using transient input signals. As a first step towards the above objective, a generalized performance integral will be derived.

2.2 Generalized Performance Integral

Let $u=u(t)$ and $v=v(t)$ be the actual and desired system outputs, respectively, of a feedback control system. A functional F can be defined by the integral

$$F = \int_0^\infty uv\,dt \tag{2.1}$$

which is a measure of the correlation between u and v. By introducing weighting factors F can be modified to a functional I as follows

$$I = \int_0^\infty c_0 uv\,dt + \int_0^\infty c_1 uv\,dt + \ldots\ldots\ldots + \int_0^\infty c_n uv\,dt \tag{2.2}$$

where c_0, c_1, ..., c_n are functions of time. Defining the weighting functions c_0, c_1, ..., c_n as follows

$$c_0 = (qt)^0 = 1; \quad c_1 = -(qt)^1; \quad c_2 = (qt)^2/2!; \; \ldots\ldots\ldots$$
$$c_n = (-1)^n (qt)^n/n!$$

where q is a positive number, Eq. (2.2) becomes

$$I = \int_0^\infty uv\,dt - \int_0^\infty qtuv\,dt + \ldots\ldots\ldots + (-1)^n \int_0^\infty q^n t^n uv\,dt/n!$$

or $$I = \int_0^\infty uvdt\left[1 - qt + \ldots\ldots\ldots + (-1)^n q^n t^n/n!\right]$$

$$= \int_0^\infty uv\exp(-qt)dt\,, \qquad \text{for large values of } n \quad (2.3)$$

The functional I is the generalized performance integral which will be studied and it may be used to define other functionals. Differentiating the right hand member of Eq. (2.3) with respect to q the performance integral I_1 is obtained as

$$I_1 = dI/dq = -\int_0^\infty tuv\exp(-qt)dt \qquad (2.4)$$

Differentiating the right hand member of Eq. (2.3) with respect to q k times the performance integral I_k is obtained:

$$I_k = d^kI/dq^k = (-1)^k\int_0^\infty t^k uv\ \exp(-qt)dt \qquad (2.5)$$

Eq. (2.5) is similar in form to the functional F_5 given by Eq. (1.5) of the previous chapter. It can be seen that Eqs. (2.3) and (2.4) are particular cases of Eq. (2.5) for values of k equal to zero and k equal to 1, respectively. Eqs. (2.3) and (2.5) will be denoted in the following forms:

$$\int_0^\infty uv\ \exp(-qt)dt = (u,v) = I_{mn} \qquad (2.6)$$

and
$$(-1)^k \int_0^\infty t^k uv \exp(-qt)dt = (u,v)_k = I_{mnk} \tag{2.7}$$

where the subscript k in I_{mnk} denotes the kth derivative of I_{mn} with respect to q. The meaning of the subscripts m and n will be explained later in this chapter. For k = 0, Eq. (2.7) reduces to

$$\begin{aligned}\int_0^\infty uv \exp(-qt)dt &= (u,v)_0 = I_{mn0} \\ &= (u,v) = I_{mn}\end{aligned} \tag{2.6}$$

where, for reasons of convenience, the subscript zero has been omitted.

Though Eq. (2.7) appears to be the most general form from which Eq. (2.6) and other performance integrals having various forms of time weighting can be obtained, Eq. (2.6) will be considered as the equation giving the generalized performance integral I_{mn}. The reason for doing so will now be considered.

Linear system design is often carried out in the domain of the complex frequency variable s. The functions considered are then the Laplace transforms U(s) and V(s), and in the majority of cases, these are rational functions of s. The integral forms can be evaluated by using Parseval's theorem to replace the integral by one taken along the imaginary s-axis, the integrand being a product of transforms. By using Parseval's theorem, Eq. (2.6) can be written as

$$I_{mn} = \int_0^{\infty} uv \exp(-qt)dt$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} U(s)V(-s+q)ds \qquad (2.8)$$

Since I_{mnk} is the kth derivative of I_{mn},

$$I_{mnk} = d^k I_{mn}/dq^k = \frac{d^k}{dq^k}\,\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} U(s)V(-s+q)ds \qquad (2.9)$$

Thus it is only necessary to evaluate Eq. (2.8) to be able to express I_{mnk} in a suitable algebraic form in terms of the coefficients of U(s) and V(s). Hence, I_{mn} is chosen as the generalized performance integral instead of I_{mnk}.

When the integrand in Eq. (2.8) is in a symmetrical form, the known properties of symmetrical functions make the desired result possible. This may be achieved by properly selecting the contour of integration so that the integral taken along the imaginary s-axis is replaced by one taken along a contour C [3,4]. This is justified as long as U(s) and V(s) have poles in the left-half plane only. Thus, I_{mn} can be written as

$$I_{mn} = \frac{1}{2\pi j}\int_C U(s+p)V(-s+p)ds \qquad (2.10)$$

where $p = q/2$.

It is now possible to derive standard forms by solving the integral I_{mn} in terms of the coefficients of U(s) and V(s) and the parameter p for any given order of denominator polynomials of U(s) and V(s) with the obvious restriction that the order of the numerators must be one less than that of the denominators. Finally, performing the required operations on I_{mn}, I_{mnk} is obtained. The differentiations will now be performed with respect to p instead of q.

If the Laplace transforms of the response functions U(s) and V(s) are expressed as ratios of two polynomials

$$U(s) = \frac{A'(s)}{C'(s)} \tag{2.11}$$

where

$$A'(s) = \sum_{k=0}^{m-1} a'_k s^k$$
$$C'(s) = \sum_{k=0}^{m} c'_k s^k \tag{2.12}$$

and

$$V(-s) = \frac{B'(s)}{D'(s)} \tag{2.13}$$

where

$$B'(s) = \sum_{k=0}^{n-1} b'_k s^k$$
$$D'(s) = \sum_{k=0}^{n} d'_k s^k \tag{2.14}$$

then U(s+p) and V(-s+p) can be expressed as

$$U(s+p) = \frac{A'(s+p)}{C'(s+p)} = \frac{A(s)}{C(s)} \tag{2.15}$$

where

$$A'(s+p) = \sum_{k=0}^{m-1} a_k'(s+p)^k$$

$$C'(s+p) = \sum_{k=0}^{n} c_k'(s+p)^k \qquad (2.16)$$

$$A(s) = \sum_{k=0}^{m-1} a_k s^k$$

$$C(s) = \sum_{k=0}^{n} c_k s^k \qquad (2.17)$$

$$a_k = \sum_{i=0}^{k+m-1} \binom{k+i}{i} a_{k+i}' p^i$$

$$c_k = \sum_{i=0}^{k+m} \binom{k+i}{i} c_{k+i}' p^i \qquad (2.18)$$

and

$$V(-s+p) = \frac{B'(s+p)}{D'(s+p)} = \frac{B(s)}{D(s)} \qquad (2.19)$$

where

$$B'(s+p) = \sum_{k=0}^{n-1} b_k'(s+p)^k$$

$$D'(s+p) = \sum_{k=0}^{n} d_k'(s+p)^k \qquad (2.20)$$

$$B(s) = \sum_{k=0}^{n-1} b_k s^k$$

$$D(s) = \sum_{k=0}^{n} d_k s^k \tag{2.21}$$

$$b_k = \sum_{i=0}^{k+n-1} \binom{k+i}{i} b'_{k+i} p^i$$

$$d_k = \sum_{i=0}^{k+n} \binom{k+i}{i} d'_{k+i} p^i \quad . \tag{2.22}$$

It is shown in Appendix A that the solution of the integral I_{mn} is given by

$$I_{mn} = \Delta'/c_m\Delta \tag{2.23}$$

where Δ is the determinant of the (m+n)-rowed square matrix M and Δ' is the determinant obtained from M on replacing its last column by the column F, where

$$M = \begin{bmatrix} c_0 & \cdots & 0 & d_0 & \cdots & 0 \\ c_1 & \ddots & c_0 & d_1 & \ddots & d_0 \\ \vdots & & \vdots & \vdots & & \vdots \\ c_m & \ddots & \vdots & d_n & \ddots & \vdots \\ & 0 & c_m & & 0 & d_n \end{bmatrix} \tag{2.24}$$

$$F = (f_0,\ f_1,\ \ldots\ldots,\ f_k,\ 0,\ \ldots\ldots\ 0) \tag{2.25}$$

and where
$$f_j = \sum_{i=0}^{j} a_i b_{j-i} , \qquad \text{for } 0 \leqq j \leqq m+n-2 .$$

In the integral I_{mn}, the subscripts m and n denote the orders of the denominator polynomials U(s) and V(s), respectively.

Letting v = u in Eqs. (2.6) and (2.7), yields

$$\int_0^{\infty} u^2 \exp(-qt)dt = (u,u) = I_{mm} = J_m \tag{2.26}$$

and
$$(-1)^k \int_0^{\infty} t^k u^2 \exp(-qt)dt = (u,u)_k = I_{mmk} = J_{mk} \tag{2.27}$$

Letting u = v in Eqs. (2.6) and (2.7), yields

$$\int_0^{\infty} v^2 \exp(-qt)dt = (v,v) = I_{nn} = K_n \tag{2.28}$$

and
$$(-1)^k \int_0^{\infty} t^k v^2 \exp(-qt)dt = (v,v)_k = I_{nnk} = K_{nk} \tag{2.29}$$

The subscripts m, n, and k have the same meaning as before and again for convenience the subscript zero in Eqs. (2.26) and (2.28) has been omitted. Using Parseval's theorem and properly choosing the contour of integration, J_m can be represented in the following symmetrical form

$$J_m = \frac{1}{2\pi j} \int_{p-j\infty}^{p+j\infty} U(s+p)U(-s+p)ds \tag{2.30}$$

where $\quad p = q/2$

and Eqs. (2.11), (2.12), and (2.15) to (2.18) hold for U(s+p); and the solution of the integral J_m is obtained as (see Appendix A)

$$J_m = (-1)^{m+1} \Delta'/c_m\Delta \tag{2.31}$$

where Δ is the determinant of the (mxm) square matrix M' and Δ' is the determinant obtained from M' on replacing its last column by the column L, where

$$M' = \begin{bmatrix} c_0 & & & 0 & \\ c_2 & c_1 & c_0 & & \\ \cdot & & & & \\ \cdot & c_3 & c_2 & c_1 & c_0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{2m-2} & \cdot & \cdot & \cdot & c_{m-1} \end{bmatrix}$$

and $\qquad L = (L_0, L_2, \ldots, L_{2m-2})$

where
$$2L_j = \sum_{i=0}^{j} (-1)^{j-i} a_i a_{j-i} \text{ , for } 0 \leqq j \leqq m-1$$

$$= \sum_{i=j-m+1}^{m-1} (-1)^{j-i} a_i a_{j-i} \text{ , for } m \leqq j \leqq 2m-2 \text{ .}$$

Since J_{mk} is the kth derivative of J_m with respect to q, replacing q by p and performing the required operations on J_m, J_{mk} is obtained. The solution of K_n has the same form as J_m where a and c are replaced by b and d, respectively.

Thus, there are three performance integrals, a generalized performance integral I_{mn}, and two derived from I_{mn}, J_m and K_n. Starting from these performance integrals other performance integrals I_{mnk}, J_{mk}, and K_{nk} can be obtained. These performance integrals when evaluated appear as algebraic functions of system parameters.

2.3 Generalized Performance Indices

In designing a feedback control system for a specific application, the designer usually has a definite goal in mind. The control system is to perform some given operation subject to physical constraints on its response. The designer is, therefore, faced with the problem of translating this essentially physical information into a mathematical definition of the desired performance which then becomes a criterion for synthesizing the system. There sometimes arises the problem of designing the best possible control system of a given order of complexity to meet a given requirement. However, it is not easy to give a precise criterion of best performance. If it is desired to design a best (in some sense) possible control system, it is necessary to define mathematically a criterion of performance. One commonly used criterion is the minimum integral of error-squared criterion for transient inputs. While mathematically convenient, this criterion has the disadvantage of giving too great an emphasis to large momentary errors. However, for comparison of performance between widely dissimilar systems, this comprehensive error criterion is likely to be much more consistent than any empirical ones. The question then arises of the possibility of devising a criterion of a comprehensive type that will correspond to the accepted empirical criteria in straight forward cases, but will be available for wider use as systems become more complicated and diverse. To be convenient to apply, such a criterion must also be capable of allowing a simple algebraic representation of the error measure directly in terms of system parameters. This is possible for the generalized performance functionals chosen since they are all capable of algebraic representation in the desired form.

2.3.1 Generalized Performance Index Based on System Error

The performance integrals J_m and K_n and J_{mk} and K_{nk} can be used as performance indices when $u = u(t)$ and $v = v(t)$ are replaced by the system error function $e = e(t)$. Then Eqs. (2.26) and (2.28) give the performance indices as quadratic measure of error and Eqs. (2.27) and (2.29) give the performance indices as time-weighted measure of error. Any desired performance index based on error measure can be obtained by proper choice of q and k. For the specific case of q=0, k=0 Eqs. (2.27) and (2.29) reduce to the familiar form of the quadratic measure of error, the integral-squared-error (ISE). The time-weighted measures of error, the integral-time multiplied-squared error (ITSE) and the integral-squared time multiplied-squared error (ISTSE) are obtained for the cases q=0, k=1 and q=0, k=2, respectively. Since J_m and K_n have similar forms, it is sufficient to consider only one of them. The following form of J_m is considered here as the generalized performance index based on system error.

$$J_m = \int_0^\infty e^2(t)\ \exp(-qt)dt \tag{2.32}$$

Minimization of J_m or its derivatives with respect to q can be used as the criterion for optimum or best design.

2.3.2 Generalized Performance Index Based on the Correlation Between the Response of Two Systems

It has been shown that when u and v are replaced by the system error, e, I_{mn}, J_m and K_n all reduce to the form given by Eq. (2.32) and can be used as a performance index, the basis on which the system performance can be judged. It will now be shown that I_{mn}, J_m, and K_n, given by Eqs. (2.6), (2.26) and (2.28), respectively, can be used to define a generalized performance index based on the correlation between the response of two systems.

Regarding Eqs. (2.6), (2.26) and (2.28) as the correlation equations between (u,v), (u,u) and (v,v), respectively, and using Schwarz's inequality yields

$$[(u,v)]^2 \leqq (u,u)(v,v)$$

$$\text{or } \left|(u,v)\right| \leqq \sqrt{(u,u)(v,v)} \;. \tag{2.33}$$

Rearranging Eq. (2.33), and defining

$$P = \left|(u,v)\right| / \sqrt{(u,u)(v,v)} \leqq 1 \tag{2.34}$$

a performance index relating I_{mn}, J_m, and K_n is obtained. The performance index P given by Eq. (2.34) can be regarded as a normalized measure of the correlation between the two responses u and v. P can be regarded as a performance index in the best match sense when maximization of P is considered as the design objective. Maximization of P, therefore, is a meaningful approach and can be used as a basis for the optimum or best design of a control system. When evaluated, P appears as an algebraic function of the system parameters. By using a suitable maximization procedure, values for the unknown parameters which maximize P can be found. Using time-weighted forms $(u,v)_k$, $(u,u)_k$

and $(v,v)_k$, P takes the new form P_k given by

$$P_k = \left|(u,v)_k\right| / \sqrt{(u,u)_k(v,v)_k} \leqq 1 \quad . \tag{2.35}$$

2.3.3 Minimization and Maximization Procedure

The performance index, chosen on the basis of either the error measure or the best match measure, when evaluated, will be an algebraic function of the system parameters. In minimizing or maximizing the performance index the usual procedure is to differentiate the performance index with respect to each of the k design variables, equating each derivative to zero. It is evident from the form of Eq. (2.23) that the derivatives, which are the sensitivity coefficients, are expressed as algebraic functions of the system parameters. That is, explicit relations are obtained for the sensitivity coefficients which are determined by analog means by the method of Bingulac and Kokotovic [1]. However, the solution of the k simultaneous nonlinear equations for the parameter presents some computational difficulties. The procedure selected here avoids the differentiation problem and its associated difficulties and has the advantage of being an automatic method for dealing with the problem of minimization or maximization of the performance index with the aid of a digital computer [22].

3. A PERFORMANCE FUNCTION APPROACH TO LINEAR SINGLE VARIABLE SYSTEM DESIGN

3.1 Outline

The following examples demonstrate the use of the performance indices P and J_{mk} in the design of linear single input - single output control systems. In the first example, a linear third order system is designed on the basis of a response correlation with the response of a known second order system. In the second example, a comparison between unweighted and time-weighted error criteria is given for the case of a simple second order system.

3.2.1 Design of a Third Order System

A position control servomechanism having two time constants and an integration and compensated by a phase-lead network, as shown in Figure 3.1, is designed on the basis of maximizing the correlation of its response with that of a reference second order system for unit impulse input to both systems.

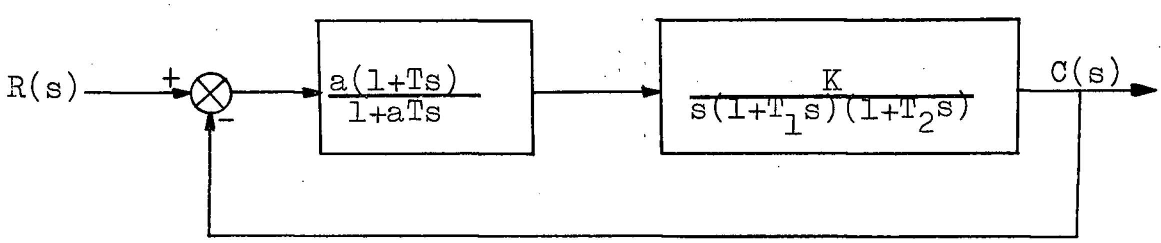


Figure 3.1 Phase-Lead Compensated Position Control Servomechanism.

The closed-loop transfer function of the reference second order system is

$$H_c(s) = \frac{\omega_c^2}{s^2 + 2\zeta\omega_c s + \omega_c^2} \tag{3.1}$$

where ω_c, the natural frequency of oscillation of the system in rad./sec., and ζ, the damping ratio of the system, are known.

The closed-loop transfer function of the system shown in Figure 3.1 is

$$H(s) = \frac{KaTs + Ka}{\left[aTT_1T_2s^4 + \{aT(T_1+T_2)+T_1T_2\}s^3 + (aT+T_1+T_2)s^2 \quad (KaT+1)s + Ka\right]} \tag{3.2}$$

where T_1=4 secs. and T_2=1 sec. and K, a, and T are regarded as the design variables with the constraint on a that $a \geqq 0.1$.

The values of K, a, and T are chosen to obtain the maximum correlation between the two system responses for a unit impulse input to both systems.

