On the Existence of Weak Solutions of the Navier-Stokes Equations

by

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The existence of a weak solution \( u(x, t) \), in the sense of J. Leray ([7]), is established for the initial-boundary value problem for the Navier-Stokes equations:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{k=1}^{n} u_k D_k u + \text{grad}_x p &= f \\
\text{div} u &= 0.
\end{aligned}
\]

The solution is required to satisfy the initial condition \( u(x, 0) = u_0(x) \) for \( x \in \Omega \) and the boundary condition \( u(x, t) = 0 \) on \( \partial \Omega \times [0, T] \), where \( \Omega \) is an open bounded domain in \( \mathbb{R}^n \), with \( 2 \leq n \leq 4 \). Galerkin's method is employed to find a weak solution \( u \) as the limit of approximate solutions \( \{u_m\} \). The convergence of the \( \{u_m\} \) is guaranteed by some compact embedding theorems, which depend on a priori estimates for the \( \{u_m\} \) and their fractional time derivatives of order \( \gamma, 0 < \gamma < \frac{1}{4} \).
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Chapter I

Introduction

1.1 Classical Navier-Stokes Equations.

In the study of hydrodynamics, one is interested in determining the subsequent velocity, within a fixed domain \( \Omega \subset \mathbb{R}^n \), of a viscous incompressible fluid, which has been initially set into motion and is governed by the Navier-Stokes equations. To be more precise, one seeks the velocity \( u = u(x, t) \) and the pressure \( p = p(x, t) \) which, for \( x \in \Omega \) and \( t > 0 \), satisfy:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{k=1}^{n} u_k D_k u + \nabla_x p = f \\
\text{div } u = 0
\end{cases}
\]

and obey the initial and boundary conditions

\[
u(x, 0) = u_0(x),
\]

\[
u(x, t) = 0 \quad \text{on} \quad \partial \Omega \times [0, T].
\]

Here \( f \) is a given force density; \( \nu > 0 \) is a fixed constant; and \( \Delta = \Delta_x = D_1^2, \ldots, D_n^2 \).
1.2 Notation

i) Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, with $2 \leq n \leq 4$.

ii) $H^1(\Omega) = \{ u : u \in L^2(\Omega), \; D_i u = \frac{\partial u}{\partial x_i} \in L^2(\Omega) \}$. Here $u$ is real valued and the derivatives are taken in the distribution sense. $H^1(\Omega)$ is a Hilbert space with:

$$||u||_{H^1(\Omega)}^2 = ||u||_{L^2(\Omega)}^2 + \sum_{i=1}^{n} ||D_i u||_{L^2(\Omega)}^2$$

and $(f, g)_{L^2(\Omega)} = \int_{\Omega} f(x)g(x)dx$ for $f, g \in L^2(\Omega)$.

iii) $H^1_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, where $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$.

iv) $H(\Omega) = H = \left( L^2(\Omega) \right)^n$ (taken as the nth product space). Thus $f \in H$ if $f = (f_1, \cdots, f_n)$ where $f_i \in L^2(\Omega)$, $i = 1, \cdots, n$. For $f, g \in H$ we get $(f, g)_H = \sum_{i=1}^{n} (f_i, g_i)_{L^2(\Omega)}$.

v) $V$ will be the closure in the product space $(H^1(\Omega))^n$ of the subspace of functions $\psi = (\psi_1, \cdots, \psi_n)$, where $\psi_i \in \mathcal{D}(\Omega)$ and $\text{div} \psi = 0$. For $u, v \in V$, we set

$$(u, v)_V = \sum_{i=1}^{n} \sum_{j=1}^{n} (D_i u_j, D_i v_j)_{L^2(\Omega)}$$

and $||u||_V = (u, u)_V^{1/2}$.

vi) The trilinear form $b(u, v, w) = \sum_{i,k=1}^{n} \int_{\Omega} u_k D_k v_i w_i dx$ is defined on $L^4 \times V \times L^4$, where $L^4 = \left( L^4(\Omega) \right)^n$. 

vii) \( H_\gamma(a,b; V; H) = \{ u : u \in L^2(a,b; V) ; |\tau|^\gamma \widehat{u}(\tau) \in L^2(a,b; H) \} \)

where:

a) \( 0 < \gamma < 1 \); 

b) \( L^2(a,b; V) = \{ u : u(x, t) \in V \text{ for each } t \in (a,b) \text{ and } \int_a^b \| u \|_V^2 dt < \infty \} \); 

c) \( L^2(a,b; H) = \{ u : u(x, t) \in H \text{ for each } t \in (a,b) \text{ and } \int_a^b \| u \|_H^2 dt < \infty \} \); 

d) \( \widehat{u}(\tau) \) is the \( L^2 \) Fourier transform of \( u \) with respect to \( t \); 

e) we say that \( u \in L^2(a,b; H) \) has a fractional time derivative \( D_t^\gamma u \in L^2(a,b; H) \) if and only if \( |\tau|^\gamma \| \widehat{u}(\tau) \|_H \in L^2(a,b) \); 

f) \[ \| u \|_{H_\gamma(a,b; V; H)} = \left( \int_a^b \| u \|_V^2 dt + \int_a^b |\tau|^{2\gamma} \| \widehat{u}(\tau) \|_H^2 dt \right)^{\frac{1}{2}} \]

and \( H_\gamma(a,b; V; H) \) is a Hilbert space.

