TRANSIENT ANALYSIS OF NONLINEAR NON-AUTONOMOUS SECOND ORDER SYSTEMS USING JACOBIAN ELLIPTIC FUNCTIONS

by

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ABSTRACT

A method is presented for determining approximate solutions to a class of grossly nonlinear, non-autonomous second order differential equations characterized by

$$\frac{d^2x}{d\tau^2} + m^2 (x + rx^3) + \mu f (x, \frac{dx}{d\tau}, \tau) = 0 , \quad (-1 < r < \infty)$$

with the restriction that resonance effects be negligible. Solutions are developed in terms of the Jacobian elliptic functions, and may be related directly to the degree of non-linearity in the differential equation. An integral error definition, which can be applied to any particular differential equation, is used to portray regions of validity of the approximate solution in terms of equation parameters. In practice the approximate solution is shown to be of greater accuracy than would be expected from the error analysis, and use of the error diagram leads to a pessimistic estimate of solution accuracy. Two autonomous equations are considered to facilitate comparison between the elliptic function approximation and that obtained from the method of Kryloff and Bogoliuboff. The elliptic function solution is shown to be accurate even for heavily damped nonlinear autonomous equations, when the quasi-linear approximation of Kyrloff-Bogoliuboff cannot with validity be applied. Four examples are chosen, from the fields of astrophysics, mechanics, circuit theory and control systems to illustrate some areas to which the general approximation method relates.

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1. INTRODUCTION

" ... practically all of the problems in mechanics simply <u>are</u> nonlinear from the outset, and the linearizations commonly practiced are an approximating device which is often simply a confession of defeat in the face of the challenge presented by the nonlinear problems as such".

J.J. Stoker [22]

These words, which appear in the introduction of Stoker's book "NONLINEAR VIBRATIONS", may well be taken as an indication of the complexity with which engineers and scientists are confronted when endeavouring to describe the operation of physical systems. In the analysis of nonlinear systems the investigator has three basic tools at his disposal:

a) graphical analysis, which, although it may be applied to grossly nonlinear problems, usually implies a lack of precision.

b) digital or analog computer simulation techniques which, in common with graphical analysis, can only be applied to a particular problem, and

c) techniques of theoretical analysis.

Of these three approaches only the last enables results to be obtained in terms of the system parameters. The system may then be designed to conform with specifications, and therein lies its value to the engineer. Until recently, however, little research has been conducted into the behaviour of systems which are grossly nonlinear, and existing analytical techniques are restricted to investigations of quasi-linear systems.

As an example of a quasi-linear differential equation, B. Van der Pol [23] considered

$$\ddot{x} + \omega^2 x = \mu f(x,\dot{x})$$

$$\dot{x} \triangleq \frac{dx}{dt}$$

which is the equation of a simple harmonic oscillator perturbed by a small

function $f(x,\dot{x})$, the degree of "smallness" being determined by the constant term μ $(0 < \mu \ll 1)$.

This equation was also considered by N. Kryloff and N. Bogoliuboff [13], and the solution method which they developed, commonly called the K-B method, has become a standard technique in the analysis of quasi-linear systems. Because of its relevance to the present work, a brief description of the K-B method will now be given.

When the constant ε is zero in the equation

$$\ddot{x} + \omega^2 x + \varepsilon f(x,\dot{x}) = 0$$

the exact solution is $x(t) = A \cos(\omega t + \phi)$. Kryloff and Bogoliuboff postulated that, for small ε, the solution of the quasi-linear equations departs only slightly from the linear solution for which A and ϕ are constant quantities. By considering the solution of the quasi-linear equation to be

$$x(t) = A(t) \cos(\omega t + \phi(t))$$

and defining $\dot{A}_{av}$ as the average value of $\dot{A}$ over one period, and $\dot{\phi}_{av}$ as the average value of $\dot{\phi}$ over one period, they derived relationships for $\dot{A}_{av}$ and $\dot{\phi}_{av}$ by integrating the Fourier series expansions for

$$f(x,\dot{x}) \sin\theta \quad \text{and} \quad f(x,\dot{x}) \cos\theta$$

over the interval $(t,\ t + \frac{2\pi}{\omega})$, where $\theta = \omega t + \phi$.

Thus they obtained an average value for $\dot{A}$ and $\dot{\phi}$ over one period in the form

$$\dot{A}_{av} = \frac{\varepsilon}{2\pi\omega} \int_0^{2\pi} f(A\cos\theta,\ -A\omega\sin\theta) \sin\theta \, d\theta$$

$$\dot{\phi}_{av} = \frac{\varepsilon}{2\pi\omega A} \int_0^{2\pi} f(A\cos\theta,\ -A\omega\sin\theta) \cos\theta \, d\theta \ .$$

These integrals are, in general, readily evaluated, but the approach is limited to systems which are quasi-linear.

There does, however, exist a nonlinear differential equation having a periodic solution in terms of the Jacobian Elliptic functions [7], [11], [21], of which the harmonic oscillator equation is a special case. This equation:

$$\ddot{x} + m^2(x + px^3) = 0$$

was considered in a previous paper [1], where an approximate solution of the equation

$$\ddot{x} + m^2(x + px^3) + \mu f(x,\dot{x}) = 0 \qquad (p > 0)$$

was determined by a method comparable to the K-B method.

The importance of this equation in describing nonlinear oscillatory systems stems from the natural occurrence of odd, in preference to even, nonlinearities. Often a cubic polynomial in the dependent variable is sufficient to model such a nonlinearity; a higher order polynomial may still, however, be accommodated either by incorporating the high order terms in the function $f(x,\dot{x})$ or by approximating the polynomial by a cubic. An exhaustive analysis of this latter technique is to be found in Soudack [20]. The oscillation frequency of a nonlinear system may be strongly amplitude dependent, and this provides further motivation for the use of elliptic functions which have the same property.

In 1955, Bogoliuboff and Mitropolsky [4] presented an extension of the K-B method, which will be called the B-M method, and in particular presented a firm mathematical foundation for the approximation technique. Starting from the basic equation

$$\ddot{x} + \omega^2 x = \mu f(x,\dot{x}) \ , \qquad (0 < \mu \ll 1)$$

they assumed the following form for the solution $x(t)$:

$$x(t) = a \cos\psi + \mu u^{(1)}(a,\psi) + \mu^2 u^{(2)}(a,\psi) + \ \ldots.$$

where

$$\dot{a} = \mu A^{(1)}(a) + \mu^2 A^{(2)}(a) + \ldots.$$

$$\dot{\psi} = \omega + \mu B^{(1)}(a) + \mu^2 B^{(2)}(a) + \ldots.$$

where the $u^{(i)}$ are functions periodic in 2π, and the $A^{(i)}$ and $B^{(i)}$ are functions to be determined. The major advantage of this method is that successive approximations of increasing accuracy can be obtained recursively. The same basic approach may be adopted for non-autonomous systems of the form

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}, t)$$

where the $u^{(i)}$ become functions of a, ψ and t, but the non-autonomous term can only be periodic, i.e. perturbation terms of the form tx, $t\dot{x}$ cannot be handled. The B-M method is essentially that of "variation of parameters" [6], but it is still only applicable to quasi-linear systems.

Nonlinear, non-autonomous systems involving no time delay may broadly be divided into two categories: those which exhibit resonance phenomena, and those which do not. Time delay phenomena, although of great interest, will not be considered in the present work. Resonance effects in quasi-linear systems lead to particularly interesting and unusual phenomena, such as jump resonance [6] and sub-harmonic resonance [9], and it is possible to analyse these effects by the B-M method.

The research presented in this thesis takes, as its starting point, perturbation series similar to those chosen by Bogoliuboff and Mitropolsky, in an analysis of the differential equation

$$\frac{d^2x}{d\tau^2} + m^2 (x + rx^3) + \rho g \left(x, \frac{dx}{d\tau}, \tau\right) = 0 \qquad (-1 < r < \infty)$$

using the Jacobian elliptic functions. This is a non-autonomous equation, but, because of the analytical difficulties associated with resonance

effects in grossly nonlinear systems, the analysis is restricted to a consideration of non-resonant systems.

Equations of this general form arise in the fields of, for example, astrophysics [12], circuit analysis [35] - [45] and control systems [46] - [54].

1.1 Thesis Outline

Much of the work presented in chapter 2 appears in a paper [2] where the first-approximation solution of the equation

$$\ddot{x} + x + px^3 + \beta f(x,\dot{x},t) = 0 \qquad (p > 0)$$

is developed under the assumption that solution frequency remains approximately constant. Three examples, relating to mechanical systems and frequency-modulation circuits, are chosen to demonstrate application of the approximation method. It is shown that the approximate solution can be obtained in a state-equation form where, for a particular differential equation, all the coefficients are known. This enables solution error to be predicted, and error results are shown for the three examples considered. Three specific differential equations, corresponding to the earlier examples, are solved using the first-approximation method. The graphical solutions of figures 2.7 to 2.10 demonstrate the accuracy which can be maintained up to relatively large values of the parameter β, which determines the magnitude of the non-autonomous term.

In chapter 3 the same basic equation is considered, but the assumption that solution frequency remain approximately constant is removed. The same three examples are considered, new error results are obtained and refined solutions of the specific equations are shown in figures 3.7 to 3.9. A partial cancellation of certain first order terms is related to three different approaches for calculating the approximate solution, which enhance

the accuracy of the resulting solution.

Chapter 4 extends the refined analysis of chapter 3 to consider the equation

$$\ddot{x} + x - px^3 + \beta f(x,\dot{x},t) = 0 \qquad (0 < p < 1).$$

Comparable examples to those of chapter 3 are chosen to derive error results and to facilitate comparison of the solutions shown in figures 4.7 to 4.9 with the corresponding figures of chapter 3.

Two autonomous differential equations, a modified Van der Pol equation [30] and the damped Duffing equation [34], are considered in chapter 5. Because neither the K-B method or the B-M method could be applied to non-autonomous equations, a comparison of the results of the solution method with those obtained from classical methods was not possible in previous chapters. For the autonomous case, however, such a comparison is possible and is made in section 5.5, the results being shown in figures 5.1 and 5.3. It is also shown that use of the elliptic function solution for the heavily damped Duffing equation results in a first-order approximation of considerable accuracy which is, at the same time, in a simple form.

In chapter 6 examples are chosen from the fields of mechanics, astrophysics, circuit theory and control systems to demonstrate some areas of application of the general analysis presented in the earlier chapters.

The thesis is concluded in chapter 7, where some possible extensions of the approximation technique are suggested. There are two sections in an Appendix, the first containing some pertinent relationships for elliptic functions, and the second being a tabulation of two constants defined in chapters 2 and 4.

1.2 Notation

It is usual to express an elliptic function as a function of two variables, the first being the independent variable u (say), and the second

being a constant associated with the elliptic function, called its modulus, k [5]. When the modulus is zero the elliptic sine (Sn) and cosine (Cn) reduce to the circular functions sin and cos respectively. As k increases to 1 the elliptic sine and cosine depart from the circular functions and eventually become hyperbolic functions. The generality of elliptic functions can thus be seen to depend on this modulus k.

The elliptic function solution of the equation

$$\ddot{x} + x + px^3 = 0 \qquad (p > 0)$$

with $x(0) = a$ and $\dot{x}(0) = 0$ is $x(t) = a\ \mathrm{Cn}\ [(1+pa^2)^{1/2}t, k]$ using the notation $\mathrm{Cn}[u,k]$. However, the modulus k can also be expressed in terms of 'p' and 'a' as

$$k^2 = \frac{pa^2}{2(1+pa^2)} .$$

Because the value of k^2 is expressible in this form, the notation $\mathrm{Cn}[u,k]$ will subsequently be replaced by the simpler form Cnu.

2. THE ELLIPTIC FUNCTION APPROXIMATION.

2.1 Introduction

Taking the approach of Bogoliuboff-Mitropolsky [4] as a basis for the present analysis, approximate solutions to a class of non-autonomous, grossly nonlinear second order differential equations are developed, under the assumption that solution frequency remains approximately constant and with the restriction that resonance effects be negligible or non-existent.

The equation analysed in this chapter is of the general form

$$\frac{d^2x}{d\tau^2} + m^2(x + px^3) + \rho g(x, \frac{dx}{d\tau}, \tau) = 0$$

where $p > 0$, and ρ is a small constant coefficient which may be either positive or negative. This equation represents an oscillatory system with a "hardening" [6] characteristic, but if the characteristic is of a higher order than cubic in the dependent variable it may still be approximated by a cubic polynomial [14], [20].

Existing techniques for the analysis of non-autonomous differential equations include variation of parameters [17] and the WKBJ method [6], but are restricted to quasi-linear systems. The analysis presented in this chapter now makes possible the investigation of nonlinear non-autonomous, in addition to grossly nonlinear autonomous, systems.

2.2 Development of the approximation

Using well known techniques [3], [6], it can be shown that any differential equation of the form

$$\ddot{x} + m^2(x + px^3) + \rho g(x,\dot{x},\tau) = 0$$

where $p > 0$, ρ is a small constant parameter, and with a general initial condition $x(0) = x_0$ and $\dot{x}(0) = 0$, can be expressed in the normalized form

$$\ddot{x} + x + px^3 + \beta f(x,\dot{x},t) = 0 \qquad (1)$$

where $x(0) = 1.0$, $\dot{x}(0) = 0$, $p > 0$ and β is a small constant coefficient.

The solution of equation (1), when $\beta = 0$, is $x(t) = Cn\ \omega t$, where Cn is the elliptic cosine function [5], $\omega = (1+p)^{1/2}$ and $k^2 = \frac{p}{2(1+p)}$ [6].

For $\beta \neq 0$, consider an approximate solution

$$\tilde{x}(t) = a(t)\ Cn\ (\omega t + \theta)\ ,$$

where $a(t)$ is the solution amplitude envelope, and θ is a phase-modifying term. For ease of notation in the subsequent analysis, $x(t)$ will be used instead of $\tilde{x}(t)$. In this case

$$\omega^2 = 1+pa^2$$

and

$$k^2 = \frac{pa^2}{2(1+pa^2)}$$

If

$$\psi = \omega t + \theta = \psi(a,t)$$

then

$$x = aCn\psi = x(a,\psi) \tag{2}$$

Differentiating equation (2):

$$\dot{x} = \dot{a}\,\frac{\partial x}{\partial a} + \dot{\psi}\,\frac{\partial x}{\partial \psi} \tag{3}$$

and from equation (3):

$$\ddot{x} = \ddot{a}\,\frac{\partial x}{\partial a} + \dot{a}\,\frac{d}{dt}\left[\frac{\partial x}{\partial a}\right] + \dot{\psi}\,\frac{d}{dt}\left[\frac{\partial x}{\partial \psi}\right] + \ddot{\psi}\,\frac{\partial x}{\partial \psi} \tag{4}$$

The approximation of Bogoliuboff-Mitropolsky [4] is based on the following polynomial representation of $\dot{a}$ and $\dot{\theta}$:

$$\dot{a} = \mu A^{(1)}(a,t) + \mu^2 A^{(2)}(a,t) + \ldots + \mu^i A^{(i)}(a,t) + \ldots$$

$$\dot{\theta} = \mu B^{(1)}(a,t) + \mu^2 B^{(2)}(a,t) + \ldots + \mu^i B^{(i)}(a,t) + \ldots$$

where μ is a small constant parameter, $A^{(i)}$ and $B^{(i)}$ $(i=1,2,3,\ldots)$ are functions to be determined, and both polynomials are assumed to be convergent in μ and contain no small divisors so that $\forall$ i:

$$\mu^{i+1} A^{(i+1)} \ll \mu^{i} A^{(i)}$$

$$\mu^{i+1} B^{(i+1)} \ll \mu^{i} B^{(i)}$$

Differentiating the expressions for $\dot{a}$ and $\dot{\theta}$, and discarding terms of $O(\mu^3)$:

$$\ddot{a} = \mu A_t^{(1)} + \mu^2 \left[A^{(1)} A_a^{(1)} + A_t^{(2)} \right]$$

$$\ddot{\theta} = \mu B_t^{(1)} + \mu^2 \left[A^{(1)} B_a^{(1)} + B_t^{(2)} \right]$$

where the subscripts denote partial differentiation with respect to amplitude (a) and time (t).

If now the following assumption is made:

$$\frac{d}{dt} (\omega t) = \omega$$

i.e. the frequency of the solution x(t) is assumed constant over the time interval of interest, then equation (4) may be written as:

$$\ddot{x} = \ddot{a}\, Cn\psi - 2\dot{a}\, (\omega+\dot{\theta})\, Sn\psi\, Dn\psi - a\, Sn\psi\, Dn\psi\, \ddot{\theta} - aR\, (\omega+\dot{\theta})^2 \qquad (5)$$

where $R \triangleq \frac{\partial}{\partial\psi}(Sn\psi\, Dn\psi)$

i.e. $R = (1-2k^2)\, Cn\psi + 2k^2\, Cn^3\psi$

and hence $R = \frac{1}{1+pa^2} \left[Cn\psi + pa^2\, Cn^3\psi \right]$

Equation (5) can now be expressed as a polynomial function in μ plus one other term (which is independent of μ). As primarily only a first order approximation is being considered, the polynomial function will be concluded at terms of $O(\mu)$. Note that, in general, the expansion can be continued to any order of accuracy by including higher order terms in μ.

The expression for $\ddot{x}$ becomes:

$$\ddot{x} = -aR\,(1+pa^2)$$
$$- \mu\, Sn\psi\, Dn\psi \left[aB_t^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} \right] \tag{7}$$
$$+ \mu \left[A_t^{(1)}\, Cn\psi - (Cn\psi + pa^2 Cn^3\psi)\, \frac{2aB^{(1)}}{(1+pa^2)^{1/2}} \right] .$$

But (from equation (6))

$$R\,(1+pa^2) = Cn\psi + pa^2\, Cn^3\psi$$

and from equation (1):

$$\ddot{x} = -(aCn\psi + pa^3 Cn^3\psi) - \beta f(x,\dot{x},t)$$

From equation (7) it now follows directly that

$$\beta f(x,\dot{x},t) = \mu\, Sn\psi\, Dn\psi \left[aB_t^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} \right] \tag{8}$$
$$-\mu \left[A_t^{(1)}\, Cn\psi - (Cn\psi + pa^2 Cn^3\psi)\, \frac{2aB^{(1)}}{(1+pa^2)^{1/2}} \right]$$

A convenient simplification of equation (8) can be obtained by taking

$$\int_{-K}^{K} Cn^3\psi\, d\psi = \varepsilon \int_{-K}^{K} Cn\psi\, d\psi \tag{9}$$

i.e. $Cn^3\psi$ is replaced by a term $\varepsilon Cn\psi$ having the same integral on the interval

$$-K \leq \psi \leq K ,$$

where K is the complete elliptic integral of the first kind (see Appendix 1).

Now $\frac{d}{d\psi}(Sn\psi\, Dn\psi) = (1-2k^2)\, Cn\psi + 2k^2\, Cn^3\psi$

and hence

$$2k^2 \int Cn^3\psi\, d\psi = Sn\psi\, Dn\psi - (1-2k^2) \int Cn\psi\, d\psi \quad .$$

By evaluating the integral of $Cn\psi$ [5], the expression for ε may now be obtained as:

$$\varepsilon = \left[\frac{(2k^2-1)2\phi + \sin 2\phi}{4k^2\phi} \right] \tag{10}$$

where $\phi = \sin^{-1} k$. A tabulation of p, k^2, ε and K is given in Appendix 2, Table 1.

If the integral defined by equation (9) is evaluated for $\cos^3\psi$, instead of $Cn^3\psi$, a value of ε of 2/3 results. By applying l'Hôpital's rule to equation (10), the same value is obtained for k = 0, when $Cn\psi = \cos\psi$ [5].

As both μ and β are constant small parameters, no generality is lost if, in equation (8), μ is taken equal to β [4], [16]. Substituting for $Cn^3\psi$, equation (8) then becomes:

$$\begin{aligned} f(x,\dot{x},t) = {} & Sn\psi\, Dn\psi \left[aB_t^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} \right] \\ & - Cn\psi \left[A_t^{(1)} - 2aB^{(1)} \frac{(1+\varepsilon pa^2)}{(1+pa^2)^{1/2}} \right] \end{aligned} \tag{11}$$

Now from Milne-Thompson [16] the Fourier series expansion of $Sn\psi\, Dn\psi$ may be derived as:

$$Sn\psi\, Dn\psi = -\frac{d}{d\psi}(Cn\psi) = -\frac{d}{d\psi}\left[\frac{2\pi}{Kk} \sum_{s=0}^{\infty} \left[\frac{q^{(s+1/2)}}{1+q^{(2s+1)}} \right] \cos(2s+1)\frac{\pi}{2K}\psi \right]$$

$$\text{or} \quad Sn\psi\, Dn\psi = \frac{\pi^2}{kK^2} \sum_{s=0}^{\infty} \left[\frac{q^{(s+1/2)}}{1+q^{(2s+1)}} \right] (2s+1) \sin(2s+1)\frac{\pi}{2K}\psi$$

where $q = \exp\left[\frac{-\pi K'}{K} \right]$, and K' is defined in Appendix 1.

The expansion of $Sn\psi\, Dn\psi$ can thus be expressed as an infinite sum of sine terms; and $Cn\psi$ as an infinite sum of cosine terms.

This property suggests that $Cn\psi$ and $Sn\psi\, Dn\psi$ may be treated in the same way as are $\cos\psi$ and $\sin\psi$ in the principle of harmonic balance [6].

If equation (11) could be solved, then the unknown functions $A^{(1)}$ and $B^{(1)}$ could be determined.

Two terms prevent the direct solution of these equations, namely the partial derivatives with respect to time of $A^{(1)}$ and $B^{(1)}$.

