A STUDY OF CERTAIN TYPES OF SURFACE WAVEGUIDES

by

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We accept this thesis as conforming to the
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May, 1968
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ABSTRACT

This work consists of two parts. The first part is a comprehensive study of surface-wave propagation along dielectric tube waveguides. It includes the derivation of the characteristic equations and expressions for group velocity and attenuation coefficient, the latter by a perturbation method. Mode designations are justified and the physical distinction between the HE_{11} and EH_{11} modes is further illustrated by showing three-dimensional plots of the field configurations.

Computed characteristics are given for a wide range of parameters, and are compared with those of standard rectangular waveguides. Finally, a method of shielding the tube from weather conditions is proposed and the resulting changes in characteristics are noted.

The second part of this work is essentially a unified analysis of all slow-wave modes in eight cylindrical waveguides. Characteristic equations are derived and expressions are obtained for the group velocity and the attenuation coefficients by a perturbation method. Accurate propagation characteristics for the dominant TM_{01} mode are computed for four waveguides with no restrictions on their radial dimensions. These guides are the Goubau line and a coaxial cable with dielectric linings on the inner, outer, or both conductors.
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LIST OF SYMBOLS

\[ a_i, \ b_i \] = constants

\[ A_n(p_{ij}), \ B_n(p_{ij}) \] = functions of Bessel functions

\[ c_i \] = \( jb_i / a_i \)

\[ C_i \] = constant

\[ C_f \] = field extension ratio (see definition of \( r_f \))

\[ C_p \] = fraction of power transmitted within \( r=r_p \)

\[ C_n(p_{ij}), \ D_n(p_{ij}) \] = functions of modified Bessel functions

\[ d_i \] = \( (\nu_{ri} \ c_i \ k_o \ Z_o) / \beta \)

\[ e_i \] = \( (\epsilon_{ri} \ k_o) / (c_i Z_o \ \beta) \)

\[ E_zi, \ E_{ri}, \ E_{\theta i} \] = longitudinal, radial and azimuthal components of electric field, respectively, in medium \( i \)

\[ E_{zi}, \ E_{ri}, \ E_{\theta i} \] = radial variation of \( E_zi, \ E_{ri} \) and \( E_{\theta i} \), respectively

\[ f \] = frequency

\[ F_i \] = constant

\[ F_n(p_{ij}), \ G_n(p_{ij}) \] = functions of Bessel functions

\[ h_i \] = wave number of medium \( i \)

\[ H^{(1)}_n(p_{ij}), \ H^{(2)}_n(p_{ij}) \] = Hankel functions of the first and second kinds

\[ H_zi, \ H_{ri}, \ H_{\theta i} \] = longitudinal, radial and azimuthal components of magnetic field, respectively, in medium \( i \)

\[ H_{zi}, \ H_{ri}, \ H_{\theta i} \] = radial variation of \( H_zi, \ H_{ri} \) and \( H_{\theta i} \), respectively

\[ I_n(p_{ij}) \] = modified Bessel function of the first kind

\[ J_n(p_{ij}) \] = Bessel function of the first kind

\[ k_o \] = phase coefficient of free space
\( K_n(p_{ij}) \) = modified Bessel function of the second kind

\( L \) = length of a resonant waveguide

\( m, n \) = mode subscripts

\( N, N_i \) = total power carried and power carried in medium \( i \), respectively

\( p_i \) = \( h_i r \)

\( p_{ij} \) = \( h_i r_j \)

\( P, P_i \) = total power loss per unit length and power loss per unit length in medium \( i \), respectively

\( P_p, \overline{P} \) = end-plate losses and waveguide losses of a resonant waveguide

\( Q \) = quality factor of a resonant waveguide

\( r \) = radial co-ordinate

\( r_i \) = radius of an interface between two different homogeneous media

\( r_f \) = radius at which the longitudinal field components become \( C_f \) times their values at the tube outer boundary

\( r_p \) = see definition of \( C_p \)

\( S_A, S_B, S_{AB}, S_F \)

\( S_G, S_{FG}, T_A, T_F \) = integrals of functions of Bessel functions

\( S_C, S_D, S_{CD}, S_I \)

\( S_K, T_C, T_I, T_K \) = integrals of functions of modified Bessel functions

\( t \) = time

\( \tan \delta_i \) = loss tangent of medium \( i \)

\( v_o, v_g, v_p \) = speed of light in free space, group velocity and phase velocity

\( W, W_i \) = total energy storage per unit length and energy storage per unit length in medium \( i \), respectively

\( \bar{W} \) = total energy stored by a resonant waveguide

\( x, y, z, x', y' \) = cartesian co-ordinates

\( Y_n(p_{ij}) \) = Bessel function of the second kind
$Z, \tilde{Z}$ = accurate and approximate waveguide impedance
$Z_0$ = impedance of free space
$\alpha, \tilde{\alpha}$ = accurate and approximate attenuation coefficient, respectively
$\alpha_i, \tilde{\alpha}_i$ = accurate and approximate attenuation coefficients of medium $i$, respectively
$\alpha_0$ = attenuation coefficient of a TEM wave in medium 2
$\bar{\alpha}$ = $\alpha_2/\alpha_0$
$\beta$ = phase coefficient
$\gamma$ = $\alpha + j\beta$
$\Delta v_p$ = percent phase-velocity reduction
$\varepsilon_{ri}$ = relative permittivity of medium $i$
$\theta$ = azimuthal co-ordinate
$\lambda$ = free-space wavelength
$\lambda_c$ = cutoff wavelength
$\lambda_g$ = guide wavelength
$\mu_{ri}$ = relative permeability of medium $i$
$\rho$ = $r_1/r_2$
$\sigma$ = conductivity of rectangular waveguide
$\sigma_i$ = conductivity of medium $i$
$\omega$ = angular frequency
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GENERAL INTRODUCTION

The increasing demand for high-capacity long-distance communications systems has necessitated the development of wide-bandwidth transmission systems, capable of handling large numbers of channels. This can only be achieved by operating at extremely high frequencies, and two promising types of systems are presently being investigated. The \( \text{TE}_{01} \) overmoded circular waveguide is being developed to operate from 40 to 110 GHz. Optical waveguides (beam, lens and dielectric fibre) are also the subject of intensive study. These would be capable of handling extremely large numbers of channels. In both cases, there are still numerous difficulties to be overcome.

The continuing development of very low-loss dielectric materials could make guides, such as the dielectric tube or rod, competitive as transmission media both at millimeter-wave and at optical frequencies. Part I of this work is undertaken to provide a comprehensive study of the dielectric tube waveguide, with a view to its possible application to low-loss transmission.

One serious limitation of dielectric guides, being open structures, is that they are subject to radiation and interference. It would be useful if this difficulty could be overcome while still maintaining the desirable characteristics of the unshielded guides. For this reason, a study of screened surface waveguides is undertaken in Part II of this work, which examines several types of these structures.
PART I

THE DIELECTRIC TUBE WAVEGUIDE
1. INTRODUCTION

The first analysis of wave propagation along a dielectric tube was carried out by Zachoval\(^1\) in 1932. The characteristic equation for TM modes was obtained and solved graphically for several tube sizes and dielectric constants. Two years later, the existence of these modes was verified by Liska\(^2\), whose measurements of guide wavelength showed good agreement with Zachoval's theory. It was not until 1949 that Astrahan\(^3\) obtained the characteristic equations of TE and hybrid modes. Experimental values of guide wavelength for the HE\(_{11}\), TE\(_{01}\) and TM\(_{01}\) modes for various sizes of polystyrene tubes agreed very well with theory. Astrahan also gave data for TM\(_{01}\) and TE\(_{01}\) cutoff wavelengths, plots of the radial dependence of the field components and a field configuration of the HE\(_{11}\) mode in a transverse plane. Jakes\(^4\) obtained the attenuation coefficients of TM and TE modes by expanding the characteristic equation in a truncated Taylor series about the lossless solution, thus separating the equation into its real and imaginary parts. The real part was solved for the lossless phase coefficient, while the imaginary part yielded the attenuation coefficient valid for small losses. Comparisons were made of the attenuation characteristics of these modes with those of the TE\(_{01}\) mode in a circular waveguide having the same cutoff wavelength. The measured attenuation of TM\(_{01}\) and TE\(_{01}\) modes for polystyrene tubes agreed favourably with Jakes' theory. Coleman and Becker\(^5\) gave a theoretical attenuation characteristic which was obtained earlier by Beam\(^6\) in the vicinity of 100 GHz. A technique for obtaining the attenuation coefficient of any mode was outlined by Unger\(^7\) in 1954 using a method similar to Jakes', but the analysis was completed only for the HE\(_{11}\) mode on tubes with small diameter-to-wavelength ratios. Unger also derived cutoff conditions for HE\(_{1m}\) and EH\(_{1m}\) modes and gave a comparison of dispersion and guide radii between tubes and circular waveguides with the same attenuation. Some numerical results were also given and these were extended in a later paper\(^8\). Mallach\(^9\) made a rough
estimate of $HE_{11}$ mode attenuation by measuring the radius at which the magnitude of the electric field fell to $1/e$ of its value at the tube surface. In 1954, apparently unaware of the work of Mallach and Unger, workers at the Bell Telephone Laboratories published the results of a study of the relative merits of the tube and the rod as long-distance waveguides. Calculations showed that the guide wavelength of the tube was more slowly varying with radius than that of the rod, thus requiring a less strict dimensional tolerance. However, this was considered insufficient to justify the additional problem of tube fabrication, and a decision was made in favour of solid-dielectric waveguides of rectangular cross-section which have the advantage of supporting modes of known polarization.

The dielectric-tube antenna has been investigated by Jakes and Kiely, who have calculated and measured radiation patterns of these devices.

In the field of millimetre-wave generation, Pantell and others have used dielectric tube cavities resonating in TM and TE modes to couple electromagnetic waves and electron beams. Extensive data on cavity Q and field variation with radius for various sizes and dielectric constants of tubes have been given by these workers.

From the above review, it can be seen that, to date, no comprehensive study of the possible usefulness of the dielectric tube as a low-loss waveguide has been made. The purpose of the present work is to provide such a study, particularly regarding the possible use of the tube as a low-loss waveguide at microwave and millimetre-wave frequencies.

The field components for TM, TE and hybrid modes are given in Chapter 2 in terms of eight arbitrary constants. Relations among these constants are obtained and the lossless characteristic equations for all modes are derived in Chapter 3. Numerical solutions of these equations are obtained for various
sizes of polyethylene tubes. Computer programs are also written to calculate and plot the radial variation of the field components. The verification of mode designations is established from these graphs, and cutoff conditions for \( n=0 \) and \( n=1 \) modes are given in terms of the behaviour of the longitudinal field components as cutoff is approached. Also shown in Chapter 3 are three-dimensional plots of the field configurations for the first two hybrid modes, obtained by numerical solution of the differential equations, with calculations and plotting performed by computer.

In Chapter 4, the equations for group velocity and attenuation coefficient for all modes are derived by integrating the appropriate combinations of field components to obtain power flow, energy storage per unit length and power loss per unit length. Numerical results are given for the \( \text{TM}_{01} \), \( \text{TE}_{01} \) and \( \text{HE}_{11} \) modes. The concentration of energy in the various regions and the decay of fields from the tube surface are studied in Chapter 5. In Chapter 6, a comparison is made between the tube characteristics and those of rectangular waveguides. Finally, a proposed practical system of embedding the tube in a polyfoam medium is considered, and the resulting changes in characteristics are noted.
2. FIELD COMPONENTS

The tube configuration of interest is shown in Figure 2.1. It consists of two coaxial dielectric regions of infinite length and relative permittivities $\varepsilon_{r1}$ and $\varepsilon_{r2}$ embedded in a third infinite dielectric of relative permittivity $\varepsilon_{r3}$, where

$$\begin{align*}
\varepsilon_{r2} &> \varepsilon_{r1} \\
\text{and} \quad \varepsilon_{r2} &> \varepsilon_{r3}
\end{align*}$$

\hspace{2cm} 2.1

![Figure 2.1 The Dielectric Tube Waveguide](image)

In all cases, it will be assumed that the relative permeability of the $i^{th}$ region, $\mu_{ri}$, is unity; however, the relative permeabilities of the various media will be retained in the equations for completeness. Propagation is assumed in the $z$-direction, with $t\theta-z$ dependence of the form $\exp(j(\omega t - n\theta - \beta z))$ in the lossless case.

Under these conditions, omitting the factor $\exp(j(\omega t - n\theta - \beta z))$, the field components are given by
\[
\begin{align*}
E_{z1} &= a_1 I_n(h_1 r) \\
E_{r1} &= j \frac{\beta}{h_1} a_1 I_n'(h_1 r) + \frac{n \mu r_1 k_o Z_0}{h_1^2 r} b_1 I_n(h_1 r) \\
E_{\theta 1} &= \frac{n \beta}{h_1^2 r} a_1 I_n(h_1 r) - j \frac{\mu r_1 k_o Z_0}{h_1} b_1 I_n'(h_1 r) \\
H_{z1} &= b_1 I_n(h_1 r) \\
H_{r1} &= -\frac{n \varepsilon r_1 k_o}{h_1^2 r} a_1 I_n(h_1 r) + j \frac{\beta}{h_1} b_1 I_n'(h_1 r) \\
H_{\theta 1} &= j \frac{\varepsilon r_1 k_o}{h_1 Z_0} a_1 I_n'(h_1 r) + \frac{n \beta}{h_1^2 r} b_1 I_n(h_1 r)
\end{align*}
\]

\[
E_{z2} = a_2 \left[ J_n(h_2 r) + \frac{a_4}{a_2} \frac{\alpha}{\alpha} \gamma_n(h_2 r) \right] \triangleq a_2 A_n(h_2 r)
\]

\[
E_{r2} = -j \frac{\beta}{h_2} a_2 A_n'(h_2 r) - \frac{n \mu r_2 k_o Z_0}{h_2^2 r} b_2 B_n(h_2 r)
\]

\[
E_{\theta 2} = -\frac{n \beta}{h_2^2 r} a_2 A_n(h_2 r) + j \frac{\mu r_2 k_o Z_0}{h_2} b_2 B_n'(h_2 r)
\]

\[
H_{z2} = b_2 \left[ J_n(h_2 r) + \frac{b_4}{b_2} \gamma_n(h_2 r) \right] \triangleq b_2 B_n(h_2 r)
\]

\[
H_{r2} = \frac{n \varepsilon r_2 k_o}{h_2^2 r} a_2 A_n(h_2 r) - j \frac{\beta}{h_2} b_2 B_n'(h_2 r)
\]

\[
H_{\theta 2} = -j \frac{\varepsilon r_2 k_o}{Z_0 h_2^2} a_2 A_n'(h_2 r) - \frac{n \beta}{h_2^2 r} b_2 B_n(h_2 r)
\]

\[0 \leq r \leq r_1\]

\[r_1 \leq r \leq r_2\]
\( E_{z3} = a_3 K_n(h_3 r) \)

\[
E_{r3} = j \frac{\varepsilon}{h_3} a_3 K_n(h_3 r) + \frac{n \mu r_3 k_0 Z_0}{h_3^2 r} b_3 K_n(h_3 r) \]

\[
E_{\theta3} = \frac{\mu}{h_3^2 r} a_3 K_n(h_3 r) - j \frac{\mu r_3 k_0 Z_0}{h_3} b_3 K_n(h_3 r) \]

\[
H_{z3} = b_3 K_n(h_3 r) \]

\[
H_{r3} = - \frac{n \varepsilon r_3 k_0}{h_3^2 r Z_0} a_3 K_n(h_3 r) + j \frac{\varepsilon}{h_3} b_3 K_n(h_3 r) \]

\[
H_{\theta3} = j \frac{\varepsilon r_3 k_0}{h_3 Z_0} a_3 K_n(h_3 r) + \frac{\mu}{h_3^2 r} b_3 K_n(h_3 r) \]

\( r_2 \leq r \leq \infty \)

where, from the wave equation

\[
h_1^2 = \beta^2 - \mu r_1 \varepsilon r_1 k_0^2 \]

\[
h_2^2 = \mu r_2 \varepsilon r_2 k_0^2 - \beta^2 \]

\[
h_3^2 = \beta^2 - \mu r_3 \varepsilon r_3 k_0^2 \]

The symbols appearing in equations 2.2 and 2.3 are defined in the list of symbols.

Upon setting \( n=0 \) (no \( \theta \)-variation), equations 2.2 separate into two sets corresponding to the circularly symmetric modes designated \( TM_{0m} \) and \( TE_{0m} \). The corresponding field components are given by
TM MODES

\[ E_{z1} = a_1 I_0(h_1r) \]
\[ E_{r1} = j \frac{\beta}{h_1} a_1 I_0^{'}(h_1r) \]
\[ H_{\theta 1} = \frac{\epsilon_{r1} k_0}{Z_0 \beta} E_{r1} \]
\[ E_{z2} = a_2 A_0(h_2r) \]
\[ E_{r2} = -j \frac{\beta}{h_2} a_2 A_0^{'}(h_2r) \]
\[ H_{\theta 2} = \frac{\epsilon_{r2} k_0}{Z_0 \beta} E_{r2} \]
\[ E_{z3} = a_3 K_0(h_3r) \]
\[ E_{r3} = j \frac{\beta}{h_3} a_3 K_0^{'}(h_3r) \]
\[ H_{\theta 3} = \frac{\epsilon_{r3} k_0}{Z_0 \beta} E_{r3} \]

TE MODES

\[ H_{z1} = b_1 I_0(h_1r) \]
\[ H_{r1} = j \frac{\beta}{h_1} b_1 I_0^{'}(h_1r) \]
\[ E_{\theta 1} = - \frac{\mu_{r1} k_0 Z_0}{\beta} H_{r1} \]
\[ H_{z2} = b_2 B_0(h_2r) \]
\[ H_{r2} = -j \frac{\beta}{h_2} b_2 B_0^{'}(h_2r) \]
\[ E_{\theta 2} = - \frac{\mu_{r2} k_0 Z_0}{\beta} H_{r2} \]
\[ H_{z3} = b_3 K_0(h_3r) \]
\[ H_{r3} = j \frac{\beta}{h_3} b_3 K_0^{'}(h_3r) \]
\[ E_{\theta 3} = - \frac{\mu_{r3} k_0 Z_0}{\beta} H_{r3} \]

For \( n \neq 0 \), equations 2.2 describe inseparable combinations of TE and TM modes which are designated hybrid modes. In general, one or the other of the component parts of a hybrid mode is dominant. If the TE portion is dominant, the mode is designated \( HE_{nm} \); if the TM component is dominant, it is termed \( EH_{nm} \). The nature of TE or TM-dominance and the significance of the subscript \( m \) in the mode designation will be discussed in Section 3.2.
3. MODE SPECTRUM

The field properties of the various modes are investigated in this section. The characteristic equations for all modes are derived, and solutions are given for several modes with \( n = 0 \) and \( n = 1 \). The radial variations of the fields are shown, and the results used to give a consistent mode designation for the tube and the rod. Finally, three-dimensional field configurations are given for two of the lower-order hybrid modes.

3.1 Characteristic Equations

By using equations 2.2 and equating the tangential field components in media 1 and 2 at \( r = r_1 \), and those in media 2 and 3 at \( r = r_2 \), eight homogeneous equations can be obtained in the eight unknowns \( a_i, b_i, i = 1,4 \).

The most commonly-used method of obtaining the characteristic equation is to set the determinant of this system of equations equal to zero. However, by algebraic manipulation, it is possible to obtain two equations, one from the matched tangential fields at each boundary. These equations contain two unknowns, \( a_4/a_2 \) and \( b_4/b_2 \), which are found by applying Cramer's rule to the original eight equations. The two equations are then solved simultaneously to yield the phase characteristics of the tube. This approach has the following advantages over the alternative method of setting the determinant equal to zero:

(i) The algebraic manipulation yields certain ratios of the arbitrary constants which are required in later parts of the work.

(ii) The functions \( A_n(p_{22}) \) and \( B_n(p_{22}) \) defined by equations 2.2.b appear in the characteristic equation explicitly. This proves useful in deriving cutoff conditions for the various modes in terms of zeroes of these functions (Section 3.3).

