LINEAR TRANSFORMATIONS ON ALGEBRAS OF MATRICES
OVER THE CLASS OF INFINITE FIELDS

by

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ABSTRACT

The problem of determining the structure of linear transformations on the algebra of $n$-square matrices over the complex field is discussed by M. Marcus and B. N. Moyls in the paper "Linear Transformations on Algebras of Matrices". The authors were able to characterize linear transformations which preserve one or more of the following properties of $n$-square matrices; rank, determinant and eigenvalues.

The problem of obtaining a similar characterization of transformations as given by M. Marcus and B. N. Moyls but for a wider class of fields is considered in this thesis. In particular, their characterization of rank preserving transformations holds for an arbitrary field. One of the results on determinant preserving transformations obtained by M. Marcus and B. N. Moyls states that if a linear transformation $T$ maps unimodular matrices into unimodular matrices, then $T$ preserves determinants. Since this result does not necessarily hold for algebras of matrices over finite fields, the discussion on the characterization of determinant preserving transformations is limited to algebras of matrices over infinite fields.
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## BIBLIOGRAPHY
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CHAPTER ONE

RANK PRESERVERS

1.1 INTRODUCTION

The purpose of this chapter is to establish some notation to be used in this paper and to present a characterization of linear transformations which preserve the rank of n-square matrices. Although the paper [1]* of M. Marcus and B. N. Moyls is concerned with the algebra of matrices over the complex field, their proofs of the lemmas and the theorem on rank preservers are not dependent on the characteristic nor the algebraic closure of the field. In fact, the proofs hold word for word for the algebra of matrices over an arbitrary field. No proofs are given in this chapter but the main result obtained in [1] is stated below in Theorem 1.3.2.

1.2 NOTATION

The notation used in [1] will be adopted. Let $F$ denote a field. Let the following symbols denote the respective sets:

*Numbers in square brackets refer to the bibliography
2.

\[ M_n \] - the algebra of \( n \)-square matrices over \( F \)

\[ U_n \] - the unimodular group of matrices in \( M_n \)

(i.e., matrices in \( M_n \) with determinant 1)

Finally, \( A^t \) denotes the transpose of \( A \).

1.3 RANK PRESERVERS

1.3.1 Definition Let \( \sigma(A) \) denote the rank of the matrix \( A \). A linear transformation \( T \) on \( M_n \) is said to be a rank preserver if \( \sigma(T(A)) = \sigma(A) \) for all \( A \in M_n \).

1.3.2 Theorem Let \( T \) be a linear transformation of \( M_n \) into \( M_n \). \( T \) is a rank preserver if and only if there exist non-singular matrices \( U \) and \( V \) such that either:

\[ T(A) = UAV \quad \text{for all } A, \]

or \( T(A) = UATV \quad \text{for all } A \).
2.1 INTRODUCTION

The main result of this chapter is given in Theorem 2.3.8 which states that the following conditions are equivalent:

(1) $T$ maps $U_n$ into $U_n$.

(2) $T$ preserves determinant; i.e., $\det T(A) = \det A$ for all $A \in M_n$.

(3) There exist non-singular matrices $U$ and $V$ with $\det UV = 1$ such that either:

$$T(A) = UAV \quad \text{for all } A$$

or

$$T(A) = U{A}^tV \quad \text{for all } A.$$

When the base field $F$ is an arbitrary infinite field, the major difficulty encountered in obtaining the above results lies in showing that condition (1) implies condition (2). The solution of this problem is to a large extent the point of this thesis. When $F$ is a finite field, condition (1) does not necessarily imply condition (2). A counterexample will be given to show this.
2.2 A COUNTEREXAMPLE FOR FINITE FIELDS

The following example is a linear transformation which maps unimodular matrices into unimodular matrices but does not preserve determinant.

Consider the algebra $M_2$ of $2 \times 2$ matrices over the finite field of characteristic two with elements $\{0, 1\}$. Let $G_1$ and $G_2$ be the following subspaces of $M_2$:

$$G_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

and

$$G_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Let $T$ be a linear transformation on $M_2$ such that $T(G_1) = G_1$ and kernel $T = G_2$. Such a transformation is possible since $M_2$ is the direct sum of $G_1$ and $G_2$ (that is, $G_1 \cap G_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ and dimension $M_2 = \text{dimension } G_1 + \text{dimension } G_2$).