By definition:

$$A^{(1)} = A^{(1)}(a,t)$$
$$B^{(1)} = B^{(1)}(a,t)$$

and hence:

$$\dot{A}^{(1)} = A^{(1)}_{a}\,\dot{a} + A^{(1)}_{t}$$

$$\dot{B}^{(1)} = B^{(1)}_{a}\,\dot{a} + B^{(1)}_{t}$$

Substituting for $\dot{a}$, to order μ^2:

$$\dot{A}^{(1)} = \mu A^{(1)} A^{(1)}_{a} + A^{(1)}_{t}$$

$$\dot{B}^{(1)} = \mu A^{(1)} B^{(1)}_{a} + B^{(1)}_{t}$$

If the term in $\mu A^{(1)}$ can be neglected in each case, then $A^{(1)}_{t}$ and $B^{(1)}_{t}$ can be approximated by $\dot{A}^{(1)}$ and $\dot{B}^{(1)}$ respectively. Finally equation (11) can be written as:

$$f(x,\dot{x},t) = \mathrm{Sn}\psi\,\mathrm{Dn}\psi\left[a\dot{B}^{(1)} + 2A^{(1)}(1+pa^2)^{1/2}\right] - \mathrm{Cn}\psi\left[\dot{A}^{(1)} - 2aB^{(1)}\frac{(1+\varepsilon pa^2)}{(1+pa^2)^{1/2}}\right] \tag{12}$$

For a particular function $f(x,\dot{x},t)$, the unknown functions $A^{(1)}$ and $B^{(1)}$ can now be evaluated. To clarify the application of equation (12), three examples will now be considered. Complicated equations can be avoided by taking a, $(1+pa^2)$, $(1+\varepsilon pa^2)$ etc. as constant [13] when evaluating the functions $A^{(1)}$ and $B^{(1)}$. If the amplitude is slowly time-varying, then the error incurred by this approximation will be slight. The examples each relate to systems involving a time-varying component. The equation of example 1 contains a linearly-varying phase term, and is a nonlinear form of Hermite's

equation [10], [18]. A linearly time-varying damping term is considered in example 2, where the equation might describe oscillatory motion in a fluid of time-varying viscosity. Applications of the equations of both examples 1 and 2 are treated in greater detail in chapter 5.

The equation chosen for the third example describes the steady-state operation of a nonlinear frequency-modulated negative-resistance oscillator, and, because of its practical interest, a brief derivation is given below.

Cunningham [6] considers the nonlinear oscillator circuit shown in figure 2.0.

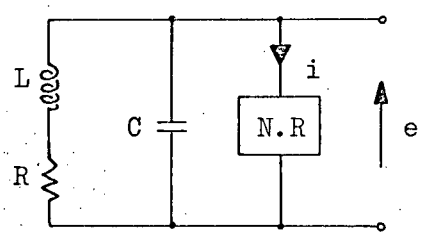

Fig. 2.0 Negative - resistance oscillator .

where the negative resistance device has the current-voltage characteristic $i = -ae + be^3$, a and b being positive constants. If the resistance R is not negligible, the differential equation describing this circuit is:

$$\ddot{e} + \frac{e}{LC}(1 - aR) + \frac{Rbe^3}{LC} + \dot{e}\left[\frac{R}{L} - \frac{a}{C} + \frac{3be^2}{C}\right] = 0$$

Now suppose the capacitance C is varied so that

$$C = C_0\,(1+m\cos\omega_0 t) \qquad [\ 6\ ],$$

where C_0 is the mean value and m is a small constant. After some manipulation it can be shown that the differential equation of the system takes the form

$$\ddot{e} + \frac{e}{LC_0}(1 - aR) + \frac{bRe^3}{LC_0} - me\left[\left[\frac{1 - aR}{LC_0} + \omega_0^2\right]\cos\omega_0 t + \omega_0\,\frac{R}{L}\sin\omega_0 t\right]$$

$$- \dot{e}f(e,\omega_0 t) + O(mbR,m^2) = 0$$

where $f(e,\omega t)$ is a function involving constants, e and periodic terms in $\omega_0 t$.

This is a Van der Pol - type equation [6], and if terms of order

mBR, m^2 may be neglected, its steady state solution may be obtained from the equation

$$\ddot{e} + \frac{e}{LC_0}(1-aR) + \frac{bRe^3}{LC_0} - me\left[\left[\frac{1-aR}{LC_0} - \omega_0^2\right]\cos\omega_0 t + \omega_0 \frac{R}{L}\sin\omega_0 t\right] = 0$$

A corresponding, and slightly simpler, form was chosen for analysis in example 3, i.e.

$$\ddot{x} + x + px^3 - \beta x \cos\omega_0 t = 0 .$$

The same equation is also considered by Minorsky [17] and, in connection with a vibrating string problem, by McLachlan [15].

Example 1.

Consider the equation

$$\ddot{x} + x + px^3 + \beta tx = 0$$

where $f(x,\dot{x},t) = tx = atCn\psi$.

From equation (12), comparing coefficients of $Cn\psi$ and $Sn\psi Dn\psi$:

$$a\dot{B}^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} = 0 \tag{13}$$

$$\dot{A}^{(1)} - 2aB^{(1)}\frac{(1+\varepsilon pa^2)}{(1+pa^2)^{1/2}} = -at \tag{14}$$

Differentiating equation (14):

$$a\dot{B}^{(1)} = (\ddot{A}^{(1)} + a)\frac{(1+pa^2)^{1/2}}{2(1+\varepsilon pa^2)}$$

Substituting for $a\dot{B}^{(1)}$ in equation (13), and rearranging:

$$\ddot{A}^{(1)} + 4A^{(1)}(1+\varepsilon pa^2) = -a$$

or, expressing this as a Laplace transform:

$$A^{(1)}(s) = \frac{a}{4(1+\varepsilon pa^2)}\left[\frac{s}{s^2 + 4(1+\varepsilon pa^2)} - \frac{1}{s}\right]$$

assuming $A^{(1)}(0^-) = \dot{A}^{(1)}(0^-) = 0$.

$A^{(1)}(t)$ will consist of a periodic oscillation plus a particular integral term. By considering only the particular integral:

$$A^{(1)}(t) = \frac{-a}{4(1+\varepsilon pa^2)}$$

Note that this process is equivalent to the "averaging principle" employed by Kryloff and Bogoliuboff [13].

The expression for $\dot{a}$ is then:

$$\dot{a} = \frac{-\beta a}{4(1+\varepsilon pa^2)}$$

and the solution for $a(t)$ becomes:

$$a(t) = \exp\left[\frac{-\beta t}{4(1+\varepsilon p)}\right] \tag{15}$$

To determine $B^{(1)}$, the partial derivative of $A^{(1)}$ with respect to time is substituted for $\dot{A}^{(1)}$ in equation (14) (this removes one of the approximations made earlier), giving

$$B^{(1)} = \frac{t(1+pa^2)^{1/2}}{2(1+\varepsilon pa^2)}$$

and finally

$$\theta(t) = \frac{\beta t^2(1+pa^2)^{1/2}}{4(1+\varepsilon pa^2)} \tag{16}$$

<u>Example 2</u>

$$\ddot{x} + x + px^3 + \beta t\dot{x} = 0$$

In this case $f(x,\dot{x},t) = t\dot{x} = -at(1+pa^2)^{1/2}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi + O(\beta t)$.

Note that terms of order βt in $f(x,\dot{x},t)$ are equivalent to terms of order $\beta^2 t$ in $\beta f(x,\dot{x},t)$, and may consequently be neglected in a first order analysis.

From equation (12):

$$a\dot{B}^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} = -at(1+pa^2)^{1/2} \tag{17}$$

$$\dot{A}^{(1)} - 2aB^{(1)}\frac{(1+\varepsilon pa^2)}{(1+pa^2)^{1/2}} = 0 \tag{18}$$

Differentiating equation (18), and substituting for $a\dot{B}^{(1)}$ in equation (17) yields

$$\ddot{A}^{(1)} + 4A^{(1)}(1+\varepsilon pa^2) = -2at\,(1+\varepsilon pa^2)$$

Taking the particular integral of this equation:

$$A^{(1)} = \frac{-at}{2}$$

and $a(t) = \exp\left[-\beta t^2/4\right]$. (19)

Substitution for $A^{(1)}_t$ in equation (18) gives finally:

$$\theta(t) = \frac{-\beta t(1+pa^2)^{1/2}}{4(1+\varepsilon pa^2)} \tag{20}$$

<u>Example 3</u>

$$\ddot{x} + x + px^3 - \beta x \cos\omega_0 t = 0$$

where $f(x,\dot{x},t) = -x\cos\omega_0 t = -aCn\psi\cos\omega_0 t$.

To preserve the condition that resonance effects be negligible, it is necessary to require that $\omega_0^2 \ll 1+pa^2$.

From equation (12):

$$a\dot{B}^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} = 0 \tag{21}$$

$$\dot{A}^{(1)} - 2aB^{(1)}\frac{(1+\varepsilon pa^2)^{1/2}}{(1+pa^2)^{1/2}} = a\cos\omega_0 t \tag{22}$$

By following exactly the same procedure as before:

$$a(t) = \exp\left[\frac{\beta(\cos\omega_0 t - 1)}{4(1+\varepsilon p) - \omega_0^2}\right]$$

or, if the exponent is small:

$$a(t) = 1 + \left[\frac{\beta(\cos\omega_0 t - 1)}{4(1+\varepsilon p) - \omega_0^2}\right] \tag{23}$$

Finally

$$\theta(t) = \frac{-2\beta\,(1+pa^2)^{1/2}\,\sin\omega_0 t}{\omega_0\left[4(1+\varepsilon pa^2) - \omega_0^2\right]} \quad . \tag{24}$$

2.3 Error Analysis

The approximation technique has been developed in its entirety, but with no discussion of the assumptions made during the analysis. As the usefulness of any approximation depends on its range of validity, an investigation of solution error to determine these regions of validity is a necessary adjunct to the approximation method.

From equations (3) and (4):

$$\dot{x} = \dot{a}\,\mathrm{Cn}\psi - a\dot{\psi}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi \tag{25}$$

and

$$\ddot{x} = \ddot{a}\,\mathrm{Cn}\psi - 2\dot{a}\dot{\psi}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi - a\ddot{\psi}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi - \frac{a\dot{\psi}^2}{(1+pa^2)}(\mathrm{Cn}\,\psi + pa^2\mathrm{Cn}^3\psi) \tag{26}$$

Now $\quad x = a\,\mathrm{Cn}\psi$

i.e., $\quad \mathrm{Cn}\psi = \dfrac{x}{a}$

and from equation (25):

$$\dot{a}\,\mathrm{Cn}\psi - \dot{x} = a\dot{\psi}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi$$

or

$$a\dot{\psi}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi = \frac{\dot{a}}{a}x - \dot{x}$$

Substituting for $\mathrm{Cn}\psi$ and $\mathrm{Sn}\psi\,\mathrm{Dn}\psi$ in equation (26):

$$\ddot{x} = \frac{\ddot{a}}{a}x + \left[\dot{x} - \frac{\dot{a}}{a}x\right]\left[\frac{\ddot{\psi}}{\dot{\psi}} + 2\frac{\dot{a}}{a}\right] - \frac{\dot{\psi}^2}{(1+pa^2)}(x+px^3) \tag{27}$$

All the coefficients in equation (27) are known (for a particular differential equation), and consequently integration of the two state equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{\ddot{a}}{a} x_1 + \left[x_2 - \frac{\dot{a}}{a} x_1\right]\left[\frac{\ddot{\psi}}{\dot{\psi}} + 2\frac{\dot{a}}{a}\right] - \frac{\dot{\psi}^2}{(1+pa^2)}(x_1 + px_1^3)$$

yields the approximate solution $x_1(t)$.

The accuracy of the approximate solutions for the three examples considered earlier was determined by comparison with solutions obtained from a digital simulation, using the following integral error definition:

$$I_e = \left[\frac{\int_0^t \left| x_{ex} - x_{app} \right| dt}{\int_0^t \left| x_{ex} \right| dt}\right] \tag{28}$$

Here x_{ex} is the solution obtained by numerical integration of the equation $\ddot{x} + x + px^3 + \beta f(x,\dot{x},t) = 0$, which was assumed for all practical purposes to be the exact solution, and x_{app} is the approximate solution obtained from equation (27). The numerator of equation (28) then represents the area enclosed between the true and approximate solutions. The area under the exact solution was chosen as a normalizing factor, and I_e is then a normalized error function. Error in an approximation is often the result of phase inaccuracy, and this error function is particularly sensitive to such a condition.

The error integral I_e is shown as a function of time (t) and β for the three examples in figures 2.1, 2.2 and 2.3, with the parameter p (which determines the degree of non-linearity) held constant at 2.0. The total time of integration is 20 seconds, which corresponds to about four complete periods of the solution (depending on the particular function $f(x,\dot{x},t)$). Figures 2.4, 2.5 and 2.6 show the variation of the error integral with p, holding the parameter β constant. It is interesting to observe that the error for the second example is almost independent of p (as shown in figure 2.5).

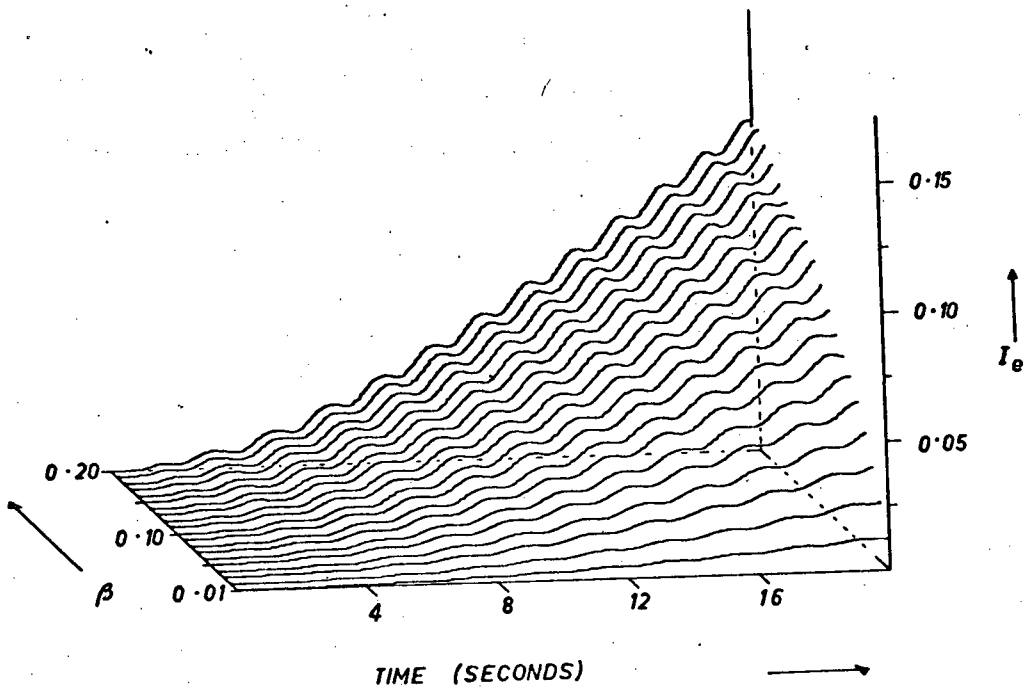

Fig. 2.1 Error integral as a function of time and β for $f(x,\dot{x},t) = tx$, with $p = 2.0$.

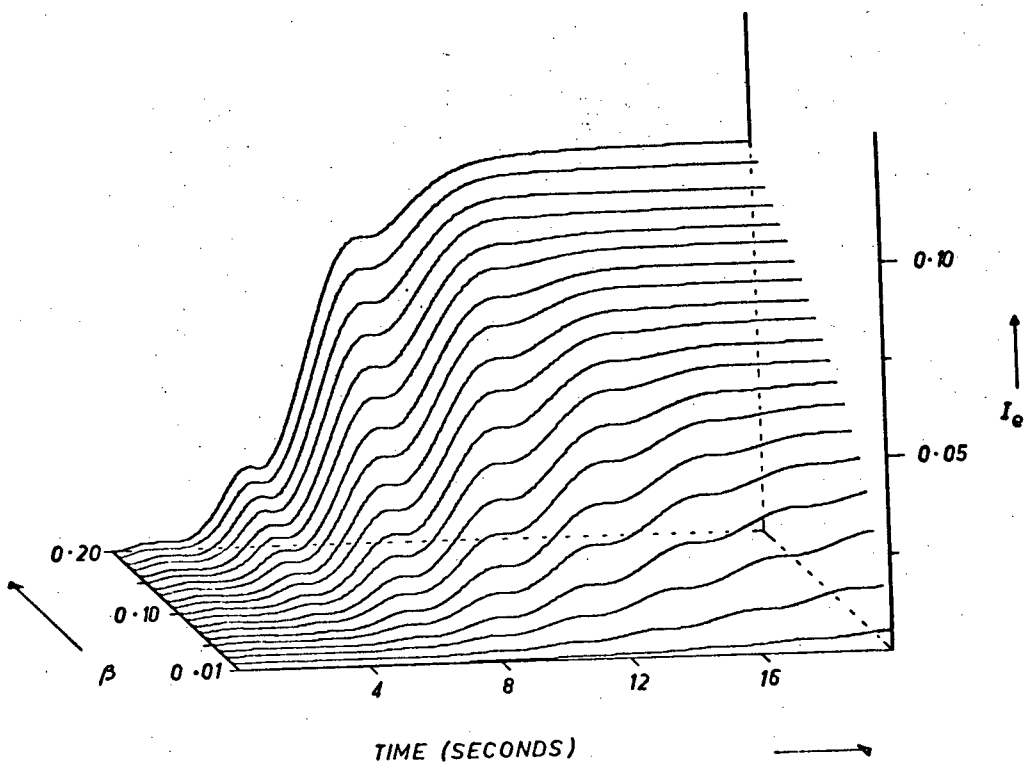

Fig. 2.2 Error integral as a function of time and β for $f(x,\dot{x},t) = t\dot{x}$, with $p = 2.0$.

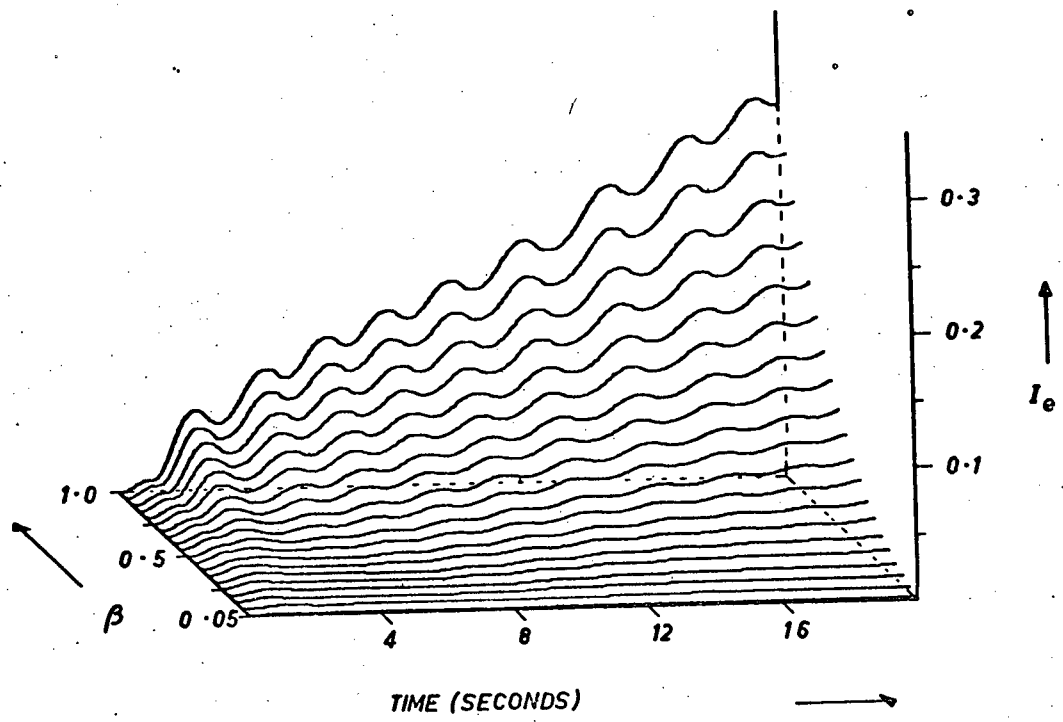

Fig. 2.3 Error integral as a function of time and β for $f(x,\dot{x},t) = -x \cos t/m$, with $p = 2.0$.

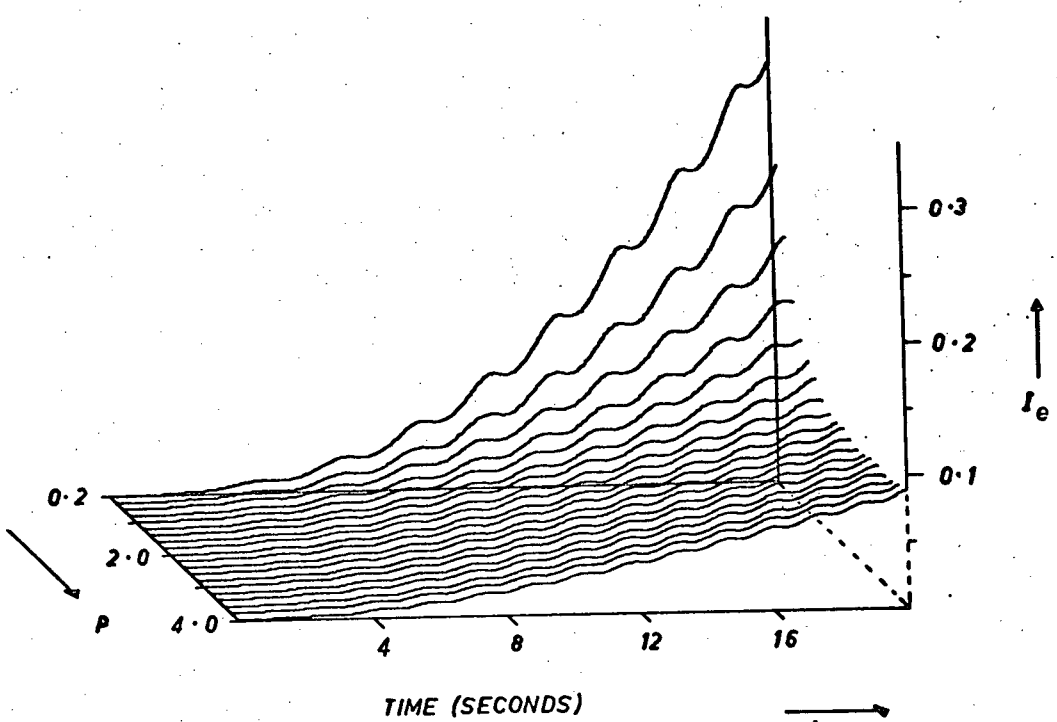

Fig. 2.4 Error integral as a function of time and p for $f(x,\dot{x},t) = tx$, with $\beta = 0.10$.