Letting \( p_{ij} \triangleq h_i r_j \), and matching tangential fields at \( r = r_2 \) yields
\[ \frac{a_2}{a_3} = \frac{K_n(p_{32})}{A_n(p_{22})}, \quad \frac{b_2}{b_3} = \frac{K_n(p_{32})}{B_n(p_{22})} \]

\[ \frac{b_3}{a_3} = -j \left( \frac{r_2 A_n'(p_{22})}{p_{22} A_n(p_{22})} + \frac{r_3 K_n'(p_{32})}{p_{32} K_n(p_{32})} \right) \left[ \frac{n_8 Z_0}{k_0} \left( \frac{1}{p_{22}} + \frac{1}{p_{32}} \right) \right]^{-1} \]

\[ = -j \left( \frac{r_2 B_n'(p_{22})}{p_{22} B_n(p_{22})} + \frac{r_3 K_n'(p_{32})}{p_{32} K_n(p_{32})} \right) \left[ \frac{n_8 Z_0}{k_0} \left( \frac{1}{p_{22}} + \frac{1}{p_{32}} \right) \right]^{-1} \]

\[ \Delta = -j c_3 \]

\[ \frac{b_2}{a_2} = \frac{b_3}{a_3} \frac{a_3}{a_2} \frac{b_2}{b_3} = -j c_2 \]

and from matching fields at \( r = r_1 \), we obtain

\[ \frac{a_2}{a_1} = \frac{I_n(p_{11})}{A_n(p_{21})}, \quad \frac{b_2}{b_1} = \frac{I_n(p_{11})}{B_n(p_{21})} \]

\[ \frac{b_1}{a_1} = -j \left( \frac{r_2 A_n'(p_{21})}{p_{21} A_n(p_{21})} + \frac{r_1 I_n'(p_{11})}{p_{11} I_n(p_{11})} \right) \left[ \frac{n_8 Z_0}{k_0} \left( \frac{1}{p_{21}} + \frac{1}{p_{11}} \right) \right]^{-1} \]

\[ = -j \left( \frac{r_2 B_n'(p_{21})}{p_{21} B_n(p_{21})} + \frac{r_1 I_n'(p_{11})}{p_{11} I_n(p_{11})} \right) \left[ \frac{n_8 Z_0}{k_0} \left( \frac{1}{p_{21}} + \frac{1}{p_{11}} \right) \right]^{-1} \]

\[ \Delta = -j c_1 \]

\[ \frac{b_2}{a_2} = \frac{b_1}{a_1} \frac{a_1}{a_2} \frac{b_2}{b_1} = -j c_2 \]

Equating the two expression for \( c_3 \) from 3.1.a, and those for \( c_1 \) from 3.1.b, yields the following two equations

\[ \left( \frac{r_2 A_n'(p_{22})}{p_{22} A_n(p_{22})} + \frac{r_3 K_n'(p_{32})}{p_{32} K_n(p_{32})} \right) \left( \frac{r_2 B_n'(p_{22})}{p_{22} B_n(p_{22})} + \frac{r_3 K_n'(p_{32})}{p_{32} K_n(p_{32})} \right) = \left[ \frac{n_8 Z_0}{k_0} \left( \frac{1}{p_{22}} + \frac{1}{p_{32}} \right) \right]^2 \] 3.2.a

and

\[ \left( \frac{r_2 A_n'(p_{21})}{p_{21} A_n(p_{21})} + \frac{r_1 I_n'(p_{11})}{p_{11} I_n(p_{11})} \right) \left( \frac{r_2 B_n'(p_{21})}{p_{21} B_n(p_{21})} + \frac{r_1 I_n'(p_{11})}{p_{11} I_n(p_{11})} \right) = \left[ \frac{n_8 Z_0}{k_0} \left( \frac{1}{p_{21}} + \frac{1}{p_{11}} \right) \right]^2 \] 3.2.b

Equations 3.2 contain two ratios of arbitrary constants, \( a_4/a_2 \) and \( b_4/b_2 \), which have yet to be determined. Applying Cramer's rule to the original eight equations yields
The ratio $b_4/b_2$ is obtained from equation 3.3 by interchanging the $\varepsilon_r$ and $\mu_r$.

Solutions for hybrid modes are obtained by solving equations 3.2 simultaneously for $\beta$ using the method of False Position, where $a_4/a_2$ and $b_4/b_2$ are given by the appropriate form of equation 3.3, and the three wavenumbers are then given by equations 2.3.

Upon setting $n=0$, both left-hand sides of equations 3.2 separate into two factors, each of which equals zero. These are given by

$$
\left( \frac{\varepsilon_{r2} A_0(p_{22})}{p_{22} A_0(p_{22})} + \frac{\varepsilon_{r3} K_0(p_{32})}{p_{32} K_0(p_{32})} \right) = 0 \quad \cdots \cdots \cdots 3.4
$$

$$
\left( \frac{\mu_{r2} B_0(p_{22})}{p_{22} B_0(p_{22})} + \frac{\mu_{r3} K_0(p_{32})}{p_{32} K_0(p_{32})} \right) = 0 \quad \cdots \cdots \cdots 3.5
$$

$$
\left( \frac{\varepsilon_{r2} A_0(p_{21})}{p_{21} A_0(p_{21})} + \frac{\varepsilon_{r1} I_0(p_{11})}{p_{11} I_0(p_{11})} \right) = 0 \quad \cdots \cdots \cdots 3.6
$$

$$
\left( \frac{\mu_{r2} B_0(p_{21})}{p_{21} B_0(p_{21})} + \frac{\mu_{r1} I_0(p_{11})}{p_{11} I_0(p_{11})} \right) = 0 \quad \cdots \cdots \cdots 3.7
$$

Solving equation 3.6 for $a_4/a_2$ and equation 3.7 for $b_4/b_2$ yields

$$
a_4 \frac{a_4}{a_2} = -\left( \frac{J_0(p_{21})}{Y_0(p_{21})} \left( \frac{\varepsilon_{r1} I_0'(p_{11})}{p_{11} I_0(p_{11})} + \frac{\varepsilon_{r2} J_0'(p_{21})}{p_{21} J_0(p_{21})} \right) \left( \frac{\varepsilon_{r1} I_0(p_{11})}{p_{11} I_0'(p_{11})} + \frac{\varepsilon_{r2} Y_0'(p_{21})}{p_{21} Y_0(p_{21})} \right) \right)^{-1} \quad \cdots \cdots \cdots 3.8
$$

$$
b_4 \frac{b_4}{b_2} = -\left( \frac{J_0(p_{21})}{Y_0(p_{21})} \left( \frac{\mu_{r1} I_0'(p_{11})}{p_{11} I_0(p_{11})} + \frac{\mu_{r2} J_0'(p_{21})}{p_{21} J_0(p_{21})} \right) \left( \frac{\mu_{r1} I_0(p_{11})}{p_{11} I_0'(p_{11})} + \frac{\mu_{r2} Y_0'(p_{21})}{p_{21} Y_0(p_{21})} \right) \right)^{-1} \quad \cdots \cdots \cdots 3.9
$$

The characteristic equation for TM modes is given by equation 3.4 with $a_4/a_2$ defined by equation 3.8, while that for TE modes is given by equation 3.5 with $b_4/b_2$ defined by equation 3.9.

Solutions of equations 3.4 for TM modes, 3.5 for TE modes and 3.2 for $n=1$ hybrid modes are shown in figures 3.1.a, 3.1.b and 3.1.c for three sizes of polyethylene tubes in free space ($\varepsilon_{r1}=\varepsilon_{r3}=\mu_{r1}=\mu_{r2}=\mu_{r3}=1$, $\varepsilon_{r2}=2.26$, and $\rho_1/\rho_2=0, 0.5, 0.95$). The following main features of the mode spectrum may be
FIGURE 3.1.a Mode Spectrum of the Polyethylene Rod

FIGURE 3.1.b Mode Spectrum of the Polyethylene Tube, $\rho=0.5$
FIGURE 3.1.c Mode Spectrum of the Polyethylene Tube, $\rho=0.95$
noted from these graphs:

(i) For the dielectric rod \( (\rho=0) \), \( \text{TE}_{0m} \) and \( \text{TM}_{0m} \) modes have the same value of \( r_2/\lambda \) at cutoff, as do \( \text{EH}_{1m} \) and \( \text{HE}_{1,m+1} \) modes \( (m\geq 1) \).

(ii) The \( \text{HE}_{11} \) mode has no lower cutoff.

(iii) If \( \rho\neq 0 \), the \( \text{TE}_{0m} \) and \( \text{TM}_{0m} \) modes no longer have the same value of \( r_2/\lambda \) at cutoff. The same is true for \( \text{HE}_{1,m+1} \) and \( \text{EH}_{1m} \) modes.

(iv) As \( \rho+1 \), the phase characteristics of the \( \text{H}_{0m} \) and \( \text{HE}_{1m} \) modes become indistinguishable, as do those of the \( \text{E}_{0m} \) and \( \text{EH}_{1m} \) modes, thus providing the physical distinction between \( \text{HE} \) and \( \text{EH} \) modes.

(v) As \( \rho+1 \), the \( n=0 \) and \( n=1 \) modes appear in widely separated clusters, each cluster consisting of four modes \( (\text{HE}_{1m}, \text{TE}_{0m}, \text{TM}_{0m} \) and \( \text{EH}_{1m} \)).

(vi) The \( \text{HE}_{1m} \) and \( \text{H}_{0m} \) phase characteristics intersect at some value of \( r_2/\lambda \). In most cases, for values of \( r_2/\lambda \) greater than that at the intersection, the differences in the two curves are too small to be seen graphically. However, the degeneracy of the \( \text{HE}_{12} \) and \( \text{H}_{02} \) modes for \( \rho=0.5 \) can be seen in figure 3.1.b. The intersection of \( \text{HE}_{1m} \) and \( \text{TE}_{0m} \) mode transmission characteristics will be observed in Chapter 4.

Points (i), (ii) and (iii) are well-established facts. Point (iv) provides the distinction between \( \text{HE} \) and \( \text{EH} \) modes as well as the basis for their description. It is believed that (iv), (v) and (vi) have not been previously observed.
3.2 Radial Variation of Fields and Mode Designations

Except for the mode designation for the dielectric rod given by Clarricoats\textsuperscript{21}, which was based upon the sequence of solutions of the characteristic equation, other attempts to provide a consistent designation have met with limited success\textsuperscript{16-19}. In this section, the mode designation for the dielectric tube is obtained by examining the sequence of solutions as well as plots of the radial dependence of the field components for various values of $\rho$ and $r_2/\lambda$. It is shown that this designation agrees with that of Clarricoats for the special case of the rod. The functions describing these radial variations are given by

\[
\begin{align*}
\bar{E}_{zi} &= E_{zi}/(\cos \theta \cos \beta z) & \bar{E}_{ri} &= E_{ri}/(\cos \theta \sin \beta z) & \bar{E}_{\theta i} &= E_{\theta i}/(\sin \theta \sin \beta z) \\
\bar{H}_{zi} &= H_{zi}/(\sin \theta \cos \beta z) & \bar{H}_{ri} &= H_{ri}/(\sin \theta \sin \beta z) & \bar{H}_{\theta i} &= H_{\theta i}/(\cos \theta \sin \beta z),
\end{align*}
\]

(3.10)

where $E_{zi}$, $E_{ri}$, $E_{\theta i}$, $H_{zi}$, $H_{ri}$ and $H_{\theta i}$ are given by equations 2.2. The normalized field components were plotted for several lower-order modes on polyethylene tubes in free space ($\varepsilon_{r2}=2.26$, $\varepsilon_{r1}=\varepsilon_{r3}=1.0$). The plots for eight representative cases are shown in figures 3.2 to 3.5. The conditions for these cases are listed in Table 3.1. In all cases, $r_2/\lambda = 2.0$.

<table>
<thead>
<tr>
<th>figure</th>
<th>mode</th>
<th>$\rho$</th>
<th>$k_0/\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.a</td>
<td>HE$_{11}$</td>
<td>0.1</td>
<td>0.6729</td>
</tr>
<tr>
<td>3.2.b</td>
<td>HE$_{11}$</td>
<td>0.0</td>
<td>0.6701</td>
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<td>3.3.a</td>
<td>TE$_{02}$</td>
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</tr>
<tr>
<td>3.3.b</td>
<td>HE$_{12}$</td>
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<td>0.7759</td>
</tr>
<tr>
<td>3.4.a</td>
<td>EH$_{11}$</td>
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<td>3.4.b</td>
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<td>3.5.a</td>
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</tr>
<tr>
<td>3.5.b</td>
<td>EH$_{12}$</td>
<td>0.5</td>
<td>0.8146</td>
</tr>
</tbody>
</table>

TABLE 3.1 Parameters for Radial Field-dependence Plots
FIGURE 3.2.a Radial Dependence of Fields, \( HE_{11} \) Mode, \( \rho = 0.1 \)

FIGURE 3.2.b Radial Dependence of Fields, \( HE_{11} \) Mode, \( \rho = 0.0 \)
FIGURE 3.3.a Radial Dependence of Fields, TE_{02} Mode, \( \rho = 0.5 \)

FIGURE 3.3.b Radial Dependence of Fields, HE_{12} Mode, \( \rho = 0.5 \)
FIGURE 3.4.a Radial Dependence of Fields, EH$_{11}$ Mode, $\rho = 0.1$

FIGURE 3.4.b Radial Dependence of Fields, EH$_{11}$ Mode, $\rho = 0.0$
FIGURE 3.5.a Radial Dependence of Fields, $TM_{02}$ Mode, $\rho=0.5$

FIGURE 3.5.b Radial Dependence of Fields, $EH_{12}$ Mode, $\rho=0.5$
Examination of these graphs reveals the following properties of the modes:

(i) All of the modes $HE_{nm}$, $H_{0m}$, $E_{0m}$ or $EH_{nm}$ can be supported by the dielectric tube, where the first subscript indicates the number of harmonic variations of the fields with $\theta$.

(ii) For $H_{0m}$ and $HE_{1m}$ modes, $m$ is the number of zeros of $H$ in $r_1 < r < r_2$ (figures 3.2 and 3.3). Here, equality is required for the $HE_{1m}$ modes because the first zero of $H$ in $r_1 < r < r_2$ moves to $r=0$ as $r_1 \to 0$ (figures 3.2.a and 3.2.b), to become coincident with the zero already existing there.

(iii) For $E_{0m}$ and $EH_{1m}$ modes, $m$ is the number of zeros of $E$ in $r_1 < r < r_2$ (figures 3.4 and 3.5). Equality is not required here since there is no zero shift to the origin as $r_1 \to 0$ (figures 3.4.a and 3.4.b).

(iv) Strong similarities exist between the $H_{0m}$ and $HE_{1m}$ modes for large $r_2/\lambda$ (figures 3.3.a and 3.3.b). This effect becomes increasingly pronounced as $p \to 1$, but is not easily detected as $p \to 0$. The same behaviour exists for $E_{0m}$ and $EH_{1m}$ modes (figures 3.5.a and 3.5.b), thus providing the distinction between EH and HE modes in the mode designation.

Although the above properties are derived from a study of $n=0$ and $n=1$ modes, there appears to be no reason why they would not be valid for all $n$. 
3.3 Cutoff Conditions

Lossless surface-wave propagation on the dielectric tube requires that all quantities appearing in equations 2.3 be real and positive. If \( \mu_r \text{ and } \epsilon_r \), then cutoff occurs when \( h_3 = 0 \) and \( h_1 = 0 \), or more generally, when \( p_{32} = 0 \) and \( p_{11} = 0 \). These requirements are used in Appendix A to derive the cutoff conditions given by equations A2, A7, A9 and A12, for all modes with \( n = 0 \) and \( n = 1 \). Letting \( \mu_r = \mu_r \), these conditions reduce to

\[
\begin{align*}
J_0(p_{22}) &= \frac{ \epsilon_r J_0(p_{21}) - 2 \epsilon_r J_1(p_{21}) }{ \epsilon_r J_0(p_{21}) - 2 \epsilon_r J_1(p_{21}) } \quad \text{TM modes} \\
J_1(p_{22}) &= \frac{ J_1(p_{21}) }{ J_1(p_{22}) } \quad \text{HE modes} \\
\end{align*}
\]

Equations 3.11.a for TE and TM modes yield the same results as obtained by Astrahan\(^3\), while equation 3.11.b for HE modes has been derived by Unger\(^7\), and equation 3.11.c reduces to Unger's condition for EH modes if \( \epsilon_r = \epsilon_r = 1 \).

For the dielectric rod \( r_1 = 0 \) (or \( p_{21} = 0 \)), and equations 3.11 reduce to

\[
\begin{align*}
J_0(p_{22}) &= 0 \quad \text{TM and TE modes} \\
J_1(p_{22}) &= 0 \quad \text{HE \( m+1 \) and EH \( m \) modes, } m > 1 \\
p_{22} &= 0 \quad \text{HE \( 11 \) mode} \\
\end{align*}
\]

From these conditions, it can be seen that, for the dielectric rod, the E\(_{0m}\) and H\(_{0m}\) modes have the same value of \( p_{22} \) at cutoff for given \( m \). This is also true
for EH\(_{lm}\) and HE\(_{1,m+1}\) modes.

At cutoff \(p_{22}\) is given by

\[
p_{22} = 2\pi \frac{r^2}{\lambda^2_c} \sqrt{\varepsilon_r 2 \mu_r 2 - \varepsilon_r 3 \mu_r 3},
\]

(3.13)

from which the value of \(r^2/\lambda^2_c\) may be determined. If \(p_{22}=0\), then \(r^2/\lambda^2_c = 0\) and there is no lower cutoff frequency. Solutions of equations 3.11 to 3.13 for polyethylene tubes and rods in free space are shown in figure 3.6 for the \(H_{01}, E_{01}, EH_{11}\) and \(HE_{12}\) modes.

![Figure 3.6 Cutoff Conditions, H\(_{01}\), E\(_{01}\), EH\(_{11}\) and HE\(_{12}\) Modes](image-url)
3.4 Field Configurations

While the field configurations of TM and TE modes on the dielectric tube have been fully discussed in the literature\textsuperscript{12, 14}, only limited information is available for hybrid modes. It was only recently that the correct field configuration in a transverse plane of the dielectric rod for the \( \text{HE}_{11} \) mode has been given\textsuperscript{22}.

In this section, an attempt is made to present a clear picture of the field configurations of the first two hybrid modes. This is achieved by obtaining three-dimensional computer plots of the field lines. The method involves solving the differential equations of the field lines using numerical methods. These equations are formed by taking ratios of the field components in cartesian co-ordinates and equating them to the corresponding ratios of differentials. These field components are converted from cylindrical co-ordinates as shown in figure 3.7 by using the transformations.

\[
\begin{align*}
E_{xi} &= E_{ri} \cos \theta - E_{\theta i} \sin \theta \\
E_{yi} &= E_{ri} \sin \theta + E_{\theta i} \cos \theta \\
E_{zi} &= E_{zi}
\end{align*}
\]  \hspace{1cm} \text{3.14}

The expressions for these field components in cartesian co-ordinates are given in Appendix B for \( n=1 \) modes.
The computational procedure for obtaining E-field lines is as follows:

(i) Given an initial condition \((x_0, y_0, z_0)\) through which the desired field line is to pass, convert the point to \((r_0, \theta_0, z_0)\) and calculate the three field components.

(ii) Take the field component which is largest in magnitude, say \(E_x\), and form the two differential equations

\[
\frac{dy}{dx} = \frac{E_y}{E_x} \quad \text{and} \quad \frac{dz}{dx} = \frac{E_z}{E_x},
\]

thus assuring that all slopes are less than one in magnitude.

(iii) From the initial condition \((x_0, y_0, z_0)\) determine a new point \((x_1, y_1, z_1)\) on the field line by numerical solution of the two differential equations using a fourth-order Runge Kutta method.

(iv) Using \((x_1, y_1, z_1)\) as the new initial condition, repeat the procedure until the field line touches a boundary of the medium considered or until the field line closes upon itself.

(v) Choose a new initial condition and repeat (i) to (iv).

(vi) Repeat the procedure for each of the three regions of the tube, and for H-fields as well.

Plots of the electric and magnetic field configurations for the HE\(_{11}\) and EH\(_{11}\) modes are shown in figures 3.8 and 3.9, respectively, where the transformations used for the conversion from three to two-dimensional co-ordinates are given by

\[
\begin{align*}
x' &= z - (0.7/\sqrt{2})y \\
y' &= x - (0.7/\sqrt{2})y
\end{align*}
\]

The dotted lines shown in figures 3.8 and 3.9 represent the field lines in medium 2, while the solid lines are those in media 1 and 3.
FIGURE 3.8.a Electric Field Configuration for the Polyethylene Tube, HE\textsubscript{11} Mode, $\rho=0.9$, $r_2/\lambda=1.68$, $\lambda_g/\lambda=0.8804$

FIGURE 3.8.b Magnetic Field Configuration for the Polyethylene Tube, HE\textsubscript{11} Mode, $\rho=0.9$, $r_2/\lambda=1.68$, $\lambda_g/\lambda=0.8804$
FIGURE 3.9.a Electric Field Configuration for the Polyethylene Tube, $EH_{11}$ Mode, $\rho=0.9$, $r_2/\lambda=1.68$, $\lambda_g/\lambda=0.9654$

FIGURE 3.9.b Magnetic Field Configuration for the Polyethylene Tube $EH_{11}$ Mode, $\rho=0.9$, $r_2/\lambda=1.68$, $\lambda_g/\lambda=0.9654$
4. PROPAGATION CHARACTERISTICS

In this section, the phase-velocity characteristics are obtained by solving the dispersion equation. The group velocity and attenuation coefficient are obtained by integrating the appropriate combinations of field components. The standard perturbation technique is used where it is assumed that the fields are only slightly perturbed by the small losses, thus permitting the use of lossless field components in obtaining expressions for power flow, power loss per unit length and energy storage per unit length. Numerical results are given for the phase and group velocities normalized to the speed of light in free space and for the attenuation coefficient normalized to that of a TEM wave in medium 2. For polyethylene tubes in free space, \( \varepsilon_{r1}=\varepsilon_{r3}=1.0, \varepsilon_{r2}=2.26, \tan\delta_1=\tan\delta_3=0 \) and \( \tan\delta_2 \) is arbitrary.

4.1 Phase Velocity

Given values of \( \mu_{ri}, \varepsilon_{ri}, \rho \) and \( r_2/\lambda \), equations 3.2, 3.4 and 3.5 (for hybrid, TM and TE modes, respectively) can be solved numerically for \( k_0/\beta \). This also yields the normalized phase velocity and the normalized guide wavelength, since

\[
k_0/\beta = v_p/v_o = \lambda_g/\lambda \quad \text{............................................. 4.1}
\]

The solutions for \( HE_{11} \) and \( TE_{01} \) modes for "thick" tubes (\( \rho=0.1 \) to 0.9) and for "thin" tubes (\( \rho=0.91 \) to 0.99) are given in figures 4.1, and for the \( TM_{01} \) mode in figures 4.2.

By inspecting figures 4.1 and 4.2, the following properties may be noted:

(i) The phase velocity decreases with increasing \( r_2/\lambda \) from the free space value at cutoff and asymptotically approaches that of a TEM wave in medium 2.