1.3 Formulation of Solutions.

In the research of J. Leray during 1933-34 ([7]) the concept of a weak solution of the initial boundary value problem for the Navier-Stokes equations (1) was formed for \( \Omega \subset \mathbb{R}^n \), with \( n = 2, 3 \). In 1951 E. Hopf ([5]) extended this formulation to any dimension \( n > 1 \). For \( u_0(x) \) belonging to the closure \( \overline{V} \) of \( V \) in \( H \), and \( f \) given in \( L^2(0,T; H) \), we say \( u \) is a weak solution of (1) if \( u \in L^2(0,T; V) \) and
\[ (2) \int_0^T \{ \nabla(u(t), \phi(t))_V + b(u(t), u(t), \phi(t)) - (u(t), \phi_t(t))_H \} \, dt \]

\[ = \int_0^T (f(t), \phi(t))_H \, dt + (u_0, \phi(0))_H \]

for all \( T > 0 \) and all \( \phi \in (C_0^\infty([0, \infty) \times \Omega))^n \) with \( \text{div} \phi = 0 \).

In 1957, A. Kicelev and O. Ladyzenskaya ([13]) proved uniqueness within the class of weak solutions such that \( \|u\|_V \) and \( \|u_t\|_H \) are uniformly bounded on finite time intervals for dimensions \( n = 2 \) or \( 3 \). In the case of small data they proved existence within this class. J. Serrin ([16]) extended the results of uniqueness and, for small data, existence, within this class to dimension \( n = 4 \). For the two dimensional problem, Ladyzenskaya has demonstrated the existence of such solutions for arbitrarily large data ([14]). It is not yet known if a solution in this class exists for dimensions larger than 4, or in dimensions 3 and 4 for large data.

This thesis will be an exposition of part of the paper ([9]) by J.L. Lions. Lions showed the existence of a weak solution \( u \) of (1) belonging to \( L(0, \infty; V) \) and having in addition a fractional time derivative \( D^\gamma_t u \in L^2(0, \infty; H) \) for any \( \gamma, \ 0 < \gamma < \frac{1}{4} \). Here we consider only the case of a bounded domain \( \Omega \) in \( \mathbb{R}^n \), with \( 2 \leq n \leq 4 \).

The rest of the chapter contains some preliminary lemmas and a statement of the existence theorem. The second chapter will concern itself with a proof of the existence theorem.
1.4 Preliminary Theory.

Proposition 1: Let $\Omega$ be a bounded, open and connected subset of $\mathbb{R}^n$, with $n \leq 4$. Then there exists a natural continuous injection mapping from $V$ into $(L^4(\Omega))^n$.

The proposition follows as a special case of the Sobolev embedding theorem ([10]). The natural injection mapping of $H^1_0(\Omega)$ into $L^4(\Omega)$ is continuous.

Corollary 1: Let $u, v, w \in V$, then $b(u, v, w)$ is continuous on $V \times V \times V$.

Proof: $|b(u, v, w)| \leq \sum_{i,j=1}^{n} \int_{\Omega} |u_i||D_i v_j||w_j|dx$, for any $u, v, w \in V$.

From Proposition 1,

$|b(u, v, w)| \leq C ||u||_{L^4} ||v||_V ||w||_{L^4} \leq C ||u||_V ||v||_V ||w||_V$.

Q.E.D.

Proposition 2: Let $u(x, t) \in L^2(0, \infty; V)$ and $\phi \in C^\infty_0([0, \infty); V)$, then $b(u, u, \phi) \in L^1(0, \infty)$.

Proof: By Corollary 1 of Proposition 1, we have

$|b(u, u, \phi)| \leq C ||u(t)||^2_\mathbf{V} \phi ||_\mathbf{V}$.
Thus \( b(u, u, \phi) \in L^1(0, \infty) \). Q.E.D.

**Proposition 3:** Suppose \( u, v, w \in V \), then \( b(u, v, w) + b(u, w, v) = 0 \).

**Proof:** For \( u, v, w \in D(\Omega) \) and divergence free, we can integrate by parts:

\[
b(u, v, w) = \sum_{i,j=1}^{n} \int_{\Omega} u_i D_i v_j w_j \, dx
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ (u_i v_j w_j) \right]_{\partial \Omega} - \int_{\Omega} D_i u_i v_j w_j \, dx
\]

\[- \int_{\Omega} u_i v_j D_i w_j \, dx . \]

Since \( v_j, w_j \in D(\Omega) \) and div \( u = 0 \),

\[
b(u, v, w) = -b(u, w, v)
\]

Hence for any \( u, v, w \in V \) the desired result follows by passing to the limit using Corollary 1 of Proposition 1.

**Proposition 4:** For \( 0 < \gamma < \frac{1}{4} \), there exists \( \beta > \frac{1}{2} \) and \( C_1 \in \mathbb{R} \) such that

\[
|\tau|^{2\gamma} \leq \frac{C_1(1 + |\tau|)}{1 + |\tau|^\beta}
\]

for all \( \tau \in \mathbb{C} \).