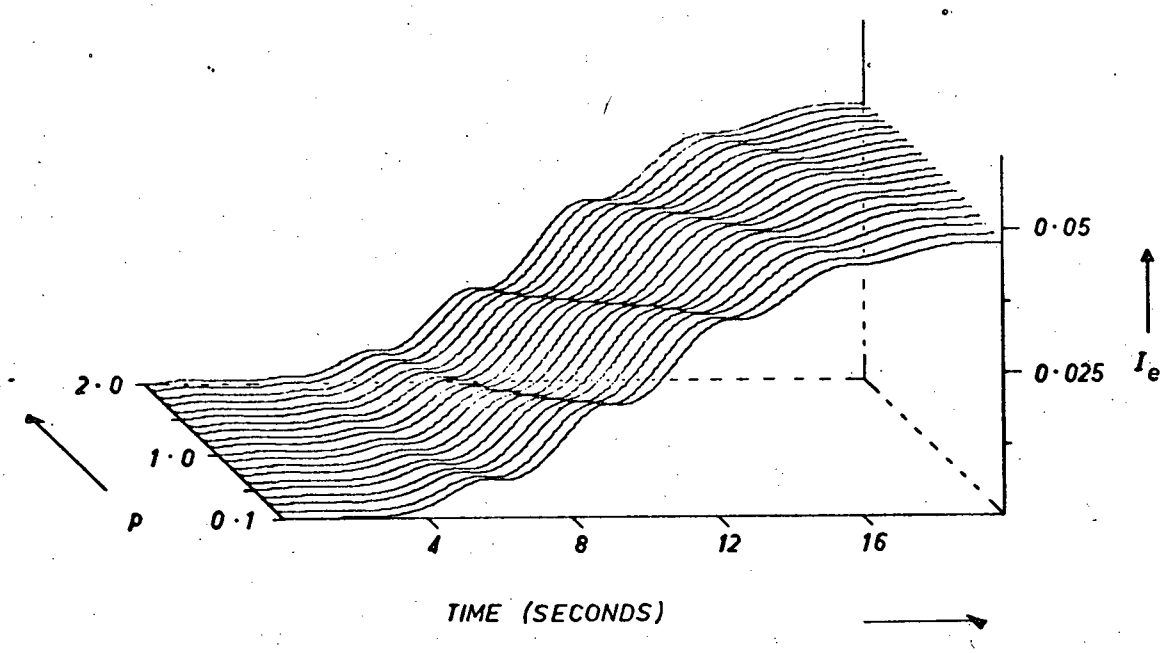

Fig. 2.5 Error integral as a function of time and p for $f(x,\dot{x},t) = t\dot{x}$, with $\beta = 0.06$.

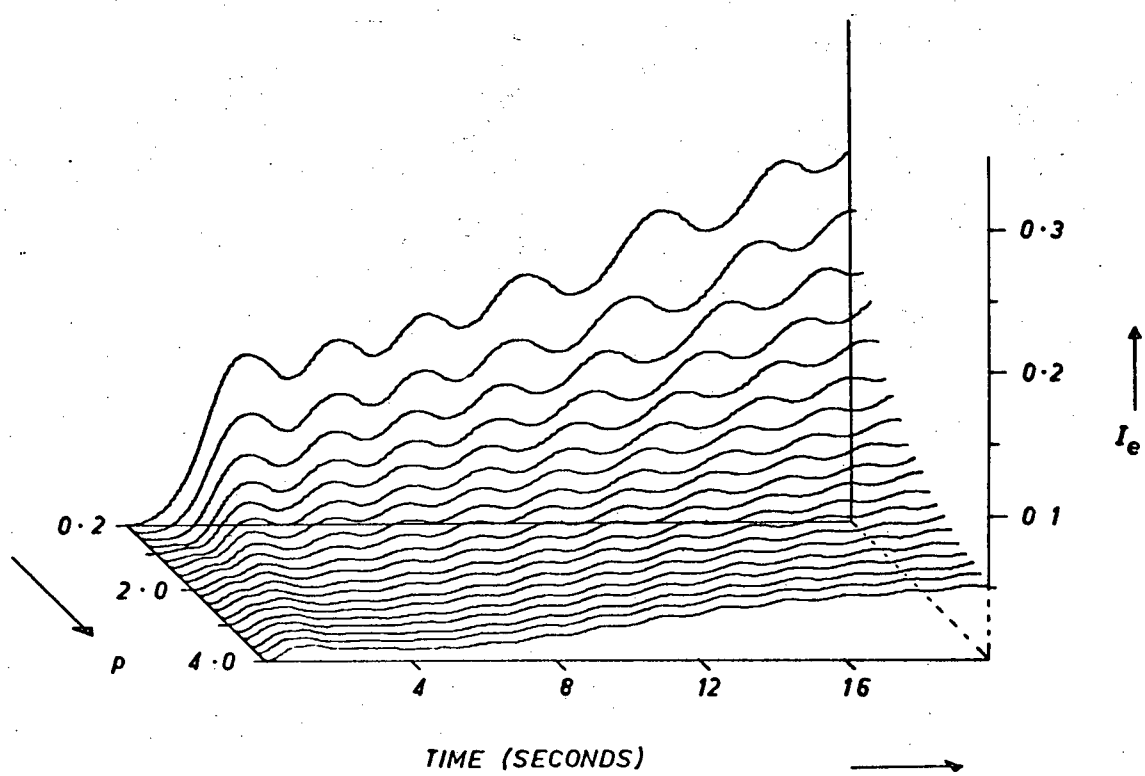

Fig. 2.6 Error integral as a function of time and p for $f(x,\dot{x},t) = -\,x \cos t/m$, with $\beta = 0.60$.

The approximation derived for this case indicates that the amplitude envelope is independent of p (equation (19)), and also shows the phase-variation term to be small; this practical result is consequently in agreement with predicted solution behaviour.

When determining the approximate solution to a particular differential equation (from equation (12)), an auxiliary second order linear differential equation must be solved. This equation is always conservative, and its unforced solution is discarded. This is directly equivalent to the "averaging principle". It might be anticipated that this approximation would lead to an oscillatory error in the final solution, and indeed such behaviour is readily observed in figures 2.1 - 2.6.

2.4 Application of the method

Now that the approximation technique has been fully developed, and solution error determined, three equations corresponding to the earlier examples will be solved to illustrate its application.

Taking solution time as τ, if t is the independent variable of the normalized equations, then

$$t = m\tau$$

where m is a constant. Substituting for t, the differential equations of examples 1, 2 and 3 become:

$$\frac{d^2x}{d\tau^2} + m^2 (x + px^3) + \begin{cases} m^3\beta\tau x = 0 & \text{(Ex. 1)} \\ m^2\beta\tau\frac{dx}{d\tau} = 0 & \text{(Ex. 2)} \\ -m^3\beta x \cos m\omega_0\tau = 0 & \text{(Ex. 3)} \end{cases}$$

For ease of notation, let $\mu = m^3\beta$ in the first case, and $\mu = m^2\beta$ for the remaining examples. After substitution for β and t in equations (15) and (16), the approximate solution to the equation

$$\frac{d^2x}{d\tau^2} + m^2\,(x + px^3) + \mu\tau x = 0$$

becomes:

$$x(\tau) = \exp\left[\frac{-\mu\tau}{4m^2(1+\varepsilon p)}\right] \mathrm{Cn}\left[(1+pa^2)^{1/2}\tau\left[m + \frac{-\mu\tau}{4m(1+\varepsilon pa^2)}\right]\right]$$

Similarly, from equations (19) and (20), for the equation

$$\frac{d^2x}{d\tau^2} + m^2(x + px^3) + \mu\tau\frac{dx}{d\tau} = 0$$

$$x(\tau) = \exp\left[-\mu\frac{\tau^2}{4}\right] \mathrm{Cn}\left[(1+pa^2)^{1/2}\tau\left[m - \frac{\mu}{4m(1+\varepsilon pa^2)}\right]\right]$$

and, from equations (23) and (24), the approximate solution to the equation

$$\frac{d^2x}{d\tau^2} + m^2(x + px^3) - \mu x \cos m\,\omega_0\tau = 0$$

where $m\,\omega_0 = 1.0$ is:

$$x(\tau) = \left[1 + \frac{-\mu(\cos\tau - 1)}{m^2\left[4(1+\varepsilon p) - \omega_0^{\,2}\right]}\right] \mathrm{Cn}\left[(1+pa^2)^{1/2}\left[m\tau \frac{-2\mu\sin\tau}{m\left[4(1+\varepsilon pa^2) - \omega_0^{\,2}\right]}\right]\right]$$

The most direct method of obtaining a graphical solution is first to plot the solution envelope, and then to determine the time instants at which the approximate solution is either at a local maximum, minimum, or zero. The first of these steps presents no problem; the second, however, usually dictates that a graphical approach be used to determine the phase.

Basically, the following equation must be solved:

$$\psi(\tau) = n\,K$$

where $\psi(\tau)$ is the argument of the elliptic cosine function, n is an integer (n=0,1,2,....)

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{-1/2} d\phi\ ,$$

the complete elliptic integral of the first kind (see [5] and Appendix 1),

and $k^2 = \dfrac{pa^2}{2(1+pa^2)}$

For the first example an allowance was made for the decrease in K with amplitude by solving the following equation graphically:

$$nK(5.0) = \psi(\tau),$$

where K(5.0) is the value of K corresponding to the envelope amplitude at $\tau = 5.0$ (the time for which the approximate solution was determined). If 'a' is this amplitude:

$$k^2(5.0) = \frac{pa^2}{2(1+pa^2)}$$

and K(5.0) may be determined from Appendix 2, Table 1 by finding the value of K corresponding to this k^2.

A solution of the equation

$$\frac{d^2x}{d\tau^2} + 5(x + 2x^3) + 1.118\tau x = 0$$

is shown in figure 2.7.

The phase correction term for the second example (where $f(x,\dot{x},t) = tx$) is small compared with $m(1+pa^2)^{1/2}\tau$, and may be neglected. The decrease in K with amplitude must, however, be taken into consideration. The following algorithm was used to determine the approximate solutions of the equation

$$\frac{d^2x}{d\tau^2} + 5(x + 2x^3) + \mu\tau\frac{dx}{d\tau} = 0 ,$$

with $\mu = 0.2$ and 0.5, shown in figures 2.8 and 2.9.

1. Assume the average value of the amplitude to be 1.0 over the first quarter period, and let 'a' denote this average amplitude.
2. Determine k^2 from the relationship

$$k^2 = \frac{pa^2}{2(1+pa^2)}$$

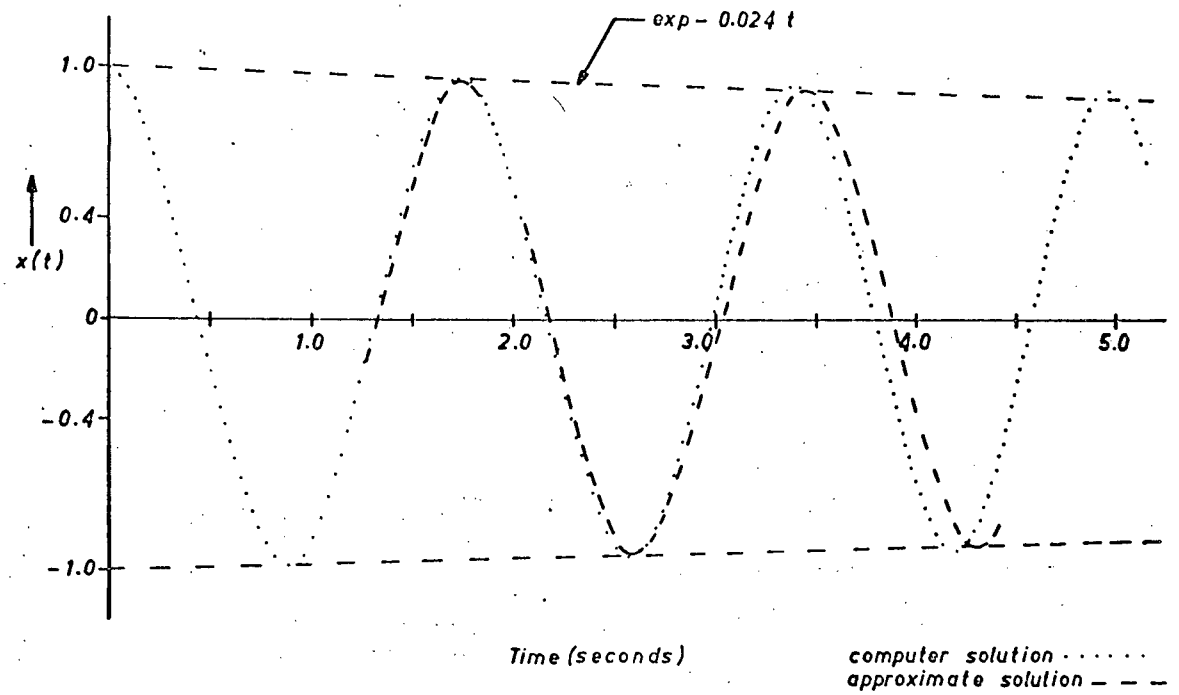

Fig. 2.7 Approximate and exact solutions of $\frac{d^2x}{d\tau^2} + 5x + 10x^3 + 1.118\tau x = 0$

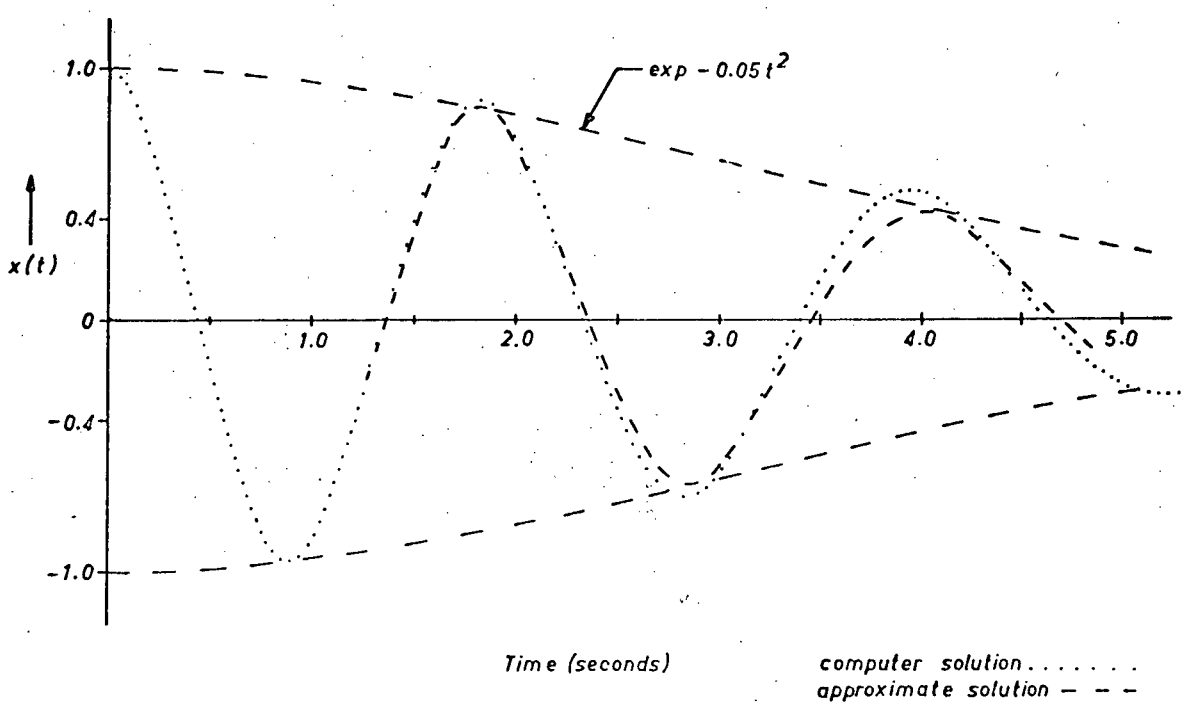

Fig. 2.8 Approximate and exact solutions of $\frac{d^2x}{d\tau^2} + 5x + 10x^3 + 0.2\tau\frac{dx}{d\tau} = 0$

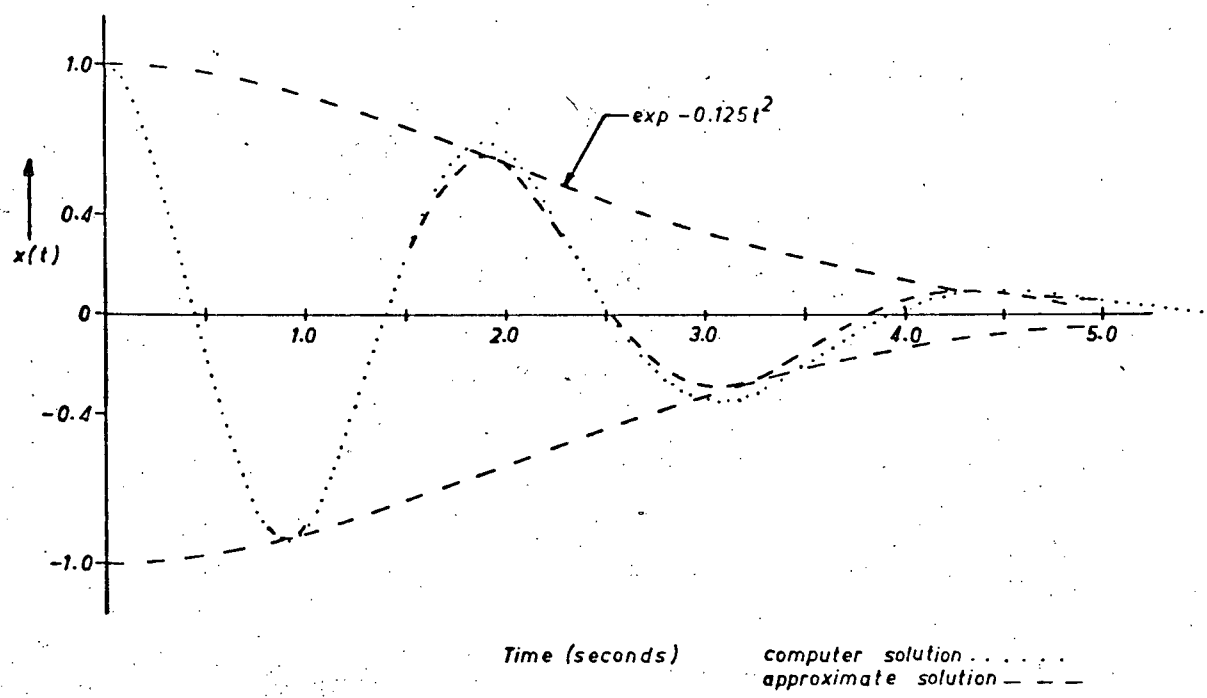

Fig. 2.9 Approximate and exact solutions of $\frac{d^2x}{d\tau^2} + 5x + 10x^3 + 0.5\tau\frac{dx}{d\tau} = 0$

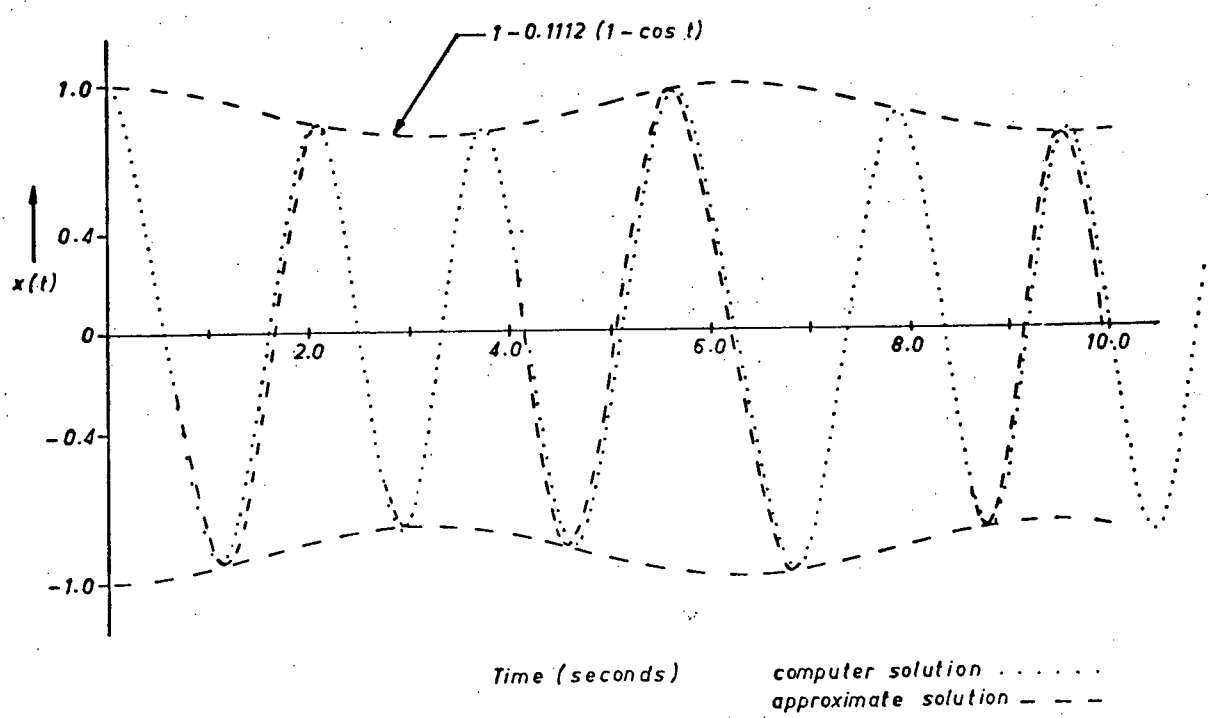

Fig. 2.10 Approximate and exact solutions of $\frac{d^2x}{d\tau^2} + 5x + 10x^3 - 5x\cos\tau = 0$

3. Determine the value of K corresponding to k^2 from Appendix 2, Table 1.

4. Calculate the quarter period T from the relationship

$$T = \frac{K}{\omega} \qquad \text{where } \omega = m(1+pa^2)^{1/2} .$$

5. Determine the average amplitude over the next quarter period (assuming it to be of duration T seconds) from the solution envelope (i.e. in this case $\exp\left[-\frac{\mu\tau^2}{4}\right]$), and return to step 2.

For the final example:

$$\psi(\tau) = (1+pa^2)^{1/2}\left[m\tau - \frac{2\mu\sin\tau}{m\left[4(1+\varepsilon pa^2) - \omega_0^2\right]}\right] = nK$$

where K is calculated for the average value of the solution envelope (from Appendix 2, Table 1), i.e. an amplitude

$$a = 1 - \frac{\mu}{m^2\left[4(1+\varepsilon p) - \omega_0^2\right]} .$$

The maxima, minima and zeroes of the approximate solution are then given by

$$\tau - \frac{nK}{m(1+pa^2)^{1/2}} = \frac{2\mu\sin\tau}{m^2\left[4(1+\varepsilon pa^2) - \omega_0^2\right]}$$

where 'a' denotes the average amplitude. This equation is readily solved graphically.

An approximate solution of the equation

$$\frac{d^2x}{d\tau^2} + 5(x + 2x^3) - 5\,x\cos\tau = 0$$

is shown in figure 2.10.