Denoting the output response of the reference second order system as V(s) and that of the unknown system as U(s), for a unit impulse input, Eqs. (3.1) and (3.2) yield

$$V(s) = \frac{\omega_c^2}{s^2 + 2\zeta\omega_c s + \omega_c^2} \tag{3.3}$$

and

$$U(s) = \frac{KaTs + Ka}{\left[aTT_1T_2s^4 + \{aT(T_1+T_2)+T_1T_2\}s^3 + (aT+T_1+T_2)s^2 \quad (KaT+1)s + Ka\right]} \tag{3.4}$$

The maximum correlation between the two system responses is obtained by maximizing the performance index P such that

$$P_{max} = \frac{|(u,v)|}{\sqrt{(u,u)(v,v)}} \leqq 1 \tag{3.5}$$

where

$$(u,v) = \int_0^\infty uv\,dt$$

$$(u,u) = \int_0^\infty u^2 dt$$

and $$(v,v) = \int_0^\infty v^2 dt$$

Using Eqs. (2.24) and (2.32) and regarding $q = 0$, that is $p = 0$, (u,v), (u,u) and (v,v) can be written in the following determinant forms.

$$(u,v) = \frac{1}{c_4}\,\frac{\begin{vmatrix} c_0 & 0 & \omega_c^2 & 0 & 0 & a_0\omega_c^2 \\ c_1 & c_0 & -2\zeta\omega_c & \omega_c^2 & 0 & a_1\omega_c^2 \\ c_2 & c_1 & 1 & -2\zeta\omega_c & \omega_c^2 & 0 \\ c_3 & c_2 & 0 & 1 & -2\zeta\omega_c & 0 \\ c_4 & c_3 & 0 & 0 & 1 & 0 \\ 0 & c_4 & 0 & 0 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} c_0 & 0 & \omega_c^2 & 0 & 0 & 0 \\ c_1 & c_0 & -2\zeta\omega_c & \omega_c^2 & 0 & 0 \\ c_2 & c_1 & 1 & -2\zeta\omega_c & \omega_c^2 & 0 \\ c_3 & c_2 & 0 & 1 & -2\zeta\omega_c & \omega_c^2 \\ c_4 & c_3 & 0 & 0 & 1 & -2\zeta\omega_c \\ 0 & c_4 & 0 & 0 & 0 & 1 \end{vmatrix}}$$

$$(u,u) = \frac{(-1)}{2c_4} \frac{\begin{vmatrix} c_0 & 0 & 0 & a_0^2 \\ c_2 & c_1 & c_0 & -a_1^2 \\ c_4 & c_3 & c_2 & 0 \\ 0 & 0 & c_4 & 0 \end{vmatrix}}{\begin{vmatrix} c_0 & 0 & 0 & 0 \\ c_2 & c_1 & c_0 & 0 \\ c_4 & c_3 & c_2 & c_1 \\ 0 & 0 & c_4 & c_3 \end{vmatrix}}$$

$$(v,v) = \frac{(-1)}{2} \frac{\begin{vmatrix} \omega_c^2 & \omega_c^4 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} \omega_c^2 & 0 \\ 1 & 2\zeta\omega_c \end{vmatrix}}$$

where $c_0 = Ka$; $c_1 = (KaT+1)$; $c_2 = (aT+T_1+T_2)$

$c_3 = aT(T_1+T_2) + T_1T_2$; $c_4 = aTT_1T_2$

$a_0 = Ka$; $a_1 = KaT$

Results obtained by substituting (u,v), (u,u), and (v,v) in Eq. (3.5) and maximizing P for various values of ω_c are shown in Table 3.1. A brief summary of the method [22] used for maximizing P is given in Appendix B. This method avoids differentiating P with respect to each of the design variables, equating each derivative to zero, and solving a set of non-linear algebraic equations. The method works directly with the expression for P and searches for the maximum automatically on a digital computer once the search is initiated [23]. In terms of the notations explained in Appendix B values of $\alpha = 3.0$, $\beta = 0.5$ and $\varepsilon = 0.1$ were used, along with starting values of

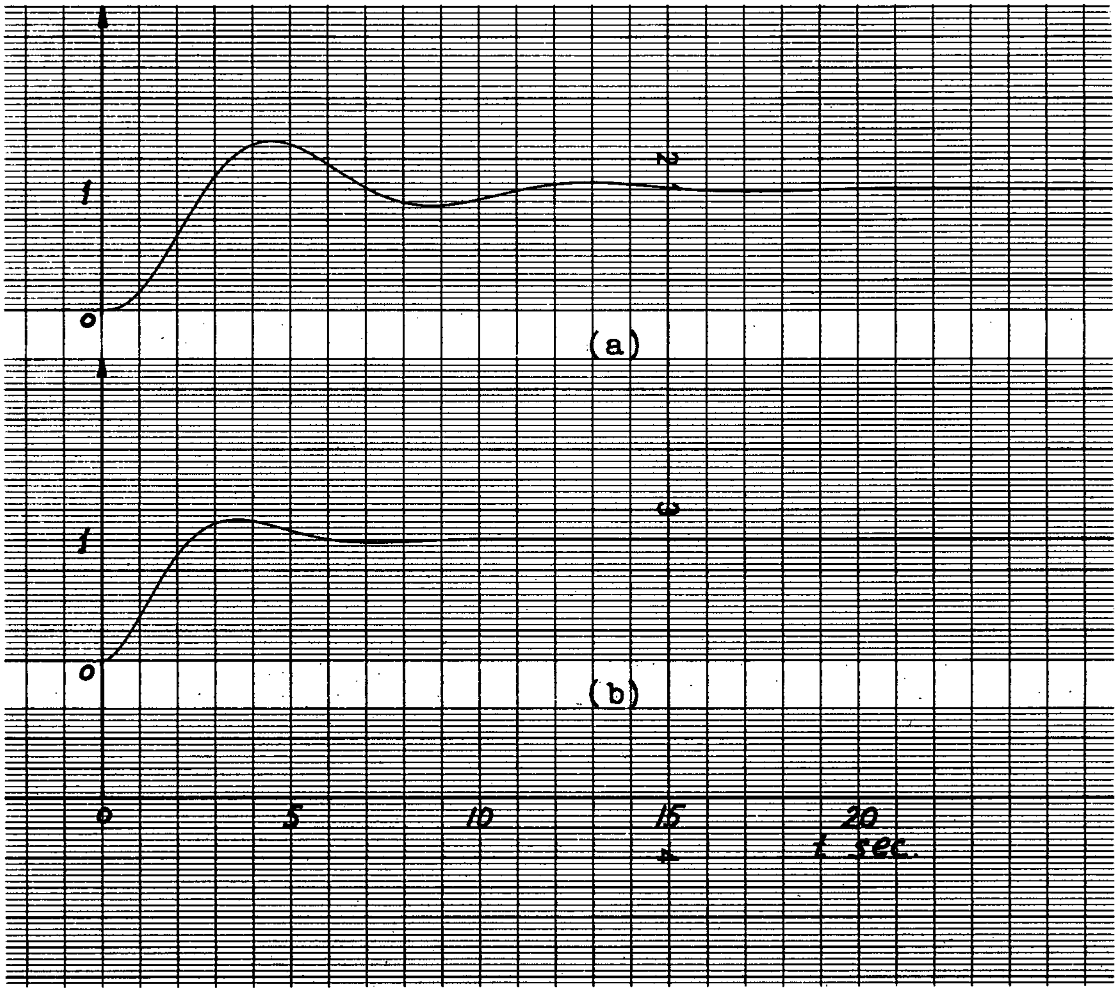
$K = 3.0$, $a = 0.2$, and $T = 3.0$, to initiate the search for maximizing P. Given in Section 3.3 are methods of obtaining approximate values of design variables to initiate a search on the computer. The method is fast and reliable especially when a large scale digital computer is used. The meaning of the number of trials and the number of stages given in Table 3.1 is explained in Appendix B.

Table 3.1 Values of K, a, T and P_{max} for Known Values of ω_c and ζ.

ω_c rad/sec	ζ	P_{max}	K	a	T	No. of Trials	No. of Stages	Approx. Time of Solution
5.0	0.5	0.070	2.917	0.100	0.235	1972	60	200 sec.
4.0	0.5	0.092	11.936	0.100	1.860	1990	60	200 sec.
3.0	0.5	0.239	52.827	0.100	1.763	1738	60	175 sec.
2.0	0.5	0.505	24.609	0.100	2.517	2231	60	235 sec.
1.0	0.5	1.000	7.115	0.156	2.825	223	7	25 sec.

The step response of the system illustrated in Figure 3.1 for values of K, a, and T corresponding to $P_{max} = 1.0$, as given in Table 3.1, is shown in Figure 3.2. The step response of the reference second order system for $\omega_c = 1.0$ and $\zeta = 0.5$ is also shown in Figure 3.2.

From the results given in Table 3.1 it can be inferred that the performance index P could be used to provide a flexible criterion capable of being adjusted to suit system performance. A performance index based on system error when minimized becomes insensitive to parameter variation but the time domain response of the system, such as the overshoot, could still be sensitive to parameter variations. The minimization of such a fixed



(a) Compensated Third Order System Response

(b) Reference Second Order System Response

Figure 3.2 Unit-Step Responses of the System Shown in Figure 3.1 and the Reference Second Order System.

performance index, therefore, cannot always ensure meaningful results and can lead to system designs that are unstable or physically unrealizable [23]. Furthermore, it is usually more important to minimize the time domain response sensitivity to parameter changes than it is to optimize the performance index.

To overcome these disadvantages, it is desirable to allow a degree of flexibility in the choice of a performance function. The performance function P used here, since it is based on the response correlation with a class of reference systems, gives this flexibility. If $\alpha = (\alpha_1, \ldots\ldots\ldots, \alpha_n)$ represents the parameters of the system to be designed and $\beta = (\beta_1, \ldots\ldots\ldots, \beta_m)$ represents the parameters of a reference system having a known response, then it is possible to determine the parameters α and β by the operation

$$\underset{\{\beta\}}{\text{Min}}\ S\left[\underset{\{\alpha\}}{\text{Max}}\ P(\alpha,\beta)\right] \qquad (3.6)$$

where S represents some suitably chosen time-domain sensitivity function for the system and where, in general, the parameters β_k have specified upper and lower bounds. For example, S could be the sensitivity of the maximum overload with respect to variations in K. Further improvement could be made if the class could, for example, be the class of systems having a delayed second order response for step inputs. The reference system variables then become τ, the time delay, ζ, the damping ratio, and ω_c, the natural frequency of oscillation of the system. The use of this idea will be considered in the next Chapter. By using a class of reference systems, it becomes possible to investigate the parameter sensitivity of the time domain response, for example the overshoot to a unit step input. Thus, instead of choosing the system given by Eq. (3.6), it may be more meaningful to choose the system which has least sensitivity to variations in its time domain response.

3.2.2 Design of a System With and Without Time Weighting

To compare the merits of different performance indices a second order position control servomechanism will now be designed on the basis of an unweighted and two time weighted error criteria. The system is shown in Figure 3.3.

The closed-loop transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + as + 1} \tag{3.7}$$

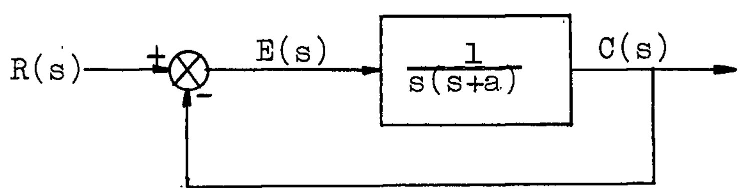


Figure 3.3 Second Order Position Control Servomechanism.

where a is regarded as the design variable. It may be noted that if the natural frequency of oscillation of the system is considered as unity, a equals twice the value of the damping ratio, ζ, of the system.

Consider that the system is subjected to a unit step input and that the value of a is to be found so as to minimize the integral of error squared, J_2, the first time moment of error squared, J_{21}, and the second time moment of error squared, J_{22}, in that order.

The error transform of the given system for a unit step input is given by

$$E(s) = \frac{s + a}{s^2 + as + 1} \tag{3.8}$$

Using the standard form from Table A.2, Appendix A, for p = 0,

$$J_2 = (1 + a^2)/2a$$

J_2 is a minimum for a = 1, when J_2 = 1.

Using the standard form from Table A.3, Appendix A, for p = 0,

$$J_{21} = (2 + a^4)/2a^2$$

J_{21} is a minimum for a = 1.19, when J_{21} = 0.718.

Using the standard form from Table A.4, Appendix A, for p = 0,

$$J_{22} = (a^6 - a^4 + a^2 + 4)/a^3$$

J_{22} is a minimum for a = 1.334, when J_{22} = 1.737.

The results are summarized in Table 3.2

Table 3.2 Comparison of Unweighted and Time-weighted Error Criteria.

J_{mk}	Minimum value of J_{mk}	a	ζ
J_2	1.000	1.000	0.500
J_{21}	0.718	1.190	0.595
J_{22}	1.737	1.334	0.667

This example shows how the integral performance criteria can be used to determine unknown system parameters. In general, however, one weakness of these criteria is that there is no direct relationship between the performance integrals and the time-domain response. To overcome this weakness a method for determining algebraic relationships between system parameters and the time-domain response will be developed in Chapter 4.

3.3 Methods of Obtaining Approximate Values of Design Parameters

The search procedure selected for minimizing or maximizing a chosen performance index on a digital computer requires a starting value to initiate the search. While it is possible to arrive at a reasonable initial guess for simple systems, the problem becomes increasingly difficult as more complex systems are encountered.

The following sections deal with two methods of arriving at reasonable starting values for the design variables to initiate a digital computer solution. The methods to be discussed use a comparison system and give relationships between the design variables and known parameters of the system to be designed and the comparison system. The comparison system is assumed to be a second order system. However, this is not a restriction imposed on the method; other comparison systems could be used. The theoretical development will be illustrated by an example in Section 3.4.

3.3.1 Routh Array Approximation[24]

Consider the system of Figure 3.4 where K and K_T are regarded as the design variables. The method to be discussed is based on choosing values of K and K_T to give an optimum correlation with respect to the response of the comparison system

$$H_c(s) = \frac{\omega_c^2}{s^2 + 2\zeta\omega_c s + \omega_c^2} \tag{3.9}$$

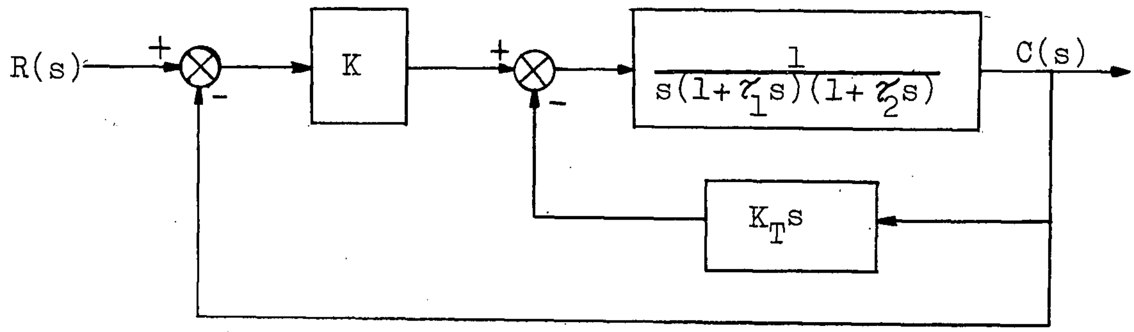


Figure 3.4 Third Order Control System With Tachometer Feedback.

The closed-loop transfer function of the system shown in Figure 3.4 is given by

$$H(s) = \frac{1}{\frac{\tau_1\tau_2}{K} s^3 + \frac{\tau_1+\tau_2}{K} s^2 + \frac{1+K_T}{K} s + 1} \tag{3.10}$$

The Routh array for the characteristic equation of this system is shown in Table 3.3. The characteristic equation is obtained by equating the denominator of Eq. (3.10) to zero. If an approximating transfer function, called an associated function, of second order is constructed by using the last

Table 3.3 Routh Array for the Characteristic Equation of the Given Control System.

$\frac{\tau_1 \tau_2}{K}$	$\frac{1+K_T}{K}$
$\frac{\tau_1 + \tau_2}{K}$	1
$\frac{1+K_T}{K} - \frac{\tau_1 \tau_2}{\tau_1 + \tau_2}$	
1	

three elements of the first column of the Routh array, then this transfer function will have the same integral squared impulse response as the system transfer function, Eq. (3.10). The integral squared impulse response is computed from the last two elements in the Routh array. The associated function is given by

$$A(s) = \frac{1}{\frac{\tau_1 + \tau_2}{K} s^2 + \left[\frac{1+K_T}{K} - \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \right] s + 1} \tag{3.11}$$

The natural frequency of oscillation of the system given by Eq. (3.11) is the same as that of the system given by Eq. (3.10). As a first approximation in designing the unknown system, its natural frequency of oscillation is equated to that of the comparison system. Eqs.(3.9) and (3.11) yield

$$\sqrt{\frac{K}{\tau_1+\tau_2}} = \omega_c \ .$$

As a first approximation, therefore,

$$K = (\tau_1 + \tau_2)\omega_c^2 \ . \tag{3.12}$$

The energy ratio of the impulse responses of the two systems H(s) and $H_c(s)$ will now be considered. This ratio is computed from the first order coefficients of $H_c(s)$ and A(s) which yield the energy ratio

$$E_1 = \frac{2\zeta/\omega_c}{\frac{1+K_T}{K} - \frac{\tau_1\tau_2}{\tau_1+\tau_2}}$$

This ratio will be unity for a correlation match in the ideal sense. Therefore, as a second approximation, the following equation is obtained.

$$1 + K_T = K\left[\frac{2\zeta}{\omega_c} + \frac{\tau_1\tau_2}{\tau_1+\tau_2}\right] \tag{3.13}$$

Eqs. (3.12) and (3.13) are the required equations from which K and K_T are obtained in terms of the known parameters τ_1, τ_2, ζ, and ω_c.

Additional energy ratios can be defined between the elements of the first column of the Routh array and the coefficients of the characteristic equation to obtain further relationships between the system parameters. These energy ratios are listed in Table 3.4.

The energy ratio method is based on the concept of obtaining a simple response approximation for a system by comparing it with a known comparison system. The approximation is restricted to systems described by a lumped-constant linear differential equation whose transform is of the following form.

$$H(s) = \frac{b_0}{a_n s^n + \ldots\ldots\ldots + a_1 s + a_0}$$

The response approximation is performed by placing a constraint on the ratio between the integrated square of the system impulse response and the corresponding integral of the comparison system.

Since the response of a second order system is easy to visualize it is convenient to choose a second order comparison system. However, by means of a Padé approximation, a delayed second-order response could also be used. Many systems of high order have responses which can be accurately approximated by such a delayed response.

3.3.2 <u>Correlation Function Approximation</u>

For the two system transfer functions $H_c(s)$ and $H(s)$, given by Eqs. (3.9) and (3.10), the following three relationships are obtained using Parseval's theorem.

Table 3.4 Energy Ratios Defined From the First Column Elements of a Routh Array and the Coefficients of the Characteristic Equation.

Energy Ratio	Routh Array
$E_{n-2} = a_{n-2}/R_{3,1}$	$a_n \quad a_{n-2} \quad \cdots\cdot \quad a_2 \quad a_0$
....	$a_{n-1} \quad a_{n-3} \quad \cdots\cdot \quad a_1$
....	$R_{3,1} \quad R_{3,2} \quad \cdots\cdot$
....	$\cdot\cdot \quad \cdot\cdot \quad \cdot\cdot$
$E_3 = a_3/R_{n-2,1}$	$R_{n-2,1} \quad R_{n-2,2} \quad \cdots\cdot$
$E_2 = a_2/R_{n-1,1}$	$R_{n-1,1} \quad R_{n-1,2}$
$E_1 = a_1/R_{n,1}$	$R_{n,1}$
	$R_{n+1,1}$

$$\overline{h_c^2} = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} H_c(s)H_c(-s)\, ds \tag{3.14}$$

$$\overline{h^2} = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} H(s)H(-s)\, ds \tag{3.15}$$

and

$$\overline{hh_c} = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} H(s)H_c(-s)\, ds \quad . \tag{3.16}$$

Each of the above three equations can be solved from a set of linear equations. Corresponding to Eq. (3.14) the set of linear equations is

$$\begin{aligned}
\omega_c^2 z_1 &= -\omega_c^4/2 \\
z_1 + 2\zeta\omega_c z_2 &= 0
\end{aligned} \tag{3.17}$$

where

$$z_2 = \overline{h_c^2}$$

Corresponding to Eq. (3.15) the set of linear equations is

$$\begin{aligned}
Ky_1 &= K^2/2\tau_1\tau_2 \\
(\tau_1 + \tau_2)y_1 + (1+K_T)y_2 + Ky_3 &= 0 \\
\tau_1\tau_2 y_2 + (\tau_1+\tau_2)y_3 &= 0
\end{aligned} \tag{3.18}$$

where

$$y_3 = \overline{h^2}$$

Corresponding to Eq. (3.16) the set of linear equations is

$$\begin{aligned}
\omega_c^2 x_1 + Kx_4 &= K\omega_c^2 \\
-2\zeta\omega_c x_1 + \omega_c^2 x_2 + (1+K_T)\, x_4 + Kx_5 &= 0 \\
x_1 - 2\zeta\omega_c x_2 + \omega_c^2 x_3 + (\tau_1+\tau_2)x_4 + (1+K_T)x_5 &= 0 \\
x_2 - 2\zeta\omega_c x_3 + \tau_1\tau_2 x_4 + (\tau_1+\tau_2)x_5 &= 0 \\
x_3 + \tau_1\tau_2 x_5 &= 0
\end{aligned} \tag{3.19}$$

where

$$x_5 = \overline{hh_c}$$

In the above equations the variables z_1, z_2, y_1, y_2, etc., represent the coefficients of powers of s occurring in Eq. (A.9) in Appendix A.