(ii) The dispersion characteristics improve as \( r_2/\lambda \) is increased.
FIGURE 4.1.a Normalized Phase Velocity of Thick Tubes,\[\text{--- HE}_{11} \text{ Mode, \cdots\cdots\cdots TE}_{01} \text{ Mode}\]

FIGURE 4.1.b Normalized Phase Velocity of Thin Tubes,\[\text{--- HE}_{11} \text{ Mode, \cdots\cdots\cdots TE}_{01} \text{ Mode}\]
FIGURE 4.2.a Normalized Phase Velocity of Thick Tubes, TM$_{01}$ Mode

FIGURE 4.2.b Normalized Phase Velocity of Thin Tubes, TM$_{01}$ Mode
(iii) The separation of the TM\(_{01}\) and TE\(_{01}\) modes increases as \(\rho\) increases.

(iv) The phase characteristics of the HE\(_{11}\) mode become indistinguishable from those of the TE\(_{01}\) mode for values of \(r_2/\lambda\) remote from cutoff of the TE\(_{01}\) mode. This behaviour becomes increasingly pronounced as \(\rho\) increases. Similar behaviour of group velocity and attenuation characteristics will be noted in Sections 4.2 and 4.3, respectively.

The significance of these properties will be considered when dealing with possible practical systems in Chapter 6.
4.2 Group Velocity

The group velocity of a given mode in a waveguide is normally found by taking the slope of the $\omega-B$ diagram. This may be found graphically, or by differentiating the characteristic equation. In the case of the dielectric tube this procedure is both lengthy and tedious. Alternatively, the group velocity may be found by determining the rate of transport of energy, which for a lossless waveguide is given by

$$v_g = \frac{N}{W} = \frac{3}{i=1} \frac{N_i}{3} \frac{w_i}{i=1}$$

where $N_i$ = power flow in the $i^{th}$ medium
and $w_i$ = energy storage per unit length in the $i^{th}$ medium.

For hybrid modes, these quantities are given by

$$N_i = \frac{\pi}{2} \int_{r_{i-1}}^{r_i} (E_{ri} H_{\theta i}^* - E_{\theta i} H_{ri}^*) r \, dr , \quad i = 1, 2, 3 \ldots$$

$$W_i = \left( \frac{\varepsilon_i \pi}{2v_0 Z_0} \right) \int_{r_{i-1}}^{r_i} (E_{zi} E_{z i}^* + E_{ri} E_{r i}^* + E_{\theta i} E_{\theta i}^*) r \, dr , \quad i = 1, 2, 3 \ldots$$

where $r_0 = 0$ and $r_3 = \infty$.

Substituting from equations 2.2 yields
The functions $S_I, S_A, S_{AB}, S_K, T_I, T_A$ and $T_K$ are integrals of functions of Bessel functions which are defined and evaluated as follows:

\[ T_I = 2 \int_0^{P_{11}} I_n^2(p_1) p_1 dp_1 = (p_{11}^2 + n^2) I_n^2(p_{11}) - p_{11}^2 I_n(p_{11}) \]  \[ S_I = 2 \int_0^{P_{11}} \left[ I_n^2(p_1) p_1 + \frac{n^2}{p_1} I_n^2(p_1) \right] dp_1 = -T_I + 2p_{11} I_n(p_{11}) I_n'(p_{11}) \]  \[ T_K = 2 \int_{P_{32}}^{\infty} K_n^2(p_3) p_3 dp_3 = -(p_{32}^2 + n^2) K_n^2(p_{32}) + p_{32}^2 K_n(p_{32}) \]  \[ S_K = 2 \int_{P_{32}}^{\infty} \left[ K_n^2(p_3) p_3 + \frac{n^2}{p_3} K_n^2(p_3) \right] dp_3 = -T_K - 2p_{32} K_n(p_{32}) K_n'(p_{32}) \]
\[ T_A = 2 \int_{p_{21}}^{p_{22}} A_n^2(p_2) p_2 \, dp_2 \]

\[ = (p_{22}^2 - n^2) A_n^2(p_{22}) + p_{22}^2 A_n^2(p_{22}) - (p_{21}^2 - n^2) A_n^2(p_{21}) + p_{21}^2 A_n^2(p_{21}) \]

\[ S_A = 2 \int_{p_{21}}^{p_{22}} \left[ A_n^2(p_2) + \frac{n^2}{p_2^2} A_n^2(p_2) \right] \, dp_2 \]

\[ = T_A + 2p_{22} A_n(p_{22}) A_n(p_{22}) - 2p_{21} A_n(p_{21}) A_n(p_{21}) \]

\[ S_B = \text{SAME AS } S_A \text{ WITH } A \text{ REPLACED BY } B \]

\[ S_{AB} = - \int_{p_{21}}^{p_{22}} \left[ A_n(p_2) B_n(p_2) + B_n(p_2) A_n(p_2) \right] \, dp_2 \]

\[ = A_n(p_{21}) B_n(p_{21}) - A_n(p_{22}) B_n(p_{22}) \]

The corresponding \( N_i \) and \( W_i \) for TE and TM modes are obtained by using equations 2.4 for the field components in equations 4.3 and 4.4. This yields for TE modes

\[ N_1 = \frac{\mu_r \beta_{2} \beta_{0} \beta_{0} b_{2} b_{2}^{*}}{2 h_{1}^{4}} \left( \frac{B_{0}(p_{21}^2)}{B_{0}(p_{11}^2)} \right) S_I \]

\[ N_2 = \frac{\mu_r \beta_{2} \beta_{0} \beta_{0} b_{2} b_{2}^{*}}{2 h_{2}^{4}} S_B \]

\[ N_3 = \frac{\mu_r \beta_{2} \beta_{0} \beta_{0} b_{2} b_{2}^{*}}{2 h_{3}^{4}} \left( \frac{B_{0}(p_{22}^2)}{B_{0}(p_{32}^2)} \right) S_K \]

\[ W_i = \frac{\mu_r \epsilon_r \beta_{0} N_i}{\beta_{0}} \quad i=1,2,3 \]

For TM modes the corresponding relations are

\[ ..4.7.c \]
The normalized group velocity characteristics for the same conditions as used for figures 4.1 and 4.2 are given in figures 4.3 and 4.4, respectively.

From figures 4.3 and 4.4, it can be seen that for large $r_2/\lambda$ the normalized group velocity approaches $1/\sqrt{\varepsilon r_2}$, but unlike the phase velocity, may become less than this asymptotic value at some intermediate $r_2/\lambda$. A similar phenomenon has been observed in the case of wave propagation along dielectric-coated conductor surface-wave transmission lines.
FIGURE 4.3.a Normalized Group Velocity of Thick Tubes, HE_{11} Mode, TE_{01} Mode

FIGURE 4.3.b Normalized Group Velocity of Thin Tubes, HE_{11} Mode, TE_{01} Mode
FIGURE 4.4.a Normalized Group Velocity of Thick Tubes, TM_{01} Mode

FIGURE 4.4.b Normalized Group Velocity of Thin Tubes, TM_{01} Mode
4.3 Attenuation Coefficient

The attenuation coefficient of the dielectric tube may be found by assuming a complex permittivity for medium 2 in the characteristic equation and solving for the propagation coefficient, $\gamma$. Jakes has solved the equations for TM and TE modes by expanding the resulting complex equation in a truncated Taylor's series about the lossless solution, thus separating the equation into its real and imaginary parts. The real part yields essentially the lossless characteristic equation which is solved for $\beta$, and the imaginary part may then be solved for $\alpha$. Unger has applied this method to the HE$_{11}$ mode, but his result is only valid for small phase-velocity reductions since he used small-argument approximations of the Bessel functions. A disadvantage of this method is that it involves differentiating the lossless characteristic equation with respect to three variables. Also, losses in regions 1 and 3 cannot be included unless two additional derivatives are obtained.

An alternative approach is to use the well-known perturbation method. The small-argument approximation of $\exp(-2\alpha)$ is used to obtain an expression for the attenuation coefficient which is valid for small losses. This method is employed in this work, yielding for the attenuation coefficient

$$\alpha = P/2N = (1/2) \sum_{i=1}^{3} \frac{P_i}{N_i} \tag{4.12}$$

where $P_i$ = power loss per unit length in the $i^{th}$ medium, and is given by

$$P_i = \omega \tan \delta_i W_i \tag{4.13}$$

The equations for $N_i$ and $W_i$ are given in section 4.2 for all modes.

The attenuation coefficients of the HE$_{11}$, TE$_{01}$ and TM$_{01}$ modes, normalized to that of a TEM wave in medium 2 ($\alpha_0$) are shown in figures 4.5 and 4.6 where $\alpha_0$ is given by

$$\alpha_0 = (1/2) \sqrt{\epsilon_r} k_0 \tan \delta_2 \tag{4.14}$$
FIGURE 4.5.a Normalized Attenuation Coefficient of Thick Tubes,
\[ HE_{11} \text{ Mode}, \quad TE_{01} \text{ Mode} \]

FIGURE 4.5.b Normalized Attenuation Coefficient of Thin Tubes
\[ HE_{11} \text{ Mode}, \quad TE_{01} \text{ Mode} \]
FIGURE 4.6.a Normalized Attenuation Coefficient of Thick Tubes, TM$_{01}$ Mode

FIGURE 4.6.b Normalized Attenuation Coefficient of Thin Tubes, TM$_{01}$ Mode
From figures 4.5 and 4.6, the following points are noted:

(i) The attenuation characteristics of all modes become more slowly varying with $r_2/\lambda$ as $\rho$ increases.

(ii) For any particular value of $r_2/\lambda$, the attenuation coefficient decreases with increasing $\rho$.

(iii) For large $r_2/\lambda$ the attenuation coefficient approaches that of a TEM wave in medium 2.
5. **POWER CONCENTRATION AND FIELD EXTENSION**

Of interest in any open surface waveguide is the degree of power concentration in the vicinity of the structure. A measure of this concentration can be taken as the fraction, \( C_p \), of total transmitted power which is contained within a given radius \( r_p \). This is given by

\[
C_p = 1 - \frac{\pi}{2N} \int_{r_p}^{\infty} (E_{r3}H_{\theta3}^* - E_{\theta3}H_{r3}^*)rdr
\] .......................... 5.1

Conversely, it might be desirable to know the value of \( r_p \) for some fixed value of \( C_p \). This may be determined numerically.

An alternative measure may be taken as the ratio, \( C_f \), of the field strength, either \( E_{z3} \) or \( H_{z3} \), at some radius \( r_f \) to that at the surface of the guide. This is given by

\[
C_f = \frac{K_n(h_3r_f)}{K_n(p_{32})}
\] .......................... 5.2

Again, this equation can be solved for \( r_f \), given a specified value of \( C_f \).

Solutions of equation 5.1 for the \( \text{HE}_{11} \) mode are shown in figures 5.1 for \( C_p = 0.999 \).

Similar curves for \( C_f = 0.001 \) are shown in figures 5.2. If maximum permissible values of \( r_p \) and \( r_f \) are set, then a lower operating frequency is established for a given tube size. This will be considered in Chapter 6.

The behaviour of the dielectric tube waveguide can be further understood by examining the division of transmitted power among the three regions. Since the variations of \( N_i/N \) with \( r_2/\lambda \) for all modes are very similar, only those results for the \( \text{HE}_{11} \) mode are plotted in figures 5.3 and 5.4. From these plots, it is seen that the dominant feature is the power shift from outside to within the walls of the tube as \( r_2/\lambda \) increases from zero, until for very large \( r_2/\lambda \), essentially all power is transmitted within the tube walls. This accounts for the asymptotic behaviour of the propagation characteristics discussed in Chapter 4.
FIGURE 5.1.a Power Concentration Characteristics of Thick Tubes, HE_{11} Mode

FIGURE 5.1.b Power Concentration Characteristics of Thin Tubes, HE_{11} Mode
FIGURE 5.2.a Field Extension Characteristics of Thick Tubes, HE_{11} Mode

FIGURE 5.2.b Field Extension Characteristics of Thin Tubes HE_{11} Mode
FIGURE 5.3.a Fraction of Power Carried in Medium 1 of Thick Tubes, HE_{11} Mode

FIGURE 5.3.b Fractions of Power Carried in Media 2 and 3 of Thick Tubes, HE_{11} Mode

--- N_3/N, ------ N_2/N
FIGURE 5.4.a Fraction of Power Carried in Medium 1 of Thin Tubes, HE$_{11}$ Mode

FIGURE 5.4.b Fractions of Power Carried in Media 2 and 3 of Thin Tubes, HE$_{11}$ Mode

--- $N_3/N$, --- $N_2/N$
6. POSSIBLE PRACTICAL SYSTEMS

The results of Chapters 4 and 5 suggest the possible use of dielectric tubes as low-loss waveguides in regions where conventional waveguide systems are too impractical. In an attempt to show possible superiority of the dielectric tube waveguide over other types, a series of tube sizes is proposed corresponding to the standard rectangular waveguides in the various frequency ranges. The performance of the dielectric tube is adversely affected by accumulation of snow, ice and moisture, and by the presence of nearby obstacles which may cause radiation. Also, supporting the guide presents a problem. A method is proposed for overcoming these difficulties by embedding the tube in a low-density, low-loss material of sufficient extent that a negligible portion of the wave is carried outside this medium. The effects of such a shield upon the propagation characteristics are also investigated in this chapter.

6.1 A Series of "Standard" Dielectric Tube Waveguides

The choice of size of dielectric tube for a given frequency range is an arbitrary one depending upon which restrictions are more important in a particular application, i.e. attenuation, dispersion, field extension or higher-order modes. The choice given here is by no means optimum. The emphasis is on improving the dispersion characteristics over those of rectangular guides and achieving a lower attenuation for guides used above 30 GHz.

The propagating mode is the \( \text{HE}_{11} \) (fundamental) mode, which requires the smallest diameter tubes. A moderately thin wall tube \( (\rho=0.9) \) is chosen since the attenuation is smaller and the dispersion characteristics are better. The same bandwidth as for rectangular waveguides is achieved by using \( r^2/\lambda \) from 0.6 to 0.9. This range provides a reasonable compromise between excessive attenuation and excessive field extension. The normalized characteristics for these condi-
tions are given in Table 6.1.

<table>
<thead>
<tr>
<th>( r_2/\lambda )</th>
<th>0.6</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_p/v_0 )</td>
<td>0.9839</td>
<td>0.9590</td>
</tr>
<tr>
<td>( v_g/v_0 )</td>
<td>0.9399</td>
<td>0.8839</td>
</tr>
<tr>
<td>( a/a_0 )</td>
<td>0.0836</td>
<td>0.1950</td>
</tr>
<tr>
<td>( r_p/r_2 )</td>
<td>3.81</td>
<td>2.34</td>
</tr>
<tr>
<td>( r_f/r_2 )</td>
<td>9.05</td>
<td>4.60</td>
</tr>
</tbody>
</table>

**TABLE 6.1 Normalized Characteristics of Polyethylene Tubes in Free Space, HE_{11} Mode, \( \rho = 0.9 \).**

The outer radius of each tube is given by

\[
 r_2 = 0.6 \frac{v_0}{f} \quad \text{......................................................... 6.1}
\]

where \( f \) is the recommended lower operating frequency of the corresponding rectangular waveguide. For the conditions given in Table 6.1, the \( \text{TM}_{01} \) mode is cut off, but the \( \text{TE}_{01} \) mode has its cutoff at \( r_2/\lambda = 0.66 \). At the upper limit of each band, \( r_2/\lambda = 0.9 \) and \( v_p/v_0 = 0.9654 \) for the \( \text{TE}_{01} \) mode, compared with 0.9590 for the \( \text{HE}_{11} \) mode, indicating that the two modes still have an appreciable difference in phase velocity. It is assumed that the \( \text{TE}_{01} \) mode is not excited, either at the source or due to bends and irregularities in the tube.

Straight line approximations to the attenuation characteristics of the rectangular waveguides and the corresponding tubes are shown in figure 6.1. The tube characteristics are based upon a loss tangent for polyethylene of \( \tan \delta_2 = 0.00050 \). There is a range of published loss tangents for polyethylene from 0.00030 to 0.00060. The corresponding shift in attenuation for these values is also shown in figure 6.1 for the smallest tube size. Because of the normalization, the curves of figures 4.5 and 4.6 are valid for any value of \( \tan \delta_2 \), and the attenuation coefficient is readily found for any loss tangent.
FIGURE 6.1 Comparison of Rectangular Waveguide and Dielectric Tube Attenuation Characteristics

- Polyethylene tubes in free space, $\tan\delta = 0.0005$;
- Polyethylene tubes in polyfoam, $\tan\delta = 0.0003$ & 0.0006;
- Standard aluminium rectangular waveguides, $a = 3.54 \times 10^{-5}$ mhos/cm;
- Standard silver rectangular waveguides, $a = 6.17 \times 10^{-5}$ mhos/cm;

Frequency, GHz

Attenuation coefficient, dB/100ft
6.2 Comparison of Dielectric Tubes and Standard Rectangular Waveguides

In this section, the characteristics of rectangular waveguides are compared with those of the dielectric tubes chosen in Section 6.1. The factors considered are dispersion, attenuation, size and support.

The normalized group velocity of standard rectangular waveguides varies from 0.6 to 0.848 over the recommended frequency range. The dielectric tubes have corresponding values of 0.9399 and 0.8839. Hence, not only is the velocity of energy propagation higher for the tubes, but also the change in group velocity over each band is only 23% of the change for rectangular waveguides.

The attenuation characteristics for dielectric tubes (tanδ₂=0.0005) are given in Table 6.2, along with those for rectangular waveguides. From these results, it can be seen that there is a steady improvement of attenuation limits above 10 GHz of the proposed tubes over the metallic guides. These results are also shown graphically in figure 6.1 for those guides operated above 1 GHz.

The dimensions of both types of guide are given in Table 6.2, from which it can be seen that each tube diameter is approximately twice the broad dimension of the corresponding rectangular waveguide. This is an advantage for the tubes at very high frequencies, where the guides are small and a strict dimensional tolerance is required.

