ON THE ATTITUDE DYNAMICS OF SLOWLY SPINNING
AXISYMMETRIC SATELLITES UNDER THE
INFLUENCE OF GRAVITY GRADIENT
TORQUES

by

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ABSTRACT

The dynamics of slowly spinning axisymmetric satellites under the influence of gravity gradient torque is investigated using analytical and numerical techniques. Particular emphasis is on motion near the equilibrium position in which the spin axis is normal to the orbital plane. The problem is studied in increasing orders of difficulty. Phase I deals with the response and stability of a simplified model free to librate in roll while the more general problem is treated in Phase II.

Phase I serves as a proving ground for techniques to be used in subsequent analysis. A closed form solution is obtained in terms of elliptic functions for the autonomous case. In general, for non-circular orbits, motion in the large is studied using the concept of the invariant solution surface. These surfaces, obtained numerically, reveal the nature of motion in the large in terms of the dominant periodic solutions and allow one to determine the limits of oscillatory motion in terms of the state parameters. Floquet theory is employed in conjunction with numerical solutions of the linearized equations of motion to study stability in the small. This technique is extended to assess the variational stability of the dominant periodic motions in the large.

Phase II investigates a more general model with three degrees of freedom in attitude motion. The presence of an ignorable coordinate gives a fourth order, non-autonomous system for an elliptic trajectory. Motion in the small is studied extensive-

ly, again using Floquet theory, and stability charts suitable for design purposes are presented. The invariant surface concept is successfully extended to the study of the autonomous case in the large. Methods are developed for determining the maximum response to a given disturbance resulting in a set of charts which are useful in assessing the effects of non-linearities and the validity of the analysis in the small. Procedures are explained for determining periodic solutions of the problem, as well as their stability, for arbitrary eccentricity.

The analysis suggests the possibility of attitude instability during spin-up operations. It is shown that stable motion can be established by providing either a positive or negative spin to the satellite with the former preferable. Given sufficient spin any configuration, even those with an adverse gravity gradient effect, can be stabilized. Eccentricity affects the attitude motion of a satellite adversely as regions of unstable motion increase in size and number with it.

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LIST OF SYMBOLS

A	$I(\sigma+1)$
B	$3I-4$
C_1, C_2, C_3, C_4	Constants defined in equation (2.45)
$\bar{C}_1, \bar{C}_2$	Constants defined in equation (2.42)
D_1, D_2	Constants defined in equation (3.11) and (3.12)
E	Total energy, $T+U$
F	Function defined in equation (2.41)
G	Function defined in equation (2.39)
H	Hamiltonian
I	Inertia parameter, I_x/I_y
I_x, I_y, I_z	Principal moments of inertia
K_j	Coefficient in equation (2.52)
L	Lagrangian
L_j	Coefficient in equation (2.53)
O	Center of force
P	Pericenter
R	Distance between satellite center of mass and center of force
S	Satellite center of mass
T	Kinetic energy
U	Potential energy
V	Liapounov testing function
X	State vector
a	$h_0^2/\mu(1-e)^2$
b_j	Element of characteristic matrix

c_{ij}	Periodic coefficient of equations of motion in state vector form
e	Orbit eccentricity
f	Function defined in (I.5)
g	Function defined in (I.9)
h_α, h_0	Constants of motion
k	Modulus of elliptic functions
k_x, k_y, k_z	Principal radii of gyration
k_i	Constants in equations (2.29) and (2.30)
l	Spin parameter
m	Number of oscillations involved in periodic motion
m_s	Mass of the satellite
n	Number of orbits over which motion is periodic
r	Distance from center of force to a mass element
s	Root of characteristic equation
t	Time
u	$A + B \cos \delta$
u_1, u_2, v_3, u_4	Constants defined in equations (2.28)
v	$(1/R) - (\mu/h_0^2)$
v_0	$\mu e / h_0^2$
x, y, z	Spinning body coordinates
x_0, y_0, z_0	Inertial coordinates
x_p, y_p, z_p	Principal body coordinates
x_s, y_s, z_s	Intermediate body centered coordinates
Δe	Incremental change in e per orbit
$\Delta \phi$	Incremental change in ϕ per orbit
Θ	Normal solution basis

Φ^i	i^{th} normal solution vector
Φ	Solution basis
Φ^i	i^{th} solution vector
Φ_j^i	j^{th} element of i^{th} solution vector
α, β, γ	Spin, yaw and roll rotations, respectively
γ_A	Maximum amplitude of roll oscillation
$\gamma_0, \gamma_1, \gamma_2, \dots$	Perturbation elements of roll solution
γ	$\gamma(1 + e \cos \theta)$
$\delta_\beta, \delta_{\beta'}, \delta_\gamma, \delta_{\gamma'}$	Iteration changes in $\beta_i, \beta'_i, \gamma_i$ and γ'_i respectively
ϵ	Perturbation parameter
θ	True anomaly
λ_i	i^{th} characteristic multiplier
μ	Gravitational constant
ξ	$\int G(\theta) d\theta$
ρ	Size parameter
σ	Spin parameter
τ	Period of librational motion
ϕ	Perturbation of the pericenter
ψ	$\theta + \phi$
ω	Librational frequency
ω_f	Fundamental librational frequency
$\omega_x, \omega_y, \omega_z$	Angular velocity components in principal body coordinates
ω_0	Basic librational frequency, $[I(\sigma+4)-4]^{1/2}$
$\omega_1, \omega_2, \dots$	perturbation frequencies

Subscripts

c	Critical value
i	Initial value
p	Periodic solution
v	Variation from periodic solution

Dots and primes indicate differentiation with respect to t and θ respectively.

1. INTRODUCTION

1.1 Preliminary Remarks

Interest in the attitude dynamics of rigid orbiting bodies dates back to the eighteenth century when astronomers studied motions of natural satellites, e.g., lunar librations. In recent years and particularly since the launching of the first artificial satellite research in this field has measurably accelerated. Modern satellite systems, capable of performing sophisticated on-board experiments, usually demand a corresponding degree of sophistication in attitude control to overcome a variety of disturbing influences, e.g., solar radiation pressure, gravity gradient and magnetic field effects, and micrometeorite impacts. Ideally, however, the attitude control of a satellite should be accomplished with the minimum expenditure of energy since space and weight are at a premium aboard instrument packed space vehicles.

In many applications where attitude control requirements are not too severe, passive techniques involving no expenditure of stored energy have proven to be adequate. Among the methods belonging to this category, those utilizing gravity gradient and/or gyroscopic effects are commonly used.

The former utilizes the moment due to the gravitational gradient across a satellite so that its long axis (i.e., the axis of least inertia) points in the direction of the attracting body. The latter, in its simplest form, turns the entire

satellite into a gyroscope by permitting it to spin about a suitable axis. When this axis coincides with an axis of symmetry, the system is in equilibrium with the spin vector normal to the orbital plane.

For certain satellite configurations, the gravity gradient torque tends to reinforce spin stabilization. For example, a thin disk or ring shaped satellite would be in a position of stable equilibrium under the influence of the gravity torque if its axis of symmetry were normal to the plane of the orbit. On the other hand, the gravity torque may work against spin stabilization as in the case of a slender, pencil-shaped satellite in the identical orientation. The gravity gradient torque is always present, except for the case of a spherical satellite, and its effect is particularly significant where the rate of spin is small. Consequently, an investigation of the behavior of slowly spinning satellites should take the gravity gradient torque into account.

One may ask: Why not simply spin the satellite at a high rate so that gravity effects are minimized? While this approach may be permissible in certain specific applications, high rates of spin are usually not compatible with other design objectives. Moreover, during injection into orbit and subsequent spin-up for stabilization, there is a transitory period during which gravitational and gyroscopic effects are of comparable magnitudes. Thus a study of the attitude dynamics of a slowly spinning satellite in a gravitational field should lead to information of considerable practical significance.

1.2 Historical Background

A survey of the pertinent literature suggests that compared to gravity gradient stabilized systems the dynamical analysis of slowly spinning satellites has received little attention. Research in the field of gravity orientated systems has, until recently, been confined to the study of the idealized dumbbell satellite configuration. Much of this work pertains to a simplified model of the system with a single degree of freedom in attitude allowing for planar libration only. Klemperer ¹ obtained the exact solution of this system in terms of elliptic functions for circular orbital motion. Schechter ² attempted, with limited success, to extend this solution to non-circular orbital motion by perturbation methods while Baker ³ found periodic solutions of the problem for small orbit eccentricity. Brereton ⁴ has presented an excellent review of this work. More recently, Zlatousov, et al ⁵ and Brereton and Modi ⁶ employed numerical methods involving the use of stroboscopic phase planes to study motion in the large for orbits of arbitrary eccentricity.

The study of the more general problem involving out of plane librations was undertaken by Modi and Brereton ⁷ in which a rotational constraint was involved and by DeBra ⁸ allowing three degrees of freedom. In both cases numerical methods were used to determine satellite response.

In the field of slowly spinning satellites, early work has been restricted to the study of systems undergoing circular

orbital motion. Thomson ⁹ presented a stability criterion in terms of spin and inertia parameters using linearized analysis. Subsequently Kane et al ¹⁰ applied this criterion to obtain a stability chart in terms of these parameters.

The stability of a spinning unsymmetric satellite was investigated by Kane and Shippey ¹¹ applying Floquet theory to a linearized model of the system.

The large amplitude librational motion of a spinning satellite in a circular orbit was studied by Pringle ¹² using a Liapounov type of analysis. Positions of equilibrium as well as bounds of motion, i.e. separatrices, about these positions were obtained.

The attempt by Kane and Barba ¹³ to analyze motion in an elliptic orbit should be mentioned here. By applying Floquet theory to a linearized system, they developed a procedure for testing the stability, in the small, of a spinning satellite undergoing orbital motion of arbitrary eccentricity.

In a recent paper, Wallace and Meirovitch ¹⁴ studied the same problem using asymptotic analysis in conjunction with Liapounov's direct method. A normalized Hamiltonian function was employed as the testing function. Much of this work involved the use of linearized analysis although perturbation methods were used in an attempt to investigate the effects of first order non-linearities. Non-linear stiffening effects and resonance regions were observed; however, there is an element of doubt about the validity of these results since even the conclusions based on the linearized analysis were not in general agreement with those of Kane and Barba ¹³, particularly in the

negative spin regime.

1.3 Purpose and Scope of the Investigation

From the foregoing it is clear that, although researchers have been active in this field for sometime there is still much to be learned about the behavior of spinning, gravity influenced satellites. The main purpose of this investigation is to develop techniques by which one may explore regions of parameter space that are of particular interest during the feasibility study and design stage of a satellite program. It is also intended here to apply these techniques to a number of specific situations so as to obtain a fundamental understanding of the dynamics of slowly spinning satellites.

Due to the complex nature of the problem, it was judged advisable to approach it in two stages. Phase I studies a simplified system consisting of a spinning satellite free to librate in roll. The model would serve as a test bed for methods of analysis. Such a system, although it fails to represent the physical situation accurately, does possess several properties of a more general model incorporating three degrees of freedom. This approach roughly parallels that taken by investigators in studying the attitude dynamics of a dumbbell satellite where the early work restricted librational motion to the plane of the orbit.

Phase II of the analysis treats a more general situation involving a model with three degrees of freedom in attitude, viz., roll, yaw and spin. In this case, attempts have been

made to generalize and extend those methods which proved most fruitful in Phase I.

In each phase of the study, it was considered appropriate to approach the problem in stages representing increasing degrees of complexity. In the beginning, a linearized model of the system is used to study motion in the small. Since practical applications such as communications satellites usually demand considerable pointing accuracy, a knowledge of the behavior in the small would be of great value. Subsequently, studies of motion in the large are undertaken since gravity torques and spin coupling effects are inherently non-linear.

A further subdivision of the problem is possible. It is shown that to a high degree of accuracy orbital motion, i.e., motion of the center of mass, and librational motion may be considered to be uncoupled. Thus, the equations governing attitude dynamics are autonomous for circular orbital motion while those for elliptical orbits are non-autonomous. A considerable simplification is, therefore, realized by initially restricting the study to those systems in which orbital motion is circular. This also helps provide a firm basis for subsequent studies involving non-circular orbital motion.

Figure 1-1 schematically illustrates the proposed method of attack. In each phase the investigation begins with the simplest model based upon a linear, autonomous representation and progresses through to the most complex system governed by non-linear, non-autonomous equations of motion. It is felt that this approach, by the nature of the escalating complexities

Figure 1-1 Schematic diagram of the proposed program

of the various models, provides a coherent program to explore the subject.

2. PHASE I-ROLL DYNAMICS OF A SPINNING AXISYMMETRIC SATELLITE

2.1 Preliminary Remarks

This chapter investigates the attitude dynamics of a rigid, axisymmetric, spinning satellite free to librate in roll. The effects of orbit eccentricity and the gravity gradient torque are included in the analysis. Librational motion is studied both in the small (i.e. linearized analysis) and in the large.

Particular attention is paid to the development of techniques suitable for the investigation of the more general problem of a satellite free to librate in both yaw and roll. The analysis of stability in the small is performed using a numerical method based on Floquet theory which is subsequently adapted to the variational analysis of a large amplitude periodic motion. A closed form solution in terms of elliptic function is obtained for the particular case of circular orbital motion. Suitability of several approximate methods is investigated for cases involving orbital motion of arbitrary eccentricity. The WKBJ solution is shown to lead to the concept of invariant surfaces in three dimensional state space. This concept, an extension of the stroboscopic phase plane method, is utilized in the numerical analysis of the problem to obtain limiting surfaces for stable (non-tumbling) motion.

2.2 Formulation of the Problem

Consider a rigid, axisymmetric, spinning satellite with center of mass at **S** librating about the local horizontal while moving in orbit about the center of force at **O** (Figure 2-1). The coordinates **R** and **θ** define the position of the center of mass with respect to the inertial frame x_o, y_o, z_o . Let x_p, y_p, z_p represent the principal body coordinates with origin at the center of mass (**S**) and the coordinate x_p coinciding with the axis of symmetry. Let x_s, y_s, z_s be another set of orthogonal coordinates with origin at **S** but orientated such that x_s is normal to the orbital plane and y_s lies along the extension of the radius vector **R**. The angular orientation of the satellite is specified by the Euler angles γ and α relative to the non-inertial frame x_s, y_s, z_s . The first rotation, γ , about the local horizontal is referred to as roll while the second rotation, α , about the axis of symmetry represents the spin of the satellite.

Owing to the symmetry of the satellite, formulation of the problem is most easily accomplished using the coordinate frame x, y, z (Figure 2-1) in which the satellite spins with angular velocity $\dot{\alpha}$ about its axis of symmetry lying along the x axis.

The kinetic and potential energies of the system may be written as

$$T = \frac{m_s}{2} \left(\dot{R}^2 + (R\dot{\theta})^2 \right) + \frac{1}{2} \left(I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 \right)$$

(2.1)

Figure 2-1 Geometry of Phase I model

$$U = - \int_{m_s} \frac{\mu}{r} dm_s \quad (2.2)$$

where the angular velocities are given by

$$\begin{aligned} \omega_x &= \dot{\alpha} + \dot{\theta} \cos \delta \\ \omega_y &= -\dot{\theta} \sin \delta \\ \omega_z &= \dot{\delta} \end{aligned} \quad (2.3)$$

and

$$r = R \left[1 + \left\{ \left(\frac{x}{R} \right)^2 + \left(\frac{y}{R} \right)^2 + \left(\frac{z}{R} \right)^2 + 2 \left(\frac{x}{R} \right) \sin \delta + 2 \left(\frac{y}{R} \right) \cos \delta \right\} \right]^{\frac{1}{2}} \quad (2.4)$$

Expanding equation (2.4) using the binomial theorem gives

$$\begin{aligned} \frac{1}{r} &= \frac{1}{R} \left[1 - \left(\frac{1}{R} \right) \{ x \sin \delta + y \cos \delta \} - \frac{1}{2} \left(\frac{1}{R} \right)^2 \{ (1 - \right. \\ &\quad \left. 3 \sin^2 \delta) x^2 + (1 - 3 \cos^2 \delta) y^2 + z^2 - 3xy \sin 2\delta \} \right. \\ &\quad \left. + O \left\{ \left(\frac{1}{R} \right)^3 \right\} \right]. \end{aligned} \quad (2.5)$$

Since the coordinates x, y, z were chosen with origin at S and the x axis along the axis of symmetry, the following relations hold:

$$\int_{m_s} x dm_s = \int_{m_s} y dm_s = \int_{m_s} z dm_s = 0 \quad (2.6)$$

and

$$\int_{m_s} x^2 dm_s = \frac{1}{2} (I_y + I_z - I_x)$$

$$\int_{m_s} y^2 dm_s = \frac{1}{2} (I_x + I_z - I_y) \quad (2.7)$$

$$\int_{m_s} z^2 dm_s = \frac{1}{2} (I_x + I_y - I_z)$$

$$\int_{m_s} xy dm_s = 0.$$

Using equations (2.5), (2.6) and (2.7), neglecting $O\left\{\left(\frac{1}{R}\right)^3\right\}$ and noting that $I = I_x/I_y = I_x/I_z$ gives

$$U = -\frac{\mu m_s}{R} - \frac{\mu I_x}{2R^3} \left(\frac{I-1}{I} \right) (1 - 3 \sin^2 \gamma) \quad (2.8)$$

and hence the Lagrangian function, L , can be written as

$$L = \frac{m_s}{2} \left[\dot{R}^2 + (R\dot{\theta})^2 \right] + \frac{I_x}{2} \left[(\dot{\alpha} + \dot{\theta} \cos \gamma)^2 + \right.$$

$$\left. \frac{1}{I} (\dot{\gamma}^2 + \dot{\theta}^2 \sin^2 \gamma) \right] + \frac{\mu m_s}{R} + \frac{\mu I_x}{2R^3} * \left(\frac{I-1}{I} \right) (1 - 3 \sin^2 \gamma). \quad (2.9)$$

As both α and θ are cyclic coordinates, there are two first integrals of motion given by

$$h_\alpha = \frac{1}{m_s} \frac{\partial L}{\partial \dot{\alpha}} = \frac{I_x}{m_s} (\dot{\alpha} + \dot{\theta} \cos \gamma) \quad (2.10)$$

and
$$h_\theta = \frac{1}{m_s} \frac{\partial L}{\partial \dot{\theta}} = \left[R^2 + \frac{I_x}{m_s} \left(\cos^2 \gamma + \frac{1}{I} \sin^2 \gamma \right) \right] \dot{\theta} + \left[\frac{I_x}{m_s} \cos \gamma \right] \dot{\alpha} \quad (2.11)$$

where h_α and h_θ are constants of motion.