2.5 Discussion

The major innovation of this analysis is a consequence of the defining equation for the unknown functions $A^{(1)}$ and $B^{(1)}$ (see equation (12)), which leads (for a specific example) to a linear, undamped second order

differential equation for $A^{(1)}(t)$. The complementary solution of this equation is discarded, leaving a particular integral term; this approach (as mentioned in section 2.2) is equivalent to the "averaging method" of Kryloff and Bogoliuboff. The K-B method and also that of Bogoliuboff-Mitropolsky, however, both reach this averaging procedure after a Fourier series expansion which assumes periodicity in the natural, i.e. unmodified, frequency of the unforced system differential equation. For systems where the solution frequency is rapidly varying this approach might be expected to result in inaccuracy. Note that this problem does not arise in the present analysis, as no such assumption about periodicity of the equations defining the amplitude and phase variation terms is necessary to the derivation of the expressions $A^{(1)}$ and $B^{(1)}$.

The anticipated oscillatory behaviour of solution error is shown in figures 2.1 to 2.6, but the graphical solutions (see figures 2.7 to 2.10) indicate that greater accuracy is obtained in practice than would be expected from the error analysis. This apparent increase in accuracy is a consequence of the expedients employed, when deriving the graphical solutions, which could not be incorporated in the digital computer simulation. The main value of the integral error results is the portrayal of qualitative behaviour, but it is not disadvantageous to have a pessimistic estimate of solution error. Actual error in the graphical solutions is much less than that predicted, and hence the error results provide a conservative aid in determining regions of validity of an approximate solution.

2.6 Conclusion

A method for determining approximate solutions to a class of non-linear, non-autonomous differential equations characterized by

$$\ddot{x} + m^2(x + px^3) + \mu f(x,\dot{x},t) = 0$$

has been developed using the Jacobian elliptic functions; the approximation is easy to apply, and, although only a first order approximation is employed, the three examples considered demonstrate the accuracy which can be obtained. An expression for solution error makes it possible to determine the accuracy of the approximate solution for any function $f(x,\dot{x},t)$ once the amplitude envelope and phase relationships have been derived. This approximation method is refined in the following chapter, to take into consideration the variation of solution frequency with amplitude.

3. REFINEMENT OF THE FIRST APPROXIMATION

3.1 Introduction

In the previous chapter first-approximation solutions were developed under the assumption that the frequency of the non-linear oscillation remained constant (at least to first order). Although the accuracy of the solutions obtained demonstrated that this assumption was valid over a short time interval, more accurate solutions over longer time intervals might be anticipated if allowance were made for the variation of frequency with amplitude. In the present chapter, the analysis of chapter 2 is extended to account for this variation of frequency. New error-integral results are shown, and refined solutions of the three examples considered in chapter 2 demonstrate that an appreciable improvement in accuracy can be obtained.

3.2 The refined approximation

The solution of the normalized equation

$$\ddot{x} + x + px^3 + \beta f(x,\dot{x},t) = 0 ,$$

with $x(0) = 1.0$, $\dot{x}(0) = 0$ and where $p > 0$, is again taken as

$$x(t) = a(t)\ \mathrm{Cn}\ (\omega t + \theta) = a\ \mathrm{Cn}\,\psi$$

where

$$\omega = 1 + pa^2$$

$$\psi = \omega t + \theta.$$

However, in the present case, the assumption $\frac{d}{dt}[\omega t] = \omega$ is no longer made.

As before, writing

$$x = x(a,\psi)$$

and differentiating with respect to time, the following equations are obtained:

$$\dot{x} = \dot{a}\frac{\partial x}{\partial a} + \dot{\psi}\frac{\partial x}{\partial \psi}$$

$$\ddot{x} = \ddot{a}\frac{\partial x}{\partial a} + \dot{a}\frac{d}{dt}\left[\frac{\partial x}{\partial a}\right] + \dot{\psi}\frac{d}{dt}\left[\frac{\partial x}{\partial \psi}\right] + \ddot{\psi}\frac{\partial x}{\partial \psi}$$

Also
$$\dot{a} = \mu A^{(1)}(a,t) + \mu^2 A^{(2)}(a,t) + \ldots.$$
$$\dot{\theta} = \mu B^{(1)}(a,t) + \mu^2 B^{(2)}(a,t) + \ldots.$$
$$\ddot{a} = \mu A^{(1)}_t + O(\mu^2)$$
$$\ddot{\theta} = \mu B^{(1)}_t + O(\mu^2)$$

where the subscript denotes partial differentiation with respect to time.

Now
$$\dot{\psi} = \omega + t\dot{\omega} + \dot{\theta}$$
where
$$\omega = (1 + pa^2)^{1/2}$$
and
$$\dot{\omega} = \frac{pa\dot{a}}{(1+pa^2)^{1/2}}$$

Retaining only first order terms in μ, the expression for $\dot{\psi}$ becomes:
$$\dot{\psi} = (1+pa^2)^{1/2} + \mu\left[B^{(1)} + \frac{patA^{(1)}}{(1+pa^2)^{1/2}}\right]. \tag{29}$$

Similarly
$$\dot{\psi}^2 = 1+pa^2 + \mu\left[2B^{(1)}(1+pa^2)^{1/2} + 2patA^{(1)}\right] \tag{30}$$

where terms of $O(\mu^2)$, $O(\mu^2 t)$ and $O(\mu^2 t^2)$ are neglected. It should be noted that this assumption is likely to cause a deterioration in solution accuracy for large t, i.e. when $\mu^2 t^2$ is no longer negligible.

For $\ddot{\psi}$:
$$\ddot{\psi} = 2\dot{\omega} + t\frac{d}{dt}\left[\frac{pa\dot{a}}{(1+pa^2)^{1/2}}\right] + \ddot{\theta}$$

or
$$\ddot{\psi} = \ddot{\theta} + \frac{2pa\dot{a}}{(1+pa^2)^{1/2}} + pt\left[\frac{a\ddot{a}}{(1+pa^2)^{1/2}} + \frac{\dot{a}^2}{(1+pa^2)^{3/2}}\right].$$

The term $\dot{a}^2$ is of order μ^2, and hence to first order in μ :
$$\ddot{\psi} = \mu\left[B^{(1)}_t + \frac{2paA^{(1)}}{(1+pa^2)^{1/2}} + \frac{patA^{(1)}_t}{(1+pa^2)^{1/2}}\right] \tag{31}$$

The refined relationship corresponding to equation (5) is therefore

$$\ddot{x} = \ddot{a}Cn\psi - 2\dot{a}\dot{\psi}Sn\psi Dn\psi - aSn\psi Dn\psi\ddot{\psi} - aR\dot{\psi}^2 \qquad (32)$$

By appropriate substitution into equation (32), the first-order approximation to $\ddot{x}$ becomes:

$$\ddot{x} = -(aCn\psi + pa^3Cn^3\psi)$$

$$-\mu Sn\psi Dn\psi\left[aB_t^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} + \left[\frac{2pa^2A^{(1)} + pa^2tA_t^{(1)}}{(1+pa^2)^{1/2}}\right]\right]$$

$$+\mu\left[A_t^{(1)}Cn\psi - (Cn\psi + pa^2Cn^3\psi)\left[\frac{2aB^{(1)}(1+pa^2)^{1/2} + 2pa^2tA^{(1)}}{(1+pa^2)}\right]\right]$$

If the term $(Cn\psi + pa^2Cn^3\psi)$ is approximated by $Cn\psi(1+\varepsilon pa^2)$ (where the quantity ε is defined by equation (10)), then a refined relationship corresponding to equation (11) may finally be written as:

$$f(x,\dot{x},t) = Sn\psi Dn\psi\left[aB_t^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} + \left[\frac{2pa^2A^{(1)} + pa^2tA_t^{(1)}}{(1+pa^2)^{1/2}}\right]\right] \qquad (33)$$

$$- Cn\psi\left[A_t^{(1)} - \frac{(1+\varepsilon pa^2)}{(1+pa^2)}\left[2aB^{(1)}(1+pa^2)^{1/2} + 2pa^2tA^{(1)}\right]\right]$$

A comparison of this equation and equation (11) reveals that the major modification is to the term involving $Sn\psi Dn\psi$, but terms involving t explicitly are now incorporated in the coefficients of both $Cn\psi$ and $Sn\psi Dn\psi$. To investigate the effect of this refinement on equation solutions, examples 1, 2, and 3 of chapter 2 will briefly be reconsidered.

<u>Example 1.</u>

Consider the equation

$$\ddot{x} + x + px^3 + \beta tx = 0$$

with $x(0) = 1.0$ and $\dot{x}(0) = 0$. For this equation $f(x,\dot{x},t) = tx = atCn\psi$. Applying equation (33), and comparing coefficients of $Cn\psi$ and $Sn\psi Dn\psi$:

$$aB_t^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} + \left[\frac{2pa^2A^{(1)} + pa^2tA_t^{(1)}}{(1+pa^2)^{1/2}}\right] = 0 \tag{34}$$

$$A_t^{(1)} - \frac{(1+\varepsilon pa^2)}{(1+pa^2)}\left[2aB^{(1)}(1+pa^2)^{1/2} + 2pa^2tA^{(1)}\right] = -at \tag{35}$$

Taking the partial derivative of equation (35) with respect to time:

$$aB_t^{(1)} = (A_{tt}^{(1)} + a)\frac{(1+pa^2)^{1/2}}{2(1+\varepsilon pa^2)} - \frac{pa^2}{(1+pa^2)^{1/2}}(A^{(1)} + tA_t^{(1)})$$

Substitution for $aB_t^{(1)}$ in equation (34) then yields

$$A_{tt}^{(1)} + 4A^{(1)}(1+\varepsilon pa^2)\left[1 + \frac{pa^2}{2(1+pa^2)}\right] = -a\ . \tag{36}$$

At this juncture in chapter 2, the equation corresponding to equation (36) had been derived as

$$\ddot{A}^{(1)} + 4A^{(1)}(1+\varepsilon pa^2) = -a\ ,$$

and this expression was then written as a Laplace transform. Inversion of the transform gave $A^{(1)}(t)$ as the superposition of a complementary solution and particular integral, of which only the particular integral was retained.

The particular integral of equation (36) may be obtained by setting $A_{tt}^{(1)}$ 0, and hence

$$A^{(1)}(a,t) = \frac{-a}{4(1+\varepsilon pa^2)\left[1 + \frac{pa^2}{2(1+pa^2)}\right]}$$

Also, the term $\frac{pa^2}{2(1+pa^2)}$ is equal to k^2, where k is the modulus of the elliptic function. Integrating the expression for $A^{(1)}$, and assuming that the amplitude 'a' [13] may be treated as a constant without incurring gross

inaccuracy:

$$a(t) = \exp\left[\frac{-\beta t}{4(1+\varepsilon pa^2)(1+k^2)}\right] \tag{37}$$

Setting $A_t^{(1)} = 0$ in equation (35) yields, after some manipulation:

$$B^{(1)}(a,t) = \frac{t(1+pa^2)^{1/2}}{2(1+\varepsilon pa^2)} + \frac{pa^2 t}{4(1+\varepsilon pa^2)(1+k^2)(1+pa^2)^{1/2}}$$

and finally:

$$\theta(t) = \frac{\beta t^2(1+pa^2)^{1/2}}{4(1+\varepsilon pa^2)} + \frac{\beta pa^2 t^2}{8(1+\varepsilon pa^2)(1+k^2)(1+pa^2)^{1/2}} \tag{38}$$

It is particularly interesting to observe that the refinement introduces terms involving k^2, a parameter which increases with the degree of non-linearity and which, for this example, decreases the rate of decay of the solution envelope (by comparison with the unrefined approximation).

From figures 2.7 to 2.10 it might be anticipated that a decrease in the rate of decay of the amplitude envelope would improve solution accuracy substantially. This, in section 3.4, is seen to be the case and the refined approximation predicts solution behaviour accurately.

<u>Example 2</u>.

$$\ddot{x} + x + px^3 + \beta t\dot{x} = 0$$

with $x(0) = 1.0$, $\dot{x}(0) = 0$, and for which

$$f(x,\dot{x},t) = t\dot{x} = -at\,(1+pa^2)^{1/2}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi + O(\beta t) \quad .$$

Note that terms of $O(\beta t)$ in $f(x,\dot{x},t)$ are equivalent to terms of order $\beta^2 t$ in $\beta f(x,\dot{x},t)$, and may consequently be neglected.

From equation (33):

$$aB_t^{(1)} + 2A^{(1)}\,(1+pa^2)^{1/2} + \left[\frac{2pa^2A^{(1)} + pa^2tA_t^{(1)}}{(1+pa^2)^{1/2}}\right] = -at(1+pa^2)^{1/2} \tag{39}$$

$$A_t^{(1)} - \frac{(1+\varepsilon pa^2)}{(1+pa^2)}\left[2aB^{(1)}(1+pa^2)^{1/2} + 2pa^2tA^{(1)}\right] = 0 \qquad (40)$$

Taking the partial derivative with respect to time of equation (40) and substituting for $aB_t^{(1)}$ in equation (39) gives:

$$A_{tt}^{(1)} + 4A^{(1)}(1+\varepsilon pa^2)(1+k^2) = -2at(1+\varepsilon pa^2)$$

The particular integral of this equation is

$$A^{(1)} = \frac{-at}{2(1+k^2)},$$

and by integration:

$$a(t) = \exp\left[\frac{-\beta t^2}{4(1+k^2)}\right] \qquad (41)$$

Substitution for $A_t^{(1)}$ in equation (40) yields:

$$B^{(1)}(a,t) = \frac{pa^2t^2}{2(1+k^2)(1+pa^2)^{1/2}} - \frac{(1+pa^2)^{1/2}}{4(1+k^2)(1+\varepsilon pa^2)}$$

In this example, it is no longer valid to assume that the amplitude remains approximately constant, and the phase of the solution is more easily determined in an approximate manner as described in section 3.4.

<u>Example 3</u>.

$$\ddot{x} + x + px^3 - \beta x \cos \omega_0 t = 0$$

with $x(0) = 1.0$, $\dot{x}(0) = 0$, and where $f(x,\dot{x},t) = -x \cos \omega_0 t = -aCn\psi \cos \omega_0 t$.

From equation (33):

$$aB_t^{(1)} + 2A^{(1)}(1+pa^2)^{1/2} + \left[\frac{2pa^2A^{(1)} + pa^2tA_t^{(1)}}{(1+pa^2)^{1/2}}\right] = 0$$

$$A_t^{(1)} - \frac{(1+\varepsilon pa^2)}{(1+pa^2)}\left[2aB^{(1)}(1+pa^2)^{1/2} + 2pa^2tA^{(1)}\right] = a \cos \omega_0 t \qquad (42)$$

and hence:

$$A^{(1)}(a,t) = \frac{-a\omega_0 \sin\omega_0 t}{4(1+\varepsilon pa^2)(1+k^2) - \omega_0^2}$$

Taking the partial derivative of $A^{(1)}$ with respect to time, and substituting in equation (42):

$$B^{(1)}(a,t) = \frac{1}{\left[4(1+\varepsilon pa^2)(1+k^2) - \omega_0^2\right]}\left[\frac{pa^2 t\omega_0 \sin\omega_0 t}{(1+pa^2)^{1/2}} - 2(1+pa^2)^{1/2}(1+k^2)\cos\omega_0 t\right]$$

The expression for a(t) may finally be obtained as:

$$a(t) = 1 + \left[\frac{\beta(\cos\omega_0 t - 1)}{4(1+\varepsilon pa^2)(1+k^2) - \omega_0^2}\right] \tag{43}$$

The phase-modifying term $\theta(t)$ will be discussed later (in section 3.4), but at this juncture it is sufficient to observe that integration of the equation defining $B^{(1)}(a,t)$ is unnecessary.

A comparison of these results with those obtained in chapter 2 shows that the main consequence of the refinement is the introduction of factors involving k , the modulus of the elliptic function. As k increases with the parameter p (the coefficient governing the degree of non-linearity), the effect of the refinement should be most noticeable for high values of p. This hypothesis is investigated further in the following section.

3.3 Error Analysis

The equations defining the approximate solution in algebraic terms, derived in section 2.3, are still valid, but the quantities a, $\dot{a}$, $\ddot{a}$, $\dot{\psi}$ and $\ddot{\psi}$ are now different. The major source of error in any approximation is usually to be found in the phase-modifying term, or, more generally, in the argument of the periodic function. In the un-refined case considered in chapter 2 this argument (ψ) was taken as

$$\psi = \omega t + \theta$$

and

$$\dot{\psi} = \omega + \dot{\theta}$$

or, substituting for ω and $\dot{\theta}$:

$$\dot{\psi} = (1+pa^2)^{1/2} + \mu B^{(1)} .$$

In the present (refined) case, from equation (29):

$$\dot{\psi} = (1+pa^2)^{1/2} + \mu \left[B^{(1)} + \frac{-patA^{(1)}}{(1+pa^2)^{1/2}} \right]$$

i.e., the refinement has introduced a term

$$\frac{-\mu\, patA^{(1)}}{(1+pa^2)^{1/2}}$$

into the expression for $\dot{\psi}$. To demonstrate the effect of this modification, consider the values of $\dot{\psi}$ for example 1 (where $f(x,\dot{x},t) = tx$). From chapter 2, the un-refined expression for $\dot{\psi}$ is:

$$\dot{\psi} = (1+pa^2)^{1/2} + \frac{\beta t(1+pa^2)^{1/2}}{2(1+\varepsilon pa^2)}$$

Substitution for $A^{(1)}$ and $B^{(1)}$ into equation (29) gives the refined expression for $\dot{\psi}$ as:

$$\dot{\psi} = (1+pa^2)^{1/2} + \beta \left[\frac{t(1+pa^2)^{1/2}}{2(1+\varepsilon pa^2)} + \frac{pa^2 t}{4(1+\varepsilon pa^2)(1+k^2)(1+pa^2)^{1/2}} - \frac{pa^2 t}{4(1+\varepsilon pa^2)(1+k^2)(1+pa^2)^{1/2}} \right]$$

which yields the interesting result that, to a first-order approximation, the change in $\dot{\psi}$ due to $\frac{\partial}{\partial a}(\omega)$ is cancelled by the change in $\dot{\psi}$ due to $\frac{\partial}{\partial a}(\theta)$. A similar result can readily be obtained for the remaining two examples. Any improvement in solution accuracy will, as a consequence, only arise from an improved approximation to the solution amplitude and should be more apparent for high values of the parameter p.

Families of error curves for the refined case, corresponding to those obtained earlier, are shown in figures 3.1 to 3.6. Figures 3.1, 3.2 and

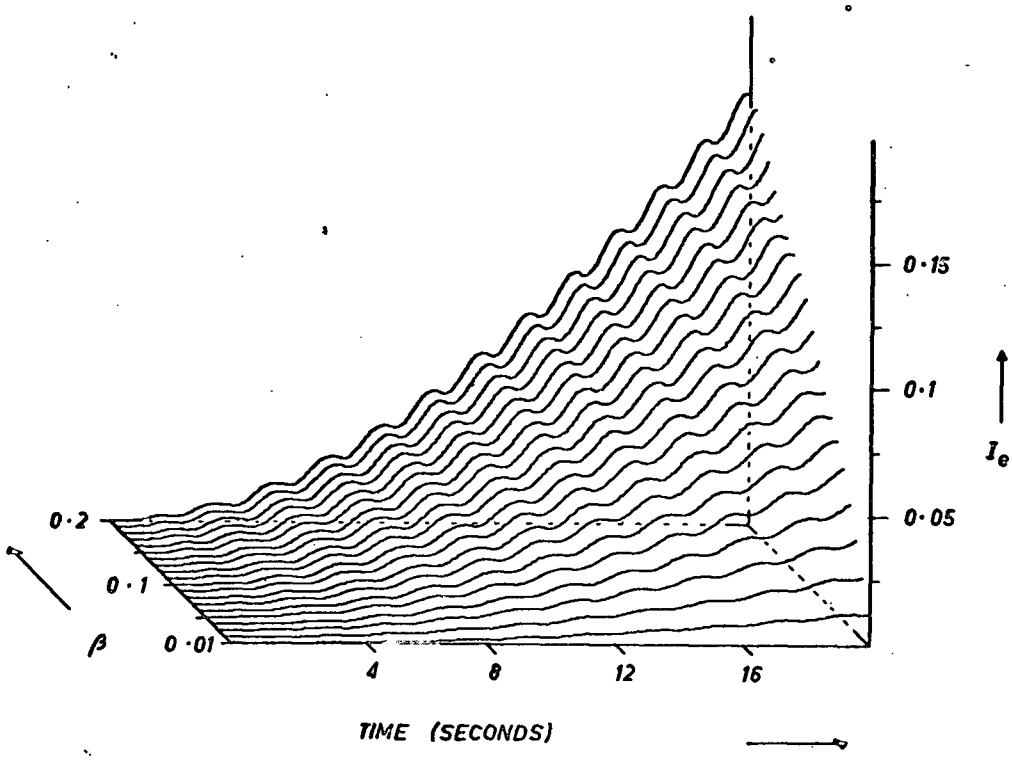

Fig. 3.1 Error integral as a function of time and β for $f(x,\dot{x},t) = tx$, with $p = 2.0$.

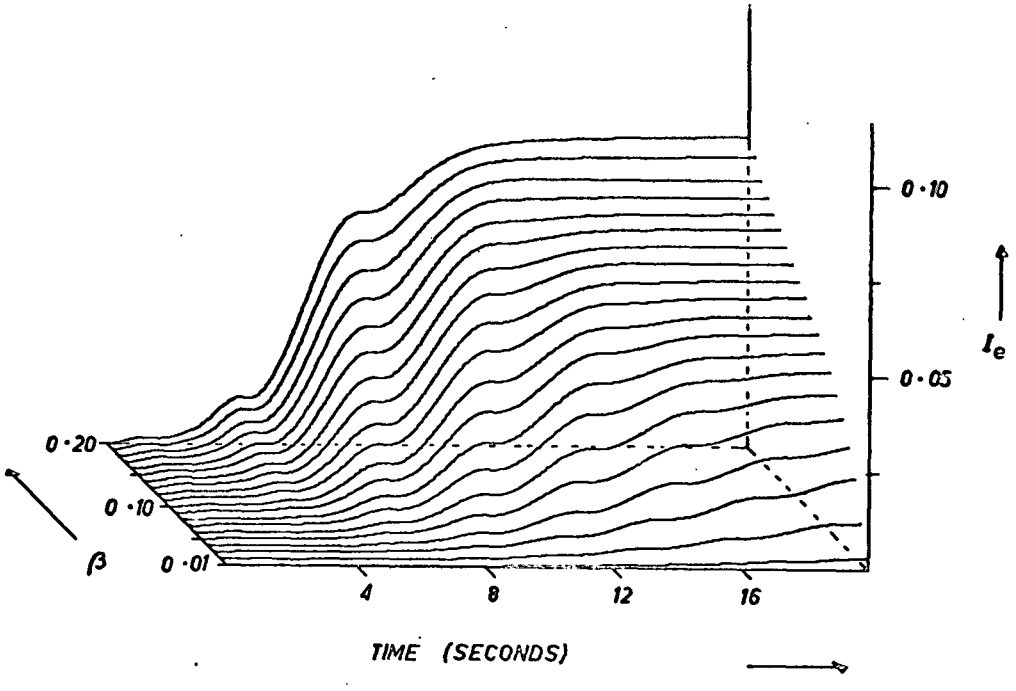

Fig. 3.2 Error integral as a function of time and β for $f(x,\dot{x},t) = t\dot{x}$, with $p = 2.0$.

