THE EFFECTIVENESS OF SIMPLE ENUMERATION AS A STRATEGY FOR DISCOVERY

by

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ABSTRACT

Leask, I.C. The effectiveness of simple enumeration as a strategy for discovery.

Problem

This study is related to the controversy surrounding the relative merits of teaching by discovery and expository methods. Specifically, it investigated the effectiveness of treatment with simple enumeration as a strategy for discovery compared to treatment using an expository method. It was hypothesized that the two treatments would yield the same mathematical achievement, but the simple enumeration treatment would yield more mathematical and non-mathematical transfer effect than the expository treatment.

Procedure

The subjects comprised six classes in Mathematics 12. They had been randomly assigned to classes at the beginning of the school year and three classes were assigned to each treatment group. All classes were taught a unit on arithmetic and geometric progressions by the experimenter. Equivalence of the groups was established in terms of the covariates I. Q. and previous term mark.

The measuring instruments consisted of the Lorge-Thorndike Intelligence Test, Form 1, Level H of the 1964

Multi-Level Edition; a mathematical content test; and a mathematical transfer test. In addition, the Nonverbal Battery of Form 1 of the Lorge-Thorndike Test was used as a pretest to measure the ability of students to generalize and discover principles from examples. Form 2 was used as a posttest measure to determine whether any improvement in ability to generalize had occurred as a result of the experience with the unit on progressions.

The generalized t-test was used to compare means of achievement on all tests. All results were analysed at the University of British Columbia Computing Centre.

Conclusions

On the basis of results on the tests, the following conclusions were reached:

1. Treatment with simple enumeration yielded the same level of mathematical achievement as treatment with an expository method.
2. Treatment with simple enumeration yielded significantly greater effect on a mathematical transfer test than treatment with an expository method. An examination of I. Q. levels showed that the superiority in performance was largely located at the medium I. Q. level.

3. Treatment with simple enumeration was no more effective than treatment with an expository method when the criterion measured general transfer. Both groups showed significant improvement in ability to generalize after studying the unit on arithmetic and geometric progressions. The improvement was mainly located at the medium and low I. Q. levels and was independent of teaching method.

The implication of this study is that if concern is centred on acquisition of facts, simple enumeration is no more effective than an expository teaching method. However, if there is concern for pupil participation and for training students to advance independently to related but more difficult material, then discovery-orientated lessons are advantageous.

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TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

THE PROBLEM

Background of the Problem

During the past three decades much attention has been directed towards the various ramifications of teaching by discovery as opposed to those of teaching by expository methods. Indicative of the importance attached to discovery in the teaching of mathematics is the fact that it has been widely accepted in the basic philosophy of textbook writers. Evidence of this is seen in such statements as "this textbook guides the student in discovering mathematical principles and furnishes him with extensive exercise material and applied problems to strengthen his comprehension of these principles and of their usefulness."[1] The fact that this is part of the avowed philosophy in many textbooks does not necessarily mean that it is always present in the content material. All too frequently the textbooks follow a definition, illustration, and practice format.

It seems clear from the literature that discovery is not just a method but that it embodies many methods and strategies. Hendrix cites four methods of discovery which

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[1] Mary P. Dolciani, Simon L. Berman, and William Wooton, Modern Algebra and Trigonometry Structure and Method, Book 2, (Boston: Houghton Mifflin Company, 1963), p. 2.

she labels inductive, nonverbal awareness, incidental and deductive.[2] Henderson has identified the following seven strategies for discovery: analogy, simple enumeration, agreement, difference, difference and agreement, concomitant variation, and independent action.[3] Schaaf classifies methods of generalization as empirical and rational procedures. He states that simple enumeration, analogy, continuity of form, and statistical procedures belong to the empirical category, while deduction, variation, formal analogy, and inverse deduction are rational procedures.[4]

Among the arguments advanced in support of discovery are that it is the essence of mathematical thinking; it can be applied to other fields; and it creates greater involvement and interest among students. Beberman, for example, states that "the discovery method develops interest in mathematics and power in mathematical thinking. Because of the students' independence of rote rules and routines, it also develops versatility in applying mathematics."[5] Bruner

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[2]Gertrude Hendrix, "A New Clue to Transfer of Training," Elementary School Journal, XLVIII (December, 1947), p. 197.

[3]Kenneth B. Henderson, "Strategies for Teaching by the Discovery Method," Updating Mathematics, I (November, 1958), pp. 57-60, and I (April, 1959), pp. 61-64.

[4]Oscar Schaaf, "Student Discovery of Algebraic Principles as a Means of Developing Ability to Generalize," The Mathematics Teacher, XLVIII (May, 1955), pp. 324-27.

[5]Max Beberman, An Emerging Program of Secondary School Mathematics (Cambridge: Harvard University Press, 1958), p. 83.

lists the benefits which accrue from discovery as increase in intellectual potency, a shift from extrinsic to intrinsic rewards, a provision for learning the heuristics of discovery, and an aid in the conservation of memory.[6] Suchman, who has experimented in the field of inquiry training, postulates that "some have been prompted to reformulate their methods to capitalize on the intense motivation and deep insight that seem to accrue from the 'discovery' approach to concept attainment."[7] Kersh claims that students acquire what psychologists call a "learning set" or strategy for discovery which assists them in the solution of new problems.[8] Others who advocate the use of the discovery method concur with these claims and consistently expound the superiority of discovery-orientated learning.

Ausubel is one of the educators who question the extensive claims made for discovery. He concedes that it can be valuable in the early stages of learning and in the teaching of the scientific approach to problem solving. Nevertheless, he feels that the crucial issue is not whether

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[6] Jerome S. Bruner, Essays for the Left Hand (Cambridge: Harvard University Press, 1963), p. 83.

[7] J. Richard Suchman, "Inquiry Training: Building Skills for Autonomous Discovery," Merrill-Palmer Quarterly of Behavior and Development, VII (April, 1961), p. 147.

[8] Bert Y. Kersh, "Learning by Discovery: Instructional Strategies," The Arithmetic Teacher, XII (October, 1965), pp. 414-17.

learning by discovery enhances learning, retention, and transferability, but whether it does so sufficiently for those who are capable of learning meaningfully without it to warrant the time spent. In addition, he questions the feasibility of discovery as a technique for transmitting content to students who have mastered the rudiments and vocabulary of a subject. It is his contention that such students can accomplish as much, in as proficient a manner, and in less time, by means of good expository teaching.[9]

The obvious lack of unanimity of opinion regarding the relative merits of discovery and expository methods is also prevalent in the results of empirical research. This is substantiated in the review of the literature in the second chapter. Wittrock has cited several reasons for the equivocal nature of the results of studies based on discovery. The first of these is the semantic inconsistency in labeling different treatments. Some of the researchers were concerned with the amount and kind of external guidance, some with the role of verbalization, and some with the rate of presentation. Wittrock points out that it is necessary to identify more accurately the relevant teaching-learning variables and to direct research based on interactions of the methods with different types of teachers, pupils, and subject matter.

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[9] David P. Ausubel, "Learning by Discovery: Rationale and Mystique," Bulletin of the National Association of Secondary School Principals, XLV (December, 1961), pp. 18-56.

Perhaps the most salient factor in the contradictory evidence is the differing specifications of discovery, guided discovery, and exposition. These terms have not been reduced to uniformly operational definitions by the experts.[10]

If one accepts the premise that various strategies for discovery do exist, and that they constitute unique approaches to the discovery process, then it should be possible to investigate whether these strategies can be taught to students, whether some of the strategies are more applicable to the teaching of mathematics than others, and whether certain topics are more amenable to specific strategies than others. It may be possible to analyse such instructional procedures and techniques and to identify the behavior elicited by each. Such analysis could lead to more accurate prediction of learning outcomes and to more perceptive discernment of the comparative effects of different strategies.

The Problem

The present study is concerned with the use of the strategy of simple enumeration as a technique for discovery in two ways. The first concern is with the effectiveness of this strategy in learning specific mathematical material.

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[10] M.C. Wittrock, "Verbal Stimuli in Concept Formation: Learning by Discovery," Journal of Educational Psychology, LIV, No. 4, (1963), pp. 183-90.

The second concern is whether students who have used this strategy will display marked superiority on transfer tasks which may or may not be related to the mathematical material. Specifically, the present study will attempt to answer the following questions: Can students be taught to discover generalizations using the strategy of simple enumeration? Are students who have acquired this strategy superior to those who have been taught by an expository method when mastery of subject matter is measured? Is there any significant difference in the ability of the two groups to transfer their knowledge to similar but unfamiliar mathematical materials? Does the group which has used the strategy of simple enumeration achieve at a significantly higher level than the group which has been taught by an expository method when the task requires generalization from examples which are not directly related to mathematical materials?

Definition of Terms

Discovery. Bruner describes discovery as "a matter of rearranging or transforming evidence in such a way that one is enabled to go beyond the evidence so reassembled to new insights."[11] The crux of discovery in the present study is the recognition and understanding of relationships among

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[11] Bruner, op. cit., p. 82.

concrete examples, the application of this recognition, and the operation of putting it into a compact rule. The students are presented with an ordered, structured series of examples which are designed to maximize the opportunity for discovery of a generalization associated with the examples. By studying examples and answering questions, the students are expected to discover the underlying principle and formulate a symbolic generalization. Although students are expected to use the generalization in the solution of practice problems, they are not expected to verbalize the principle before using it.

Expository method. The initial step in instruction by the expository method is the derivation and presentation of the rule by the instructor. A complete explanation of the rule is given both verbally and symbolically. This is followed by the working of several examples illustrating the principle. To minimize rote memorization, emphasis is placed on the relation of the examples to the principle involved.

Strategy. This is a plan of action designed to lead the students to knowledge, skills, and attitudes. The measurement of the degree to which the strategy has been acquired is in terms of increase in mean scores between a pretest and a posttest involving generalization from examples.

Simple enumeration. This consists of the presentation of many instances of the generalization to be discovered. The students form hypotheses based on the examples and test these to determine which is correct. One counter-example is sufficient to warrant rejection of a hypothesis.

Hypotheses to be Tested

In general, it is assumed that there can be sufficient distinction made in teaching methods so that it is possible to compare the methods in terms of student performance on mathematical and transfer tasks. To acquire experimental evidence for this study, the investigator spent three weeks during the month of May, 1968 teaching six Grade XII classes at Delbrook Senior Secondary School in S. D. No. 44 (North Vancouver). The following hypotheses were tested in the study.

1. Treatment with simple enumeration as a strategy for discovery will yield the same mathematical achievement effect as treatment by an expository method.

2. Treatment with simple enumeration will yield greater mathematical transfer effect to unfamiliar mathematical materials than treatment with an expository method.

3. Treatment with simple enumeration will yield more transfer effect to non-mathematical materials than treatment with an expository method.

Justification for the Study

Since there are entire programs being developed in the fields of science and mathematics which are based on the philosophy of discovery, it would seem expedient to investigate specific facets of this approach. One of the crucial related problems or questions which is classified as "unanswered" by researchers in mathematics is, "What are the optimum methods for inducing and utilizing discovery methods?"[12] The present study deals with one aspect of the discovery process and is an effort to determine whether the use of simple enumeration to lead to generalizations is an effective procedure. If students who have been exposed to this strategy for discovery during the study of a specific unit in mathematics show evidence of superiority in dealing with a transfer task, then it can be interpreted as evidence that they have acquired a "learning set" or strategy for discovering generalizations which is superior to that of students who have been exposed to an expository approach. Since the study is conducted in the classroom environment, using materials from the curriculum of Grade XII mathematics, it should be of interest to teachers involved in teaching at this level, as well as to textbook writers and curriculum consultants.

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[12] Kenneth E. Brown and Theodore L. Abell, Analysis of Research in the Teaching of Mathematics, U.S. Department of Health, Education and Welfare (Washington: 1965), p. 19.

CHAPTER II

A REVIEW OF THE LITERATURE

The purpose of this study is to investigate the effectiveness of simple enumeration as a strategy for discovery. Much has been written concerning the discovery method and many studies have been conducted in an effort to establish empirical verification for the claims made for it. This chapter constitutes a review of some of the pertinent materials related to the discovery method. For organizational purposes the literature is classified according to studies having general relevance to the problem, studies having specific relevance to the problem, literature related to opinions regarding discovery methods, and a brief review of mathematical programs based on the discovery principle.

Studies Having General Relevance to the Problem

Katona conducted intensive studies involving the learning of principles for problem solving. His investigations were directed to the solution of card-trick and match-stick pattern problems and demonstrated the relative ineffectiveness of memorizing verbal principles compared to understanding. He considered that learning which involves meaningful wholes favors transfer to problem solving situations. The subjects who had the greatest success in solving card-trick problems were those who listened to an explanation of the basic problem and watched a step-by-step demonstration. This group was followed in order by the group which learned verbal principles, the group which memorized steps, and the group which had no training.[1]

As a follow-up to Katona's studies Hilgard, Edgren, and Irvine investigated the cause of errors made by the most successful students who apparently understood the work. They designed five methods of training for understanding with a view to determining which method would best eliminate errors in transfer problems. Overall differences in methods were slight. The prevalence of careless errors suggested that it is important in teaching for understanding to devise methods that are open to review or checking. The authors concluded that attitudes of caution or carelessness are more important in determining error level than lack of understanding. The best results were obtained with the method which was easiest to check.[2]

Ray conducted a study comparing directed discovery and "tell-and-do" methods for learning micrometer skills.