Dielectric tubes, being open waveguides, cannot be approached too closely, since a major portion of the energy is transmitted in the space surrounding the structures. It is noted from Table 6.1 that, at the lower frequency limit, the field strength falls to 0.1% of its surface value at \( r_f = 9.05 \, r_2 \). These values of "field radius" for the tubes considered are given in Table 6.2. In order that the tubes may be shielded from adverse weather conditions and supported without causing reflections or radiation, it is suggested that they may be embedded in a polyfoam medium (\( \varepsilon_r = 1.03 \),
<table>
<thead>
<tr>
<th>Frequency* (Ghz)</th>
<th>Rectangular Waveguide Inner Dimensions (inches)</th>
<th>Dielectric Tube Diameter (inches)</th>
<th>Rectangular Waveguide** Theoretical Attenuation Lowest to Highest Frequency (dB/100ft)</th>
<th>Dielectric Tube Theoretical Attenuation Lowest to Highest Frequency (dB/100ft)</th>
<th>Dielectric Tube Field Radius for $C_{p}^{0.001}$ (inches)</th>
</tr>
</thead>
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<td>0.320 - 0.490</td>
<td>27.000 - 11.500</td>
<td>44.291</td>
<td>0.039 - 0.027</td>
<td>0.056 - 0.195</td>
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<tr>
<td>0.350 - 0.530</td>
<td>21.000 - 10.500</td>
<td>40.491</td>
<td>0.046 - 0.031</td>
<td>0.061 - 0.213</td>
<td>183.</td>
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<td>0.410 - 0.625</td>
<td>18.000 - 9.000</td>
<td>34.569</td>
<td>0.056 - 0.038</td>
<td>0.071 - 0.250</td>
<td>156.</td>
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<tr>
<td>0.480 - 0.750</td>
<td>15.000 - 7.500</td>
<td>28.923</td>
<td>0.069 - 0.050</td>
<td>0.085 - 0.299</td>
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<tr>
<td>0.640 - 0.960</td>
<td>11.500 - 5.750</td>
<td>22.146</td>
<td>0.128 - 0.075</td>
<td>0.151 - 0.390</td>
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<td>0.750 - 1.120</td>
<td>9.750 - 4.875</td>
<td>18.800</td>
<td>0.137 - 0.095</td>
<td>0.165 - 0.457</td>
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<td>7.700 - 3.810</td>
<td>14.191</td>
<td>0.201 - 0.136</td>
<td>0.216 - 0.557</td>
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<td>6.500 - 2.500</td>
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<td>0.195 - 0.663</td>
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<td>5.100 - 2.150</td>
<td>9.725</td>
<td>0.350 - 0.255</td>
<td>0.255 - 0.834</td>
<td>44.3</td>
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<td>1.700 - 2.600</td>
<td>4.300 - 2.110</td>
<td>8.337</td>
<td>0.561 - 0.320</td>
<td>0.296 - 1.04</td>
<td>37.7</td>
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<tr>
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<td>3.400 - 1.700</td>
<td>6.442</td>
<td>0.669 - 0.466</td>
<td>0.383 - 1.34</td>
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<td>2.840 - 1.340</td>
<td>5.451</td>
<td>0.940 - 0.641</td>
<td>0.453 - 1.58</td>
<td>24.7</td>
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<tr>
<td>3.300 - 4.900</td>
<td>2.290 - 1.145</td>
<td>4.295</td>
<td>1.110 - 0.645</td>
<td>0.575 - 2.01</td>
<td>19.4</td>
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<td>3.950 - 5.850</td>
<td>1.872 - 0.872</td>
<td>3.588</td>
<td>1.370 - 1.22</td>
<td>0.660 - 2.41</td>
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<tr>
<td>4.900 - 7.050</td>
<td>1.590 - 0.795</td>
<td>2.892</td>
<td>1.840 - 1.42</td>
<td>0.854 - 2.99</td>
<td>12.1</td>
</tr>
<tr>
<td>5.850 - 8.200</td>
<td>1.332 - 0.622</td>
<td>2.423</td>
<td>2.46 - 1.94</td>
<td>1.02 - 3.57</td>
<td>11.0</td>
</tr>
<tr>
<td>7.050 - 10.00</td>
<td>1.132 - 0.497</td>
<td>2.010</td>
<td>3.50 - 2.74</td>
<td>1.23 - 4.30</td>
<td>9.10</td>
</tr>
<tr>
<td>8.200 - 12.40</td>
<td>0.900 - 0.400</td>
<td>1.728</td>
<td>5.49 - 3.83</td>
<td>1.42 - 5.00</td>
<td>7.82</td>
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<tr>
<td>10.00 - 15.00</td>
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<td>1.417</td>
<td>6.65 - 4.50</td>
<td>1.76 - 6.10</td>
<td>6.43</td>
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<tr>
<td>12.40 - 18.00</td>
<td>0.622 - 0.311</td>
<td>1.141</td>
<td>8.14 - 5.26</td>
<td>2.18 - 7.56</td>
<td>5.17</td>
</tr>
<tr>
<td>15.00 - 22.00</td>
<td>0.510 - 0.255</td>
<td>0.945</td>
<td>9.87 - 6.10</td>
<td>2.63 - 9.14</td>
<td>4.20</td>
</tr>
<tr>
<td>18.00 - 26.50</td>
<td>0.420 - 0.170</td>
<td>0.787</td>
<td>13.3 - 9.5</td>
<td>3.14 - 11.0</td>
<td>3.56</td>
</tr>
<tr>
<td>22.00 - 33.00</td>
<td>0.340 - 0.130</td>
<td>0.644</td>
<td>16.3 - 11.2</td>
<td>3.83 - 13.4</td>
<td>2.92</td>
</tr>
<tr>
<td>26.50 - 40.00</td>
<td>0.250 - 0.100</td>
<td>0.535</td>
<td>21.8 - 15.0</td>
<td>4.62 - 16.2</td>
<td>2.42</td>
</tr>
<tr>
<td>33.00 - 50.00</td>
<td>0.274 - 0.112</td>
<td>0.450</td>
<td>21.0 - 20.9</td>
<td>5.75 - 20.1</td>
<td>1.94</td>
</tr>
<tr>
<td>40.00 - 60.00</td>
<td>0.198 - 0.094</td>
<td>0.354</td>
<td>30.8 - 22.7</td>
<td>6.97 - 24.4</td>
<td>1.60</td>
</tr>
<tr>
<td>50.00 - 75.00</td>
<td>0.148 - 0.074</td>
<td>0.283</td>
<td>52.9 - 39.1</td>
<td>8.71 - 30.5</td>
<td>1.26</td>
</tr>
<tr>
<td>60.00 - 90.00</td>
<td>0.122 - 0.061</td>
<td>0.236</td>
<td>93.3 - 52.2</td>
<td>10.5 - 36.6</td>
<td>1.07</td>
</tr>
<tr>
<td>75.00 - 110.00</td>
<td>0.100 - 0.050</td>
<td>0.199</td>
<td>100. - 70.4</td>
<td>13.1 - 45.7</td>
<td>0.66</td>
</tr>
<tr>
<td>90.00 - 140.00</td>
<td>0.080 - 0.040</td>
<td>0.157</td>
<td>152. - 69.9</td>
<td>15.7 - 54.9</td>
<td>0.21</td>
</tr>
<tr>
<td>110.00 - 170.00</td>
<td>0.065 - 0.035</td>
<td>0.129</td>
<td>163. - 137.</td>
<td>19.2 - 67.0</td>
<td>0.58</td>
</tr>
<tr>
<td>140.00 - 220.00</td>
<td>0.051 - 0.025</td>
<td>0.103</td>
<td>308. - 193.</td>
<td>24.4 - 65.3</td>
<td>0.46</td>
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<tr>
<td>170.00 - 260.00</td>
<td>0.043 - 0.021</td>
<td>0.083</td>
<td>384. - 254.</td>
<td>29.6 - 103.</td>
<td>0.38</td>
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<tr>
<td>210.00 - 325.00</td>
<td>0.034 - 0.017</td>
<td>0.064</td>
<td>512. - 340.</td>
<td>34.3 - 124.</td>
<td>0.29</td>
</tr>
</tbody>
</table>

* The upper frequency of each dielectric tube is taken as 1.5 times the lower frequency. The upper frequencies shown apply to the rectangular waveguides only.

** The first nineteen rectangular waveguides are aluminum, the remaining fifteen are silver.

TABLE 6.2 Rectangular Waveguide and Dielectric Tube Characteristics
which extends to $r=r_f$. The modified characteristics of these shielded tubes are considered in Section 6.3. The rectangular waveguides are closed structures, and obviously no such problems of support or radiation exist.

6.3 The Dielectric Tube in a Polyfoam Medium

The characteristics of the $HE_{11}$ mode on polyfoam-shielded polyethylene tubes are shown in figures 6.2 to 6.5, where the normalized quantities $v_p/v_0$, $v_g/v_0$, $\alpha/\alpha_0$, $\alpha_2/\alpha$ and $\alpha_3/\alpha$ are plotted for the same ranges of $\rho$ and $r_2/\lambda$ as considered previously for thick tubes in free space. Inspection of these graphs yields the following information:

(i) The $HE_{11}$ mode still has no lower cutoff frequency.

(ii) The phase-velocity reduction has a non-zero lower limit dependent upon $\varepsilon_r3$, and an upper limit which is unchanged from the case of the unshielded tube.

(iii) The attenuation is increased because of the outer lossy medium. The increase is most significant at low phase-velocity reductions.

(iv) Since at cutoff, all the power is carried in the outer medium, there is a lower asymptote to the normalized attenuation given by

$$\frac{\alpha/\alpha_0}{\varepsilon_r3/\varepsilon_r2} = \frac{\tan \delta_3}{\tan \delta_2}$$

... 6.2
FIGURE 6.2 Normalized Phase Velocity of Thick Shielded Tubes, HE_{11} Mode

FIGURE 6.3 Normalized Group Velocity of Thick Shielded Tubes, HE_{11} Mode
FIGURE 6.4 Normalized Attenuation Coefficient of Thick Shielded Tubes, HE_{11} Mode

FIGURE 6.5 Division of Attenuation Between Media 2 and 3 of Thick Shielded Tubes, HE_{11} Mode

--- \( \alpha_2/\alpha \), \( \alpha_3/\alpha \), \( \alpha_4=0.0 \)
The characteristics of the HE_{11} mode for $p=0.9$ and $r_2/\lambda$ of 0.6 and 0.9 are listed in Table 6.3. A comparison with the data in Table 6.1 reveals little significant change from the case of the tube in free space, except for the increase in attenuation, particularly at the lower frequency.

<table>
<thead>
<tr>
<th>$r_2/\lambda$</th>
<th>0.6</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_p/v_0$</td>
<td>0.9746</td>
<td>0.9528</td>
</tr>
<tr>
<td>$v_g/v_0$</td>
<td>0.9392</td>
<td>0.8826</td>
</tr>
<tr>
<td>$\alpha/\alpha_0$</td>
<td>0.2116</td>
<td>0.2852</td>
</tr>
</tbody>
</table>

**TABLE 6.3** Normalized Characteristics of Polyethylene Tubes Surrounded by an Infinite Polyfoam Medium, HE_{11} Mode, $p=0.9$.

The straight line approximations to the attenuation coefficients of the shielded tubes are shown in figure 6.1, where it is noted that, although the attenuation is increased, its range is reduced below that of the unshielded tubes. The proposed shielded tubes also have lower attenuations than the corresponding rectangular waveguides for frequencies above 30 GHz.
APPENDIX A  Mode Cutoff Conditions for the Dielectric Tube

(i)  $E_{0m}$ and $H_{0m}$ Modes

From equation 3.4, the characteristic equation of $E_{0m}$ modes can be written

$$\frac{\varepsilon_{r2} A_0'(p_{22})}{B_{22} A_0(p_{22})} = \frac{\varepsilon_{r3} K_1(p_{32})}{B_{32} K_0(p_{32})} = \frac{\varepsilon_{r3}}{B_{32} \ln \left( \frac{1}{0.89 p_{32}} \right)}$$  \hspace{1cm} A1

as $p_{32} \to 0$. At cutoff $p_{32}=0$, and equation A1 is satisfied if $A_0(p_{22}) = 0$. It can be shown by applying small-argument approximations to the left-hand side of A1, that $p_{22} \to 0$ is an unacceptable solution to the limiting process. Now $A_0(p_{22}) \approx J_0(p_{22}) + a_4/a_2 Y_0(p_{22})$, and by letting $p_{11} \to 0$ in equation 3.8, it can be proven that the condition $A_0(p_{22}) = 0$ for $E_{0m}$ modes corresponds to

$$\frac{J_0(p_{22})}{Y_0(p_{22})} = \left( \frac{\varepsilon_{r1} p_{21}}{\varepsilon_{r1} p_{21} Y_0(p_{21}) - 2 \varepsilon_{r2} Y_1(p_{21})} \right), \hspace{1cm} E_{0m} \text{ modes} \hspace{1cm} A2$$

Similarly, it can be shown that the cutoff condition for $H_{0m}$ modes is given by equation A2 with $\varepsilon_{r1}$ replaced by $\mu_{ri}$.

(ii)  $HE_{1m}$ Modes

From the theory of Bessel functions, it is readily proven that

$$\frac{A_n'(p_{22}) B_n'(p_{22})}{p_{22}^2 A_n(p_{22}) B_n(p_{22})} = \frac{n^2}{p_{22}^2} - \frac{A_{n-1}(p_{22}) B_{n-1}(p_{22})}{p_{22}^2 A_n(p_{22}) B_n(p_{22})} + \frac{n}{p_{22}} \left[ \frac{B_{n-1}(p_{22})}{B_n(p_{22})} - \frac{A_{n-1}(p_{22})}{A_n(p_{22})} \right]$$  \hspace{1cm} A3.a

and

$$\frac{K_n'(p_{32})}{p_{32}^2 K_n'(p_{32})} = \frac{n^2}{4} \left( \frac{K_{n-1}(p_{32}) K_{n+1}(p_{32})}{p_{32}^2 K_n'(p_{32})} \right)$$  \hspace{1cm} A3.b

Multiplying out equation 3.2.a and using equations A3 yields
\[
\varepsilon r^2 u n_2 \left[ \frac{1}{3} \frac{B_{n-1}(p_{22})}{B_n(p_{22})} - \frac{A_{n-1}(p_{22})}{A_n(p_{22})} \right] - \left( \frac{A_{n-1}(p_{22})B_{n-1}(p_{22})}{p_{22}^2 A_n(p_{22})B_n(p_{22})} \right)
\]
\[
+ \varepsilon r^3 u r^3 \left( \frac{K_{n-1}(p_{32})K_{n+1}(p_{32})}{p_{32}^2 K_n(p_{32})} \right)
\]
\[
+ \left[ \frac{n - K_{n+1}(p_{32})}{p_{32}^2 K_n(p_{32})} \right] \left[ \varepsilon r^2 u r^3 \left( \frac{A_{n-1}(p_{22})}{p_{22}^2 A_n(p_{22})} \right) + \varepsilon r^3 u r^2 \left( \frac{B_{n-1}(p_{22})}{p_{22}^2 B_n(p_{22})} \right) - \frac{n}{p_{22}^2} (\varepsilon r^2 u r^3 + \varepsilon r^3 u r^2) \right]
\]
\[
= \left[ \frac{n^2}{k_0} \left( \frac{1}{p_{22}} + \frac{1}{p_{32}} \right) \right]^2 - n^2 \left[ \frac{\varepsilon r^2 u r^2}{4 p_{22}} + \frac{\varepsilon r^3 u r^3}{4 p_{32}} \right] = \frac{n^2(\varepsilon r^2 u r^2 + \varepsilon r^3 u r^3)}{p_{22}^2 p_{32}^2} \quad \text{...... A4}
\]

For small \(p_{32}\)
\[
\frac{K_{n+1}(p_{32})}{p_{32}^2 K_n(p_{32})} = \frac{2n}{p_{32}^2} \quad \text{and} \quad \frac{K_0(p_{32})}{p_{32}^2 K_1(p_{32})} = \ln \left( \frac{1}{0.89 p_{32}} \right) \quad \text{...... A5}
\]

Applying these approximations, letting \(n=1\) and multiplying throughout by \(p_{32}^2\) yields
\[
p_{32}^2 \varepsilon r^2 u r^2 \left[ \frac{1}{3} \frac{B_0(p_{22})}{B_1(p_{22})} - \frac{A_0(p_{22})}{A_1(p_{22})} \right] - \left( \frac{A_0(p_{22})B_0(p_{22})}{p_{22}^2 A_1(p_{22})B_1(p_{22})} \right)
\]
\[
- \left[ \frac{\varepsilon r^2 u r^3 A_0(p_{22})}{p_{22}^2 A_1(p_{22})} + \frac{\varepsilon r^3 u r^2 B_0(p_{22})}{p_{22}^2 B_1(p_{22})} \right]
\]
\[
= -2 \varepsilon r^3 u r^3 \ln \left( \frac{1}{0.89 p_{32}} \right) + \frac{(\varepsilon r^2 u r^3)(\varepsilon r^2 - \varepsilon r^3)}{p_{22}^2} \quad \text{...... A6}
\]

From this equation, it can be seen that as \(p_{32}\) approaches zero, the right-hand side becomes infinite, and \(p_{22}\) must approach a zero of \(A_1(p_{22})\), a zero of \(B_1(p_{22})\), or must itself approach zero. Hence, the cutoff condition for the \(HE_{11}\) mode is given by
\[
p_{22} = 0 \quad \text{...... A7}
\]
If \( A_1(p_{22}) = 0 \), then from equations 2.2 \( a_4/a_2 = \frac{J_1(p_{22})}{Y_1(p_{22})} \), ........ A8.a

Similarly, if \( B_1(p_{22}) = 0 \), then \( b_4/b_2 = \frac{J_1(p_{22})}{Y_1(p_{22})} = a_4/a_2 \) ............ A8.b

Hence, at cutoff for all \( n=1 \) modes, \( A_1(p_{22}) = B_1(p_{22}) = 0 \). By letting \( p_{32} \to 0 \) and \( p_{11} \to 0 \) in equation 3.3, it can be shown that equations A8 are satisfied if

\[
\frac{J_1(p_{22})}{Y_1(p_{22})} = \frac{J_1(p_{21})}{Y_1(p_{21})}, \quad \text{..................... A9}
\]

which is the cutoff condition for \( HE_{1m} \) modes, \( m > 1 \).

(iii) \( EH_{1m} \) Modes

By applying a similar limiting procedure on equation 3.2.b as was used on equation 3.2.a, the following result is obtained

\[
\varepsilon r_2^\mu r_1 \left( \frac{A'_1(p_{21})}{p_{21} A_1(p_{21})} \right) + \varepsilon r_1^\mu r_2 \left( \frac{B'_1(p_{21})}{p_{21} B_1(p_{21})} \right) = \frac{\varepsilon r_2^\mu r_2 + \varepsilon r_1^\mu r_1}{p_{21}^2} - \frac{\varepsilon r_1^\mu r_1}{2} \ldots \text{A10}
\]

But from equations A8,

\( B_1(p_{21}) = A_1(p_{21}) \), and equation A10 becomes

\[
\frac{A'_1(p_{21})}{A_1(p_{21})} = \frac{1}{p_{21}} \left( \frac{\varepsilon r_2^\mu r_2 + \varepsilon r_1^\mu r_1}{\varepsilon r_2^\mu r_1 + \varepsilon r_1^\mu r_2} \right) - \frac{p_{21} \varepsilon r_1^\mu r_1}{2(\varepsilon r_2^\mu r_1 + \varepsilon r_1^\mu r_2)} \ldots \text{A11}
\]

Applying equation A8.a yields

\[
\begin{bmatrix}
J'_1(p_{21}) - Y'_1(p_{21}) \\
J'_1(p_{22}) - Y'_1(p_{22})
\end{bmatrix} = \begin{bmatrix}
J_1(p_{21}) - Y_1(p_{21}) \\
J_1(p_{22}) - Y_1(p_{22})
\end{bmatrix} \begin{bmatrix}
1 \left( \frac{\varepsilon r_2^\mu r_2 + \varepsilon r_1^\mu r_1}{\varepsilon r_2^\mu r_1 + \varepsilon r_1^\mu r_2} \right) - \frac{p_{21} \varepsilon r_1^\mu r_1}{2(\varepsilon r_2^\mu r_1 + \varepsilon r_1^\mu r_2)}
\end{bmatrix}
\]

which is the cutoff condition for \( EH_{1m} \) modes, \( m > 1 \).
APPENDIX B Dielectric Tube Field Components in Cartesian Co-ordinates

Using equations 3.14 and 2.2, the field components for the dielectric tube in cartesian co-ordinates are given by

\[0 \leq r \leq r_1\]

\[E_{x1} = \sin \beta \left( \frac{\theta}{h_1} \right) \left( \cos^2 \theta - d_1 \sin^2 \theta \right) I_0(p_1) + \left( \sin^2 \theta - \cos^2 \theta \right) (1 + d_1) \frac{I_1(p_1)}{p_1} \]

\[E_{y1} = \sin \theta \cos \theta \sin \beta \left( \frac{\theta}{h_1} \right) (1 + d_1) \left( I_0(p_1) - \frac{2I_1(p_1)}{p_1} \right) \]

\[E_{z1} = \cos \theta \cos \beta \cdot I_1(p_1) \]

\[H_{x1} = \sin \theta \cos \theta \sin \beta \left( \frac{\theta}{h_1} \right) (1 + e_1) \left( I_0(p_1) - \frac{2I_1(p_1)}{p_1} \right) \]

\[H_{y1} = \sin \beta \left( \frac{\theta}{h_1} \right) \left( \sin^2 \theta - e_1 \cos^2 \theta \right) I_0(p_1) + \left( \cos^2 \theta - \sin^2 \theta \right) (1 + e_1) \frac{I_1(p_1)}{p_1} \]

\[H_{z1} = \sin \theta \cos \beta \cdot I_1(p_1) \]

\[r_1 \leq r \leq r_2\]

\[E_{x2} = -\sin \beta \left( \frac{\theta}{h_2} \right) \left( \cos^2 \theta - d_2 \sin^2 \theta B_0(p_2) + \left( \sin^2 \theta - \cos^2 \theta \right) \frac{1}{p_2} (A_1(p_2) + d_2 B_1(p_2)) \right) \]

\[E_{y2} = -\cos \theta \sin \theta \sin \beta \left( \frac{\theta}{h_2} \right) \left( A_0(p_2) + d_2 B_0(p_2) - \frac{2}{p_2} (A_1(p_2) + d_2 B_1(p_2)) \right) \]

\[E_{z2} = \cos \theta \cos \beta \cdot A_1(p_2) \]

\[H_{x2} = -\cos \theta \sin \theta \sin \beta \left( \frac{\theta}{h_2} \right) \left( e_2 A_0(p_2) + B_0(p_2) - \frac{2}{p_2} (e_2 A_1(p_2) + B_1(p_2)) \right) \]

\[H_{y2} = -\sin \beta \left( \frac{\theta}{h_2} \right) \left( \sin^2 \theta - e_2 \cos^2 \theta \right) A_0(p_2) + \left( \cos^2 \theta - \sin^2 \theta \right) \frac{1}{p_2} (B_1(p_2) + e_2 A_1(p_2)) \]

\[H_{z2} = \sin \theta \cos \beta \cdot B_1(p_2) \]
\( r_2 \leq r \leq \infty \)

\[
E_{x3} = \sin \beta \left( \frac{\beta}{\hbar} \right) \left( -\left( \cos^2 \theta - d_3 \sin^2 \theta \right) K_0 (p_3) + \left( \sin^2 \theta - \cos^2 \theta \right) (1 + d_3) K_1 (p_3) \right) \frac{p_3}{p_3} \\
E_{y3} = -\sin \theta \cos \theta \sin \beta \left( \frac{\beta}{\hbar} \right) (1 + d_3) \left( K_0 (p_3) + \frac{2K_1 (p_3)}{p_3} \right) \\
E_{z3} = \cos \theta \cos \beta \ K_1 (p_3) \\
H_{x3} = -\sin \theta \cos \theta \sin \beta \left( \frac{\beta}{\hbar} \right) (1 + e_3) \left( K_0 (p_3) + \frac{2K_1 (p_3)}{p_3} \right) \\
H_{y3} = \sin \beta \left( \frac{\beta}{\hbar} \right) \left( -\left( \sin^2 \theta - e_3 \cos^2 \theta \right) K_0 (p_3) + \left( \cos^2 \theta - \sin^2 \theta \right) (1 + e_3) \frac{K_1 (p_3)}{p_3} \right) \\
H_{z3} = \sin \theta \cos \beta \ K_1 (p_3)
\]

where
\[ d_i \triangleq \mu r_i \frac{c_i}{\beta} \beta_0 Z_i , \quad e_i \triangleq \frac{e \mu r_i \beta_0}{c_i Z_0 \beta} \]

and
\[ p_i \triangleq h_i r \]

\[ \text{B1.c} \]

\[ \text{B2} \]

\[ \text{B3} \]
PART II

SCREENED SURFACE WAVEGUIDES
7. **INTRODUCTION**

In 1965 Barlow suggested a technique for screening the Goubau surface-wave transmission line\(^{25}\). The proposed waveguide consisted of a coaxial cable with thin dielectric layers on both conductors. The prediction was made that the dual surface wave supported by this structure would have a lower attenuation than that of the coaxial cable with no loading. In a later paper\(^{26}\), Barlow completed the exact analysis for quasi-TM modes, and discovered that one of these modes had no lower cutoff frequency. Calculations also revealed that, for a fixed frequency, it was possible to obtain a minimum of attenuation at some combination of thickness of the dielectric layers, this minimum being significantly lower than the attenuation of the bare coaxial cable. The validity of Barlow's technique of obtaining numerical data has since been questioned by Millington\(^{27}\) whose calculated results show a very gradual increase in attenuation in the case of the lined stripline. Subsequent experimental results\(^{28,29,30,31}\) on lined coaxial cables apparently indicated that a minimum in attenuation existed for loading on the inner, outer, or both conductors. In a recent paper\(^{32}\), Barlow outlines the approximations used in obtaining his numerical results.