The transformation $T$ maps $U_n$ into itself; that is, $T(U_n) \subseteq U_n$. For each unimodular matrix $A$ there exists a matrix $B \in G_2$ and a matrix $C \in \left[ G_1 - \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \right]$. 

such that \( A = B + C \) (\(+\) denotes the usual matrix addition).

Since \( T \) is linear and \( B \in G_2 \), then \( T(A) = T(B+C) = T(B) + T(C) = T(C) \). But by the hypothesis \( T(C) \in \left[ G_1 - \{ (0,0) \} \right] \) all of whose members are unimodular. Thus \( T(A) \) is unimodular, and \( T(U_n) \subseteq U_n \).

On the other hand, \( T \) does not preserve zero determinant. Consider \( E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Since \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). And since \( T \) preserves unimodular matrices, \( \det T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \det T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \).

But \( \det E = 0 \), therefore \( T \) does not preserve zero determinant. Thus the transformation \( T \) maps unimodular matrices into unimodular matrices but does not preserve determinant.

2.3 DETERMINANT PRESERVERS

2.3.1 Definition A linear transformation \( T \) on \( M_n \) is said to be a determinant preserver if \( \det T(A) = \det A \) for all \( A \in M_n \).

2.3.2 Lemma If \( F \) is an infinite field, then the \( n \)-th root of an element of \( F \) exists in \( F \) for an infinite number of elements of \( F \).
Proof: Let \( a_1 \) be a non-zero element of \( F \) and \( b_1 = a_1^n \). Consider the equation \( x^n = b_1 \). Now \( a_1 \) is one solution and there are at most \( n \) solutions of the equation in \( F \). But since there are an infinite number of elements in \( F \), there exists a non-zero element \( a_2 \in F \) such that \( a_2^n = b_2 \) and \( b_2 \neq b_1 \). Suppose \( N \) distinct elements \( b_1, b_2, \ldots, b_N \) have been found, each of which has an \( n \)-th root. Then there are at most \( nN \) solutions of the equations \( x^n = b_i \), \( i = 1, 2, \ldots, N \). Since \( F \) is an infinite field, there exist non-zero elements \( a_{N+1} \) and \( b_{N+1} \) belonging to \( F \) such that \( a_{N+1}^n = b_{N+1} \) and \( b_{N+1} \neq b_i \), \( i = 1, 2, \ldots, N \). Therefore there are an infinite number of elements in \( F \) with an \( n \)-th root in \( F \).

2.3.3 Notation Let \( A = (a_{ij}) \in M_n \).

(1) \( A_{ij} \) denotes the cofactor of \( a_{ij} \) in \( A \).

(2) \( A(s,t)(x) = (a_{ij}) \) where \( a_{ij} = a_{ij} \) when \( (i,j) \neq (s,t) \) and \( a_{st} = x \). (That is, \( A(s,t)(x) \) is the \( nxn \) matrix obtained from \( A \) by replacing the entry \( a_{st} \) by the indeterminate \( x \).)
2.3.4 Lemma If $A = (a_{ij}) \in M_n$ and $\det A \neq 0$, then for some $j$, $a_{lj}A_{lj} \neq 0$. Consequently, $a_{lj} \neq 0$ and $A_{lj} \neq 0$ for this $j$.

Proof: Suppose $a_{lj}A_{lj} = 0$ for all $j$, $1 \leq j \leq n$. Then $\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{lj}A_{lj} = 0$ which contradicts the hypothesis that $\det A \neq 0$. Therefore $a_{lj}A_{lj} \neq 0$ for some $j$. And since $a_{lj}$ and $A_{lj}$ are elements of $F$, then $a_{lj}A_{lj} \neq 0$ implies that $a_{lj} \neq 0$ and $A_{lj} \neq 0$.

2.3.5 Lemma If $\det A \neq 0$, then for some fixed $j$ and for each $b \in F$ there exists an element $a \in F$ such that $\det A(l,j)(a) = b$.