Assuming that $\overline{h^2} = \overline{h_c^2}$,

that is, $$y_3 = z_2 = \omega_c/4\zeta \tag{3.20}$$

Eqs. (3.18) and (3.20) yield

$$1 + K_T = K\left[\frac{\tau_1\tau_2}{\tau_1+\tau_2} + \frac{2\zeta}{\omega_c}\right] \tag{3.21}$$

Solving for x_5, from the set of Eq. (3.19),

$$x_5 = \frac{AK + BK(1+K_T)}{K^2 + EK + FK(1+K_T) + G(1+K_T) + H(1+K_T)^2 + J} \tag{3.22}$$

where

$$A = \omega_c^4\tau_1\tau_2(1-4\zeta^2) - 2\zeta\omega_c^3(\tau_1+\tau_2)$$

$$B = -\omega_c^2$$

$$E = 2\omega_c^2(\tau_1+\tau_2)(2\zeta^2-1) + 2\zeta\omega_c^3\tau_1\tau_2(4\zeta^2-3)$$

$$F = 2\zeta\omega_c$$

$$G = 2\omega_c^4\tau_1\tau_2(2\zeta^2-1) + 2\zeta\omega_c^3(\tau_1+\tau_2)$$

$$H = \omega_c^2$$

and $$J = \omega_c^6\tau_1^2\tau_2^2 + \omega_c^4(\tau_1+\tau_2)^2 + 2\zeta\omega_c^5\tau_1\tau_2(\tau_1+\tau_2)$$

To maximize the correlation function given by Eq. (3.16), Eq. (3.22) is differentiated with respect to K. Equating the derivative to zero yields

$$\left[A+B(1+K_T) \right]\left[-K^2+G(1+K_T)+H(1+K_T)^2 + J \right] = 0$$

Therefore, either

$$A + B(1+K_T) = 0$$

which is a trivial solution,

or $$K^2 = G(1+K_T) + H(1+K_T)^2 + J \tag{3.23}$$

Substituting Eq. (3.23) in Eq. (3.22), yields

$$x_5 = \frac{A + B(1+K_T)}{2K + E + F(1+K_T)} \tag{3.24}$$

Assuming that $\overline{hh}_c = \overline{h_c^2}$,

$$x_5 = \omega_c/4\zeta . \tag{3.25}$$

Letting $$\alpha = \omega_c\tau_1\tau_2/(\tau_1+\tau_2) \tag{3.26}$$

and substituting Eqs. (3.21), (3.25) and (3.26) in Eq. (3.24) yields

$$K = \frac{a_1\alpha^3 - a_2\alpha^2}{a_3\alpha + a_4} \tag{3.27}$$

where

$$a_1 = \frac{(\tau_1+\tau_2)^3}{\tau_1^2\tau_2^2}(5\zeta-12\zeta^3)$$

$$a_2 = \frac{(\tau_1+\tau_2)^3}{\tau_1^2\tau_2^2}(6\zeta^2-2)$$

$$a_3 = 3\zeta$$

and

$$a_4 = 1 + 6\zeta^2\tau_1\tau_2$$

If ω_c of the comparison system is regarded as an unknown quantity, instead of assigning an arbitrary fixed value to it, Eq. (3.27) can be used to obtain an approximate starting value. From Eq. (3.27) it is seen that for K to be positive,

$$a_1\alpha - a_2 > 0$$

or

$$\alpha > a_2/a_1$$

that is,

$$\alpha > \frac{6\zeta^2 - 2}{5\zeta - 12\zeta^3} \tag{3.28}$$

Eq. (3.28) provides the relationship between the known parameters of the system to be designed and ζ, the known parameter, and ω_c, the unknown parameter, of the comparison system. It will now be shown how the results obtained in this and the previous section can be used to obtain approximations to system parameters which can then be used as initial estimates for the digital computer solution.

3.4 Illustrative Example

Consider the system of Figure 3.4 for which the system transfer function is given by Eq. (3.10). Let the comparison system be a second order system and its transfer function have the form given by Eq. (3.9). It is assumed that the time constants τ_1 and τ_2 in Eq. (3.10) and ζ in Eq. (3.9) are known and have the following values.

$$\tau_1 = 1.174 \text{ sec.}$$

$$\tau_2 = 0.426 \text{ sec.}$$

$$\zeta = 0.6$$

Based on these choices and considering K and K_T as unknowns, values of K and K_T will be calculated using the approximation methods described before.

3.4.1 Routh Array Approximation

To obtain a suitable value of ω_c consider that the rise time of the unknown system is the same as that of the comparison system and from physical considerations let this, for example, be 2.02 seconds. Therefore, for the comparison system, the maximum value of the impulse response is

$$h_{max} = 1/(\text{Rise Time}) = 0.495$$

In terms of ζ and ω_c, h_{max} is given by

$$h_{max} = \frac{\omega_c}{\sqrt{1-\zeta^2}} \exp\left[\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \cdot \frac{\tan^{-1}(\sqrt{1-\zeta^2}/\zeta)}{180} \right] \quad (3.29)$$

Substitution of the value of h_{max} into Eq. (3.29) yields

$$\omega_c = 0.786 \text{ rad./sec.}$$

For this value of ω_c, Eqs. (3.12) and (3.13) yield

$$K = 0.99$$

$$1 + K_T = 1.82$$

Results obtained using these values of K and K_T as initial estimates for the design variables and maximizing the correlation type performance index P are shown in Table 3.5. The results correspond to the comparison system parameters

$$\zeta = 0.6$$

$$\omega_c = 0.786 \text{ rad./sec.}$$

3.4.2 <u>Correlation Function Approximation</u>

For $\zeta = 0.6$, Eq. (3.28) yields

$$\alpha > 0.392$$

From Eq. (3.27) it is seen that for $\alpha = 0.392$, $K = 0$ and for small values of α, K is also small. For example, if $\alpha = 0.4$ $K = 0.0034$, and Eq. (3.26) gives $\omega_c = 1.28$ rad./sec. Choosing a larger value of α, for example $\alpha = 1$, Eqs. (3.26) and (3.27) yield

$$\omega_c = 3.2 \text{ rad./sec.}$$

$$K = 1.051$$

Substitution of these values of ω_c and K in Eq. (3.21) yields

$$1 + K_T = 0.723$$

The negative value of K_T, as an initial estimate, is the result of the choice of τ_1 and τ_2 and ζ. The value of ω_c also depends on the choice of these parameters.

Maximizing the performance index P for the above initial estimates of the unknown parameters K, K_T and ω_c it is found that the value of P increases as ω_c is reduced from the estimated value of 3.2 rad./sec. For comparison with the Routh array approximation results, obtained in Section 3.4.1, values of K and K_T, corresponding to ω_c = 0.786 rad./sec., obtained by this method are given in Table 3.5.

Table 3.5 Results Obtained by Maximizing the Correlation Type Performance Index P on an IBM 7040 Digital Computer.

	Routh Array Approximation	Correlation Function Approximation
<u>Approximate Values</u>		
K	0.99	1.051
K_T	0.82	-0.277
ω_c	0.786	3.2
<u>Maximum Correlation Values</u>		
K	1.00074	1.00083
K_T	0.91208	0.91198
ω_c	0.786	0.786
P_{max}	0.98733	0.98733

3.4.3 <u>Remarks</u>

Results given in Table 3.5 indicate that irrespective of the initial value used to obtain a maximum correlation between the unknown system and the comparison system the digital computer program yielded almost identical results. It can, however, be seen that the Routh array approximation gave initial estimated values closer to the maximum correlation values of the design variables. Furthermore, the correlation function approximation method becomes difficult to handle as the complexity of the unknown system increases. From this point of view the Routh array approximation has an advantage over the correlation function approximation.

4. ALGEBRAIC EXPRESSIONS RELATING THE TIME-DOMAIN RESPONSE WITH SYSTEM PARAMETERS

4.1 Outline.

A continuing problem in systems design is to determine the relationships between time-domain response characteristics and system parameters. The root-locus and the parameter-plane methods are graphical means of establishing numerical relationships between the characteristic roots of linear time-invariant systems and system parameters.[25] Computer solutions of system differential equations, whether obtained by analog or digital means, are essentially numerical in nature and can only be used to determine empirical relationships between system parameters and system response by curve fitting techniques. It has been suggested that the initial value theorem and a Taylor series could be used to obtain analytical relationships.[26] However, little use has been made of this suggestion because of the poor convergence of the Taylor series. Better methods for the evaluation of system response are the state-space approach[27] and the use of moments and special sets of polynomials.[28] However, these methods are again numerical in nature. The following sections deal with a technique which determines algebraic relations between system parameters and the time-domain system response which is superior to the Taylor series approach both in the accuracy achieved with a given number of terms as well as in its computational convenience. Application of this technique to the analysis of piece-wise linear systems will be discussed in Chapter 5.

4.2 Generalized Time Domain Design Method

Consider the feedback control system of Figure 4.1 with the closed-loop transfer function having the general form

$$H(s) = \frac{a_{n-1}s^{n-1} + \ldots\ldots\ldots\ldots\ldots + a_1 s + a_0}{c_n s^n + \ldots\ldots\ldots\ldots\ldots\ldots + c_1 s + c_0}$$

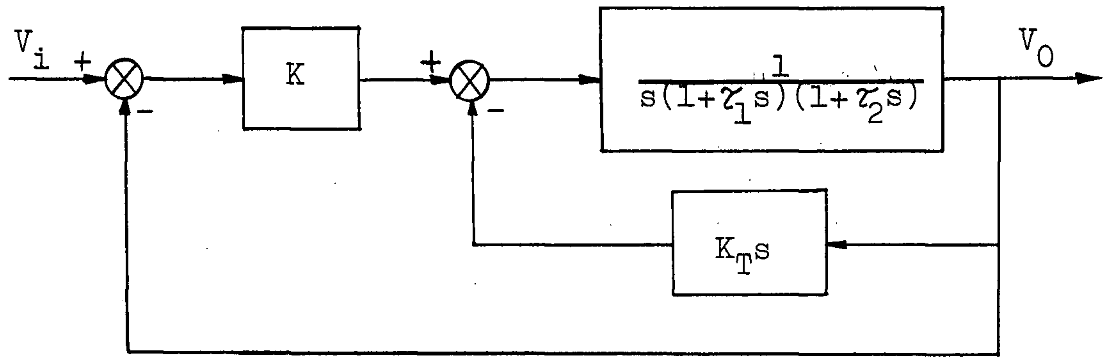


Figure 4.1 A Feedback Control System.

Taking the inverse Laplace transform of H(s) yields

$$h(t) = \frac{1}{2\pi j}\int_C H(s)\exp(st)ds \qquad (4.1)$$

Eq. (4.1) is inconvenient in control system design because of the fact that it does not yield any direct relationship between system parameters and system response. One such relationship is obtained by relating a Taylor series of H in powers of 1/s in the frequency domain to a Taylor series of h in powers of t in the time domain[26]. The expansion of h in the Maclaurin series form yields

$$h(t) = h(0) + th'(0) + t^2h''(0)/2! + \ldots\ldots + t^k h^{(k)}(0)/k!$$

where
$$h(0) = a_1$$
$$h'(0) = a_2 - c_1 h(0)$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$h^{(k)}(0) = a_{k+1} - c_1 h^{(k-1)}(0) - \ldots\ldots\ldots - c_k h(0).$$

The usefulness of this approach of time domain design is, however, lost because of the necessity of employing a large number of terms even for simple systems.

A new method for obtaining direct algebraic relations between system response and system parameters through use of Eq. (4.1) will be presented in Section 4.3. It appears from Eq. (4.1) that if a system with a transfer function $H_c(s) = \exp(-st)$ could be used, it would be possible to obtain h(t) in terms of the system parameters and t following the procedure for evaluating I_{mn} in Eq. (2.23). This, however, requires $H_c(s)$ to be a ratio of polynomials in s, and that the denominator of $H_c(s)$ be a Hurwitz polynomial. The required representation for exp(-st) can be obtained by means of a rational fractional approximation. A Taylor series approximation to exp(-st) is one such possibility. However, it is commonly accepted that Padé approximations are superior to a Taylor series approximation. Padé approximations for exp(st) are given by

$$P_{uv} = \exp(st) = \lim_{(u+v)\to\infty} \frac{F_{uv}(st)}{G_{uv}(st)}$$

where
$$F_{uv}(st) = 1 + \frac{u(st)}{u+v} + \frac{u(u-1)(st)^2}{2!(u+v)(u+v-1)} + \ldots\ldots$$
$$+ \frac{u(u-1)\ldots\ldots(2)(1)(st)^u}{u!(u+v)\ldots\ldots\ldots\ldots(v+1)}$$

and
$$G_{uv}(st) = 1 - \frac{v(st)}{v+u} + \frac{v(v-1)(st)^2}{2!(v+u)(v+u-1)} + \ldots\ldots$$
$$+ (-1)^v \frac{v(v-1)\ldots\ldots\ldots(2)(1)(st)^v}{v!(v+u)\ldots\ldots\ldots\ldots(u+1)}$$

Padé approximations of exp(st) for various values of u and v are shown in Table 4.1.

4.3 The Derivation of Algebraic Relations Between System Response and System Parameters

The technique to be discussed can be applied to time-invariant linear and piece-wise linear systems and is based on Parseval's identity

$$I \triangleq \int_0^{\infty} u(t)v(t)dt = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} V(s)U(-s)ds \tag{4.2}$$

and on a method given by Talbot[4] for evaluating Eq. (4.2). Talbot's method requires that u(t) and v(t) be the output response of stable time-invariant linear systems and is a generalization of the well known technique for evaluating mean-square integrals.

Table 4.1 Padé Approximations of exp(st) for Various Values of u and v.

u	v	$P_{uv} = \exp(st)$
1	1	$\dfrac{2 + st}{2 - st}$
1	2	$\dfrac{6 + 2st}{6 - 4st + s^2t^2}$
2	2	$\dfrac{12 + 6st + s^2t^2}{12 - 6st + s^2t^2}$
2	3	$\dfrac{60 + 24st + 3s^2t^2}{60 - 36st + 9s^2t^2 - s^3t^3}$
3	3	$\dfrac{120 + 60st + 12s^2t^2 + s^3t^3}{120 - 60st + 12s^2t^2 - s^3t^3}$
3	4	$\dfrac{840 + 360st + 60s^2t^2 + 4s^3t^3}{840 - 480st + 120s^2t^2 - 16s^3t^3 + s^4t^4}$
4	4	$\dfrac{1680 + 840st + 180s^2t^2 + 20s^3t^3 + s^4t^4}{1680 - 840st + 180s^2t^2 - 20s^3t^3 + s^4t^4}$

Let $$V(s) = \frac{A(s)}{C(s)} \quad ; \qquad U(-s) = \frac{B(s)}{D(s)}$$

where
$$A(s) \triangleq \sum_{k=0}^{m-1} a_k s^k \; ; \qquad B(s) \triangleq \sum_{k=0}^{n-1} b_k s^k$$
$$C(s) \triangleq \sum_{k=0}^{m} c_k s^k \; ; \qquad D(s) \triangleq \sum_{k=0}^{n} d_k s^k \tag{4.3}$$

and where C(s) and D(-s) are Hurwitz polynomials. The evaluation of Eq.(4.2) could be performed by a partial fraction expansion of the form

$$\frac{A(s)B(s)}{C(s)D(s)} = \frac{Q(s)}{D(s)} + \frac{R(s)}{C(s)} \tag{4.4}$$

where

$$\frac{R(s)}{C(s)} = \sum_{k=0}^{m} \frac{A(s_k)B(s_k)}{D(s_k)C'(s_k)} \cdot \frac{1}{s-s_k} \tag{4.5}$$

and by completing the path of integration along an infinitely large semicircle in the left-half s-plane. Thus,

$$I = \sum_{k=0}^{m} \frac{A(s_k)B(s_k)}{D(s_k)C'(s_k)} \tag{4.6}$$

$$= \lim_{s \to \infty} \frac{sR(s)}{C(s)} \tag{4.7}$$

However, the evaluation of I by Eq.(4.6) requires the numerical determination of the characteristic roots, which can be avoided if suitable use is made of Eq.(4.7). Let

$$R(s) = \sum_{k=0}^{m-1} r_k s^k \; ; \qquad Q(s) = \sum_{k=0}^{n-1} q_k s^k \tag{4.8}$$

$$F(s) \triangleq A(s)B(s) = \sum_{k=0}^{n+m-2} f_k s^k \tag{4.9}$$

It follows from Eq.(4.7) that

$$I = \frac{r_{m-1}}{c_m} \tag{4.10}$$

and from Eq.(4.4) that

$$F(s) = Q(s)C(s) + R(s)D(s) \tag{4.11}$$

Substituting Eqs.(4.3) and (4.8) into Eq.(4.11) and comparing like coefficients of s yields

$$\begin{array}{llllll}
c_0 q_0 + & & + d_0 r_0 & & = f_0 \\
c_1 q_0 + c_0 q_1 + & & + d_1 r_0 + d_0 r_1 & & = f_1 \\
\vdots & & \vdots & & \vdots \\
 & & + d_n r_0 + & & \\
c_m q_0 + & \vdots & & \vdots & = f_m \\
 & c_m q_{n-1} + & & d_n r_{m-1} & = 0
\end{array} \qquad (4.12)$$

where $f_k = 0$ if $k > n+m-2$. The system of Eq.(4.12) can be solved for r_{m-1} and substituted into Eq.(4.10). Thus

$$I = \frac{1}{c_m} \cdot \frac{\Delta'}{\Delta} \qquad (4.13)$$

where

$$\Delta = \left| \begin{array}{cccccccc}
c_0 & & & & d_0 & & & \\
\vdots & \ddots & & 0 & \vdots & \ddots & & 0 \\
\vdots & & \ddots & & \vdots & & \ddots & \\
\vdots & & & c_0 & \vdots & & & d_0 \\
c_m & & & \vdots & d_n & & & \vdots \\
 & \ddots & & \vdots & & \ddots & & \vdots \\
 & 0 & \ddots & \vdots & & 0 & \ddots & \vdots \\
 & & & c_m & & & & d_n
\end{array} \right| \qquad (4.14)$$

and where Δ' is obtained from Δ by replacing the last column by the right hand column of Eq. (4.12). The result given by Eq.(4.13) expresses I in terms of system parameters. Consider now the possibility of expressing I in terms of a time-domain response. If $u(\tau) = \delta(\tau - t)$, where $\delta(t)$ is the unit impulse, then

$$U(-s) = \exp(st) \; ; \qquad (t > 0) \tag{4.15}$$

and Eq.(4.2) reduces to the conventional inverse Laplace transform. However, Talbot's method does not apply for Eq.(4.15). On the otherhand, it is known that Eq.(4.15) can be approximated by means of rational fractions, for example, the Padé approximation[29]

$$P_{23}(st) = \frac{60 - 24st + 3(st)^2}{60 + 36st + 9(st)^2 + (st)^3} \tag{4.16}$$

may be used to approximate the ideal delay exp(-st). Let $u_{mn}(\tau)$ be the impulse response of a system whose transfer function is $P_{mn}(st)$. For the all-pass case where m = n,

$$u_{nn}(\tau) = (-1)^n \delta(\tau) + \delta_{nn}(\tau - t), \quad (t > 0) \tag{4.17}$$

and for the low-pass case where m = n-1

$$u_{mn}(\tau) = \delta_{mn}(\tau - t), \qquad (t > 0) \tag{4.18}$$

It is a consequence of the Padé approximation that

$$\lim_{n \to \infty} \delta_{mn}(\tau) = \delta(\tau) \tag{4.19}$$

For the all-pass case, let B(s) and D(s) be polynomials in s which have no common divisor and which are defined by

$$P_{nn}(-st) = (-1)^n + \frac{B(s)}{D(s)} \tag{4.20}$$

Substituting Eq.(4.17) and Eq.(4.20) into Eq.(4.2) yields

$$v_{nn}(t) = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} V(s)\frac{B(s)}{D(s)}\, ds \tag{4.21}$$

where

$$v_{nn}(t) \triangleq \int_0^{\infty} v(\tau)\ \delta_{nn}(\tau - t)d\tau , \qquad (t>0) \tag{4.22}$$

For the low-pass case, let B(s) and D(s) be polynomials in s which have no common divisor and which are defined by

$$P_{mn}(-st) = \frac{B(s)}{D(s)} \tag{4.23}$$

Substituting Eqs.(4.18) and (4.23) into Eq.(4.2) yields

$$v_{mn}(t) = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} V(s)\frac{B(s)}{D(s)}\, ds \tag{4.24}$$

where

$$v_{mn}(t) \triangleq \int_0^{\infty} v(\tau)\ \delta_{mn}(\tau - t)d\tau , \qquad (t>0) \tag{4.25}$$

The integrals in Eqs.(4.21) and (4.24) can be expressed in the form of Eq.(4.13), consequently

$$v_{mn}(t) = \frac{1}{c_m}\cdot\frac{\Delta'}{\Delta} , \qquad (t>0) \tag{4.26}$$

It follows from Eqs.(4.19), (4.22) and (4.25) that

$$\lim_{n\to\infty} v_{mn}(t) = v(t) , \qquad (t>0) \tag{4.27}$$

Thus, Eq.(4.26) gives the desired algebraic relation between the time-domain response of a system and its parameters.