Barlow was not, however, the first to consider slow-wave modes propagating in such structures. The lossless characteristic equations for TM modes in these three coaxial cables had been derived earlier by Prache\(^{33}\), but no numerical results were given. The screened Goubau line had been studied by Kaplunov\(^ {34,35,36}\), and by Yoshida\(^{37,38}\) as early as 1952. Both derived the expression for the attenuation coefficient by perturbation for the TM modes, and Kaplunov has given some design data\(^{35}\).

In this work, a perturbation analysis is given for these configurations for TM, TE and hybrid slow-wave modes. It is shown that the analysis is easily extended to include five additional waveguides which have been studied
previously to various degrees of detail by other workers. These structures are
the shielded and unshielded dielectric rod, the dielectric rod in a lined
circular waveguide, the dielectric-coated conductor and the circular waveguide
with a dielectric lining.

The previous work on these waveguides will now be outlined. The
dielectric-coated conductor is the most common form of surface waveguide and it
has been quite thoroughly studied. The original analysis was carried out by
Harms\textsuperscript{39} in 1907. It was not until 1950 that Goubau\textsuperscript{40} demonstrated that the
device had potential as a practical transmission line. He undertook extensive
experimental work\textsuperscript{41} to determine the effects of bends and weather conditions
upon the propagation characteristics, and published a design chart\textsuperscript{42} valid for
small phase-velocity reductions. Extensive experimental work followed by
others\textsuperscript{43,44} to verify the characteristics of the \textsuperscript{TM}01 mode. Other workers gave
approximations\textsuperscript{24,45} for large phase-velocity reductions. In the present work,
a design chart is given which has no restrictions imposed upon the phase
velocity\textsuperscript{46}. Semenov\textsuperscript{24,47} has analysed the \textsuperscript{TM}01 mode exactly, but, in order to
solve the equations, he made approximations for thin coatings, for thick
coatings and for low frequency. The characteristic equation for hybrid modes
has been obtained by several authors\textsuperscript{48,49,50}. The theory of bends in the
surface-wave line has been developed by Suzuki\textsuperscript{51}, and experimental verification
has recently been obtained\textsuperscript{52}.

The dielectric rod has been studied intensively because of its
usefulness both as a waveguide and as an antenna. The characteristic equation
was obtained by Hondros and Debye\textsuperscript{53} in 1910, and its attenuation constant was
derived much later by Elsasser\textsuperscript{54}. Experimental confirmation of his results
was simultaneously given by Chandler\textsuperscript{55}. Measurements on lower-order TM, TE
and hybrid modes followed in 1952 by Horton and McKinney\textsuperscript{56}. Detailed theoretical
studies of higher-order hybrid modes were later published by several authors\textsuperscript{19,21,57}.
The theoretical aspects of bends have recently been investigated by Bohme\textsuperscript{58}.
The case of a dielectric rod in a waveguide has been considered in detail by Clarricoats and his colleagues\textsuperscript{21,59}, both for fast and slow-wave modes. Their work includes the derivation of the characteristic equations for all modes and of the attenuation coefficient for all hybrid modes. Extensive numerical results are given in these publications. Other workers\textsuperscript{33,60} have given results for TM and TE modes in this structure.

The case of a dielectric rod inserted in a circular waveguide with a dielectric lining has not been mentioned in the literature.

The problem of propagating slow waves in a dielectric lined circular waveguide has received only limited attention by a few authors. Beam and Wachowski\textsuperscript{60} have derived and solved the characteristic equations for TM and TE modes, while Loshakov\textsuperscript{61} has derived cutoff conditions for the \textsuperscript{TM01} mode and has given additional numerical data.

The general case of a coaxial cable with dielectric linings on both conductors is analysed in Chapters 8-11. The lossless characteristic equations are derived, and expressions for the attenuation coefficient and group velocity are obtained for all slow-wave modes. The results are extended to seven other configurations as shown in Chapter 8. In Chapter 12, numerical results are given for four of the surface waveguides considered.
The most general form of surface waveguide considered in this work consists of a coaxial cable with dielectric linings on both conductors as shown in figure 8.1. Media 1 and 5 are conductors, media 2 and 4 are dielectric layers with relative permittivities \( \varepsilon_{r2} \) and \( \varepsilon_{r4} \), respectively, and medium 3 is a dielectric region of relative permittivity \( \varepsilon_{r3} \), such that

\[
\begin{align*}
\varepsilon_{r2} &> \varepsilon_{r3} \\
\text{and } \varepsilon_{r4} &> \varepsilon_{r3}
\end{align*}
\]

Additional surface waveguides can be obtained from this configuration by removing one or more of the five media shown in figure 8.1. The eight structures shown in figure 8.2 are some of the possible configurations, with the common characteristic of having at least one interface between two dielectric media of unequal permittivities. These cases have been chosen because the radial dependence of the z-components of electric and magnetic fields can be determined by inspection. This is possible when the following conditions are satisfied by the dielectric media:
FIGURE 8.2 Types of Surface Waveguides

1. Coaxial cable with two dielectric linings
2. Coaxial cable with dielectric lining on inner conductor
3. Dielectric-lined conductor
4. Coaxial cable with dielectric lining on outer conductor
5. Dielectric rod in a lined circular waveguide
6. Dielectric rod in a circular waveguide
7. Dielectric rod
8. Lined circular waveguide
(i) extension to $r=0$ of medium 3 in case 8, and of medium 2 in cases 5, 6, 7;

or (ii) extension to $r=\infty$ of medium 3 in cases 3, 7;

or (iii) adjacency to a conducting boundary of medium 4 in cases 1, 4, 5, 8, and of medium 2 in cases 1, 2, 3, and of medium 3 in cases 2, 4, 6;

or (iv) existence of some radius within the region at which a conducting boundary could be placed (i.e. tangential electric fields are assumed equal to zero at $r=r_m$ in cases 1 and 5).

The analysis of each of the eight waveguides may be carried out independently as a separate boundary-value problem. However, it is also possible to obtain the solutions for waveguides 2-8 from that of 1 by using the step(s) shown schematically in figure 8.3, which involves allowing a radius to assume a limiting value. In addition, certain cases may be obtained from others which are less simple by using the same procedure, e.g. case 5 reduces to case 8 by letting $r_2=0$ in the solution for case 5.

FIGURE 8.3 Steps for Obtaining Solutions for Cases 2-8 from Case 1
9. FIELD COMPONENTS

In this chapter, the field components of hybrid, TM and TE modes are given for the eight configurations shown in figure 8.2. Lossless propagation is assumed in the z-direction, with t-θ-z dependence of the form \( \exp(j(\omega t - n\phi - \beta z)) \). The radial variation of the z-components of electric and magnetic fields are described by Bessel functions of the first and second kinds in media 2 and 4, and by modified Bessel functions of the first and second kinds in medium 3. The remaining field components are found using Maxwell's curl equations. Omitting the factor \( \exp(j(\omega t - n\phi - \beta z)) \), these are given by

\[
\begin{align*}
E_{z2} &= a_2 A_n(h_2^2 r) \\
E_{r2} &= -j \frac{\beta}{h_2^2} a_2 A_n(h_2^2 r) - \frac{n r_2^2 k_o Z_o}{h_2^2 r} b_2 B_n(h_2^2 r) \\
E_{\theta 2} &= - \frac{n\epsilon}{h_2^2} a_2 A_n(h_2^2 r) + j \frac{\mu r_2^2 k_o Z_o}{h_2^2} b_2 B_n(h_2^2 r) \\
H_{z2} &= b_2 B_n(h_2^2 r) \\
H_{r2} &= \frac{n \epsilon r_2^2 k_o}{Z_o h_2^2 r} a_2 A_n(h_2^2 r) - j \frac{\beta}{h_2^2} b_2 B_n'(h_2^2 r) \\
H_{\theta 2} &= - \frac{\epsilon r_2^2 k_o}{Z_o h_2^2 r} a_2 A_n'(h_2^2 r) - \frac{n\epsilon}{h_2^2 r} b_2 B_n(h_2^2 r)
\end{align*}
\]

\[ r_1 \leq r \leq r_2 \]

............... 9.1.a
\[ E_{z3} = a_3 C_n(h_3 r) \]
\[ E_{r3} = j \left( \frac{\beta}{h_3} a_3 C_n(h_3 r) + \frac{n_{\nu r3} k_0 Z_0}{h_3^2} b_3 D_n(h_3 r) \right) \]
\[ E_{\theta 3} = \frac{n_{\beta}}{h_3^2} a_3 C_n(h_3 r) - j \frac{\nu r3}{h_3^2} b_3 D_n'(h_3 r) \]
\[ H_{z3} = b_3 D_n(h_3 r) \]
\[ H_{r3} = -\frac{n_{\epsilon r3} k_0}{Z_0 h_3^2} a_3 C_n(h_3 r) + j \frac{\beta}{h_3^2} b_3 D_n'(h_3 r) \]
\[ H_{\theta 3} = j \frac{\epsilon r3}{h_3^2} a_3 C_n'(h_3 r) + \frac{n_{\beta}}{h_3^2} b_3 D_n(h_3 r) \]
\[ E_{z4} = a_4 F_n(h_4 r) \]
\[ E_{r4} = -j \left( \frac{\beta}{h_4} a_4 F_n(h_4 r) - \frac{n_{\nu r4} k_0 Z_0}{h_4^2} b_4 G_n(h_4 r) \right) \]
\[ E_{\theta 4} = -\frac{n_{\beta}}{h_4^2} a_4 F_n(h_4 r) + j \frac{\nu r4}{h_4^2} b_4 G_n'(h_4 r) \]
\[ H_{z4} = b_4 G_n(h_4 r) \]
\[ H_{r4} = \frac{n_{\epsilon r4} k_0}{Z_0 h_4^2} a_4 F_n(h_4 r) - j \frac{\beta}{h_4^2} b_4 G_n'(h_4 r) \]
\[ H_{\theta 4} = -j \frac{\epsilon r4}{h_4^2} a_4 F_n'(h_4 r) + \frac{n_{\beta}}{h_4^2} b_4 G_n(h_4 r) \]

\[ r_2 \leq r \leq r_3 \]

\[ r_3 \leq r \leq r_4 \]

where, from the wave equation

\[ h_2^2 = \nu r2 \epsilon r2 k_0^2 - \beta^2 \]
\[ h_3^2 = \beta^2 - \nu r3 \epsilon r3 k_0^2 \]
\[ h_4^2 = \nu r4 \epsilon r4 k_0^2 - \beta^2 \]

\[ \text{......... 9.2} \]
The functions $A_n(h_2r)$, $B_n(h_2r)$, $F_n(h_4r)$ and $G_n(h_4r)$ are linear combinations of Bessel functions which are chosen in such a way that the following conditions are satisfied:

(i) fields are finite at $r=0$

(ii) $E_{zi}$, $E_{q1}$ and $H_{ri}$ vanish at $r=r_1$, $r=r_m$ and $r=r_4$.

Also $C_n(h_3r)$ and $D_n(h_3r)$ are linear combinations of modified Bessel functions satisfying the same requirements. These functions for the eight waveguides can be determined by inspection, and are listed in Table 9.1, where

$$p_i \triangleq h_ir$$

and $p_{ij} \triangleq h_ir_j$ .............................. 9.3

<table>
<thead>
<tr>
<th>MEDIUM NUMBER</th>
<th>GUIDE NUMBER</th>
<th>$E_{zi}$ DESCRIBING FUNCTIONS</th>
<th>$H_{zi}$ DESCRIBING FUNCTIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1,2,3</td>
<td>$A_n(p_2) = J_n(p_2) - \frac{J_n(p_{21})}{Y_n(p_{21})} Y_n(p_2)$</td>
<td>$B_n(p_2) = J_n(p_2) - \frac{J_n(p_{21})}{Y_n(p_{21})} Y_n(p_2)$</td>
</tr>
<tr>
<td></td>
<td>5,6,7</td>
<td>$A_n(p_2) = J_n(p_2)$</td>
<td>$B_n(p_2) = J_n(p_2) = A_n(p_2)$</td>
</tr>
<tr>
<td>3</td>
<td>1,5</td>
<td>$C_n(p_3) = K_n(p_3) - \frac{K_n(p_{3m})}{I_n(p_{3m})} I_n(p_3)$</td>
<td>$D_n(p_3) = K_n(p_3) - \frac{K_n(p_{3m})}{I_n(p_{3m})} I_n(p_3)$</td>
</tr>
<tr>
<td></td>
<td>2,6</td>
<td>$C_n(p_3) = K_n(p_3) - \frac{K_n(p_{34})}{I_n(p_{34})} I_n(p_3)$</td>
<td>$D_n(p_3) = K_n(p_3) - \frac{K_n(p_{34})}{I_n(p_{34})} I_n(p_3)$</td>
</tr>
<tr>
<td></td>
<td>3,7</td>
<td>$C_n(p_3) = K_n(p_3)$</td>
<td>$D_n(p_3) = K_n(p_3) = C_n(p_3)$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>$C_n(p_3) = K_n(p_3) - \frac{K_n(p_{31})}{I_n(p_{31})} I_n(p_3)$</td>
<td>$D_n(p_3) = K_n(p_3) - \frac{K_n(p_{31})}{I_n(p_{31})} I_n(p_3)$</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>$C_n(p_3) = I_n(p_3)$</td>
<td>$D_n(p_3) = I_n(p_3) = C_n(p_3)$</td>
</tr>
<tr>
<td></td>
<td>1,4,5</td>
<td>$F_n(p_4) = J_n(p_4) - \frac{J_n(p_{44})}{Y_n(p_{44})} Y_n(p_4)$</td>
<td>$G_n(p_4) = J_n(p_4) - \frac{J_n(p_{44})}{Y_n(p_{44})} Y_n(p_4)$</td>
</tr>
</tbody>
</table>

**TABLE 9.1** Functions Describing $E_{zi}$ and $H_{zi}$
Upon setting \( n=0 \) (no e-variation), equations 9.1 separate into the two independent sets of circularly symmetric modes designated \( \text{TM}_{0m} \) and \( \text{TE}_{0m} \). The corresponding field components are given by:

**TM MODES**

\[
\begin{align*}
E_{z2} &= a_2 A_0(h_2 r) \\
E_{r2} &= -j \frac{b}{h_2} a_2 A_0'(h_2 r) \\
H_{\theta 2} &= \frac{e_{r2}}{Z_0} \frac{k_0}{\beta} E_{r2}
\end{align*}
\]

**TE MODES**

\[
\begin{align*}
H_{z2} &= b_2 B_0(h_2 r) \\
H_{r2} &= -j \frac{b}{h_2} b_2 B_0'(h_2 r) \\
E_{\theta 2} &= -\frac{\mu r2}{k_0 Z_0} H_{r2}
\end{align*}
\]

\[ r_1 \leq r \leq r_2 \]

\[
\begin{align*}
E_{z3} &= a_3 C_0(h_3 r) \\
E_{r3} &= j \frac{b}{h_3} a_3 C_0'(h_3 r) \\
H_{\theta 3} &= \frac{e_{r3}}{Z_0} \frac{k_0}{\beta} E_{r3}
\end{align*}
\]

**TE MODES**

\[
\begin{align*}
H_{z3} &= b_3 D_0(h_3 r) \\
H_{r3} &= j \frac{b}{h_3} b_3 D_0'(h_3 r) \\
E_{\theta 3} &= -\frac{\mu r3}{k_0 Z_0} H_{r3}
\end{align*}
\]

\[ r_2 \leq r \leq r_3 \]

\[
\begin{align*}
E_{z4} &= a_4 F_0(h_4 r) \\
E_{r4} &= -j \frac{b}{h_4} a_4 F_0'(h_4 r) \\
H_{\theta 4} &= \frac{e_{r4}}{Z_0} \frac{k_0}{\beta} E_{r4}
\end{align*}
\]

**TE MODES**

\[
\begin{align*}
H_{z4} &= b_4 G_0(h_4 r) \\
H_{r4} &= -j \frac{b}{h_4} b_4 G_0'(h_4 r) \\
E_{\theta 4} &= -\frac{\mu r4}{k_0 Z_0} H_{r4}
\end{align*}
\]

\[ r_3 \leq r \leq r_4 \]

If \( n \neq 0 \), equations 9.1 describe inseparable combinations of TE and TM modes which are designated hybrid modes. The hybrid modes may be of the EH or HE type and a mode designation could be devised for each of the waveguides under consideration.
10. **CHARACTERISTIC EQUATIONS**

The characteristic equations are obtained by matching the field components at $r=r_2$ and $r=r_3$, and eliminating the arbitrary constants from the resulting equations. Matching tangential field components at $r=r_2$ using equations 9.1 yields four homogeneous equations in the four unknowns $a_2$, $b_2$, $a_3$, and $b_3$, one of which is arbitrary. These are expressed in the following convenient form

$$
\begin{align*}
\frac{a_2}{a_3} &= \frac{C_n(p_{32})}{A_n(p_{22})} \\
\frac{b_2}{b_3} &= \frac{D_n(p_{32})}{B_n(p_{22})} \\
\frac{b_3}{a_3} &= -j \left( \frac{C_n(p_{32})}{D_n(p_{32})} \right) \left[ \frac{\varepsilon_{r2}}{\varepsilon_{r3}} \frac{A_n'(p_{22})}{A_n(p_{22})} + \frac{\varepsilon_{r3}}{\varepsilon_{r3}} \frac{C_n'(p_{32})}{C_n(p_{32})} \right]^{-1} \left[ \frac{n \beta}{k_0} \left( \frac{1}{p_{22}} + \frac{1}{p_{32}} \right) \right] \\
\frac{b_2}{a_2} &= \frac{b_3}{a_3} \frac{a_3}{a_2} \frac{b_2}{b_3} \triangleq -jc_2
\end{align*}
$$

In a similar manner, matching tangential fields at $r=r_3$, one obtains
\[
\frac{a_4}{a_3} = \frac{C_n(p_{33})}{F_n(p_{43})}
\]
\[
\frac{b_4}{b_3} = \frac{D_n(p_{33})}{G_n(p_{43})}
\]
\[
\frac{b_3}{a_3} = -j \left( \frac{C_n(p_{33})}{D_n(p_{33})} \right) \left[ \frac{\varepsilon_r 4}{p_{43}} \frac{F_n(p_{43})}{F_n(p_{43})} + \frac{\varepsilon_r 3}{p_{33}} \frac{C_n(p_{33})}{C_n(p_{33})} \right]^{-1} \left[ \frac{n_\beta z_0}{k_0} \left( \frac{1}{2} + \frac{1}{2} \right) \right]^{-1}
\]
\[
\frac{b_4}{a_4} = \frac{b_3}{a_3} \frac{a_3}{a_4} \frac{b_4}{b_3} \triangleq -jc_3
\]

Equating the two expressions for \( c_2 \) from equations 10.1, and those for \( c_4 \) from equations 10.2, yields the following two equations

\[
\begin{bmatrix}
\varepsilon_{r2} \frac{A_n(p_{22})}{A_n(p_{22})} + \varepsilon_{r3} \frac{C_n(p_{32})}{C_n(p_{32})} \\
\varepsilon_{r2} \frac{B_n(p_{22})}{B_n(p_{22})} + \varepsilon_{r3} \frac{D_n(p_{32})}{D_n(p_{32})}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{r2} \frac{C_n(p_{32})}{C_n(p_{32})} + \varepsilon_{r3} \frac{C_n(p_{33})}{C_n(p_{33})} \\
\varepsilon_{r2} \frac{D_n(p_{32})}{D_n(p_{32})} + \varepsilon_{r3} \frac{D_n(p_{33})}{D_n(p_{33})}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{r4} \frac{G_n(p_{43})}{G_n(p_{43})} + \varepsilon_{r3} \frac{G_n(p_{43})}{G_n(p_{43})} \\
\varepsilon_{r4} \frac{D_n(p_{43})}{D_n(p_{43})} + \varepsilon_{r3} \frac{D_n(p_{43})}{D_n(p_{43})}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{r4} \frac{C_n(p_{43})}{C_n(p_{43})} + \varepsilon_{r3} \frac{D_n(p_{33})}{D_n(p_{33})} \\
\varepsilon_{r4} \frac{D_n(p_{43})}{D_n(p_{43})} + \varepsilon_{r3} \frac{D_n(p_{33})}{D_n(p_{33})}
\end{bmatrix}
\begin{bmatrix}
\frac{n_\beta z_0}{k_0} \left( \frac{1}{2} + \frac{1}{2} \right) \\
\frac{n_\beta z_0}{k_0} \left( \frac{1}{2} + \frac{1}{2} \right)
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} + \frac{1}{2} \\
\frac{1}{2} + \frac{1}{2}
\end{bmatrix}
\end{array}
\]

Cases 1 and 5 require the simultaneous solution of both equations for \( \beta \) and \( r_m \). However, \( r_m \) does not exist for the remaining cases and only equation 10.3 is required for cases 2, 3, 6 and 7, while equation 10.4 is applicable to cases 4 and 8. In all cases, the relationships among the various wave-numbers are given by equations 9.2.