**Proof:** Since \( 0 < 2\gamma < \frac{1}{2} \) we have

\[
2 \tau \geq \frac{|\tau|^{2\gamma}}{1 + |\tau| + 1} .
\]
Hence,

\[ 2(1 + |\tau|) \geq |\tau|^{2\gamma} + |\tau| + 1 \]

\[ \geq |\tau|^{2\gamma} + |\tau| \]

\[ = |\tau|^{2\gamma}(1 + |\tau|^{-2\gamma}) , \]

and so

\[ |\tau|^{2\gamma} \leq \frac{2(1 + |\tau|)}{1 + |\tau|^{-1-2\gamma}} . \]

Thus we may let \( c_1 = 2 \) and \( \beta = 1 - 2\gamma \). Q.E.D.

Proposition 5: \( V \) has a countable basis consisting of elements \( w = (w_1, \ldots, w_n) \), where \( w_i \in D(\Omega) \), \( i = 1, \ldots, n \); and \( \text{div } w = 0 \).

**Proof:** Let \( S = \{ w : w = (w_1, \ldots, w_n); w_i \in D(\Omega), i = 1, \ldots, n; \text{div } w = 0 \} . \)

Since \( (C(\Omega))^n \) is separable, \( S \) is separable, ([3], Th. I.6.12). Hence \( S \) has a countable basis \( \{ w_1, \ldots, w_k \ldots \} \), ([15]).

Since \( V \) is the closure \( \overline{S} \) of \( S \) in \( (H^1(\Omega))^n \), for any element \( v \in V \), there exists a sequence \( \{ s_p \} \) \( S \) such that \( s_p \to v \) as \( p \to \infty \), in \( (H^1(\Omega))^n \). Hence the Proposition follows. Q.E.D.

Proposition 6: A weak solution of the initial-boundary value problem satisfies the Navier-Stokes equations (1) in the distribution sense. Conversely, a distribution solution \( u \in L^2(0,\infty; V) \) of the Navier-Stokes equations (1)
in $\Omega \times (0, \infty)$ satisfies the integral identity (2) for all $\phi \in C^0_0((0, \infty); V)$.

**Proof:** Suppose $u \in L^2(0, \infty; V)$ satisfies (1). Let $\phi \in (C^0_0(\Omega \times (0, \infty)))^n$ and $\text{div } \phi = 0$, then

$$\int_0^\infty (\Delta u, \phi)_H \, dt = \int_0^\infty \phi(t), \phi(t))_V \, dt$$

and

$$\int_0^\infty (u_t, \phi)_H \, dt = -\int_0^\infty u(t), \phi(t)_H \, dt - (u_0, \phi(0))_H.$$

By Proposition 2, $b(u, u, \phi) \in L^1(0, \infty)$ and hence

$$\int_0^\infty \left( \sum_{k=1}^n u_k D_k u, \phi \right)_H = \int_0^\infty b(u(t), u(t), \phi(t))_H \, dt.$$

Since $\phi \in (C^0_0(\Omega \times (0, \infty)))^n$ and $\text{div } \phi = 0$ we have

$$\int_0^\infty (\text{grad}_x p, \phi)_H \, dt = \int_0^\infty \left( \sum_{i=1}^n \int_\Omega \frac{\partial p}{\partial x_i} \cdot \phi_i \, dx \right) \, dt$$

$$= \int_0^\infty \sum_{i=1}^n \left( p \phi_i \right|_\Omega - \int_\Omega \frac{\partial \phi_i}{\partial x_i} \, dx \right) \, dt$$

$$= \int_0^\infty \sum_{i=1}^n \left( \int_\Omega \frac{\partial \phi_i}{\partial x_i} \, dx \right) \, dt$$

$$= \int_\Omega \left( p \sum_{i=1}^n \frac{\partial \phi_i}{\partial x_i} \right) \, dx \, dt = 0.$$

Conversely, let $u$ satisfy (2). Define a distribution $T$ in $D^1(\Omega \times (0, \infty))^n$ by

$$T = \frac{\partial}{\partial t} u + v \Delta u + \sum_{k=1}^n u_k D_k u - f.$$
By the preceding arguments,

\[
\int_0^\infty (T, \phi)_H dt = 0 \quad \text{for all } \phi \in (C_0^\infty(\Omega \times (0, \infty)))^n \text{ with } \text{div } \phi = 0 .
\]

Hence \( T = \text{grad } p \), ([6], Th.1). \hfill \text{Q.E.D.}

1.5. Existence Theorem.

Let \( \Omega \) be an open, bounded and connected subset of \( \mathbb{R}^n \), with \( 2 < n \leq 4 \). For any \( f \in L^2(0, \infty; H) \), \( \nu > 0 \), and \( u_0(x) \in \overline{V} \), there exists a function \( u \in L^2(0, \infty; V) \) which satisfies

(5) \[
\int_0^\infty \{ \nu(u(t), \phi(t))_V + b(u(t), u(t), \phi(t)) - (u(t), \phi_t(t))_H \} dt
\]

\[
= \int_0^\infty (f(t), \phi(t))_H dt + (u_0 + \phi(0))_H
\]

for all \( \phi \in C_0^1([0, \infty); V) \) with \( \phi_t \in L^2(0, \infty; H) \). Moreover \( D_t^\gamma u \in L^2(0, \infty; H) \), with \( 0 < \gamma < \frac{1}{4} \).
The proof of the existence theorem will be given in a series of lemmas which will be broken into three sections. In the first section the existence of approximate solutions \( \{u_m\} \) is proven. In the second section a priori estimates for the \( \{u_m\} \) and their fractional time derivatives are established. Finally we use these estimates along with some compact embedding theorems to show the convergence of the \( \{u_m\} \) to a weak solution \( u \) of the initial-boundary value problem for the Navier-Stokes equations.