Using Lagrangian formulation the equations of motion for the remaining coordinates, viz., R and γ , can be written as

$$\ddot{R} - R\dot{\theta}^2 + \frac{\mu}{R^2} + \frac{3\mu I_x}{2m_s R^4} \left(\frac{I-1}{I} \right) (1 - 3\sin^2 \gamma) = 0 \quad (2.12)$$

and
$$\ddot{\gamma} + (I-1) \left(\dot{\theta}^2 + \frac{3\mu}{R^3} \right) \sin \gamma \cos \gamma + I \dot{\alpha} \dot{\theta} \sin \gamma = 0. \quad (2.13)$$

These non-linear, non-autonomous, coupled equations of motion do not possess any known closed form solution. Some simplification of these equations can be achieved by neglecting the effects of librational motion on orbital motion. An analysis, using the method of variation of parameters, is performed in Appendix I which verifies this approach.

Neglecting attitude motion effects on the orbital motion leads to the classical equations for a particle undergoing

central face motion,

$$h_0 = R^2 \dot{\theta}$$

(2.14)

and
$$\ddot{R} - R\dot{\theta}^2 + \frac{\mu}{R^2} = 0.$$

These equations, in turn, yield the Keplerian relation for R and θ ,

$$R = h_0^2 / \mu (1 + e \cos \theta). \quad (2.15)$$

Note that h_α , which is a constant of motion, is a measure of spin. Consequently a dimensionless spin parameter, σ , defined as

$$\sigma = \left. \frac{\dot{\alpha}}{\dot{\theta}} \right|_{\substack{\theta=0 \\ \gamma=0}} = \left. \frac{m_s h_\alpha}{I_x \dot{\theta}} \right|_{\theta=0} \quad (2.16)$$

may be used to eliminate the cyclic coordinate α .

Recognizing that $\frac{d}{dt} = \frac{h_0}{R^2} \frac{d}{d\theta}$,

$$\frac{d^2}{dt^2} = -\frac{2h_0^2}{R^5} \frac{dR}{d\theta} \frac{d}{d\theta} + \frac{h_0^2}{R^4} \frac{d^2}{d\theta^2}$$

the governing equation of motion can be written as

$$\gamma'' - \left(\frac{2e \sin \theta}{1 + e \cos \theta} \right) \gamma' + \left[\left(\frac{3I - 4 - e \cos \theta}{1 + e \cos \theta} \right) \cos \gamma + I(\sigma + 1) \left(\frac{1 + e}{1 + e \cos \theta} \right)^2 \right] \sin \gamma = 0. \quad (2.17)$$

It should be noted that, in equation (2.17), the spin parameter σ is based upon $\theta = \gamma = 0$ such that $\sigma = \alpha'$. If initial conditions other than these are chosen, the spin parameter is related to α' by

$$\sigma = \left(\frac{1 + e \cos \theta}{1 + e} \right)^2 (\alpha' + \cos \gamma) - 1 \quad .$$

Analysis of the non-linear, non-autonomous equation governing motion, i.e. equation (2.17), involving three system parameters, I , σ and e , forms the subject of this study.

2.3 Motion in the Small

Investigation of motion in the small is an important phase in the general study of any dynamical systems. In most satellite applications only small amplitude librational motion is permissible. Consequently, the question of the stability of the linearized system is of considerable significance.

Linearization of the equation governing attitude motion (i.e., equation 2.17) results in

$$\gamma'' - \left\{ \frac{2e \sin \theta}{1 + e \cos \theta} \right\} \gamma' + \left\{ \frac{3I - 4 - e \cos \theta}{1 + e \cos \theta} + I(\sigma + 1) \left(\frac{1 + e}{1 + e \cos \theta} \right)^2 \right\} \gamma = 0 \quad . \quad (2.18)$$

It should be noted that although equation (2.18) is linear, it is not in general autonomous since the independent

variable θ is present explicitly.

Let us consider the special case where $e=0$, i.e., circular orbital motion. In this case, equation (2.18) becomes autonomous and stability in the small is determined by the sign of the coefficient of δ . The stability criterion for this situation is simply,

$$3I - 4 + I(\sigma + 1) \begin{cases} > 0, \text{ stable} \\ = 0, \text{ critical} \\ < 0, \text{ unstable.} \end{cases} \quad (2.19)$$

Expressed in terms of a critical spin parameter these relations become

$$\sigma \begin{cases} > \sigma_c, \text{ stable} \\ = \sigma_c, \text{ critical, } \sigma_c = 4(1-I)/I \\ < \sigma_c, \text{ unstable.} \end{cases} \quad (2.20)$$

A chart showing the regions of stable and unstable configurations in I, σ -space is presented in Figure 2-2.

The next logical step would be to extend the analysis to cases involving non-circular orbital motion. Since the coefficients in equation (2.18) are periodic in θ (period 2π), Floquet theory may be employed to study stability.

Floquet theory¹⁵ asserts that for an " n th" order system governed by a set of linear, homogeneous, differential equations having coefficients of a common periodicity (say τ) in the independent variable, there exists a basis consisting of n

Figure 2-2 Stability chart for motion in the small; $e=0$

normal solution vectors. Further, from the theory of linear equations, we know that any solution can be constructed from a linear combination of basis solutions. Normal solutions have the property that $\Theta^i(t+\tau) = \lambda_i \Theta^i(t)$ where the constants λ_i are referred to as characteristic multipliers. Here t is used as the independent variable and the superscript i denotes specific solution vectors.

For a second order system, such as the one with which we are dealing, the stability criterion can be expressed as follows,

$$|\lambda_i| \begin{cases} \leq 1, i=1,2; \text{stable} \\ > 1, i=1,2; \text{unstable.} \end{cases} \quad (2.21)$$

In order to construct a basis of normal solutions, it is necessary that some basis, say $\Phi(t)$, of linearly independent solutions be known. Since $\Phi^i(t+\tau)$ must also be a solution due to the periodic nature of the coefficients of the governing equations, one can express $\Phi^i(t+\tau)$ as a linear combination of the basis solutions, $\Phi^j(t)$. Thus both $\Theta^i(t)$ and $\Theta^i(t+\tau)$ can be written in terms of $\Phi^j(t)$, and a relation for the characteristic multipliers λ_i in terms of the combinative constants relating $\Phi^i(t+\tau)$ to $\Phi^j(t)$ can be obtained as follows:

$$\text{For } \Phi^i(t+\tau) = \sum_{j=1}^n b_{ij} \Phi^j(t),$$

$$\text{Det} \begin{pmatrix} b_1 - \lambda & b_1^2 & \cdot & \cdot & \cdot & b_1^n \\ b_2 & b_2^2 - \lambda & \cdot & \cdot & \cdot & b_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_n & b_n^2 & \cdot & \cdot & \cdot & b_n^n - \lambda \end{pmatrix} = 0. \quad (2.22)$$

Here the subscript j denotes a specific element within a solution vector. In particular, if $\Phi_j^i(0) = \delta_{ij}$, we see that $b_j^i = \Phi_j^i(\tau)$ since $\Phi^i(\tau) = \sum_{j=1}^n b_j^i \Phi^j(0)$.

Thus the procedure is clear; first obtain $\Phi(\tau)$ from $\Phi(0)$ equal to the identity matrix and second, find the eigenvalues of $\Phi(\tau)$, these being the characteristic multipliers of the system. The difficulty lies in the execution of the first step. Numerical solution of the governing differential equations offers a relatively easy and accurate means to this end.

The criterion for stability can be further simplified in this case since it may be shown that the product of the characteristic multipliers is unity. This follows from a consideration of the Wronskian and its derivatives yielding the relation,

$$\prod_{i=1}^n \lambda_i = \exp\left(\int_0^\tau \left[\sum_{i=1}^n c_{ii}(t)\right] dt\right) \quad (2.23)$$

where the functions c_{ii} are the periodic coefficients of the

governing equations arranged in state vector form, i.e.,

$\dot{X}_i = \sum_{j=1}^n c_{ij}(t) X_j$, $i = 1, 2, \dots, n$. For the case in hand,

$$\lambda_1 \lambda_2 = \exp \left(\int_0^{2\pi} \left(\frac{2e \sin \theta}{1 + e \cos \theta} \right) d\theta \right) = 1. \quad (2.24)$$

Thus it is seen that the characteristic multipliers lie on either the unit circle or on the real axis. In the former case, the system is stable and the trace of the matrix $\Phi(2\pi)$ lies between -2 and $+2$. In the latter case, the system is unstable and the trace of $\Phi(2\pi)$ is either greater than $+2$ or less than -2 . Thus the stability criterion may be stated in terms of the trace of $\Phi(2\pi)$ as,

$$\left| \text{Tr}[\Phi(2\pi)] \right| \begin{cases} \leq 2, \text{ stable} \\ > 2, \text{ unstable.} \end{cases} \quad (2.25)$$

Using the above criterion and integrating equation (2.18) numerically using the initial conditions $\delta_i = 1$, $\delta_i' = 0$ and $\delta_i = 0$, $\delta_i' = 1$ for various values of I , σ and e , the stability charts shown in Figures 2-3, 2-4 and 2-5 were obtained.

2.4 Motion in the Large

Ideally a satellite should be stable in the small and should be able to withstand large disturbances without tumbling.

Figure 2-3 Stability chart for motion in the small; $e = .1$

Figure 2-4 Stability chart for motion in the small; $e = .3$

Figure 2-5 Stability chart for motion in the small; $e = .5$

During large amplitude motions, the non-linear terms in equation (2.17) become significant and, hence, the linearized analysis may lead to erroneous conclusions. It is, therefore, necessary to study the question of stability in the large. In this sense, oscillatory motion will be referred to as stable and non-oscillatory or tumbling motion as unstable.

Motion in the large is treated here in four parts: circular orbital motion, non-circular orbital motion (analytical approach), non-circular orbital motion (numerical approach) and, finally, the variational analysis of periodic motion.

2.4.1 Circular Orbital Motion

For circular orbital motion ($e=0$), the governing equation becomes autonomous:

$$\gamma'' + (A + B \cos \gamma) \sin \gamma = 0 \quad (2.26)$$

where $A = I(\sigma + 1)$

and $B = 3I - 4$.

Multiplying equation (2.26) by $2\gamma'$ and integrating with respect to θ , gives the first integral

$$(\gamma')^2 - 2A \cos \gamma + B \sin^2 \gamma = 2E. \quad (2.27)$$

Here E is a constant of motion and a measure of the total librational energy of the system.

Substituting $u = A + B \cos \gamma$ into equation (2.27) leads to

$$d\theta = \frac{-\sqrt{|B|} du}{\sqrt{\frac{1}{4}(u-u_1)(u-u_2)(u-u_3)(u-u_4)}} \quad (2.28)$$

where

$$u_1 = -\sqrt{(A+B)^2 - B(\gamma'_i)^2}$$

$$u_2 = A - B$$

$$u_3 = \sqrt{(A+B)^2 - B(\gamma'_i)^2}$$

$$u_4 = A + B$$

for $\gamma(0) = 0$ and $\gamma'(0) = \gamma'_i$.

Equation (2.28) can be integrated¹⁶ to give two distinct solutions corresponding to real and imaginary values of u_1 and u_3 .

i) if u_1 and u_3 are real,

$$\gamma = \cos^{-1} \left\{ \frac{k_1 + k_2 \operatorname{sn}^2(k_3 \theta + k_4 | k)}{k_3 + k_4 \operatorname{sn}^2(k_3 \theta + k_4 | k)} \right\} \quad (2.29)$$

ii) if u_1 and u_3 are imaginary,

$$\gamma = \cos^{-1} \left\{ \frac{k_1 \left(\frac{\operatorname{sn}^2(k_3 \theta + k_4 | k)}{1 + \operatorname{cn}^2(k_3 \theta + k_4 | k)} \right) - 1}{k_1 \left(\frac{\operatorname{sn}^2(k_3 \theta + k_4 | k)}{1 + \operatorname{cn}^2(k_3 \theta + k_4 | k)} \right) + 1} \right\}. \quad (2.30)$$

It may be pointed out that the modulus k and the constants k_i are dependent upon the parameters I and σ as well as the initial velocity γ'_i . Moreover, the appropriate relations defining them depend upon the relative magnitudes of u_1, u_2, u_3 and u_4 . Hence, application of the solutions involves a considerable amount of computation which tends to reduce their effectiveness.

Typical satellite response obtained using the exact closed form solutions presented here is shown in Figures 2-6 to 2-9 inclusive. It is apparent that the resulting librational motion can be stable or unstable depending on the magnitude of the impulsive disturbance. Effect of non-linearities is also evident through a strong dependence of the libration frequency on initial condition.

In studying the question of stability in the large, the first integral obtained previously, equation (2.27), offers a more direct approach. It can be written as

$$T + U = E \quad (2.31)$$

where $T = (\gamma')^2 / 2$

and $U = -A \cos \gamma + \frac{B}{2} \sin^2 \gamma.$

For maximum or minimum values of the potential energy, $dU/d\gamma = 0.$

$$\therefore (A + B \cos \gamma) \sin \gamma = 0 \quad (2.32)$$

i.e., $\gamma = 0, \pm \pi$

Figure 2-6 Librational response in a circular orbit; $l=2, \sigma=-1.5$

Figure 2-7 Librational response in a circular orbit; $I=2, \sigma=2$

Figure 2-8 Librational response in a circular orbit; $I = .5, \sigma = -.5$

Figure 2-9 Librational response in a circular orbit; $I=.5, \sigma=2$

and if $|A/B| \leq 1$, $\gamma = \pm \cos^{-1}(-A/B)$.

Thus $U(0) = -A$, $U(\pm\pi) = A$ and if $|A/B| \leq 1$,
 $U(\pm \cos^{-1}(-A/B)) = (A^2 + B^2)/2B$.

Now if either $U(\pm\pi)$ or $U(\pm \cos^{-1}(-A/B))$ is greater than $U(0)$, constant total energy curves exist in the γ, γ' -phase plane which enclose the origin. Thus motion in the large can be considered stable (i.e., oscillatory) if $A > 0$ or if $|A/B| \leq 1$ and $B \geq 0$, i.e., $B \geq |A|$. Further if $B \geq |A|$, $U(\pm \cos^{-1}(-A/B)) \geq U(\pm\pi)$ since $(A-B)^2/2B \geq 0$. Noting that a separatrix between oscillatory and non-oscillatory motion passes through the points of maximum potential energy, the relations for stability bounds in the phase plane are

$$(\gamma')^2 = \begin{cases} 2A \cos \gamma - B \sin^2 \gamma + 2A; A \geq 0, A \geq B \\ 2A \cos \gamma - B \sin^2 \gamma + \frac{A^2 + B^2}{B}; B > |A|. \end{cases} \quad (2.33)$$

In particular for $\gamma_i = 0$, the critical or maximum initial velocity, γ'_c , for stable motion is

$$(\gamma'_c)^2 = \begin{cases} 4A; A \geq 0, A \geq B \\ (A+B)^2/B; B > |A|. \end{cases} \quad (2.34)$$

Thus for large amplitude stable motion,

$$A \geq \begin{cases} 0, & \text{for } B \leq 0 \\ -B, & \text{for } B > 0. \end{cases} \quad (2.35)$$

In terms of the system parameters I and σ these conditions are

$$\sigma > \sigma_c \quad (2.36)$$

$$\sigma_c = \begin{cases} -1; & I \leq 4/3 \\ 4(1-I)/I; & I > 4/3. \end{cases}$$

The corresponding expressions for the critical value of δ' are

$$\delta'_c = \begin{cases} 2\sqrt{I(\sigma+1)}; & I \leq 4/3 \text{ or } \sigma \geq 2(I-2)/I \\ \sqrt{\frac{I^2}{3I-4}}(\sigma-\sigma_c); & I > 4/3 \text{ and } \sigma < 2(I-2)/I. \end{cases} \quad (2.37)$$

Figure 2-10 summarizes these findings in the form of a chart showing the necessary conditions for stability in the large for $e=0$ and $\delta_i=0$. A carpet plot of δ'_c versus I and σ is shown in Figure 2-11.