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[1] G. Katona, Organizing and Memorizing: Studies in Psychology of Learning and Teaching (New York: Columbia University Press, 1940.)

[2] Ernest R. Hilgard, Robert D. Edgren, Robert P. Irvine, "Errors in Transfer Following Learning with Understanding: Further Studies with Katona's Card-Trick Experiments," Journal of Experimental Psychology, XLVII, No. 6, (1954), pp. 457-64.

While there was no significant difference in manipulative performance based on knowledge of the micrometer or in ability to solve problems, there was significant difference in retention and effective application after one week and six weeks. This difference favored the directed-discovery method.[3]

Haslerud and Myers were concerned with discovery learning as applied to decoding sentences from concrete instances, as opposed to using specific instructions for decoding. The directed procedure was better for original learning but there was no significant difference on the transfer task.[4] On the basis of recorded results the conclusions of the experimenters supporting the contention that derived principles transfer more readily than given principles seems somewhat questionable. Cronbach is highly critical of this study, maintaining that it suffers from prejudice in the analysis of the data and stating that inferences were made on differences in test scores without comparing each test at the discovery-non-discovery level.[5]

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[3]Willis, E. Ray, "Pupil Discovery vs, Direct Instruction," Journal of Experimental Education, XXVI (March, 1961), pp. 271-80.

[4]G. Haslerud, Shirley Myers, "The Transfer Value of Given and Individually Derived Principles," Journal of Educational Psychology, XLIX (December, 1958), pp. 293-97.

[5]Lee Cronbach, "The Logic of Experiments in Discovery," in L.S. Shulman & E.H. Keislar (Ed.) Learning by Discovery: A Critical Approach (Chicago: Rand McNally, 1966), pp. 76-92.

Craig attempted to determine the effect of directing learners' discovery of established relations upon retention and ability to discover new relations. Groups were given different amounts of direction during discovery of the bases determining solution of multiple-choice verbal items. The group receiving greater direction learned more relations on three trials. Both of the groups were discovery groups. Craig interpreted his results as evidence that experimenters should be liberal with information designed to assist learners in the discovery of principles. Large amounts of external direction insure that the learner will have more knowledge to direct future discovery.[6]

Wittrock experimented with deciphering sentences using groups in which the rule and answer were both given, the rule was given without the answer, the answer was given without the rule, and neither answer nor rule was given. The superior groups were those given the rule. The group which had neither rule nor answer required more time to learn and showed higher retention scores than learning scores. The converse was true of the other groups.[7]

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[6] R.C. Craig, "Directed versus Independent Discovery of Established Relations," Journal of Educational Psychology, XLVII (April, 1956), pp. 223-34.

[7] M.C. Wittrock, "Verbal Stimuli in Concept Formation: Learning by Discovery," Journal of Educational Psychology, LIV, No. 4, (1963), pp. 183-90.

Kittell employed a word task with sixth-grade subjects. The three groups were designated as having minimum, intermediate, and maximum direction. The groups received varying amounts of practice with the principles involved. Because the task was difficult, the discovery group averaged less than three out of fifteen principles discovered. The intermediate group had little discovery experience but had practice in application of principles. The third group had little practice time and no opportunity for discovery. The intermediate group was superior in applying principles and in discovering new principles from examples. It was thought that the discovery behavior of the first group may have been repressed rather than reinforced because of the extreme difficulty of the task.[8]

Studies Having Specific Relevance to the Problem

Hendrix investigated to what extent the way in which one learns a generalization affects his ability to recognize an opportunity to use it. She used the concept that the sum of the first 'n' terms of an odd integer sequence is 'n^2'. The findings indicated that the group which discovered the principle independently and left it unverbalized exceeded those who discovered and verbalized, while both of

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[8] Jack E. Kittell, "An Experimental Study of the Effect of External Direction During Learning on Transfer and Retention of Principles," Journal of Educational Psychology, XLVIII (November, 1957), pp. 391-405.

these groups exceeded in transfer those for whom the principle was stated and illustrated. Her claim that the immediate flash of unverbalized awareness is what actually accounts for transfer power, and her separation of the discovery phenomena from the process of composing sentences which express the discoveries provided a new and startling proposition in learning theory. She stated that the dawning of a generalization on an unverbalized awareness level seemed to be an internal process and the indication that the process has occurred is the organism's new power for self-direction. Her experiments were significant at the .12 level but she admits that part of the results were invalidated by the control group. There was also great difficulty in designing an appropriate instrument to test achievement of unverbalized awareness and a transfer test which would present varying degrees of remoteness from the original examples used.[9]

Cummins used discovery in teaching first year calculus students. In his study each group was given two achievement tests, one of which was designed especially for the discovery group and the other for the traditional. The discovery group had significantly better results on the

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[9]Gertrude Hendrix, "A New Clue to Transfer of Training," Elementary School Journal, XLVII (December, 1947), pp. 197-208.

first test but there were no significant differences on the traditional tests.[10]

Retzer and Henderson conducted a study based on the conjecture that if a group of students are taught such concepts as variable, open sentence, universal set, generalization, instances and counter-instances of generalizations, and are given practice in applying the concepts and in writing generalizations, they will be able to state correctly the relations they discover when taught by the method of guided discovery. The experiment involved the use of a Sentences of Logic unit by the treatment group. This group was able to verbalize universal generalizations involved in the criterion measure. The authors feel that the research suggests that an alternative to delaying verbalization of discoveries would be to include the teaching of logical components of universal generalizations as an explicit part of the curriculum. In this way it would be possible to ask for immediate verbalization of a discovered generalization and to expect a great deal of precision on the part of the students.[11] Szabo expresses a similar viewpoint when he says, "In fact, there is evidence to support

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[10] Kenneth Cummins, "A Student Experience-Discovery Approach to Teaching Calculus," The Mathematics Teacher, LIII (March, 1960), pp. 162-70.

[11] Kenneth A. Retzer, Kenneth B. Henderson, "Effect of Teaching Concepts of Logic on Verbalization of Discovered Mathematical Generalizations," The Mathematics Teacher, LX (November, 1967), pp. 707-10.

the fact that too-early verbalization of discovered generalizations with mathematically immature children can be damaging to the learning, due mainly to lack of verbal facility. When students become more mature, they should be encouraged to give precise verbalization of generalizations after they demonstrate an awareness of those generalizations."[12]

Gagné and Bassler did a study of retention on non-metric elementary geometry and found that a narrowing of practice had a negative effect on retention. They also asserted that the major concepts which had been discovered were well retained but the subordinate knowledge used to develop the concepts was quickly forgotten.[13]

Worthen prepared two methods of task presentation at the fifth and sixth-grade levels. His was a classroom experiment extending over a period of six weeks. He measured initial learning, retention, transfer of concepts, and transfer of heuristics. He also conducted an analysis of teacher behavior to ensure adherence to the models in each treatment. Expository groups were superior in initial learning at the .01 level. The discovery group was superior on retention and transfer of heuristics and slightly superior on the transfer task. The superiority of the discovery group

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[12] Steven Szabo, "Some Remarks on Discovery," The Mathematics Teacher, LX (December, 1967), p. 839.

[13] Robert M. Gagné, Otto C. Bassler, "Study of Retention of some Topics of Elementary Non-Metric Geometry," Journal of Educational Psychology, LIV, No. 3, (1963), pp. 123-31.

on the majority of inter-treatment comparisons varied from the .025 to the .08 level.[14]

Gagné and Brown conducted a study on the summation of a series task. The results of their study favored the discovery method. All groups followed self-instructional programs designed to give three approaches. The discovery group was required to discover the rules for the problem series and had no practice in application. The guided discovery group carried through a series of steps to lead to formulation of the rules but had no formal practice in application. The directed group was told the rules and given formal practice. The test measured only the learners' ability to discover new rules from different problem series and was not a test of recall or application. The results showed the best performance by the guided discovery group and the least effective performance by the rule and example group with the discovery group between them.[15] This study is open to criticism on the basis of lack of sound didactic teaching to the non-discovery or rule and example group. The guided discovery group was taught to look for a structural relationship based on the terms of the series. The non-

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[14] Blaine R. Worthen, "Discovery and Expository Task Presentation in Elementary Mathematics," Journal of Educational Psychology Monograph Supplement, LIX (February, 1968), No. 1, Part 2.

[15] Robert M. Gagné, Larry T. Brown, "Some Factors in the Programming of Conceptual Learning," Journal of Experimental Psychology, LXII (October, 1961), pp. 313-21.

discovery group should have been made aware of the existence of the relationship even though they were not required to discover it.

Since the present study is based on one of Schaaf's premises concerning methods of generalizing, a fairly detailed treatment of his original study will be given. He proposed a course in grade nine algebra and conducted a class based on his designed program. The theme chosen as a guideline for the direction of the course was improvement in ability to generalize, and this determined to a large degree the nature of class procedures. Schaaf hypothesized that students would learn to generalize if they were given sufficient practice, and if they discovered for themselves as many as possible of the mathematical principles involved. This required a program which provided guidance and mathematical experiences designed to make students aware of concepts and principles. The tasks which Schaaf set himself in his study were to analyse different processes of generalizing, determine the characteristics of a superior generalizer, formulate lessons and procedures to aid students in developing ability to generalize in mathematical and non-mathematical situations, and evaluate in terms of specified criteria of generalizing ability and mathematical achievement. His criteria of a superior generalizer consisted of fifteen items. Methods of generalizing were classified as empirical or rational. Empirical methods included simple enumeration, analogy, continuity of form, and statistical procedures, while rational methods included deduction, variation, formal analogy, and inverse deduction. Various approaches involving all methods were used in the presentation of the lessons. Evidence from observers' reports, students' notebooks, teachers' notes, and responses on students' reactions indicated that the course did lead students to acquire the desired abilities. Schaaf concluded that his experimental group made significantly greater improvement in ability to draw conclusions, to recognize non-justifiable conclusions, and to interpret graphical data than the status group which was used for comparative purposes. Although less time was spent on the study of algebra than in average classes, the achievement of the experimental group as measured by the Lankton First Year Algebra Test was significantly greater than was predicted by the results on the Iowa Algebra Aptitude Test.[16]

Kersh was concerned in his studies with the motivating effect of learning by discovery. In his studies he used the odd-numbers rule and the constant-difference rule for series as learning tasks. He accepted the premise that learning by discovery is superior and investigated whether a possible explanation was that discovery made learning more meaningful

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[16] Oscar Schaaf, "Student Discovery of Algebraic Principles as a Means of Developing Ability to Generalize," The Mathematics Teacher, XLVIII (May, 1955), pp. 324-27.

in the cognitive sense of understanding or organization. His data suggested the inadequacy of the meaning theory but revealed that students were motivated to continue learning and practising after the formal period of instruction was over. Analyses were made of differences in treatment, differences in test periods, and differences attributed to interaction of treatments and time periods. The guided group was superior to the unguided group in use of rules, retention, and transfer. The rote group was superior to others in all respects but was not included in the results.[17]

Tuckman and others recently reported a study, the purpose of which was to induce a search set in individuals through prior experience and to determine the conditions which would allow the search set to transfer. The task was a four by six matrix of two-digit numbers which were to be added but in each case a shortcut method could be used. They performed three experiments to investigate the effects of appropriate and inappropriate practice experiences on students' tendency to search for and find shortcut solutions to problems. The strategy of looking for, and skill in finding a shortcut, were termed a search set. In the first experiment where the criterion problems resembled practice

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[17] Bert Y. Kersh, "The Adequacy of Meaning as an Explanation for the Superiority of Learning by Directed Discovery," Journal of Educational Psychology, XLIX (October, 1958) pp. 282-92; "The Motivating Effect of Learning by Discovery," Journal of Educational Psychology, LIII (April, 1962), pp. 65-71.

problems, the subjects having search experience were more likely to search for and find correct solutions. In the second and third experiments where the problems were dissimilar to practice problems, the subjects who had search experience were more likely to search for shortcuts but were singularly unsuccessful in finding correct solutions. The researchers concluded that search skill has limited transfer possibilities as compared to search strategy. They recommend a conceptual distinction between searching and finding. Although they started the experiment with the idea of combining these concepts, they concluded that this was not a sound idea and recommended the use of the term "search set" to refer to the strategy of search and perhaps the term "learning set" to the skill of finding a shortcut solution. Another implication of the study was that limited exposure to a problem-solving approach might induce students to adopt a strategy but leave them without the skill to apply it. On this assumption the researchers suggest that it is necessary to provide a level of skill commensurate with the commitment to the strategy or the latter cannot be used effectively. This implies that it is essential to give extensive practice sequences.[18]

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[18] Bruce W. Tuckman and others, "Induction and Transfer of Search Sets," Journal of Educational Psychology, LIX (April, 1968), pp. 59-68.

Literature Concerning Opinions of Discovery

As mentioned in the first chapter, Ausubel is one of the most severe and vociferous critics of the discovery method. In his articles he concedes that, under certain conditions, there is a defensible rationale for discovery. Unfortunately there has been a tendency to use it as a panacea and to attempt to extrapolate its advantages to all age levels, to all levels of subject sophistication, to all educational objectives, and to all kinds of learning. He refutes claims made by Bruner[19] and Hendrix.[20] Hendrix says that verbalization is not only unnecessary for transfer of ideas and understanding but is harmful if used for these purposes, and that language only enters the picture because of a need to attach a symbol or label to subverbal insight so that it can be recorded and communicated to others. Ausubel says that the function of language is not just to label, and that verbalization does more than attach a symbol to thought. It constitutes part of the process of abstraction. An individual who is using language to express an idea is engaged in an intellectual process of generating a level of insight which transcends the subverbal awareness stage in every respect. In explaining the apparent success of programs

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[19] Bruner, loc. cit.