Proof of the Existence Theorem

2.1. **Existence of an approximate solution** \( u_m \) **of the Navier-Stokes equations.**

Let \( w_1, \ldots, w_s \ldots \) be a basis for \( V \) as defined in Proposition 5. Then \( w_j = (w_{j1}, \ldots, w_{jn}), w_{jk} \in \mathcal{D}(\Omega), \) with \( k = 1, \ldots, n; \) and \( \text{div } w_j = 0, \ j = 1, \ldots. \) Since \( u_0 \in \overline{V}, \) there exists real \( \alpha_i \) such that

\[
\sum_{i=1}^{m} \alpha_i w_i \longrightarrow u_0 \text{ in } H \text{ as } m \to \infty.
\]

**Lemma 1:** Let \( u_m = \sum_{j=1}^{n} g_{jm}(t)w_j(x). \) Then the initial value problem:

\[
\begin{align*}
(u_m', w_j)_H + v(u_m, w_j)_V + b(u_m, u_m, w_j) &= (f, w_j)_H, \ j = 1, \ldots, m, \\
g_{jm}(0) &= \alpha_i, \ i = 1, \ldots, m
\end{align*}
\]
NOTE: \( u' = \frac{\partial u}{\partial t} \)

has a local solution \( u_m(t) \in L^2(0, T_m; V) \).

**Proof**: To prove the lemma we shall apply the Caratheodory Existence Theorem ([2], p. 43). Rewriting (6), for \( j = 1, \ldots, m \),

\[
\frac{d}{dt}(u_m, w_1)_H = (f, w_1) - v(u_m, w_1)_V + b(u_m, u_m, w_1)
\]

\[
\vdots
\]

\[
\frac{d}{dt}(u_m, w_m)_H = (f, w_m)_H - v(u_m, w_m)_V + b(u_m, u_m, w_m)
\]

(7)

Since \( u_m = \sum_{i=1}^{n} g_{im}(t) w_i(x) \) we have

\[
\begin{pmatrix}
(w_1, w_1)_H & \cdots & (w_m, w_1)_H \\
\vdots & \ddots & \vdots \\
(w_1, w_m)_H & \cdots & (w_m, w_m)_H
\end{pmatrix}
\begin{pmatrix}
g_{im}(t) \\
\vdots \\
g_{mn}(t)
\end{pmatrix}
\begin{pmatrix}
(f, w_1)_H \\
\vdots \\
(f, w_m)_H
\end{pmatrix}
\]

(8)

\[
= \begin{pmatrix}
(f, w_1)_H \\
\vdots \\
(f, w_m)_H
\end{pmatrix} - v \begin{pmatrix}
(w_1, w_1)_V & \cdots & (w_m, w_1)_V \\
\vdots & \ddots & \vdots \\
(w_1, w_m)_V & \cdots & (w_m, w_m)_V
\end{pmatrix} \begin{pmatrix}
g_{1m}(t) \\
\vdots \\
g_{mn}(t)
\end{pmatrix}
\]
We will now let (8) be written in the matrix-vector form:

\[
\frac{d}{dt}AX(t) = F(t) + vB\cdot X(t) + P(X(t))
\]

with

\[
A = \begin{bmatrix}
(w_1, w_1)_H & \cdots & (w_m, w_1)_H \\
\vdots & & \vdots \\
(w_m, w_1)_H & \cdots & (w_m, w_m)_H
\end{bmatrix};
\]

\[
B = \begin{bmatrix}
(w_1, w_1)_V & \cdots & (w_1, w_m)_V \\
\vdots & & \vdots \\
(w_m, w_1)_V & \cdots & (w_m, w_m)_V
\end{bmatrix};
\]
\[
F(t) = \begin{pmatrix}
(f, w_1)_H \\
\vdots \\
\vdots \\
\vdots \\
(f, w_m)_H
\end{pmatrix};
\]

\[
X(t) = \begin{pmatrix}
g_{1m}(t) \\
\vdots \\
\vdots \\
\vdots \\
g_{nm}(t)
\end{pmatrix};
\]

\[
P(X(t)) = \begin{pmatrix}
b(u_m, u_m, w_1) \\
\vdots \\
\vdots \\
\vdots \\
b(u_m, u_m, w_m)
\end{pmatrix} = \begin{pmatrix}
b\left(\sum_{i=1}^{m} g_{im} w_i, \sum_{i=1}^{m} g_{im} w_i, w_1\right) \\
\vdots \\
\vdots \\
\vdots \\
b\left(\sum_{i=1}^{m} g_{im} w_i, \sum_{i=1}^{m} g_{im} w_i, w_m\right)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\sum_{i,j=1}^{m} g_{im}(t)g_{jm}(t)b(w_i, w_j, w_1) \\
\vdots \\
\vdots \\
\vdots \\
\sum_{i,j=1}^{m} g_{im}(t)g_{jm}(t)b(w_i, w_j, w_m)
\end{pmatrix}.
\]
Since \{w_n\} is a basis for \( V \subseteq H \), \{w_n\} is linearly independent in \( H \), hence determinant \( A \) is the Gram-Schmidt determinant. Therefore \( A \) is nonsingular and \( A^{-1} \) exists.