In a more general situation involving arbitrary initial conditions, stability in the large can be determined utilizing equations (2.33) in the form,

$$(\delta')^2 = \begin{cases} 2I(\sigma+1)(1+\cos\delta) - (3I-4)\sin^2\delta; & I < 4/3, \sigma > -1 \text{ or} \\ & I > 4/3, \sigma > 2(I-2)/I \\ 2I(\sigma+1)\cos\delta - (3I-4)\sin^2\delta + \frac{I^2(\sigma+1)^2 + (3I-4)^2}{3I-4}; & \\ & I > 4/3, \sigma_c \leq \sigma \leq 2(I-2)/I. \end{cases} \quad (2.38)$$

Figure 2-10 Stability chart for motion in the large; $e=0, \delta_1=0$

Figure 2-11 Variation of the critical velocity with inertia and spin parameters; $e = 0$

Figure 2-12 Effect of spin on the stability bounds; $e=0, I=.5$

These relations define the stability bounds in the γ, γ' -phase plane. Figure 2-12 illustrates the effect of the spin parameter on stability for a specific value of the inertia parameter I .

2.4.2 Non-Circular Orbital Motion (Analytical Approach)

For the general case where $e \neq 0$ there are no known exact solutions to equations (2.17). Further, if such a solution did exist, its interpretation would likely be difficult. Moreover, in this case there is no first integral of motion and consequently no direct method of establishing stability bounds as there was in the preceding section. Numerical techniques in such a situation can be used to advantage. On the other hand, approximate methods can provide information suitable for preliminary design purposes. In order to be useful approximate analyses should yield solutions which are:

- a) acceptably accurate
- b) simple in form to facilitate interpretation.

(i) WKBJ method ¹⁷

To apply the WKBJ method it is necessary to linearize the equation of motion and remove the first derivative term. Consider the linearized system, i.e., equation (2.18). The term containing γ' may be eliminated by introducing the transformation $\hat{\gamma} = \gamma(1 + e \cos \theta)$ giving

$$\hat{\gamma}'' + G^2(\theta) \hat{\gamma} = 0 \quad (2.39)$$

where $G^2(\theta) = \frac{3I-4}{1+e \cos \theta} + I(\sigma+1) \left(\frac{1+e}{1+e \cos \theta} \right)^2$.

Equation (2.39) is now in a form amenable to treatment by the WKBJ method provided

$$|F| \ll 1 \quad (2.40)$$

where

$$F = \frac{G''}{2G^3} - \frac{3}{4} \left(\frac{G'}{G^2} \right)^2.$$

Using the definition of $G^2(\theta)$, the function F may be written as

$$F = -\frac{5}{16} \frac{e^2 \sin^2 \theta}{1 + e \cos \theta} \left[\frac{\left(\frac{3I - 4 + 2I(\sigma + 1)(1 + e)^2}{1 + e \cos \theta} \right)^2}{\left(\frac{3I - 4 + I(\sigma + 1)(1 + e)^2}{1 + e \cos \theta} \right)^3} \right] +$$

$$\frac{e \cos \theta}{4} \left[\frac{\frac{3I - 4 + 2I(\sigma + 1)(1 + e)^2}{1 + e \cos \theta}}{\left(\frac{3I - 4 + I(\sigma + 1)(1 + e)^2}{1 + e \cos \theta} \right)^2} \right] +$$

$$\frac{e^2 \sin^2 \theta}{4(1 + e \cos \theta)} \left[\frac{\frac{6I - 8 + 6I(\sigma + 1)(1 + e)^2}{1 + e \cos \theta}}{\left(\frac{3I - 4 + I(\sigma + 1)(1 + e)^2}{1 + e \cos \theta} \right)^2} \right]. \quad (2.41)$$

Typical plots of this function are shown in Figures 2-13 and 2-14 which indicate the applicability of the method for a wide range of values of I , σ and e .

Rigorous application of the WKBJ method yields the solution

$$\delta = \frac{G^{-1/2}}{1+e\cos\theta} \left[\bar{C}_1 \sin \xi + \bar{C}_2 \cos \xi \right] \quad (2.42)$$

where $\xi = \int G(\theta) d\theta$ and the constants

$\bar{C}_1$ and $\bar{C}_2$ are determined by the initial conditions. Although equation (2.42) appears to be simple, its evaluation, in general, is not straightforward. Substitution for G can create difficulties. Though not imperative, it is convenient to simplify the solution by introducing the assumption of small orbit eccentricity. This is justifiable, as for most situations of practical importance, e is indeed small. It is, therefore, useful to develop an approximate WKBJ solution involving only first order terms in eccentricity.

Writing

$$G^2 = \omega_o^2 \left[1 + 2eI(\sigma+1)(1-\omega_o^2 \cos\theta)/\omega_o^2 \right]$$

$$G = \omega_o \left[1 + eI(\sigma+1)(1-\omega_o^2 \cos\theta)/\omega_o^2 \right] \quad (2.43)$$

$$G^{-1/2} = \omega_o^{-1/2} \left[1 - eI(\sigma+1)(1-\omega_o^2 \cos\theta)/2\omega_o^2 \right]$$

Figure 2-13 Variation of F with σ and θ ; $e=.5, I=2$

Figure 2-14 Variation of F with e and θ ; $I=2, \sigma=0$

where $\omega_0^2 = I(\sigma+4) - 4$,

equation (2.41) can be reduced to

$$F = \frac{e \cos \theta}{4\omega_0^4} \left[3I - 4 + 2I(\sigma+1) \right] + O(e^2). \quad (2.44)$$

Thus the condition of applicability, equation (2.40), for small values of e is related to the basic frequency of librational motion, ω_0 . Except for situations involving extremely low frequency motion ($\omega_0 \rightarrow 0$), the method is applicable.

Substituting from (2.43) into equation (2.42) and dropping $O(e^2)$, the following solution is obtained for $\gamma(0) = 0$ and $\gamma'(0) = \gamma'_i$:

$$\gamma = (C_1 + C_2 \cos \theta) \sin \left\{ \omega_0 (C_3 \theta + C_4 \sin \theta) \right\}. \quad (2.45)$$

Here $C_1 = \frac{\gamma'_i}{\omega_0} \left[1 + \frac{e}{\omega_0^2} \left(\frac{I}{4} (2\sigma+17) - 5 \right) \right],$

$$C_2 = -\frac{\gamma'_i}{\omega_0} \frac{e}{\omega_0^2} \left(\frac{I}{4} (2\sigma+11) - 3 \right),$$

$$C_3 = 1 + \frac{e}{\omega_0^2} I(\sigma+1)$$

and $C_4 = -\frac{e}{\omega_0^2} \left(\frac{I}{2} (2\sigma+5) - 2 \right).$

The nature of the motion is now clear. Both the amplitude and the frequency of motion are modulated for $e \neq 0$. Moreover, there are first order effects of eccentricity on both the mean amplitude and frequency. It is possible to extract further information about the motion by examining the coefficients of equation (2.45). For example, the maximum amplitude is related to γ'_i , I , σ and e by

$$\gamma_\lambda = \gamma'_i \left[1 + e \left(\frac{I(\sigma+7)-8}{I(\sigma+4)-4} \right) \right] / \sqrt{I(\sigma+4)-4}. \quad (2.46)$$

Furthermore, the mean or fundamental frequency of motion is given by

$$\omega_f = C_3 \omega_0$$

$$\text{i.e.,} \quad \omega_f = (I(\sigma+1)(1+e)+3I-4)/(I(\sigma+4)-4)^{1/2}. \quad (2.47)$$

While the WKBJ solution satisfies the requirement of facility in interpretation, there are limitations in its application due to the simplifications introduced. Since the solution is based upon the linearized system and terms of $O(e^2)$ have been neglected, it may not agree closely with the exact solution if either the amplitude of librational motion or the orbital eccentricity is large. Furthermore, in order to yield

meaningful results $\omega_0^2 > 0$,

$$\text{i.e., } \sigma > 4(1-I)/I.$$

To assess the accuracy of the method, the WKBJ solution given by equation (2.45) was compared with the "exact" numerical solution of the governing equation, (2.17). Typical comparisons are shown in Figures 2-15 and 2-16. In general, agreement is quite good considering the amplitude of motion and the magnitude of e , however, it is clear that studies of stability in the large would require a more precise solution.

It may be pointed out that the results obtained using the WKBJ method are identical to those given by the method of Kryloff and Bogoliuboff involving linearization with respect to e and δ .

(ii) Perturbation analysis

In order to overcome the shortcomings of the WKBJ solution, it is necessary to include the non-linear effects. While in principal this can be done utilizing equation (2.45) as a generating solution, its unwieldly form together with the fact the non-linear terms in the governing equation are trigonometric results in considerable difficulties. The usefulness of such an approach is, therefore, somewhat questionable.

An alternate approach would be to represent the non-linear trigonometric terms in series form directly and work from a simpler generating solution. Consider equation (2.17) in the form

Figure 2-15 Comparison between the exact and WKBJ solutions for librational response; $\gamma_1=0, \gamma_1'=1.5, I=2, \sigma=1, e=.2$

Figure 2-16 Comparison between the exact and WKBJ solutions
for librational response; $\gamma_1=0, \gamma'_1=1, I=2, \sigma=0, e=.5$

$$\begin{aligned}
& \gamma'' + [I(\sigma+4) - 4] \gamma - \epsilon \left[\left\{ \frac{2}{3} (3I-4) + \right. \right. \\
& \left. \left. \frac{I}{6} (\sigma+1) \right\} \gamma^3 + \left\{ \right\} \gamma^5 + \left\{ \right\} \gamma^7 + \dots \right. \\
& \left. - \left\{ 2\gamma' \sin \theta + [3(I-1) \cos \theta - 2I(\sigma+1)] * \right. \right. \\
& \left. \left. (1 - \cos \theta) \right\} \gamma \right] e \left. \right] + O(\epsilon^2) = 0. \quad (2.48)
\end{aligned}$$

Here ϵ is a perturbation parameter, equal to unity, assigned to the orbital eccentricity, e , and to all non-linear terms arising from the series expansion of the trigonometric functions. Neglecting $O(\epsilon^2)$ and writing $\gamma = \gamma_0 + \epsilon \gamma_1$, and $\omega^2 = \omega_0^2 + \epsilon \omega_1^2$ gives

$$\gamma_0'' + \omega^2 \gamma_0 = 0 \quad (2.49)$$

which for $\gamma(0) = 0, \gamma'(0) = \gamma'_i$ yields

$$\gamma_0 = \frac{\gamma'_i}{\omega} \sin(\omega \theta) \quad (2.50)$$

and

$$\gamma_1'' + \omega^2 \gamma_1 = \omega_1^2 \gamma_0 +$$

$$\left\{ \frac{2}{3} (3I-4) + \frac{I}{6} (\sigma+1) \right\} \gamma_0^3 + \left\{ \right\} \gamma_0^5 + \dots$$

$$+ \left\{ 2\gamma_0' \sin \theta + (3(I-1) \cos \theta - 2I(\sigma+1)(1-\cos \theta)) \gamma_0 \right\} e.$$

(2.51)

$$\begin{aligned} \therefore \gamma_1 = \frac{e}{2} \left(\frac{\gamma_1'}{\omega} \right) & \left[\left(\frac{2\omega + 3(I-1) + 2I(\sigma+1)}{2\omega + 1} \right) \left(\frac{\omega+1}{\omega} \right) \sin(\omega\theta) - \right. \\ & \left. \sin((\omega+1)\theta) \right] + \left(\frac{2\omega - 3(I-1) - 2I(\sigma+1)}{2\omega - 1} \right) * \\ & \left[\left(\frac{\omega-1}{\omega} \right) \sin(\omega\theta) - \sin((\omega-1)\theta) \right] + \\ & \sum_{j=1}^{j_{\max}} K_j [(2j+1) \sin(\omega\theta) - \sin((2j+1)\omega\theta)] \quad (2.52) \end{aligned}$$

where $j_{\max}$ represents the number of non-linear terms included in the series representation of the trigonometric functions and

$$K_j = \frac{(-1)^j}{4_j(j+1)\omega^2} \sum_{i=j}^{j_{\max}} (-1)^{i+1} \frac{(3I-4+I(\sigma+1)/2^{2i})}{(i+j+1)!(i-j)!} \left(\frac{\gamma'_i}{\omega}\right)^{2i+1}.$$

The elimination of secular terms requires that

$$\omega_i^2 = 2e I(\sigma+1) - \sum_{j=1}^{j_{\max}} L_j \left(\frac{\gamma'_i}{\omega}\right)^{2j} \quad (2.53)$$

where

$$L_j = (-1)^{j+1} \frac{(3I-4+I(\sigma+1)/2^{2j})}{(j+1)! j!}.$$

Clearly the practical usefulness of this solution is limited unless one restricts the number of terms in the series representations of the trigonometric functions. Even in the case of $j_{\max}=1$, i.e., when $O(\gamma^5)$ terms are neglected, the resulting solution is considerably more complex than the WKBJ solution as it contains terms of the form $\sin(\omega\theta)$, $\sin(3\omega\theta)$, $\sin((\omega+1)\theta)$ and $\sin((\omega-1)\theta)$ where a quadratic is involved in defining ω^2 .

Before attempting to compare this solution with the WKBJ method, an investigation was performed with the aim of establishing the best compromise between accuracy and simplicity of form. This was done by comparing the responses as obtained using $j_{\max}=1, 2, 3$ and 4 against accurate numerical solutions. Typical examples are shown in Figures 2-17 and 2-18. On the basis of these tests, it was decided to select the perturbation solution based upon $j_{\max}=2$ to compare with the WKBJ method using an accurate numerical standard. The example given in Figure 2-19 indicates that, in general, little, if any, improvement in accuracy was attained over the simpler WKBJ method.

Figure 2-17 Comparison between the exact and perturbation solutions for librational response; $\delta_1=0, \delta_1'=1, I=1, \sigma=2, e=2$

Figure 2-18

Comparison between the exact and perturbation solutions for librational response; $\delta=0$, $\delta'=2$, $I=2$, $\sigma=1$, $e=.15$

Figure 2-19 Comparison between the exact, WKBJ and perturbation solutions for librational response; $\delta_1 = 0, \delta_1' = 1, I = 1, \sigma = 2, e = 0.2$

In order to answer the question of stability in the large, it is necessary to have a solution accurate for large amplitude motion, particularly in the neighbourhood of $\gamma = \pm \pi$. As indicated by Figure 2-18 the correlation between the best ($j_{\max} = 3$) perturbation and the numerical solutions is relatively poor even for γ as small as one radian. Thus it seems fair to conclude that, in order to attain sufficient accuracy for an analytical study of stability in the large, a highly sophisticated and complex perturbation procedure is required. This, in turn, would make such a study so involved that it would defeat the purpose of searching for an analytical solution.

(iii) Invariant surface concept

Returning to the WKBJ solution, viz. equation (2.45), and differentiating it with respect to θ gives

$$\gamma' = (C_1 + C_2 \cos \theta)(C_3 + C_4 \cos \theta) \omega_0^*$$

$$\cos [\omega_0(C_3 \theta + C_4 \sin \theta)] - C_2 \sin \theta^*$$

$$\sin [\omega_0(C_3 \theta + C_4 \sin \theta)]. \quad (2.54)$$

The argument $\omega_0(C_3 \theta + C_4 \sin \theta)$ appearing in equations (2.45) and (2.54) can be eliminated giving

$$\gamma' = (C_1 + C_2 \cos \theta)(C_3 + C_4 \cos \theta) \omega_0 *$$

$$\left[1 - (\gamma / (C_1 + C_2 \cos \theta))^2 \right]^{1/2} - (C_2 \sin \theta /$$

$$(C_1 + C_2 \cos \theta)) \gamma. \quad (2.55)$$

Equation (2.55) relating the state variables γ , γ' and θ suggests the existence of a time invariant solution surface in a three dimensional state space. Since θ enters this relation by way of trigonometric functions periodic in 2π , the state space may be truncated to cover any interval of 2π in θ without loss of generality. In this analysis the interval $(0, 2\pi)$ is used.

A schematic diagram of an invariant solution surface is shown in Figure 2-20. A trajectory emanating from point 1 follows the surface to point 2 at $\theta = 2\pi$. Continuation of the trajectory beyond this point can be represented by a succession of trajectories emanating at $\theta = 0$ having the same values of γ and γ' as the preceding trajectory had when terminated at $\theta = 2\pi$. This procedure when repeated over a large number of orbits should define a surface referred to as an invariant surface or integral manifold. Moreover, for any specified value

Figure 2-20 Schematic of an invariant solution surface in state space.

of θ , it should be possible to generate a cross section of the invariant surface simply by using the values of γ and γ' obtained from a solution. Hence a meaningful comparison between methods should be possible in terms of the cross sections generated by them.

It should be pointed out that the existence of an invariant surface implies ordered intersections of the trajectories with a given plane resulting in a well defined cross section. On the other hand, an ergodic distribution of points of intersection would suggest the absence of such a surface.

The concept of an invariant surface in state space is not new. Theoretical work by Moser¹⁸ and recent numerical experiments by Hénon and Heiles¹⁹ and by Jeffreys²⁰ have demonstrated the existence and usefulness of such surfaces. Brereton^{4,6} showed, numerically, that similar surfaces can be generated for a specific, conservative, non-linear, non-autonomous system with periodic coefficients. To verify the application of this concept to the problem in hand, an attempt was made to generate cross sections at $\theta = 0$ using an accurate numerical solution. Typical examples shown in Figures 2-21 and 2-22 clearly suggest the existence of invariant solution surfaces for this system. The figures also compare cross sections obtained using the WKBJ solution with those obtained numerically.