Upon setting \( n=0 \), each of equations 10.3 and 10.4 separates into two factors, each of which equals zero. These correspond to TM and TE modes.
The equations for TM modes are given by
\[
\begin{align*}
\left[ \frac{\varepsilon_r^2 A'_0(p_{22})}{p_{22} A_0(p_{22})} + \frac{\varepsilon_r^3 C'_0(p_{32})}{p_{32} C_0(p_{32})} \right] = 0 \quad \cdots \quad 10.5
\end{align*}
\]
\[
\begin{align*}
\left[ \frac{\varepsilon_r^4 F'_0(p_{43})}{p_{43} F_0(p_{43})} + \frac{\varepsilon_r^3 C'_0(p_{33})}{p_{33} C_0(p_{33})} \right] = 0 \quad \cdots \quad 10.6
\end{align*}
\]
and for TE modes
\[
\begin{align*}
\left[ \frac{\mu_r^2 B'_0(p_{22})}{p_{22} B_0(p_{22})} + \frac{\mu_r^3 D'_0(p_{32})}{p_{32} D_0(p_{32})} \right] = 0 \quad \cdots \quad 10.7
\end{align*}
\]
\[
\begin{align*}
\left[ \frac{\mu_r^4 C'_0(p_{43})}{p_{43} C_0(p_{43})} + \frac{\mu_r^3 D'_0(p_{33})}{p_{33} D_0(p_{33})} \right] = 0 \quad \cdots \quad 10.8
\end{align*}
\]
For TM modes, solutions in cases 2, 3, 6 and 7 are obtained by using equation 10.5, while equation 10.6 is applicable to cases 4 and 8. The simultaneous solution of both equations is necessary in cases 1 and 5. A similar situation exists when dealing with equations 10.7 and 10.8 for TE modes.
11. GROUP VELOCITY AND ATTENUATION COEFFICIENT

In this chapter, expressions for group velocity and attenuation coefficient are derived by employing the same techniques as were used in Part I. The phase coefficients for hybrid, TM and TE modes can be found by solving the characteristic equations obtained in Chapter 10 for $\beta$, from which the phase velocity is readily found.

11.1 Group Velocity

Expressions for group velocity of all slow-wave modes are found by integrating the appropriate combinations of the lossless field components to obtain the power flow and energy storage per unit length, which determines the rate of transport of energy given by

$$v_g = \frac{N}{W} = \sum_{i=2}^{4} N_i / \sum_{i=2}^{4} W_i , \quad \dot{11.1}$$

where $N_i$ and $W_i$ are the power carried in medium $i$ and the energy storage per unit length in medium $i$, respectively. These quantities are given by

$$N_i = \frac{\pi}{2} \int_{r_{i-1}}^{r_i} (E_{ri} H_{0i}^* - E_{0i} H_{ri}^*) r \, dr , \quad i=1, 2, 3 \quad \dot{11.2}$$

and

$$W_i = \left(\frac{\pi}{2} \frac{e_{ri}}{\varepsilon_0 Z_0} \right) \int_{r_{i-1}}^{r_i} (E_{zi} E_{zi}^* + E_{ri} E_{ri}^* + E_{0i} E_{0i}^*) r \, dr , \quad i=1,2,3 \quad \dot{11.3}$$

For hybrid modes, by substituting equations 9.1 into 11.2 and 11.3, one obtains
\[ N_2 = \frac{a_3^{3, \pi}}{4h_2} \left( \frac{C_2(p_{32})}{A_n(p_{32})} \right) \left[ \frac{\varepsilon r^2 k_0 \beta}{Z_0} \right] \frac{s_A + \mu r_3 k_0 Z_0 \gamma c_2}{S_B + 2nc_2(\beta^2 + \mu r_2 \gamma r_2 k_0^2)} \] 

\[ N_3 = \frac{a_3^{a_3 \pi}}{4h_3} \left[ \frac{\varepsilon r^3 k_0 \beta}{Z_0} \right] \frac{s_C + \mu r_3 k_0 Z_0 \gamma c_2}{S_D + 2nc_3(\beta^2 + \mu r_3 \gamma r_3 k_0^2)} \] 

\[ N_4 = \frac{a_3^{a_3 \pi}}{4h_4} \left[ \frac{C_4(p_{33})}{F_n(p_{43})} \right] \left[ \frac{\varepsilon r^4 k_0 \beta}{Z_0} \right] \frac{s_F + \mu r_4 k_0 Z_0 \gamma c_4}{S_G + 2nc_4(\beta^2 + \mu r_4 \gamma r_4 k_0^2)} \] 

\[ W_2 = \frac{\varepsilon r^2 a_3 \pi}{4z_0 h_2} \left[ \frac{C_2(p_{32})}{A_n(p_{22})} \right] \frac{h^2 T_A + \beta^2 S_A + (\mu r_2 k_0 Z_0 \gamma c_2)^2}{S_B + 4nc_2 \mu r_2 \gamma r_2 k_0^2} \] 

\[ W_3 = \frac{\varepsilon r^3 a_3 \pi}{4z_0 h_3} \left[ \frac{C_3(p_{33})}{A_n(p_{33})} \right] \frac{h^2 T_c + \beta^2 S_C + (\mu r_3 k_0 Z_0 \gamma c_3)^2}{S_D + 4nc_3 \mu r_3 \gamma r_3 k_0^2} \] 

\[ W_4 = \frac{\varepsilon r^4 a_3 \pi}{4z_0 h_4} \left[ \frac{C_4(p_{34})}{F_n(p_{44})} \right] \frac{h^2 T_F + \beta^2 S_F + (\mu r_4 k_0 Z_0 \gamma c_4)^2}{S_G + 4nc_4 \mu r_4 \gamma r_4 k_0^2} \] 

The integrals \( S_A, S_B, S_C, S_D, S_F, S_G, S_{AB}, S_{CD}, S_{FG}, T_A, T_C \) and \( T_F \) are functions of Bessel functions which are defined by

\[ S_A = 2 \int_{p_{21}}^{p_2} \left( A_n(p_2) p_2 + \frac{n^2 A_n(p_2)}{p_2} \right) dp_2, \quad T_A = 2 \int_{p_{21}}^{p_2} A_n(p_2) p_2 dp_2 \] 

\[ S_B = \text{SAME AS } S_A \text{ WITH } A \text{ REPLACED BY } B \] 

\[ S_{AB} = - \int_{p_{21}}^{p_2} \left( A_n(p_2) B_n(p_2) + A_n(p_2) B_n(p_2) \right) dp_2 \] 

\[ \ldots 11.6.a \]
\[ S_c = 2 \int_{p_{32}}^{p_{33}} \left( C_n^2(p_3)p_3 + \frac{n^2 C_n^2(p_3)}{p_3} \right) dp_3, \quad T_c = 2 \int_{p_{32}}^{p_{33}} C_n^2(p_3)p_3 dp_3 \]

\[ S_d = \text{SAME AS } S_c \text{ WITH } C \text{ REPLACED BY } D \]

\[ S_{cd} = - \int_{p_{32}}^{p_{33}} \left( C_n(p_3)D_n(p_3) + C_n^2(p_3)D_n(p_3) \right) dp_3 \]

\[ S_f = 2 \int_{p_{43}}^{p_{44}} \left( F_n^2(p_4)p_4 + \frac{n^2 F_n^2(p_4)}{p_4} \right) dp_4, \quad T_f = 2 \int_{p_{43}}^{p_{44}} F_n^2(p_4)p_4 dp_4 \]

\[ S_g = \text{SAME AS } S_f \text{ WITH } F \text{ REPLACED BY } G \]

\[ S_{fg} = - \int_{p_{43}}^{p_{44}} \left( F_n(p_4)G_n(p_4) + F_n^2(p_4)G_n(p_4) \right) dp_4 \]

All of the integrals defined by equations 11.6 can be evaluated in closed form. The limits of integration for the individual structures are determined from figures 8.2 and 8.3. These are summarized in Table 11.1, where an asterisk shown in a certain case denotes the absence of a region in that case.

<table>
<thead>
<tr>
<th>CASE</th>
<th>LIMITS OF INTEGRATION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( p_{21} )</td>
</tr>
<tr>
<td>1</td>
<td>( p_{21} )</td>
</tr>
<tr>
<td>2</td>
<td>( p_{21} )</td>
</tr>
<tr>
<td>3</td>
<td>( p_{21} )</td>
</tr>
<tr>
<td>4</td>
<td>*</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>*</td>
</tr>
</tbody>
</table>

TABLE 11.1 Limits of Integration for the Eight Waveguides
All of the integrals defined by equations 11.6 vanish at \( r=0 \) and \( r=\infty \), and in addition, at metallic boundaries, certain functions contained in the evaluated integrals also vanish. These can be determined by inspection and result in obtaining the simplified integrals listed in Tables 11.2 and 11.3 for the eight waveguides.

<table>
<thead>
<tr>
<th>CASES</th>
<th>EVALUATED INTEGRALS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,2,3</td>
<td>( T_A = p_{22}^2 A_n^{12}(p_{22}) + (p_{22}^2-n^2)A_n^{2}(p_{22}) - p_{21}^2 A_n^{12}(p_{21}) )</td>
</tr>
<tr>
<td>5,6,7</td>
<td>( T_A = p_{22}^2 A_n^{12}(p_{22}) + (p_{22}^2-n^2)A_n^{2}(p_{22}) )</td>
</tr>
<tr>
<td>1,2,3</td>
<td>( S_B = p_{22}^2 B_n^{12}(p_{22}) + (p_{22}^2-n^2)B_n^{2}(p_{22}) + 2p_{22}B_n(p_{22})B_n^{1}(p_{22}) - (p_{21}^2-n^2)B_n^{2}(p_{21}) )</td>
</tr>
<tr>
<td>5,6,7</td>
<td>( S_B = S_A = T_A + 2p_{22} A_n(p_{22})A_n^{1}(p_{22}) )</td>
</tr>
<tr>
<td>1,2,3</td>
<td>( S_A = T_A + 2p_{22} A_n(p_{22})A_n^{1}(p_{22}) )</td>
</tr>
<tr>
<td>1,4,5,8</td>
<td>( T_F = p_{44}^2 F_n^{12}(p_{44}) - \left( p_{43}^2 F_n^{12}(p_{43}) + (p_{43}^2-n^2)F_n^{2}(p_{43}) \right) )</td>
</tr>
<tr>
<td>1,4,5,8</td>
<td>( S_G = (p_{44}^2-n^2)G_n^{2}(p_{44}) - \left( p_{43}^2 G_n^{12}(p_{43}) + (p_{43}^2-n^2)G_n^{2}(p_{43}) + 2p_{43}G_n(p_{43})G_n(p_{43}) \right) )</td>
</tr>
<tr>
<td>1,4,5,8</td>
<td>( S_F = T_F - 2p_{43} F_n(p_{43})F_n^{1}(p_{43}) )</td>
</tr>
</tbody>
</table>

**TABLE 11.2 Evaluated Integrals for Media 2 and 4**
### Table 11.3 Evaluated Integrals for Medium 3

The equations for $N_i$ and $W_i$ for TE and TM modes ($n=0$) are obtained by using equations 9.4 for the field components in equations 11.2 and 11.3. This yields for TE modes

<table>
<thead>
<tr>
<th>CASES</th>
<th>EVALUATED INTEGRALS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,5</td>
<td>$T_C = (p_{33}^2+n^2)c_n^2(p_{33}) - p_{33} c_n^2(p_{33}) - \left( (p_{32}^2+n^2)c_n^2(p_{32}) - p_{32} c_n^2(p_{32}) \right)$</td>
</tr>
<tr>
<td>2,6</td>
<td>$T_C = -p_{34} c_n^2(p_{34}) - \left( (p_{32}^2+n^2)c_n^2(p_{32}) - p_{32} c_n^2(p_{32}) \right)$</td>
</tr>
<tr>
<td>3,7</td>
<td>$T_C = -\left( (p_{32}^2+n^2)c_n^2(p_{32}) - p_{32} c_n^2(p_{32}) \right)$</td>
</tr>
<tr>
<td>4</td>
<td>$T_C = (p_{33}^2+n^2)c_n^2(p_{33}) - p_{33} c_n^2(p_{33}) + p_{31} c_n^2(p_{31})$</td>
</tr>
<tr>
<td>8</td>
<td>$T_C = (p_{33}^2+n^2)c_n^2(p_{33}) - p_{33} c_n^2(p_{33})$</td>
</tr>
<tr>
<td>1,5</td>
<td>$S_D = p_{33} d_n^2(p_{33}) - (p_{33}^2+n^2)d_n^2(p_{33}) + 2p_{33} d_n(p_{33})d_n'(p_{33})$ [ - \left( p_{32} d_n^2(p_{32}) - (p_{32}^2+n^2)d_n^2(p_{32}) + 2p_{32} d_n(p_{32})d_n'(p_{32}) \right)$</td>
</tr>
<tr>
<td>2,6</td>
<td>$S_D = -\left( p_{34}^2+n^2\right)d_n^2(p_{34})$ [ - \left( p_{32} d_n^2(p_{32}) - (p_{32}^2+n^2)d_n^2(p_{32}) + 2p_{32} d_n(p_{32})d_n'(p_{32}) \right)$</td>
</tr>
<tr>
<td>3,7</td>
<td>$S_D = S_C = -T_C - 2p_{32} c_n(p_{32})c_n'(p_{32})$ [ S_{CD} = c_n^2(p_{32})$</td>
</tr>
<tr>
<td>4</td>
<td>$S_D = p_{33} d_n^2(p_{33}) - (p_{33}^2+n^2)d_n^2(p_{33}) + 2p_{33} d_n(p_{33})d_n'(p_{33})$</td>
</tr>
<tr>
<td>8</td>
<td>$S_D = S_C = -T_C + 2p_{33} c_n(p_{33})c_n'(p_{33})$ [ S_{CD} = -c_n^2(p_{33})$</td>
</tr>
<tr>
<td>1,5</td>
<td>$S_C = -T_C + 2p_{33} c_n(p_{33})c_n'(p_{33}) - 2p_{32} c_n(p_{32})c_n'(p_{32})$</td>
</tr>
<tr>
<td>1,5</td>
<td>$S_{CD} = -c_n(p_{33})d_n(p_{33}) + c_n(p_{32})d_n(p_{32})$</td>
</tr>
<tr>
<td>2,6</td>
<td>$S_C = -T_C - 2p_{32} c_n(p_{32})c_n'(p_{32})$ [ S_{CD} = c_n(p_{32})d_n(p_{32})$</td>
</tr>
<tr>
<td>4</td>
<td>$S_C = -T_C + 2p_{33} c_n(p_{33})c_n'(p_{33})$ [ S_{CD} = -c_n(p_{33})d_n(p_{33})$</td>
</tr>
</tbody>
</table>
For TM modes, the corresponding relations are

\[ N_2 = \frac{\mu r e_i k_o e b_z b_3^* \left( d_0^2(p_{32}) \right)}{2 h_2^4 \left( b_0^2(p_{22}) \right)} S_B \] .......................... 11.7

\[ N_3 = \frac{\mu r e_i k_o e b_z b_3^*}{2 h_3^4} S_D \]

\[ N_4 = \frac{\mu r e_i k_o e b_z b_3^* \left( d_0^2(p_{33}) \right)}{2 h_4^4 \left( a_0^2(p_{43}) \right)} S_G \]

\[ W_i = \frac{\mu r e_i k_o N_i}{\beta v_0} \] .......................... 11.8

11.2 Attenuation Coefficient

The attenuation coefficients of the eight waveguides are found by using the perturbation method employed in Part I. The attenuation
coefficient of any mode is given by

\[ \alpha = \frac{P_a}{2N} = (1/2) \sum_{i=1}^{5} P_i / \sum_{i=2}^{4} N_i, \]  

................. 11.12

where the dielectric power loss per unit length is given by

\[ P_i = k_0 \nu \tan \delta_i W_i, \quad i=2,3,4 \]  

................. 11.13

The expressions for \( N_i, W_2, W_3 \) and \( W_4 \) are given in Section 11.1. The power loss per unit length in the \( i^{th} \) conducting medium is given by

\[ P_i = (1/4) R_i \int_0^{2\pi} \left( H_z j H_z^* + H_{0j} H_{0j}^* \right) r d\theta \]  

\[ r = \begin{cases} r_1 & \text{if } i=1 \\ r_4 & \text{if } i=5 \end{cases} \]  

................. 11.14

\( i \neq 1 \) or 5 and \( j \) is the dielectric medium number bounded by \( r_1 \) or \( r_4 \), respectively, and

\[ R_i = \sqrt{(k_0 l_0 / 2 \sigma_i)} \]  

.......................... 11.15

Substituting equations 9.1 into 11.14 yields for hybrid modes

\[ P_1 = \frac{r_1 R_1 a_3 a_3^*}{2h_2} \left[ \frac{C_n(p_{32})^2}{A_n(p_{22})} \right] \left[ c_2^2 \left( h_2^2 + \frac{n_2^2 \beta^2}{p_{21}} \right) B_n^2(p_{21}) + \left( \frac{\epsilon_r^2 k_0 A_n'(p_{21})}{Z_0} \right)^2 \right] 

- \frac{2nc_2 \epsilon_r^2 k_0 \beta}{Z_0 p_{21}} A_n(p_{21}) B_n(p_{21}) \right] \]  

CASES 1,2,3

\[ P_1 = \frac{r_1 R_1 a_3 a_3^*}{2h_3} \left[ \frac{C_n(p_{31})^2}{A_n(p_{31})} \right] \left[ c_3^2 \left( h_3^2 + \frac{n_3^2 \beta^2}{p_{31}} \right) D_n^2(p_{31}) + \left( \frac{\epsilon_r^3 k_0 C_n'(p_{31})}{Z_0} \right)^2 \right] 

- \frac{2nc_3 \epsilon_r^3 k_0 \beta}{Z_0 p_{31}} C_n(p_{31}) D_n(p_{31}) \right] \]  

CASE 4

\[ P_5 = \frac{r_4 R_5 a_3 a_3^*}{2h_4} \left[ \frac{C_n(p_{33})^2}{F_n(p_{43})} \right] \left[ c_4^2 \left( h_4^2 + \frac{n_4^2 \beta^2}{p_{44}} \right) G_n^2(p_{44}) + \left( \frac{\epsilon_r^4 k_0 F_n'(p_{44})}{Z_0} \right)^2 \right] 

- \frac{2nc_4 \epsilon_r^4 k_0 \beta}{Z_0 p_{44}} F_n(p_{44}) G_n(p_{44}) \right] \]  

CASES 1,4,5,8
and

\[ P_5 = \frac{r_4 R_5 a_3 a_3^\pi}{2h_3^2} \left[ \frac{2}{3} \left( \frac{2 \pi}{n_b^2} \frac{n_b^2}{p_{34}} \right) D_n^2(p_{34}) + \left( \frac{\epsilon r_3 k_0 C_n(p_{34})}{Z_0} \right)^2 \right] \]

\[ - \frac{2nc_3\epsilon r_3 k_0}{Z_0 p_{34}} C_n(p_{34}) D_n(p_{34}) \]

CASES 2,6 11.16

The corresponding equations for TE modes are obtained from equations 9.4 and 11.14.

\[ P_1 = r_1 R_1 b_3 b_3^\pi D_0^2(p_{32}) \left( \frac{B_0^2(p_{21})}{B_0^2(p_{22})} \right) \]

CASES 1,2,3

\[ P_1 = r_1 R_1 b_3 b_3^\pi D_0^2(p_{31}) \]

CASE 4

\[ P_5 = r_4 R_5 b_3 b_3^\pi D_0^2(p_{33}) \left( \frac{G_0^2(p_{43})}{G_0^2(p_{44})} \right) \]

CASES 1,4,5,8

\[ P_5 = r_4 R_5 b_3 b_3^\pi D_0^2(p_{34}) \]

CASES 2,6 11.17

and for TM modes

\[ P_1 = r_1 R_1 a_3 a_3^\pi \left( \frac{\epsilon r_2 k_0 C_0(p_{32})}{Z_0 h_2} \frac{A_0(p_{21})}{A_0(p_{22})} \right)^2 \]

CASES 1,2,3

\[ P_1 = r_1 R_1 a_3 a_3^\pi \left( \frac{\epsilon r_3 k_0 C_0(p_{31})}{Z_0 h_3} \right)^2 \]

CASE 4

\[ P_5 = r_4 R_5 a_3 a_3^\pi \left( \frac{\epsilon r_4 k_0 C_0(p_{33})}{Z_0 h_4} \frac{F_0(p_{44})}{F_0(p_{43})} \right)^2 \]

CASES 1,4,5,8

\[ P_5 = r_4 R_5 a_3 a_3^\pi \left( \frac{\epsilon r_3 k_0 C_0(p_{34})}{Z_0 h_3} \right)^2 \]

CASES 2,6 11.18
In this chapter, numerical results are given for four of the eight waveguides described previously. These are the Goubau line (case 3), and the three configurations of the dielectric-lined coaxial cables (cases 1, 2 and 4).

Calculations were made on the Goubau line with no restrictions on its radial dimensions, and the results are displayed in the form of design charts. Detailed calculations were also carried out on the three coaxial waveguides studied by Barlow, since they, like case 3, have no lower cutoff frequency when operated in the TM_{01} mode. Accurate evaluation of $\alpha$ and $\beta$ was made in order to ascertain Barlow's prediction of a reduced attenuation below that of the unloaded coaxial cable.