4.3.1. Illustrative Example

To illustrate the proposed method consider

$$V(s) = \frac{2s^2 + 3.5s + 1.75}{s^3 + 3s^2 + 2.75s + 0.75} = \frac{A(s)}{C(s)} \tag{4.28}$$

and choose Eq.(4.16) so that Eq.(4.23) is used. The choice of Eq.(4.28) is made so that a comparison can be made with the state-space method proposed by Liou which is claimed to be superior to classical methods. Eq.(4.14) yields

$$\Delta = \begin{vmatrix} 0.75 & 0 & 0 & 60 & 0 & 0 \\ 2.75 & 0.75 & 0 & -36t & 60 & 0 \\ 3 & 2.75 & 0.75 & 9t^2 & -36t & 60 \\ 1 & 3 & 2.75 & -t^3 & 9t^2 & -36t \\ 0 & 1 & 3 & 0 & -t^3 & 9t^2 \\ 0 & 0 & 1 & 0 & 0 & -t^3 \end{vmatrix} \tag{4.29}$$

and Eq.(4.9) yields

$$f_0 = 105\ ,\ f_1 = 210 + 42t,\ f_2 = 120 + 8.4t + 5.25t^2$$

$$f_3 = 48t + 10.5t^2\ ,\ f_4 = 6t^2\ ,\ f_5 = 0 \tag{4.30}$$

The response $v_{23}(t)$, given by Eq.(4.26), is expressed as the ratio of a fifth order polynomial in t and a sixth order polynomial in t. Table 4.2 compares the results given by Liou with Eq.(4.26). A direct comparison is not possible. However, Liou's method requires the computation of ninth-order matrix products and is essentially based on a Taylor expansion which includes terms up to the ninth order. For the initial portion of the response Eq.(4.26) is not only a simpler representation, but has the further advantage that system parameters enter in a simple way. This is readily seen by replacing the numerical entries in Eq.(4.29) by parameters. The response given by Eq.(4.26) then consists of a ratio of polynomials in t with the

Table 4.2 The Exact and Approximate Solutions of the System of Eq.(4.28).

$t = nT$	$v(t)$	$v_{23}(t)$	Exact Solution
0	2.00000	2.000000	2.00000
0.1	1.76781	1.767809	1.76781
0.2	1.56775	1.567742	1.56774
0.3	1.39515	1.395146	1.39515
0.4	1.24604	1.246038	1.24604
0.5	1.11701	1.117022	1.11700
0.6	1.00515	1.005196	1.00515
0.7	0.907982	0.908084	0.907979
0.8	0.823383	0.823582	0.823379
0.9	0.749542	0.749889	0.749538
1.0	0.684914	0.685474	0.684912

system parameters entering in a simple algebraic manner. It is evident from this representation that time-domain response sensitivity to parameter variations can be readily evaluated. This as well as other possibilities will now be discussed.

4.4 Applications to the Time-Domain Analysis of Linear Time-Invariant Systems

Consider a feedback system whose closed-loop transfer function is given by (Figure 4.1)

$$H(s) = \frac{K}{\tau_1\tau_2 s^3 + (\tau_1+\tau_2)s^2 + (1+K_T)s + K} \tag{4.31}$$

It is of interest to determine how the gain K and the tachometer feedback parameter K_T affect the maximum overshoot for a unit step input. Let h(t) and g(t) be the unit-impulse and unit-step response, respectively. The maximum overshoot occurs at the first zero of h(t). This can be found from Eq.(4.26) by choosing V(s) and, for example, $U(s) = P_{23}(st)$:

$$h_{23}(t) = \frac{1}{\tau_1\tau_2} \cdot \frac{\begin{vmatrix} K & 0 & 0 & 60 & 0 & 60K \\ 1+K_T & K & 0 & -36t & 60 & 24tK \\ \tau_1+\tau_2 & 1+K_T & K & 9t^2 & -36t & 3t^2K \\ \tau_1\tau_2 & \tau_1+\tau_2 & 1+K_T & -t^3 & 9t^2 & 0 \\ 0 & \tau_1\tau_2 & \tau_1+\tau_2 & 0 & -t^3 & 0 \\ 0 & 0 & \tau_1\tau_2 & 0 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} K & 0 & 0 & 60 & 0 & 0 \\ 1+K_T & K & 0 & -36t & 60 & 0 \\ \tau_1+\tau_2 & 1+K_T & K & 9t^2 & -36t & 60 \\ \tau_1\tau_2 & \tau_1+\tau_2 & 1+K_T & -t^3 & 9t^2 & -36t \\ 0 & \tau_1\tau_2 & \tau_1+\tau_2 & 0 & -t^3 & 9t^2 \\ 0 & 0 & \tau_1\tau_2 & 0 & 0 & -t^3 \end{vmatrix}} \qquad (4.32)$$

Equating the numerator determinant of Eq.(4.32) to zero yields

$$t^6K^2 - 8t^5K(1+K_T) - 128t^4K(\tau_1+\tau_2) - 744t^3K\tau_1\tau_2 + 20t^4(1+K_T)^2$$
$$- 840t^2(1+K_T)\tau_1\tau_2 + 120t^3(1+K_T)(\tau_1+\tau_2) + 1200t^2(\tau_1+\tau_2)^2$$
$$+ 9600t(\tau_1+\tau_2)\tau_1\tau_2 + 3600\tau_1^2\tau_2^2 = 0 \qquad (4.33)$$

If τ_1, τ_2 and $t = t_m$ are specified, Eq.(4.33) is a quadratic form in K and $1+K_T$ and shows how these parameters must be related if the maximum overshoot is to occur at the instant t_m. The maximum output amplitude $g_m = g(t_m)$ can also be expressed in the form of Eq.(4.26). To obtain the desired result let A(s) and C(s) be polynomials in s which have no common factor and which are defined by

$$G(s) = \frac{1}{s} + \frac{H(s) - 1}{s} = \frac{1}{s} + \frac{A(s)}{C(s)} \tag{4.34}$$

If $V(s) \triangleq A(s)/C(s)$, the inverse Laplace transform of Eq. (4.34) yields

$$g(t) = 1 + v(t) , \qquad (t > 0) \tag{4.35}$$

and from the previous discussion it follows that v(t) can be approximated by Eq.(4.26). Thus

$$g_{mn}(t) = 1 + v_{mn}(t) ; \qquad (t > 0) \tag{4.36}$$

is an approximation to the unit step response where

$$v_{mn}(t) = \frac{1}{\tau_1\tau_2} \cdot \frac{\begin{vmatrix} K & 0 & 0 & 60 & 0 & f_0 \\ 1+K_T & K & 0 & -36t & 60 & f_1 \\ \tau_1+\tau_2 & 1+K_T & K & 9t^2 & -36t & f_2 \\ \tau_1\tau_2 & \tau_1+\tau_2 & 1+K_T & -t^3 & 9t^2 & f_3 \\ 0 & \tau_1\tau_2 & \tau_1+\tau_2 & 0 & -t^3 & f_4 \\ 0 & 0 & \tau_1\tau_2 & 0 & 0 & f_5 \end{vmatrix}}{\begin{vmatrix} K & 0 & 0 & 60 & 0 & 0 \\ 1+K_T & K & 0 & -36t & 60 & 0 \\ \tau_1+\tau_2 & 1+K_T & K & 9t^2 & -36t & 60 \\ \tau_1\tau_2 & \tau_1+\tau_2 & 1+K_T & -t^3 & 9t^2 & -36t \\ 0 & \tau_1\tau_2 & \tau_1+\tau_2 & 0 & -t^3 & 9t^2 \\ 0 & 0 & \tau_1\tau_2 & 0 & 0 & -t^3 \end{vmatrix}} \tag{4.37}$$

and where

$$f_0 = -60(1+K_T) \; ; \; f_1 = -60(\tau_1+\tau_2) - 24(1+K_T)t;$$

$$f_2 = -60\tau_1\tau_2 - 24(\tau_1+\tau_2)t - 3(1+K_T)t^2 \; ;$$

$$f_3 = -24\tau_1\tau_2 t - 3(\tau_1+\tau_2)t^2 \; ; \quad f_4 = -3\tau_1\tau_2 t^2 \; ;$$

$$f_5 = 0.$$

Figure 4.2 illustrates the type of data that can be obtained from Eqs. (4.33) and (4.36) where the choice $\tau_1 = 1.174$, $\tau_2 = 0.46$ has been made. By choosing $1+K_T$ and t_m, Eq.(4.33) can be solved for K and Eq.(4.36) can be solved for the maximum output amplitude g_m. The time-domain sensitivity of g_m and t_m to variation in K and K_T can be determined from Figure 4.2. The stability boundary (SB), defined by the values of K and K_T which result in an unstable system, is also shown in Figure 4.2. It is interesting to note that with suitable restrictions, the proposed method determines the initial response of unstable systems as shown in Figure 4.3. To discuss the method for an unstable system, direct use must be made of the inverse Laplace transform

$$g(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{H(s)}{s} \exp(st)ds \; ; \quad (t > 0) \qquad (4.38)$$

where the line $c+j\omega$ is chosen so that all poles of H(s) are to the left. Provided that the poles of $P_{mn}(-st)$ are to the right of this line, the exponential function in Eq.(4.38) can be approximated by $P_{mn}(-st)$ and the integral

$$g_{mn}(t) \triangleq \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{H(s)}{s} P_{mn}(-st)ds \; ; \quad (t>0) \tag{4.39}$$

evaluated by the method of residues, that is, Eqs. (4.6), (4.7) and consequently Eq.(4.13) then remain valid and can be used as an alternative method for evaluating Eq.(4.39).

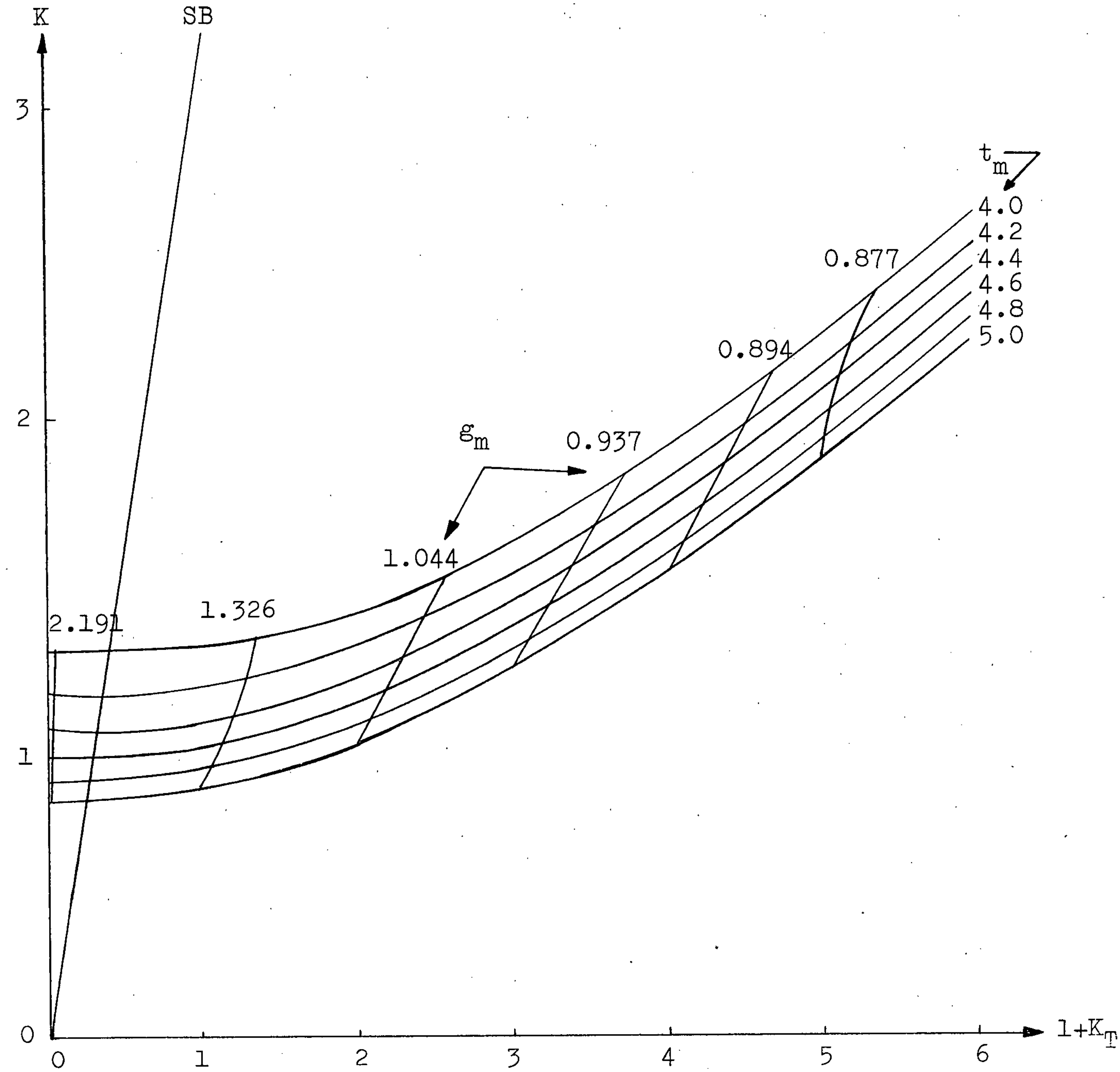
Figure 4.4 illustrates a plot of Eq.(4.36) for the case $K = 1$, $K_T = 1.7$, compared with the exact response. The closed-loop transfer function for this case is

$$H(s) = \frac{2}{s^3 + 3.2s^2 + 3.4s + 2} \tag{4.40}$$

The accuracy can be improved by choosing a larger value of n. However, even for the choice $m = 2$, $n = 3$, it is seen that reasonable accuracy is maintained up to the first overshoot.

4.5 Method of Residues

If most of the system parameters are specified numerically eq.(4.26) can be readily evaluated by a digital computer, even for systems of high order. However, if most of the system parameters are initially unspecified, the algebraic forms obtained from Eq.(4.26) could become unwieldy. An alternative approach, based on the method of residues applied to a form such as Eq.(4.39), could then be considered. The conventional method for evaluating Eq.(4.38) is to complete the path of integration in the left-half s-plane and requires that the poles of H(s) be determined. However, if Eq.(4.39) is used, the path of



<u>Figure 4.2</u> The Gains K and K_T as Functions of Time t_m of the First Maximum Amplitude g_m.

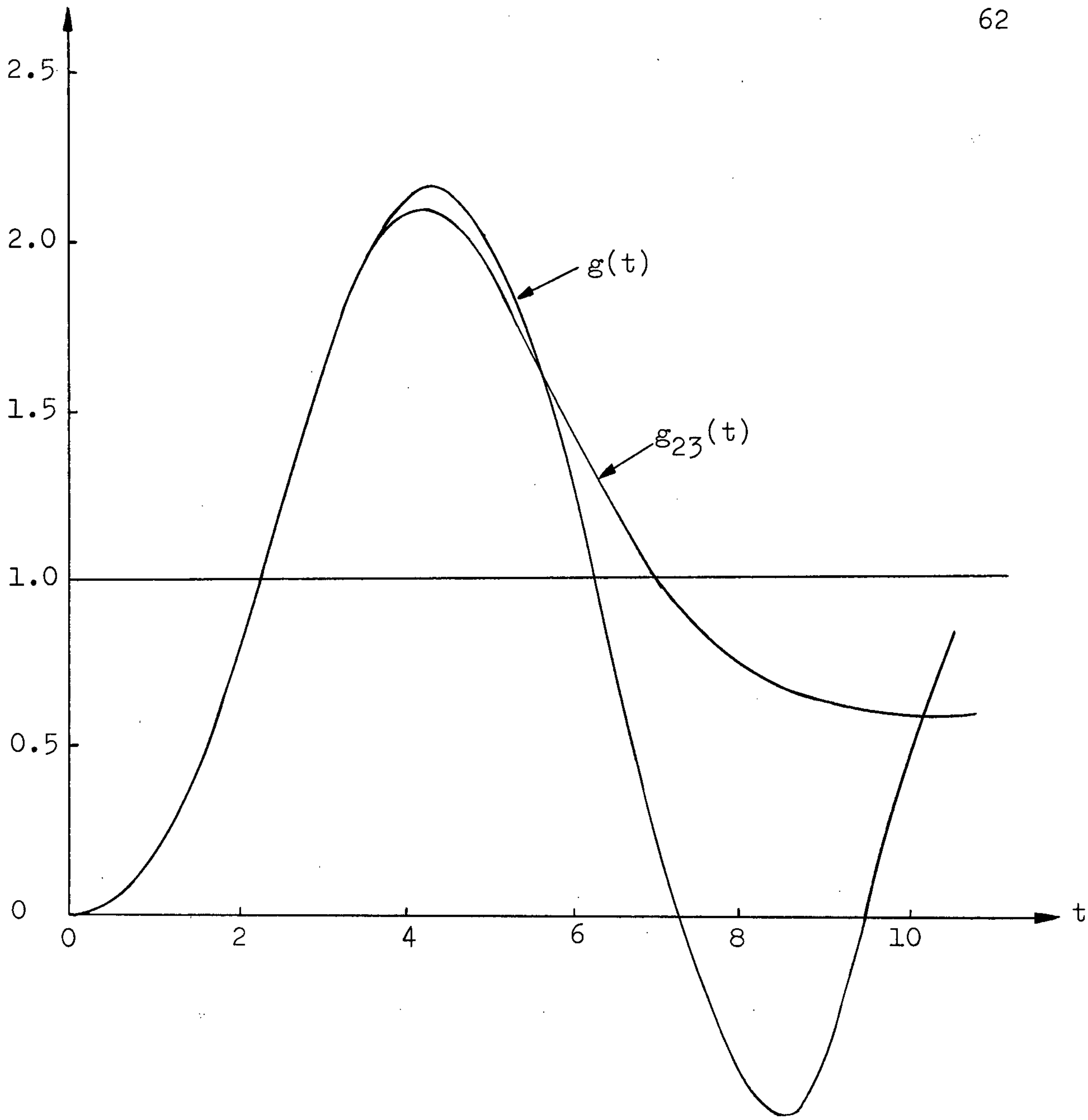


Figure 4.3 The Exact and Approximate Unit-Step Responses of the System Shown in Figure 4.1 in the Case of Instability.

integration can be completed in the right-half s-plane and yields

$$g_{mn}(t) = \sum_{k=1}^{n} \frac{H(s_k)\ B(s_k)}{s_k\ \ D'(s_k)} \tag{4.41}$$

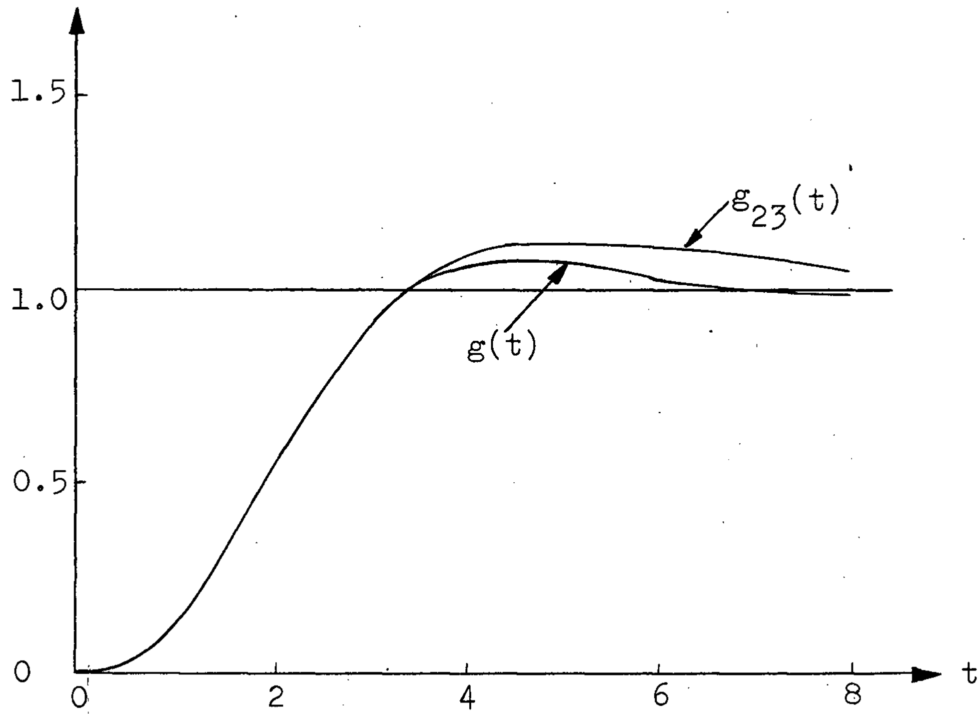


Figure 4.4 The Exact and Approximate Unit-Step Responses of the System Shown in Figure 4.1.

where the form of Eq.(4.23) has been used and where s_k are the poles of $P_{mn}(-st)$. If, for example, n = 3 is chosen, Eq.(4.41) contains only three terms irrespective of the order of the system. Figure 4.5 illustrates the response obtained from Eq.(4.41) where H(s) is given by Eq.(4.40).