So (6) may be written as

\[
(6') \quad \frac{d}{dt}X(t) = A^{-1}F(t) - vA^{-1}B.X(t) + A^{-1}P(X(t))
\]

\[
= A^{-1}F(t) - vA^{-1}B.X(t) + A^{-1} \sum_{i=1}^{m} g_{im}C_iX(t),
\]

where \[ C_i = \begin{pmatrix}
    b(w_i, w_1, w_1) & \ldots & b(w_i, w_m, w_1) \\
    \vdots & \ddots & \vdots \\
    \vdots & \ddots & \vdots \\
    b(w_i, w_1, w_m) & \ldots & b(w_i, w_m, w_m)
\end{pmatrix} \]

Let the right hand side of (6') be denoted by \( G(X(t)) \). For fixed \( X(t) \), \( G(X(t)) \) is certainly measurable in \( t \), as \( (f, w_i)_H \) is measurable in \( t \) by hypothesis. For fixed \( t \), the first two terms of \( G(X(t)) \) are linear in \( X(t) \) and since \( P(X(t)) = \sum_{i=1}^{m} g_{im}C_iX(t) \) is a continuous function of \( X(t) \), \( G(X(t)) \) is continuous in \( X(t) \).

Let \( R \) be the rectangle \( 0 \leq t \leq T_m \), with \( T_m > 0 \), and

\[
|X(t) - X(0)| = \sum_{i=1}^{m} |g_{im}(t) - a_{im}| \leq a, \text{ with } a > 0. \text{ Then using the matrix norm, } || \cdot || = \text{absolute value of the largest component of } (\cdot),
\]
\[ |G(X(t))| \leq ||A^{-1}|| |F(t)| + v||A^{-1}|| |B| |X| + ||X|| |P(X(t))| \]

\[ \leq \beta_1 |F(t)| + v\beta_1\beta_2 \rho + \xi \rho^2, \]

where \( \beta_1 = ||A^{-1}||; \beta_2 = ||B||; \frac{\rho}{\mu} = \max_{i,t \leq T_m} |g_{im}(t)|; \) and \( \frac{\xi}{\mu} = \max_{i} ||C_i||. \)

Hence on \( 0 < t \leq T_m \), \( (\beta_1 |F(t)| + v\beta_1\beta_2 \rho + \xi \rho^2) \) is Lebesgue integrable. Thus by Caratheodory's theorem, there exists a solution \( u_m \), defined on \( (0, T_m) \). Q.E.D.

2.2. A priori estimates for \( u_m \) and its fractional time derivatives.

The following two lemmas contain a priori estimates for the \( \{u_m\} \) and their fractional time derivatives.

Lemma 2: The solutions \( \{u_m\} \) of the system (6) are uniformly bounded in the norms of \( L^2(0, \infty; V) \) and \( L^\infty(0, T; H) \), with finite \( T \).

Proof: Multiply both sides of (6) by \( g_{jm}(t) \) and sum with respect to \( j = 1, \ldots, m \). Since \( u_m(t) = \sum_{i=1}^{m} g_{im}(t)w_i(x) \), (6) is replaced by

\[ \left( \frac{d}{dt} u_m(t), u_m(t) \right)_H + \nu(u_m(t), u_m(t))_V + b(u_m, u_m, u_m) \]

\[ = (f, u_m(t))_H. \]

(By Proposition 3, \( b(u_m, u_m, u_m) = 0 \)).
We now show the \( \{u_m\} \) are uniformly bounded in \( L^\infty(0, T; H) \), for all finite \( T \). From (9)

\[
\frac{1}{2} \frac{d}{dt} \| u_m(t) \|_H^2 \leq (f(t), u_m(t))_H.
\]

Thus

\[
\| u_m(t) \|_H \leq \| f \|_H \| u_m(t) \|_H.
\]

Since \( \sum_{i=1}^m a_i w_i \rightarrow u_0 \) in \( H \) we have \( \| u_m(0) \|_H \leq C \| u_0 \|_H \) and hence integrating from 0 to \( t, \ t \leq T < \infty \),

\[
\| u_m(t) \|_H \leq C \| u_0 \|_H + \int_0^t \| f(\sigma) \|_H \, d\sigma.
\]

Since \( f \in L^2(0, \infty; H) \) we have

\[
\sup_{t \in [0, T]} \| u_m(t) \|_H \leq J_2 < \infty.
\]

Thus we may appeal to the global existence theorem ([17], p.122) for ordinary differential equations to conclude that \( u_m \) exists for all \( t \in (0, \infty) \). Hence

\[
\sup_{t \in [0, T]} \| u_m(t) \|_H \leq J_2 < \infty, \text{ for all finite } T.
\]

We show the \( \{u_m\} \) are uniformly bounded in \( L^2(0, \infty; V) \). From the definition of \( (\cdot, \cdot)_H \)
Integration of (9) from 0 to t yields

$$\frac{1}{2} || u_m(t) ||_H^2 - \frac{1}{2} || u_m(0) ||_H^2 + \nu \int_0^t || u_m(\sigma) ||_H^2 \, d\sigma$$