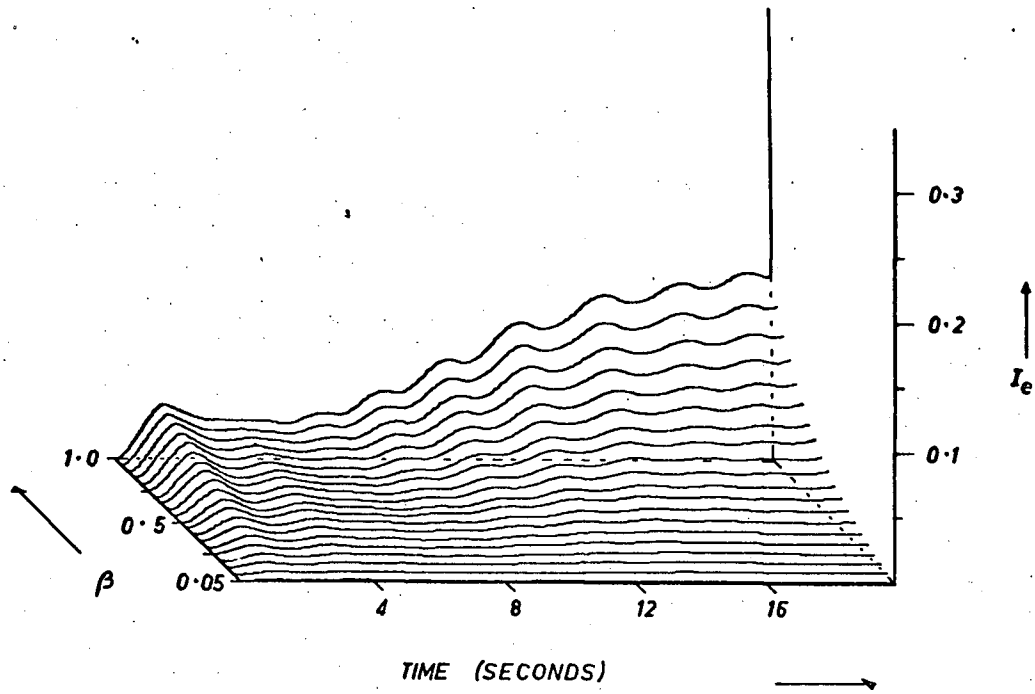

Fig. 3.3 Error integral as a function of time and β for $f(x,\dot{x},t) = -x \cos t/m$, with $p = 2.0$.

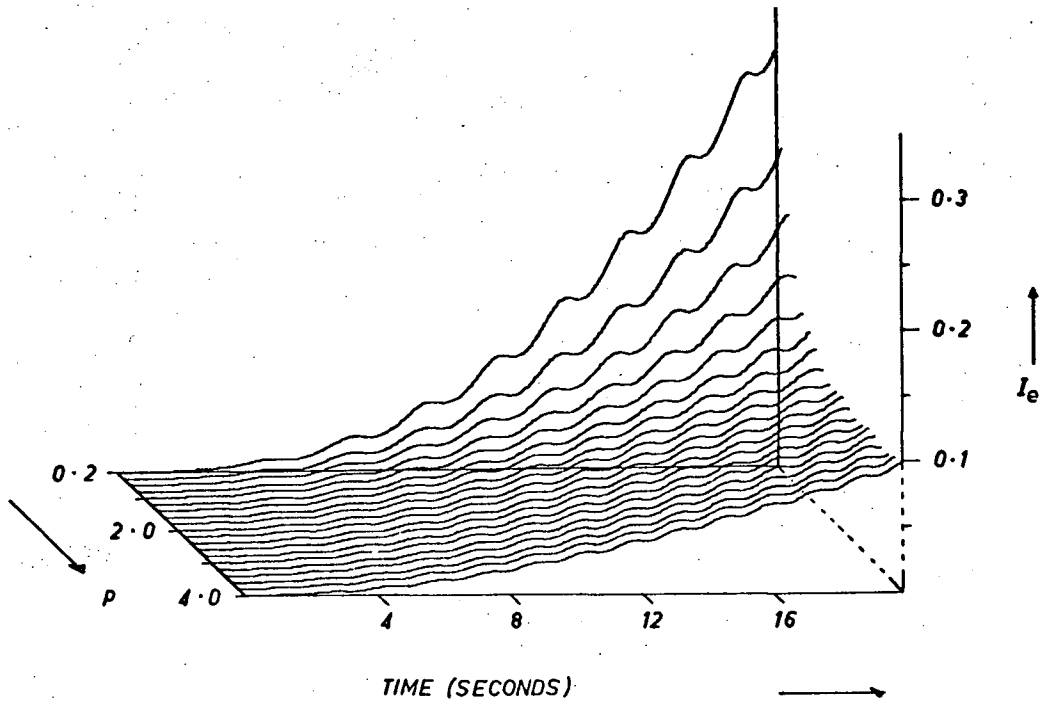

Fig. 3.4 Error integral as a function of time and p for $f(x,\dot{x},t) = tx$, with $\beta = 0.10$.

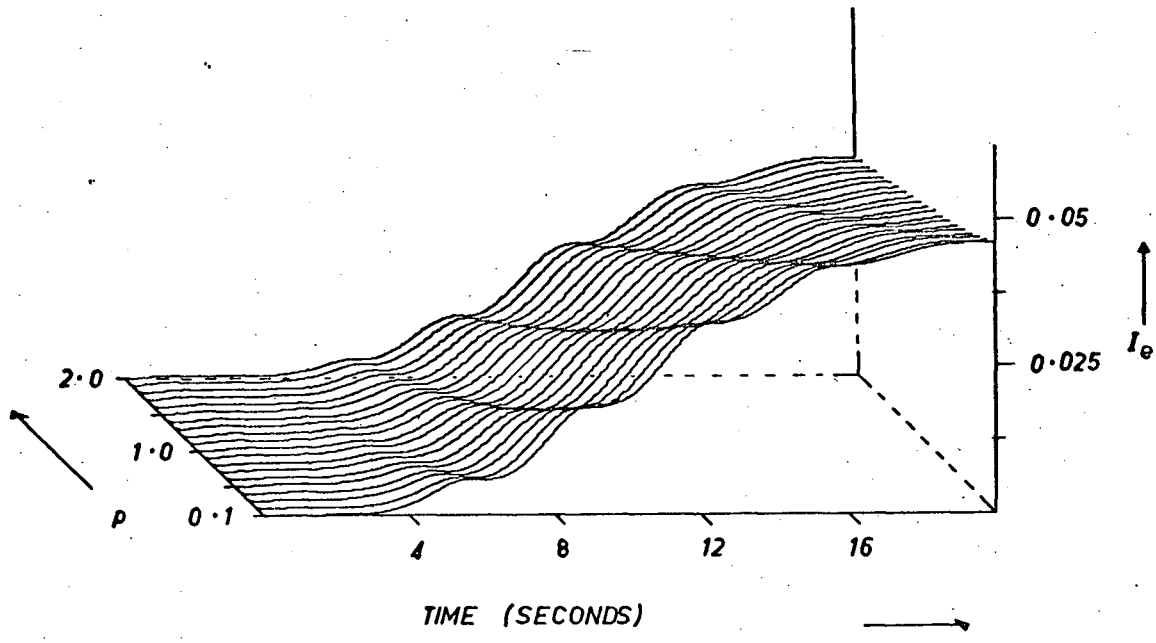

Fig. 3.5 Error integral as a function of time and p for $f(x,\dot{x},t) = t\dot{x}$, with $\beta = 0.06$.

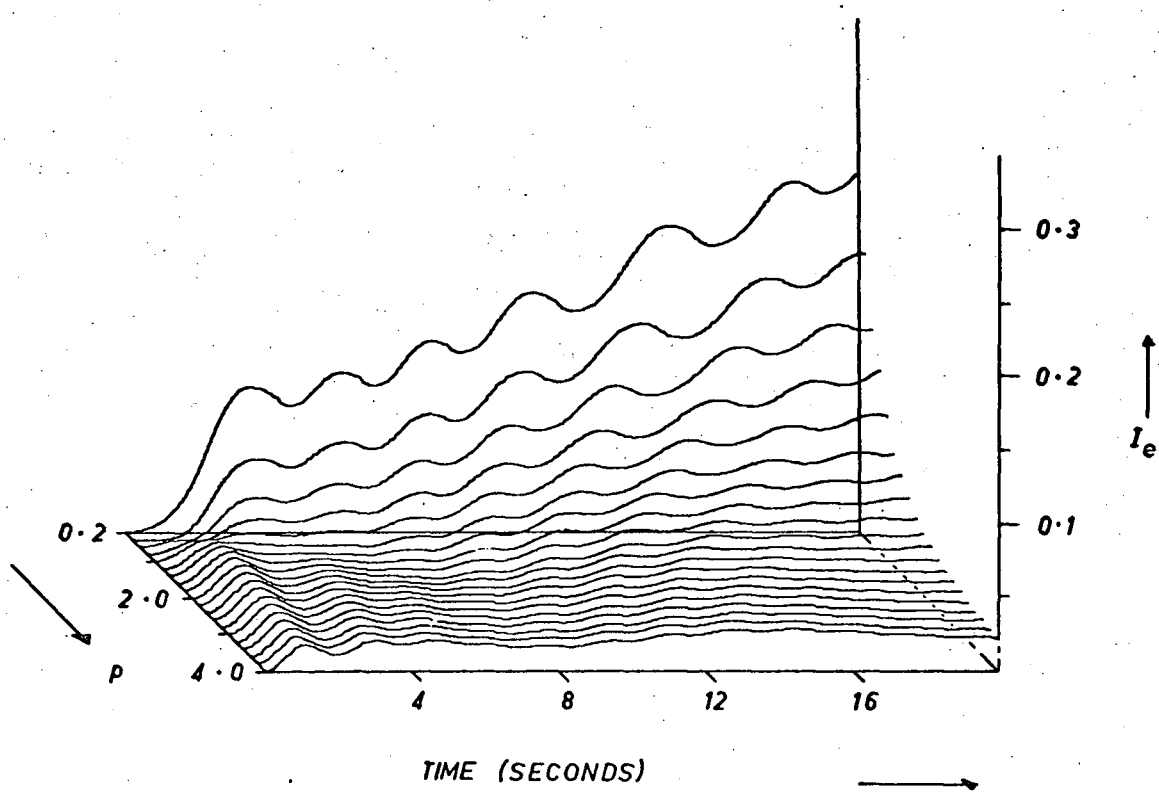

Fig. 3.6 Error integral as a function of time and p for $f(x,\dot{x},t) = -x\cos t/m$, with $\beta = 0.60$.

3.3 refer to the variation of the error integral with β for the three examples, with p held constant at 2.0; figures 3.4, 3.5 and 3.6 show error integral variation with the parameter p, holding β constant.

A comparison of figures 2.1 and 3.1 and 2.4 and 3.4 reveals that less error is incurred by assuming ω to remain constant for the first example, where $f(x,\dot{x},t) = tx$. The deterioration in accuracy when using the refinement (which is small, but not insignificant) is the consequence of neglecting terms of order $\mu^2 t^2$ when deriving the approximation (see section 3.2). This particular case, it should be noted, contains no energy-dissipative term (e.g. $\dot{x}$), so that any change in amplitude can result only from a change in solution frequency. The time varying function tx alters the solution frequency or, more specifically, the total phase of the solution, with the result that substantial phase variation is the dominant feature of the solution. This factor, coupled with the slow change in amplitude envelope with time, indicates that, of the three examples considered, the first will be the most sensitive to phase error.

The remaining two cases show that considerable improvement in solution accuracy is obtained by using the refined approximation (see figures 2.5, 2.6, 3.5 and 3.6) at high values of the parameter p, as had been anticipated.

It should be emphasized that error predicted from the error-integral results tends to be pessimistic when compared with the final graphical solution, as many expedients may then be employed which cannot be incorporated in the digital computer simulation. This becomes particularly apparent in the next section, where approximate solutions corresponding to the three examples of section 3.2 are obtained, and it becomes evident that an improvement in solution accuracy results from use of the refined approximation for each example, including the first.

3.4 Application of the refinement

For a non-autonomous function $f(x,\dot{x},t)$ equation (29) will always be of the form $\dot{\psi} = (1+pa^2)^{1/2} + \beta g(t)$, where $g(t)$ is a function containing t explicitly, i.e., the cancellation of terms of order β is not complete. It will be shown later (in chapter 5) that for a linear autonomous function $f(\dot{x})$ this cancellation of first order terms is complete. This is an important result, but it will be treated in detail in section 5.2. In the non-autonomous case, however, the form of the expression for $\dot{\psi}$ can be used to indicate the method of approach when determining the graphical solution of a particular equation.

By comparison with the phase-modifying term $\theta(t)$, the amplitude envelope $a(t)$ is, for a specific example, readily determined and hence the amplitude behaviour of the solution can be predicted. Also the function $B^{(1)}$ can readily be found once $A^{(1)}$ is known, and the expression

$$\dot{\psi} = (1+pa^2)^{1/2} + \beta\left[B^{(1)} + \frac{-patA^{(1)}}{(1+pa^2)^{1/2}}\right] \qquad (29)$$

can be evaluated to determine the function $g(t)$ in $\dot{\psi} = (1+pa^2)^{1/2} + \beta g(t)$, where

$$g(t) = B^{(1)} + \frac{-patA^{(1)}}{(1+pa^2)^{1/2}} .$$

The choice of approach should then depend on the nature of $a(t)$ (the amplitude envelope), and the quantity $g(t)$.

a) If, over the time interval of interest, the amplitude envelope is approximately constant, then the phase of the approximate solution can be obtained from $\theta(t)$. When integrating the expression for $B^{(1)}$ to obtain $\theta(t)$ the amplitude may then be assumed constant, and the parameter K (the complete elliptic integral of the first kind) in the expression

$$\psi(t) = nK ,$$

which is used to determine local maxima, minima and zeroes of the solution, takes a constant value corresponding to the initial solution amplitude a(0). The evaluation of the quantity $\psi(t)$ from $\psi(t) = (1+pa^2)^{1/2}t + \theta(t)$ (where $\theta(t)$ is given by, for example, equation (38)), should, however, take into account variation in the amplitude. This provision is necessary to ensure that equation (29) is not violated, i.e. the partial cancellation of variations in $\dot{\psi}$ caused by changes in a(t) and $\theta(t)$ must be taken into consideration, although in a circuitous manner. Note that it is much less tedious to integrate the expression for $B^{(1)}$ assuming the amplitude to be constant than it is to integrate equation (29) where this assumption cannot be made. The solution of the normalized equation (e.g. equation 1) is taken in the form:

$$x(t) = a(t)\ \mathrm{Cn}\ [(1+pa^2(t))^{1/2}\ t + \theta(t)]$$

and the local maxima, minima and zeroes of the solution are determined from

$$\psi(t) = (1+pa^2(t))^{1/2}\ t + \theta(t) = nK(0)\ .$$

This approach is adopted for the first example of this section.

b) If the mean value of the amplitude envelope is constant, and hence g(t) is periodic, then equation (29) may be integrated assuming the amplitude 'a' to take its mean value. The value of K in $\psi(t) = nK$ should correspond to this mean-value of the amplitude envelope. This approach makes it unnecessary to evaluate $\theta(t)$, as $\psi(t)$ may be obtained directly from equation (29). The approximate solution is then of the form

$$x(t) = a(t)\ \mathrm{Cn}\,\psi\ .$$

This method is applicable to the final example considered.

c) If the amplitude varies substantially over the time interval of interest neither of these approaches can be adopted, and values for the local maxima, minima and zeroes of the approximation solution must be found by considering variation in the parameter K (see the algorithm of

section 2.4). Note that the existence of g(t) in $\dot{\psi} = (1+pa^2)^{1/2} + \beta g(t)$ for the non-autonomous case implies that the argument ψ of the elliptic function is both time and amplitude dependent, which provides additional justification for considering the variation of K with amplitude (again see the algorithm of section 2.4).

In this case it is unnecessary to evaluate either $B^{(1)}$ or $\theta(t)$ although, depending on the particular system, some inaccuracy may be incurred. The solution is obtained in the form

$$x(t) = a(t)\ \mathrm{Cn}\ [(1+pa^2(t))^{1/2}]\ .$$

Exactly the same equations are those chosen in section 2.4 will now be used to demonstrate practical application of the refined approximation.

For the first example, setting $t = m\tau$ and $\mu = m^3\beta$:

$$\frac{d^2x}{d\tau^2} + m^2(x + px^3) + \mu\tau x = 0$$

and from equations (36) and (37):

$$x(\tau) = \exp\left[\frac{-\mu\tau}{4m^2(1+\varepsilon p)(1+k^2)}\right]\mathrm{Cn}\left[(1+pa^2)^{1/2}\tau\left[m + \frac{\mu\tau}{4m\ (1+\varepsilon pa^2)}\right.\right.$$

$$\left.\left. + \frac{\mu pa^2\tau^2}{8m(1+\varepsilon pa^2)(1+k^2)(1+pa^2)^{1/2}}\right]\right.$$

This solution is shown in figure 3.7 with $m^2 = 5$, $p = 2$ and $\mu = 1.118$. The local maxima, minima and zeroes of the approximate solution were determined by solving graphically the following equation:

$$\psi(\tau) = nK(0) \qquad (n=0,1,2,\ \ldots)$$

where K(0) is the value of K (from Appendix 2, Table 1) corresponding to the initial amplitude. Note that this approach differs from that in section 2.4, because the change in frequency of the solution is taken into account by the refined analysis; additional compensation for this frequency

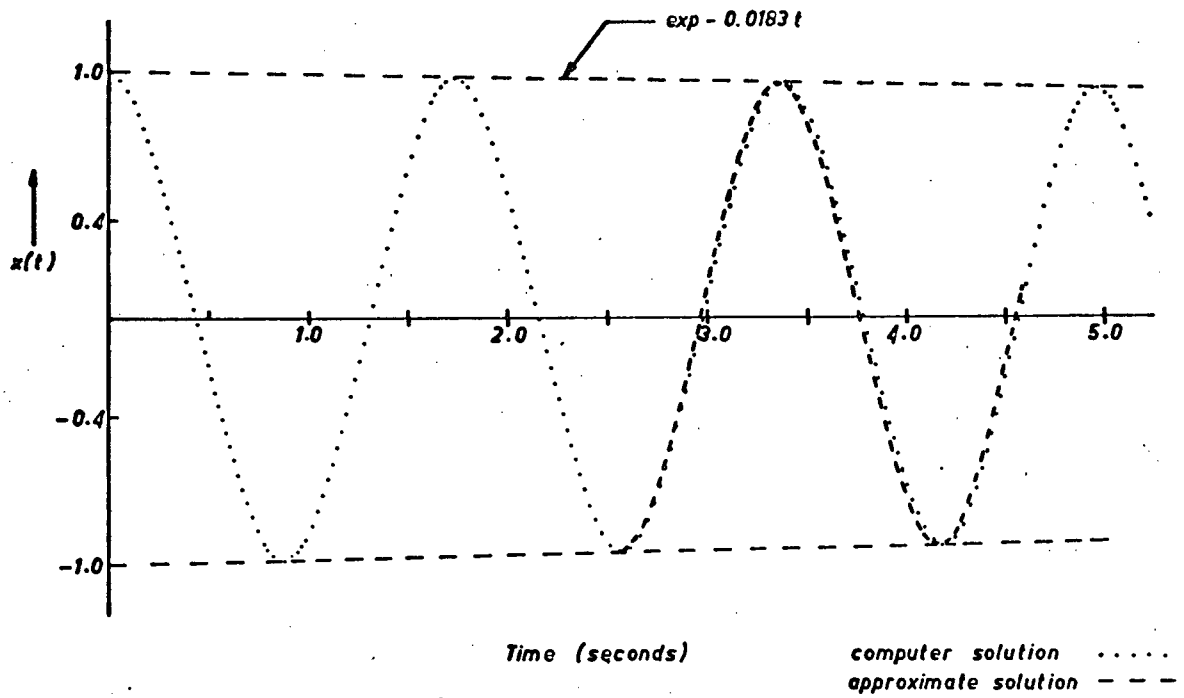

Fig. 3.7 Approximate and exact solutions of $\frac{d^2x}{d\tau^2} + 5x + 10x^3 + 1.118\tau x = 0$

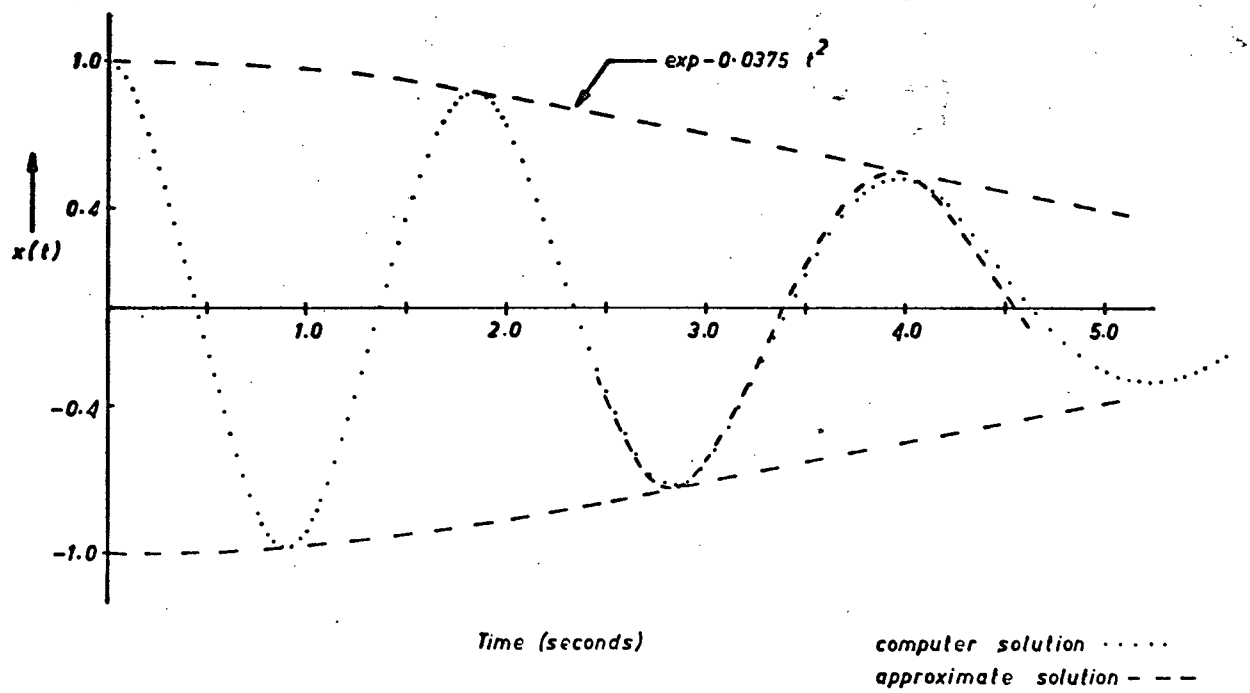

Fig. 3.8 Approximate and exact solutions of $\frac{d^2x}{d\tau^2} + 5x + 10x^3 + 0.2\tau\frac{dx}{d\tau} = 0$