[20] Hendrix, loc. cit.

based on the discovery method, Ausubel hypothesizes that they succeed because they are highly organized and systematic and because they use discovery judiciously in the early stages and gradually attenuate it. The courses have been taught by well-trained and enthusiastic teachers and are a testimonial to didactic verbal exposition.[21] Ausubel's criticisms are based on personal opinion and have not been subjected to empirical verification.

Taba states that learning by discovery as presently pursued pertains to the cognitive aspects of learning and is limited in content to mathematics and science. She feels that there are two aspects of the transactional process--assimilation of content and operation of cognitive processes to organize and use the content. She favors a discovery approach because it is the chief mode of intellectual productivity and autonomy. The individual is better equipped to move into unknown areas, gather data, and abstract concepts. The learner develops an attitude of search and a set to learn and becomes freed from extrinsic rewards. While Taba is a proponent of discovery she does state that there should be a balance between receptive and assimilative

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[21] David P. Ausubel, "Some Psychological and Educational Limitations of Learning by Discovery," The Arithmetic Teacher, XI (May, 1964), pp. 290-302. "Learning by Discovery: Rationale and Mystique," Bulletin of the National Association of Secondary School Principals, XLV (December, 1961), pp. 18-56.

learning and views the task of the educator and curriculum planner as securing this balance in instruction.[22]

Programs Based on the Discovery Method

Interest in research has not been confined to individuals. The University of Illinois has prepared materials for teaching secondary mathematics and for training teachers. In discussing the basic philosophy on which the program is based, Beberman says, "We believe that a student will come to understand mathematics when his text-book and teacher use unambiguous language and when he is enabled to discover generalizations for himself."[23] The technique of delaying verbalization of important discoveries is a prominent feature of the UICSM program. It is also characterized by careful sequencing and structuring of concepts and by pupil involvement in the discovery of these concepts. Davis, with the Madison Project, is also attempting to put the discovery aspect into the mathematics curriculum for all grades.[24] Although the philosophy of the authors is discovery-orientated, the apparent success of the

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[22] Hilda Taba, "Learning by Discovery, Psychological and Educational Rationale," Elementary School Journal, LXIII (March, 1963), pp. 308-15.

[23] Beberman, op. cit., p. 4.

[24] Robert B. Davis, "Madison Project of Syracuse University," The Mathematics Teacher, LIII (November, 1960) 571-75.

programs cannot be interpreted as confirmation of the claims made for discovery. Many factors are operative and no accurate measure of any of the contributing factors has been attempted.

Summary

The majority of the research would appear to favor the use of some form of discovery. While the superiority of discovery techniques is not always evident in the immediate learning situation, most experimenters concede that these techniques are more effective with reference to transfer and retention. Because of the wide diversity of materials used, the range in the age of subjects, and lack of uniformity in definition of methods, the precise areas in which discovery techniques produce maximum positive effects on learning have not been established. Much of the research is general in nature and little attention has been directed to the analysis of the many strategies which are involved in the discovery process. In addition, a great deal of the current literature is based on opinion and has not been subjected to rigidly controlled research. Therefore, it cannot be accepted as factual information concerning the relative merits of discovery and expository methods.

CHAPTER III

PROCEDURE AND DESIGN

This chapter is concerned with the design of the study and the procedures used to implement the design. It contains information pertaining to instructional methods and materials, subjects, tests and measures, statistical procedures, and limitations of the study. Complete details of lesson procedures with written exercises and copies of tests constructed by the experimenter are contained in Appendices A and B respectively.

Design

As previously indicated, two treatments were used in this study. One treatment involved a lecture or expository method of presenting specific mathematical material and the other involved the presentation of the same material by the use of simple enumeration as a strategy for discovery. In both treatments seven one-hour periods were devoted to the study of arithmetic and geometric progressions. In the expository treatment each period commenced with a formal presentation by the instructor of pertinent principles and definitions related to a particular phase of the unit. During the development of the lesson, no attempt was made to have the students participate in, or contribute to a discussion. If questions were posed by students, they were immediately answered by the instructor. At the completion of the lecture-type presentation, the students were given written exercises comprised of two basic types of questions--those involving a direct application of the rules which had been developed in the lesson, and those which were problem-type questions designed to show a practical application of mathematical concepts. Some of the written exercises were completed during class time with the instructor moving about the room and answering any questions which were asked by students. The solutions to the exercises were made available only after the exercises had been completed. Students were expected to check their own work and any difficulties which arose out of the exercises were dealt with either with the class as a whole or with individuals at an appropriate time.

The second treatment made use of the strategy of simple enumeration to lead to the discovery of generalizations. Students were presented with examples of the principles to be discovered, followed by a series of questions structured to maximize the opportunity for discovery. Because there is a tendency for only a few students to be involved in the discovery process when this type of lesson is presented orally, the students were given individual copies of the examples and questions in an effort to ensure that the maximum number of students actually discovered the desired principles. After students had discovered the principle and worked several examples, they proceeded to the written exercises. For those who were unable to discover the principles from the original examples, further illustrations were provided on the blackboard. The only direct information given to students was concerned with notation or labeling of concepts. It was felt that precise terms were essential for effective communication in the written exercises. As in the case of the previous treatment, the instructor moved about the room but responded to questions in a different manner. Instead of giving direct answers, the instructor asked further questions which were designed to promote understanding. The solutions were provided in the same manner as for the expository treatment.

Subjects

At the outset of the study, 158 twelfth-grade students in 6 classes at Delbrook Senior Secondary School in S. D. No. 44 (North Vancouver) made up the sample population for the two treatments. This is an academically-orientated school with a total population of approximately 750 students, and is located in an upper middle class socio-economic area. During the course of the study, thirty-one of the students were eliminated because of absence from a period of instruction or a test. The students had been randomly assigned to classes by computer at the beginning of the school term, and with the exception of one class, all had been taught by one teacher throughout the year. For the present study three classes were randomly assigned to each treatment and all classes were taught by one experimenter.

Control

It is conceded that many factors are operating in a classroom situation, and that rigid control of sources of extraneous variation over a period of time in such circumstances is virtually impossible. However, an attempt was made to eliminate as many sources of extraneous variation as possible. The fact that the classes were all taught by the experimenter eliminated the teacher variable and any interaction due to the presence of a different instructor in the classroom was comparable for all classes. Since the experimenter had previously taught at Delbrook and was known to the students, the 'Hawthorne effect' would be minimal if it existed at all. In addition, there was no difference in the time allocated to each group, in the examples, or in written exercises. The only major difference in the conduct of the classes was in the diversification of method of presenting the materials. Because all other aspects of the learning situation were treated alike, it should be possible to attribute any major differences in results on criterion tests to the variation in teaching method.

Instructional Materials

The unit on arithmetic and geometric progressions is part of the mathematics curriculum at the Grade XII level in the Province of British Columbia. Although no textbook was used during the course of the experiment, the lessons were based on Chapter XIII of the prescribed textbook which is "Modern Algebra and Trigonometry, Book 2" written by Dolciani, Berman, and Wooton and published by Houghton Mifflin Company. This unit was chosen because, in the opinion of the experimenter, it is suitable for the use of the strategy of simple enumeration, and because it is new material and should be less subject to effects created by previous knowledge.

Measuring Instruments

Lorge-Thorndike Intelligence Test. Before any content material was taught, the Lorge-Thorndike Intelligence Test, Form 1, Level H of the 1964 Multi-Level Edition was administered to all students. The first part of the test is composed of one hundred items in five subtests classified as vocabulary, verbal classification, sentence completion, arithmetic reasoning, and verbal analogy. The remainder of the test is a Nonverbal Battery of eighty items subdivided into three tests which are categorized as pictorial classification, pictorial analogy, and numerical relationships. Separate differential I. Q. scores are given for Verbal and Nonverbal Batteries but these can be combined to form a composite I. Q. score which is the simple unweighted average of these two scores. On the basis of the composite score, students were assigned to high, medium, and low I. Q. subgroups.

The Multi-Level test was standardized on a nation-wide sample of Grade XII students in the fall of 1963 and bears a 1964 copyright date. The mean is set at 100 with a standard deviation of 16. It is stated that the average college freshman class may have a mean as high as 115 to 130 and the spread of scores about these values will be considerably more restricted than in the general population. The statistical information contained in the administration manual is limited because of the recency of the date of revision of the test. It does indicate that preliminary data obtained when the order of presentation of Form 1 and Form 2 was rotated yielded reliability coefficients of .90 for the Verbal Battery and .92 for the Nonverbal Battery. Because the sample used to establish these coefficients was small, the authors state that the results should only be regarded as suggestive. No information is available on the validity of the Multi-Level Edition. This test represents a revision and refinement of the Separate Level Edition published in 1954. Information with respect to predictive validity of the latter is meager in that a correlation coefficient of .67 at the ninth-grade level is the only statistic quoted. Concurrent validity of this edition as measured by correlation with three other well-known group intelligence tests was .77, .79, and .84 for the Verbal Battery and .65, .71, and .74 for the Nonverbal Battery. The authors feel that the Multi-Level Edition has a higher reliability than the Separate Level Edition and that correlations with other tests would be at least as high as those of the latter.

There is general concurrence of opinion among the reviewers of the Lorge-Thorndike tests that they are among the best of the group intelligence tests. They are well-designed and constructed, and they provide reliable measures of verbal and nonverbal reasoning. The only criticism proffered is that there is a lack of adequate data on predictive validity.[1]

The Nonverbal Battery of Form 1 was used as a pretest score to measure the ability of the students to see relationships among examples, discover a principle, and apply the principle. At the end of the instruction period the Nonverbal Battery of Form 2 was used as a posttest. These pretest and posttest scores were used to determine whether there was any significant growth in ability to generalize within the groups, and also to compare any differences between groups which might be attributed to the variation in treatment.

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[1] Oscar K. Buros, (ed.), The Fifth Mental Measurements Yearbook, (Highland Park, New Jersey: The Gryphon Press, 1959), pp. 478-84.

Mathematical Content Test. This test consisted of twenty-five multiple choice items constructed by the experimenter and administered during a period of forty minutes. The items were designed to measure the degree to which the students had acquired a knowledge of the principles involved in the unit on arithmetic and geometric progressions. The content validity of each item was reviewed by the regular classroom teacher and the experimenter. Upon completion of data collection, the internal consistency of each item as an index of item reliability was examined by means of point biserial correlation coefficient between the total score and responses to each item. The correlation coefficients ranged from .04 to .49, with a mean correlation of .30. Three items had correlation coefficients less than .20. Although it is apparent that some of the items were non-discriminatory, all were included in the analysis of results.

Mathematics Transfer Test. This instrument was also constructed by the experimenter. It was a twenty-item test based on mathematical material somewhat related to sequences but requiring a transfer and extension of the knowledge acquired. The items were comprised of examples from which students were required to generalize. Point biserial correlation coefficients ranged from .12 to .54 with a mean correlation value of .39. Only one item of the test had a correlation coefficient less than .20. The time allocated for the test was thirty minutes.

Statistical Procedures

The processing of data was done at the University of British Columbia Computing Centre using the generalized t-test. The basic computations performed by this test are the calculation of means and standard deviations of sets of variables, and the calculation of 't' values for specified combinations of these variables.

Limitations of the Study

A clear and unmistakable limitation of this study lies in the experimenter-constructed measuring instruments. From the information concerning the point biserial correlation coefficients, it is obvious that the tests require refinement. In the case of the Mathematics Content Test, the experimenter feels that it could be improved by increasing the number of items with a corresponding increase in the time allotted. The fact that there were many omissions in the solutions of the Mathematics Transfer Test would indicate that insufficient time was given for this test. It is not suggested that all items could or should have been completed by all students but the task appears to have been more difficult than anticipated for the time assigned to it.

A second limitation of the study is one which is commonly held to be true of all classroom experiments. McDonald states that "even with the best intentions on the part of school personnel, ordinary school and class conditions are not highly suitable for experimentation." He suggests that task and method variables ought to be tested under controlled conditions and that it is better to do a small study in which a few well-defined variables may be manipulated efficiently.[2] It could, of course, be argued that an experiment conducted within the time limits and learning conditions representative of typical school behavior and curriculum might well be generalized to classroom situations with a greater degree of confidence than could be assumed from a short-term laboratory experiment.

A further limitation of the study is that it is based on a comparatively short unit of instruction and is done with what might be considered a select group of students. The experimenter feels that this group is representative of the population taking the Grade XII mathematics course in the lower mainland or any other urban area of British Columbia. It is doubtful that the results are applicable to populations in outlying districts because students in these schools may be taking mathematics because of a lack of choice of options, whereas, in more populated areas, only students who are on the academic-technical program and want mathematics for a major are likely to enrol in the course.

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[2]Frederick J. McDonald, "Meaningful Learning and Retention: Task and Method Variables," Review of Educational Research, XXXIV (1964), p. 542.

This study is limited to the use of only one of a possible eight strategies which have been suggested for the purpose of discovery. It is obvious that most students will have had previous experience with this strategy but it is hoped that the newness of the material may help to counteract the previous experience.

Summary

In this study, six Grade XII mathematics classes were assigned to two treatment groups. During the course of seven class periods the groups were taught a unit on arithmetic and geometric progressions. An expository method was used in teaching one group while the strategy of simple enumeration to lead to discovery was used in the second group. On the basis of results from criterion measures, comparisons were made with respect to acquisition of knowledge and transfer ability by using the t-test. The major limitations of the study are inherent in the experimenter-constructed tests and in the classroom environment in which the study was conducted.