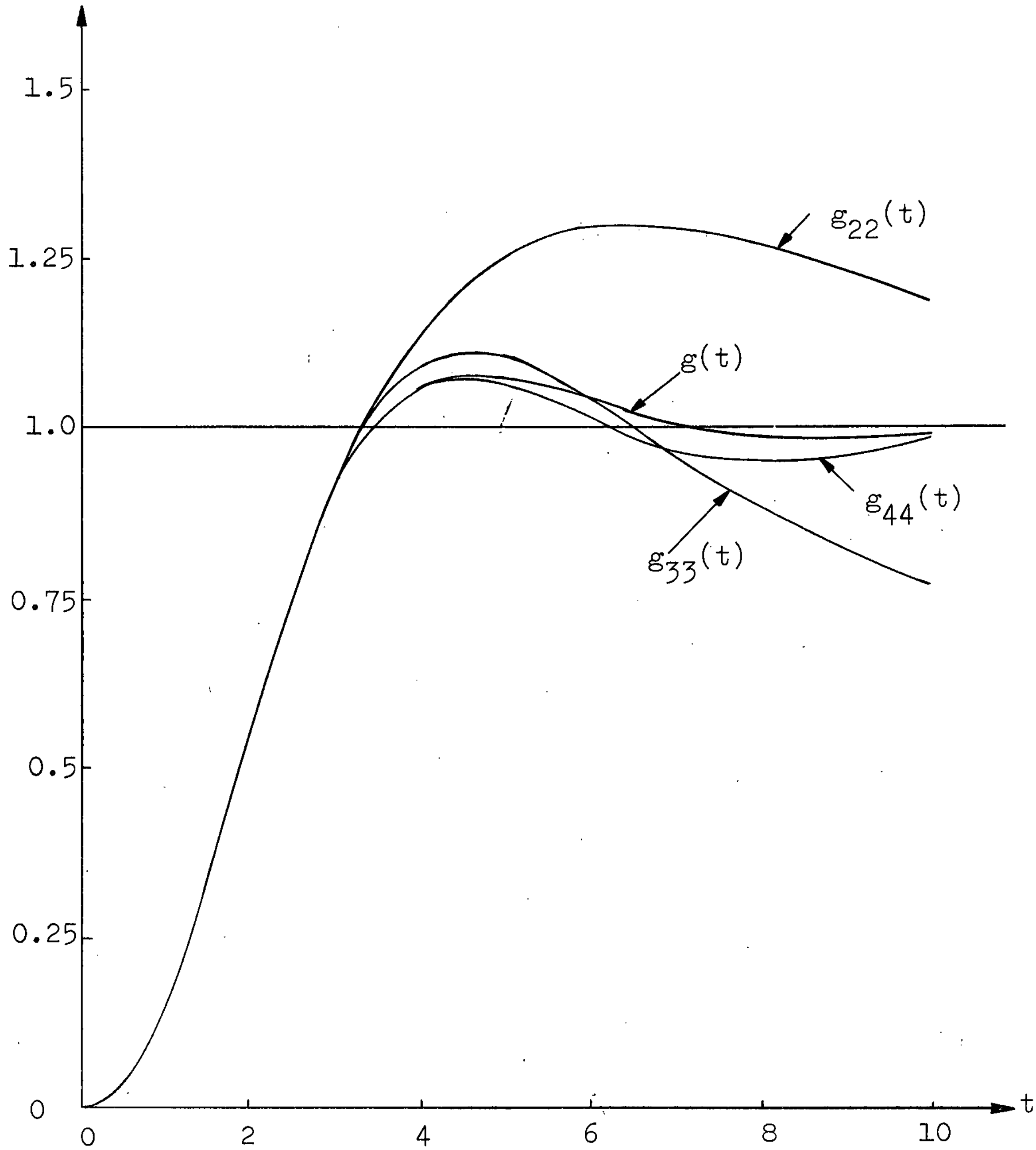


Figure 4.5 The Exact and Approximate Unit-Step Responses of the System Shown in Figure 4.1.

5. NONLINEAR SYSTEM DESIGN

5.1 Outline

Any system with any number of loops and linear elements can be reduced to an equivalent system having the block diagram representation shown in Figure 5.1 provided that the system contains only one nonlinear element.

The characteristic of the nonlinear element is taken to have the form shown in Figure 5.2 so that it lies in the

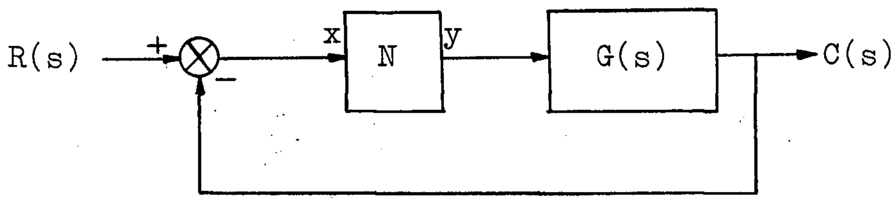


Figure 5.1 Nonlinear Control System.

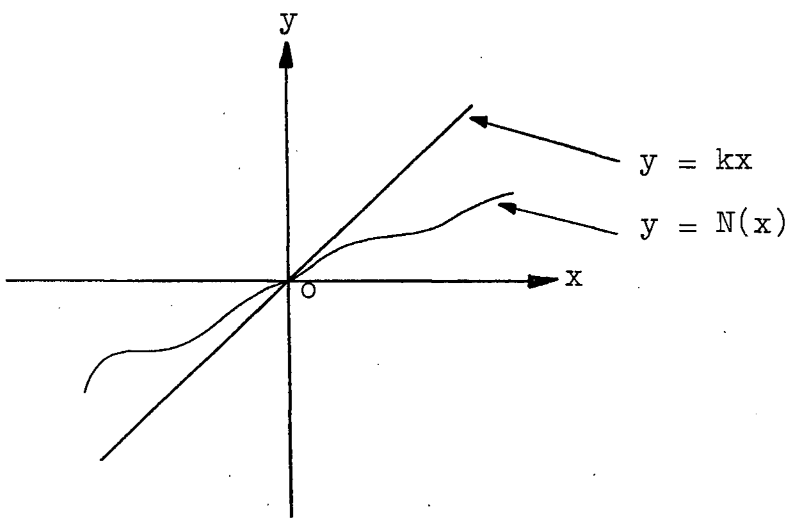


Figure 5.2 Characteristic of the Nonlinear Element and its Linear Bounds.

sector formed by the x-axis and the line

$$y = kx$$

where $k > 0$.

The linear part, given by G(s), in Figure 5.1 can be designed on the basis of an optimum output correlation of the closed loop system with respect to the output of a closed loop comparison system where G is replaced by G_c. The parameters of the comparison system are assumed to be known except for the optimum gain K. K can, however, be expressed as a function of the slope k and the parameters of the comparison system. For example, for a second order system K could be expressed as

$$K = K(k, \zeta_c, \omega_c)$$

where ζ_c is the damping ratio and ω_c is the natural frequency of oscillation of the comparison system.

5.2 The Design Principle

Replacing the nonlinearity in Figure 5.1 by k and the linear part by the comparison system $G_c(s)$ the following equivalent system of Figure 5.3 is obtained.

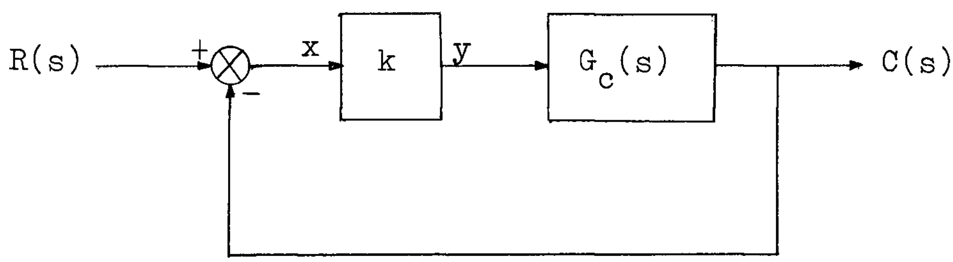


Figure 5.3 Equivalent Linear System for the System of Figure 5.1.

In order to define the optimum gain in terms of k and the known parameters of the comparison system, Popov's[30] criterion for absolute stability will be used. Popov's criterion for absolute stability for the system of Figure 5.3 requires that

$$\mathrm{Re}\left[(1+j\omega q)\ G_c(j\omega)\right] + 1/k > 0 \qquad (5.1)$$

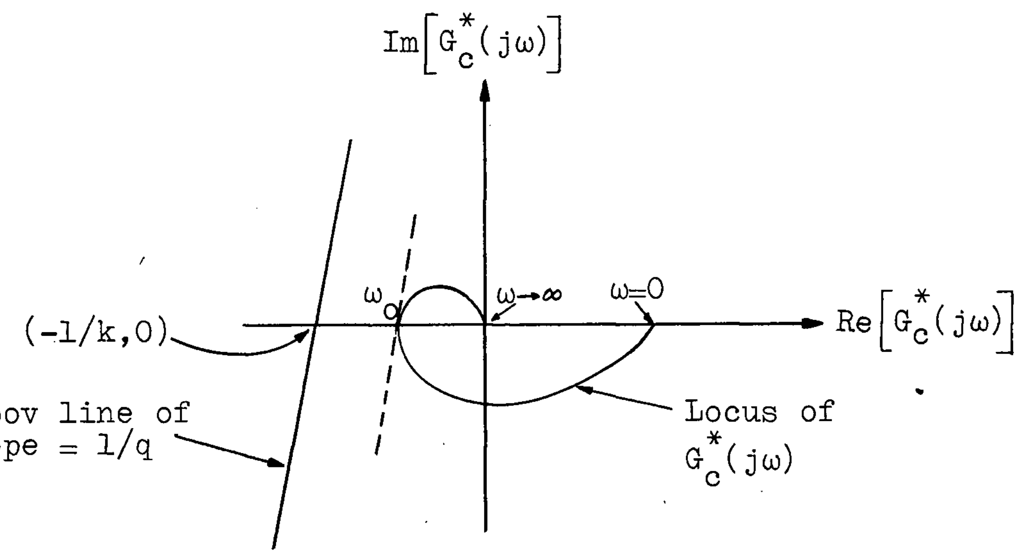


Figure 5.4 Popov Line and Locus of $G_c^*(j\omega)$.

where q, an arbitrary real parameter, determines the slope of the Popov line shown in Figure 5.4. The system is stable provided that the locus of $G_c^*(j\omega)$ lies to the right of the Popov line passing through the point (-1/k,0). In Figure 5.4 the dotted line represents the tangent to the locus of $G_c^*(j\omega)$ at the critical frequency ω_0 and

$$\mathrm{Re}\left[G_c^*(j\omega)\right] = \mathrm{Re}\left[G_c(j\omega)\right]$$

and

$$\mathrm{Im}\left[G_c^*(j\omega)\right] = \omega\mathrm{Im}\left[G_c(j\omega)\right]$$

If the arbitrary variable q is chosen so that the

Popov line passing through the point (-1/k,0) is parallel to the tangent to the locus of $G_c^*(j\omega)$ at the critical frequency, then the stability of the system in the Popov sense is ensured. The above value of q when substituted in Eq. (5.1) will then define a range of the optimum gain K of the comparison system.

5.2.1 Choice of q for the Popov Line and the Range of K

Consider the following comparison system given by

$$G_c(s) = \frac{K}{(s+b)(s^2+2\zeta_c\omega_c s+\omega_c^2)} \tag{5.2}$$

Substituting s = jω and separating the real and imaginary parts yields

$$\mathrm{Re}\left[G_c^*(j\omega)\right] = \mathrm{Re}\left[G_c(j\omega)\right] = \frac{K\left[b\omega_c^2 - \omega^2(b+2\zeta_c\omega_c)\right]}{(\omega^2+b^2)\left[(\omega_c^2-\omega^2)^2+4\zeta_c^2\omega_c^2\omega^2\right]} \tag{5.3}$$

and

$$\mathrm{Im}\left[G_c^*(j\omega)\right] = \omega\mathrm{Im}\left[G_c(j\omega)\right] = \frac{K\omega^2\left[(\omega^2-\omega_c^2)-2\zeta_c\omega_c b\right]}{(\omega^2+b^2)\left[(\omega_c^2-\omega^2)^2+4\zeta_c^2\omega_c^2\omega^2\right]} \tag{5.4}$$

The locus of $G_c^*(j\omega)$, shown in Figure 5.4, cuts the real axis in the left half plane at $\omega = \omega_0$, the critical frequency. Equating Eq. (5.4) to zero yields

$$\omega = \omega_0 = \sqrt{\omega_c^2 + 2\zeta_c\omega_c b}$$

The positive value of ω_0 is taken since the locus of $G_c^*(j\omega)$ is plotted for positive values of ω only.

The slope of the tangent to the locus of $G_c^*(j\omega)$ at ω_0 is given by the expression

$$\frac{\omega_c^2 + 2\zeta_c\omega_c b}{2\zeta_c\omega_c + b} \tag{5.5}$$

Substituting the value for $G_c^*(j\omega)$ into the inequality (5.1) yields

$$\frac{K\left[b\omega_c^2 - \omega^2\left\{(b+2\zeta_c\omega_c) - q(\omega_c^2+2\zeta_c\omega_c b) + q\omega^2\right\}\right]}{(\omega^2+b^2)\left[(\omega_c^2-\omega^2)^2 + 4\zeta_c^2\,\omega_c^2\omega^2\right]} + \frac{1}{k} > 0 \tag{5.6}$$

Choosing the slope of the Popov line equal to the slope of the tangent to the locus of $G_c^*(j\omega)$ at ω_0 and using Eq. (5.5) results in the following expression for q:

$$q = \frac{2\zeta_c\omega_c + b}{\omega_c^2 + 2\zeta_c\omega_c b}$$

Substituting this value for q into Eq. (5.6) and evaluating it at the critical frequency yields

$$\frac{-K}{2\zeta_c\omega_c(\omega_c^2 + 2\zeta_c\omega_c b + b^2)} + \frac{1}{k} > 0$$

Thus the range of K is given by

$$0 < kK < 2\zeta_c\omega_c(\omega_c^2 + 2\zeta_c\omega_c b + b^2)$$

where k, ζ_c, ω_c and b are known quantities.

The linear part G(s) can now be designed on the basis of an optimum correlation with respect to the comparison system $G_c(s)$ by maximizing the performance index P, given by Eq. (2.35), where

$$U(s) = \frac{kG(s)}{1 + kG(s)}$$

and

$$V(s) = \frac{kG_c(s)}{1 + kG_c(s)}$$

The critical frequency ω_0, the slope of the Popov line $\frac{1}{q}$, and the range of the optimum gain K for several comparison systems are given in Table 5.1.

The stability of a system designed by the above technique cannot be unconditionally guaranteed. However, since the responses of the linearized systems are similar which is a consequence of maximizing P, it follows that $G(j\omega)$ must approximate $G_c(j\omega)$ over a range in values of ω. Thus if $G_c(j\omega)$ does not cross the Popov line it can be anticipated that $G(j\omega)$ will not cross the Popov line. This approach cannot, therefore, guarantee a suitable response in the time-domain. It does, however, result in a comparatively simple way for choosing system parameters. This method could be used to determine suitable initial estimates for the time-domain approach discussed in the following section.

Table 5.1 Critical Frequency, Popov Line Slope and Range of Optimum Gain in Terms of Known Comparison System Parameters.

Comparison System $G_c(s)$	Critical Frequency ω_0	Popov Line Slope $1/q$	Range of Optimum Gain K
$\dfrac{K}{s^2 + 2\zeta_c\omega_c s + \omega_c^2}$	——	$2\zeta_c\omega_c$	$0 < kK < \infty$
$\dfrac{K}{s(s^2 + 2\zeta_c\omega_c s + \omega_c^2)}$	ω_c	$\omega_c/2\zeta_c$	$0 < kK < 2\zeta_c\omega_c{}^3$
$\dfrac{K}{(s+b)(s^2 + 2\zeta_c\omega_c s+\omega_c^2)}$	$\sqrt{\omega_c^2 + 2\zeta_c\omega_c b}$	$\dfrac{\omega_c^2 + 2\zeta_c\omega_c b}{2\zeta_c\omega_c + b}$	$0 < kK < 2\zeta_c\omega_c(\omega_c^2+2\zeta_c\omega_c b+b^2)$
$\dfrac{K}{s(s+b)(s^2+2\zeta_c\omega_c s+\omega_c^2)}$	$\sqrt{\dfrac{b\omega_c^2}{2\zeta_c\omega_c + b}}$	$\dfrac{b\omega_c(2\zeta_c\omega_c+b)}{2\zeta_c(\omega_c^2+2\zeta_c\omega_c b+b^2)-b\omega_c}$	$0 < kK < \dfrac{2\zeta_c\omega_c^3 b(\omega_c^2+2\zeta_c\omega_c b+b^2)}{(2\zeta_c\omega_c + b)^2}$
$\dfrac{K(s+a)}{(s+b)(s^2+2\zeta_c\omega_c s+\omega_c^2)}$	$\sqrt{\dfrac{(a-b)\omega_c^2+2\zeta_c\omega_c b}{a - (b+2\zeta_c\omega_c)}}$	$\dfrac{\left[\{(a-b)\omega_c^2+2\zeta_c\omega_c b\} \cdot\{a-(b+2\zeta_c\omega_c)\}\right]}{\left[(a-b)\omega_c^2+2\zeta_c\omega_c b + \left[\{a(b+2\zeta_c\omega_c)\} \cdot\{a-(b+2\zeta_c\omega_c)\}\right]\right]}$	$0 < kK < \dfrac{2\zeta_c\omega_c\left[\omega_c^2 - 1 -b\{a-(b+2\zeta_c\omega_c)\}\right]}{a - b - 2\zeta_c\omega_c}$

5.3 Time-Domain Analysis of Piece-wise Linear Systems

Consider the piece-wise linear feedback system illustrated in Figure 5.5 where the nonlinear element has a saturation type of characteristic. This type of non-linearity approximates the saturation characteristics of many energy conversion devices at high signal levels. The saturation level in this example is taken as $v_s = 1.4$. The method to be discussed is, however, also applicable to the case of amplifier saturation where the nonlinear element precedes the linear element.