Proof: By Lemma 2.3.4 there exists a $j$, $1 \leq j \leq n$, such that $A_{lj} \neq 0$. Let this value of $j$ be $k$. If $\sum_{j=1, j \neq k}^{n} (-1)^{1+j} a_{lj}A_{lj} = c$, then $\det A(l,k)(x) = xA_{lk} + c$.

For any $b \in F$, the polynomial $xA_{lk} + c - b$ in the indeterminate $x$ has a root, namely $x = (b - c)A_{lk}^{-1}$. Denote this root by $a$. Thus, for any $b \in F$ there exists an $a \in F$ such that $\det A(l,k)(a) = b$. 
2.3.6 Lemma  Let $F$ be an infinite field. If $T(U_n) \subseteq U_n$, then for any $A \in M_n$ such that $\det A \neq 0$, $\det T(A) = \det A$.

Proof: Let $A \in M_n$ and $\det A \neq 0$. Let $B$ be an infinite subset of $F$ such that if $b \in B$, then $b \neq 0$ and $b$ has an $n$-th root in $F$. By Lemma 2.3.2 such a set exists. Since $\det A \neq 0$, by Lemma 2.3.5 there exists a $k$ such that for each $b \in B$ there is an $a \in F$ for which $\det A(1,k)(a) = b$. Let $S$ be the subset of $F$ such that for each $a \in S$ there corresponds a $b \in B$ for which $b = \det A(1,k)(a)$. In other words, for each $a \in S$ $\det [A(1,k)(a)]^{1/n} \in F$. Since $B$ contains an infinite number of distinct elements then so does $S$. For each $a \in S$, $\det [A(1,k)(a)]^{1/n} = 1$. And since $T(U_n) \subseteq U_n$, for each $a \in S$, $\det T(A(1,k)(a)) = \det A(1,k)(a)$.

Denote by $E_{ij}$ the $n \times n$ matrix whose $(i,j)$-th position contains the element 1 and is zero elsewhere. Then $\det T(A(1,k)(x)) = \det T(xE_{1k} + \sum_{(i,j) \neq (1,k)} a_{ij} E_{ij}) = \det (xE_{1k} + \sum_{(i,j) \neq (1,k)} a_{ij} T(E_{ij})) = p(x)$, where $p(x)$ is a polynomial in $x$ of degree $\leq n$. In the proof of Lemma 2.3.5 it was shown that $\det A(1,k)(x) = xA_{1k} + c$ for some $c \in F$. Since for each $x \in S$ $\det A(1,k)(x) = \det T(A(1,k)(x))$, then $xA_{1k} + c = p(x)$ for each $x \in S$. 
But $S$ contains an infinite number of distinct elements, therefore $xA_{lk} + c = p(x)$ identically in $x$. Thus for any $x \in F$, $\det T(A(l,k)(x)) = \det A(l,k)(x)$.

In particular, the equality holds for $x = a_{lk}$; and since $A(l,k)(a_{lk}) = A$, then $\det T(A) = \det A$.

2.3.7 Corollary. Let $F$ be an infinite field. If $T(U) \subseteq U_n$, then for any $A \in M_n$, $\det T(A) = \det A$.

Proof: Let $A$ be any matrix belonging to $M_n$, and let $A(u)$ be the matrix obtained by replacing $a_{ii}$ in $A$ by $a_{ii} + u$, for $i = 1, 2, 3, \ldots, n$, where $u$ is an indeterminate.

$$\det A(u) = u^n + p_1(u) = p(u),$$

where $p_1(u)$ is a polynomial in $u$ of degree $\leq n-1$. But $\det T(A(u)) = \det \left( \sum_{i=1}^{n} (u+a_{ii}) T(E_{ii}) + \sum_{i \neq j} a_{ij} T(E_{ij}) \right)$, thus $\det T(A(u)) = q(u)$ where $q(u)$ is a polynomial in $u$ of degree $\leq n$.