It is interesting to note that for $I = 2$, $\sigma = 0$ and $e = .5$, there is good agreement between the analytical and numerical methods (Figure 2-22) despite relatively poor agreement in dynamical response (Figure 2-16). Thus the disagreement in

Figure 2-21 Comparison of cross sections at $\theta = 0$ of invariant surfaces generated by the WKBJ and numerical analyses; $l=2$, $\sigma=-1$, $e=.4$

Figure 2-22 Comparison of cross sections at $\theta = 0$ of invariant surfaces generated by the WKBJ and numerical analyses; $I=2, \sigma=0, e=.5$

phase between the responses predicted by the two methods does not seem to affect the geometry of the surface for relatively small amplitude motion.

The principal limitation of the WKBJ method appears to arise as a result of linearizing the equation of motion. Differences between approximate and "exact" results grow as the amplitude of motion gets larger. Nonetheless, the WKBJ method yields results of sufficient accuracy for preliminary design purposes.

2.4.3 Non-Circular Orbital Motion (Numerical Approach)

Fundamentally the problem of obtaining a numerical solution to a system governed by a set of ordinary differential equations is comparatively easy as several techniques are available. Interpretation of these results, on the other hand, is often difficult. It is here that the invariant surface concept introduced previously proves to be most useful.

As shown in the foregoing, the WKBJ method applied to the linearized system gives rise to invariant solution surfaces in state space. In addition, it was seen that for $e=0$, the energy relation, equation (2.27), defines curves in the δ', δ -phase plane which are, in effect, equivalent to uniform cross sections of solution surfaces in δ', δ, θ - state space. Thus the existence of invariant surfaces for both large amplitude motion in a circular orbit and for small amplitude motion in an elliptic orbit has been demonstrated. As well, the numerical cross section tests discussed in the previous section strongly suggest the existence of such surfaces under more general conditions.

In effect, this is the basis of Minorsky's method¹⁵ involving a stroboscopic phase plane which, in essence, is just a $\theta =$ constant cross section of state space.

Minorsky asserts that a stationary point in the stroboscopic phase plane is associated with a periodic solution. This is evident in γ', γ, θ - state space as for $e \neq 0$, a periodic solution must be of period $2n\pi$, where n is a finite integer. Thus the solution is represented by a finite number of trajectories in the truncated state space whose cross sections are a finite number of stationary points. Furthermore, in the stroboscopic phase plane, periodic solutions appear as singular points of an ordinary phase plane. For the case under consideration, it can be shown, by Floquet theory, that periodic solutions appear as either centers (considered stable) or saddles. It is clear, therefore, that closed cross sections about stable periodic solutions give rise to tube-like invariant surfaces surrounding the periodic motion.

Since the governing equation (2.17) satisfies the Lipshitz condition for uniqueness, invariant surfaces cannot intersect as this would imply non-unique solutions. In principle, beginning with a stable periodic solution, a succession of nested invariant surfaces can be generated by progressively choosing initial conditions exterior to the preceding surface. Floquet theory does not predict a limit to this exercise but since equation (2.17) is non-linear, a breakdown should eventually occur. For example, tumbling motion cannot generate a closed surface about an oscillatory periodic solution. Thus it would

seem appropriate to devise a series of "numerical experiments" ¹⁹ to determine the outermost or limiting invariant surface as a function of the system parameters, I , σ and e .

In practice, limiting surfaces can be obtained to almost any desired degree of accuracy by numerically integrating the governing equation and assessing a resulting cross section, for example, see Figures 2-21 and 2-22. In most of the cases investigated, the eventual breakdown occurred dramatically with an abrupt fragmentation of the cross section indicating tumbling or unstable oscillatory motion. On a few occasions uncertainty existed near the limiting surface where fragmentation of the cross section occurred but tumbling was either not observed within the period of integration, generally 30 orbits, or only observed belatedly. In general, however, the envelope of stable motion in the large was defined to a highly satisfactory degree of accuracy by the outermost unfragmented surface.

Isometric views of typical limiting invariant surfaces, representing the bounds of stable motion, are shown in Figures 2-23 and 2-24. Note that for the case of $I = .5$, $\sigma = 2$ and $e = .16$, three such surfaces are defined: a major envelope of stable motion (mainland) and two subsidiary envelopes associated with periodic solutions, subsequently referred to as islands. Such secondary surfaces wrap around the mainland in close proximity to it and represent small zones of stability of little practical significance.

Aside from the actual envelope of motion, the characteristics of a system are better represented in the stroboscopic

Figure 2-23 Typical example of limiting invariant surface; $l=5, \nu=2, e=.35$

Figure 2-24 Typical example of limiting invariant surfaces; $I=5, \sigma=2, e=.16$

phase plane or θ cross section rather than in state space. The terms mainland and island may be seen to take on more practical significance in this context. Generally, mainlands have large cross sections centered on the origin while islands have smaller cross sections and are peripheral to the mainland.

Since all trajectories span the interval $(0, 2\pi)$ in θ , it is immaterial which cross section is chosen for a study of system characteristics. It is, however, advisable to choose a cross section which will provide maximum information for a given amount of integration. The cross section at $\theta = 0$ was selected because, as shown by the governing equation, it is symmetric with respect to both δ and δ' resulting in a considerable saving in computer time.

Using this concept of cross sectioning, studies of the characteristics of large amplitude motion were readily accomplished. Typical examples are shown in Figures 2-25 and 2-26. The role of periodic solutions as backbones or spines of invariant surfaces is evident. The sphere of influence of any given periodic solution is apparent as well. This suggests that many of the salient features of large amplitude motion can be ascertained through the variational analysis of relevant periodic solutions. This is analogous to the study of singular points in an autonomous system.

Yet another useful extension of the invariant surface concept is the development of a stability chart using some specific intercept of the limiting surfaces as a measure of stability. This results in a considerable condensation of data.

Figure 2-25 Study of invariant surface cross sections; $I=2, \sigma=1, e=3$

Figure 2-26 Study of invariant surface cross sections; $I=5, \sigma=2, e=.33$

It is clear, however, that any point on a limiting invariant surface is sufficient to define the entire envelope of stable motion in state space. Figures 2-27 and 2-28 are examples of such charts utilizing $|\delta'|$ at $\delta-\theta=0$ as a measure of stability showing the destabilizing effects of orbit eccentricity. The ragged nature of the stability bound with spike-like features can be attributed to the emergence of periodic solutions and related islands from the mainland limiting surface. This again underlines the need to pursue the analysis of periodic solutions.

It should be emphasized at this time that, although the cross sectioning concept is relatively simple and yields much insight into the nature of motion in the large, the numerical character of this approach requires a considerable amount of computer time. For example, a typical cross section generated over 30 orbits requires nearly one minute on an IBM 7044 computer. On occasion, double precision arithmetic and as many as 720 integration steps per orbit were required to give accurate results. Since this phase of the analysis was primarily intended for the development and testing of techniques, a large scale parametric study was considered unwarranted.

2.4.4 Variational Analysis of Periodic Solutions

In the foregoing the need to perform a variational analysis of periodic motion became apparent. To obtain the variational equation for periodic motion, let $\delta = \delta_p + \delta_v$, where δ_p is the periodic solution and δ_v represents a small perturbation. Substituting into equation (2.17) and linearizing with respect to δ_v yields:

Figure 2-27 Variation of critical velocity with orbit eccentricity; $\gamma_i = 0$, $l = .5$, $\sigma = 2$

Figure 2-28 Variation of critical velocity with orbit eccentricity; $\gamma_i = 0$, $l = 2$, $\sigma = -1$

$$\begin{aligned}
& \gamma_v'' - \left\{ \frac{2e \sin \theta}{1 + e \cos \theta} \right\} \gamma_v' + \left\{ \left(\frac{3I - 4 - e \cos \theta}{1 + e \cos \theta} \right)^* \right. \\
& \left. \left(\cos \gamma_p - \sin \gamma_p \right) + I(\sigma + 1) \left(\frac{1 + e}{1 + e \cos \theta} \right)^2 \right. \\
& \left. \cos \gamma_p \right\} \gamma_v = 0.
\end{aligned} \tag{2.56}$$

Since γ_p must be periodic in $2n\pi$, where n is an integer, Floquet theory can be applied as the coefficients of equation (2.56) have a common period namely that of the periodic solution. Furthermore, as in the study of motion in the small, the product of the characteristic multipliers is unity because, once again, equation (2.24) applies. This constitutes a proof of the statement made previously that periodic solutions behave as either centers or saddles in the stroboscopic phase plane since the characteristic multipliers lie either on the unit circle or on the real axis.

In order to utilize this analysis the following procedure was adopted. First, initial condition space was scanned in the area of interest for potential periodic solutions. This was done by integrating equation (2.17) numerically over a number of orbits and observing the state of the system at the end of each orbit. When a periodic solution was bracketed in this manner, an interactive technique based on the variable secant method was used to match initial and final conditions. With

this, equation (2.56) could be integrated simultaneously with equation (2.17) to obtain $\Phi(2n\pi)$ from $\Phi(0) = 1$. The criterion for the variational stability of any particular solution is then

$$\left| \text{Tr}[\Phi(2n\pi)] \right| \begin{cases} > 2, & \text{unstable} \\ \leq 2, & \text{stable.} \end{cases} \quad (2.57)$$

This method was applied to study the stability charts presented in Figures 2-27 and 2-28. Results of the analysis are shown in Figures 2-29 and 2-30. The notation m/n is used to indicate periodic motion of m oscillations in n orbits. As pointed out before, the appearance of spikes is associated with stable periodic solutions separating from the mainland. On the other hand, it should be emphasized that unstable periodic solutions have little effect on the stability boundary. Termination of the spikes or islands at certain definite values of orbit eccentricity as predicted by the variational analysis agrees well with the results obtained by cross sectioning. In fact, the variational analysis would be expected to be more accurate in this regard as less computation is involved and the criterion (2.57) is far less subjective than an assessment of cross section data.

It should be stressed here that although this type of variational approach appears to resemble an analysis of singularities of a conservative autonomous system, it does not, in fact, offer a "quick look" at the features of motion in the

Figure 2-29 Variational analysis of periodic solutions; $\gamma_i = 0, I = .5, \sigma = 2$

Figure 2-30 Variational analysis of periodic solutions; $\gamma_i = 0, I = 2, \sigma = 1$

large since periodic solutions must first be found. Despite this deficiency, however, the analysis is useful in seeking bifurcation values of certain parameters.

2.5 Concluding Remarks

As the purpose of this investigation was to serve as a testing ground for Phase II, it would be appropriate to emphasize the important aspects of the analyses and the conclusions based on them:

(i) Librational motion of a spinning satellite in roll is coupled to the orbital motion. Fortunately, the coupling effects are small and can be neglected. In the resulting non-linear, non-autonomous equations of motion, orbital characteristics are specified by a single parameter (e).

(ii) Stability in the small can be successfully treated using a numerical approach involving Floquet theory. The method yields stability charts delineating stable and unstable regions in parameter space.

(iii) For the particular case of $e=0$, the first integral of motion (2.27) as well as a closed form solution are available. The first integral proves to be of greater use in analyzing motion in the large as the closed form solution is difficult to interpret.

(iv) Approximate analytical solutions have limited value as they are not sufficiently accurate to assess stability in the large. Among the methods developed, the WKBJ solution yields results of greatest value. For small oscillations, it predicts the amplitude and frequency with sufficient accuracy

to be useful in preliminary design.

(v) Bounded (oscillatory) motion was found to generate closed invariant solution surfaces in δ, δ', θ - state space. The envelopes of motion can be determined easily and accurately using numerical techniques. This approach to the study of motion in the large was found to be most useful and definitive.

(vi) Stable periodic solutions form backbones or spines of invariant surfaces. Variational analysis using Floquet theory establishes the stability of periodic motion and hence, the salient features of system behavior.

A comment should be made at this stage on the use of Liapounov's direct method for determining stability in the large. The method involves determination of a suitable testing function V for which no definite procedure is available when the system is non-linear and non-autonomous. The very existence of closed invariant solution surfaces, however, implies neutral stability, i.e. $dV/dt = 0$. Thus the testing function is a constant when the motion is bounded and, in fact, could be defined by the relation describing the invariant surface. The complicated nature of limiting surfaces suggests, however, that the analytical form of the Liapounov function would be highly complex thus virtually ruling out any possibility of obtaining it by trial and error. On the other hand, numerical methods yield the desired information with relative ease.

3. PHASE II - ATTITUDE DYNAMICS OF A SPINNING AXISYMMETRIC SATELLITE

3.1 Preliminary Remarks

This chapter investigates the attitude dynamics of a rigid, axisymmetric satellite free to librate in both roll and yaw. The effects of orbit eccentricity and the gravity gradient torque are included in the study. As indicated in the introductory chapter most of the research in this field involves a linear or quasi-linear representation of the system. In this analysis, librational motion is studied both in the small and in the large.

The pattern established in the previous chapter is followed with extensive use being made of the numerical methods employed in that study, i.e., the invariant surface and cross sectioning concepts and Floquet theory in the variational analysis of both motion in the small and periodic motion.

Stability charts are presented which are of considerably greater detail than those found in current literature. Through the use of the Hamiltonian function, a variation of the invariant surface concept developed in Chapter 2 is used to investigate motion in the large for the case of $e = 0$. The concept of principal cross sections is introduced enabling the maximum response of the system to be ascertained given an arbitrary disturbance. Methods are given for determining the initial conditions for and the variational stability of periodic solutions for either circular or elliptical orbital motion.

3.2 Formulation of the Problem

Consider a satellite with center of mass S moving in a Keplerian orbit about the center of force at O (Figure 3-1). As before, the coordinates R and θ define the position of the center of mass with respect to an inertial frame x_o, y_o, z_o .

Let x_p, y_p, z_p represent the principal body coordinates with origin at S and the coordinate x_p coinciding with the axis of symmetry. Further, let x_s, y_s, z_s be another set of orthogonal coordinates with origin at S but orientated such that the x_s axis is normal to the orbital plane and y_s lies along the extension of the radius vector R . The coordinates γ, β and α are modified Euler angles defining the attitude of the satellite relative to the non-inertial frame x_s, y_s, z_s . The first rotation, γ , about the local horizontal, z_s axis, is referred to as roll; the second rotation, β , about the y axis represents yaw while the third rotation, α , about the axis of symmetry is the spin of the satellite.

Assuming librational motion to have a negligible effect on the motion of the center of mass, a Lagrangian formulation need only involve those terms directly related to attitude motion. This approach is deemed justifiable in the light of the results of the analysis of Appendix I and is consistent with the methods employed by other researchers in this field. ^{9,11,12,13,14}

Again owing to the symmetry of the satellite, the coordinate frame x, y, z (Figure 3-1) is used in which the satellite spins with angular velocity $\dot{\alpha}$ about the x axis.

The librational kinetic energy of the satellite can be

Figure 3-1 Geometry of Phase II model

written as

$$T = \frac{1}{2} (I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2) \quad (3.1)$$

where the angular velocities are given by

$$\omega_x = \dot{\alpha} - \dot{\gamma} \sin \beta + \dot{\theta} \cos \beta \cos \gamma$$

$$\omega_y = \dot{\beta} - \dot{\theta} \sin \gamma$$

$$\omega_z = \dot{\gamma} \cos \beta + \dot{\theta} \sin \beta \cos \gamma. \quad (3.2)$$

Following the method used previously the potential energy can be written as

$$U = -\frac{\mu I_x}{2R^3} \left(\frac{I-1}{I} \right) (1 - 3 \cos^2 \beta \sin^2 \gamma). \quad (3.3)$$

Noting that $I = I_x / I_y = I_x / I_z$, the Lagrangian function for the system becomes

$$\begin{aligned} L = & \frac{I_x}{2I} \left[I (\dot{\alpha} - \dot{\gamma} \sin \beta + \dot{\theta} \cos \beta \cos \gamma)^2 + \right. \\ & (\dot{\beta} - \dot{\theta} \sin \gamma)^2 + (\dot{\gamma} \cos \beta + \dot{\theta} \sin \beta \cos \gamma)^2 + \\ & \left. \frac{\mu}{R^3} (I-1) (1 - 3 \sin^2 \gamma \cos^2 \beta) \right]. \quad (3.4) \end{aligned}$$

As before, α is a cyclic coordinate with the following first integral defining $\dot{\alpha}$:

$$h_{\alpha} = \frac{1}{m_s} \frac{\partial L}{\partial \dot{\alpha}} = \frac{I_x}{m_s} (\dot{\alpha} - \dot{\gamma} \sin \beta + \dot{\theta} \cos \beta \cos \gamma). \quad (3.5)$$

The spin parameter, σ , is again defined to be

$$\sigma = \left. \frac{\dot{\alpha}}{\dot{\theta}} \right|_{\theta=\beta=\gamma=0} = \left. \frac{m_s h_{\alpha}}{I_x \dot{\theta}} \right|_{\theta=0}. \quad (3.6)$$

Changing the independent variable from τ to θ through the use of equations (2.14) and (2.15) and making use of the spin parameter, the governing equations for librational motion become:

$$\beta'' - \gamma' \cos \gamma + I(\sigma+1) \left(\frac{1+e}{1+e \cos \theta} \right)^2 (\gamma' \cos \beta + \sin \beta \cos \gamma)$$

$$- (\gamma' \cos \beta + \sin \beta \cos \gamma) (\cos \beta \cos \gamma - \gamma' \sin \beta) -$$

$$\frac{3(I-1)}{1+e \cos \theta} \sin \beta \cos \beta \sin^2 \gamma -$$

$$\left(\frac{2e \sin \theta}{1+e \cos \theta} \right) (\beta' - \sin \gamma) = 0, \quad (3.7)$$

$$\gamma'' \cos \beta - 2\beta' \gamma' \sin \beta + 2\beta' \cos \beta \cos \gamma -$$

$$I(\sigma+1) \left(\frac{1+e}{1+e \cos \theta} \right)^2 (\beta' - \sin \gamma) - \cos \beta \sin \gamma \cos \gamma +$$

$$\frac{3(I-1)}{1+e \cos \theta} \cos \beta \sin \gamma \cos \gamma - \left(\frac{2e \sin \theta}{1+e \cos \theta} \right) *$$

$$(\gamma' \cos \beta + \sin \beta \cos \gamma) = 0. \quad (3.8)$$

This fourth order, non-linear, non-autonomous system is clearly more complex than that analyzed in the previous chapter. As analytical methods in such a formidable situation are not likely to be successful, the numerical approach is used extensively in the analysis of this system.