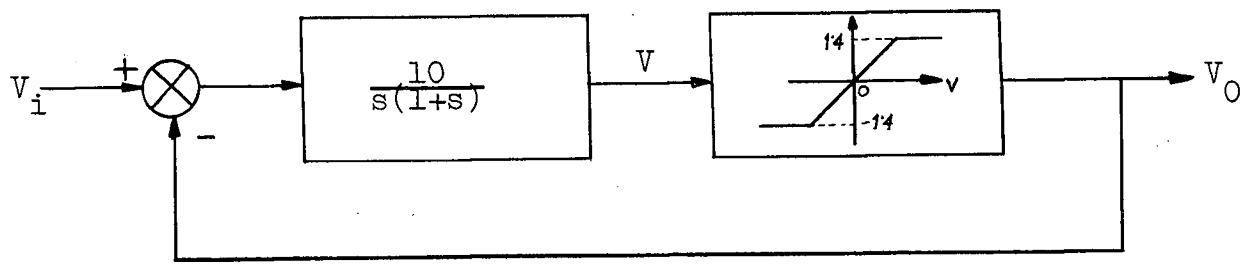


Figure 5.5 A Piece-Wise Linear Feedback System.

Let $v(0) = 0 = \dot{v}(0)$ be the initial conditions and let the input be a unit step. Let $v_1 = v$, $v_2 = \dot{v}$. Before saturation occurs, the state equations are

$$\begin{aligned} \dot{v}_1 - v_2 &= 0 \\ \dot{v}_2 + v_2 + 10v_1 &= 10 \end{aligned} \tag{5.7}$$

Solving Eq. (5.7) by Laplace transform methods yields

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} h_1+h_2 & h_1 \\ -10h_1 & h_2 \end{pmatrix} \begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} + \begin{pmatrix} 1-h_1-h_2 \\ 10h_1 \end{pmatrix} \quad (5.8)$$

where

$$h_1 \triangleq \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{\exp(st)}{s^2+s+10}\, ds; \quad h_2 = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{s \exp(st)}{s^2+s+10}\, ds \quad (5.9)$$

For the given initial conditions Eq. (5.8) reduces to

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} 1-h_1-h_2 \\ 10h_1 \end{pmatrix} , \quad 0 \leqq t \leqq t_1 \quad (5.10)$$

At the instant t_1 of saturation, the state equations change to

$$\begin{aligned} \dot{v}_1 - v_2 &= 0 \\ \dot{v}_2 + v_2 &= y \end{aligned} \quad (5.11)$$

where $$y \triangleq 10(1-v_s)$$

and where v_s is the saturation level of the nonlinear element. Solving Eq. (5.11) by Laplace transform methods yields

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1-h_3 \\ 0 & h_3 \end{pmatrix} \begin{pmatrix} v_1(t_1) \\ v_2(t_1) \end{pmatrix} + \begin{pmatrix} y(t-1+h_3) \\ y(1-h_3) \end{pmatrix} ; \; t_1 \leqq t \leqq t_2 \quad (5.12)$$

where

$$h_3 \triangleq \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{\exp(st)}{s+1}\, ds \qquad (5.13)$$

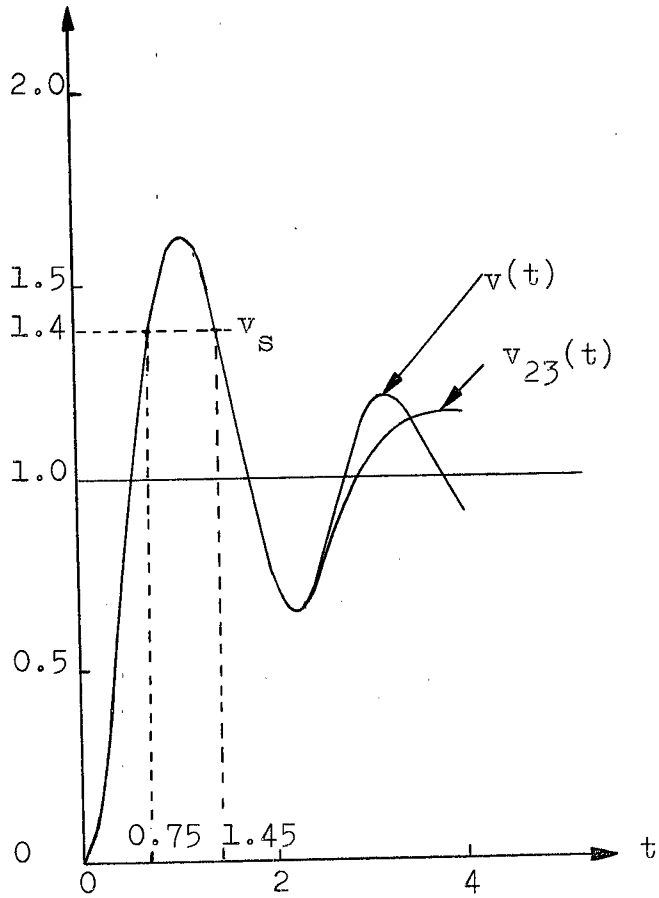
At $t = t_3$, the state equations are again given by Eq. (5.7) and Eq. (5.8) can be used provided that $v_1(0)$ and $v_2(0)$ are replaced by $v_1(t_2)$ and $v_2(t_2)$, respectively. The functions h_1, h_2, h_3 can be evaluated to any degree of accuracy in the form of Eq. (4.26) by means of a rational fraction approximation to e^{-st}. The choice of a rational fraction approximation is based on a compromise between the desired accuracy over a given period of time and computational simplicity. If, for example, $P_{23}(st)$ is used and if $v_s = 1.4$, Eq. (4.26) yields

$$h_1 \cong \frac{300t^5-240t^4-3780t^3+360t^2+3600t}{1000t^6+900t^5+4500t^4+2640t^3+2700t^2+2160t+3600}$$

$$h_2 \cong \frac{5100t^4+480t^3-15660t^2-1440t+3600}{1000t^6+900t^5+4500t^4+2640t^3+2700t^2+2160t+3600}$$

$$h_3 \cong \frac{3t^2-24t+60}{t^3+9t^2+36t+60} \qquad (5.14)$$

Figure 5.6 shows the response v(t) obtained from Eqs. (5.8), (5.10), (5.11) and (5.14) compared with the exact response. Since the matrices in Eqs. (5.8) and (5.12) are state-transition matrices, it is seen that the elements of the state-transition matrices can be represented in the form of Eq. (4.26), that is, the elements can be expressed as the ratio of polynomials in t with coefficients which are algebraic



<u>Figure 5.6</u> The Exact and Approximate Response v(t) of the System Shown in Figure 5.5 for a Unit Step Input.

functions of the system parameters. By means of these algebraic forms, the parameters of a piece-wise linear control system can be directly related to its time-domain response. The application of these forms to system design and to the determination of response sensitivity to parameter variations is similar to that given in Section 4.4 and will not be discussed further.

6. MULTIVARIABLE CONTROL SYSTEM DESIGN

6.1 Outline

The design of multivariable control systems utilizing matrix formulation has been considered by many authors.[31,32,33] While some of them have been concerned with the question of physical realizability, the problem of relating system parameters to time-domain response and interaction in the time-domain are not considered by these authors. Interaction within a multivariable control system may, in some applications, be desirable, the interaction being controlled rather than removed. Recognizing that the physical construction of a completely noninteracting control system is impossible, a root-locus design method[34] applying the techniques of single-variable system design has been suggested. However, the practical advantage of the root-locus method applied to multivariable control system design could be realized only if a rapid transition from the pole-zero locations to the time-domain characteristics could be made. Two methods of designing multivariable control systems, based on the methods of linear single-variable control system design discussed in Chapters 2 and 4, are given in the following sections.

6.2 Design Method Based on Performance Functionals

Consider the system of Figure 6.1 which represents an interacting plant with facility for compensation to be inserted in as G_{11} and G_{22}.

Assume that $H_{11} = 1/(s+1)$, $H_{12} = H_{21} = 1/(s+20)$,

$H_{22} = 1/(s+2)$,

$G_{11} = K_1(s+\alpha_1)/(s+n_1\alpha_1)$, $n_1 < 1$

$G_{22} = K_2(s+\alpha_2)/(s+n_2\alpha_2)$, $n_2 < 1$.

The group of design variables K_1, K_2, α_1, α_2, n_1 and n_2 are positive real numbers.

The equations describing the block diagram of the system are

$$C_1 = \frac{G_{11}H_{11}(1+G_{22}H_{22}) - G_{11}G_{22}H_{12}H_{21}}{\Delta}R_1 + \frac{G_{22}H_{12}}{\Delta}R_2$$

$$C_2 = \frac{G_{11}H_{21}}{\Delta}R_1 + \frac{G_{22}H_{22}(1+G_{11}H_{11}) - G_{11}G_{22}H_{12}H_{21}}{\Delta}R_2$$

where $\Delta = (1+G_{11}H_{11})(1+G_{22}H_{22}) - G_{11}G_{22}H_{12}H_{21}$

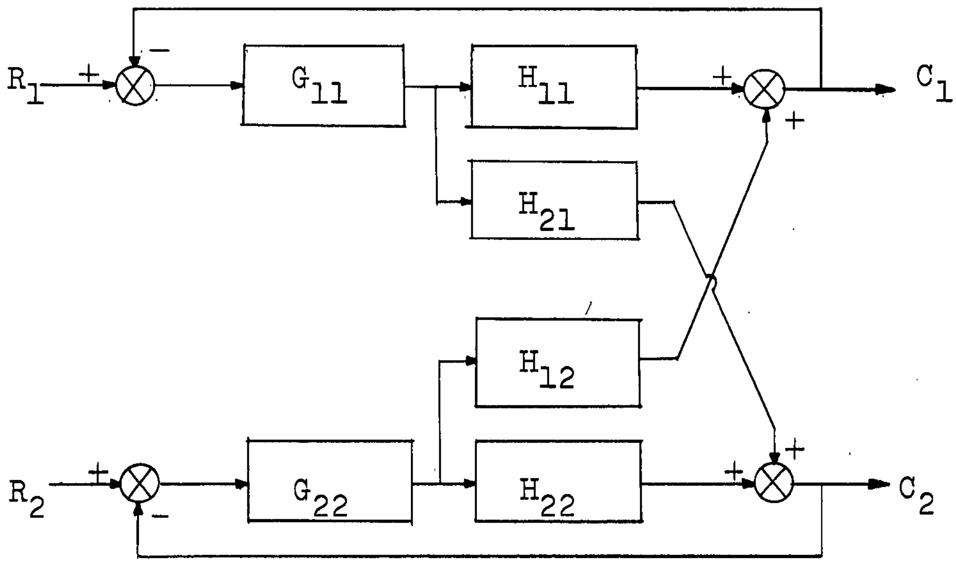


Figure 6.1 Multivariable Control System.

Consider the case when $R_2 = 0$,

$$\frac{C_1}{R_1} = \frac{A_{11}}{1+A_{11}} \tag{6.1}$$

where $A_{11} = G_{11}\left[H_{11} - H_{12}H_{21}\{G_{22}/(1+G_{22}H_{22})\}\right]$

Eq. (6.1) can be represented by the block diagram shown in Figure 6.2. Similarly, if $R_1 = 0$,

$$\frac{C_2}{R_2} = \frac{A_{22}}{1 + A_{22}} \tag{6.2}$$

where $\quad A_{22} = G_{22}\left[H_{22} - H_{12}H_{21}\{G_{11}/(1+G_{11}H_{11})\}\right]$

then the block diagram representing Eq. (6.2) is shown in Figure 6.3.

Considering Eqs. (6.1) and (6.2) as the transfer functions of single variable systems it becomes possible to determine two sets of parameters of the compensating networks G_{11} and G_{22} by maximizing the correlations of C_1 in Figure 6.2

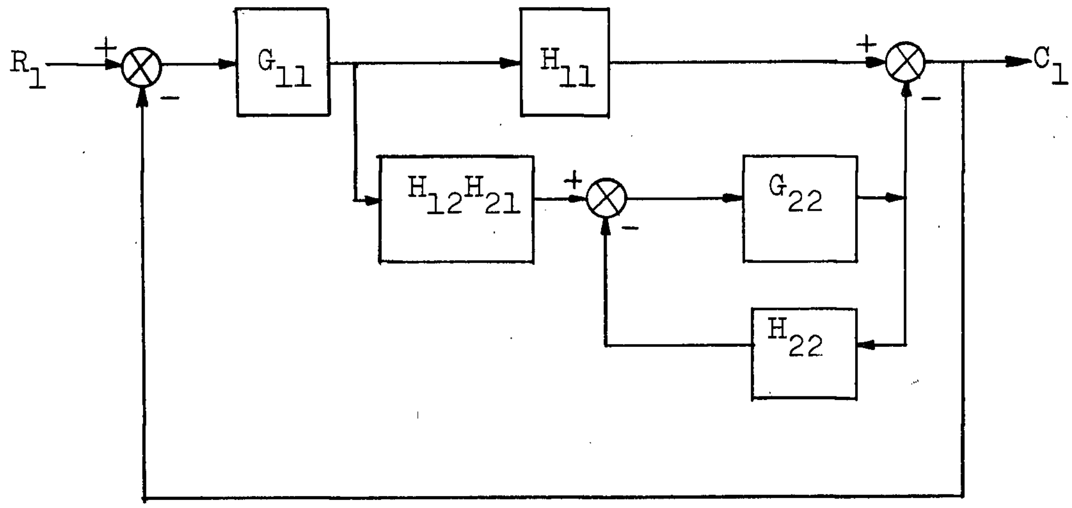


Figure 6.2 Block Diagram Representation of Eq. (6.1).

and C_2 in Figure 6.3 with the output responses of two known reference systems. The design technique has been described in Section 2.3.2 of Chapter 2. The choice of which set of parameters to be used may be made on the basis of satisfying

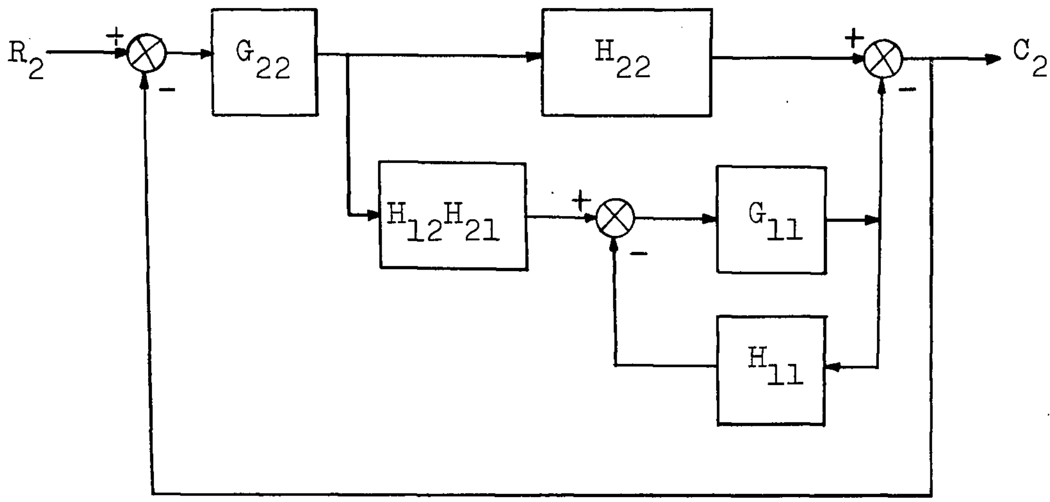


Figure 6.3 Block Diagram Representation of Eq. (6.2).

a given interaction constraint[34] such as

$$\left.\frac{C_i}{R_j}\right|_{\substack{R_i = 0 \\ \omega = 0}} \leqq \varepsilon_{ij} \left.\frac{C_i}{R_i}\right|_{\substack{R_j = 0 \\ \omega = 0}} \tag{6.3}$$

or on the basis of minimizing a suitable time domain sensitivity function for the system similar to that given by Eq. (3.6) in Chapter 3.

6.3 Time Domain Design Method

If the design specifications are given in terms of the transient response of the multivariable system then the design technique described in Section 6.2 cannot be employed. A method will now be described which employs time domain specifications for the design of a multivariable control system.

Consider the following design specifications for the system shown in Figure 6.1. Here u(t) represents the unit step and all initial conditions are zero.

1. $c_1(t)\Big|_{\substack{r_1(t) = u(t) \\ r_2(t) = 0}}$ reaches a maximum

overshoot of $x_1\%$ in t_1 seconds.

2. $c_2(t)\Big|_{\substack{r_1(t) = 0 \\ r_2(t) = u(t)}}$ reaches a maximum

overshoot of $x_2\%$ in t_2 seconds.

3. $$\frac{C_1}{R_2}\Bigg|_{\substack{R_1 = 0 \\ \omega = 0}} = \frac{C_2}{R_1}\Bigg|_{\substack{R_2 = 0 \\ \omega = 0}}$$

4. $$c_1(t_1)\Bigg|_{\substack{r_1(t)=0 \\ r_2(t)=u(t)}} = c_2(t_2)\Bigg|_{\substack{r_1(t)=u(t) \\ r_2(t)=0}}$$

where t_1 and t_2 are the times defined in (1) and (2) above.

The last design specification defines an additional interaction constraint. In the general form it is defined as[34]

$$c_i\Bigg|_{\substack{r_j=f(t) \\ r_i=0}} \leqq \varepsilon'_{ij}, \text{ for } j \neq i$$

at some time, for example at the time for which the response c_i is a maximum.

In terms of the above definition the last design specification yields $\varepsilon'_{12} = \varepsilon'_{21}$.

For the set of design specifications outlined above the required design equations will now be obtained. Recognizing that the time of maximum overshoot for a step input corresponds to the time when the impulse response equals zero, the design specifications (1) and (2) yield

$$
\begin{aligned}
h_1(t_1) &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} H_1(s)\ \exp(st_1)ds = 0 \\
h_2(t_2) &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} H_2(s)\ \exp(st_2)ds = 0
\end{aligned}
\tag{6.4}
$$

where $H_1(s)$ and $H_2(s)$ are given by Eqs. (6.1) and (6.2). For the system under consideration $H_1(s)$ and $H_2(s)$ have sixth order denominator polynomials and fifth order numerator polynomials in s. Using Padé approximation of the desired order for the exponential functions in Eq. (6.4) and evaluating Eq. (6.4) two design equations, in terms of the known and unknown system parameters and real time, are obtained. The interaction constraint defined by the design specification (4) yields

$$
\begin{aligned}
c_1(t_1) &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left[H_1(s)/s\right]\exp(st_1)ds \leqq \varepsilon'_{12} \\
c_2(t_2) &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left[H_2(s)/s\right]\exp(st_2)ds \leqq \varepsilon'_{21}
\end{aligned}
\tag{6.5}
$$

where $\varepsilon'_{12} = \varepsilon'_{21}$. The design specification (3) yields the interaction constraint relations given by Eq. (6.6)

$$\begin{aligned} 2K_1n_2 &\leqq \varepsilon_{21}K_2(20n_1+19.9K_1) \\ 2K_2n_1 &\leqq \varepsilon_{12}K_1(40n_2+19.9K_2) \end{aligned} \qquad (6.6)$$

where $$\varepsilon_{21} = \varepsilon_{12}.$$

The set of relations given by Eqs. (6.4), (6.5) and (6.6) constitute the required design equations for the evaluation of the six design variables K_1, K_2, α_1, α_2, n_1, and n_2.