Since $p(u) = 0$ has at most $n$ solutions, then $\det A(u) = p(u) \neq 0$ for infinitely many values of $u$ in $F$. But if $\det A(u) \neq 0$, then by Lemma 2.3.6 $\det A(u) = \det T(A(u))$ and consequently $p(u) = q(u)$ for infinitely many values of $u$ in $F$. Thus $p(u) = q(u)$ identically in $u$ and $\det A(u) = \det T(A(u))$ for all values of $u$ in $F$. In particular, the equality holds for $u = 0$. But since $A(0) = A$,
then \[ \det A = \det T(A) \]. \( A \) was an arbitrary element in \( M_n \) and therefore \( \det A = \det T(A) \) for all \( A \in M_n \).

The proofs of Lemma 2.3.8 and Lemma 2.3.9 are those given by M. Marcus and B. N. Moyls in the paper [1]. The proofs are included in this paper for the sake of completeness.

2.3.8 Lemma If \( T \) preserves determinant, then \( T \) is non-singular and hence onto.

Proof: Suppose \( T(A)' = 0 \). Since \( \det A = \det T(A) = 0 \), then the rank of \( A \), denoted by \( \sigma(A) \), is less than \( n \).
There exist non-singular matrices \( M \) and \( N \) such that \( MAN = I_r \cdot 0_{n-r} \) where \( r = \sigma(A) \), \( I_r \) the \( r \times r \) unit matrix and \( 0_{n-r} \) the \((n-r)\times(n-r)\) zero matrix and \( + \) denotes the direct sum. For any \( X \in M_n \), \[ \det(MAN+X)[\det M^{-1}N^{-1}] = \]
\[ \det M^{-1}(MAN+X)N^{-1} = \det(A+M^{-1}XN^{-1}) = \det T(M^{-1}XN^{-1}) = \]
\[ \det M^{-1}XN^{-1} = \det X \det M^{-1}N^{-1} . \] Therefore \( \det(MAN+X) = \det X \).
Set \( X = O_r + I_{n-r} \). Then \( \det(MAN+X) = \det I = 1 \). But \( \det X = 0 \) unless \( r = 0 \). But \( r = 0 \) implies \( A = 0 \). Thus \( T \) is non-singular.

2.3.9 Lemma Let \( F \) be an infinite field. If \( T \) preserves determinant, then \( T \) preserves rank.
Proof: Let $A \in M_n$ be an arbitrary matrix. There exist non-singular matrices $M_1, N_1, M_2$ and $N_2$ such that

$M_1A N_1 = Y_1 = I_r + O_{n-r}$ and $M_2 T(A) N_2 = Y_2 = I_s + O_{n-s}$ where $r = \sigma(A)$ and $s = \sigma(T(A))$.

Define $\varnothing : M_n \rightarrow M_n$ by $\varnothing(X) = M_2 T(M_1^{-1} X N_1^{-1}) N_2$.

The mapping $\varnothing$ has the following properties:

1. $\varnothing$ is linear since $T$ is linear.

2. $\det \varnothing(X) = k \det X$, where $k = \det(M_2 M_1^{-1} N_1^{-1} N_2)$.

This results from $\det \varnothing(X) = \det M_2 \cdot \det(T(M_1^{-1} X N_1^{-1})) \cdot \det N_2 = \det M_2 \cdot \det M_1^{-1} X N_1^{-1} \cdot \det N_2 = \det(M_2 M_1^{-1} N_1^{-1} N_2) \cdot \det X$.

3. $\varnothing(Y_1) = Y_2$ since $\varnothing(Y_1) = M_2 T(M_1^{-1} (M_1 A N_1) N_1^{-1}) N_2 = M_2 T(A) N_2 = Y_2$.

Set $Y_3 = O_r + I_{n-r}$. For each scalar $\lambda$, $\det(\lambda Y_1 + Y_3)$

$= \lambda^r$ and $\det(\varnothing(\lambda Y_1 + Y_3)) = \det(\lambda \varnothing(Y_1) + \varnothing(Y_3)) = \det(\lambda Y_2 + \varnothing(Y_3)) = p(\lambda)$, where $p(\lambda)$ is a polynomial in $\lambda$ of degree $\leq s$. But $\det \varnothing(\lambda Y_1 + Y_3) = k \det(\lambda Y_1 + Y_3) = k \lambda^r$, therefore $p(\lambda) = k \lambda^r$ for any $\lambda \in F$. Since
k \neq 0 , \; p(\lambda) = k\lambda^r \text{ identically in } \lambda \, . \text{ Therefore } r \leq s \text{ and } \sigma(A) \leq \sigma(T(A)) \, .