3.3 Motion in the Small

Following the pattern established in the preceding phase, a study of the linear system is first attempted. Linearization of the governing equations of motion, (3.7) and (3.8), with respect to the coordinates β and γ results in the following:

$$\beta'' = \left[1 - I(\sigma+1) \left(\frac{1+e}{1+e \cos \theta} \right)^2 \right] \beta +$$

$$\left[\frac{2e \sin \theta}{1+e \cos \theta} \right] \beta' - \left[\frac{2e \sin \theta}{1+e \cos \theta} \right] \gamma +$$

$$\left[2 - I(\sigma+1) \left(\frac{1+e}{1+e \cos \theta} \right)^2 \right] \gamma' \quad (3.9)$$

$$\begin{aligned}
\gamma'' = & \left[\frac{2e \sin \theta}{1+e \cos \theta} \right] \beta + \left[I(\sigma+1) \left(\frac{1+e}{1+e \cos \theta} \right)^2 - \right. \\
& \left. 2 \right] \beta' + \left[1 - I(\sigma+1) \left(\frac{1+e}{1+e \cos \theta} \right)^2 - \right. \\
& \left. \frac{3(I-1)}{1+e \cos \theta} \right] \gamma + \left[\frac{2e \sin \theta}{1+e \cos \theta} \right] \gamma' . \quad (3.10)
\end{aligned}$$

For the autonomous case of $e=0$, i.e., circular orbital motion, equations (3.9) and (3.10) reduce to:

$$\beta'' = -(D_1 - 1)\beta - (D_1 - 2)\gamma' \quad (3.11)$$

$$\gamma'' = (D_1 - 2)\beta' - (D_1 + D_2)\gamma \quad (3.12)$$

where

$$D_1 = I(\sigma+1),$$

$$D_2 = 3I - 4; (-4 \leq D_2 \leq 2).$$

The characteristic equation is given by

$$\text{Det} \begin{pmatrix} -s & 1 & 0 & 0 \\ -(D_1 - 1) & -s & 0 & -(D_1 - 2) \\ 0 & 0 & -s & 1 \\ 0 & (D_1 - 2) & -(D_1 + D_2) & -s \end{pmatrix} = 0$$

or

$$s^4 + \left[(D_1 - 2)^2 + (D_1 + D_2) + (D_1 - 1) \right] s^2 + \left[(D_1 - 1)(D_1 + D_2) \right] = 0. \quad (3.13)$$

For stability, the coefficients of the characteristic equation must be of the same sign. Beginning with the coefficient of s^2 , for stability

$$D_1^2 - 2D_1 + (3 + D_2) \geq 0. \quad (3.14)$$

$$\text{i.e., } D_{1c} = 1 \pm \left[1 - (3 + D_2) \right]^{1/2}. \quad (3.15)$$

For D_{1c} to be meaningful, it must be real. Thus $(1 - (3 + D_2)) \geq 0$ or $D_2 \leq 2$, i.e. $I \leq 2/3$. Hence the stability requirement involving the coefficient of s^2 is met iff

$$\begin{aligned} & \text{i) } I \geq 2/3 \\ \text{or} \quad & \text{ii) } I < 2/3 \quad \text{and} \quad \begin{cases} \text{a) } \sigma < (1 - \sqrt{2 - 3I}) / (I - 1) \\ \text{b) } \sigma > (1 + \sqrt{2 - 3I}) / (I - 1). \end{cases} \end{aligned} \quad (3.16)$$

Turning to the coefficient of s^0 , stability requires that

$$D_1^2 + (D_2 - 1)D_1 - D_2 \geq 0. \quad (3.17)$$

Solving (3.17) for the critical values of D_1 , yields

$$D_{1c} = -\left(\frac{D_2-1}{2}\right) \pm \sqrt{\left(\frac{D_2-1}{2}\right)^2 + D_2} \quad (3.18)$$

$$\text{i.e., } D_{1c} = \begin{cases} 1 \\ \text{or} \\ -D_2. \end{cases} \quad (3.19)$$

Thus, the stability requirement involving the coefficient of s^0 is met iff

$$\text{i) } I \geq 1 \text{ and } \begin{cases} \text{a) } \sigma > (1-I)/I \\ \text{or} \\ \text{b) } \sigma < 4(1-I)/I \end{cases}$$

or

$$\text{ii) } I < 1 \text{ and } \begin{cases} \text{a) } \sigma < (1-I)/I \\ \text{or} \\ \text{b) } \sigma > 4(1-I)/I. \end{cases} \quad (3.20)$$

Furthermore, it is clear that, since the characteristic equation (3.13) is a quadratic in s^2 , the origin will be stable iff its roots are negative and real. Otherwise some of the characteristic roots will have positive real parts. Thus, for stability yet another requirement must be met, viz.,

$$D_2^2 + 2 [D_1^2 - 4 D_1 + 5] D_2 +$$

$$[D_1^4 - 4 D_1^3 + 6 D_1^2 - 8 D_1 + 9] \gg 0. \quad (3.21)$$

Solving (3.21) for the critical values of D_2 results in

$$D_{2c} = -[D_1^2 - 4 D_1 + 5] \pm 2 \sqrt{-D_1^3 + 5 D_1^2 - 8 D_1 + 4}. \quad (3.22)$$

For real and hence meaningful values of D_{2c} ,

$$-D_1^3 + 5 D_1^2 - 8 D_1 + 4 \geq 0. \quad (3.23)$$

As the zeros of this expression occur at 1, 2 and 2, it is clear that there is a bound on D_2 iff $D_1 \leq 1$ i.e., $\sigma \leq (1-1)/1$. In that case, equation (3.22) can be used to define the region of stability in I, σ -parameter space.

Utilizing the relations (3.16), (3.20) and (3.22), it is possible to produce a stability chart involving the parameters I and σ (Figure 3-2). This chart when compared with Figure 2-2 illustrates the greater complexity of this phase of the study.

Turning now to the more general situation of $e \neq 0$, it is seen that the governing equations, (3.9) and (3.10), have periodic coefficients and hence, can be studied using Floquet theory. Recalling that the method involves the determination of a final condition matrix $\Phi(2\pi)$ from an initial condition matrix equal to the identity matrix with the subsequent evalu-

Figure 3-2 Stability chart for motion in the small; $e=0$

ation of the corresponding eigenvalues or characteristic multipliers, λ_i , the stability criterion can be expressed as follows:

$$|\lambda_i| \begin{cases} \leq 1; \text{ stable} \\ > 1; \text{ unstable.} \end{cases} \quad (3.24)$$

$i=1,2,3,4$

Although it can be shown, by equation (2.23), that $\prod_{i=1}^4 \lambda_i = 1$, the trace of $\Phi(2\pi)$ cannot be used as the sole measure of stability as the characteristic multipliers are not, in this case, restricted to be on the unit circle or the real axis. It is clear, however, that if $|\text{Tr} [\Phi(2\pi)]| > 4$, the system is unstable. Thus the stability criterion becomes,

$$|\text{Tr}[\Phi(2\pi)]| \begin{cases} > 4; \text{ unstable} \\ \leq 4; \begin{cases} |\lambda_i| \leq 1; \text{ stable} \\ |\lambda_i| > 1; \text{ unstable.} \end{cases} \end{cases} \quad (3.25)$$

$i=1,2,3,4$

Using the above criterion and integrating equations (3.9) and (3.10) numerically to generate $\Phi(2\pi)$ from the following initial conditions:

- i) $\beta(0) = 1, \beta'(0) = \gamma(0) = \gamma'(0) = 0$
- ii) $\beta'(0) = 1, \beta(0) = \gamma(0) = \gamma'(0) = 0$
- iii) $\gamma(0) = 1, \beta(0) = \beta'(0) = \gamma'(0) = 0$
- iv) $\gamma'(0) = 1, \beta(0) = \beta'(0) = \gamma(0) = 0$

the stability charts shown in Figures 3-3, 3-4, 3-5 and 3-6 were obtained. Although these charts differ considerably in detail from their counterparts obtained in Phase I (Figures 2-3, 2-4 and 2-5), they have some similarity in basic structure. Furthermore, the usefulness of Phase I in developing suitable analyses is demonstrated here by the relative ease with which these results were obtained.

3.4 Motion in the Large

Until this stage of the analysis, motion was readily classified as stable or unstable. Motion in the small never presents a problem in this regard. In Phase I, motion in the large was seen to be either oscillatory or non-oscillatory. With the inclusion of the β degree of freedom, the stability of motion in the large is difficult to define. Rather, for practical purposes, the response of the system may or may not be acceptable and would then be referred to as stable or unstable. For example, a configuration may be deemed acceptable if the axis of symmetry does not deviate from the normal to the orbital plane

Figure 3-3 Stability chart for motion in the small; $e=.1$

Figure 3-4 Stability chart for motion in the small; $e = .2$

Figure 3-5 Stability chart for motion in the small; $e = .3$

Figure 3-6 Stability chart for motion in the small; $e = .4$

by more than some prescribed angle. Since the matter of stability is not well defined in the case, no attempt is made here to impose an arbitrary standard. This, however, does make it difficult to condense results in a manner similar to that in Phase I.

Motion in the large is treated in two parts: The first pertains to circular orbital motion and the latter to elliptical trajectories. In each case, the analysis involves the extension of the numerical techniques which proved useful in Phase I.

3.4.1 Circular Orbital Motion (Analytical Approach)

As in Phase I, the restriction of $e = 0$ reduces the governing equations to autonomous form:

$$\begin{aligned} \beta'' + [I(\sigma+1)\cos\beta - 2\cos^2\beta\cos\delta]\delta' - \\ [1 + (3I-4)\sin^2\delta - (\delta')^2]\sin\beta\cos\beta + \\ [I(\sigma+1)]\sin\beta\cos\delta = 0, \end{aligned} \quad (3.26)$$

$$\begin{aligned} \delta''\cos\beta + [2\cos\beta\cos\delta - I(\sigma+1) - 2\delta'\sin\beta]\beta' + \\ [(3I-4)\cos\beta]\sin\delta\cos\delta + I(\sigma+1)\sin\delta = 0. \end{aligned} \quad (3.27)$$

Prospects of finding a closed form solution to these equations are dim indeed. However, as seen in Phase I, considerable in-

sight into the problem can be obtained by examining the first integral. Noting that, for $e = 0$, $\mu/R^3 = \dot{\theta}^2$ and the Lagrangian of the system can be expressed as

$$L = I(\dot{\alpha} - \dot{\gamma} \sin \beta + \dot{\theta} \cos \beta \cos \gamma)^2 + (\dot{\beta} - \dot{\theta} \sin \gamma)^2 +$$

$$(\dot{\gamma} \cos \beta + \dot{\theta} \sin \beta \cos \gamma)^2 + (I-1)\dot{\theta}^2(1-3\cos^2 \beta \sin^2 \gamma) \quad (3.28)$$

where $\dot{\theta} = \text{constant}$.

Since $\partial L / \partial t = 0$, $dH/dt = 0$, i.e. the Hamiltonian is a first integral or constant of motion. By definition,

$$H = \frac{\partial L}{\partial \dot{\alpha}} \dot{\alpha} + \frac{\partial L}{\partial \dot{\beta}} \dot{\beta} + \frac{\partial L}{\partial \dot{\gamma}} \dot{\gamma} - L. \quad (3.29)$$

Thus, within a multiplicative constant, the Hamiltonian can be written as

$$H = I[(\alpha' - \gamma' \sin \beta)^2 - \cos^2 \beta \cos^2 \gamma] + (\beta')^2 +$$

$$(\gamma')^2 \cos^2 \beta + \cos^2 \beta \cos^2 \gamma + 3(I-1) \cos^2 \beta \sin^2 \gamma. \quad (3.30)$$

Furthermore, from equations (3.5) and (3.6), the variable α' can be eliminated by introducing the spin parameter σ giving

$$H = -2I(\sigma+1) \cos \beta \cos \gamma + (\beta')^2 +$$

$$[(\gamma')^2 + 3(I-1) - (3I-4) \cos^2 \gamma] \cos^2 \beta. \quad (3.31)$$

The above equation can be rewritten as

$$(\beta')^2 + (\gamma')^2 \cos^2 \beta = H + 2I(\sigma+1) \cos \beta \cos \gamma - [3(I-1) - (3I-4) \cos^2 \gamma] \cos^2 \beta. \quad (3.32)$$

For real motion,

$$H + 2I(\sigma+1) \cos \beta \cos \gamma - [3(I-1) - (3I-4) \cos^2 \gamma] \cos^2 \beta \geq 0. \quad (3.33)$$

For the critical situation where the equality is satisfied, the relation yields zero velocity curves representing the absolute bounds of motion in β, γ -space.

Solving for β in terms of γ gives

$$\beta = \cos^{-1} \left\{ \frac{I(\sigma+1) \cos \gamma}{3(I-1) - (3I-4) \cos^2 \gamma} \pm \sqrt{\left[\left(\frac{I(\sigma+1) \cos \gamma}{3(I-1) - (3I-4) \cos^2 \gamma} \right)^2 + \frac{H}{3(I-1) - (3I-4) \cos^2 \gamma} \right]} \right\}. \quad (3.34)$$

It is evident that zero velocity curves are symmetrical with respect to both β and γ . In addition, due to the periodic nature of the $\cos^{-1}$ function and the manner in which γ enters the equation (3.34), zero velocity curves are not only periodic in 2π in both β and γ but possess symmetry about the point π, π . Thus it is sufficient to present these curves in the

region covering the interval $(0, \pi)$ in γ and $(0, 2\pi)$ in β ,
 It may be pointed out that if the argument of $\cos^{-1}$ exceeds unity in absolute value or is complex, the expression does not give a real value of β indicating the absence of zero velocity.

Zero velocity curves for several configurations (I, σ) over a variety of initial conditions (H) are shown in Figures 3-7 to 3-10. Interpretation of these plots is relatively simple. Real motion is possible only on the side of decreasing H .

Of particular interest are the curves about the origin enclosing a region of real motion. From equation (3.34) it follows that such curves exist iff $\sigma > -1$. However, even for $\sigma > -1$, the origin does not represent the minimum realizable H if $I < 1$. In these cases the zero velocity curves, near the origin but not necessarily enclosing it, cover relatively large regions as shown in Figure 3-10.

It should be emphasized at this point that although curves closing around the origin yield the maximum possible amplitude of β and γ motion for a given value of H , they cannot predict the actual maximum amplitude of motion for a given disturbance. Furthermore, curves which do not enclose the origin do not necessarily imply large amplitude motion but merely indicate its possible occurrence. Note that for a given value of H , the disturbance is not uniquely determined, i.e., there are an infinite number of values of the state parameters which satisfy equation (3.31).

It is possible to obtain some further information from the Hamiltonian. Since the Hamiltonian is a constant of motion,

Figure 3-7 Zero velocity curves; $I=2$, $\sigma=2$

Figure 3-8 Zero velocity curves; $I = 2$, $\sigma = -2$

Figure 3-9 Zero velocity curves; $I = 1.25$, $\sigma = 2$

Figure 3-10 Zero velocity curves; $I = .5$, $\sigma = 1$

it can be utilized to eliminate any one of the four state parameters and thus three dimensional state space can be used to describe the response of the system. Within the bounds imposed by the zero velocity curves, relations defining surfaces in β , β' , δ or δ , δ' , β space can be obtained by equating the ignored velocity to zero. Thus motion is bounded by

$$(\beta')^2 = H + 2I(\sigma+1)\cos\beta\cos\delta -$$

$$[3(I-1) - (3I-4)\cos^2\delta]\cos^2\beta \quad (3.35)$$

in β , β' , δ space and by

$$(\delta')^2 = \left\{ H + 2I(\sigma+1)\cos\beta\cos\delta - \right.$$

$$\left. [3(I-1) - (3I-4)\cos^2\delta]\cos^2\beta \right\} / \cos^2\beta \quad (3.36)$$

in δ , δ' , β space.

Examples of such surfaces are shown in Figures 3-11 to 3-14. They represent envelopes of possible motion in state space for a given value of the Hamiltonian. Note that the zero velocity curves are merely cross sections of these surfaces showing the extent of maximum possible motion.

It should be emphasized that the actual motion of a system is dependent upon the initial conditions and not merely the value of the Hamiltonian. Thus, in order to establish its characteristics, such as amplitude and frequency, it is necessary

Figure 3-11 Motion envelope in β' -space; $I=2, \sigma=2, H=-3.66$

Figure 3-12 Motion envelope in Y' -space; $I=2$, $\sigma=2$, $H=-3.66$

Figure 3-13 Motion envelope in β' -space; $I=.5, \sigma=1, H=-.5$

Figure 3-14 Motion envelope in δ' -space; $I=.5, \sigma=1, H=-.5$

to solve the governing equations (3.26) and (3.27). The concept of an invariant surface obtained numerically is used to study the system in the following section.