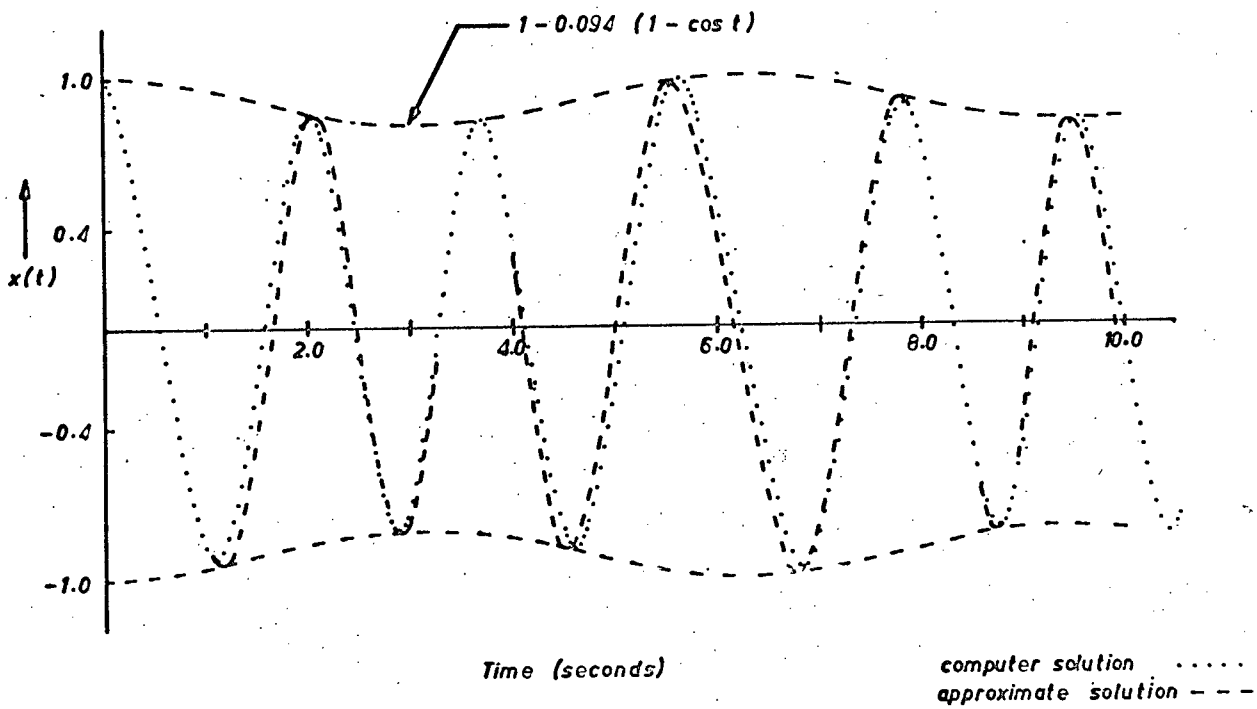

Fig. 3.9 Approximate and exact solutions of $\frac{d^2x}{d\tau^2} + 5x + 10x^3 - 5x \cos\tau = 0$

change is therefore unnecessary.

For the second example, where $\mu = m^2\beta$ and

$$\frac{d^2x}{d\tau^2} + m^2\,(x + px^3) + \mu\tau\frac{dx}{d\tau} = 0\ ,$$

the approximate solution is:

$$x(\tau) = \exp\left[\frac{-\mu\tau^2}{4(1+k^2)}\right] \mathrm{Cn}\left[m(1+pa^2)^{1/2}\tau\right] .$$

In this case phase cannot be determined explicitly in a simple form, and approach (c) is necessary.

In the analysis of the unrefined approximation the term $\theta(t)$ was found to be negligible, and the variation of phase with amplitude was determined from the variation of K, the complete elliptic integral of the first kind. A suitable algorithm is given in section 2.4.

A solution of the equation

$$\frac{d^2x}{d\tau^2} + 5(x + 2x^3) + 0.2\,\tau\,\frac{dx}{d\tau} = 0$$

obtained from this approach is shown in figure 3.8.

For the final example

$$\frac{d^2x}{d\tau^2} + m^2(x + px^3) - \mu x \cos m\omega_0\tau = 0$$

where $\mu = m^2\beta$ and $m\omega_0 = 1.0$, the solution envelope is given by:

$$a(\tau) = 1 + \mu U(\cos\tau - 1)$$

where

$$U = \frac{1}{m^2[4(1+\varepsilon pa^2)(1+k^2) - \omega_0{}^2\ ]}\ ,$$

which can be obtained directly from equation (43). Note that U is itself a function of 'a' (which is assumed to be constant), and that U defines the depth of modulation of the solution amplitude. The following short algorithm

may be used to determine U:

1. Set a = 1
2. Evaluate U, taking $k^2 = \dfrac{pa^2}{2(1+pa^2)}$
3. Calculate an improved value of the average amplitude 'a' from the relationship $a = 1 - \mu U$.
4. Return to step 2, and repeat until the desired accuracy is obtained.

The algorithm converges rapidly, e.g. three iterations were sufficient to give three figure accuracy in the example subsequently considered.

As the average solution amplitude may be assumed constant, the phase of the approximation solution (ψ) can be determined from an integration of the expression for $\dot{\psi}$. From equation (29):

$$\dot{\psi} = (1+pa^2)^{1/2} + \beta\left[B^{(1)} + \frac{pat A^{(1)}}{(1+pa^2)^{1/2}}\right]$$

Substitution for $A^{(1)}$ and $B^{(1)}$ yields:

$$\dot{\psi} = (1+pa^2)^{1/2} - \frac{2\beta(1+pa^2)^{1/2}\,(1+k^2)\cos\omega_0 t}{\left[4(1+\varepsilon pa^2)(1+k^2) - \omega_0^{\,2}\right]}$$

Note particularly that the term with a 't' multiplier has disappeared. The expression for $\psi(\tau)$ can now be obtained as:

$$\psi(\tau) = m(1+pa^2)^{1/2}\tau - \mu U\left[\frac{2(1+pa^2)^{1/2}(1+k^2)\sin\tau}{\omega_0}\right]$$

where 'a' is the average solution amplitude defined by $a = 1 - \mu U$, and k^2 corresponds to this value of 'a'. The local maxima, minima and zeroes of the solution are determined from a graphical solution of the equation

$$nK = \psi(\tau)$$

where n is an integer (n=0,1,2, ...) and K is obtained from Appendix 2, Table 1 for the value of k^2 corresponding to the average amplitude.

A solution of the equation

$$\frac{d^2x}{d\tau^2} + 5(x + 2x^3) - 5\,x\cos\tau = 0$$

is shown in figure 3.9.

3.5 Discussion

A comparison of the results of the unrefined and refined approximations indicates that substantial improvement in both amplitude and phase accuracy can be obtained when the refinement is employed. The first example (where $f(x,\dot{x},t) = tx$) is the case most susceptible to phase error, and, as such, is a severe test of an approximation method. It is encouraging that the approximate solution for this example should demonstrate such a noticeable improvement in accuracy, but perhaps the most interesting aspect of the refinement is the connection between amplitude and phase variation which was demonstrated algebraically in section 3.3 and discussed in section 3.4.

3.6 Conclusion

The refinement of the first approximation of chapter 2 is mathematically no more difficult to apply, and yields approximate solutions which are significantly better than those derived by the un-refined method. The results of the error analysis show that the greatest improvement is obtained at high values of the parameter p (which determines the degree of non-linearity).

There are three aspects of the refined approximation of particular significance.

a) The use of elliptic functions in the representation of non-linear oscillations results in a zero-order approximation which is versatile and inherently more accurate than a corresponding circular function approximation.

b) Consideration of solution-frequency variation with amplitude leads to the introduction of terms involving k^2, a parameter which varies with the degree of non-linearity and which can therefore accommodate, with accuracy,

wide variations in non-linearity. This feature is unique to the present analysis, and is an important contribution to the accuracy of the method.

c) The equation defining $A^{(1)}$ requires no assumption of solution frequency. $A^{(1)}$ is obtained as the particular integral of a linear conservative differential equation, and should therefore, on average, be an exact representation of the behaviour of the amplitude envelope. This is in contrast to either the Kryloff-Bogoliuboff method [13] or the Bogoliuboff Mitropolski method [4], which both assume the amplitude and frequency remain constant when deriving the expression for a(t). Although in the present method a constant amplitude may be taken when deriving the phase-variation term, neither constant amplitude or frequency need be assumed when calculating the amplitude envelope.

In the following chapter this refinement is extended to consider equations of the form

$$\ddot{x} + x - px^3 + \beta f(x,\dot{x},t) = 0 ,$$

which relate to systems exhibiting saturation or limiting phenomena, e.g. electronic circuits involving saturating amplifiers, inductors and capacitors, negative-resistance devices, or oscillatory motion with a restoring force F of the form $F = ax - bx^3$. For the parameter p in the range $0 < p < 1$, and $\beta = 0$, this equation has an exact solution in terms of the elliptic sine function.

4. SATURATING NONLINEARITIES

4.1 Introduction

The analysis, so far, has been concerned with the transient response of non-resonant nonlinear systems where the parameter p in the equation

$$\ddot{x} + x + px^3 + \beta f(x,\dot{x},t) = 0$$

is positive, i.e. the non-linearity is of the "hardening" type [6]. Many systems of practical interest, however, contain non-linear elements of a "softening" type [6] (e.g. saturation effects in inductors and capacitors) [17], negative resistance devices in electronic circuits [6]), and may be described by an equation of the form

$$\ddot{x} + x - px^3 + \beta f(x,\dot{x},t) = 0 \qquad (p > 0)$$

This equation, for $\beta = 0$, is still satisfied by an elliptic function, but the form of the solution is now dependent on the quantity pa^2, where 'a' is the solution amplitude [21]. If pa^2 is less than 1.0, then the solution is oscillatory and is given by

$$x(t) = a\ Sn(\omega t + K)$$

when $x(0) = 1.0$, $\dot{x}(0) = 0$; otherwise (i.e., if $pa^2 > 1$) the solution is unbounded. The present analysis will be confined to a consideration of oscillatory motion, and, as before, resonance effects are required to be negligible or non-existent. The parameter p ($p > 0$) is retained, to facilitate comparison with the results of chapter 3; when a more general differential equation is considered i.e., where the nonlinearity can be either hardening or softening , the following form is used:

$$\ddot{x} + x + rx^3 + \beta f(x,\dot{x},t) = 0$$

where $-1 < r < \infty$.

4.2 Development of the first approximation

From Hsu [11], the exact solution of the non-linear differential equation

$$\ddot{x} + x - px^3 = 0 \quad (p > 0) \quad .$$

with $x(0) = 1.0$, $\dot{x}(0) = 0$ is:

$$x(t) = a\ \mathrm{Sn}(\omega t + K)$$

where $\omega = \left[1 - \frac{pa^2}{2}\right]^{1/2}$, K is the complete elliptic integral of the first kind, and the modulus of the elliptic function is now given by

$$k^2 = \frac{pa^2}{2\left[1 - \frac{pa^2}{2}\right]}$$

Accordingly, when $\beta \neq 0$ in the equation

$$\ddot{x} + x - px^3 + \beta f(x,\dot{x},t) = 0$$

let the solution for $x(0) = 1$, $\dot{x}(0) = 0$ be taken as

$$x(t) = a(t)\ \mathrm{Sn}(\omega t + \theta + K) \quad = a\ \mathrm{Sn}\,\psi$$

As before:

$$x = x(a,\psi)$$

$$\dot{x} = \dot{a}\frac{\partial x}{\partial a} + \dot{\psi}\frac{\partial x}{\partial \psi} \tag{44}$$

and hence

$$\ddot{x} = \ddot{a}\frac{\partial x}{\partial a} + \dot{a}\frac{d}{dt}\left[\frac{\partial x}{\partial a}\right] + \dot{\psi}\frac{d}{dt}\left[\frac{\partial x}{\partial \psi}\right] + \ddot{\psi}\frac{\partial x}{\partial \psi} \tag{45}$$

Evaluating the derivatives in this expression:

$$\frac{\partial x}{\partial a} = \mathrm{Sn}\,\psi$$

$$\frac{d}{dt}\left[\frac{\partial x}{\partial a}\right] = \dot{\psi}\,\mathrm{Cn}\,\psi\ \mathrm{Dn}\,\psi$$

$$\frac{d}{dt}\left[\frac{\partial x}{\partial \psi}\right] = \dot{a}\,\mathrm{Cn}\,\psi\ \mathrm{Dn}\,\psi + a\dot{\psi}\frac{\partial}{\partial \psi}(\mathrm{Cn}\,\psi\ \mathrm{Dn}\,\psi)$$

where $\frac{\partial}{\partial \psi}(\mathrm{Cn}\,\psi\ \mathrm{Dn}\,\psi) = -\frac{(\mathrm{Sn}\,\psi - pa^2\mathrm{Sn}^3\psi)}{\left[1 - \frac{pa^2}{2}\right]}$

and $\frac{\partial x}{\partial \psi} = a\,\mathrm{Cn}\,\psi\ \mathrm{Dn}\,\psi$.

By substitution, equation (45) may now be written as:

$$\ddot{x} = \ddot{a}\mathrm{Sn}\,\psi + 2\dot{a}\dot{\psi}\,\mathrm{Cn}\,\psi\ \mathrm{Dn}\,\psi - a\dot{\psi}^2\frac{(\mathrm{Sn}\,\psi - pa^2\mathrm{Sn}^3\psi)}{\left[1 - \frac{pa^2}{2}\right]} + a\ddot{\psi}\,\mathrm{Cn}\,\psi\ \mathrm{Dn}\,\psi \tag{46}$$

The polynomial representation of $\dot{a}$ and $\dot{\theta}$ is unchanged from chapter 2, i.e.

$$\dot{a} = \mu A^{(1)}(a,t) + \mu^2 A^{(2)}(a,t) + \ldots\ldots \mu^i A^{(i)}(a,t) + \ldots$$

$$\dot{\theta} = \mu B^{(1)}(a,t) + \mu^2 B^{(2)}(a,t) + \ldots\ldots \mu^i B^{(i)}(a,t) + \ldots$$

and, retaining only terms of $O(\mu)$:

$$\ddot{a} = \mu A^{(1)}{}_t$$

$$\ddot{\theta} = \mu B^{(1)}{}_t$$

where the subscripts denote partial differentiation with respect to time. Following the approach of the refined approximation (where the variation of solution frequency with amplitude is taken into consideration):

$$\dot{\psi} = \omega + t\dot{\omega} + \dot{\theta}$$

where
$$\omega = \left[1 - \frac{pa^2}{2}\right]^{1/2}$$

and
$$\dot{\omega} = \frac{-pa\dot{a}}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}}$$

To first order terms in μ, the expression for $\dot{\psi}$ then becomes:

$$\dot{\psi} = \left[1 - \frac{pa^2}{2}\right]^{1/2} + \mu\left[B^{(1)} - \frac{pat\,A^{(1)}}{2\left[1-\frac{pa^2}{2}\right]^{1/2}}\right] \tag{47}$$

Similarly:

$$\dot{\psi}^2 = \left[1 - \frac{pa^2}{2}\right] + \mu\left[2B^{(1)}\left[1 - \frac{pa^2}{2}\right]^{1/2} - pat\,A^{(1)}\right] \tag{48}$$

where terms of $O(\mu^2)$, $O(\mu^2 t)$ and $O(\mu^2 t^2)$ are neglected.

For $\ddot{\psi}$:

$$\ddot{\psi} = 2\dot{\omega} + t\frac{d}{dt}\left[\frac{-pa\dot{a}}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}}\right] + \ddot{\theta}$$

or

$$\ddot{\psi} = \ddot{\theta} \quad \frac{-\ pa\dot{a}}{\left[1 - \frac{pa^2}{2}\right]^{1/2}} - \frac{pt}{2}\left[\frac{a\ddot{a}}{\left[1 - \frac{pa^2}{2}\right]^{1/2}} + \frac{\dot{a}^2}{\left[1 - \frac{pa^2}{2}\right]^{3/2}}\right]$$

Now the term $\dot{a}^2$ is of order μ^2, and hence to first order in μ:

$$\ddot{\psi} = \mu\left[B_t^{(1)} - \frac{pa\ A^{(1)}}{\left[1 - \frac{pa^2}{2}\right]^{1/2}} - \frac{pat\ A_t^{(1)}}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}}\right] \tag{49}$$

Substitution for $\dot{\psi}$, $\dot{\psi}^2$ and $\ddot{\psi}$ in equation (46) yields the following expression for $\ddot{x}$:

$$\begin{aligned}\ddot{x} = &-(aSn\psi - pa^3\ Sn^3\psi\ ) \\ &+ \mu\ Cn\psi Dn\psi\left[2A^{(1)}\left[1 - \frac{pa^2}{2}\right]^{1/2} + aB_t^{(1)} - \frac{(2pa^2\ A^{(1)} + pa^2t\ A_t^{(1)})}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}}\right] \\ &+ \mu\left[A_t^{(1)}Sn\psi - \frac{(Sn\psi - pa^2\ Sn^3\psi)}{\left[1 - \frac{pa^2}{2}\right]}\left[2aB^{(1)}\left[1 - \frac{pa^2}{2}\right]^{1/2} - pa^2t\ A^{(1)}\right]\right]\end{aligned} \tag{50}$$

But, from equation (44), $\ddot{x}$ may also be expressed as

$$\ddot{x} = -(aSn\psi - pa^3Sn^3\psi + \beta f(x,\dot{x},t))$$

and, setting $\beta = \mu$, it now follows directly from equation (50) that

$$\begin{aligned}f(x,\dot{x},t) = &- Cn\psi Dn\psi\left[2A^{(1)}\left[1 - \frac{pa^2}{2}\right]^{1/2} + aB_t^{(1)} - \frac{(2pa^2\ A^{(1)} + pa^2t\ A_t^{(1)})}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}}\right] \\ &- \left[A_t^{(1)}\ Sn\psi - \frac{(Sn\psi - pa^2\ Sn^3\psi)}{\left[1 - \frac{pa^2}{2}\right]}\left[2aB^{(1)}\left[1 - \frac{pa^2}{2}\right]^{1/2} - pa^2t\ A^{(1)}\right]\right]\end{aligned} \tag{51}$$

Following a similar approximation of chapter 2 let

$$\delta\int_0^{2K} Sn\psi d\psi = \int_0^{2K} Sn^3\psi\, d\psi \tag{52}$$

Now $\frac{d}{d\psi}(Cn\psi Dn\psi) = -Sn\psi(1 + k^2) + 2k^2\ Sn^3\psi$

and hence $2k^2 \int Sn^3\psi \, d\psi = Cn\psi Dn\psi + (1+k^2) \int Sn\psi \, d\psi$.

By evaluating the integral of $Sn\psi$ [5], the expression for δ may now be obtained as

$$\delta = \frac{1}{k^2 \ln\left[\frac{1+k}{1-k}\right]} \left[\frac{(1+k^2)}{2} \ln\left[\frac{1+k}{1-k}\right] - k\right] . \qquad (53)$$

A tabulation of p, k^2, δ and K is to be found in Appendix 2, Table 2.

Integration of equation (52) when k = 0, i.e. when $Sn\psi$ reduces to $\sin\psi$, gives the value of δ as 2/3. An identical value is obtained for equation (53) (when k = 0) after the application of L'Hôpital's rule.

Equation (51) may now be simplified to:

$$f(x,\dot{x},t) = -Cn\psi Dn\psi \left[2A^{(1)} \left[1 - \frac{pa^2}{2}\right]^{1/2} + aB^{(1)}_t - \frac{(2pa^2 A^{(1)} + pa^2 t A^{(1)}_t)}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}}\right]$$

$$-Sn\psi \left[A^{(1)}_t - \frac{(1 - \delta pa^2)}{\left[1 - \frac{pa^2}{2}\right]} \left[2aB^{(1)} \left[1 - \frac{pa^2}{2}\right]^{1/2} - pa^2 t A^{(1)}\right]\right] \qquad (54)$$

From Milne-Thompson [16], the Fourier series expansion of $Cn\psi Dn\psi$ may be derived as

$$Cn\psi Dn\psi = \frac{d}{d\psi}(Sn\psi) = \frac{d}{d\psi}\left[\frac{2\pi}{kK} \sum_{s=0}^{\infty} \left[\frac{q^{(s+1/2)}}{1 - q^{(2s+1)}}\right] \sin(2s+1)\frac{\pi\psi}{2K}\right]$$

$$\text{i.e.} \quad Cn\psi Dn\psi = \frac{\pi^2}{kK^2} \sum_{s=0}^{\infty} \left[\frac{q^{(s+1/2)}}{1 - q^{(2s+1)}}\right] (2s+1) \cos(2s+1)\frac{\pi\psi}{2K}$$

where $q = \exp\left[\frac{-\pi K'}{K}\right]$, and hence $Sn\psi$ and $Cn\psi Dn\psi$ may be expressed as infinite sums of sine and cosine terms respectively.

As for the previous case (where r > 0 in the equation $\ddot{x} + x + rx^3 + \beta f(x,\dot{x},t) = 0$), this property indicates that $Sn\psi$ and $Cn\psi Dn\psi$ may be treated as analogous to $\sin\psi$ and $\cos\psi$ in the principle of harmonic balance. The three examples of chapter 3 are again chosen to demonstrate the

application of equation (54).