The above method is also applicable to the design of noninteracting multivariable control system by making $\varepsilon_{ij} = 0$ and $\varepsilon'_{ij} = 0$. An illustrative example is given in the following section.

6.4 <u>An Illustrative Design Example</u>

Consider the system shown in Figure 6.4 where

$$H_{11} = -2/(s+1), \quad H_{12} = 3/(s+1),$$

$$H_{21} = 4/(s+1), \quad H_{22} = (8s+2)/(s+1)$$

The equations of the system are

$$\begin{aligned} C_1 &= (R_1+C_1)G_{11}H_{11} + (R_2-C_2)G_{22}H_{12} \\ C_2 &= (R_1+C_1)G_{11}H_{21} + (R_2-C_2)G_{22}H_{22} \end{aligned}$$

In matrix form the above equations can be written as

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \qquad (6.7)$$

where

$$A_{11} = \left[G_{11}H_{11}(1+G_{22}H_{22}) - G_{11}G_{22}H_{12}H_{21}\right]/\Delta$$

$$A_{12} = (G_{22}H_{12})/\Delta$$

$$A_{21} = (G_{11}H_{21})/\Delta$$

$$A_{22} = \left[G_{22}H_{22}(1-G_{11}H_{11}) + G_{11}G_{22}H_{12}H_{21}\right]/\Delta$$

and where $\Delta = (1-G_{11}H_{11})(1+G_{22}H_{22}) + G_{11}G_{22}H_{12}H_{21}$.

Taking $r_1(t)$, $r_2(t)$ as impulses at time $t=0$ of areas r_1 and r_2 respectively, the inverse Laplace transform of Eq. (6.7) yields

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

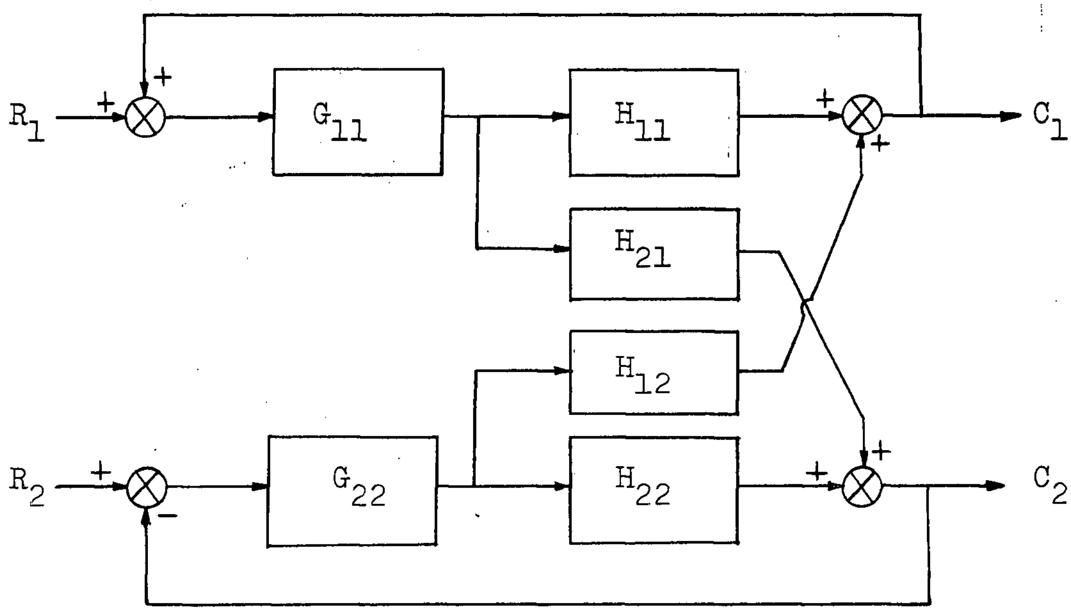


Figure 6.4 Block Diagram of a Multivariable Control System.

where

$$a_{11} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{G_{11}H_{11}(1+G_{22}H_{22})-G_{11}G_{22}H_{12}H_{21}}{\Delta}\exp(st)ds$$

$$a_{12} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{G_{22}H_{12}}{\Delta}\exp(st)ds$$

$$(6.8)$$

$$a_{21} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{G_{11}H_{21}}{\Delta}\exp(st)ds$$

$$a_{22} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{G_{22}H_{22}(1-G_{11}H_{11})+G_{11}G_{22}H_{12}H_{21}}{\Delta}\exp(st)ds$$

Let $G_{11} = K_1$ and $G_{22} = K_2$, where K_1 and K_2 are the design variables.

Substituting the values for G_{11} and G_{22} into Eq. (6.7) yields

$$\begin{aligned}
\Delta &= (a+bs+cs^2)/(1+s)^2 \\
A_{11} &= -2K_1(1+s)(1+8K_2)/(a+bs+cs^2) \\
A_{12} &= 3K_2(1+s)/(a+bs+cs^2) \\
A_{21} &= 4K_1(1+s)/(a+bs+cs^2) \\
A_{22} &= 2K_2\left[(1+8K_1)+s(5+8K_1)\right]/(a+bs+cs^2)
\end{aligned} \tag{6.9}$$

where

$$\begin{aligned}
a &= 1+2K_1+2K_2+16K_1K_2 \\
b &= 2(1+K_1+5K_2+8K_1K_2) \\
c &= 1+8K_2
\end{aligned}$$

Substituting Eq. (6.9) into Eq. (6.8) and using the Padé approximation for exp(st), a_{11}, a_{12}, a_{21}, and a_{22} can be expressed as ratios of polynomials in t with coefficients as algebraic functions of the system parameters. Using the fifth order Padé approximation

$$e^{st} = \frac{60 + 24st + 3s^2t^2}{60 - 36st + 9s^2t^2 - s^3t^3},$$

solving for a_{12} and a_{21} from Eq. (6.8) and equating the results to zero, the following relations are obtained:

$$720K_2\begin{bmatrix} A+BK_1+CK_2+DK_1^2+EK_2^2+FK_2^3+GK_1K_2+HK_1K_2^2 \\ +JK_1K_2^3+LK_1^2K_2+MK_1^2K_2^2+NK_1^2K_2^3 \end{bmatrix} = 0$$

$$960K_1\begin{bmatrix} A+BK_1+CK_2+DK_1^2+EK_2^2+FK_2^3+GK_1K_2+HK_1K_2^2 \\ +JK_1K_2^3+LK_1^2K_2+MK_1^2K_2^2+NK_1^2K_2^3 \end{bmatrix} = 0 \tag{6.10}$$

where

$$A = -0.0125t^5 - 0.1125t^4 + 0.4t^3 + 0.6t^2 - 3t - 15$$

$$B = -0.05t^5 - 0.55t^4 + 0.6t^3 + 4.2t^2 + 12t$$

$$C = -0.15t^5 - 1.05t^4 + 7.8t^3 + 1.8t^2 - 108t - 360$$

$$D = -0.05t^5 - 0.65t^4 - 1.4t^3 - 3t^2$$

$$E = -0.45t^5 - 1.05t^4 + 35.4t^3 - 113.4t^2 - 1152t - 2880$$

$$F = -0.4t^5 + 1.2t^4 + 296t^3 + 552t^2 - 3840t - 7680$$

$$G = -0.9t^5 - 9.3t^4 + 22.8t^3 + 118.8t^2 + 288t$$

$$H = -4.8t^5 - 43.2t^4 + 147.2t^3 + 672t^2 + 2304t$$

$$J = -6.4t^5-32t^4+844.8t^3+3302.4t^2+6144t$$

$$L = -1.2t^5-15.6t^4-33.6t^3-72t^2$$

$$M = -9.6t^5-125.4t^4-268.8t^3-576t^2$$

$$N = -25.6t^5-332.8t^4-716.8t^3-1536t^2$$

Figure 6.5 illustrates the type of data that can be obtained from Eq. (6.10). By choosing K_2 and t, Eq. (6.10) can be solved for K_1 and Eq. (6.7) can be solved for the maximum output amplitude for specified inputs. The time-domain sensitivity of the maximum output amplitude to variation of K_1 and K_2 can be determined from Figure 6.5. This information is similar to that obtained from Figure 4.2 of Chapter 4. A_{11} and A_{22} in Eq. (6.9) can be expressed as follows:

$$A_{11} = \frac{-2K_1(1+8K_2)}{3K_2}A_{12}$$

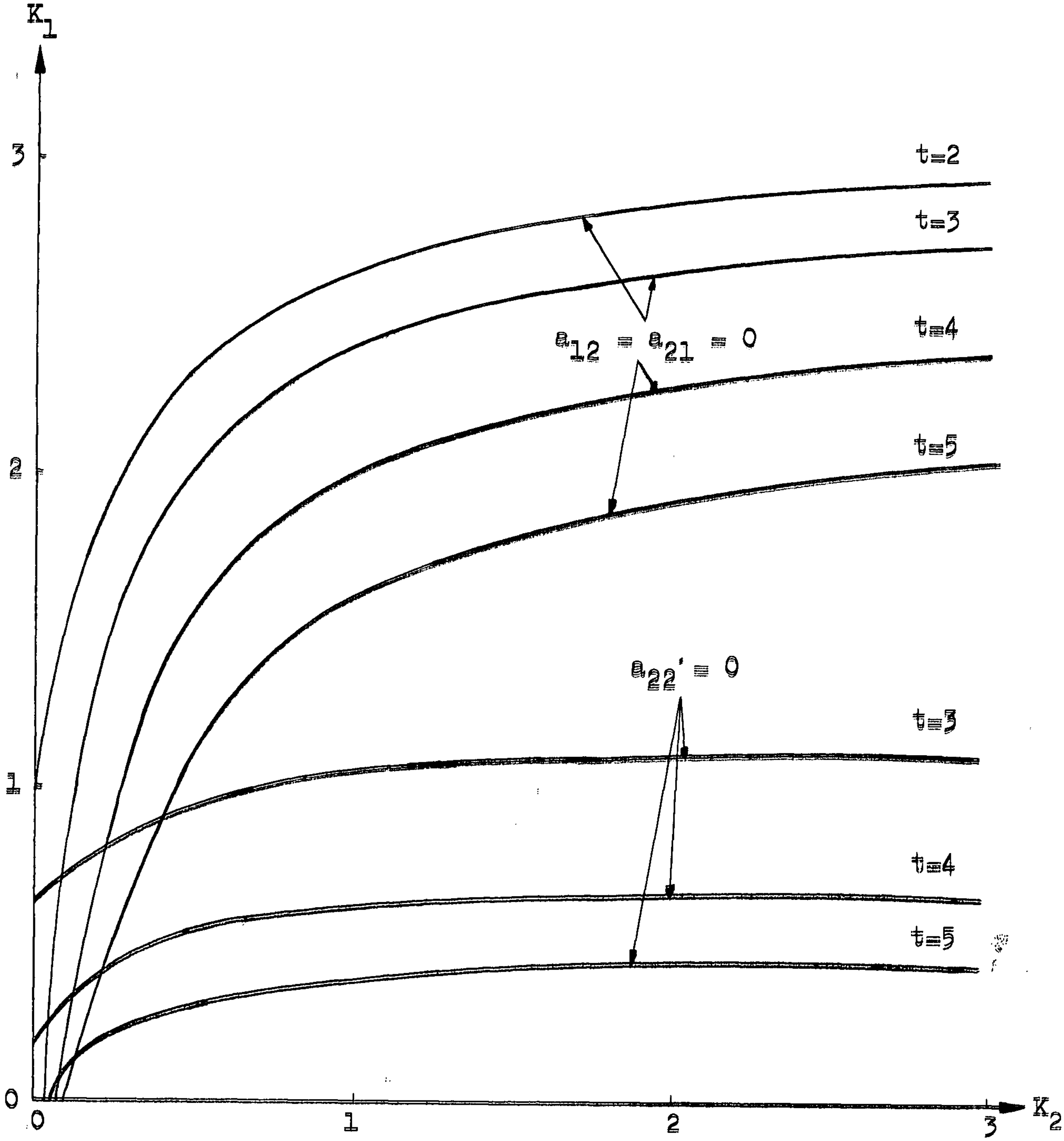
$$A_{22} = \frac{2K_2(1+8K_1)}{3K_2}A_{12} + \frac{8K_2s}{a + bs + cs^2}$$

When $A_{12}= 0$, $A_{11} = 0$, and

$$A_{22} = \frac{8K_2s}{a + bs + cs^2}$$

Solving for a_{22}, using the fifth order Padé approximation for exp(st), and equating to zero yields

$$2400K_2 \begin{bmatrix} A'+B'K_1+C'K_2+D'K_1^2+E'K_2^2+F'K_2^3+G'K_1K_2 \\ +H'K_1K_2^2+J'K_1K_2^3+L'K_1^2K_2+M'K_1^2K_2^2+N'K_1^2K_2^3 \end{bmatrix} = 0 \tag{6.11}$$



<u>Figure 6.5</u> The Gains K_1 and K_2 as Functions of Time t for the System Shown in Figure 6.4.

where

$$A' = 0.17t^4+0.16t^3-2.88t^2-9.6t+12$$

$$B' = 0.68t^4+0.48t^3-5.76t^2-9.6t$$

$$C' = 2.04t^4+2.4t^3-51.84t^2-201.6t+288$$

$$D' = 0.68t^4+0.32t^3+2.4t^2$$

$$E' = 6.12t^4+10.26t^3-254.88t^2-1382.4t+2304$$

$$F' = 5.44t^4+12.83t^4+12.83t^3-1209.6t^2-3072t+6144$$

$$G' = 12.24t^4+9.6t^3-138.24t^2-230.4t$$

$$H' = 65.28t^4+61.44t^3-998.4t^2-1843.2t$$

$$J' = 87.04t^4+122.88t^3-3870.72t^2-4915.2t$$

$$L' = 16.32t^4+7.68t^3+57.6t^2$$

$$M' = 130.56t^4+61.44t^3+460.8t^2$$

$$N' = 348.16t^4+163.84t^3+1228.8t^2$$

The relationship between K_1, K_2 and the instant of time when $a_{22}= 0$, given by Eq. (6.11) for $t=5$, is also shown in Figure 6.5. The point of intersection of the two curves gives the values of K_1 and K_2 for which the impulse responses a_{11}, a_{12}, a_{21} and a_{22} reach zero value at $t=5$. In other words, the amplitude of the outputs for unit step inputs reach their respective maximum values at that instant of time. This could be considered a desirable effect in some applications. The example discussed above only illustrates the principle of the proposed design technique and the possibilities of getting useful information from the algebraic relations.

In general, the output and input signals of a multivariable control system can be related by the following matrix representation.

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \cdots\cdots & a_{1n}(t) \\ \vdots & & \\ a_{n1}(t) & \cdots\cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

where the $c_i(t)$ are the outputs and the r_i are the areas of impulse inputs or the amplitudes of step inputs and where each $a_{jk}(t)$ is a ratio of polynomials in t with coefficients which are algebraic functions of system parameters.

It is interesting to note that the interaction constraints[34] ε_{jk} and ε'_{jk}, discussed in Sections 6.2 and 6.3 can easily be expressed as ratio of polynomials in t with coefficients which are algebraic functions of system parameters. It, therefore, becomes possible to investigate the interaction effects of a multivariable control system with parameter variations in the time-domain along with a sensitivity investigation.

7. CONCLUSIONS

A method has been presented for obtaining algebraic relations between system parameters and system response based on the frequency domain evaluation of an integral performance criterion. The performance criterion P, defined as a correlation measure between the responses of a known reference system and the system to be designed, provides a flexible criterion. Unlike the minimization of performance criteria based on error measures, this method allows a choice of different values of P to be made. Within the class of systems defined by the maximization technique, the particular system with the smallest parameter sensitivity can then be chosen. This is often more important than minimizing or maximizing a fixed performance function.

A method has also been presented for obtaining algebraic relations between the parameters of linear and piece-wise linear systems and their time-domain response characteristics. Since the method is based on the solution of systems of linear equations, the computations required are easily performed, and the difficult problem of relating characteristic roots to several system parameters is avoided. The algebraic relations obtained or the systems of linear equations used are well suited for time-domain sensitivity calculations by digital computer means. As is done in the sensitivity analysis of networks [35], the unspecified parameters can be tagged and derivatives with respect to these parameters obtained by simply deleting the parameters in the systems of linear equations. This is possible since the parameters enter the equations in a linear manner. A graphical display of the algebraic relations allows one to see the effect of several parameters on the time-domain response. The method augments very effectively other parameter plane methods since it avoids dealing directly with the characteristic roots which is an essential feature of these other methods.

APPENDIX A

Evaluation of Performance Integrals I_{mn} and J_m

A.1 Outline

The evaluation of performance integrals

$$I_{mn} = \frac{1}{2\pi j}\int_C U(s+p)V(-s+p)ds \qquad (A.1)$$

and

$$J_m = \frac{1}{2\pi j}\int_C U(s+p)U(-s+p)ds \qquad (A.2)$$

can be reduced to the solution of a system of linear algebraic equations. The following derivations are based on a proof given by Talbot[4].

A.2 Evaluation of Performance Integral I_{mn}

When U(s+p) and V(s+p) are rational functions of s vanishing at infinity, with the poles of U(s+p) all to the left of the poles of V(-s+p), the contour C may be completed by an infinite semicircle on either side of C. Taking it to the left and expressing U(s+p) and V(-s+p) in the form given by equations (2.15) and (2.19), respectively, equation (A.3) is obtained from Cauchy's residue theorem.

$$I_{mn} = \sum_i \left[F(s)/C'(s)D(s)\right]_{s=s_i} \qquad (A.3)$$

where F(s) denotes A(s)B(s) and where s_i are the zeros of C(s). Equations (2.16) to (2.18) and (2.20) to (2.22) give the numerator and denominator polynomials of U(s+p) and V(-s+p), respectively. Since all zeros of C(s) are to the left of the zeros of

D(s), C(s) and D(s) have no common factors.

In order to evaluate I_{mn} in equation (A.3), the following identity is considered:

$$\frac{F(s)}{C(s)D(s)} = \frac{R(s)}{C(s)} + \frac{Q(s)}{D(s)} \tag{A.4}$$

Since B(s) and D(s) have no common factor and the degree of F(s), or A(s)B(s), is less than the degree of C(s)D(s), R(s)/C(s) is the sum of those partial fraction terms of F(s)/C(s)D(s), which belong to C(s); and similarly for Q(s)/D(s). Thus,

$$\frac{R(s)}{C(s)} = \sum_{i} \left.\frac{[F(s)/C'(s)D(s)]}{s - s_i}\right|_{s=s_i} \tag{A.5}$$

It follows that I_{mn} is the coefficient of 1/s in R(s)/C(s) if this is expanded in descending powers of s, that is,

$$I_{mn} = \lim_{s\to\infty} \frac{sR(s)}{C(s)}$$

Thus, if

$$R(s) = r_{m-1}s^{m-1} + \ldots\ldots\ldots + r_0 \tag{A.6}$$

and

$$Q(s) = q_{n-1}s^{n-1} + \ldots\ldots\ldots + q_0 \tag{A.7}$$

then

$$I_{mn} = \frac{r_{m-1}}{c_m} \tag{A.8}$$

Equation (A.4) is equivalent to the polynomial equation

$$F(s) = R(s)D(s) + Q(s)C(s) \tag{A.9}$$

By equating terms containing the same powers of s in equation (A.9), a set of simultaneous equations are obtained for the coefficients in R(s) and Q(s) which may be written as

$$M\ w = F \tag{A.10}$$

where, w is the (m+n)-rowed column $(q_0, \ldots q_{n-1}, r_0, \ldots r_{m-1})$

F is the (m+n)-rowed column $(f_0, \ldots f_k, 0, \ldots 0)$

$$\text{where, } f_j = \sum_{i=0}^{j} a_i b_{j-i} \quad , \qquad \text{for } 0 \leqq j \leqq m+n-2$$

and M is the (m+n)-rowed square matrix given by equation (A.11).

$$M = \begin{bmatrix} c_0 & & 0 & d_0 & & 0 \\ c_1 & \ddots & & d_1 & \ddots & \\ \vdots & & c_0 & \vdots & & d_0 \\ \vdots & & \vdots & d_n & & \vdots \\ c_m & & \vdots & & \ddots & \vdots \\ & 0 \ddots & c_m & & 0 & d_n \end{bmatrix} \tag{A.11}$$

The solution of I_{mn} is given by

$$I_{mn} = \frac{\Delta'}{c_m \Delta} \tag{A.12}$$

where Δ is the determinant of M and Δ' is the determinant of M on replacing its last column by F.