3.4.2 Circular Orbital Motion (Numerical Approach)

From equation (3.31) it is seen that although any one of the state elements can be expressed in terms of the other three and the Hamiltonian, there is an ambiguity as to its sign. As pointed out by Hénon and Heiles¹⁹, it is necessary to delineate between these two possibilities to utilize the invariant surface concept in the study of such a system. Hence, two surfaces must be used to describe the state of the system; one for positive values of the eliminated state element and the other for negative values. That is to say, once a state parameter is chosen for elimination, its sign should be used to determine the portions of the trajectory pertaining to a given surface. For example, if γ' is eliminated, the system is described in β, β', δ state space which may or may not be bounded by a closed envelope of possible motion (Figures 3-11 and 3-13). It should be pointed out that for finite values of H , the angular velocities β' and δ' are bounded but not necessarily the angular displacements.

As pointed out before, motion of the representative point occurs in two state spaces; one for situations where $\delta' > 0$ and the other for $\delta' < 0$. For a given initial condition, the trajectory defined by β, β' and δ lies in a specific state space depending upon the value of δ' . Switching points between the

spaces occur when $\gamma' = 0$. This means that the trajectory terminates in a given space when it meets the envelope of possible motion and then continues in the other space.

A parallel may be drawn between the state space β, β', γ or γ, γ', β and the one used in Phase I (γ, γ', θ). They are similar as each involves a phase plane stretched into a third dimension by a coordinate. Furthermore, in each case a representative point moves from one switching point to the next in a given space in such a manner that the stretching coordinate, i.e., β, γ or θ , is either monotonically increasing or decreasing with time. This was evident in Phase I as $\dot{\theta}$ was always positive. It also holds for this phase of the study as the cases corresponding to positive and negative values of the eliminated velocity are treated separately. This property of a monotonically changing state element facilitates the interpretation of cross sections of the invariant surface. It should be emphasized that such monotonic behavior of the stretching coordinate is not available in either β, β', γ' or γ, γ', β' space. For this reason, the discussion of invariant surfaces is restricted to those in β, β', γ and γ, γ', β state spaces, referred to as β' and γ' space respectively.

Consider now the symmetry properties of invariant surfaces in β' and γ' spaces. The governing equations of motion are invariant during the transformation $\theta = -\theta$, $\gamma = \gamma$ and $\beta = -\beta$. Since this also implies that $\gamma' = -\gamma'$ and $\beta' = \beta'$, it is clear that:

- i) for β' of the same sign, symmetry exists about the

γ axis in γ' space.

ii) for γ' of opposite sign, there is symmetry about the β', γ - plane.

Similarly with the transformation $\theta = -\theta$, $\gamma = -\gamma$ and $\beta = \beta$, the equations of motion are unchanged. Since this implies that $\gamma' = \gamma'$ and $\beta' = -\beta'$, it follows that:

iii) for γ' of the same sign, symmetry exists about the β axis in β' space.

iv) for β' of opposite sign, there is symmetry about the γ, β - plane.

It should be noted that the above symmetry properties only imply the existence of an image solution but this does not necessarily mean that the image surface is coincident with that generated by the given solution. Coincidence of these surfaces can be guaranteed only if initial conditions are chosen to lie along an axis of symmetry.

Turning now to the problem of invariant surface generation; due to symmetry properties (ii) and (iv), it is sufficient to discuss only those surfaces pertaining to $\gamma' > 0$ in β' space and to $\beta' > 0$ in γ' space. Thus the problem is similar to that encountered in Phase I except that the stretching coordinate, γ or β , is not the independent variable. Furthermore, in this case switching points are unknown in terms of the independent variable.

In Phase I, cross sections of the invariant surface were easily obtained since the numerical integration of the equation of motion was performed using a fixed step-size in θ , the stretching coordinate. Here, however, to obtain cross sections

in β' or γ' space, it was necessary to develop an interpolation scheme so that the state of the system could be ascertained for any given value of the stretching coordinate. This was achieved by using the Adams Bashforth predictor-corrector method ²¹ in conjunction with a polynomial fit to the past history of the state coordinates and their derivatives. Having fitted the numerical solution with polynomials in θ , the state of the system could readily be computed using Newton Raphson iteration.

The mechanics of generating invariant surfaces is now clear. At a number of preselected values of the stretching coordinate, cross sections were obtained by determining the state of the system each time the trajectory intersected the section plane. The symmetry properties discussed above were utilized to minimize computational effort.

It should be noted that during the integration of the equations of motion no attempt was made to reduce the order of the system using the Hamiltonian. Rather, the Hamiltonian, which is a constant of motion, was computed along with cross section data and was used as a check on the overall accuracy of the method.

Typical examples of surfaces generated in this manner are shown in Figures 3-15 to 3-18. Note the presence of open cross sections. These openings are analogous to the ends of the invariant surfaces obtained in Phase I (Figures 2-23 and 2-24). As against a common value of the stretching coordinate for switching points, here they occur over a range of values of the stretching coordinate. As mentioned previously, switching

Figure 3-15 Invariant solution surface in β' -space;
 $I=2, \sigma=0, \beta_1=\gamma_1'=0, \beta_1'=2.46, \gamma_1=.5$

Figure 3-16 Invariant solution surface in δ' -space;
 $I=2, \sigma=0, \beta_1=\delta'_1=0, \beta'_1=2.46, \delta_1=.5$

Figure 3-17 Invariant solution surfaces in β' -space;
 $I = 2, \sigma = 1, \beta'_i = \delta_i = 0, \beta_i = -.1, \delta'_i = 2.002$

Figure 3-18 Invariant solution surfaces in δ' -space;
 $I=2$, $\sigma=1$, $\beta'_1=\delta_1=0$, $\beta_1=-.1$, $\delta'_1=2.002$

points occur at bounding surfaces similar to those shown in Figures 3-11 to 3-14. It is clear, therefore, that invariant surfaces are, in fact, terminated by the associated bounding surface or envelope.

In Phase I of the study, much economy of effort was realized by determining only the relevant cross sections rather than the entire surface. Cross sections clearly revealed the structure of the nesting surfaces and the role that the periodic solutions played in determining these structures.

Having chosen β' and γ' space for this study, the invariant surfaces associated with these spaces are of a similar type with the stretching coordinate behaving in a monotonic fashion. Consequently, it should be expected that $\gamma, \beta = \text{constant}$ cross sections in β' and γ' space respectively yield similar information to that obtained for $\theta = \text{constant}$ sections in Phase I. The difficulty here lies in the question of where to section the surface. In Phase I of the study, position of the cross section was immaterial as the switching points were known a priori, i.e., at $\theta = 0$ and 2π . On the other hand, in this case switching points are not known a priori. In absence of such knowledge it would seem necessary to use several sections to obtain a complete picture of motion in the large. However, since in most practical applications the primary interest is in the motion about the origin, i.e., near $\beta = \gamma = 0$, sections were taken at $\gamma = 0$ in β' space and $\beta = 0$ in γ' space. Furthermore, additional information can be obtained by sectioning the surface at $\beta' = 0$ in β' space and at $\gamma' = 0$ in γ' space. These sections give the maximum

amplitude of motion in the β and γ modes.

The foregoing cross sections are collectively referred to as the principal cross sections. A summary of the properties and uses of these sections is presented in the following table:

TABLE I
Principal Cross Sections

Type	Zero Element	Relevant Space	Use
1	β	$\gamma'(\gamma, \gamma', \beta)$	γ mode study
2	β'	$\beta'(\beta, \beta', \gamma)$	β amplitude
3	γ	$\beta'(\beta, \beta', \gamma)$	β mode study
4	γ'	$\gamma'(\gamma, \gamma', \beta)$	γ amplitude

A mention should be made of the two problems encountered at this stage. The first involves the difficulty imposed by the singularity in the γ equation (3.27). At $|\beta| = \pi/2 \pm 2n\pi$, where n is an integer, it is seen that γ'' is infinite. This difficulty is inherent with the use of Euler angles and while it can be overcome by the use of a different set of Euler angles, in addition to the set already used, it would seem practical to simply limit the study to motion of $|\beta| < \pi/2$. This is justifiable as the motion of larger amplitude is of little practical interest.

The second difficulty arises when apparent breakdown of the invariant surfaces occurs. In Phase I, breakdown represented

unstable or tumbling motion. In this case, however, the concept of stability is undefined. Nonetheless, breakdown was observed to occur, and occasionally even when the associated bounding surface was closed. It was not clear whether the breakdown was real, i.e., indicating ergodic behavior of the system, or if it was simply the result of the limitations of the numerical approach. Its occurrence was always associated with relatively large amplitude motion and consequently the uncertainty of this data was of little importance.

The study of the maximum amplitude of motion for a given system was performed using cross sections of types 2 and 4. Unfortunately, symmetry properties could not guarantee that a given cross section, say in γ' space with $\beta' > 0$, would have its mirror image generated by the same trajectory for $\beta' < 0$. It was, therefore, necessary to consider the sections for both velocity conditions to establish the maximum response.

Figures 3-19 and 3-20 show sections of types 2 and 4 respectively for the specified values of the system parameters and initial conditions. These sections reveal that the ensuing motion is bounded $(-0.5 < \beta < 0.5; -0.76 < \gamma < 0.76)$. Similarly Figures 3-21 and 3-22 show, for different values of the initial conditions, the motion to be bounded by $-1.0 < \beta < 1.0$ and $-1.185 < \gamma < 1.185$. Note that while in the former case, bounded motion is guaranteed by the closed zero velocity curve, in the latter, bounded motion occurs despite the fact that the envelope as derived from the Hamiltonian is open. Thus the point made earlier, that bounding surfaces indicate only

Figure 3-19 Type 2 cross sections of invariant surface;
 $I = 2, \sigma = 0, \beta' = \gamma = 0, \beta = .5, \gamma' = 1.503$

Figure 3-20 Type 4 cross sections of invariant surface;
 $I=2, \sigma=0, \beta'_1=\gamma_1=0, \beta_1=.5, \gamma'_1=1.503$

Figure 3-21 Type 2 cross sections of invariant surface;
 $I=2, \sigma=0, \beta'_1=\delta_1=0, \beta_1=1, \delta'_1=3.645$

Figure 3-22 Type 4 cross sections of invariant surface;
 $I=2, \sigma=0, \beta_1'=\gamma_1=0, \beta_1=1, \gamma_1'=3.645$

the region of possible motion, is clear.

Consider now the problem of studying the structure of the invariant surfaces using cross sections of types 1 and 3. As in Phase I, it is evident that uniqueness assures that surfaces do not intersect. As well, it is clear that periodic solutions appear as stationary points in cross section and that sections of surfaces closing about such points are nested. Consequently, relatively few, appropriately chosen, solutions are required to give considerable insight into the nature of motion in the large.

Typical examples of cross section studies are shown in Figures 3-23 to 3-27. In each case several solutions having the same values of I , σ and H are sectioned to show the structural properties of the solution surfaces. In Figures 3-23 to 3-26, cross sections of both types, i.e., types 1 and 3, are shown. Figure 3-27, on the other hand, contains type 1 cross sections for $\beta' > 0$ and $\beta' < 0$ showing the almost complete lack of symmetry in the behavior of the system ($I = .5$, $\sigma = 1$, $H = -.9$). In this case the majority of solutions were such that the coordinate γ did not change sign and, for these solutions, the corresponding type 3 sections were non-existent.

These studies show the existence of stationary points which appear as either centers or saddles as did periodic

Figure 3-23 Invariant surface section studies of types 1 and 3; $I=2, \sigma=1, H=-3$

Figure 3-24 Invariant surface section studies of types 1 and 3; $I=2$, $\sigma=1$, $H=1$

Figure 3-25 Invariant surface section studies of types 1 and 3; $I=2$, $\sigma=1$, $H=5$

Figure 3-26 Invariant surface section studies of types 1 and 3; $I = 1.5, \sigma = -2, H = 4$

Figure 3-27 Invariant surface section studies-type 1;
 $I = .5, \sigma = 1, H = -.9$

solutions in Phase I. When tested for stability using Floquet theory, the former proved to be variationally stable while the latter were found to be unstable.

Referring to Figures 3-23, 3-24 and 3-25, it is seen that varying H for a given configuration materially affects the initial conditions required for periodic motion. Hence a study of periodic solutions, similar to that conducted in Phase I but using H rather than e as the varied parameter, is useful. In this manner it is possible to relate variational stability, initial conditions, and the period of the motion to the Hamiltonian.

It may also be pointed out that, for closed envelopes of possible motion, cross section studies suggested the existence of several periodic solutions for a given value of H (Figures 3-23 and 3-27). However, for positive values of the Hamiltonian (i.e., open envelopes of possible motion) only one of these appeared to persist (Figures 3-24 and 3-25). This tendency was observed in all the cases studied. The periodic solution existing over the largest range of the Hamiltonian is referred to as "fundamental" in the subsequent discussion.

Figure 3-23 shows sections of invariant surfaces, for a variety of initial conditions, bounded by a closed envelope. As pointed out before, for $H > 0$, the zero velocity curves are open. Even in these cases, closed cross sections can exist as illustrated in Figures 3-24 and 3-25. Similarly, for the case of $I = 1.5$, $\sigma = -2.0$ studied in Figure 3-26 in which no closed envelope exists, the analysis of motion in the small

correctly predicted the nature of the response to small disturbances. Figure 3-27 studies a configuration which was found unstable in the small despite the fact that envelopes of motion as determined by suitably small values of the Hamiltonian are closed indicating motion of finite amplitude. These cross sections show that the system is characterized by an unstable origin but motion remains finite, provided the Hamiltonian is small, due to the influence of stable periodic solutions nearby. Nonetheless, the analysis of motion in the small correctly predicted the unstable nature of the origin as truly small amplitude motion of this system is not possible.

Cross section studies were carried out for a variety of system parameters but for conciseness only representative results are presented here.

A useful application of the cross sectioning method should be mentioned at this time. Cross sections of types 2 and 4 were shown to give the maximum amplitude of the motion. As an accurate estimate of the maximum response usually involves relatively little computational effort, a parametric study can be undertaken quite readily. It would be useful to obtain response charts, similar to the stability charts obtained by the linearized analysis, showing the maximum response for a given disturbance as a function of the system parameters I and σ (Figures 3-28 to 3-31). Figures 3-28 and 3-29 pertain to systems exposed to a disturbance characterized by the initial conditions:

$$\beta_i = \beta'_i = \gamma_i = 0$$

Figure 3-28 Maximum β response chart; $\beta_i = \beta'_i = \gamma_i = 0, \gamma'_i = .01$

Figure 3-29 Maximum δ response chart; $\beta_i = \beta'_i = \gamma_i = 0, \gamma'_i = .01$

Figure 3-30 Maximum β response chart; $\beta_i = \beta'_i = \delta_i = 0, \delta'_i = .5$

Figure 3-31 Maximum γ response chart; $\beta_i = \beta'_i = \gamma_i = 0, \gamma'_i = .5$

and $\gamma'_i = 0.01$.

Similarly, Figures 3-30 and 3-31 pertain to systems exposed to a larger initial disturbance:

$$\beta_i = \beta'_i = \gamma_i = 0$$

and $\gamma'_i = 0.5$.

It is interesting to compare Figures 3-28 and 3-29 to the stability chart shown in Figure 3-2. While details differ, there is a remarkable agreement in the fundamental regions of stability and instability indicating the validity of the linearized analysis.

The primary purpose of the second set of response charts (Figures 3-30 and 3-31) is to demonstrate the effects of the inherent non-linearities upon system response. It should be noted that the contour levels shown in these charts are for responses fifty times greater than the lowermost levels shown in Figures 3-28 and 3-29. Noting that the disturbance for the charts shown in Figures 3-30 and 3-31 is fifty times greater than that used for the preceding charts, a comparison of the corresponding contours shows the "hardening" or "softening" effect of the non-linearities. For example, a large discrepancy in position is seen between the 0.02 contour in Figure 3-28 and the 1.0 contour in Figure 3-30, particularly in the region of $\sigma > 0$ and $|\zeta| < 1$. This behavior may also be noted in the corresponding γ response charts indicating that a non-linear stiffening effect is present in this region of parameter space.

Consider now the question of periodic motion. By studying a series of cross sections, similar to those shown in Figures 3-23 to 3-27, it is apparent that some form of periodic motion exists for most system configurations. As periodic solutions continue to determine the character of invariant surfaces as in Phase I, their study in relation to the parameter H is important.

For a circular orbit the equations of motion are autonomous and hence, periodic motion can be specified in terms of an infinite variety of initial conditions. In most cases, however, the examination of cross section data revealed the existence of periodic motion for initial conditions of the form:

$$\beta'_i = \delta_i = 0, \beta_i \neq 0, \delta'_i \neq 0$$

$$\text{or} \quad \delta'_i = \beta_i = 0, \delta_i \neq 0, \beta'_i \neq 0$$

A scheme for determining the non-zero initial conditions in terms of the relevant system parameters is now developed.

Consider a β_i, δ - plane as shown in Figure 3-32. For given I, σ, H and a specified value of β_i , the corresponding δ'_i can be determined using the expression for the Hamiltonian with $\beta'_i = \delta_i = 0$. Using these initial conditions a value of δ can be determined by integrating the governing equations until β executes some prescribed number of oscillations. Repeating the process for a range of values of β_i results in a plot shown in Figure 3-32 where the initial conditions represented by the β_i axis are mapped into final condition space utilizing the equations of motion. Clearly, intersections of the

Figure 3-32 Mapping scheme for locating periodic solutions; $e = 0$

resultant mapping with the β_i axis represent initial conditions for which, after some specified number of β oscillations, at least β' and γ attain their initial values. Note that $\beta' = 0$ whenever β completes an oscillation. All that is necessary then is to determine whether β and γ' are also matched to their initial values. If so, periodic motion has been found and its period is equal to the time taken for β to execute the specified number of oscillations. If β or γ' are not matched with initial values, periodic motion is still indicated but will occur over a different number of β oscillations.