Thus

$$\int_0^t \langle f(\sigma), u_m(\sigma) \rangle_H \, d\sigma.$$

Using the inequality

$$(10') \quad f \cdot g \leq \frac{1}{2} \left( \frac{f^2}{\eta^2} + \eta^2 g^2 \right), \text{ with } \eta > 0 \text{ we get}$$

$$\int_0^t || f(\sigma) ||_H || u_m(\sigma) ||_H \, d\sigma \leq \frac{1}{2} \int_0^t \frac{1}{\eta^2} || f(\sigma) ||_H^2 \, d\sigma$$

$$+ \frac{1}{2} \int_0^t \eta^2 || u_m(\sigma) ||_H^2 \, d\sigma.$$

By the Poincaré inequality ([1], p.73),
Taking $\eta^2 = \frac{\nu}{d^2}$ we obtain

$$\frac{1}{2} \int_0^t \eta^2 \| u_m(\sigma) \|_H^2 d\sigma \leq \frac{\nu}{2} \int_0^t \| u_m(\sigma) \|_V^2 d\sigma.$$

Hence it follows that

$$\int_0^\infty \| u_m(t) \|_V^2 dt \leq J_1 < \infty.$$

Q.E.D.

In order to consider the fractional time derivative of $u_m$ we extend its domain of definition by setting $\tilde{u}_m(t) = \begin{cases} u_m(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$ or equivalently $\tilde{u}_m(t) = u_m(t) H(t)$ where $H(t)$ is the Heaviside function. Then $\tilde{u}_m(t) \in L^2(-\infty, \infty; V)$.

Lemma 3: For fixed $\gamma$, $0 < \gamma < \frac{1}{4}$, $\tilde{u}_m(t) \in H_{\gamma}(-\infty, \infty; V; H)$ and the \{\tilde{u}_m(t)\} are uniformly bounded in $H_{\gamma}(-\infty, \infty; V; H)$.

Proof: We will show that for $k \leq m$, $u_m(t)$ satisfies:

$$\frac{d}{dt} \langle \tilde{u}_m(t), w_k \rangle_H - \nu \langle \tilde{u}_m(t), w_k \rangle_V + b(\tilde{u}_m(t), \tilde{u}_m(t), w_k)$$

$$= \langle f, w_k \rangle_H + \langle u_m(0), w_k \rangle_H.$$
where \( \tilde{f} = f \cdot H(t) \) and \( \delta \) is Dirac measure on \( \mathbb{R} \). This follows from

\[
\frac{d}{dt} (\tilde{u}_m(t), w_k)_H = \frac{d}{dt} (u_m(t)H(t), w_k)_H
\]

\[
= H(t) \frac{d}{dt} (u_m(t), w_k)_H + (u_m(t), w_k)_H \frac{d}{dt} H(t),
\]

where

\[
\frac{d}{dt} (u_m(t), w_k)_H = -\nu(u_m(t), w_k)_V - b(u_m(t), u_m(t), w_k) + (f, w_k)_H
\]

and \( \frac{d}{dt} H(t) = \delta \).

Since \( b(u, v, w) \) is continuous on \( V \times V \times V \) (Corollary 1, Proposition 1), the Riesz Representation theorem gives:

\[
b(\tilde{u}_m, \tilde{u}_m, w_k)
= (h_m(t), w_k)_V,
\]

where

\[
||h_m(t)||_V \leq C \|u_m(t)\|^2_V .
\]

Taking the Fourier transform of (13) with respect to time we obtain

\[
\text{it}(\hat{\tilde{u}}_m(\tau), w_k)_H + \nu(\hat{\tilde{u}}_m(\tau), w_k)_V + (h_m(\tau), w_k)_V
\]

\[
= (\hat{f}(\tau), w_k)_H + (u_m(0), w_k)_H .
\]

Multiplying (15) by \( \hat{g}_{im}(\tau) \) and summing with respect to \( i = 1, \ldots, m \), we get
Taking the imaginary part of (16) and applying the Hölder inequality we have:

\[
|\tau||\hat{u}_m(\tau)|^2_{H} + \nu||\hat{u}_m(\tau)|^2_{V} + (\hat{h}(\tau), \hat{u}_m(\tau))_V
\]
\[
= (\hat{f}(\tau), \hat{u}_m(\tau))_H + (u_m(0), \hat{u}_m(\tau))_V.
\]

From (17) we will show

\[
\int_{-\infty}^{\infty} |\tau|^{2\gamma} ||\hat{u}_m(\tau)||^2_{H} d\tau \leq J_3 < \infty.
\]

Indeed, since \( \hat{h}_m(\tau) = \int_{-\infty}^{\infty} e^{-i\tau t} h_m(t) dt \), we have

\[
||\hat{h}_m(\tau)||_V \leq \int_{-\infty}^{\infty} ||h_m(t)||_V dt
\]
\[
\leq C_2 \int_{-\infty}^{\infty} ||\hat{u}_m(t)||^2_{V} dt \quad \text{(by (14))}
\]
\[
\leq C_3 \quad \text{(by lemma 2)}.
\]

Hence (17) yields

\[
|\tau||\hat{u}_m(\tau)|^2_{H} \leq C_4 ||\hat{u}_m(\tau)||_V + ||\hat{f}(\tau)||_H ||\hat{u}_m(\tau)||_H,
\]
where \( C_4 = (C_3 + d \cdot \| u_0 \|_H) \).