Eqs. (A.8) and (A.12) remain valid even if the zeros of C(s) are not all simple. The procedure above holds for an arbitrary numerator F(s) and arbitrary denominator factors C(s) and D(s) having no common factor, provided the degree of F(s) is less than that of C(s)D(s). The roles of U and V may always be interchanged. Δ does not vanish since the polynomials C(s) and D(s) have no common factor. C(s) and D(s), however, must be Hurwitz polynomials.

A.3 Evaluation of Performance Integral J_m

To evaluate the integral J_m it is noted that equation (A.3) becomes

$$J_m = \sum_i \left[\frac{A(s)A(-s)}{C'(s)C(-s)}\right]_{s=s_i} \tag{A.13}$$

and in place of equation (A.9) equation (A.14) is obtained.

$$R(s)C(-s) + R(-s)C(s) = A(s)A(-s) = 2L(s) \tag{A.14}$$

J_m is the coefficient of $1/s$ in $R(s)/C(s)$, and if

$$L(s) = L_{2m-2}s^{2m-2} + \dots\dots\dots + L_0 \tag{A.15}$$

$$J_m = \frac{r_{m-1}}{c_m} \tag{A.16}$$

In terms of determinants the final solution is

$$J_m = (-1)^{m+1} \frac{\Delta'}{c_m \Delta} \tag{A.17}$$

where Δ is the determinant of the (mxm) square matrix

$$M' = \begin{bmatrix} c_0 & & 0 & & \\ c_2 & c_1 & c_0 & & \\ \cdot & c_3 & c_2 & c_1 & c_0 \\ \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{2m-2} & \cdot & \cdot & \cdot & c_{m-1} \end{bmatrix}, \tag{A.18}$$

Δ' is the determinant of M' on replacing its last column by the column $L = (L_0, L_2, \dots. L_{2m-2})$,

and where

$$2L_j = \sum_{i=0}^{j} (-1)^{j-i} a_i a_{j-i} \quad , \quad \text{for } 0 \leqq j \leqq m-1$$

$$= \sum_{i=j-m+1}^{m-1} (-1)^{j-i} a_i a_{j-i} \quad , \quad \text{for } m \leqq j \leqq 2m-2 . \tag{A.19}$$

A.4 Table of Integrals

Solution of the integrals I_{mn} for values of m from 1 to 2 and n from 1 to 3 are given in Table A.1. Solution of the integrals J_m and their derived form J_{mk} are given in Tables A.2 to A.4 for values of m from 1 to 4 and k from 1 to 2. The integral forms of I_{mn} and J_m are

$$I_{mn} = \frac{1}{2\pi j}\int_C ds\, \frac{A(s)B(s)}{C(s)D(s)} \tag{A.20}$$

$$J_m = \frac{1}{2\pi j}\int_C ds\, \frac{A(s)A(-s)}{C(s)C(-s)} \tag{A.21}$$

where

$$\begin{aligned} A(s) &= a_{m-1}s^{m-1} + \dots\dots\dots + a_0 \\ B(s) &= b_{n-1}s^{n-1} + \dots\dots\dots + b_0 \\ C(s) &= c_m s^m + \dots\dots\dots\dots\dots + c_0 \end{aligned} \tag{A.22}$$

and $D(s) = d_n s^n + \dots\dots\dots\dots\dots + d_0$

Table A.1 Values of I_{mn} in Terms of the Transform Coefficients.

$$I_{11} = \frac{a_0 b_0 c_0}{c_1(c_1 d_0 - c_0 d_1)}$$

$$I_{12} = \frac{a_0(c_1 b_0 - c_0 b_1)}{c_0^2 d_2 - c_1 c_0 d_1 + c_1^2 d_0}$$

$$I_{22} = \frac{\left[\begin{array}{c} a_1 b_1(c_0 d_1 - c_1 d_0) + a_0 b_0(c_1 d_2 - c_2 d_1) \\ + (a_1 b_0 + a_0 b_1)(c_2 d_0 - c_0 d_2) \end{array}\right]}{(c_0 d_2 - c_2 d_0)^2 + (c_1 d_2 - c_2 d_1)(c_1 d_0 - c_0 d_1)}$$

$$I_{23} = \frac{\left[\begin{array}{l} a_1 b_2\ c_0(c_0 d_2 - c_2 d_0) - c_1(c_0 d_1 - c_1 d_0) \\ \quad + (a_0 b_2 + a_1 b_1)\{c_2(c_0 d_1 - c_1 d_0) - c_0^2 d_3\} \\ \quad + (a_0 b_1 + a_1 b_0)\{c_2^2 d_0 - c_0(c_2 d_2 - c_1 d_3)\} \\ \quad + a_0 b_0\{c_1\ (c_2 d_2 - c_1 d_3) + c_2(c_0 d_3 - c_2 d_1)\} \end{array}\right]}{\left[\begin{array}{l} c_0\{c_2 c_0(d_2^2 - d_3 d_1) - c_2 c_1(d_2 d_1 - d_3 d_0) + c_2^2(d_1^2 - d_2 d_0) \\ \qquad + c_0 d_3(c_0 d_3 - c_1 d_2) + d_3 d_1(c_1^2 - c_2 c_0)\} \\ \qquad + d_0\{(c_1 d_3 - c_2 d_2)(c_2 c_0 - c_1^2) + c_2 c_1(c_0 d_3 - c_2 d_1) \\ \qquad\qquad + c_2^3 d_0\} \end{array}\right]}$$

Table A.2 Values of J_m in Terms of the Transform Coefficients

$$J_1 = \frac{a_0^2}{2c_1c_0}$$

$$J_2 = \frac{a_1^2c_0 + a_0^2c_2}{2c_2c_1c_0}$$

$$J_3 = \frac{a_2^2c_1c_0 + (a_1^2 - 2a_2a_0)c_3c_0 + a_0^2c_3c_2}{2c_3c_0(-c_3c_0 + c_2c_1)}$$

$$J_4 = \frac{\left[\begin{array}{l} a_3^2(-c_3c_0^2 + c_2c_1c_0) + (a_2^2 - 2a_3a_1)c_4c_1c_0 \\ + (a_1^2 - 2a_2a_0)c_4c_3c_0 + a_0^2(-c_4^2c_1 + c_4c_3c_2) \end{array}\right]}{2c_4c_0(-c_3^2c_0 - c_4c_1^2 + c_3c_2c_1)}$$

Table A.3 Values of J_{m1} in Terms of the Transform Coefficients

$$J_{11} = \frac{a_0^2}{2c_0^2}$$

$$J_{21} = \frac{a_0^2}{2c_0^2} - \frac{a_1 a_0}{c_1 c_0} + \frac{a_1^2 c_0^2 + a_0^2 c_2 c_0}{c_1^2 c_0^2}$$

$$J_{31} = \frac{a_0^2}{2c_0^2} - \frac{a_2 a_1 + a_1 a_0 c_2/c_0}{c_2 c_1 - c_3 c_0} + \frac{\left[\begin{array}{l} a_2^2(c_1^2+c_2c_0) + a_0^2(c_3^2c_0+c_2^3)/c_0 \\ \quad + (a_1^2-2a_2a_0)(c_3c_1+c_2^2) \end{array}\right]}{(c_2c_1 - c_3c_0)^2}$$

$$J_{41} = \frac{a_0^2}{2c_0^2} - \frac{\left[\begin{array}{l} 2a_3^2c_0 + a_3a_2c_1 + (a_2^2-2a_3a_1)c_2 \\ \quad + (a_2a_1-3a_3a_0)c_3 + 2(a_1^2-2a_2a_0)c_4 \\ \quad + a_1a_0(c_3c_2-c_4c_1)/c_0 + a_0^2(c_4c_2+c_3^2)/c_0 \end{array}\right]}{c_3c_2c_1 - c_4c_1^2 - c_3^2c_0}$$

$$+ \frac{\left[\begin{array}{l} (c_3c_1-4c_4c_0+c_2^2)\{a_3^2c_1^2 + (a_2^2-2a_3a_1)c_3c_1 \\ \qquad\qquad\qquad + (a_1^2-2a_2a_0)c_3^2 \\ \qquad\qquad\qquad + a_0^2(c_3c_2-c_4c_1)c_3/c_0\} \end{array}\right]}{(c_3c_2c_1 - c_4c_1^2 - c_3^2c_0)^2}$$

Table A.4 Values of J_{m2} in Terms of the Transform Coefficients

$$J_{12} = \frac{a_0^2 c_1}{c_0^3}$$

$$J_{22} = \frac{a_0^2 c_1}{c_0^3} + \frac{a_1^2 c_0 - 2a_1 a_0 c_1 + a_0^2 c_2}{c_1 c_0^2}$$

$$- \frac{4a_1 a_0 c_2 - 4a_0^2 c_2^2 / c_0}{c_1^2 c_0} + \frac{4a_1^2 c_2}{c_1^3}$$

$$J_{32} = \frac{a_0^2 c_1}{c_0^3} + \frac{\begin{bmatrix} a_2^2 c_0^2 + 2a_2 a_0 c_2 c_0 + a_1^2 c_2 c_0 + a_0^2 c_2^2 \\ - 2a_1 a_0 c_2 c_1 + 2a_1 a_0 c_3 c_0 \end{bmatrix}}{c_0^2 (c_2 c_1 - c_3 c_0)}$$

$$- \frac{4a_2 a_1 c_0 (c_2^2 + c_3 c_1) + 4a_1 a_0 (c_2^3 + c_3^2 c_0) + 4a_0^2 c_3 c_2^2}{c_0 (c_2 c_1 - c_3 c_0)^2}$$

$$+ \frac{\begin{bmatrix} 4a_2^2 (c_2^3 c_0 + c_3 c_2 c_1 c_0 + c_3^2 c_0^2 + c_3 c_1^3) \\ - 8a_2 a_0 (c_2^4 + 2c_3^2 c_2 c_0 + c_3^2 c_1^2) \\ + 4a_1 (c_2^4 + 2c_3^2 c_2 c_0 + c_3^2 c_1^2) \\ + 8a_0^2 (c_3^2 c_2^2 + c_3^3 c_1 + c_2^5 / c_0) \end{bmatrix}}{(c_2 c_1 - c_3 c_0)^3}$$

Table A.4 (Continued)

$$J_{42} = \frac{a_0^2 c_1}{c_0^3}$$

$$+ \frac{\begin{bmatrix} 5a_3^2 c_1 c_0^2 + 4a_3 a_2 c_2 c_0^2 + (5a_2^2 - 6a_3 a_1) c_3 c_0^2 \\ + 8(a_2 a_1 - 3a_3 a_0) c_4 c_0^2 + (a_1^2 + 2a_2 a_0)(c_3 c_2 - c_4 c_1) c_0 \\ + a_1 a_0 (4c_4 c_2 c_0 + 6c_3^2 c_0 - 2c_3 c_2 c_1 + 2c_4 c_1^2) \\ + a_0^2 (3c_4 c_3 c_0 + 8c_4 c_3^2 c_0 - c_4 c_2 c_1 - c_3^2 c_1^2) \end{bmatrix}}{c_0^2 (c_3 c_2 c_1 - c_4 c_1^2 - c_3^2 c_0)}$$

$$- \frac{\begin{bmatrix} (c_2^2 + c_3 c_1 - 4c_4 c_0)\{a_3^2 c_0 (2c_3 c_0 + 4c_2 c_1) + 3a_3 a_2 c_3 c_1 c_0 \\ + (a_2^2 - 2a_3 a_1)(4c_4 c_1 + 3c_3 c_2) c_0 \\ + 3(a_2 a_1 - 3a_3 a_0) c_3^2 c_0 \\ + 10(a_1^2 - 2a_2 a_0) c_4 c_3 c_0 \\ + 3a_1 a_0 c_3 (c_3 c_2 - c_4 c_1) \\ + a_0^2 (6c_4 c_3 c_2 c_0 - 4c_4^2 c_1 c_0 + 3c_3^3 c_0 \\ + c_3^3 - c_3^2 c_2 c_1 + c_4 c_3 c_1 + c_4 c_3 c_2)\} \\ + 8c_3 c_2 \{a_3^2 c_1 c_0 + (a_2^2 - 2a_3 a_1) c_3 c_1 c_0 \\ + (a_1^2 - 2a_2 a_0) c_3^2 c_0 + a_0^2 c_3 (c_3 c_2 - c_4 c_1)\} \end{bmatrix}}{c_0 (c_3 c_2 c_1 - c_4 c_1^2 - c_3^2 c_0)^2}$$

$$+ \frac{\begin{bmatrix} 4c_3 (c_2^2 + c_3 c_1 - 4c_4 c_0)^2 \{a_3^2 c_1^2 + (a_2^2 - 2a_3 a_1) c_3 c_1 c_0 \\ + (a_1^2 - 2a_2 a_0) c_3^2 \\ + a_0^2 (c_3 c_2 - c_4 c_1) c_3 / c_0\} \end{bmatrix}}{(c_3 c_2 c_1 - c_4 c_1^2 - c_3^2 c_0)^3}$$

APPENDIX B

MINIMIZATION AND MAXIMIZATION PROCEDURE

B.1 Outline

The search procedure used in this thesis leading to the minimum or maximum value of the performance index, was first suggested by Rosenbrock[22] and works with n orthogonal directions in which the search progresses at each stage. A stage is defined as the set of trials made with one set of directions and the subsequent change of these directions. Each attempt to find a new value of the performance index is called a trial.

Each stage is started with a step of arbitrary length ε. If the step is successful, ε is multiplied by α, where $\alpha > 1$. If the step is unsuccessful, ε is multiplied by $-\beta$, where $0 < \beta < 1$. Success is defined to mean that the new value of the performance index is less than or equal to the old value when a minimum is sought or is greater than or equal to the old value when a maximum is sought.

To change the direction of a vector V in which the steps are taken the value of ε is altered until at least one trial is successful and one trial is unsuccessful in each of the n directions.

Suppose that D_1 is the algebraic sum of all the successful steps ε_1, in the direction V_1, etc., and if

$$\begin{aligned}
A_1 &= D_1V_1^0 + D_2V_2^0 + \cdots\cdots + D_nV_n^0 \\
A_2 &= \phantom{D_1V_1^0 +{}} D_2V_2^0 + \cdots\cdots + D_nV_n^0 \\
&\;\vdots \\
A_n &= \phantom{D_1V_1^0 + D_2V_2^0 + \cdots\cdots +{}} D_nV_n^0
\end{aligned} \tag{B.1}$$

then A_1 is the vector joining the initial and final points obtained by use of orthogonal unit vectors V_1^0, V_2^0, ... , V_n^0, A_2 is the sum of all the advances made in directions other than the first, etc. The orthogonal unit vectors V_1^1, V_2^1, ..., V_n^1, are then obtained as follows:

$$
\begin{aligned}
B_1 &= A_1 \\
V_1^1 &= B_1/|B_1| \\
B_2 &= A_2 - A_2 \cdot V_1^1 V_1^1 \\
V_2^1 &= B_2/|B_2| \\
&\cdots\cdots\cdots\cdots\cdots \\
B_n &= A_n - \sum_{j=1}^{n-1} A_n \cdot V_j^1 V_j^1 \\
V_n^1 &= B_n/|B_n|
\end{aligned}
\tag{B.2}
$$

The above algorithm ensures that V_1 lies along the direction of fastest advance, V_2 along the best direction which can be found normal to V_1, and so on. An obvious advantage of this method is that no partial derivatives of the performance index with respect to the design parameters need be calculated.

B.2 IBM 7040 Digital Computer Program

The computer program incorporating the above mentioned ideas and written in the FORTRAN IV language for the IBM 7040 digital computer includes the evaluation of the performance index from the determinant form. The overall flow diagram is shown in Figure B.1.

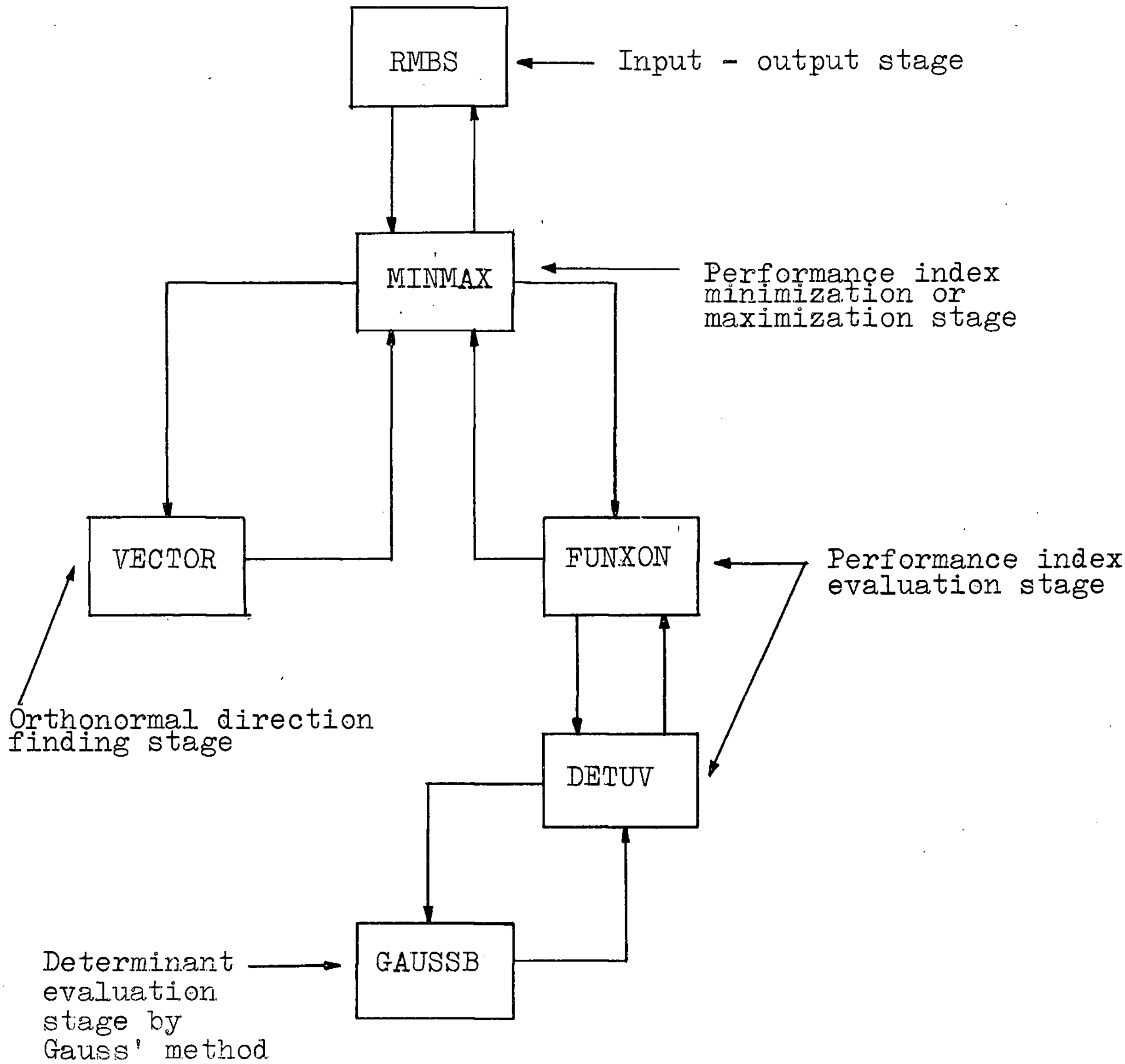


Figure B.1 Overall Flow Diagram for Minimizing or Maximizing the Performance Index on the Digital Computer.

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