Thus a method of determining periodic solutions is established. At first sight the resultant mapping would appear to be a single valued and continuous function of β_i . If such were the case, an iterative approach such as the variable secant method could be employed without difficulty to determine the initial conditions for periodic motion. In fact, such mappings are not, in general, continuous because the number of β oscillations determines the final conditions. If an oscillation is defined to occur after two successive sign changes of β' , the addition or deletion of a small ripple in the β response can lead to a sudden change in the number of oscillations over a given interval of time. This in turn would give rise to a discontinuity in the mapping. Apparent intersections of the mapping with the β_i axis due to such discontinuities giving erroneous initial conditions can be avoided by checking for a discontinuity in the period of motion.

Typical examples of periodic motion found using the

Figure 3-33 Periodic solution for a satellite in a circular orbit; $I=5$, $\sigma=1$, $\beta_1=\gamma_1=0$, $\beta_1'=.95$, $\gamma_1'=1.153$

Figure 3-34 Periodic solution for a satellite in a circular orbit; $I=5, \sigma=1, \beta=\gamma=0, \beta'=.217, \gamma'=.175$

above scheme are shown in Figures 3-33 and 3-34. They illustrate the complex nature of the solutions of this system. Thus, it is evident that highly sophisticated analytical techniques would be required to accurately describe system response.

As in Phase I, the stability of periodic solutions is studied using variational analysis. In this case, the variational equations can be written as

$$\begin{aligned}
 \beta_v'' = & - \left\{ (2 \sin^2 \beta_p - 1) (1 + (3I - 4) \sin^2 \delta_p - (\delta_p')^2 + \right. \\
 & I(\sigma + 1) \cos \beta_p \cos \delta_p + (4 \cos \beta_p \cos \delta_p - I(\sigma + 1) * \\
 & \delta_p' \sin \beta_p) \beta_v + \left\{ (6I - 8) \sin \beta_p \cos \beta_p \sin \delta_p * \right. \\
 & \cos \delta_p + I(\sigma + 1) \sin \beta_p \sin \delta_p - 2 \delta_p' \cos^2 \beta_p * \\
 & \sin \delta_p \} \delta_v - \left\{ I(\sigma + 1) \cos \beta_p + 2(\delta_p' \sin \beta_p * \right. \\
 & \left. \cos \beta_p - \cos^2 \beta_p \cos \delta_p) \right\} \delta_v' \quad (3.37)
 \end{aligned}$$

and

$$\begin{aligned}
 \delta_v'' = & \left\{ \tan \beta_p \delta_p'' + 2 \beta_p' (\delta_p' + \tan \beta_p \cos \delta_p) + \right. \\
 & (3I - 4) \tan \beta_p \sin \delta_p \cos \delta_p \} \beta_v + \left\{ I(\sigma + 1) \sec \beta_p + \right.
 \end{aligned}$$

$$2\delta'_p \tan \beta_p - 2 \cos \delta_p \} \beta'_v + \{ 2\beta'_p \sin \delta_p -$$

$$(3I-4) \cos 2\delta_p - I(\sigma+1) \cos \delta_p \sec \beta_p \} \delta'_v +$$

$$\{ 2\beta'_p \tan \beta_p \} \delta'_v . \quad (3.37)$$

It is clear that the above fourth order linear system has periodically varying coefficients of a common period. This period, say τ , is the same as that of the periodic motion since the coefficients are functions of the state elements. Thus, Floquet theory is applicable to this problem. Because the system is of fourth order, stability information would normally be in the form of four characteristic multipliers. It is not necessary, however, in this case to deduce the values of these multipliers to assess stability. Since the system is autonomous, one of the characteristic multipliers is equal to unity as the derivative of the periodic solution satisfies the variational equations. Further, it has been demonstrated that provided such systems have a first integral of motion, a second multiplier has unit value. Moreover, as the system is conservative, its state space representation in terms of Hamiltonian variables is volume preserving, i.e., it possesses an integral invariant. Under such circumstances, it may be shown that $\prod_i \lambda_i = 1$. Thus, if $\lambda_1 = \lambda_2 = 1$, then $\lambda_3 \lambda_4 = 1$. Hence, the two free characteristic multipliers, λ_3 and λ_4 , which determine the stability of the solution must lie on the unit circle or the real axis in the complex plane. A stability criterion can there-

fore be based upon the sum of the four characteristic multipliers, viz.,

$$\sum_{i=1}^4 \lambda_i \begin{cases} > 4 \text{ or } < 0; \text{ unstable} \\ \leq 4 \text{ and } \geq 0; \text{ stable.} \end{cases} \quad (3.39)$$

The application of Floquet theory involves the computation of a final condition matrix $\Phi(\tau)$ and the subsequent evaluation of its eigenvalues. Since the trace of a matrix is invariant under orthogonal transformation, it is clear that $\text{Tr}[\Phi(\tau)] = \sum_{i=1}^4 \lambda_i$. Thus the stability of periodic motion can be determined by the following criterion:

$$|\text{Tr}[\Phi(\tau)] - 2| \begin{cases} > 2; \text{ unstable} \\ \leq 2; \text{ stable.} \end{cases} \quad (3.40)$$

Utilizing this method, studies were made of the fundamental periodic solutions associated with various configurations. In all cases, the fundamental periodic motion remained variationally stable even for large disturbances, i.e., large values of the Hamiltonian. Examples shown in Figures 3-35 to 3-37 relate the period of motion, τ , and the initial condition element, β_i , to the Hamiltonian. Of course, the remaining elements of the initial condition vector are given by $\beta_i' = \delta_i = 0$ and $\delta_i' = [H / \cos^2 \beta_i + 2I(\sigma+1)/\cos \beta_i - 1]^{1/2}$.

Figure 3-35 Variation of β_i , τ and $\tau[\Phi(\tau)]$ with H for fundamental periodic motion; $I=2, \sigma=0$

Figure 3-36 Variation of β_i , τ and $\text{Tr}[\Phi(\tau)]$ with H for fundamental periodic motion; $I=2$, $\sigma=1$

Figure 3-37 Variation of β_i , τ and $Tr[\Phi(\tau)]$ with H for fundamental periodic motion; $I=2, \sigma=2$

3.4.3 Non-Circular Orbital Motion

To this stage of the study, the system was constrained in one way or another. For example, in Phase I, only a roll degree of freedom was allowed which permitted extensive analysis utilizing three dimensional state space. In the preceding section where e was restricted to zero value, similar techniques were applicable as the state of the system could again be described in three dimensions. The generalization allowing for non-zero orbit eccentricity with both roll and yaw degrees of freedom requires five dimensions $(\beta, \beta', \gamma, \gamma', \theta)$ to specify the state of the system. Reduction of this system is not possible as no known first integrals exist. Note that the Hamiltonian is no longer a constant of motion as $\partial L / \partial t \neq 0$.

While it is reasonable to expect invariant solution surfaces, more precisely hypersurfaces, to exist in this state space, it is clear that their numerical generation would involve a great deal of computation. As well, interpretation and representation of these surfaces would present formidable problems. For example, a cross section taken at $\theta = \text{constant}$ of such a surface would be a three dimensional region in four dimensional $(\beta, \beta', \gamma, \gamma')$ space. Thus it would seem that the invariant surface and cross sectioning concepts which proved so successful in the foregoing are of little practical value in this case.

There is, however, a slim chance that something worthwhile can be accomplished using this method. Instead of attempting to visualize the four dimensional space for a $\theta = \text{constant}$ cross section, consider two stroboscopic phase planes, β', β and γ', γ . In this way the problem of representing numerical

data would be eliminated although the resulting sections could not be expected to be as definitive as those obtained previously. In fact, at best, the resulting scatter of data would exhibit some degree of ordering which could be used to establish the nature of the motion. For example, quasi-periodic motion would be expected to form banded rings about the periodic conditions. These bands would be analogous to the closed curves about stationary points pertaining to periodic motion in two dimensional sections.

By integrating the equations of motion (3.7) and (3.8), and sectioning at pericenter, i.e., $\theta = 0$, attempts were made to utilize this scheme. In practically all instances, however, the resulting section data was well scattered and showed little or no evidence of ordering (Figure 3-38). It was only in a rare situation that some degree of ordering in cross section data was observed. Examples of banding about a non-trivial as well as a null periodic solution are shown in Figures 3-39 and 3-40, respectively. The failure, in this case, of the invariant surface technique or even a modified form thereof is unfortunate. Clearly, numerical studies of solution surface structures are impossible and, further, generation of response charts similar to those obtained for the autonomous case (Figures 3-28 to 3-31) would be extremely laborious. Thus, despite its great value, a parametric study of this form was not undertaken for this case.

The next logical step would be to locate and variationally analyze periodic solutions. One procedure would be to start with known periodic motion for $e = 0$ and extend it, using

Figure 3-38 Scattered intersections of a trajectory with the stroboscopic phase planes at $\theta = 0$, $I = .5$, $\sigma = 1$, $e = .1$, $\beta_i = \gamma_i = 0$, $\beta'_i = -1$, $\gamma'_i = .01$

Figure 3-39 Trajectory intersections with the stroboscopic phase planes at $\theta = 0$ showing banding about a periodic solution; $I = .5$, $\sigma = 1$, $e = .1$, $\beta = \delta = 0$, $\beta' = .5$, $\gamma = -1$

Figure 3-40 Trajectory intersections with the stroboscopic phase planes at $\theta = 0$ showing banding about the origin; $I=2$, $\sigma=1$, $e=.1$, $\beta=\gamma=0$, $\beta'=1.5$, $\gamma'=-.5$

some iterative scheme, to values of non-zero eccentricity.

While not providing a complete picture of the dynamics of the system, such an approach, if successful, would be a step in the right direction.

Replacing β by $\beta_p + \beta_v$ and γ by $\gamma_p + \gamma_v$, the governing equations yield the following variational relations:

$$\begin{aligned}
 \beta_v'' = & \left\{ I(\sigma+1) \left(\frac{1+e}{1+e\cos\theta} \right)^2 (\gamma_p' \sin \beta_p - \cos \beta_p * \right. \\
 & \cos \gamma_p) + \left(\frac{3(I-1)}{1+e\cos\theta} \sin^2 \gamma_p + \cos^2 \gamma_p - (\gamma_p')^2 \right) * \\
 & (\cos^2 \beta_p - \sin^2 \beta_p) - 4 \gamma_p' \sin \beta_p \cos \beta_p \cos \gamma_p \} * \\
 \beta_v + & \left\{ \frac{2e\sin\theta}{1+e\cos\theta} \right\} \beta_v' + \left\{ I(\sigma+1) \left(\frac{1+e}{1+e\cos\theta} \right)^2 * \right. \\
 & \sin \beta_p \sin \gamma_p + 2 \left(\frac{3(I-1)}{1+e\cos\theta} - 1 \right) \sin \beta_p \cos \beta_p * \\
 & \sin \gamma_p \cos \gamma_p - \frac{2e\sin\theta}{1+e\cos\theta} \cos \gamma_p - \gamma_p' \sin \gamma_p * \\
 & (\cos^2 \beta_p - \sin^2 \beta_p + 1) \} \gamma_v + \left\{ -(I(\sigma+1) * \right. \\
 & \left. \left(\frac{1+e}{1+e\cos\theta} \right)^2 + 2 \gamma_p' \sin \beta_p \right) \cos \beta_p + (\cos^2 \beta_p - \\
 & \sin^2 \beta_p + 1) \cos \gamma_p \} \gamma_v'
 \end{aligned}
 \tag{3.41}$$

and

$$\begin{aligned}
 \gamma_v'' \cos \beta_p = & \left\{ \frac{3(I-1)}{1+e \cos \theta} \sin \beta_p \sin \gamma_p \cos \gamma_p + \right. \\
 & \frac{2e \sin \theta}{1+e \cos \theta} (\cos \beta_p \cos \gamma_p - \gamma_p' \sin \beta_p) + \\
 & 2 \beta_p' \gamma_p' \cos \beta_p + \sin \beta_p (\gamma_p'' + \cos \gamma_p (2 \beta_p' - \\
 & \sin \gamma_p)) \left. \right\} \beta_v + \left\{ I(\sigma+1) \left(\frac{1+e}{1+e \cos \theta} \right)^2 + \right. \\
 & 2 \gamma_p' \sin \beta_p - 2 \cos \beta_p \cos \gamma_p \left. \right\} \beta_v' + \\
 & \left\{ -I(\sigma+1) \left(\frac{1+e}{1+e \cos \theta} \right)^2 \cos \gamma_p - \left(\frac{3(I-1)}{1+e \cos \theta} - 1 \right) * \right. \\
 & \cos \beta_p (\cos^2 \gamma_p - \sin^2 \gamma_p) - \sin \gamma_p (\sin \beta_p * \\
 & \left. \left(\frac{2e \sin \theta}{1+e \cos \theta} \right) - 2 \beta_p' \cos \beta_p \right\} \gamma_v + \left\{ \frac{2e \sin \theta}{1+e \cos \theta} * \right. \\
 & \left. \cos \beta_p + 2 \beta_p' \sin \beta_p \right\} \gamma_v'. \quad (3.42)
 \end{aligned}$$

Despite the complex form of the above equations, it is seen that all of the coefficients vary periodically with a period which is some multiple of 2π . Thus, Floquet theory is again applicable although, in this case, it will be necessary to evaluate the four characteristic multipliers to assess stability.

The fundamental problem here then is not in assessing the variational stability of periodic motion but is in locating the periodic motion itself. In essence, this means finding an initial state of the system, as defined by $\beta_i, \beta'_i, \gamma_i$ and γ'_i , such that after a specified number of orbits, the state matches the initial state. In previous studies of periodic motion, the problem involved the matching of fewer state elements. In those cases, the variable secant method was applied to only one element. Obviously a more sophisticated iteration scheme is required in this case.

Consider the possibility of utilizing the variational equations in such an iteration scheme. Recall that in using these equations, an initial condition matrix equal to the identity matrix is integrated to yield a final condition matrix, $\Phi(\tau)$. Consider only the first column of the identity matrix as an initial condition vector for the variation equations, i.e.,

$$\begin{pmatrix} \beta_i \\ \beta'_i \\ \gamma_i \\ \gamma'_i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \Phi^1(0).$$

Integrating over a given period τ would result, in general, in a new state vector of the variational equation:

$$\Phi^1(\tau) = \begin{pmatrix} \beta_v \\ \beta'_v \\ \gamma_v \\ \gamma'_v \end{pmatrix}_\tau.$$

Note that the elements of this vector represent the change in the state elements β, β', γ and γ' for a unit change in β_i . As the system is linear and $\beta_{v,i} = 1$,

$$\Phi^1(\tau) = \begin{pmatrix} \frac{\partial \beta}{\partial \beta_i} \\ \frac{\partial \beta'}{\partial \beta_i} \\ \frac{\partial \gamma}{\partial \beta_i} \\ \frac{\partial \gamma'}{\partial \beta_i} \end{pmatrix}_\tau.$$

Similarly,

$$\Phi^2(\tau) = \begin{pmatrix} \frac{\partial \beta}{\partial \beta'_i} \\ \frac{\partial \beta'}{\partial \beta'_i} \\ \frac{\partial \gamma}{\partial \beta'_i} \\ \frac{\partial \gamma'}{\partial \beta'_i} \end{pmatrix}_\tau.$$

Hence,

$$\Phi(\tau) = \begin{pmatrix} \frac{\partial \beta}{\partial \beta_i} & \frac{\partial \beta}{\partial \beta'_i} & \frac{\partial \beta}{\partial \gamma_i} & \frac{\partial \beta}{\partial \gamma'_i} \\ \frac{\partial \beta'}{\partial \beta_i} & \frac{\partial \beta'}{\partial \beta'_i} & \frac{\partial \beta'}{\partial \gamma_i} & \frac{\partial \beta'}{\partial \gamma'_i} \\ \frac{\partial \gamma}{\partial \beta_i} & \frac{\partial \gamma}{\partial \beta'_i} & \frac{\partial \gamma}{\partial \gamma_i} & \frac{\partial \gamma}{\partial \gamma'_i} \\ \frac{\partial \gamma'}{\partial \beta_i} & \frac{\partial \gamma'}{\partial \beta'_i} & \frac{\partial \gamma'}{\partial \gamma_i} & \frac{\partial \gamma'}{\partial \gamma'_i} \end{pmatrix} \cdot \quad (3.43)$$

Recognizing this, an iteration scheme for locating periodic motion can be developed which essentially amounts to a fourth order Newton Raphson approach.