For \( 0 < \gamma < \frac{1}{4} \), we have by Proposition 5,

\[
|\tau|^{2\gamma} \| \hat{u}_m(\tau) \|_H^2 \leq C_1 (1 + |\tau|) (1 + |\tau|^{\beta^{-1}}) \| \hat{u}_m(\tau) \|_H^2
\]

\[
= C_1 (1 + |\tau|^{\beta^{-1}}) \| \hat{u}_m(\tau) \|_H^2
\]

\[
+ C_1 |\tau| (1 + |\tau|^{\beta^{-1}}) \| \hat{u}_m(\tau) \|_H^2
\].

Now substituting (18) into the previous inequality and integrating from \(-\infty\) to \(\infty\) we have:

\[
\int_{-\infty}^{\infty} |\tau|^{2\gamma} \| \hat{u}_m(\tau) \|_H^2 d\tau \leq C_1 C_4 \int_{-\infty}^{\infty} (1 + |\tau|^{\beta^{-1}}) \| \hat{u}_m(\tau) \|_V d\tau
\]

\[
+ C_1 \int_{-\infty}^{\infty} (1 + |\tau|^{\beta^{-1}}) \| \hat{f}(\tau) \|_H \| \hat{u}_m(\tau) \|_H d\tau
\]

\[
+ C_1 \int_{-\infty}^{\infty} \| \hat{u}_m(\tau) \|_H^2 (1 + |\tau|^{\beta^{-1}}) d\tau
\].

Let \( I_1, I_2, I_3 \) be the three integrals on the right hand side of the previous inequality. Since \( \| \hat{u}_m \|_{L^2(-\infty, \infty ; V)} \leq J_1 \),

\[
I_1 \leq C_1 C_4 (\int_{-\infty}^{\infty} (1 + |\tau|^{\beta^{-1}})^{\frac{1}{2}} (\int_{-\infty}^{\infty} \| \hat{u}_m(\tau) \|_V^2 d\tau)^{\frac{1}{2}})^{\frac{1}{2}}
\]

\[
\leq M_2 < \infty .
\]
We have

\[ I_2 \leq C_1 \left( \int_{-\infty}^{\infty} (1 + |\tau|^\beta)^{-1} ||\hat{f}||_H ||\hat{u}_m(\tau)||_H^2 d\tau \right) \leq C \left( \int_{-\infty}^{\infty} ||\hat{f}||_H^2 d\tau + \int_{-\infty}^{\infty} ||\hat{u}_m(\tau)||_H^2 d\tau \right)^{\frac{1}{2}}. \]

By Plancherel's theorem and the Poicaré inequality (10") \( I_2 \leq M_3 \). Also by (10") and the Plancherel theorem \( I_3 \leq M_4 \). Thus

\[ \int_{-\infty}^{\infty} |\tau|^{2\gamma} ||\hat{u}_m(\tau)||_H^2 d\tau \leq J_3. \]

Q.E.D.

2.3. **Convergence of the approximate solutions** \( \{u_m\} \) **to a weak solution** \( u \) **of the Navier-Stokes Equations.**

In the last three lemmas of this chapter we will show that there exists a subsequence \( \{u_{m_p}\} \) of \( \{u_m\} \) which converges, in a sense to be specified, to a weak solution \( u \) of the Navier-Stokes equations.

**Lemma 4:** There exists a subsequence \( \{u_{m_p}\} \) of \( \{u_m\} \), which for simplicity we denote by \( \{u_p\} \), such that:

\[ u_p \rightharpoonup u \text{ weakly in } L^2(0, \infty; V), \]

\[ u_p \rightharpoonup u \text{ in the weak star topology of } L^{\infty}(0,T; H), \text{ for all finite } T, \]

\[ u_p \rightharpoonup u \text{ strongly in } L^2(0,T; H) \text{ for all finite } T. \]
Proof: The unit sphere in a reflexive Banach space is weakly compact. Since \( \{u_m\} \) is uniformly bounded in \( L^2(0, \infty; V) \) and \( L^\infty(0,T; H) \), there exists a subsequence \( \{u_p\} \) satisfying (19) and (20), ([3], Th. V.4.2). 

The natural injection mapping of \( V \) into \( H \) is compact since \( \Omega \) is bounded. Thus by the compactness theorem of Lions and Hormander, (Lions [10], Prop. 4.2), the natural injection mapping:

\[
H_\gamma(0,T; V; H) \longrightarrow L^2(0,T; H)
\]

is compact. By Lemma 3, \( ||u_m||_{H_\gamma(0,T; V; H)} \leq J_3 \), hence there exists a subsequence \( \{u_p\} \) of \( \{u_m\} \) such that \( u_p \longrightarrow u \) strongly in \( L^2(0,T; H) \).

Q.E.D.

Since \( u_p(t) \) is a solution of the system (6), the following equation holds:

\[
(22) \quad \int_0^B \{v(u_p(t), \phi(t))_V + b(u_p(t), u_p(t), \phi(t)) + (u_p'(t), \phi(t))_H\} dt = \int_0^B (f(t), \phi(t))_H dt,
\]

for all \( \phi \) of the form \( \phi(t) = \sum_{j=1}^\mu \psi_j(t)w_j(x) \), where \( \mu < p \) and \( \psi_j(t) \in C^1_0([0, \infty)) \).