CHAPTER IV

RESULTS OF THE STUDY

The purpose of this chapter is to present and interpret the data obtained in the study, and in so doing, to test the hypotheses enumerated in the first chapter.

Analysis of the Data

Using criteria based on mathematical content, mathematical transfer, and general transfer, the results of teaching by simple enumeration were compared with those of teaching by expository method. The specific hypotheses were tested at the .05 level by the use of the t-test. The basic assumptions associated with the use of this statistic are normal distribution, homogeneity of variance, and independent observation within each sample and between groups. Some researchers have suggested that the importance of the first two assumptions may be overrated. Lindquist says that unless variances are so heterogeneous as to be readily apparent, the effect on the test will be negligible.[1] Boneau contends that in a large number of research situations the probability statements resulting from the use of 't' tests, even when the

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[1] E. Lindquist, Design and Analysis of Experiments, (Boston: Houghton Mifflin, 1953), pp. 78-86.

first two assumptions are violated, will be highly accurate.[2] Under the conditions of the present study the t-test was deemed to be an adequate and suitable statistic.

<u>Equivalence of the Two Treatment Groups on the Basis of I. Q. and Previous Term Marks</u>

Prior to testing the hypotheses, the groups were compared with respect to I. Q. and previous term marks. Since these criteria are considered to be predictors of achievement, any major differences in these areas would have a marked effect on results and would require adjustment by means of analysis of covariance.

The Lorge-Thorndike Intelligence Test, Form 1, Level H of the 1964 Multi-Level Edition was administered to all students. Students were then arbitrarily classified into high, medium, and low I. Q. subgroups within each treatment. Those having differential I. Q. scores of 125 or more were in the high category, those with scores from 114 to 124 were in the medium category, and those less than 114 were in the low I. Q. category. This resulted in a distribution of 16 high, 33 medium, and 15 low for the discovery group and 9 high, 33 medium, and 21 low for the expository group. The reason for the classification was to enable the experimenter to determine whether any significant differences in treatment

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[2] C. Boneau, "The Effects of Violations of Assumptions Underlying the t Test," <u>Psychological Bulletin</u>, LVII (1960), pp. 49-64.

groups ranged through all I. Q. levels or whether they could be attributed to a specific level.

The results of the comparisons for equivalence of the groups are shown in Table I. It indicates that there are no significant differences in the total group with respect to either criterion. Examination of the subgroups indicates that the minor difference in I. Q. is located in the low level category. These results signify that the treatment groups are equivalent on the basis of I. Q. and previous term marks, and that analysis of covariance to adjust for initial differences is unnecessary.

<u>Hypotheses Testing</u>

There are three major hypotheses to be tested in this study. Essentially they are concerned with the effects of two different treatments in teaching a unit on arithmetic and geometric progressions at the Grade XII level. The hypotheses are first examined according to the treatment effect on the total sample in each group. Then the effects on subgroups classified according to I. Q. level are analysed where applicable.

The hypotheses that treatment with simple enumeration will yield the same level of mathematical achievement as treatment with an expository method was accepted. The results for treatment groups and I. Q. levels as shown in Table II show no significant differences between groups or at any level.

TABLE I

MEANS OF I. Q. SCORES AND OF PREVIOUS TERM MARKS
BY TREATMENTS AND I. Q. LEVELS

	I. Q.				TERM MARK			
	Discovery	Expository	df	t	Discovery	Expository	df	t
Total	119.63 (8.41)	116.70 (8.98)	125	1.895	67.41 (13.68)	65.94 (10.64)	125	0.675
High	130.88 (5.43)	130.44 (3.61)	23	0.212	73.63 (12.91)	76.33 (14.04)	23	-0.488
Medium	118.73 (3.37)	119.36 (2.79)	64	-0.836	64.61 (13.54)	66.79 (8.47)	64	-0.785
Low	109.60 (2.64)	106.62 (5.38)	34	1.977	66.93 (13.45)	60.14 (8.48)	34	1.859

Note: Figure in parentheses is standard deviation for each corresponding mean. Examination of standard deviations for treatments shows homogeneity of variances involved.

TABLE II

MEANS OF MATHEMATICS ACHIEVEMENT SCORES BY TREATMENTS AND I. Q. LEVELS

	Discovery	Expository	df	t
Total	17.81 (2.91)	17.06 (3.23)	125	1.375
High	19.06 (2.35)	19.44 (3.36)	23	-0.334
Medium	17.39 (3.05)	17.09 (2.90)	64	0.414
Low	17.40 (2.90)	16.00 (3.26)	34	1.330

Note: Figure in parentheses is standard deviation for each corresponding mean.

The inference drawn from these results is that the use of simple enumeration as a strategy for discovery is not any more effective than teaching by an expository method when the criterion is concerned with acquisition of specific knowledge or mastery of subject matter.

The information in Table III supports the hypothesis that treatment with simple enumeration will yield more mathematical transfer than treatment with an expository method. On a total sample basis the superior achievement of the simple enumeration treatment group is significant at the .01 level. The breakdown by I. Q. levels shows that, at the high I. Q. level, achievement was approximately equal in both treatments, whereas there was significant superiority for the simple enumeration treatment at the medium level, and substantial, although not significant, superiority at the low level. Since the high I. Q. subgroup in the expository treatment achieved on a similar level to the comparable group in simple enumeration, it seems clear that treatment at this level had little relevance. The explanation for this lack of change is probably that since these students had shown marked ability to generalize in the pretest, the experience gained in the mathematical unit was not as effective for them as for other groups. The results at the other levels indicate that ability to generalize from simple enumerations can be improved in a relatively short time with a minimum of practice.

TABLE III

MEANS OF MATHEMATICS TRANSFER TEST SCORES
BY TREATMENTS AND I. Q. LEVELS

	Discovery	Expository	df	t
Total	8.30 (3.30)	6.76 (2.98)	125	** 2.752
High	9.44 (3.50)	9.33 (3.50)	23	0.071
Medium	8.15 (3.28)	6.64 (2.43)	64	* 2.131
Low	7.40 (2.97)	5.86 (3.04)	34	1.516

Note: Figure in parentheses is standard deviation for each corresponding mean.

* Significant at .05 level.

** Significant at .01 level.

The hypothesis that treatment with simple enumeration will yield more transfer effect to non-mathematical materials than treatment with an expository method was rejected. Consideration of this hypothesis led to certain pertinent related questions. Did either or both treatments lead to improved ability to generalize from examples? If so, was the improvement significant? Did treatment with simple enumeration lead to greater measurable improvement than treatment with an expository method? To assess the ability of students to generalize prior to any treatment, the score of the Nonverbal Battery of the Lorge-Thorndike Intelligence Test, Form 1 was used as a pretest measure. The subtests involve pictorial classification, pictorial analogy, and numerical relationships. According to the authors they measure ability to see relationships and generalize. Following the period of instruction, the Nonverbal Battery of Form 2 was administered as a posttest. Table IV, which shows the pretest and posttest means by treatment groups and I. Q. levels, indicates that both treatment groups showed significant improvement. The conclusion based on the evidence of this table is that the improvement was not attributable to treatment. Table V, which specifically compares the posttest scores of the two treatments corroborates this conclusion. The implication is that the use of simple enumeration as a strategy for discovery is no more effective than teaching by an expository method when the transfer test is non-mathematical. This is

TABLE IV

MEANS OF POSTTEST AND PRETEST SCORES OF NONVERBAL BATTERY BY TREATMENTS AND I. Q. LEVELS

	DISCOVERY				EXPOSITORY			
	Posttest	Pretest	df	t	Posttest	Pretest	df	t
Total	**52.14 (7.93)	46.67 (7.34)	126	*4.049	49.57 (7.77)	44.79 (8.62)	124	*3.269
High	58.75 (8.84)	55.63 (5.58)	30	1.196	56.00 (6.48)	54.55 (6.58)	16	0.469
Medium	51.15 (5.94)	45.52 (4.06)	64	*4.500	51.97 (5.55)	47.61 (5.00)	64	*3.356
Low	47.27 (6.40)	39.67 (4.89)	28	*3.654	43.05 (6.94)	36.19 (6.27)	40	*3.359

* Significant at .002 level.

** Figure in parentheses is standard deviation for each corresponding mean.

TABLE V

MEANS OF POSTTEST SCORES BY TREATMENTS AND I. Q. LEVELS

	Discovery	Expository	df	t
Total	52.14 (7.93)	49.57 (7.77)	125	1.844
High	58.75 (8.84)	56.00 (6.48)	23	0.815
Medium	51.15 (5.94)	51.97 (5.55)	64	-0.578
Low	47.27 (6.40)	43.05 (6.94)	34	1.857

Note: Figure in parentheses is standard deviation for each corresponding mean.

in contrast to the results obtained when the test involved mathematical transfer. There are at least two possible explanations for these results. The first is that the unit covered in the mathematics classes was concerned with the structural relationships of numbers in sequences and this would tend to make students more aware of patterns when they wrote the posttest. Recognition of patterns is basic to the discovery of generalizations but it would appear that having relationships explained is equally as effective as discovering them for oneself. The second explanation is that there was only a time lapse of three weeks between tests and the students probably felt familiar with the format of the posttest.

Summary of Results

The findings of this study indicate the use of simple enumeration as a strategy for discovery of general principles is more effective than teaching by an expository method when the criterion is a transfer test which emphasizes mathematical content related to the specific material taught to students. However, when the criterion seeks to measure mathematical knowledge acquired in the study of a specific unit, there is no significant difference attributable to method. The significant gain in general transfer ability as measured by pretest and posttest scores, was comparable for both treatment groups. This gain appeared to be related to an awareness of the existence of patterns in sequences of numbers, rather than to any particular treatment.

CHAPTER V

SUMMARY AND CONCLUSIONS

The Problem

This study was specifically concerned with the effectiveness of the use of simple enumeration as a strategy for discovery as compared to an expository approach. It was hypothesized that the two treatments would yield the same mathematical achievement, but the simple enumeration treatment would yield more mathematical and non-mathematical transfer than the expository treatment.

Procedures

The material chosen for the study was arithmetic and geometric progressions as outlined in the Grade XII curriculum for the Province of British Columbia. The subjects were enrolled in six classes at Delbrook Senior Secondary School in S. D. No. 44 (North Vancouver). Three classes were arbitrarily assigned to each treatment, and treatment groups were subdivided on the basis of I. Q. as measured by the Lorge-Thorndike Intelligence Test, Form 1, Level H. The experimenter taught both groups for a period of three weeks. Students who missed a period of instruction or a test were eliminated from the study when the data were analysed. In the final results, there were 63 students in the expository group and 64 in the discovery group.

At the outset of the study, the groups were compared to see whether any significant differences existed with respect to I. Q. or previous term marks. The groups were regarded as equivalent with respect to these covariates since there were no significant 't' values in the comparison of means. Since all other important factors were controlled, it could, be assumed that any major differences in achievement or transfer were attributable to variation in teaching methods.

In order to investigate the effectiveness of simple enumeration as a strategy for discovery, the groups were compared on the basis of criteria which measured mathematical achievement, mathematical transfer, and non-mathematical transfer.

Findings

The mean achievement of the groups on the criteria was compared by the use of the t-test with the level of significance set at .05. Decisions on the three major hypotheses were as follows:

1. The hypothesis that simple enumeration would yield the same mathematical achievement as treatment by an expository method was accepted. The results for the total treatment samples and for I. Q. levels were approximately equal.

2. The experimental hypothesis that treatment with simple enumeration would yield more mathematical transfer than treatment with an expository method was also accepted. Analysis on the basis of I. Q. levels revealed that the majority of the difference was located at the medium level.
3. The experimental hypothesis that treatment with simple enumeration would yield more non-mathematical transfer effect was rejected. Both treatment groups showed significant improvement in ability to see relationships and generalize but the treatment had no apparent influence on the improvement.

Conclusions

The first three conclusions enumerated below are based on the observed results of the study. The remainder, while not substantiated by data, are included as observations of the experimenter and are an attempt at objective analysis of the average classroom situation.

1. There is no clear-cut advantage in the use of simple enumeration for teaching mathematics if the concern is centred on acquisition of facts.
2. The use of simple enumeration has a mathematical transfer effect which does not appear to accrue when the same material is taught by an expository method. This transfer effect does not extend to

non-mathematical material but is only prevalent when the task involves a situation with which students are familiar.

3. The results of the pretest-posttest measures should be a reminder to teachers to exercise extreme restraint in the interpretation of I. Q. test results. The particular test used in this study can be regarded as a measure of verbal and non-verbal reasoning but cannot be regarded as a measure of some intangible factor called mental capacity. It is obvious from the pretest-posttest improvement exhibited by both treatment groups that the scores are largely a function of environment and experience.

4. There is a need for greater emphasis on a discovery attitude and approach, not only in the classroom, but by textbook writers. An examination of the average textbook shows a "tell-and-do" approach which leads to rote memorization rather than understanding. It is the opinion of the experimenter that the discovery attitude must start at the earliest elementary level and be developed throughout the grades. It is difficult to change attitudes to any measurable degree at the secondary level. The stress placed on this aspect of learning in

philosophy underlying curricula is not evident in practice.

5. There is a need to de-emphasize the "mark" approach to learning.
6. It is not suggested that all mathematical material should be taught by a discovery method. A judicious mixture of methods is probably most effective. Under existing conditions in the average classroom, many students never experience the satisfaction of solving problems independently and with understanding, nor do they acquire any ability to analyse problem situations.