Let $(\beta_i, \beta'_i, \gamma_i, \gamma'_i)$ be an estimate of the required initial state giving periodic motion in $2n\pi$. Integration of the equations of motion, together with the variational equations, will, in general, result in a state differing from the initial state of the system. Let $(\beta, \beta', \gamma, \gamma')$ represent this state. To approach a matching of these states, the following linear system must be solved for the changes or corrections made to the initial conditions:

$$\beta_i + \delta_\beta = \beta + \frac{\partial \beta}{\partial \beta_i} \delta_\beta + \frac{\partial \beta}{\partial \beta'_i} \delta_{\beta'} + \frac{\partial \beta}{\partial \gamma_i} \delta_\gamma + \frac{\partial \beta}{\partial \gamma'_i} \delta_{\gamma'}$$

$$\beta'_i + \delta_{\beta'} = \beta' + \frac{\partial \beta'}{\partial \beta_i} \delta_\beta + \frac{\partial \beta'}{\partial \beta'_i} \delta_{\beta'} + \frac{\partial \beta'}{\partial \gamma_i} \delta_\gamma + \frac{\partial \beta'}{\partial \gamma'_i} \delta_{\gamma'}$$

$$\gamma_i + \delta_\gamma = \gamma + \frac{\partial \gamma}{\partial \beta_i} \delta_\beta + \frac{\partial \gamma}{\partial \beta'_i} \delta_{\beta'} + \frac{\partial \gamma}{\partial \gamma_i} \delta_\gamma + \frac{\partial \gamma}{\partial \gamma'_i} \delta_{\gamma'}$$

$$\gamma'_i + \delta_{\gamma'_i} = \gamma' + \frac{\partial \gamma'}{\partial \beta_i} \delta_\beta + \frac{\partial \gamma'}{\partial \beta'_i} \delta_{\beta'} + \frac{\partial \gamma'}{\partial \gamma_i} \delta_\gamma + \frac{\partial \gamma'}{\partial \gamma'_i} \delta_{\gamma'}$$

(3.44)

In more concise notation, these can be written as

$$\begin{pmatrix} \delta_\beta \\ \delta_{\beta'} \\ \delta_\gamma \\ \delta_{\gamma'} \end{pmatrix} = \left(\Phi(2n\pi) - 1 \right)^{-1} \begin{pmatrix} \beta_i - \beta \\ \beta'_i - \beta' \\ \gamma_i - \gamma \\ \gamma'_i - \gamma' \end{pmatrix}, \quad (3.45)$$

where δ_β , $\delta_{\beta'}$, δ_γ and $\delta_{\gamma'}$ represent the corrections to the initial state elements to reduce the matching error. It may be pointed out that this iteration scheme is particularly compatible with variational analysis since $\Phi(\tau)$ determined here can also be used to assess stability by Floquet theory. Hence, once an acceptable match is attained, the eigenvalues of Φ , i.e., the characteristic multipliers λ_i of the variational system, can be computed. The criterion for stability of the periodic motion can be stated as

$$|\lambda_i| \begin{cases} > 1, & \text{unstable} \\ \leq 1, & \text{stable.} \end{cases} \quad (3.46)$$

Figure 3-41

Periodic solution of a satellite in an elliptic orbit; $I=2$, $\sigma=1$, $e=.1$, $\beta_1=\delta_1'=0$, $\beta_1'=1.894$, $\gamma_1=-.564$

Figure 3-42 Variation of initial conditions for stable periodic motion with orbit eccentricity; $I=2, \sigma=1$

Figure 3-43 Variation of initial conditions for stable periodic motion with orbit eccentricity; $I=5, \sigma=1$

A typical example of periodic motion found by this iteration scheme is shown in Figure 3-41. Examples of the use of this method in tracing periodic motion through various values of ϵ are shown in Figures 3-42 and 3-43. Figure 3-42 illustrates the effect of ϵ upon the initial conditions for three stable types of periodic motion with $I = 2$ and $\sigma = 1$. It should be noted that other periodic solutions were also found for this case. These solutions appeared to be related to those shown in that they had the same variational characteristics, i.e. λ_i 's were identical, and were derived from the same fundamental periodic solutions at $\epsilon = 0$. The results shown suggest that, due to the multiplicity of periodic solutions, the sphere of influence of any one of them is small compared to that observed in Phase I. Figure 3-43 traces stable periodic motion for the case of $I = .5$ and $\sigma = 1$ in which the γ motion occurs near $\gamma = 1$ rather than near $\gamma = 0$.

3.5 Concluding Remarks

As stated in the beginning, the aim of this phase of the work is to extend the numerical techniques developed in Phase I. To a large degree this approach proved to be successful. Only in the study of the non-autonomous system in the large were difficulties encountered. Even in that case, it was possible to locate and variationally analyze periodic motion.

Stability in the small was studied numerically using Floquet theory. Results in terms of stability charts were obtained over a wide range of parameters which should prove useful in

system design. It is important to note that while these results are based upon a linearized analysis, they were substantiated by the studies of motion in the large. This linear analysis parallels those of Thomson⁹ for $e=0$ and Kane and Barba¹³ for $e \neq 0$ but the results obtained here are in considerably greater detail. Figure 3-44 compares results for the case of $e = .3$. Also shown in this figure is the stability boundary as determined using Liapounov's direct method applied to the linearized system.¹⁴ The example clearly shows the contribution of this investigation in presenting more detailed and accurate results. It should be noted that previous research in this field utilized a different spin parameter, which is related to σ and e in the manner illustrated in Figure 3-45. In essence, this parameter is a measure of the average spin rate over the entire orbit rather than the instantaneous spin rate at the pericenter P .

The special case of $e=0$ received considerable attention. Motion in the large was successfully treated using the invariant surface concept. Principal cross sections of solution surfaces, together with the envelopes of possible motion obtained from the Hamiltonian, represent a considerable extension of the preliminary work by Pringle¹² in this field. Invariant surface methods proved to be most fruitful as not only the nature of the dynamical response but the amplitude of motion in both β and γ modes of oscillation was easily ascertained.

The utilization of the invariant surface technique to obtain response charts, Figures 3-28 to 3-31, appears to be the first attempt to relate maximum response to a specified distur-

Figure 3-44 Comparison of stability regions as determined by several investigators; $e = .3$

Figure 3-45 Relationship between spin parameters l and σ

bance over a wide range of system parameters. As pointed out earlier, these charts are of considerable value since they not only establish the extent to which the linearized analysis can be applied but the nature and magnitude of non-linear effects.

Cross section studies gave considerable insight into the behavior of periodic solutions. Although studies of closed envelopes indicated the presence of several periodic solutions, only the fundamental continued to exist at higher values of the Hamiltonian.

Analyses of periodic motion were performed for both circular and non-circular orbital motion. Iterative techniques for locating such motion based upon numerical integration of the governing equations proved to be successful. The numerical methods involved in the application of Floquet theory to the fourth order variational system can be derived directly from those used in Phase I. Variational analysis of the fundamental periodic solution showed it to be stable for values of H as large as 41 (Figure 3-36). This method of locating and variationally analyzing periodic motion for both the autonomous and non-autonomous cases represents a considerable advance in the field.

As far as the analytical studies are concerned, zero velocity curves and the associated envelopes of motion based upon the first integral, i.e., the Hamiltonian, were obtained for the case of $e=0$. The use of Liapounov's direct method was not considered appropriate, as explained in Phase I. In addition, there is a problem of defining stability in the large.

Although the study of large amplitude motion is not extensive, it substantiates the results of the linearized analysis presented in Figures 3-2 to 3-6. Conclusions based on these charts can be summarized as follows:

i) Except for the case of $I = 1$, at least one spin regime exists in which motion in the small is unstable regardless of orbit eccentricity. Thus the spin-up operation of a satellite could lead to attitude instability.

ii) In contrast to the findings of Phase I pertaining to instability with negative σ , two fundamentally stable regions were observed in parameter space, one in the positive and the other in the negative spin regime. In general, the former would be preferred for operation since larger and/or more frequent unstable regions occur in the latter.

iii) Long, slender satellites, i.e., $I \ll 1$, while inherently less stable than thin disk-type configurations due to the adverse gravity gradient effect, can be effectively stabilized given sufficient spin.

iv) Effects of non-circular orbital motion are numerous. Orbit eccentricity plays a significant role in determining both the size and the number of unstable regions. Using l instead of σ as a measure of spin, it is seen (Figures 3-2 to 3-6 and 3-45) that increasing the value of eccentricity adversely affects the attitude motion of a satellite as regions of unstable motion increase in both size and number. Tracings of periodic solutions with orbit eccentricity have also revealed that such motion is altered considerably even for numerically small values of e .

The analysis of attitude stability in the small, together with the studies of the autonomous system in the large, provide information useful for the practical design of a satellite. The analysis of the non-autonomous system in the large proved to be less successful in this regard; however, some progress was made in the study of periodic motion. Thus, it seems fair to conclude that the techniques and results presented in this chapter provide a sound basis for the design of spinning axisymmetric satellites with the possible exception of systems in which librational motion as well as orbit eccentricity are large.

4. CLOSING COMMENTS

4.1 Summary

As noted at the outset, the primary aim of the investigation was to develop general methods of analysis by which the dynamics of any particular configuration could be studied. As the emphasis was on the development of techniques rather than the generation of numerical results, no attempt was made to present massive data, particularly in those areas where the approach involved a considerable expenditure of computer time. Studies of motion in the small, however, are quite extensive and detailed since they are felt to be of considerable general interest.

While conceptually the determination of a numerical solution to a system of differential equations is not difficult, response curves obtained in this manner normally provide relatively little insight into the nature of motion. In this study, the problem of interpretation of numerical data was largely overcome through the utilization of invariant surfaces. The generation of such surfaces and their cross sections demonstrated a most useful method of utilizing numerical techniques.

For the analysis of stability in the small, Floquet theory in conjunction with numerical methods proved to be useful. This technique was extended to study the stability of periodic solutions using variational methods.

It should be emphasized that the analysis in Phase I relates to a simplified model of the real system. Hence, its

findings merely indicate the approximate behavior of the physical configuration. In this sense, the analysis served its purpose quite well as the more general analysis usually found the results indicative of the correct trend and generally conservative in their predictions.

4.2 Recommendations for Future Work

Possibilities for extension of the work presented are numerous. Application of the methods developed to the study of a specific design would be particularly interesting. If this were done in addition to a detailed simulation, an assessment could be made of the applicability of the techniques to practical problems involving such additional factors as mass asymmetries, solar radiation pressure and satellite flexibility.

Although the use of this method for a general parametric study of the problem involves considerable computation, studies of a limited range of parameter space could be usefully undertaken. For example, the analysis of motion in the small, using Floquet theory, could be extended to cover a wider range of orbit eccentricity and/or spin rate. Moreover, a different form of stability chart with I fixed and e and σ as parameters should prove useful to the designer because, while the geometry of an orbiting satellite is usually fixed, σ and e are more amenable to change.

A useful contribution could also be made in the extension of the maximum response charts. For example, plots for β' excitation would be of interest. As mentioned previously, response charts for $e \approx 0$ would entail a massive computa-

tional effort but would be of great value.

Aside from these rather obvious proposals, future researchers in this field should investigate more elaborate models. A study of the effects of small mass asymmetries would be useful as such would undoubtedly be present in any physical system.

A recent paper by Meirovitch ²² has stressed the effects of higher-order inertial integrals on the attitude dynamics of certain gravity gradient satellite configurations. In the case of a spinning satellite, such integrals may be significant for I equal to or nearly equal to unity. Thus an investigation as to the magnitude of such effects would be of value.

An important extension suggested by this work would be the investigation of attitude behavior during transition through the unstable spin regimes. Such a study would necessarily have to take into account the history of spin rate as well as the disturbances introduced during the spin-up operation. Thus, the mechanism employed for altering spin rate would enter the picture.

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APPENDIX I

Analysis of the Effects of Librational Motion on Orbital Motion

One of the important steps in formulating problems in satellite attitude dynamics is the assumption that librational motion has a negligible effect upon orbital motion. This enables the equations governing attitude and orbital motion to be decoupled. The method of variation of parameters is employed to establish the validity of this concept for the problem in hand.

Equations (2.11) and (2.12) can be rewritten as

$$h_o = R^2 \dot{\theta} + \epsilon \left\{ h_x \cos \gamma + k_y^2 \sin^2 \gamma \dot{\theta} \right\} \quad (I.1)$$

$$\ddot{R} - R \dot{\theta}^2 + \frac{\mu}{R^2} + \epsilon \left\{ \frac{3\mu k_y^2}{2R^4} (I-1)(1-3\sin^2 \gamma) \right\} = 0 \quad (I.2)$$

where ϵ has unit value and is used to denote small terms.

Combining equations (I.1) and (I.2) and neglecting terms of $O(\epsilon^2)$ yields

$$\ddot{R} - \frac{h_o^2}{R^3} + \frac{\mu}{R^2} + \epsilon \left\{ \frac{3\mu k_y^2}{2R^4} (I-1)(1-3\sin^2 \gamma) + \right.$$

$$\left. \frac{2h_o^2}{R^3} \left(\frac{h_x}{h_o} \cos \gamma + \left(\frac{k_y}{R} \right)^2 \sin^2 \gamma \right) \right\} = 0. \quad (I.3)$$

It is convenient at this stage to introduce a change in variables. Replacing R by $v = \frac{1}{R} - \frac{\mu}{h_0^2}$ and changing the independent variable from t to θ using the relation

gives
$$\frac{d}{d\theta} = \frac{h_0}{R^2} \left[1 - \epsilon \left\{ \frac{h_\alpha \cos \gamma}{h_0} + \left(\frac{k_y}{R} \right)^2 \sin^2 \gamma \right\} + O(\epsilon^2) \right] \frac{d}{dt} \quad (I.4)$$

$$v'' + v - \epsilon f = 0 \quad (I.5)$$

where

$$\begin{aligned} f = & \frac{3}{2} \frac{\mu}{h_0^2} k_y^2 (I-1) (1 - 3 \sin^2 \gamma) \left(v + \frac{\mu}{h_0^2} \right)^2 + \\ & 2 \left(\frac{h_\alpha \cos \gamma}{h_0} + k_y^2 \left(v + \frac{\mu}{h_0^2} \right)^2 \sin^2 \gamma \right) * \\ & \left(v'' + v + \frac{\mu}{h_0^2} \right) + 2 k_y^2 \sin^2 \gamma \left(v + \frac{\mu}{h_0^2} \right) (v')^2 + \\ & \left(2 k_y^2 \sin \gamma \cos \gamma \left(v + \frac{\mu}{h_0^2} \right)^2 - \frac{h_\alpha \sin \gamma}{h_0} \right) \gamma' v'. \end{aligned}$$

As shown by the method of Kryloff and Bogoliuboff, equation (I.5) has a solution of the form

$$v = v_0 \cos(\theta + \phi) = v_0 \cos \psi \quad (\text{I.6})$$

where both v_0 and ϕ vary with θ .

In terms of more meaningful variables, the solution (I.6) may be written as

$$R = h_0^2 / \mu (1 + e \cos(\theta + \phi)) \quad (\text{I.7})$$

where
$$e = v_0 h_0^2 / \mu.$$

Taking the attitude motion and hence f to be periodic and considering ϕ to be small ($\phi = 0$ initially), gives

$$[e']_{\text{average}} \approx \frac{h_0^2}{\mu \tau} \int_0^\tau f \sin \psi \, d\psi$$

and

$$[\phi']_{\text{average}} \approx \frac{h_0^2}{\mu e \tau} \int_0^\tau f \cos \psi \, d\psi \quad (\text{I.8})$$

where $\delta(\psi)$ and $d\delta/d\psi$ are approximated by $\delta(\theta)$ and δ' respectively.

Let "a" be a parameter equivalent to the semi-major axis of a Keplerian orbit defined as $a = h_0^2 / \mu (1 - e^2)$. Introducing a size parameter $p = k_y / a$ equations (I.8) can be rewritten as

$$[e']_{\text{average}} = \left(\frac{\rho}{1-e^2}\right)^2 \frac{1}{\tau} \int_0^\tau g \sin \psi d\psi$$

and

$$[\phi']_{\text{average}} = \left(\frac{\rho}{1-e^2}\right)^2 \frac{1}{e\tau} \int_0^\tau g \cos \psi d\psi$$

(I.9)

where

$$g = (1 + e \cos \psi)^2 \left\{ \frac{3}{2} (I-1) (1 - 3 \sin^2 \gamma) + \right.$$

$$\left. 2 \sin^2 \gamma - 2e \sin \gamma \cos \gamma \sin \psi \frac{d\gamma}{d\psi} \right\} + (1 + e \cos \psi)^*$$

$$\left\{ 2e^2 \sin^2 \gamma \sin^2 \psi \right\} + \left(\frac{I(\sigma+1)(1+e)^2}{1 + \rho^2 I(\sigma+1)(1+e)^2 / (1-e^2)^2} \right)^*$$

$$\left\{ 2 \cos \gamma + e \sin \gamma \sin \psi \frac{d\gamma}{d\psi} \right\}.$$

From the foregoing, since $\rho \ll 1$ (normally $\rho < 10^{-5}$), it is apparent that for most situations changes in e and ϕ are essentially proportional to ρ^2 . Further, it is evident that although analytical solutions of equations (I.9) are not possible, the integral terms in these expressions should not be large since the integrands consist of trigonometric functions

Figure I-1 Incremental change per orbit in orbital parameters due to coupled periodic librational motion; $I=2, \sigma=-1, m=1, n=1$

Figure I-2 Incremental change per orbit in orbital parameters due to coupled periodic librational motion; $I=1.5$, $\sigma=-1$, $m=1$, $n=2$

Figure I-3 Incremental change per orbit in orbital parameters due to coupled periodic librational motion; $I=.5$, $\sigma=4$, $m=1$, $n=1$

with relatively small coefficients. Thus, at least qualitatively, equation (I.9) indicates that changes in e and ϕ due to attitude motion of the satellite are small.

Numerical solutions of (I.9) and (2.17), the equation governing librational motion, are possible. Representative examples for large amplitude, periodic, librational motion are shown in Figures I-1, I-2 and I-3. These results indicate that the effect of satellite librations on the motion of the center of mass is negligible and that decoupling of the equations of motion is indeed valid.