By Lemma 2.3.8, \( T^{-1} \) exists; and since \( T \) preserves determinant, \( \det B = \det(TT^{-1}(B)) = \det T^{-1}(B) \) for all \( B \in M_n \). Therefore \( T^{-1} \) preserves determinant. Thus \( \sigma(T(A)) \leq \sigma(T^{-1}(T(A))) = \sigma(A) \) and since it has previously been shown that \( \sigma(A) \leq \sigma(T(A)) \), then \( \sigma(A) = \sigma(T(A)) \). That is, \( T \) preserves rank.

The statement of the following theorem differs from that given by M. Marcus and B. N. Moyls only in part (3). In particular, when \( F \) is the field of complex numbers, they were able to find unimodular matrices \( U \) and \( V \) which satisfy conditions in part (3) of Theorem 2.3.10.

2.3.10 Theorem Let \( F \) be an infinite field and \( T \) a linear transformation on \( M_n \). The following conditions are equivalent:

1. \( T \) maps \( U_n \) into \( U_n \).

2. \( T \) preserves determinant.

3. There exist non-singular matrices \( U \) and \( V \) with \( \det UV = 1 \) such that either
   \[
   T(A) = UAV \quad \text{for all } A
   \]
   or
   \[
   T(A) = UA^TV \quad \text{for all } A \, .
   \]
Proof: (1) holds if and only if (2) holds by Corollary 2.3.7. (3) implies (2) since \( \det T(A) = \det UAV = \det A \cdot \det UV = \det A \). (2) implies (3). If \( T \) preserves determinant, then by Lemma 2.3.9 and Theorem 1.3.2, there exist non-singular matrices \( U \) and \( V \) such that either \( T(A) = UAV \) or \( T(A) = U^tAV \) for all \( A \in M_n \). \( \det T(I) = \det I = 1 \), therefore \( 1 = \det T(I) = \det UV = \det UV \). That is, \( \det UV = 1 \).

2.4 COMMENTS AND FURTHER PROBLEMS FOR INVESTIGATION

The hypothesis of Theorem 2.3.10 requires the field \( F \) to contain an infinite number of elements, but the infinite field is not a necessary condition. In order to show that two polynomials of degree \( \leq n \) are identically equal, it is sufficient that the equality holds for \( n + 1 \) distinct elements of the field. The proof of Lemma 2.3.6 requires these \( n + 1 \) elements to be non-zero and to possess an \( n \)-th root. If the field contains \( n(n+1) \) non-zero elements, then there exist at least \( n + 1 \) distinct non-zero elements each possessing an \( n \)-th root (see the proof of Lemma 2.3.2). Therefore, for a given \( n \), it is sufficient that the field \( F \) of Theorem 2.3.10 contains at least \( n(n+1) \) non-zero elements. Again this is not a necessary condition on the field \( F \) since for some given \( n \) there exist fields such that each element of the field has a \( n \)-th root; for example,
for \( n = 3 \) take \( F = \mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z} \).

The paper [1] gives a characterization of linear transformations which preserve eigenvalues for all matrices in the algebra of \( n \times n \) matrices over the complex numbers. A further problem would be to see if the same characterization holds for the algebra of matrices over a larger class of fields.

Another problem which may be considered is the following. In order to obtain the characterization of rank preservers given in [1] it is sufficient that the linear transformations preserve ranks 1, 2 and \( n \). It may be possible to find other sufficient conditions on a linear transformation \( T \) such that \( T \) is a rank preserver. For example, if \( \sigma(T(A)) = \sigma(A) \) for all symmetric matrices, \( A \), in the algebra of matrices over a field \( F \), then is \( T \) a rank preserver?
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