Suggestions for Future Research

1. The present study could be replicated after revision of the experimenter-constructed tests. It could also be designed at a different grade level with different instructional material.
2. There are many possibilities for enlargement of the scope of the present study to include more strategies, measures of attitude, and measures of retention.
3. It is possible that valuable information could be obtained by designing a study based on individual instruction rather than on a classroom situation. This would facilitate more rigid control and would

also provide an opportunity to analyse solution processes more accurately.

4. There is need for research designed to investigate the most suitable methods for teaching the high I. Q. students. Such methods should not only maximize their rate of progress but should provide them with challenging tasks through which they will have an opportunity to develop new skills and strategies.

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APPENDICES

APPENDIX A

PAGE

TEACHING PROCEDURES

Introduction

The concept of a function was reviewed because it is on the basis of this concept that the teaching of sequences is structured. The review was neither long nor detailed because of the students' considerable previous experience with functions. There was no differentiation in treatment in the review because it was not considered to be part of the subject matter on which the experiment was based. In both groups the review was largely conducted by questioning and examples.

Materials

Concept of a function - a set of ordered pairs.

The domain and range of a function and the function rule.

Notation - $f(x) = 2x + 1$, x 1 (the set of integers)

Examples of functions which students have previously had such as;

$f(x) = x^2$ $(x \in R)$ $f(x) = \cos x$ $(0^\circ \leq x \leq 360^\circ)$ $f(x) = x^3$ $(x \in R)$

Given the tables below, write the rule which determines f(x).

x	1	2	3	4	5	6	7	8
f(x)	1	3	5	7	9	11	13	15

x	1	2	3	4	5	6	7	8
f(x)	2	4	8	16	32	64	128	256

x	1	2	3	4	5	6
f(x)	1/2	1/4	1/8	1/16	1/32	1/64

In the examples above, the terms indicated by $f(x)$ form a sequence. A sequence is related to a function. The domain of the function is the set of counting numbers. $f(x)$ is found according to the function rule and is an ordered set in one-to-one correspondence with the counting numbers. Thus, if f is a function, a sequence could be defined as $f(1)$, $f(2)$, $f(3)$ etc.

LESSON I

Concepts

1. A sequence is a set of numbers in one-to-one correspondence with the set of natural numbers.

2. An arithmetic sequence has the special characteristic that there is always a common difference between terms.

3. The nth term of an arithmetic sequence is defined to be $t_n = a + (n - 1)d$, where 'a' is the first term, 'd' is the common difference, and 'n' is the number of terms.

4. The terms sequence and progression are synonymous in describing sets of numbers.

Method for Expository Group

1. Definition of arithmetic sequence: A sequence is <u>arithmetic</u> if and only if each term after the first can be obtained from the preceding term by adding to it a fixed number called the <u>common difference</u>.

   Examples: 1, 2, 3, 4, 5, 6, ...
   1, 3, 5, 7, 9, ...
   8, 11, 14, 17, 20, ...
   7, 4, 1, -2, -5, ...
   1, 3/2, 2, 5/2, 3, ...

   Note that the common difference can be positive or negative. It is not difficult to identify an arithmetic sequence or to continue the sequence when the first term and common difference are known.

2. To find a given term of an arithmetic sequence:

   Example: 4, 7, 10, 13, ...

   Suppose that one wished to know the 50th term of this sequence.

   Note the following pattern:

Term No.

$1 = 4$
$2 = 4 + 3$
$3 = 4 + (2)3$
$4 = 4 + (3)3$
$5 = 4 + (4)3$
$10 = 4 + (9)3$
$n = 4 + (n - 1)3$

This pattern is applicable to all arithmetic sequences. In general, if the first term is represented by 'a', the common difference by 'd', and 'n' is any given term, then the nth term is represented by t_n and is defined by $t_n = a + (n - 1)d$.

3. Examples: Given the A. P. 5, 9, 13, 17, ...

By inspection a = 5 and d = 4.

Find the 8th term and the 50th term.

$$t_8 = 5 + (8 - 1)4$$
$$= 5 + 28$$
$$= \underline{33}$$
$$t_{50} = 5 + (50 - 1)4$$
$$= 5 + (49)4$$
$$= \underline{201}$$

If any three of the terms a, d, n, or t_n are given, the other term can be found by application of the formula.

Example: In an arithmetic sequence the first term is 3 and the 24th term is 72. Find the common difference and write the first three terms of the sequence

$$t_{24} = 3 + (24 - 1)d$$
$$72 = 3 + 23d$$
$$d = 69/23$$
$$= \underline{3}$$

The first three terms of the sequence are 3, 6, and 9.

Example: Find the 12th term of the sequence 5, 3, 1, -1, ...

By inspection $a = 5$, $d = -2$, and $n = 12$.

$$t_{12} = 5 + (11)(-2)$$
$$= \underline{-17}$$

<u>Method for Discovery Group</u>

1. Establish the fact that sequences can be classified according to the method of obtaining consecutive terms. These examples were used to identify the arithmetic sequence.

   4, 7, 10, 13, ...
   1, 2, 4, 8, ...
   3, 6, 12, 24, ...
   4, 0, -4, -8, ...
   6, 3, 1 1/2, 3/4, ...
   1, 2, 3, 4, 5, ...

   Students were required to write the next three terms of each sequence, to state specifically the operation by which any term was obtained from the previous term, and to classify the sequences into two groups on the basis of different methods of obtaining consecutive terms. The instructor then stated that only those sequences in which the operation of addition was used would be considered.

   To find an expression for the nth term of such a sequence, the following method was used:

   Examine the sequence 1, 4, 7, 10, 13, 16, ...

   The first term is .....
   The second term is 1 + ...
   The third term is 1 + ... or 1 + ( ) ( )
   The fourth term is 1 + ... or 1 + ( ) ( )
   The tenth term is 1 + ...( ) ( )
   The fiftieth term is .... + ...
   The hundredth term is ...... + ...
   The nth term is .... + ...

3. To find the general expression for the nth term:

   Examine the sequence x, x + y, x + 2y, ...

   What is the 4th term? the 10th term? the 50th term? the nth term?

   Examine this sequence: a, a + d, a + 2d, a + 3d, ...

   What is the 8th term? the 100th term? the nth term?

   On the basis of the last sequence, designate the nth term as t_n and write an expression for t_n in terms of 'a' and 'd'.

   What does 'a' represent? What does 'd' represent?

4. The terminology arithmetic sequence or arithmetic progression is used to describe the above type of sequence. What do you think is the main characteristic of an arithmetic sequence?

5. The same examples were used for the discovery group as for the expository group, the difference being that the discovery group worked the examples without the instructor's help whereas the instructor worked the examples for the expository group.

LESSON II

Concepts

1. The sum of an arithmetic progression.

2. The summation notation.

Method for Expository Group

1. The series representing the sum of the first one hundred integers was first used.

$$S_{100} = 1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 100$$

$$S_{100} = 100 + 99 + 98 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 1$$

$$2S_{100} = 101 + 101 + 101 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 101$$

$$2S_{100} = 100(101)$$

$$S_{100} = \frac{100(101)}{2}$$

2. The general sequence can be represented by a, a + d, a + 2d, ... a + (n - 1)d.

The sum of n terms of this sequence can be written:

$$S_n = a + (a + d) + (a + 2d) + \ldots\ldots\ldots\ldots\ldots\ldots a + (n - 1)d$$

For convenience call the nth term the last term and designate it with the letter 'l'. Then write the sum as

$$S_n = a + (a + d) + (a + 2d) + \ldots\ldots\ldots\ldots\ldots\ldots + l$$

$$S_n = l + (l - d) + (l - 2d) + \ldots\ldots\ldots\ldots\ldots\ldots + a$$

$$2S_n = (a+l) + (a+l) + (a+l) + \ldots\ldots\ldots\ldots\ldots +(a+l)$$

$$2S_n = n(a + l)$$

$$S_n = \frac{n(a + l)}{2} \quad \text{or} \quad S_n = \frac{n[2a + (n - 1)d]}{2} \quad \text{(Substitute for l)}$$

Examples:

1. Find the sum of the sequence of 30 terms if the first term is 6 and the 30th term is 84.

2. Find the sum of the first 80 terms of a sequence whose first term is 3 and whose common difference is 2.

3. Suppose the sum of the first 30 terms of an arithmetic sequence is 1020 and the first term is 5. Find the common difference.

3. The Summation Notation

As a means of shortening the writing out of the series, the Greek letter Σ is used to denote summation. For example,

$\sum_{i=1}^{5}(3i)$ means the same as $3(1) + 3(2) + 3(3) + 3(4) + 3(5)$ or $3 + 6 + 9 + 12 + 15$
$= \underline{45}$

Note that it is not necessary to write out the complete series. One can calculate the first and last terms and the number of terms and use the formula $S_n = \frac{n(a + l)}{2}$

or $S_5 = \frac{5(3 + 15)}{2}$

$= \underline{45}$

Examples:

(a) $\sum_{i=1}^{6}(3i - 2)$ (b) $\sum_{m=1}^{6}(2 - 3m)$ (c) $\sum_{n=1}^{5}(3n - 1)$

(d) $\sum_{n=1}^{5}(7 - 2n)$

Method for Discovery Group

1. The story of Gauss solving the problem of the sum of the first 100 positive integers in an incredibly short time was used to introduce the topic of finding the sum of an arithmetic sequence. The students were then asked to try to find the clue given that $1 + 2 + 3 + 4 + \dots\dots\dots\dots + 100 = 5050$.

Subsequently the following additional examples were given:

$$1 + 2 + 3 + \dots\dots\dots\dots + 10 = 55$$
$$1 + 2 + 3 + \dots\dots\dots\dots + 20 = 210$$
$$1 + 2 + 3 + \dots\dots\dots\dots + 40 = 820$$
$$1 + 2 + 3 + \dots\dots\dots\dots + 50 = 1275$$
$$1 + 2 + 3 + \dots\dots\dots\dots + 1000 = 500500$$

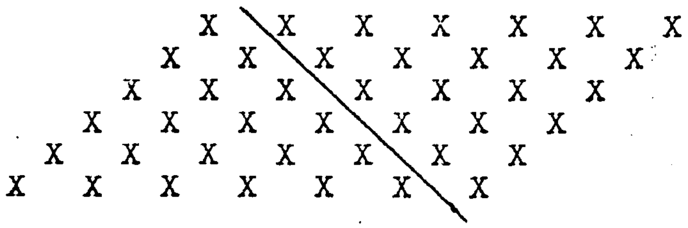
Note: The majority of the students established the formula

$$S_{100} = \frac{n(n + 1)}{2}$$

The examples $1 + 3 + 5 + 7 + \dots\dots + 49 = 625$
$5 + 8 + 11 + \dots\dots\dots + 68 = 803$

were then given and students were asked to check their formula.

The next hint consisted of a chart as illustrated.



The portion on the right of the diagonal line is a replica of the left portion turned upside down. By answering the questions: How many rows in the complete diagram? How many crosses in each row? What is the total number of crosses? What is the total number in the left section? How may the total number of crosses in each row be derived? Students were able to establish the fact that the sum was $\frac{6(6 + 1)}{2}$. Similar diagrams were used to establish the general case that $S_n = \frac{n(a + l)}{2}$ where a is the first term and l is the last term of the sequence.

2. The meaning of the summation notation was explained to this group by the instructor. In view of the fact that this is not a principle to be discovered but merely a notation device, it was not deemed necessary to differentiate between groups for this part of the lesson. The only difference was that this group was not told that it is only necessary to find the first and last terms of the sequence and use the formula but arrived at this decision by working the examples.

3. The term series was also introduced as the indicated sum of a sequence.

LESSON III

Concepts

1. An arithmetic mean between two numbers is the average of the two numbers.

2. Terms between any two given terms of an arithmetic sequence are called arithmetic means.

Method for Expository Group

1. Define an arithmetic mean between two numbers. Extend the definition to the general case. i.e. The arithmetic mean between x and y is $\frac{x + y}{2}$.

2. Define arithmetic means as stated under concepts.

   Example: In the arithmetic sequence 1, 4, 7, 10, the numbers 4 and 7 are said to be arithmetic means between 1 and 10.

3. To find the arithmetic means between any two given terms of an arithmetic sequence, use a diagram.

   Example: Insert three arithmetic means between 5 and 21.

   5, __, __, __, 21

   Note that there are 5 terms in the sequence.

   The first term is 5 and the fifth term is 21.

   $t_5 = a + 4d$

   $21 = 5 + 4d$

   $d = 4$

   Since the common difference is 4, the means to be inserted are 9, 13, and 17.

   Example: Given that the third term of an arithmetic progression is 7 and the twelfth term is 25. Find the first term and write the first 12 terms.

Method for Discovery Group

1. A series of examples was given involving the insertion of one term between two given terms with the instruction to make an arithmetic progression of three terms.

   Example: 4, __, 16

   The final example was x, __, y

2. The examples were then extended to insertion of two, three, and four arithmetic means.

   Example: Insert two arithmetic means between 3 and 15.

   3, __, __, 15.

   (Note: Most students did all examples without difficulty. Some immediately subtracted 3 from 15 in the above example and then divided by 2 and obtained a difference of 6. They quickly ascertained that this did not give an arithmetic progression and had no difficulty in arriving at the correct solution.)