Results for the unshielded and shielded dielectric rod, cases 7 and 6, are not given, since these structures have been studied in detail by Clarricoats. Also, no numerical results are given for the remaining two waveguides, cases 5 and 8, because all modes in these two structures have a lower cutoff frequency.

12.1 **Goubau's Surface-wave Transmission Line**

12.1.1 **Design Charts**

The object of the presentation of design data is to provide a graphical solution for $\alpha$ and $\beta$ of the Goubau line. The chart given here is applicable in the case of polyethylene-coated copper conductors in free space ($\epsilon_{r2}=2.26$, $\epsilon_{r3}=1.0$, $\tan\delta_3=0$, $\sigma_1=5.8 \cdot 10^5$ mhos/cm), for arbitrary values of $\tan\delta_2$. It displays four parameters: $r_2/r_1$, $\lambda/2r_1$, percent phase-velocity reduction, $\Delta v_p$, and a line impedance, $Z$, defined by

$$Z = 2N/I^*,$$

where $I$ is the total peak conduction current. The evaluation of $Z$ requires the determination of $a_3^*\delta_3$, which appears in the expression for $N$, in terms of
I. This is achieved by applying Ampère’s circuital law at \( r = r_1 \), where \( H_{02} \) is given in equations 9.4.a. Thus,

\[
I = \frac{-j2\pi \varepsilon_{r_2} k_0 a_2 A_0(p_{21})}{h^2 Z_0 r_2}, \quad \text{............... 12.2}
\]

from which \( a_2 \) can be determined in terms of \( I \). Using equations 10.1, which relate \( a_2 \) and \( a_3 \), one obtains

\[
a_3^* = \left[ \frac{h Z_0 A_0(p_{22})}{2\pi r_2 k_0 A_0(p_{21}) A_0(p_{22})} \right]^2 I^* \quad \text{............... 12.3}
\]

Substituting in equation 12.1 from 12.3 and 11.9 and making use of the relation \( C_0(p_{32}) = K_0(p_{32}) \), from Table 9.1, the line impedance becomes

\[
Z = \frac{\beta Z_0}{4\pi k_0} \left\{ \frac{1}{\varepsilon_{r_2}} \left[ \frac{A_0^2(p_{22})}{A_0^2(p_{21})} + \frac{A_0^2(p_{22})}{A_0^2(p_{21})} + \frac{2A_0(p_{22})A_0^*(p_{22})}{p_{22} A_0^2(p_{21})} - 1 \right] \right. \\
+ \left. \frac{1}{\varepsilon_{r_3}} \left( \frac{r_2 A_0^*(p_{22})}{r_1 A_0(p_{21})} \right)^2 \left[ \frac{K_0^2(p_{32})}{K_0^2(p_{32})} - \frac{2K_0(p_{32})}{p_{32} K_0(p_{32})} - 1 \right] \right\} \quad \text{............... 12.4}
\]

The percentage phase-velocity reduction of the line is given by

\[
\Delta v_p = 100(\beta - k_0)/\beta \quad \text{............... 12.5}
\]

Substituting in 10.5 from Table 9.1, and performing the differentiations, the characteristic equation for TM modes becomes

\[
\varepsilon_{r_2} \frac{J_1(p_{22}) - (J_0(p_{21})/Y_0(p_{21})) Y_1(p_{22})}{p_{22} J_0(p_{22}) - (J_0(p_{21})/Y_0(p_{21})) Y_0(p_{22})} + \frac{\varepsilon_{r_3} K_1(p_{32})}{p_{32} K_0(p_{32})} = 0 \quad \text{............... 12.6}
\]

It can be proven that the parameters \( \varepsilon_{r_2}, \varepsilon_{r_3}, \frac{r_2}{r_1} \) and \( \frac{\lambda}{2r_1} \) uniquely define \( Z \) from equation 12.4, and \( \Delta v_p \) from equations 12.6 and 12.5. Fixing \( \varepsilon_{r_2} \) and \( \varepsilon_{r_3} \), a design chart can be constructed by plotting \( r_2/r_1 \) against \( Z \), first for constant \( \Delta v_p \), and then for constant \( \lambda/2r_1 \). Figure 12.1.a shows such a chart.
FIGURE 12.1.a Surface-wave Transmission Line Design Chart

with the following ranges of parameters: \( Z = 0 \) to 500 ohms, \( r_2/r_1 = 1 \) to 100, \( \Delta v_p = 0 \) to 33\% and \( \lambda/2r_1 = 1 \) to \( 10^4 \). An expansion of that portion of figure 12.1.a for \( r_2/r_1 \) from 0 to 5 and \( Z \) from 0 to 120 ohms is shown in figure 12.1.b, which provides better accuracy for lines operated at millimeter-wave frequencies.
FIGURE 12.1.b Surface-wave Transmission Line Design Chart for the Millimeter-wave Region

$\varepsilon_r = 2.26$
$\varepsilon_r = 1.0$

$\frac{r_2}{r_1}$

$Z$, ohms
The design charts shown in figures 12.1 can be used to determine the phase characteristics of the lossless line. The attenuation of the line is calculated by considering losses in the conductor and in the dielectric layer. The attenuation coefficient due to conductor losses, \( \alpha_1 \), is given by

\[
\alpha_1 = \left(\frac{1}{2}\right) P_1/N = \frac{P_1}{(ZII^*)}
\]

Using equations 11.18 and 12.3, and letting \( \alpha_1 = 5.8 \cdot 10^5 \) mhos/cm for copper yields

\[
\alpha_1 = \frac{95.17}{(r_1\sqrt{\lambda}Z)} \text{ dB/100 ft}
\]

\( r_1 \) and \( \lambda \) in centimeters, and \( Z \) is obtained from figures 12.1. The attenuation coefficient due to dielectric losses, \( \alpha_2 \), is given by

\[
\alpha_2 = \left(\frac{1}{2}\right) P_2/N = \frac{P_2}{(ZII^*)}
\]

It is convenient to normalize \( \alpha_2 \) to the attenuation coefficient of a TEM wave in medium 2, \( \alpha_o \), where

\[
\alpha_o = \left(\frac{1}{2}\sqrt{\varepsilon_r} k_o \tan\delta_2 \right)
\]

Letting \( \tilde{\alpha} \equiv \alpha_2/\alpha_o \), and using equations 12.9, 12.10, 12.3, 11.13 and 11.9, one obtains

\[
\tilde{\alpha} = \frac{h_2^2 r_0}{4\pi(\varepsilon_2 r_2)^3 k_2^2 Z} \left\{ \left(1 + \frac{\varepsilon_2}{\varepsilon_1} \right) \left[ 4 \left( \frac{A_0^2(p_{22})}{A_0^2(p_{21})} + \frac{A_{0'}^2(p_{22})}{A_{0'}^2(p_{21})} \right)^{-1} \right] + \frac{2\varepsilon_2 r_2}{h_2^2 p_{21}r_1} \left( \frac{A_0^0(p_{22})A_{0'}^0(p_{22})}{A_0^0(p_{21})A_{0'}^0(p_{21})} \right) \right\}
\]

12.11

It can also be shown that \( \tilde{\alpha} \) is uniquely defined by \( \Delta \nu_p \) and \( Z \). A chart of \( \tilde{\alpha} \) against \( \Delta \nu_p \) with \( Z \) as parameter is shown in figure 12.2. From equations 12.10 and 12.11

\[
\alpha_2 = \left(\frac{1}{2}\sqrt{\varepsilon_r} k_o \tan\delta_2 \right) \tilde{\alpha} = 4168 f \tan\delta_2 \tilde{\alpha} \text{ dB/100 ft}
\]

where \( f \) is expressed in GHz. The total attenuation is given by

\[
\alpha = \frac{95.17}{r_1\sqrt{\lambda} Z} + 4168 f \tan\delta_2 \tilde{\alpha} \text{ dB/100 ft}
\]

12.13
FIGURE 12.2 Surface-wave Transmission Line Dielectric Attenuation Characteristics
12.1.2 Accuracy of the Design Charis

The approximate relations obtained by Goubau⁴⁰ can be derived by using small-argument approximations for the Bessel functions pertaining to medium 2 in equations 12.4 and 12.11. These are given by

\[ Z = \frac{Z_0}{2\pi k_0} \left[ \frac{1}{\varepsilon_r} \ln \left( \frac{r_2}{r_1} \right) + \frac{1}{2\varepsilon_r} \left( \frac{K_2^2(p_{32})}{K_0^2(p_{32})} - \frac{2K_0(p_{32})}{p_{32}K_0(p_{32})} - 1 \right) \right] \] .................................. 12.14

and

\[ \tilde{\alpha}_2 = \left( \frac{2Z_0 \tan \delta_2}{4\pi \varepsilon_r r_2 k_0} \right) \ln \left( \frac{r_2}{r_1} \right) \] .................................. 12.15

The approximate attenuation coefficient of the surface-wave transmission line is then given by

\[ \tilde{\alpha} = \tilde{\alpha}_1 + \tilde{\alpha}_2 = 95.17 \frac{1}{r_1 \sqrt{\lambda}} + \frac{166,347 \beta^2 f \tan \delta_2}{\varepsilon_r k_0 \frac{r_2}{r_1}} \ln \left( \frac{r_2}{r_1} \right) \text{dB/100 ft} , \] .................................. 12.16

where \( r_1, \lambda \) in centimeters, \( f \) in GHz.

It can be shown that

\[ p_{22} = 2\pi \frac{r_2}{\lambda} \left[ \varepsilon_r - \frac{1}{(1-0.01 \Delta v_p)^2} \right]^{1/2} , \] .................................. 12.17

from which it can be seen that \( p_{22} \) is small if \( r_2/\lambda \) or \( \Delta v_p \), or both, are small. Thus equations 12.14, 12.15 and 12.16 are quite accurate for small phase-velocity reductions or for cases where the radius of the guide is much smaller than a wavelength. The cases of very thin dielectric coatings, i.e. \( r_2/r_1 = 1 \), result in small phase-velocity reductions and the two conditions required for equation 12.16 to be valid are

\[ r_2 \ll \lambda \]

or

\[ r_2 = r_1 \] .................................. 12.18

A particular case will now be considered to ascertain the accuracy of Goubau's relations for the attenuation coefficients. The parameters used are
\[ f = 5.906 \text{ GHz}, \]
\[ \epsilon_r^2 = 2.26, \quad \epsilon_r^3 = 1.0, \]
\[ \tan\delta_2 = 0.0005, \quad \tan\delta_3 = 0.0, \]
\[ \sigma_1 = 5.80 \cdot 10^5 \text{ mhos/cm}, \]
\[ r_1 = 0.0254 \text{ cm and} \]
\[ r_2 \text{ varies from } 0.0 \text{ to } 2.0 \text{ cm} \]

The variation of \( \alpha, \tilde{\alpha}, \alpha_1, \tilde{\alpha}_1, \alpha_2 \) and \( \tilde{\alpha}_2 \) as a function of dielectric thickness is shown in figure 12.3.b, from which it can be seen that Goubau's equations are quite accurate for very thin dielectric linings.

**FIGURE 12.3** Attenuation Characteristics of the Goubau Line
12.1.3 Comparison with Experimental Results

The results given in this section, which are obtained from the perturbation theory, can be shown to agree better with published experimental data than those obtained from Goubau's approximate equations.

The results chosen for the purpose of comparison are given by Schiebe et. al., because they are well documented. In their experiments, the attenuation coefficients of several surface-wave transmission lines were obtained by measuring the guide wavelength and Q of the resonant waveguides, at a frequency of 9.375 GHz. The attenuation coefficient was calculated from the approximate formula

\[ \alpha \approx \frac{\pi}{Q \lambda_g} \] \hspace{1cm} 12.19

where Q is the unloaded quality factor of the resonator after correcting for end-plate losses.

The comparison of the two sets of results is shown in Table 12.1. The "corrected measured" values were obtained by using a more accurate relation than equation 12.19,

\[ \alpha = \frac{\pi v_p}{Q \lambda_g v_g} \] \hspace{1cm} 12.20*

It may be noted that Schiebe's calculated results were obtained by using Goubau's approximate semi-graphical method given in reference 42.

* This result has been derived for the dielectric tube waveguide by Mr. T. Bourk as part of a Masters Thesis project at the University of British Columbia, Electrical Engineering Department. In Appendix E, it is shown that equation 12.20 is a general result valid for any waveguide, regardless of the number of homogeneous dielectric media.
<table>
<thead>
<tr>
<th>PARAMETERS</th>
<th>$\varepsilon_{r2}$</th>
<th>2.26</th>
<th>2.26</th>
<th>2.10</th>
<th>2.10</th>
<th>2.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tan\delta_2$</td>
<td>0.0003</td>
<td>0.0003</td>
<td>0.0003</td>
<td>0.0003</td>
<td>0.0003</td>
<td></td>
</tr>
<tr>
<td>$r_1$(cm)</td>
<td>0.1295</td>
<td>0.04015</td>
<td>0.0455</td>
<td>0.0705</td>
<td>0.1295</td>
<td></td>
</tr>
<tr>
<td>$r_2$(cm)</td>
<td>0.1660</td>
<td>0.1500</td>
<td>0.1490</td>
<td>0.2365</td>
<td>0.4165</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ORIGINAL RESULTS (reference 44)</th>
<th>Q meas.</th>
<th>8790.</th>
<th>2550.</th>
<th>2750.</th>
<th>3030.</th>
<th>3470.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_g$ (cm) calc.</td>
<td>3.12</td>
<td>2.82</td>
<td>2.88</td>
<td>2.79</td>
<td>2.61</td>
<td></td>
</tr>
<tr>
<td>$\lambda_g$ (cm) meas.</td>
<td>3.12</td>
<td>2.82</td>
<td>2.92</td>
<td>2.85</td>
<td>2.79</td>
<td></td>
</tr>
<tr>
<td>$\alpha$(dB/100 ft) calc.</td>
<td>3.17</td>
<td>12.32</td>
<td>9.72</td>
<td>9.29</td>
<td>11.87</td>
<td></td>
</tr>
<tr>
<td>$\alpha$(dB/100 ft) meas.</td>
<td>3.03</td>
<td>11.55</td>
<td>10.35</td>
<td>9.61</td>
<td>8.59</td>
<td></td>
</tr>
<tr>
<td>% diff. in $\alpha$</td>
<td>+ 4.6</td>
<td>+ 6.7</td>
<td>- 6.1</td>
<td>- 3.4</td>
<td>+ 26.5</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MODIFIED RESULTS</th>
<th>$v_g/v_p$ calc.</th>
<th>0.991</th>
<th>0.952</th>
<th>0.960</th>
<th>0.942</th>
<th>0.910</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_g$ (cm) calc.</td>
<td>3.12</td>
<td>2.82</td>
<td>2.88</td>
<td>2.81</td>
<td>2.70</td>
<td></td>
</tr>
<tr>
<td>$\alpha$(dB/100 ft) calc.</td>
<td>3.16</td>
<td>11.93</td>
<td>10.39</td>
<td>9.94</td>
<td>10.44</td>
<td></td>
</tr>
<tr>
<td>$\alpha$(dB/100 ft) corrected measured</td>
<td>3.06</td>
<td>12.12</td>
<td>10.79</td>
<td>10.20</td>
<td>9.44</td>
<td></td>
</tr>
<tr>
<td>% diff. in $\alpha$</td>
<td>+ 3.2</td>
<td>- 1.6</td>
<td>- 3.9</td>
<td>- 2.6</td>
<td>+ 9.6</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 12.1 Measured Characteristics of the Goubau Line**
12.2 Dielectric-lined Coaxial Cables

In this section, the propagation coefficients are calculated for the $TM_{01}$ mode in a coaxial cable with a dielectric lining on the inner, outer or both conductors. The propagation characteristics obtained are compared with those of other workers \cite{26,27,28,29,35,37}.

The attenuation coefficients of two cables with only one dielectric lining were calculated using the perturbation method. The relations used are developed in Appendix C from the analysis given in Chapter 11. The results show a gradual increase in attenuation as the thicknesses of the dielectric linings are increased. This behaviour is similar to that found by Millington \cite{27} in the case of the loaded stripline, and it is also supported by previous work \cite{35,37} in which the case of a lined inner conductor was analysed. Millington's comments regarding the accuracy of computations in reference 26 appear to be equally valid in the coaxial case.

To check the validity of the perturbation method, the exact characteristic equations, given in Appendix D, were solved to a high degree of accuracy using an IBM 7044 computer. In these equations, the fields in the outer conductor were described by Hankel functions of the second kind, instead of the Bessel functions used in reference 26, since the fields must decay with increasing radius. No approximations were made when solving these equations; the large and small-argument approximations, used in reference 26 to replace the complex-argument Bessel functions, were avoided by employing accurate computer subroutines to generate these functions. The results for thin dielectric linings are given in Table 12.2 from which it can be seen that the attenuation coefficients obtained by the two methods differ by less than 0.1%. This ascertains the validity and establishes the accuracy of the perturbation method, which has the additional advantage of requiring less than ten percent of the computing time taken for the solution of the exact equations to the same degree of accuracy.
<table>
<thead>
<tr>
<th>Dielectric thickness $r_2-r_1$, cm</th>
<th>PERTURBATION RESULTS</th>
<th>EXACT RESULTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha$ Np/m</td>
<td>$\beta$ rad/m</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0095956</td>
<td>63.227306</td>
</tr>
<tr>
<td>0.02</td>
<td>0.0098491</td>
<td>63.608605</td>
</tr>
<tr>
<td>0.03</td>
<td>0.0100988</td>
<td>63.977169</td>
</tr>
<tr>
<td>0.04</td>
<td>0.0103452</td>
<td>64.334212</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0105888</td>
<td>64.680779</td>
</tr>
<tr>
<td>0.06</td>
<td>0.0108298</td>
<td>65.017776</td>
</tr>
<tr>
<td>0.07</td>
<td>0.0110686</td>
<td>65.345995</td>
</tr>
<tr>
<td>0.08</td>
<td>0.0113055</td>
<td>65.666131</td>
</tr>
<tr>
<td>0.09</td>
<td>0.0115406</td>
<td>65.978798</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0117743</td>
<td>66.284539</td>
</tr>
<tr>
<td>0.11</td>
<td>0.0120067</td>
<td>66.583844</td>
</tr>
<tr>
<td>0.12</td>
<td>0.0122379</td>
<td>66.877145</td>
</tr>
<tr>
<td>0.13</td>
<td>0.0124682</td>
<td>67.164837</td>
</tr>
<tr>
<td>0.14</td>
<td>0.0126976</td>
<td>67.447271</td>
</tr>
<tr>
<td>Dielectric thickness $r_4-r_3$, cm</td>
<td>PERTURBATION RESULTS</td>
<td>EXACT RESULTS</td>
</tr>
<tr>
<td>----------------------------------</td>
<td>----------------------</td>
<td>---------------</td>
</tr>
<tr>
<td></td>
<td>$\alpha$ Np/m</td>
<td>$\beta$ rad/m</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0094505</td>
<td>63.096932</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0095842</td>
<td>63.387834</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0097426</td>
<td>63.707592</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0099297</td>
<td>64.059459</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0101500</td>
<td>64.446891</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0104082</td>
<td>64.873532</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0107092</td>
<td>65.343178</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0110579</td>
<td>65.859742</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0114589</td>
<td>66.427222</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0119160</td>
<td>67.049687</td>
</tr>
</tbody>
</table>

**TABLE 12.2** Comparison of Propagation Coefficients Obtained from Perturbation Theory and from Exact Theory for Coaxial Cables with One Dielectric Lining

- $\varepsilon_r=1$, $\varepsilon_{r_4} = \varepsilon_{r_2}=2.26$, $r_1=0.157$ cm, $r_4=2.5$ cm, $f=3.0$ GHz,
- $\tan\delta_3=0.0$, $\tan\delta_4 = \tan\delta_2=0.0005$, $\sigma_1 = \sigma_5 = 1/7 \cdot 10^6$ mhos/cm
The computations using the perturbation method were extended for increasing dielectric thickness until the coaxial cables were completely filled. The results are shown in figure 12.4, where the conductor, dielectric and total attenuation coefficients are plotted against dielectric thickness on the inner or the outer conductor. The end-points of the curves were obtained from the well-known equations for the attenuation coefficient of the quasi-TEM mode in a coaxial cable. These fall in line with the intermediate points obtained from the $TM_{01}$ surface-wave mode analysis.

The parameters of the cavity used in the experimental work reported in references 28 and 29, have been given in a most recent paper, and these differ only slightly from the parameters used in obtaining the characteristics given in Table 12.2 and figure 12.4. Computations using these new parameters give attenuation characteristics which are very similar to those shown in figure 12.4.