Integration by parts yields:
Lemma 5: The limit $u$ satisfies the equation:

\[
\int_0^B \{v(u(t), \phi(t))_V + b(u(t), u(t), \phi(t)) - (u(t), \phi'(t))_H\} dt = \int_0^B (f(t), \phi(t))_H dt + (u(0), \phi(0))_H.
\]

for all $\phi$ of the form

\[
\phi = \sum_{j=1}^\mu \psi_j(t)w_j(x); \quad \psi_j \in C^1_0([0, \infty)).
\]

Proof: Let $p \to \infty$, then from (19) and (21), we have

\[
\int_0^B \psi(u_p(t), \phi(t))_V dt \to \int_0^B \psi(u(t), \phi(t))_V dt;
\]

\[
\int_0^B (u_p, \phi'(t))_H dt \to \int_0^B (u, \phi'(t))_H dt.
\]

From the initial conditions for the system (6), $(u_p(0), \phi(0))_H \to (u_0, \phi(0))_H$,

since $\sum_{i=1}^p a_iw_i \to u_0$ in $H$ as $p \to \infty$.

It remains to show that
\[ \int_0^B b(u_p(t), u_p(t), \phi(t)) dt \longrightarrow \int_0^B b(u(t), u(t), \phi(t)) dt. \]

From the definition of \( b(u, v, w) \) we have to show

\[ \int_0^\Omega u_p k(x, t)D_k u_p(x, t) \phi_1 dx dt \longrightarrow \int_0^\Omega u_k(x, t)D_k u_1(x, t) \phi_1 dx dt; \]

for \( p \to \infty; k, i = 1, \ldots, n \). Now:

\[ u_p kD_k u_1 \phi_1 - u_k D_k u_1 \phi_1 \]

\[ = (u_p k - u_k)D_k u_1 \phi_1 + (D_k u_p - D_k u_1)u_k \phi_1. \]

This implies

\[ \left| \int_0^\Omega \left( u_p kD_k u_1 \phi_1 - u_k D_k u_1 \phi_1 \right) dx dt \right| \]

\[ = \left| \int_0^\Omega \left[ (u_p k - u_p)D_k u_1 \phi_1 + (D_k u_p - D_k u_1)u_k \phi_1 \right] dx dt \right| \]

\[ \leq \left\| u_p k - u_k \right\| \left\| D_k u_p \phi_1 \right\| \left\| D_k u_1 \phi_1 \right\| \]

\[ + \int_0^\Omega \left| (D_k u_p - D_k u_1)u_k \phi_1 \right| dx dt. \]

The expression on the right tends to zero since \( u_p k \longrightarrow u_k \) strongly in \( L^2([0, B] \times \Omega) \), \( D_k u_p \longrightarrow D_k u_1 \) weakly in \( L^2([0, B] \times \Omega) \), and \( \phi_1 \in C_0^\infty([0, \infty) \times \Omega) \).
Lemma 6: The equation

\[ (27) \quad \int_0^\infty \{ v(u(t), \phi(t)) + b(u(t), u(t), \phi(t)) - (u(t), \phi'(t)) \} \, dt = \int_0^\infty (f(t), \phi(t)) \, dt + (u_0, \phi(0)) \]

holds for all functions \( \phi \) such that

\[ (28) \quad \phi \in L^2(0, \infty; V), \quad \phi' \in L^2(0, \infty; H), \]

with \( \phi(t) \) having compact support in \( t \geq 0 \).

Proof: By Lemma 5, (27) holds for all function \( \phi \) of the form (25).

Let the lefthand side of (27) be denoted by \( L(\phi) \). For any \( \phi \) satisfying (28), there exists a sequence of functions \( \phi_k \) satisfying (25) with support contained in \([0,B)\), such that \( \phi_k \to \phi \) in \( L^2(0,B; V) \) and \( \phi'_k \to \phi' \) in \( L^2(0,B; H) \) as \( k \to \infty \), ([10]).

Moreover,

\[ |L(\phi_k) - L(\phi)| \leq \int_0^B v \| u(t) \|_V \| \phi_k - \phi \|_V \, dt \]

\[ + c_1 \int_0^B \| u(t) \|_V^2 \| \phi_k - \phi \|_V \, dt \]

\[ + \int_0^B \| u(t) \|_H \| \phi_k - \phi \|_H \, dt \]
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\[\leq \nu \| u \|_{L^2(0,B; V)} \left( \int_0^B \| \phi_k - \phi \|_V^2 \, dt \right)^\frac{1}{2} + c_1 \| u \|_{L^2(0,B; H)} \left( \int_0^B \| \phi'_k - \phi' \|_H^2 \, dt \right)^\frac{1}{2}
\]

\[+ \| u \|_{L^2(0,B; V)} \left( \int_0^B \| \phi'_k - \phi' \|_H^2 \, dt \right)^\frac{1}{2} + \| u \|_{L^2(0,B; H)} \left( \int_0^B \| \phi'_k - \phi' \|_H^2 \, dt \right)^\frac{1}{2} \leq (\nu \| u \|_{L^2(0,B; V)} + c_1 \| u \|_{L^2(0,B; V)} ) + \| u \|_{L^2(0,B; H)} \left( \int_0^B \| \phi'_k - \phi' \|_V^2 + \| \phi'_k - \phi' \|_H^2 \, dt \right)^\frac{1}{2} \]

\[< \varepsilon .\]

Clearly \( \int_0^\infty (f, \phi_k) \, dt + (u_0, \phi_k(0)) \) also converges to

\[\int_0^\infty (f, \phi) \, dt + (u_0, \phi(0)) \quad \text{as} \quad k \to \infty .\]

Q.E.D.
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