LESSON IV

Concepts

1. A geometric sequence is one in which the ratio between any two consecutive terms is constant.

2. The nth term of a geometric sequence is defined to be $t_n = ar^{n-1}$, where 'a' is the first term, 'r' is the common ratio, and 'n' represents the number of a given term.

Method for Expository Group

1. A definition of a geometric sequence was given followed by these examples:

   4, 8, 16, 32, ...
   12, 6, 3, 3/2, ...
   2, -6, 18, -54, ...

   Ratios were examined between two consecutive terms to illustrate the definition and to establish the fact that the ratio may be any rational number.

2. The sequence 4, 8, 16, 32, ... was examined as follows:

$$t_1 = 4$$
$$t_2 = 4(2)$$
$$t_3 = 4(2)^2$$
$$t_4 = 4(2)^3$$
$$t_n = 4(2)^{n-1}$$

3. Examples worked by the instructor and class.

   (a) Find the 4th term of a geometric progression whose first term is 2 and whose common ratio is 4.

   (b) Which term of the geometric progression -81, 27, -3, ... is 1/9?

   (c) The seventh term of a geometric progression is 256 and the first term is 4. What is the fifth term?

<u>Method for Discovery Group</u>

1. The first, third, and fifth examples from the first lesson were used to introduce the geometric sequence. Students had previously written the next three terms of each and had specified the basis on which they had decided on the terms. It was established that this type of sequence is called geometric.

2. The sequence 3, 9, 27, 81, ... was presented with instructions to find the ratio between the second and first terms, the third and second terms, and the fourth and third terms. Ratios were also checked between terms of the preceding examples. This established that the ratio between consecutive terms is a constant.

3. The example 3, 9, 27, 81, ... was analysed in a manner similar to the analysis of an arithmetic progression in Lesson I.

   i.e. $t_1 = 3$
   $t_2 = 3(\ )$
   $t_3 = 3(\ )(\ )$ or $3\ (\ )$ or $3(\ )$
   $t_n = 3(\ )$

4. The concept was extended to the sequence x, xy, xy^2, ... for which the nth term was required.

   The final example was the sequence a, ar, ... for which the nth term was required. On the basis of the last example students were required to write an expression for t_n in terms of 'a' and 'r'.

LESSON V

Concepts

1. The sum of a finite geometric sequence.
2. The summation notation.

Method for Expository Group

1. The formula for the geometric series was established as follows:

   The general geometric progression is represented by $a, ar, ar^2, ar^3, \ldots\ldots\ldots\ldots\ ar^{n-1}$

   The sum of the above progression can be represented by the expression

   $$S_n = a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^{n-2} + ar^{n-1}$$

   $$rS_n = \quad ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots\ldots\ldots + ar^{n-1} + ar^n$$

   $$S_n - rS_n = a - ar^n$$
   $$S_n(1 - r) = a - ar^n$$
   $$S_n = \frac{a - ar^n}{1 - r} \qquad r \neq 1$$

   If the nth term is considered to be the last term and is represented by the letter 'l', this formula can also be written in the form

   $$S_n = \frac{a - lr}{1 - r} \quad r \neq 1$$

2. These two examples were worked with the class.

   (a) Find the sum of the first five terms of the geometric progression with $a = 6$, $r = 1/2$.

   (b) Given that $S_n = -14$, $n = 3$, $a = -2$, find the common ratio and write the first three terms of the geometric sequence.

3. Two examples were used to illustrate the use of the summation notation for a geometric series.

(a) $$\sum_{j=1}^{4} 4(2)^{j-1}$$ (b) $$\sum_{r=1}^{5} 9(1/3)^{r-1}$$

Method for Discovery Group

Study the three examples given below which show how to find the sum of a finite geometric progression without actually adding the terms. In each case the second statement has been obtained from the first by multiplying both sides of the equation by the same number. Identify the number by which both sides of the equation are multiplied and decide how it is related to the series.

Example 1:

Given the finite geometric progression 2, 4, 8, 16, 32. Find the sum of this sequence.

$$S_5 = 2 + 4 + 8 + 16 + 32$$
$$2S_5 = 4 + 8 + 16 + 32 + 64$$

$$S_5 - 2S_5 = 2 - 64$$
$$-S_5 = -62 \text{ or } S_5 = \underline{62}$$

Example 2:

Given the G. P. 8, 4, 2, 1, 1/2. Find the sum of the terms.

$$S_5 = 8 + 4 + 2 + 1 + 1/2$$
$$1/2S_5 = 4 + 2 + 1 + 1/2 + 1/4$$

$$S_5 - 1/2S_5 = 8 - 1/4$$
$$1/2S_5 = 8 - 1/4$$
$$S_5 = 2(8 - 1/4) \text{ or } \underline{15\ 1/2}$$

Example 3:

Find the sum of the G. P. 2, 6, 18, 54, 162, 486.

$$S_6 = 2 + 6 + 18 + 54 + 162 + 486$$
$$3S_6 = \quad 6 + 18 + 54 + 162 + 486 + 1458$$

$$S_6 - 3S_6 = 2 \qquad\qquad -1458$$
$$S_6(1 - 3) = 2 - 1458$$
$$S_6 = \frac{2 - 1458}{1 - 3} = \frac{-1456}{-2} = \underline{728}$$

After you have studied the above examples use the same process to develop a method for finding the sum below.

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \ldots\ldots\ldots\ldots + ar^{n-1}$$

Note: Examples for practice and for the use of the summation notation were the same as those for the expository group.

LESSON VI

Concepts

1. The geometric mean or mean proportional.

2. Geometric means.

Method for Expository Group

1. A single term between two terms in a geometric sequence is called a geometric mean or mean proportional. Two methods of finding a mean proportional were discussed.

   (a) Find a mean proportional between 4 and 9.

   $$4, \_\_, 9$$
   $$t_1 = 4$$
   $$t_3 = 4r^2 \quad \text{or} \quad 9 = 4r^2$$
   $$r^2 = \frac{9}{4}$$
   $$r = 3/2 \text{ or } -3/2$$

   The mean proportional is 6 or -6.

   (b) Use the idea that the ratio between consecutive terms is a constant. Let the mean proportional be 'x'.

   The sequence is 4, x, 9

   Then $\frac{x}{4} = \frac{9}{x}$ or $x^2 = 36$
   $$x = +6 \text{ or } x = -6$$

2. The geometric mean between 'a' and 'b' is represented by $\sqrt{ab}$.

3. Examples used to demonstrate how to find two or more geometric means. Geometric means are the terms between any designated terms of a geometric sequence.

   Example: Find two geometric means between 1 and 27.

   1, __, __, 27

   $a = 1$

$ar^3 = 27$
$r^3 = 27$ or $r = 3$
The means are 3 and 9

Example: Find three geometric means between 1/525 and 25/21.

$a = 1/525$ and $ar^4 = 25/21$

$r^4 = (25/21)(525/1)$

$r^4 = 25^2$ or $r^2 = 25$ and $r = +5$ or -5

The means are $\pm \frac{1}{105}$, $\frac{1}{21}$, and $\pm \frac{5}{21}$

Method for Discovery Group

In the same way as it is possible to find an arithmetic mean between two given numbers, it is possible to find a geometric mean between two terms. The single geometric mean is called a mean proportional.

The following exercises were given to the discovery group:

1. (a) Insert one number between 4 and 9 so that the resulting sequence will be in geometric progression.

   4, __, 9

   (b) Insert one geometric mean between 2 and 8.

   (c) Insert one geometric mean between x and y.

   (d) Is there a possibility of using a different number than the one you used? If so, why?

2. Find two geometric means between 1 and 27.

   1, __, __, 27

3. Find two geometric means between m^2 and m^{14}.

4. Find three geometric means between 1/525 and 25/21.

5. Find four geometric means between -7 and -224.

LESSON VII

Concepts

1. The sum of an infinite geometric series.

2. Rewriting a repeating decimal as an infinite geometric series and using this method to write a repeating decimal as a common fraction.

Method for Expository Group

1. The sum of a geometric series is given by the formula

$$S_n = \frac{a - ar^n}{1 - r}$$

Examine the sequence 1, 1/2, 1/4, 1/8, ...

$$S_3 = \frac{1 - 1(1/2)^3}{1 - 1/2} \quad \text{or} \quad \frac{1 - 1/8}{1 - 1/2}$$

$$= \underline{1\ 3/4}$$

$$S_7 = \frac{1 - 1(1/2)^7}{1 - 1/2} \quad \text{or} \quad \frac{1 - 1/128}{1 - 1/2}$$

$$= \underline{1\ 63/64}$$

$$S_n = \frac{1 - 1(1/2)^n}{1 - 1/2}$$

Note that as 'n' becomes very large the quantity represented by $(1/2)^n$ becomes very small. In fact, as 'n' increases without bound this quantity approaches zero, and the sum of the sequence approaches $\frac{1 - 0}{1 - 1/2}$ or 2. The limit of the sum of the above sequence is said to be 2. Note that the quantity represented by ar^n in the formula can only approach zero if $|r|$ is less than 1. The sum of an infinite geometric progression whose ratio has an absolute value less than 1 is defined by $S = \frac{a}{1 - r}$.

2. A repeating decimal can be written in the form of an infinite geometric series. For example 0.555 ... can be written in the following form: 0.5 + .05 + .005 + .005 + ...

The terms of the series form a sequence whose first term is 0.5 and whose common ratio is 1/10 or .1. The limit of the series is

$$\frac{0.5}{1 - .1} = \frac{5}{9}$$

3. Practice Examples:

Change the following repeating decimals to equivalent common fractions.
0.65...   0.127

Find the sum of the infinite geometric sequence with a = 3 and r = 2/3.

<u>Method for Discovery Group</u>

1. Consider the series 1 + 1/2 + 1/4 + 1/8 + ...

$S_1 = 1$
$S_2 = 1 + 1/2 = 1\ 1/2$
$S_3 = 1 + 1/2 + 1/4 = 1\ 3/4$
$S_4 = 1 + 1/2 + 1/4 + 1/8 = 1\ 7/8$
$S_5 = 1 + 1/2 + 1/4 + 1/8 + 1/16 = 1\ 15/16$
$S_6 = 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 = 1\ 31/32$

On the basis of the above examples estimate the value of

$S_{10} = 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + 1/256 + 1/512$

= ______________

Check your guess by using the formula for finding the sum of a geometric sequence.

$$S_n = \frac{a - ar^n}{1 - r}$$

Suppose n = 20

$$\text{Then } S_n = \frac{1 - \boxed{1(1/2)^{20}}}{1 - 1/2}$$

and if n = 100

$$S_n = \frac{1 - \boxed{1(1/2)^{100}}}{1 - 1/2}$$

As 'n' gets progressively larger, what is happening to the term enclosed by the rectangle?

As 'n' approaches infinity, what value does this term approach?

Would this be true for any 'r'? If not, what restriction would you place on 'r'?

What do you think is the limit of the above series?

What do you think would be an appropriate formula for evaluating the limit of the sum of an infinite geometric sequence?

2. The repeating decimal 0.555... can be rewritten in this manner:
   0.5 + 0.05 + 0.005 + ...

   As 'n' increases without bound, what is the limit of S_n for this series?

3. Practice examples as for the expository group.

WRITTEN EXERCISE - LESSON 1

1. Write the first six terms of the sequence associated with each of these functions. In each example the domain of 'n' is the set of counting numbers.

   (a) $F_1 = \left\{(n,y) \mid y = \frac{3n}{n+2}\right\}$

   (b) $F_2 = \left\{(n,y) \mid y = \sqrt{n}\right\}$

   (c) $F_3 = \left\{(n,y) \mid y = 13 - 2n\right\}$

   (d) $F_4 = \left\{(n,y) \mid y = n^2\right\}$

2. Find the next three terms in each sequence.

   (a) 87, 74, 61, ... (b) 19, 37, 55, ...

   (c) -8/3, -35/12, -19/6, ...

3. Find the indicated term of each sequence.

   (a) The 30th term of 1/3, 1, 5/3, ...

   (b) The 8th term of i, 0.8i, 0.6i, ...

   (c) The 35th term of $\sqrt{2} + 1, \sqrt{2}, \sqrt{2} - 1$, ...

   (d) The 40th term of $x - y, x, x + y$, ...

   (e) The 20th term of $-3x^2, -x^2, x^2$, ...

4. True or False:

   (a) In the sequence whose nth term is $\frac{n^2+1}{1+n}$, the 7th term is 25/4.

   (b) In a sequence whose nth term is $n^2 - 5n + 6$, the first and fourth terms are the same.

   (c) Two different arithmetic sequences always have different common differences.

   (d) It is possible for two different arithmetic sequences to have the same first three terms.

5. The 5th term of an arithmetic progression or sequence is 9. The 14th term is 45. Write the first three terms.

6. A ball which rolls off a penthouse terrace falls 16 feet in the first second, 48 feet in the second, and 80 feet in the third second. If it continues to fall in this manner, how far will it fall in the 7th second?

7. A missile fired vertically upward rises 15,840 feet in the first second, 15,808 feet in the following second, and 15,776 feet in the third second. How many feet does it rise in the 45th second? How many feet and in what direction does it move in the last second of the ninth minute after it is fired?

8. The speed of sound in air is about 332.1 meters per second at 1° C. This increases about .6 meters per second for each degree of increase in the temperature of the air. Express this in an arithmetic sequence. What is the nth term?

WRITTEN EXERCISE - LESSON 2

1. Find the indicated sums:

   (a) The sum of the first 17 terms of the sequence whose first term is 6 and whose common difference is 4.

   (b) The sum of the sequence with first term of 13, last term of 89, and difference of 4.