Similar calculations were carried out in the case of thin dielectric linings on both conductors, using both the exact and the perturbation methods. These results are shown in Table 12.3, from which it can be seen that the accuracy obtained is comparable with that found in the previous cases, and that the attenuation characteristics exhibit the same general behaviour.
FIGURE 12.4 Attenuation Characteristics of Dielectric-lined Coaxial Cables

f = 3.0 GHz
\( \varepsilon_{r2} = \varepsilon_{r4} = 2.26 \)
\( \varepsilon_{r3} = 1.0 \)

\( r_1 = 0.157 \) cm
\( r_2 = 2.5 \) cm

\( \tan \delta_2 = \tan \delta_4 = 0.0005 \), \( \tan \delta_3 = 0.0 \)

\( \sigma_1 = \sigma_5 = 1/7 \cdot 10^6 \) mhos/cm

- - - - - lining on inner conductor
- - - - - lining on outer conductor
TABLE 12.3 Comparison of Propagation Coefficients Obtained from Perturbation Theory and from Exact Theory for a Coaxial Cable with Two Dielectric Linings

<table>
<thead>
<tr>
<th>Dielectric thicknesses</th>
<th>PERTURBATION RESULTS</th>
<th>EXACT RESULTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha \ Np/m )</td>
<td>( \beta \ rad/m )</td>
</tr>
<tr>
<td>( r_2-r_1, \text{cm} )</td>
<td>( r_4-r_3 \text{ cm} )</td>
<td></td>
</tr>
<tr>
<td>0.001123</td>
<td>0.003538</td>
<td>0.0169096</td>
</tr>
<tr>
<td>0.001544</td>
<td>0.003110</td>
<td>0.0169492</td>
</tr>
<tr>
<td>0.001948</td>
<td>0.002474</td>
<td>0.0169841</td>
</tr>
<tr>
<td>0.002298</td>
<td>0.002105</td>
<td>0.0170169</td>
</tr>
<tr>
<td>0.002597</td>
<td>0.001765</td>
<td>0.0170446</td>
</tr>
<tr>
<td>0.002913</td>
<td>0.001439</td>
<td>0.0170744</td>
</tr>
<tr>
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<td>0.001176</td>
<td>0.0170907</td>
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<td>0.003298</td>
<td>0.000948</td>
<td>0.0171094</td>
</tr>
<tr>
<td>0.003500</td>
<td>0.000702</td>
<td>0.0171279</td>
</tr>
<tr>
<td>0.003655</td>
<td>0.000508</td>
<td>0.0171421</td>
</tr>
<tr>
<td>0.01123</td>
<td>0.03538</td>
<td>0.0185315</td>
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<td>0.0189336</td>
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<td>0.02474</td>
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<td>0.0204096</td>
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<td>0.00948</td>
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<td>0.0208124</td>
</tr>
<tr>
<td>0.03655</td>
<td>0.00508</td>
<td>0.0209673</td>
</tr>
</tbody>
</table>

\( \varepsilon_{r_3}=1.0, \varepsilon_{r_4}=\varepsilon_{r_2}=2.5, r_1=0.13 \text{ cm}, r_4=0.80 \text{ cm}, f=10.0 \text{ GHz}, \tan \delta_3=0.0, \tan \delta_4=\tan \delta_2=0.0004, \sigma_1=\sigma_5=6\times10^5 \text{ mhos/cm} \)
Using equations 11.9, 11.10, 11.13 and 11.18, and Tables 11.2 and 11.3, and normalizing the arbitrary constant $a_3 a_3^*$ to the peak current in the inner conductor as in Section 12.1.1, the power flow and power loss per unit length in the various media of the coaxial cable with two dielectric linings can be obtained.

$$N_2 = \frac{I^2\beta Z_0}{8\pi\varepsilon r^2 k_0} \left( \frac{p_{22}A_0'(p_{22})}{p_{21}A_0'(p_{21})} \right)^2 \left[ \frac{\left( P_{22}^2 \frac{A_0'(p_{22})}{A_0'(p_{22})} + 2A_0(p_{22}) \right)}{p_{22}A_0'(p_{22})} + 1 \right] - \left( \frac{p_{21}A_0'(p_{21})}{p_{22}A_0'(p_{22})} \right)^2$$

............... C1.a

$$N_3 = \frac{I^2\beta Z_0}{8\pi\varepsilon r^3 k_0} \left( \frac{p_{22}A_0'(p_{22})}{p_{21}A_0'(p_{21})} \right)^2 \left[ \frac{\left( \frac{p_{33}C_0'(p_{33})}{p_{32}C_0'(p_{32})} \right)^2}{\frac{C_0'(p_{33})}{C_0'(p_{33})} + \frac{2C_0(p_{33})}{p_{33}C_0(p_{33})} + 1} \right]$$

............... C1.b

$$N_4 = \frac{I^2\beta Z_0}{8\pi\varepsilon r^4 k_0} \left( \frac{p_{22}A_0'(p_{22})p_{33}C_0'(p_{33})}{p_{21}A_0'(p_{21})p_{32}C_0'(p_{32})} \right)^2 \left[ \frac{\left( \frac{p_{44}F_0'(p_{44})}{p_{43}F_0'(p_{43})} \right)^2}{\frac{F_0'(p_{43})}{F_0'(p_{43})} + \frac{2F_0(p_{43})}{p_{43}F_0(p_{43})} + 1} \right]$$

............... C1.c

$$P_1 = \frac{I^2R_1}{4\pi r_1}$$

.......................... C2.a

$$P_2 = \frac{I^2\beta Z_0}{8\pi\varepsilon r^2 k_0} \left( \frac{p_{22}A_0'(p_{22})}{p_{21}A_0'(p_{21})} \right)^2 \left\{ \left( \frac{B_0^2}{h_2^2} \left[ \frac{A_0'(p_{22})}{A_0'(p_{22})} + 1 \right] \right) \right.$$
\[ P_3 = \frac{I^2 h^2 \tan \delta}{8 \pi \epsilon r_3 k_0} \left( \frac{p_{22} A_0'(p_{22})}{p_{21} A_0(p_{21})} \right)^2 \left\{ 1 + \frac{\varepsilon^2}{h_3^2} \right\} \left( \frac{p_{33} c_0^i(p_{33})}{p_{32} c_0^i(p_{32})} \right)^2 \left( \frac{c_0^2(p_{33})}{c_0^2(p_{32})} - 1 \right) \]

\[ + \frac{2 \varepsilon^2}{h_3^2} \left( \frac{p_{33} c_0^i(p_{33})}{p_{32} c_0^i(p_{32})} \right)^2 \left( \frac{c_0(p_{33})}{c_0(p_{32})} - \frac{c_0^2(p_{33})}{c_0^2(p_{32})} \right) \]

\[ P_4 = \frac{I^2 h^2 \tan \delta}{8 \pi \epsilon r_4 k_0} \left( \frac{p_{22} A_0'(p_{22})}{p_{21} A_0(p_{21})} \right)^2 \left( \frac{p_{33} c_0^i(p_{33})}{p_{32} c_0^i(p_{32})} \right)^2 \left\{ 1 + \frac{\varepsilon^2}{h_4^2} \right\} \left( \frac{p_{44} F_0^i(p_{44})}{p_{43} F_0^i(p_{43})} \right)^2 \left( \frac{F_0^2(p_{43})}{F_0^2(p_{43}) + 1} \right) \]

\[ - \left( \frac{2 \varepsilon^2 F_0(p_{43})}{h_4^2 p_{43} F_0(p_{43})} \right) \]

\[ P_5 = \frac{I^2 R_\alpha}{4 \pi r_4} \left( \frac{p_{22} A_0'(p_{22})}{p_{21} A_0(p_{21})} \right)^2 \left( \frac{p_{33} c_0^i(p_{33})}{p_{32} c_0^i(p_{32})} \right)^4 \left( \frac{p_{44} F_0^i(p_{44})}{p_{43} F_0^i(p_{43})} \right)^4 \]  

where the functions \(A, C\) and \(F\) are defined in Table 9.1. The attenuation coefficient is given by

\[ \alpha = \sum_{i=1}^{5} \alpha_i = (1/2) \sum_{i=1}^{5} P_i / \sum_{i=2}^{4} N_i \]

where \(P_i\) and \(N_i\) are given by equations C1 and C2.

For the case of the lined inner conductor only, \(P_4\) and \(N_4\) do not exist, and \(P_1, P_2\) and \(N_2\) are given by equations C2.a, C2.b and C1.a, respectively, while \(N_3, P_3\) and \(P_5\) are given by
The attenuation coefficient of the cable is given by
\[
\alpha = \frac{1}{2} (P_1 + P_2 + P_3 + P_5) / (N_2 + N_3),
\]
where \( P_1, P_2, P_3, P_5, N_2 \) and \( N_3 \) are defined by equations C2.a, C2.b, C5.a, C5.b, C1.a and C4, respectively.

For the case of the lined outer conductor only, \( P_2 \) and \( N_2 \) do not exist, and \( P_1 \) is given by equation C2.a, while \( N_3, N_4, P_3, P_4 \) and \( P_5 \) are given by
\[
N_3 = \frac{I^2 \beta Z_0}{8 \pi e r_3 k_0} \left( \frac{p_{33} c_0'(p_{33})}{p_{31} c_0'(p_{31})} \right)^2 \left( \frac{c_0^2(p_{33})}{c_0^{12}(p_{33})} + \frac{2 c_0(p_{33})}{p_{33} c_0'(p_{33})} + 1 \right) - 1 \]
\[
N_4 = \frac{I^2 \beta Z_0}{8 \pi e r_4 k_0} \left( \frac{p_{33} c_0'(p_{33})}{p_{31} c_0'(p_{31})} \right)^2 \left( \frac{F_0'(p_{43})}{F_0'(p_{43})} \right)^2 \left( \frac{F_0^2}{F_0(p_{43})} + \frac{2 F_0(p_{43})}{p_{43} F_0'(p_{43})} + 1 \right) \]
\[
P_3 = \frac{I^2 h Z_0}{8 \pi e r_3 k_0} \tan \delta \left( \frac{p_{33} c_0'(p_{33})}{p_{31} c_0'(p_{31})} \right)^2 \left( \frac{c_0^2(p_{33})}{c_0^{12}(p_{33})} - 1 \right) + \left( \frac{p_{33} c_0'(p_{31})}{p_{33} c_0'(p_{33})} \right)^2
\]
The attenuation coefficient of the cable is given by

\[ \alpha = \frac{1}{2}(P_1 + P_3 + P_4 + P_5)/(N_3 + N_4) \]

where \( P_1, P_3, P_4, P_5, N_3 \) and \( N_4 \) are given by equations C2.a, C8.a, C8.b, C8.c, C7.a and C7.b, respectively.
APPENDIX D THE EXACT SOLUTION OF TM SURFACE-WAVE MODES IN DIELECTRIC-LINED COAXIAL CABLES

Omitting the common factor \(\exp(j\omega t - yz)\), the exact expressions for the field components of case I for the quasi-TM surface wave modes in a coaxial cable with dielectric linings on both conductors are given by

\[
E_{z1} = C_1 J_0(h_1 r)
\]

\[
E_{r1} = \frac{\gamma}{h_1} C_1 J_1(h_1 r)
\]  
0 \leq r \leq r_1 \hspace{1cm} \text{D1.a}

\[
H_{\theta 1} = \frac{\sigma_1 + jk_o/\gamma}{\gamma} E_{r1}
\]

\[
E_{z2} = C_2 \left( J_0(h_2 r) + F_2 Y_0(h_2 r) \right)
\]

\[
E_{r2} = \frac{\gamma}{h_2} C_2 \left( J_1(h_2 r) + F_2 Y_1(h_2 r) \right)
\]  
\hspace{2cm} r_1 \leq r \leq r_2 \hspace{1cm} \text{D1.b}

\[
H_{\theta 2} = j \frac{\varepsilon r_2 k_o}{\gamma Z_o} \left( 1 - j \tan \delta_2 \right) E_{r2}
\]

\[
E_{z3} = C_3 \left( H_0^{(1)}(h_3 r) + F_3 H_0^{(2)}(h_3 r) \right)
\]

\[
E_{r3} = \frac{\gamma}{h_3} C_3 \left( H_1^{(1)}(h_3 r) + F_3 H_1^{(2)}(h_3 r) \right)
\]  
\hspace{2cm} r_2 \leq r \leq r_3 \hspace{1cm} \text{D1.c}

\[
H_{\theta 3} = j \frac{\varepsilon r_3 k_o}{\gamma Z_o} \left( 1 - j \tan \delta_3 \right) E_{r3}
\]

\[
E_{z4} = C_4 \left( J_0(h_4 r) + F_4 Y_0(h_4 r) \right)
\]

\[
E_{r4} = \frac{\gamma}{h_4} C_4 \left( J_1(h_4 r) + F_4 Y_1(h_4 r) \right)
\]  
\hspace{2cm} r_3 \leq r \leq r_4 \hspace{1cm} \text{D1.d}

\[
H_{\theta 4} = j \frac{\varepsilon r_4 k_o}{\gamma Z_o} \left( 1 - j \tan \delta_4 \right) E_{r4}
\]
\[ E_{z5} = c_5 H_0^{(2)}(h_5 r) \]
\[ E_{r5} = \frac{\gamma}{h_5} c_5 H_1^{(2)}(h_5 r) \]
\[ H_{\theta5} = \frac{(\sigma_5 + j k_0 Z_0)}{\gamma} E_{r5} \]

where
\[ h_1^2 = \gamma^2 + k_0^2 - j k_0 Z_0 \sigma_1 \]
\[ h_2^2 = \gamma^2 + \varepsilon_{r2} k_0^2 (1-j \tan \delta_2) \]
\[ h_3^2 = \gamma^2 + \varepsilon_{r3} k_0^2 (1-j \tan \delta_3) \]
\[ h_4^2 = \gamma^2 + \varepsilon_{r4} k_0^2 (1-j \tan \delta_4) \]
\[ h_5^2 = \gamma^2 + k_0^2 - j k_0 Z_0 \sigma_5 \]

Matching tangential field components at \( r=r_1 \) and \( r=r_2 \) yields the following two equations:

\[ F_2 = \frac{j \left( \frac{p_{11} \varepsilon_{r2} k_0(1-j \tan \delta_2)}{p_{21} Z_0 (\sigma_1 + j k_0 Z_0)} \right) J_1(p_{21}) - \left( \frac{J_1(p_{11})}{J_0(p_{11})} \right) J_0(p_{21})}{\left( \frac{J_0(p_{11})}{J_0(p_{11})} \right) Y_0(p_{21}) - j \left( \frac{p_{11} \varepsilon_{r2} k_0(1-j \tan \delta_2)}{p_{21} Z_0 (\sigma_1 + j k_0 Z_0)} \right) Y_1(p_{21})} \]

\[ \left( \frac{p_{22} \varepsilon_{r3}(1-j \tan \delta_3)}{p_{32} \varepsilon_{r2}(1-j \tan \delta_2)} \right) \left( \frac{J_0(p_{22}) + F_2 Y_0(p_{22})}{J_1(p_{22}) + F_2 Y_1(p_{22})} \right) = \left( \frac{H_0^{(1)}(p_{32}) + F_3 H_0^{(2)}(p_{32})}{H_1^{(1)}(p_{32}) + F_3 H_1^{(2)}(p_{32})} \right) \]

Similarly, matching fields at \( r=r_3 \) and \( r=r_4 \) yields

\[ F_4 = \frac{j \left( \frac{p_{54} \varepsilon_{r4} k_0(1-j \tan \delta_4)}{p_{44} Z_0 (\sigma_5 + j k_0 Z_0)} \right) J_1(p_{44}) - \left( \frac{H_1^{(2)}(p_{54})}{H_0^{(2)}(p_{54})} \right) J_0(p_{44})}{\left( \frac{H_1^{(2)}(p_{54})}{H_0^{(2)}(p_{54})} \right) Y_0(p_{44}) - j \left( \frac{p_{54} \varepsilon_{r4} k_0(1-j \tan \delta_4)}{p_{44} Z_0 (\sigma_5 + j k_0 Z_0)} \right) Y_1(p_{44})} \]
The characteristic equation for the case of dielectric linings on both conductors can be found by eliminating the arbitrary constant, \( F_3 \), from equations D4 and D6, with the arbitrary constants \( F_2 \) and \( F_4 \) defined by equations D3 and D5, respectively.

If there is a lining on the inner, but not the outer conductor, then the fields in media 3 and 5 must be matched at \( r=r_4 \). In this case, the following equation results

\[
F_3 = \frac{j \left( \frac{p_{54}^e r_3 k_0^3 (1 - j \tan \delta_3)}{p_{34}^e r_0^3 (\sigma_5 + j k_0 Z_0)} \right) H_1^1(p_{34}) - \left( \frac{H_1^2(p_{54})}{H_0^2(p_{54})} \right) H_0^1(p_{34})}{\left( \frac{H_1^2(p_{54})}{H_0^2(p_{54})} \right) H_0^2(p_{34}) - j \left( \frac{p_{54}^e r_3 k_0^3 (1 - j \tan \delta_3)}{p_{34}^e r_0^3 (\sigma_5 + j k_0 Z_0)} \right) H_1^2(p_{34})}
\]

The characteristic equation of a coaxial cable with a lined inner conductor is given by equation D4, with the arbitrary constants \( F_2 \) and \( F_3 \) defined by equations D3 and D7, respectively.

If there is a lining on the outer, but not the inner conductor, then matching fields in media 1 and 3 at \( r=r_1 \) yields

\[
F_3 = \frac{j \left( \frac{p_{11}^e r_3 k_0^3 (1 - j \tan \delta_3)}{p_{31}^e r_0^3 (\sigma_1 + j k_0 Z_0)} \right) H_1^1(p_{31}) - \left( \frac{J_1^1(p_{11})}{J_0^1(p_{11})} \right) H_0^1(p_{31})}{\left( \frac{J_1^1(p_{11})}{J_0^1(p_{11})} \right) H_0^2(p_{31}) - j \left( \frac{p_{11}^e r_3 k_0^3 (1 - j \tan \delta_3)}{p_{31}^e r_0^3 (\sigma_1 + j k_0 Z_0)} \right) H_1^2(p_{31})}
\]

The characteristic equation of a coaxial cable with a lined outer conductor is given by equation D6, with the arbitrary constants \( F_3 \) and \( F_4 \) given by equations D8 and D5, respectively.

Although the exact characteristic equation for the unlined coaxial cable is not solved in this work, it can be easily obtained by equating
the two expressions for $F_3$ given by equations D7 and D8.

From equations D2, it can be seen that $|p_{11}|$ and $|p_{54}|$ are very large. The Bessel and Hankel functions with these arguments are extremely large, in fact, too large to be calculated accurately. However, the ratios of Bessel and Hankel functions with these arguments, which occur in equations D3, D5, D7 and D8, are approximately equal to unity in magnitude. For computational purposes, it is convenient to obtain the asymptotic series for these ratios, which are readily shown to be

$$\frac{J_1(p_{11})}{J_0(p_{11})} = \frac{1}{2p_{11}} - i \left(1 + \frac{9}{64p_{11}^2}\right) \quad \text{D9.a}$$

and

$$\frac{H_1^{(2)}(p_{54})}{H_0^{(2)}(p_{54})} = \frac{1}{2p_{54}} + i \left(1 + \frac{9}{64p_{54}^2}\right) \quad \text{D9.b}$$

Only the first three terms of each series is required, since $\left|\frac{1}{p_{54}}\right|^3 < \left|\frac{1}{p_{11}}\right|^3 < 10^{-8}$ for the conductivities and high frequencies considered.
The quality factor of a resonant low-loss waveguide is given by

\[ Q = \frac{v_o k_o \bar{W}}{2P + \bar{P}} \] \hspace{1cm} E1

where

\[ P_p = \text{power loss in one end-plate}, \]
\[ \bar{P} = \text{power loss in the waveguide} \]
and
\[ \bar{W} = \text{energy storage in the resonator}. \]

The latter two quantities are related to the corresponding quantities of the terminated waveguide by

\[ \bar{P} = 2LP = 4\alpha LN \]
\[ \bar{W} = 2LW = 2LN/v_g \] \hspace{1cm} E2

where \( v_g \) is the velocity of energy propagation, which is equal to the group velocity. Using equations E1 and E2, the following expression for \( 1/Q \) can be derived:

\[ \frac{1}{Q} = \frac{P_p}{v_o k_o WL} + \frac{2\alpha v_g}{v_o k_o} = \frac{P_p}{v_o k_o WL} + \frac{2\alpha v_g}{v_g} \] \hspace{1cm} E3

The first term in equation E3 becomes smaller as \( L \) increases, and its contribution to \( 1/Q \) can be made negligible by assuming a very long resonator. In this case, the \( Q \) of the waveguide is given by

\[ Q = \frac{\beta}{2\alpha v_g} \] \hspace{1cm} E4

from which equation 12.20 is obtained.
CONCLUSIONS

The author considers the main contributions of this work to the field of surface waveguides to be:

I. The dielectric tube waveguide

1. A mode designation based upon the properties of the field components has been shown to be consistent with a previously given designation for the dielectric rod, and the distinctions between HE and EH modes have been observed.

2. Three-dimensional field configurations, calculated and plotted by computer, have been obtained for the $\text{HE}_{11}$ and $\text{EH}_{11}$ modes.

3. Expressions for the group velocity of all surface-wave modes on the dielectric tube have been derived.

4. Expressions for the attenuation coefficients of all modes have been derived, where the tube and its surrounding media are not loss-free.

5. Accurately computed numerical results for the propagation characteristics, covering a wide range of parameters, have been given.

6. The proposal has been made that moderately thin-walled dielectric tubes may be used to advantage at the higher frequencies where standard rectangular waveguides are too impractical.

II. Screened surface waveguides

7. A unified analysis of eight waveguides has yielded the following new results:

   (i) Characteristic equations for hybrid modes in all cases but the shielded and unshielded dielectric rod and the Goubau line, which have been derived previously.

   (ii) Expressions for the group velocity of all modes except TM modes on the Goubau line, which have been given also by other workers.

   (iii) Expressions for the attenuation coefficient of hybrid modes in
all cases but the shielded and unshielded dielectric rod, which have been derived by previous workers.

8. Extensive and more accurate design data for the TM$_{01}$ mode on polyethylene-coated copper conductors in free space, with unrestricted radial dimensions, have been given.

9. In the three cases of lined coaxial cables, accurate numerical results for the TM$_{01}$ mode with no restrictions on lining thickness have shown that the attenuation cannot be reduced below that of the unlined cable.


ADDITIONAL REFERENCES