Example 1.

$$\ddot{x} + x - px^3 + \beta tx = 0$$

with $x(0) = 1.0$, $\dot{x}(0) = 0$ and where $f(x,\dot{x},t) = tx = at\ \mathrm{Sn}\psi$. Substituting for $f(x,\dot{x},t)$ in equation (54), and comparing coefficients of $\mathrm{Sn}\psi$ and $\mathrm{Cn}\psi\,\mathrm{Dn}\psi$:

$$aB^{(1)}_t + 2A^{(1)}\left[1 - \frac{pa^2}{2}\right]^{1/2} - \frac{(2pa^2A^{(1)} + pa^2tA^{(1)}_t)}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}} = 0 \qquad (55)$$

$$A^{(1)}_t - \frac{(1 - \delta pa^2)}{\left[1 - \frac{pa^2}{2}\right]}\left[2aB^{(1)}\left[1 - \frac{pa^2}{2}\right]^{1/2} - pa^2t\,A^{(1)}\right] = -at \qquad (56)$$

Taking the partial derivative of equation (56) with respect to time:

$$aB^{(1)}_t = \frac{(A^{(1)}_{tt} + a)}{2(1 - \delta pa^2)}\left[1 - \frac{pa^2}{2}\right]^{1/2} + \frac{pa^2}{2}(A^{(1)} + tA^{(1)}_t)\left[1 - \frac{pa^2}{2}\right]^{-1/2}$$

and by substitution for $aB^{(1)}_t$ in equation (55) it now follows that

$$A^{(1)}_{tt} + 4A^{(1)}(1 - \delta pa^2)\left[1 - \frac{pa^2}{4\left[1 - \frac{pa^2}{2}\right]}\right] = -a \qquad (57)$$

Setting $A^{(1)}_{tt} = 0$ (as in section 3.2 for this example), and noting that

$\dfrac{pa^2}{4\left[1 - \frac{pa^2}{2}\right]} = \dfrac{k^2}{2}$, where k is the modulus of the elliptic function $\mathrm{Sn}\psi$, the

particular integral of equation (57) may be obtained as

$$A^{(1)} = \frac{-a}{4(1 - \delta pa^2)\left[1 - \frac{k^2}{2}\right]},$$

and then, by integration:

$$a(t) = \exp\left[\frac{-\beta t}{4(1 - \delta pa^2)\left[1 - \frac{k^2}{2}\right]}\right] \qquad (58)$$

Taking $A^{(1)}_t = 0$ in equation (56), the expression for $B^{(1)}$ becomes

$$B^{(1)} = \frac{t}{2(1-\delta pa^2)}\left[1-\frac{pa^2}{2}\right]^{1/2} - \frac{pa^2 t}{8(1-\delta pa^2)\left[1-\frac{k^2}{2}\right]}\left[1-\frac{pa^2}{2}\right]^{-1/2}$$

and finally:

$$\theta(t) = \frac{\beta t^2}{4(1-\delta pa^2)}\left[1-\frac{pa^2}{2}\right]^{1/2} - \frac{\beta pa^2 t^2}{16(1-\delta pa^2)\left[1-\frac{k^2}{2}\right]}\left[1-\frac{pa^2}{2}\right]^{-1/2} \tag{59}$$

Example 2.

$$\ddot{x} + x - px^3 + \beta t\dot{x} = 0$$

with $x(0) = 1.0$, $\dot{x}(0) = 0$ and where

$$f(x,\dot{x},t) = t\dot{x} = at\left[1-\frac{pa^2}{2}\right]^{1/2} \mathrm{Cn}\psi\mathrm{Dn}\psi + O(\beta t)$$

As before (see section 3.2), terms of $O(\beta t)$ in $f(x,\dot{x},t)$ are equivalent to terms of order $\beta^2 t$ in $\beta f(x,\dot{x},t)$, and may be neglected.

From equation (54), comparing coefficients of $\mathrm{Sn}\psi$ and $\mathrm{Cn}\psi\mathrm{Dn}\psi$:

$$aB^{(1)}_t + 2A^{(1)}\left[1-\frac{pa^2}{2}\right]^{1/2} - \left(\frac{2pa^2 A^{(1)} + pa^2 t A^{(1)}_t}{2\left[1-\frac{pa^2}{2}\right]^{1/2}}\right) = -at\left[1-\frac{pa^2}{2}\right]^{1/2} \tag{60}$$

$$A^{(1)}_t - \frac{(1-\delta pa^2)}{\left[1-\frac{pa^2}{2}\right]}\left[2aB^{(1)}\left[1-\frac{pa^2}{2}\right]^{1/2} - pa^2 t A^{(1)}\right] = 0 \quad . \tag{61}$$

After taking the partial derivative of equation (61) with respect to time, and substitution for $aB^{(1)}_t$ in equation (60), the following equation is obtained:

$$A^{(1)}_{tt} + 4A^{(1)}(1-\delta pa^2)\left[1-\frac{k^2}{2}\right] = -2at(1-\delta pa^2) \quad .$$

The particular integral of this equation is

$$A^{(1)} = \frac{-at}{2\left[1-\frac{k^2}{2}\right]}$$

and hence, by integration:

$$a(t) = \exp\left[\frac{-\beta t^2}{4\left[1-\frac{k^2}{2}\right]}\right] \quad . \tag{62}$$

Substitution for $A_t^{(1)}$ in equation (61) then gives:

$$B^{(1)} = \frac{-pa^2t^2}{4\left[1 - \frac{k^2}{2}\right]}\left[1 - \frac{pa^2}{2}\right]^{-1/2} - \frac{1}{4(1 - \delta pa^2)\left[1 - \frac{k^2}{2}\right]}\left[1 - \frac{pa^2}{2}\right]^{1/2}$$

The same difficulty is found here as was encountered in section 3.2, namely that a simple closed form expression for $\theta(t)$ cannot be found if the amplitude variation is to be taken into consideration. However, as for the earlier case, it is still possible to determine solution phase by an approximate method (see sections 3.4 and 4.4).

Example 3.

$$\ddot{x} + x - px^3 - \beta x \cos \omega_0 t = 0$$

with $x(0) = 1.0$, $\dot{x}(0) = 0$ and where

$$f(x,\dot{x},t) = -x \cos \omega_0 t = -aSn\psi \cos \omega_0 t \ .$$

Applying equation (54):

$$aB_t^{(1)} + 2A^{(1)}\left[1 - \frac{pa^2}{2}\right]^{1/2} - \frac{(2pa^2 A^{(1)} + pa^2 t A_t^{(1)})}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}} = 0$$

$$A_t^{(1)} - \frac{(1 - \delta pa^2)}{\left[1 - \frac{pa^2}{2}\right]}\left[2aB^{(1)}\left[1 - \frac{pa^2}{2}\right]^{1/2} - pa^2 t A^{(1)}\right] = a \cos \omega_0 t \ . \tag{63}$$

By the same procedure as before:

$$A^{(1)} = \frac{-a\,\omega_0 \sin \omega_0 t}{4(1 - \delta pa^2)\left[1 - \frac{k^2}{2}\right] - \omega_0^2} \tag{64}$$

and, after substitution for $A_t^{(1)}$ in equation (63):

$$B^{(1)} = \frac{-1}{\left[4(1-\delta pa^2)\left[1-\frac{k^2}{2}\right] - \omega_0^2\right]}\left[\frac{pa^2 t\omega_0 \sin\omega_0 t}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}} + 2\left[1 - \frac{pa^2}{2}\right]^{1/2}\left[1 - \frac{k^2}{2}\right]\cos \omega_0 t\right].$$

Finally, integrating equation (64), and assuming 'a' to be constant:

$$a(t) = \exp\left[\frac{\beta\ (\cos\ \omega_0 t - 1)}{4(1 - \delta\, pa^2)\left[1 - \frac{k^2}{2}\right] - \omega_0^{\ 2}}\right]$$

or, if the exponent is small:

$$a(t) = 1 + \left[\frac{\beta(\cos\ \omega_0 t - 1)}{4(1 - \delta pa^2)\left[1 - \frac{k^2}{2}\right] - \omega_0^{\ 2}}\right] \tag{65}$$

As discussed in section 3.2, it is unnecessary in practice to integrate the expression for $B^{(1)}$ and determine $\theta(t)$.

Two limitations must be imposed in this particular case:

a) To avoid resonance effects $4(1 - \delta\, pa^2)\left[1 - \frac{k^2}{2}\right] \gg \omega_0^{\ 2}$.

b) For solution stability, the initial value of the expression $x - px^3 - \beta x \cos\ \omega_0 t$ must be positive [11].

when $x(0) = 1$, this last condition may be written as $1 - p > \beta$.

The equations developed above are applied to three specific examples in section 4.4, following an analysis of error incurred in applying the approximation. One final detail, however, remains to be verified. In section 3.3 it was shown that, to a first-order approximation in the case of $r > 0$ in the equation $\ddot{x} + x + rx^3 + \beta f(x,\dot{x},t) = 0$, the change in $\dot{\psi}$ due to $\frac{\partial}{\partial a}(\omega)$ was cancelled by that due to $\frac{\partial}{\partial a}(\theta)$. The first-order expression for $\dot{\psi}$ in the present case is given by equation (47) as:

$$\dot{\psi} = \left[1 - \frac{pa^2}{2}\right]^{1/2} + \beta\left[B^{(1)} - \frac{pat\ A^{(1)}}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}}\right].$$

Considering, for purposes of demonstration, example 3, and substituting for $A^{(1)}$ and $B^{(1)}$:

$$\dot{\psi} = \left[1 - \frac{pa^2}{2}\right]^{1/2} - \frac{\beta}{\left[4(1 - pa^2)\left[1 - \frac{k^2}{2}\right] - \omega_0^{\ 2}\right]}\left[\frac{pa^2 t\ \omega_0\ \sin\ \omega_0 t}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}}\right.$$

$$\left. + 2\left[1 - \frac{pa^2}{2}\right]^{1/2}\left[1 - \frac{k^2}{2}\right]\cos\ \omega_0 t - \frac{pa^2 t\ \omega_0\ \sin\ \omega_0 t}{2\left[1 - \frac{pa^2}{2}\right]^{1/2}}\right].$$

A similar cancellation to that observed in section 3.3 is thus also obtainable for the equation $\ddot{x} + x - px^3 + \beta f(x,\dot{x},t) = 0$. The above result may readily be deduced for the remaining two examples.

In the following section the method introduced in section 2.3 is applied to determine error incurred when applying the approximation.

4.3 Error Analysis

The analysis of the preceding section yields, for a specific example, relationships defining the quantities a, $\dot{a}$, $\ddot{a}$, $\dot{\psi}$ and $\ddot{\psi}$, and consequently the approximate solution can be simulated in the same manner as for the case considered in section 2.3.

The necessary expressions for $\dot{x}$ and $\ddot{x}$ are (from equations (44) and (46)):

$$\dot{x} = \dot{a}\,\mathrm{Sn}\psi + a\dot{\psi}\mathrm{Cn}\psi\mathrm{Dn}\psi \tag{66}$$

$$\ddot{x} = \ddot{a}\,\mathrm{Sn}\psi + 2\dot{a}\dot{\psi}\mathrm{Cn}\psi\mathrm{Dn}\psi - a\dot{\psi}^2\left(\frac{\mathrm{Sn}\psi - pa^2\,\mathrm{Sn}^3\psi}{\left[1 - \frac{pa^2}{2}\right]}\right) + a\ddot{\psi}\mathrm{Cn}\psi\mathrm{Dn}\psi \tag{67}$$

Now $x = a\,\mathrm{Sn}\psi$

i.e. $\mathrm{Sn}\psi = x/a$

and from equation (66):

$$\dot{x} - \dot{a}\,\mathrm{Sn}\psi = a\dot{\psi}\mathrm{Cn}\psi\mathrm{Dn}\psi$$

Substituting for $\mathrm{Sn}\psi$:

$$\mathrm{Cn}\psi\mathrm{Dn}\psi = \frac{\dot{x}}{a\dot{\psi}} - \frac{\dot{a}x}{a^2\dot{\psi}}$$

and hence, substituting for $\mathrm{Sn}\psi$ and $\mathrm{Cn}\psi\mathrm{Dn}\psi$ in equation (67):

$$\ddot{x} = \frac{\ddot{a}x}{a} + \left[\dot{x} - \frac{\dot{a}x}{a}\right]\left[\frac{\ddot{\psi}}{\dot{\psi}} + \frac{2\dot{a}}{a}\right] - \frac{\dot{\psi}^2}{\left[1 - \frac{pa^2}{2}\right]}\left(x - px^3\right) \tag{68}$$

All the coefficients of equation (68) are known (for a particular differential equation), and the approximate solution can now be obtained by integration of

the state equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{\ddot{a}}{a} x_1 + \left[x_2 - \frac{\dot{a}}{a} x_1\right]\left[\frac{\ddot{\psi}}{\dot{\psi}} + 2\frac{\dot{a}}{a}\right] - \frac{\dot{\psi}^2}{\left[1 - \frac{pa^2}{2}\right]}(x_1 - px_1^3) .$$

These equations were integrated (using a digital simulation) for the three examples of section 4.2, and the integral error definition of equation (28) was again used to obtain the results shown in figures 4.1 to 4.6. Figures 4.1, 4.2 and 4.3 refer to the variation of the error integral with β for the three examples, holding p constant at 0.5; figures 4.4, 4.5 and 4.6 show variation of the error integral with the parameter p, with β held constant.

An interesting feature of figures 4.4 to 4.6 is that solution error is shown to increase with the parameter p (c.f. figures 3.4 to 3.6 where the converse is seen to be true); note, however, that $k^2 = \frac{pa^2}{2\left[1 - \frac{pa^2}{2}\right]}$ for this case, p tends to 1.0 as the modulus k tends to 1.0, and then the elliptic function $\mathrm{Sn}\,\psi$ tends to $\tanh \psi$, which is non-oscillatory. In section 3.4, where the equation under consideration was $\ddot{x} + x + px^3 + \beta f(x,\dot{x},t) = 0$ and $p \geqslant 0$, the results shown in figures 3.4 to 3.6 indicated that solution error decreased with increasing p, although in that case the solution period was a maximum for $p = 0$. The linking factor of the two cases (which, between them, cover the entire range of oscillatory solutions of the equation $\ddot{x} + x + px^3 + \beta f(x,\dot{x},t) = 0$ where $-1 < p < \infty$) is thus seen to be the solution period, and it may be concluded that, in general, the error incurred by the approximation diminishes with decreasing solution period. As before, the oscillatory behaviour of the error integral, predicted from the analysis where an oscillatory term in $A^{(1)}$ was neglected (see, for example, sections 2.2, 3.2 and 4.2 referring to

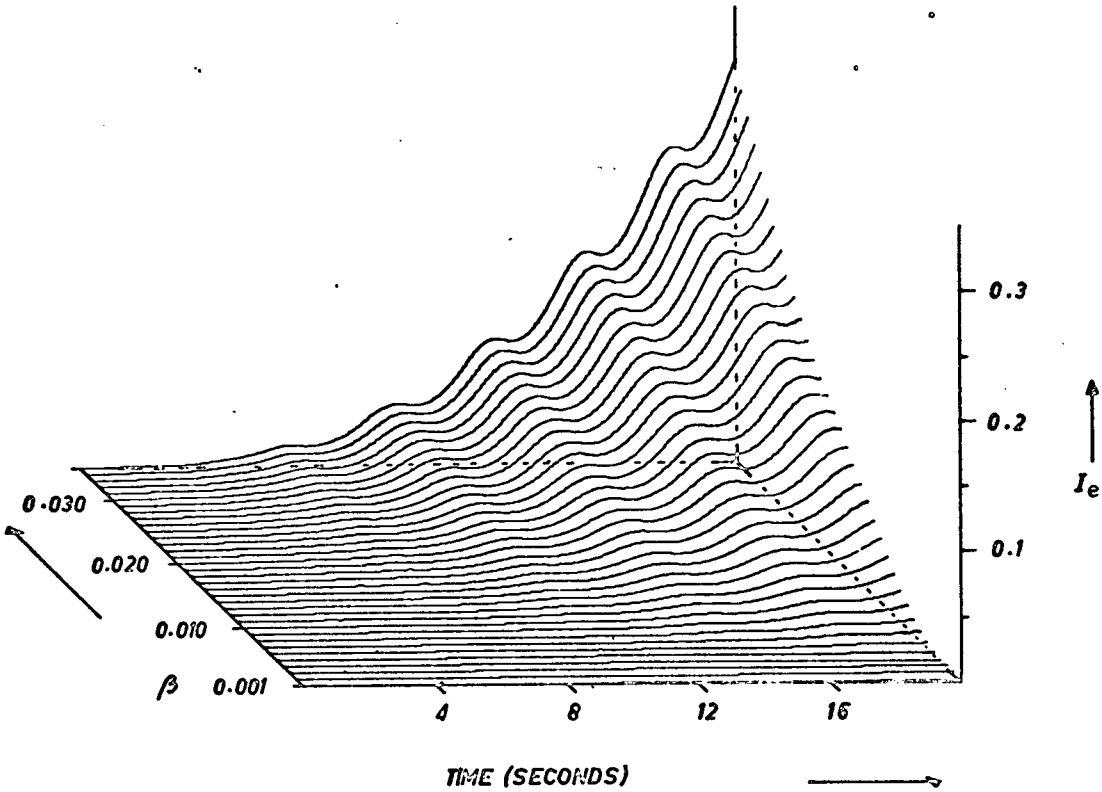

Fig. 4.1 Error integral as a function of time and β for $f(x,\dot{x},t) = tx$, with $p = 0.5$.

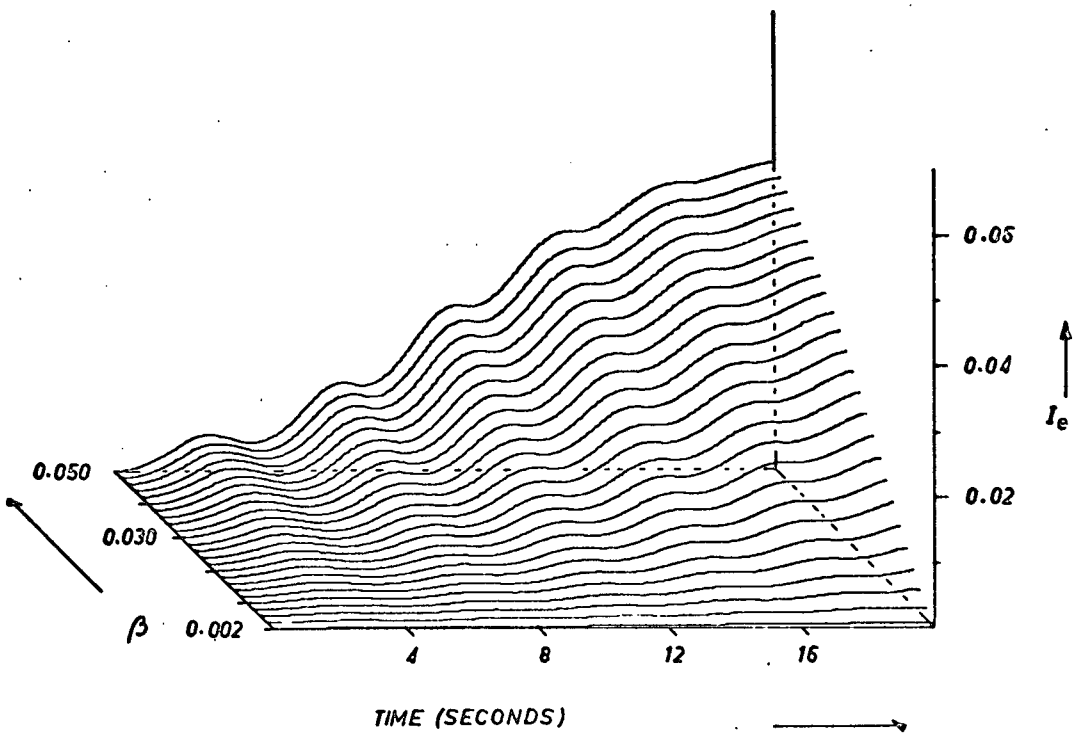

Fig. 4.2 Error integral as a function of time and β for $f(x,\dot{x},t) = t\dot{x}$, with $p = 0.5$.

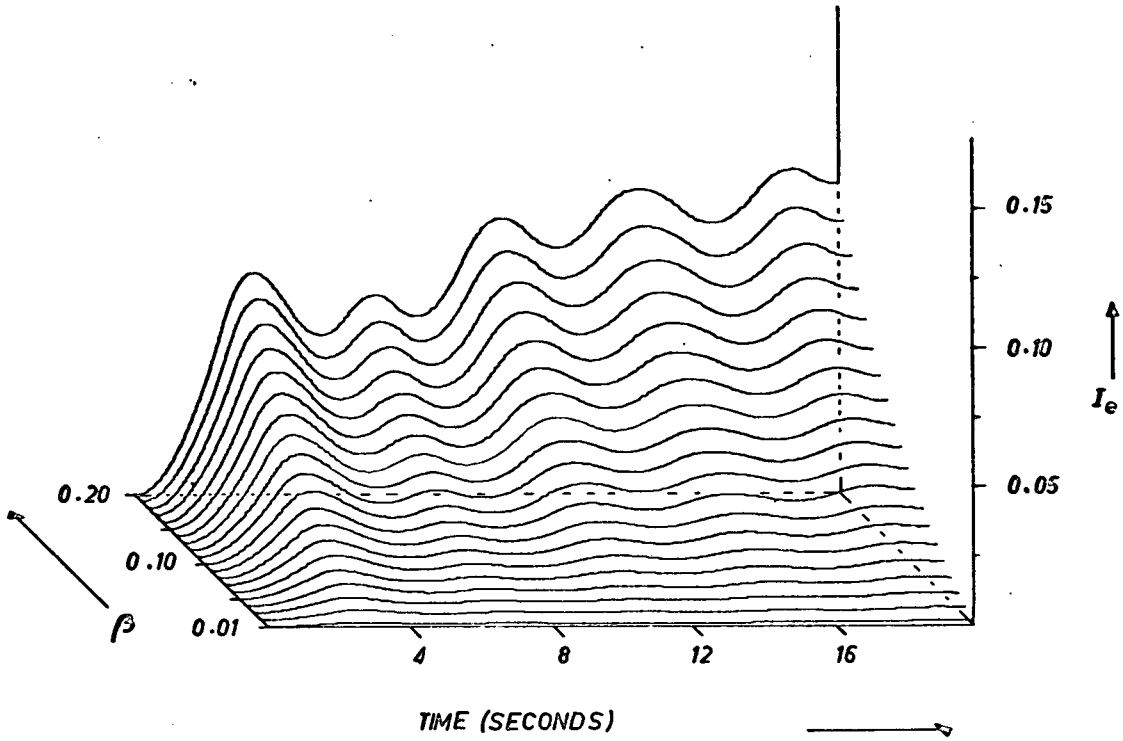

Fig. 4.3 Error integral as a function of time and β for $f(x,\dot{x},t) = -x \cos t/m$, with $p = 0.5$.