   (c) The sum of 20 terms of the sequence 2, -1, -4, -7, ...

2. Find the sums of these arithmetic series.

   (a) $\sum_{j=1}^{5} 3j$ (b) $\sum_{k=1}^{15} (3k - 1)$ (c) $\sum_{n=1}^{5} (2 - 3n)$

   (d) $\sum_{r=1}^{12} (3r - 4)$

3. Write the first three terms of each of these arithmetic progressions.

   (a) $a = 3$ $l = 17$ $S_n = 100$

   (b) $a = 8$ $n = 17$ $S_n = 183.6$

4. Use summation notation to write each of the following:

   (a) $(2 + 3\cdot 1) + (2 + 3\cdot 2) + (2 + 3\cdot 3) + (2 + 3\cdot 4)$

   (b) $(5\cdot 1 + 2) + (5\cdot 2 + 2) + (5\cdot 3 + 2) + (5\cdot 4 + 2)$

   (c) $(1 - 3\cdot 1^2) + (1 - 3\cdot 2^2) + (1 - 3\cdot 3^2) + (1 - 3\cdot 4^2)$

   (d) $2 + 5 + 8 + 11$

   (e) $1 - 1 - 3 - 5 - 7$

5. True or False:

   (a) In every arithmetic sequence with a common difference of 5, $S_{20} = S_{19} + 5$.

(b) In the arithmetic sequence -5, -12, -19, ..., $S_{11} = 0$.

(c) In any arithmetic sequence whose first term is a and whose common difference is d, $S_5 = S_3 + 2a + 7d$.

6. On a construction job a laborer is told to carry 20 joists from the lumber pile and place them on the ground at 4 foot intervals. The closest placement to the lumber pile is 60 feet. Starting at the lumber pile and finishing there as well, if he carries one joist at a time how far does he have to walk in order to place the 20 joists?

7. If the taxi rate is 70¢ for the first mile and 40¢ for each additional mile, what is the fare from a suburb to the airport which is 12 miles away?

8. Find the sum of the positive integers less than 100 which are divisible by 6.

9. The largest integer in an arithmetic progression of consecutive even integers is 9 times the smallest. The sum of the progression is 90. Find the largest and smallest integers.

WRITTEN EXERCISE - LESSON 3

1. Find:

   (a) Three arithmetic means between 10 and 16.

   (b) Five arithmetic means between -7 and 6.

   (c) Five arithmetic means between -2 and -6.

   (d) Six arithmetic means between -2 and 12.

   (e) Nine arithmetic means between -10 and 0.

   (f) One arithmetic mean between $a + bi$ and $a - bi$.

2. True or False:

   (a) One could find 100 arithmetic means between 6 and 7.

   (b) The arithmetic mean between -11 and -14 is -13.

   (c) For any three numbers x, y, and z, if z is the average of x and y, then there is an arithmetic progression which begins x, y, z....

   (d) If π is the arithmetic mean between 2 and x, then $x = 2\pi - 2$.

   (e) If a, b, c, d, e, are any five numbers in an arithmetic sequence, then $2c = ae$.

3. The seven weights in a set for analytic balance are in an arithmetic sequence. The largest is 25 grams and the smallest 1 gram. Find the weights of the other five.

4. A man driving along a road at 60 m.p.h. (88 ft. per sec.) applies the brakes and comes to a complete stop in 22 seconds. If the speeds at which he is travelling in successive seconds form an arithmetic progression, how fast did he travel the 7th second after braking?

5. A young man's salary increased for five years in arithmetic sequence. If his salary the first year was $4400 and the fifth year was $6000, what was his salary in each of the other years?

6. The reciprocal of one number is 5 times the reciprocal of another. Seven times their arithmetic mean is 4 greater than their product. Find all such pairs of numbers.

7. Write in summation notation.

(a) $2a + 4a + 8a + \ldots\ldots$ (b) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \ldots\ldots$

(c) $4 - 7 + 10 - 13 + \ldots.$

(d) $-3 - 1 + 1 + 3 + 5 + \ldots + 19.$

8. Write the first four terms of $\sum_{k=1}^{9} \frac{1}{k}$

9. Write the tenth term of $a/2,\ 3a/2,\ 5a/2,\ \ldots\ldots\ldots$

10. What is the arithmetic mean of $\frac{a}{a + b}$ and $\frac{a}{a - b}$?

WRITTEN EXERCISE - LESSON 4

1. Write the first four terms of each of these geometric sequences.

   (a) $a = -9$, $r = 2$. (b) $a = -3$, $r = -1/3$

   (c) $a = 1/12$, $r = 4$.

2. Find the indicated terms:

   (a) The 4th term of the G.P. with $a = 4$ and $r = 4$.

   (b) The 7th term of 5, 10, 20, ...

   (c) The 10th term of $-\sqrt{3}$, $\sqrt{6}$, $-2\sqrt{3}$, ...

   (d) The 9th term of 39, 13, 4 1/3, ...

3. There are two geometric progressions of real numbers with a first term of 7 and a 5th term of 112. Find the two values for r which will generate the two series.

4. Find the nth term of each sequence in Question 2. (i.e. Write an expression for the nth term)

5. The length of the arc of the first swing of a pendulum is 10 inches. The length of each succeeding swing is 1/9 less than the preceding one. How long is the seventh swing? the ninth? Write the expression for computing the answer but do not do the computations.

6. The first term of a G. P. is 27 and the common ratio is 1/3. For what value of n is $t_n = 1/3$?

7. If the value of a car depreciates 20% the first year and 5% each year after the first, what is the value of a car which is four years old and originally cost $3000?

8. A jar contains 500 cubic inches of air. On its first stroke an air pump removes 20% of the air leaving 80% of 500 cubic inches in the jar. On the second stroke it removes 20% of the remaining air and so on for the following strokes. How much air is left after the fifth stroke?

9. True or False:

   (a) The terms of a geometric sequence grow constantly smaller or larger but they never fluctuate back and forth.

(b) The (n+1) term of the geometric sequence 1/2, 1/3, 2/9,... is $(1/2)(2/3)n$.

(c) The sequence 5, 5, 5, ... is both arithmetic and geometric.

(d) A geometric sequence is uniquely determined if you know the first term and the common ratio.

(e) If the nth term of a geometric sequence is $(1/2)(4)^{n - 1}$, then the 4th term is 32.

(f) In any geometric sequence each term is divisible by all preceding terms.

(g) If a is the first term and r the common ratio, then in any geometric sequence the product of the fourth and fifth terms is always a^2r^7.

(h) If you multiply each term of a geometric sequence by 3, the resulting sequence will also be geometric.

WRITTEN EXERCISE - LESSON 5

1. Find the sums of the following geometric progressions:

   (a) $a = 1$, $r = 2$, $n = 7$.

   (b) $a = 12$, $r = 3/2$, $n = 4$.

   (c) $a = 4$, $l = 324$, $r = 3$.

   (d) 1000, 100, 10, ... when $n = 7$.

2. Find the sums of these geometric series:

   (a) $\sum_{a=1}^{7} 5(2)^{a-1}$ (b) $\sum_{j=1}^{7} 2(-\frac{1}{2})^{j-1}$

   (c) $\sum_{r=1}^{5} 9(1/3)^{r-1}$ (d) $\sum_{k=1}^{6} (-2/3)(1/2)^{k-1}$

3. In a finite geometric sequence the last term is 8,192 and the ratio is -4. If the sum of the sequence is 6554, find the first term.

4. The 5th term of a geometric sequence is 24 and the 10th term is 768. Find the sequence and the sum of the first 7 terms.

5. The side of a square is 10 inches. The midpoints of the sides are joined to form an inscribed square as shown in the diagram on the next page, with the process continued until there are five squares. Find the sum of the perimeters of the five squares.

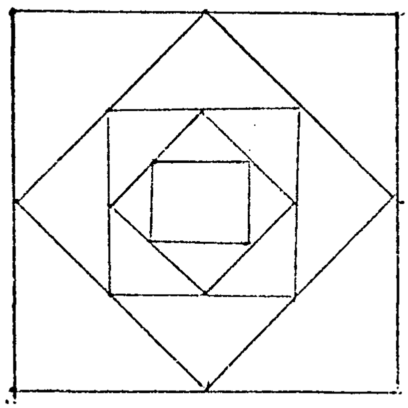
6. If the half life of the uranium 230 isotope is 20.8 days, how much of a given amount of the isotope will be left after 104 days.

7. The sum of the first and second terms of a geometric progression is -3 and the sum of the 5th and 6th terms is $-3/16$. Find the sum of the first 8 terms.

8. If a youngster decided to put 1¢ in a toy bank today, 2¢ tomorrow, 4¢ the next day, and so forth for 31 days, how many digits are there in the number of pennies he should put in on the 31st day? ($\log 2 = .301$) Is the sum for the month more than $10,000,000?

9. Solve $\sum_{k=1}^{n} 2^k = 62$ for n by testing successive values of n.

Diagram for Question 5



WRITTEN EXERCISE - LESSON 6

1. In each case find the mean proportional between the given numbers:

   (a) 5 and 20 (b) 3 and 6 (c) -7 and -189

   (d) $a + bi$ and $a - bi$

2. Insert the given number of geometric means and write the resulting finite geometric progression.

   (a) Three between 4/3 and 27/64.

   (b) Two between 1 and 27.

   (c) Seven between 3 and 48.

   (d) Three between -15 and - 1215.

3. The third term of a geometric progression is 5. The 6th term is $8/\sqrt{5}$. Find the terms between these two terms.

4. The product of three real numbers which are in geometric progression is -64. If the first number is 4 times the third, what are the numbers?

5. If -64/9 is the 6th term of a geometric progression whose common ratio is -2/3, what is t_1?

6. If $\frac{a}{b} = \frac{c}{d}$ prove that ab + cs is a mean proportional between $a^2 + c^2$ and $b^2 + d^2$.

7. Which is larger--the arithmetic mean or the geometric mean between 2 and 3? Does the arithmetic mean ever equal the geometric mean? If so, when?

8. Find x, given that $2x - 7$ is the geometric mean between $x - 5$ and $2x + 11$.

WRITTEN EXERCISE - LESSON 7

1. Find the sums of these infinite geometric sequences:

   (a) $a = 6$, $r = -1/3$.

   (b) 0.1, 0.01, 0.001, ...

   (c) 6, 2, 2/3, ...

   (d) 3, 1, 1/3, ...

   (e) 1, -1/2, 1/4, -1/8, ...

2. Find the common fraction equivalent of each of these repeating decimals:

   (a) .138138......   (b) .7373....   (c) .1494949...

3. A pendulum is brought to rest by air resistance. The first arc through which the bob of the pendulum swings is 40 cm., and each swing thereafter is .98 as long as the previous arc. Find the total distance the bob has travelled by the time it has come to rest.

4. A ship which is 101 miles from shore sustains damage and takes in water. It starts at once for shore at the rate of 10 m.p.h. but due to the damage the rate each hour decreases and is 9/10 that of the preceding hour. Will the ship reach the shore safely?

5. A rubber ball dropped 40 feet rebounds on each bounce 2/5 of the distance from which it fell. How far will it travel before coming to rest?

6. In an unending series of equilateral triangles, the vertices of each triangle after the first are the mid-points of the sides of the preceding triangle. The sides of the first triangle are each one foot long. Find the sum of the perimeters of all the triangles.

7. Find the sum of the areas of all the triangles in Question 6, given that the area of an equilateral triangle of side a is $a^2\sqrt{3}/4$.

8. Of the values a, t_n, n, r, and S_n, three are given. Find the other two.

(a) $a = 1$, $r = 2$, $n = 7$.

(b) $a = 1/3$, $r = 3$, $S_n = 40/3$.

(c) $a = 30$, $t_n = .003$, $S_n = 33.333$.

APPENDIX B

INSTRUMENTS

PAGE

MATHEMATICS CONTENT TEST

Multiple Choice — Sequences and Series

1. If a_1, a_2, a_3, is an arithmetic sequence, then $a_1 - a_2 + a_3$ is equal to:

   A. a_1 B. a_2 C. a_3 D. $a_1 + a_3$ E. $-a_2$

   1. ________

2. The set of numbers 3, 11, 19, 27, is best described as:

   A. a sequence B. a series C. an infinite sequence

   D. an infinite series E. none of these 2. ________

3. An infinite geometric progression with a ratio of 1/2 has a sum of 12. What is the sum of a geometric progression whose terms are the squares of the original progression?

   A. 12 B. 36 C. 48 D. 72 E. 144 3. ________

4. The first number of a sequence is 729, and each succeeding number is found by multiplying the preceding term by 1/3. S_4 is equal to:

   A. 1093 B. 1092 C. 1089 D. 1080 E. 29,160

   4. ________

5. The set of numbers 7, 11, 15, 19, is an example of:

   A. a geometric series B. a geometric sequence

   C. an arithmetic series D. an arithmetic sequence

   E. an algebraic sequence 5. ________

6. If a, b, c is an arithmetic progression, which of the following is a true statement?

   A. $2a = b + c$ B. $2b = c + a$ C. $2c = a + b$

   D. $b^2 = ca$ E. $c = 2(a + b)$ 6. ________

7. If the first term of an arithmetic sequence is -3, and the common difference is 2, then the nth term, t_n, is:

A. $5 - 3n$ B. $6n - 3$ C. $2n - 3$ D. $n(n - 2)$ E. $2n - 5$

7. __________

8. The 18th term of the sequence 5, 1, -3, is:

A. 77 b. 73 C. -71 D. -67 E. -63 8. __________

9. The 9th term of the series $3 - 6 + 12 - 24 + \ldots$ is:

A. 768 B. -768 C. 27 D. -15 E. -45 9. __________

10. The positive geometric mean between 3 and 54 is:

A. $3\sqrt{2}$ B. 27 C. $9\sqrt{2}$ D. $6\sqrt{3}$

E. $9\sqrt{3}$ 10. __________

11. Two arithmetic means between 11 and 29 are:

A. 17 and 23 B. 16 and 24 C. 15 and 25 D. 18 and 22

E. None of the preceding 11. __________

12. If a, b, c is a geometric progression with positive terms, which of the following is an arithmetic progression?

A. $a - 1, b - 1, c - 1$ B. a, b, c

C. $10^a, 10^b, 10^c$ D. log a, log b, log c

E. ab, bc, ca 12. __________

13. The sum of the first 20 terms of the series $(-6) + (-2) + (2) + (6) + \ldots$ is:

A. 320 B. 70 C. 1280 D. 35 E. 640

13. __________

14. Two geometric means between 7 and 189 are:

A. 68 and 129 B. 28 and 74 C. 21 and 63

D. 67 and 127 E. None of the preceding 14. __________

15. Which term of the geometric progression 1/8, -1/4, 1/2, ... is -4?

A. the 5th B. the 6th C. the 7th D. the 8th

E. -4 is not a term of this G. P. 15. __________

16. As n approaches infinity, the limit of S_n is $S = \frac{a}{1 - r}$ if:

A. $r = 1$ B. r is less than 1 C. r is greater than 1

D. $|r| > 1$ E. $|r| < 1$ 16. ________

17. The arithmetic mean between $\frac{x + a}{x}$ and $\frac{x - a}{x}$ is:

A. 2 if a = 0 B. 1 C. 2 D. x E. a/x
17. ________

18. The repeating decimal $.\overline{23}$ is equivalent to the geometric series whose common ratio and first term have the values:

A. r = 0.1, a = .23 B. r = 0.01, a = 23

C. r = 0.01, a = .23 D. r = .23, a = 0.01

E. r = 0.1, a = 23 18. ________

19. The sum of the infinite geometric series 9 + 6 + 4 + ... is:

A. 36 B. 27 C. 21 2/3 D. 54 E. 45 19. ________

20. The sum of the geometric series given by

$$\sum_{k=1}^{5} 36(-2/3)^{k-1} \text{ is:}$$

A. $\frac{220}{9}$ B. $\frac{744}{45}$ C. $\frac{-220}{9}$ D. $\frac{-744}{45}$ E. $\frac{9}{220}$
20. ________

21. The value of $\sum_{n=1}^{9} (2n - 3)$ is:

A. 14 B. 70 C. 63 D. -70 E. None of these
21. ________

22. If a, b, and c form an arithmetic progression, which of the following is not necessarily an arithmetic progression?

A. c, b, a B. a + 2, b + 2, c + 2 C. 3a, 3b, 3c

D. a^2, b^2, c^2 E. None of these 22. __________

23. If the third and seventh terms of an A. P. are 5 and 11, then the fifteenth term is:

A. 21.5 B. 22.5 C. 23.5 D. 23 E. 24

23. __________

24. The limiting sum of the series $1 + (9/10) + (9/10)^2 + \ldots$ is:

A. 1 B. 9 C. 90 D. 10 E. 10/9 24. __________

25. The number .515151... can be written as a fraction. When reduced to lowest terms, the sum of the numerator and denominator is:

A. 30 B. 50 C. 150 D. 100 E. None of these

25. __________

MATHEMATICS TRANSFER TEST

1. The numbers below are arranged in triangular form. Study the arrangement and answer the questions which follow it.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

___ ___ ___ ___ ___ ___

(a) Fill in the next row according to the pattern which has been established.

(b) If the pattern continued, what would the second term in the 1965th row be? ____________

(c) What would the third term in the 22nd row be?

____

(d) What would the sum of the terms in the 11th row be?

____

(e) How many terms would there be in the 1968th row?

____

(f) If you start with a positive sign in a row and alternate positive and negative signs in each row, what is the sum of the terms in row 769? ____________

2. Answer the questions following the triangular integer pattern below.

1

1 2 1

1 2 3 2 1

1 2 3 4 3 2 1

1 2 3 4 5 4 3 2 1

(a) What would the middle term in the 76th row be if the pattern continued? ______________

(b) What would the 6th term in the 1961st row be?

____

(c) What would the sum of the terms in the 50th row be?

____

(d) How many integers would be in the array up to and including the 12th row? ______________

3. Complete the examples below and provide a formula for the sum of n terms.

(a) $\frac{1}{1\cdot 2} =$ (b) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} =$

(c) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} =$

(d) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} =$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots\cdots\cdots + \frac{1}{n(n+1)}$$

= ______________

UNDERLINE THE CORRECT ANSWER FOR QUESTIONS 4, 5, 6, 7 and 8

4. A harmonic progression is a sequence of numbers such that their reciprocals are in arithmetic progression. i.e. If 4, 7, and 10 are in arithmetic progression, then 1/4, 1/7, and 1/10 are in harmonic progression. Let S_n represent the sum of the first n terms of a harmonic progression. For example S_3 represents the sum of the first three terms. If the first three terms of a harmonic progression are 3, 4, and 6, then:

A. $S_4 = 20$ B. $S_4 = 25$ C. $S_5 = 49$ D. $S_6 = 49$

E. $S_2 = 1/2\ S_4$

5. The next two terms in the infinite sequence 0, 1, 1, 2, 3, 5, 8, 13, 21,......are:

A. 29 and 34   B. 34 and 44   C. 34 and 55   D. 29 and 44

E. None of these

6. The sum to infinity of $\frac{1}{7} + \frac{2}{7^2} + \frac{1}{7^3} + \frac{2}{7^4} + \frac{1}{7^5} + \frac{2}{7^6} + \ldots\ldots\ldots$ is:

A. 1/5   B. 1/24   C. 5/48   D. 1/16   E. 9/48

7. The arithmetic average of the first n positive numbers is:

A. $\frac{n}{2}$ B. $\frac{n^2}{2}$ C. n   D. $\frac{n-1}{2}$ E. $\frac{n+1}{2}$

8. When simplified, the product $(1 - 1/3)(1 - 1/4)(1 - 1/5)\ldots\ldots\ldots(1 - 1/n)$ becomes:

A. $\frac{1}{n}$ B. $\frac{2}{n}$ C. $\frac{2(n-1)}{n}$ D. $\frac{2}{n(n+1)}$

E. $\frac{3}{n(n+1)}$

9. The sum of the first two positive odd integers is

____

The sum of the first three positive odd integers is

____

The sum of the first four positive odd integers is

____

The sum of the first five positive odd integers is

____

The sum of the first 100 positive odd integers is

____

The sum of the first n positive odd integers is

____

10. The diagram shows how an object would look as it falls toward the surface of three different planets. Find the pattern which seems to appear and predict how far the object will fall in 10 seconds. (t represents time and d represents distance.)

	Planet R	Planet G	Planet B
t = 0	d = 0	d = 0	d = 0
t = 1	d = 3	d = 5	d = 16
t = 2	d = 12	d = 20	d = 64
t = 3	d = 27	d = 45	d = 144
t = 4	d = 48	d = 80	d = 256
t = 10	d = ___	d = ___	d = ___

APPENDIX C

RAW DATA

Interpretation

Student Number. The first digit refers to the treatment group. The digit "1" designates expository treatment and "2" designates simple enumeration treatment.

Sex. Each student is designated "M" for male or "F" for female.

Term Mark. The mark assigned by the regular teacher in the term preceding the experimental unit.

Pretest. Raw score on Lorge-Thorndike Nonverbal Battery, Form 1, Level H.

Posttest. Raw score on Lorge-Thorndike Nonverbal Battery, Form 2, Level H.

Mathematics Test. Raw score on Mathematics Content Test.

Mathematics Transfer. Raw score on Mathematics Transfer Test.

I. Q. A composite score which is the simple unweighted average of the Verbal and Nonverbal Batteries of the Lorge-Thorndike Intelligence Test, Form 1, Level H of the 1964 Multi-Level Edition.

RAW DATA

Student Number	Sex	Term Mark	Pre-test	Post-test	Math. Test	Math. Transfer	I.Q.
101	M	63	50	47	15	10	121
102	F	69	39	53	16	4	118
103	F	62	32	42	17	4	103
104	F	76	43	45	18	8	127
105	M	60	54	57	19	10	120
106	M	60	41	44	15	3	111
107	M	71	42	44	17	5	114
108	M	56	47	47	18	5	115
109	F	72	44	45	15	8	122
110	M	69	44	56	17	5	121
111	F	60	35	41	22	4	111
112	F	67	36	52	20	8	109
113	F	59	45	56	13	9	119
114	F	57	41	38	13	5	111
115	F	74	57	63	22	9	122
116	F	67	54	58	18	7	120
117	F	72	49	45	21	6	122
118	M	50	47	51	18	10	111
119	M	58	38	47	19	2	112
120	M	65	65	65	17	11	136
121	F	55	37	42	17	9	111
122	M	73	45	52	15	10	117
123	F	55	45	57	17	5	119
124	F	83	33	39	21	9	97
125	M	81	53	51	22	11	129
126	M	57	24	40	16	3	100
127	F	72	56	53	20	3	120
128	F	62	42	49	17	4	114
129	M	58	47	48	16	9	112
130	F	83	59	57	19	7	135
131	F	79	50	52	20	6	122
132	M	68	42	40	17	8	117
133	M	48	39	43	8	5	101
134	F	58	42	47	16	8	119
135	M	45	58	51	13	4	125
136	M	65	41	37	15	6	112
137	M	79	60	63	21	5	133
138	F	52	24	35	13	3	98
139	F	79	52	59	20	11	129
140	M	84	45	48	19	8	120
141	M	66	59	61	20	11	122
142	F	58	30	34	15	1	105
143	F	88	46	47	21	8	119

Student Number	Sex	Term Mark	Pre-test	Post-test	Math. Test	Math. Transfer	I.Q.
144	F	84	46	55	22	8	123
145	M	65	32	59	15	8	100
146	M	63	43	47	11	3	122
147	M	85	50	60	20	13	130
148	F	58	33	32	11	7	109
149	M	58	47	54	11	6	117
150	F	60	38	54	17	5	102
151	F	94	51	53	25	14	130
152	M	60	52	50	18	10	122
153	M	47	38	39	18	5	102
154	M	70	43	40	14	13	112
155	F	60	56	60	14	6	121
156	F	72	48	48	20	6	123
157	F	73	31	47	16	4	110
158	M	60	44	48	15	7	116
159	M	60	46	53	12	3	114
160	F	66	45	54	17	1	115
161	M	58	52	60	17	7	122
162	M	66	46	57	17	5	120
163	F	60	49	52	17	8	121
201	M	65	58	69	17	9	127
202	M	57	46	57	16	10	118
203	M	47	42	59	14	3	117
204	M	62	53	50	17	11	125
205	F	62	45	45	19	4	124
206	F	65	36	48	19	11	105
207	M	64	45	45	19	8	117
208	F	88	45	42	13	6	114
209	M	72	40	43	16	7	108
210	F	74	53	47	20	4	117
211	F	49	41	52	8	9	115
212	M	62	51	52	16	5	128
213	F	65	42	50	22	8	121
214	M	61	44	39	16	4	108
215	M	66	48	57	15	9	118
216	M	53	45	52	19	12	117
217	F	58	44	51	17	11	120
218	M	69	46	41	21	12	118
219	F	72	52	55	18	8	122
220	M	67	39	50	13	9	114
221	M	65	46	52	18	6	121
222	M	44	30	41	13	6	106
223	M	72	42	48	17	9	119
224	M	60	58	57	20	8	130
225	F	92	38	43	21	5	111
226	F	77	46	65	17	16	123

Student Number	Sex	Term Mark	Pre-test	Post-test	Math. Test	Math. Transfer	I.Q.
227	M	71	49	64	16	12	123
228	F	50	44	54	16	10	126
229	M	63	46	51	16	5	122
230	F	72	52	35	18	7	129
231	M	58	37	48	15	5	110
232	M	85	50	65	21	5	129
233	F	67	36	41	18	6	106
234	F	75	44	52	21	11	120
235	M	36	44	44	15	4	114
236	M	78	52	61	17	14	128
237	F	84	47	52	20	9	118
238	M	90	59	67	23	9	138
239	F	86	46	44	21	10	112
240	M	74	56	58	18	10	126
241	M	92	41	59	19	13	113
242	F	81	59	62	19	12	129
243	F	84	50	54	19	7	130
244	M	29	43	46	19	6	117
245	F	63	43	47	16	7	113
246	M	67	42	48	20	4	115
247	M	90	62	57	24	5	130
248	M	68	42	43	21	8	115
249	M	58	46	58	18	8	113
250	M	58	39	53	16	7	111
251	M	77	55	57	21	16	124
252	M	79	51	61	18	8	124
253	F	72	51	47	20	7	123
254	F	78	50	47	18	3	118
255	F	78	62	70	21	17	139
256	M	44	44	53	17	10	114
257	M	64	44	51	17	7	119
258	M	60	32	44	19	2	109
259	M	65	42	45	12	9	110
260	M	90	65	67	20	14	144
261	M	63	45	56	22	11	109
262	M	57	59	62	19	8	136
263	F	65	47	54	12	8	123
264	M	45	36	50	17	7	114