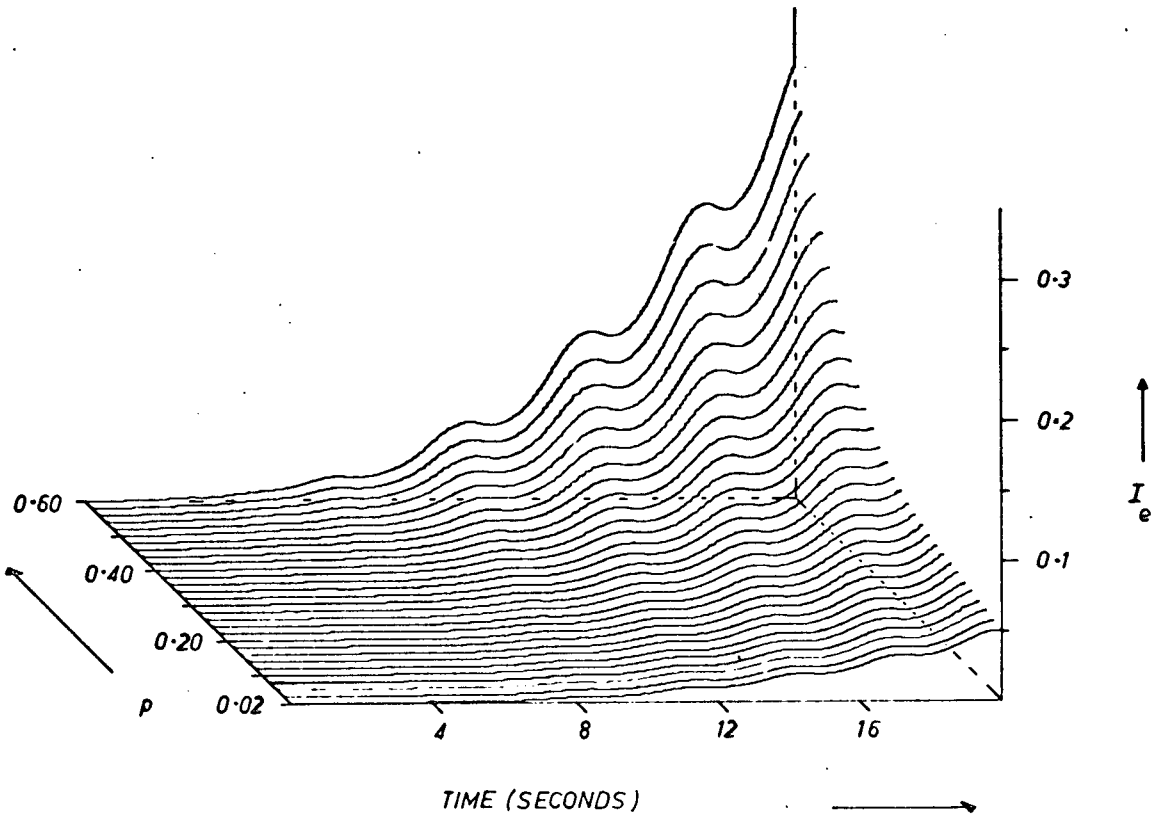

Fig. 4.4 Error integral as a function of time and p for $f(x,\dot{x},t) = tx$, with $\beta = 0.018$.

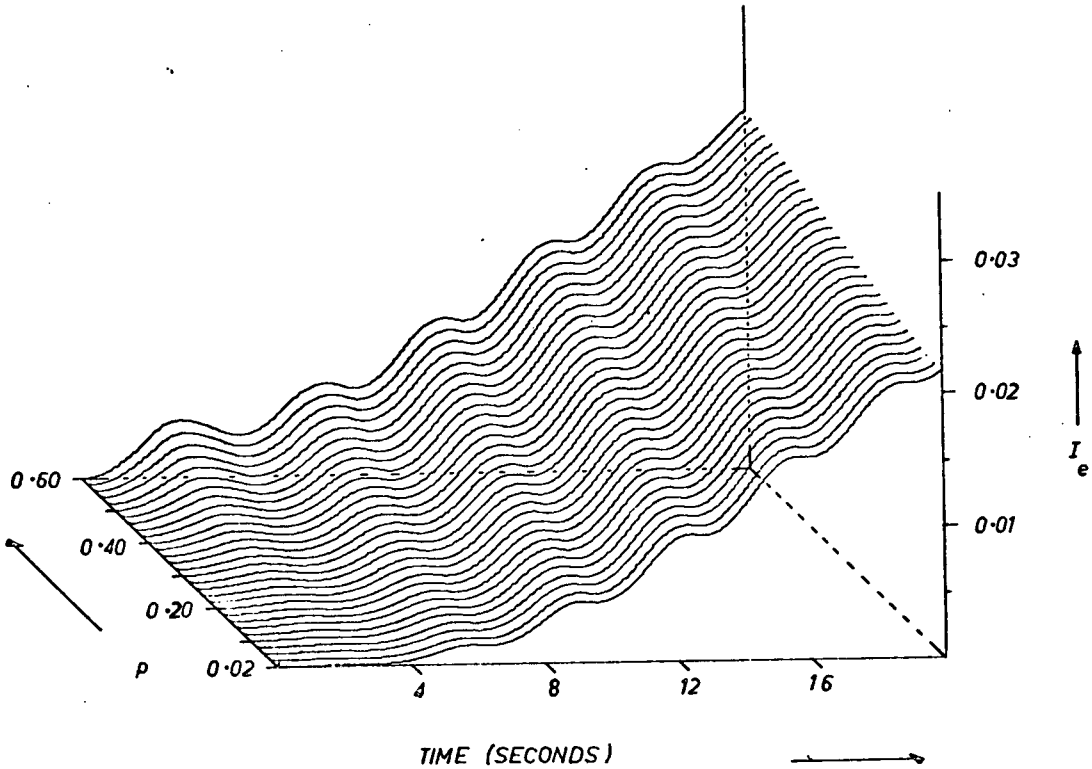

Fig. 4.5 Error integral as a function of time and p for $f(x,\dot{x},t) = t\dot{x}$, with $\beta = 0.025$.

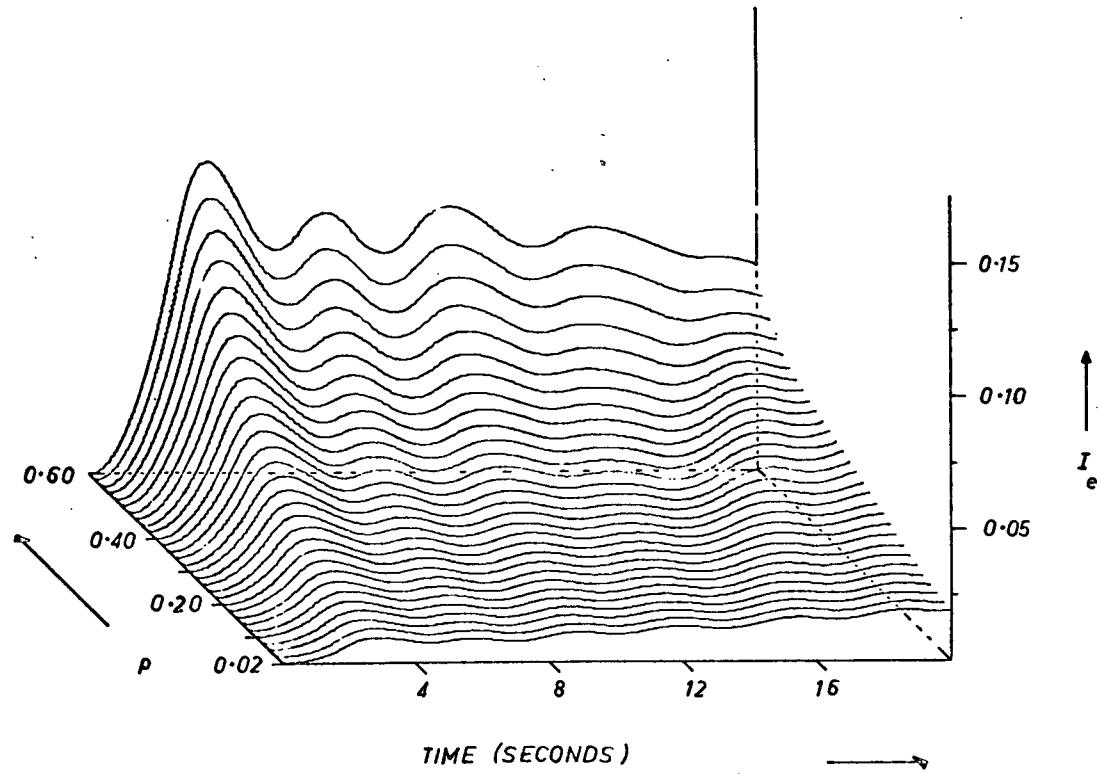

Fig. 4.6 Error integral as a function of time and p for $f(x,\dot{x},t) = -x \cos t/m$, with $\beta = 0.165$.

example 1), is readily observed.

4.4 Application

The practical application of the method will now be demonstrated by considering three equations similar to those chosen in sections 2.4 and 3.4; apart from changes in the equations defining solution amplitude and phase, however, the approaches are essentially the same as those outlined in section 3.4, and the same consideration must be given to the expression for $\dot{\psi}$ (equation (47) in this case).

By taking $t = m\tau$, the differential equations of examples 1, 2 and 3 of section 4.2 may be written as:

$$\frac{d^2x}{d\tau^2} + m^2 (x - px^3) + \begin{cases} m^3\beta\tau x = 0 & \text{(Ex. 1)} \\ m^2\beta\tau\frac{dx}{d\tau} = 0 & \text{(Ex. 2)} \\ -m^2\beta x \cos m\omega_0\tau = 0 & \text{(Ex. 3)} \end{cases}$$

where $x(0) = 1.0$ and $\frac{dx}{d\tau}(0) = 0$.

From equations (58) and (59) the approximate solution for the first example is then:

$$x(\tau) = \exp\left[\frac{-\mu\tau}{4m^2(1-\delta p)\left[1-\frac{k^2}{2}\right]}\right] Sn\left[m\left[1-\frac{pa^2}{2}\right]^{1/2}\tau + K\right.$$

$$\left. + \frac{\mu\tau^2}{4m(1-\delta pa^2)}\left[\left[1-\frac{pa^2}{2}\right]^{1/2} - \frac{pa^2}{4\left[1-\frac{k^2}{2}\right]}\left[1-\frac{pa^2}{2}\right]^{-1/2}\right]\right]$$

where K is the complete elliptic integral of the first kind corresponding to the value of p (see Appendix 2, Table 2), and $\mu = m^3\beta$.

This solution is shown in figure 4.7, with $m^2 = 5$, $p = 0.5$ and $\mu = 0.20$.

For the second example, from equation (62) the approximate solution

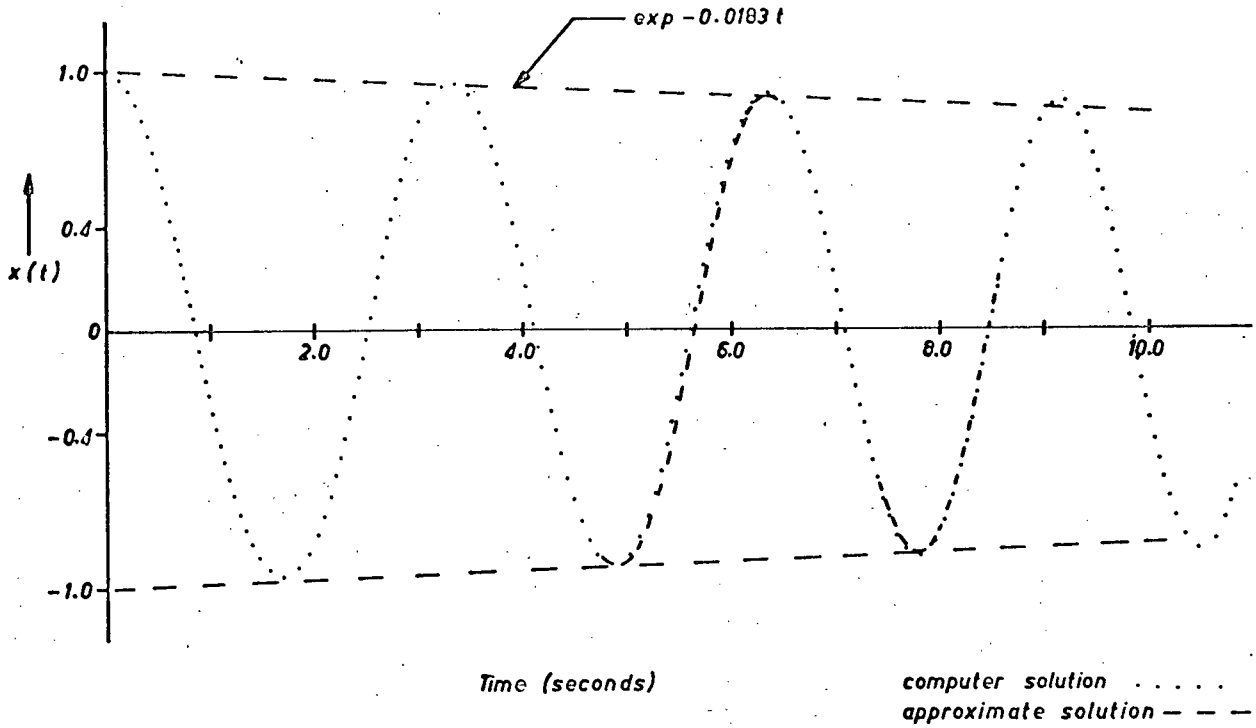

Fig. 4.7 Approximate and exact solutions of

$$\frac{d^2x}{d\tau^2} + 5x - 2.5x^3 + 0.2\tau x = 0.$$

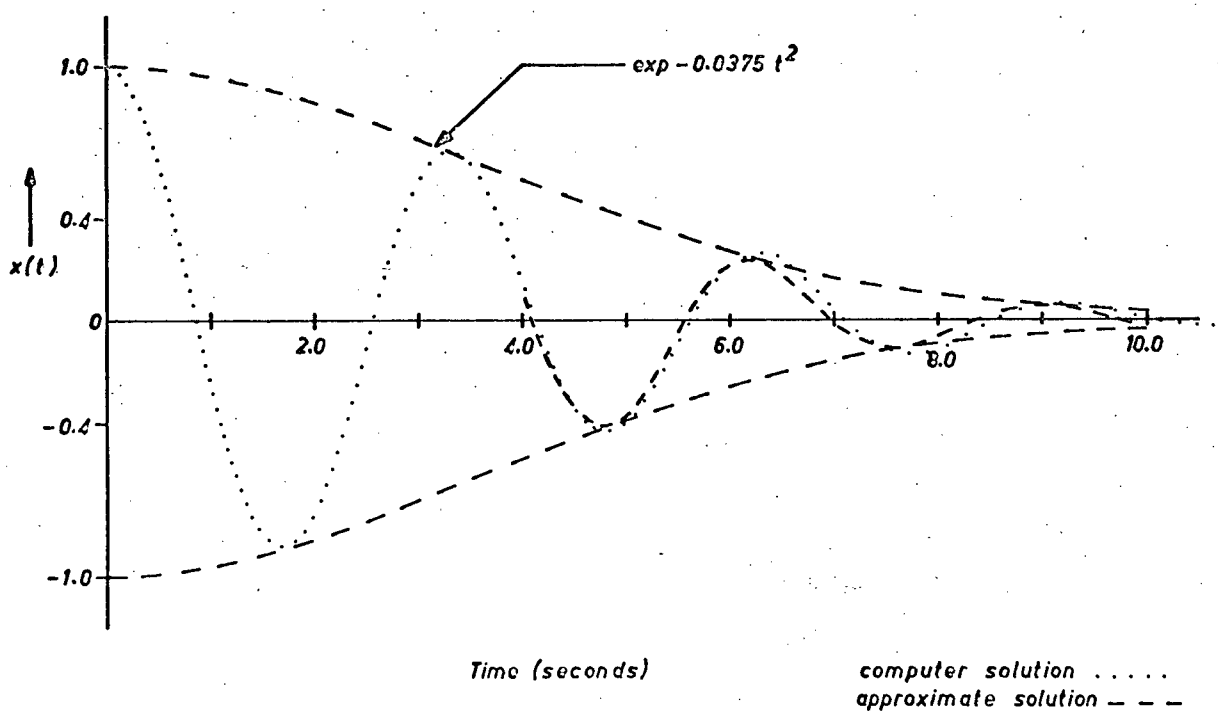

Fig. 4.8 Approximate and exact solutions of

$$\frac{d^2x}{d\tau^2} + 5x - 2.5x^3 + 0.125\tau\frac{dx}{d\tau} = 0.$$

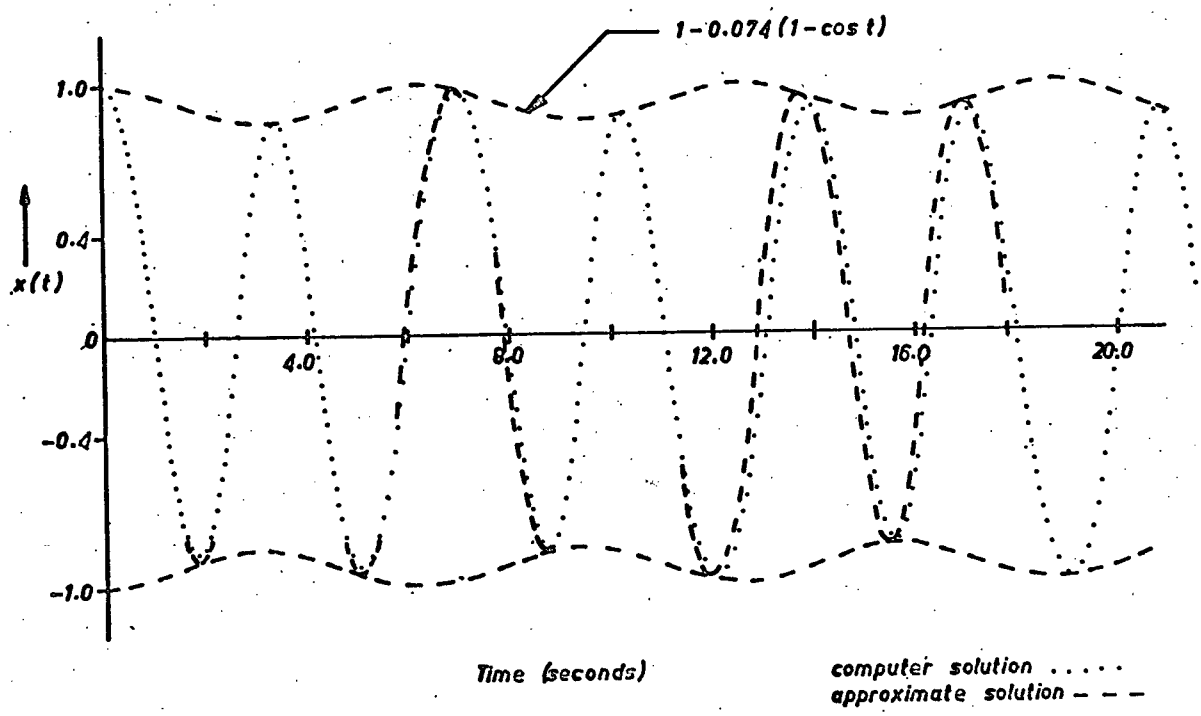

Fig. 4.9 Approximate and exact solutions of

$$\frac{d^2x}{d\tau^2} + 5x - 2.5x^3 - 0.825x\cos\tau = 0.$$

is:

$$x(\tau) = \exp\left[\frac{-\mu\tau^2}{4\left[1-\frac{k^2}{2}\right]}\right] \mathrm{Sn}\left[m\left[1-\frac{pa^2}{2}\right]^{1/2}\tau + K\right]$$

where $\mu = m^2\beta$.

Solution phase for this example must again be determined in an approximate method, and a suitable algorithm is given below.

1. Assume the average value of the amplitude envelope (denoted by 'a') to be 1.0 over the first quarter period.
2. Determine k^2 from the relationship

$$k^2 = \frac{pa^2}{2\left[1-\frac{pa^2}{2}\right]}$$

3. Determine the value of the complete elliptic integral K corresponding to this k^2 (from Appendix 2, Table 2).
4. Calculate the quarter period T from the relationship

$$T = \frac{K}{\omega}, \text{ where } \omega = m\left[1-\frac{pa^2}{2}\right]^{1/2}$$

5. Determine the average amplitude over the next quarter period (assuming it to be of duration T seconds) from the solution envelope (i.e. in this case $\exp\left[\frac{-\mu\tau^2}{4\left[1-\frac{k^2}{2}\right]}\right]$), and return to step 2.

A solution, using this algorithm, is shown in figure 4.8 for $m^2 = 5$, $p = 0.5$ and $\mu = 0.125$.

For the final example, if $m\omega_0 = 1.0$ and $\mu = m^2\beta$ the solution envelope is (from equation (65)):

$$a(t) = 1 + \mu U(\cos\tau - 1)$$

where

$$U = \frac{1}{m^2\left[4(1-\delta pa^2)\left[1-\frac{k^2}{2}\right] - \omega_0^2\right]}$$

The algorithm given for the third example (which corresponds to the present problem) of section 3.4 may again be used to determine U except that, in step 2, k^2 must be replaced by the expression $k^2 = \dfrac{pa^2}{2\left[1-\frac{pa^2}{2}\right]}$. Also, as derived in section 4.2:

$$\dot{\psi} = \left[1 - \frac{pa^2}{2}\right]^{1/2} - \frac{2\beta \cos \omega_0 t}{\left[4(1-\delta pa^2)\left[1-\frac{k^2}{2}\right] - \omega_0^2\right]} \left[1-\frac{k^2}{2}\right]\left[1-\frac{pa^2}{2}\right]^{1/2}$$

and hence

$$\psi(\tau) = m\left[1-\frac{pa^2}{2}\right]^{1/2}\tau - \mu U\left[\frac{2\sin\tau}{\omega_0}\left[1-\frac{k^2}{2}\right]\left[1-\frac{pa^2}{2}\right]^{1/2}\right]$$

where 'a' is the average solution amplitude defined by the expression $a = 1 - \mu U$, and k^2 corresponds to this value of 'a'.

An approximate solution of the equation

$$\frac{d^2x}{d\tau^2} + 5x - 2.5x^3 - 0.825x \cos\tau = 0$$

is shown in figure 4.9 (i.e. for $m^2 = 5$, $p = 0.5$ and $\mu = 0.825$).

4.5 Discussion

The coefficients m^2, p and μ chosen for the graphical solutions of section 4.4 result in amplitude envelopes which are the same (for the first two examples) or close to those of the corresponding examples in section 3.4. To include a similar number of oscillations, the solutions shown in figures 4.7 to 4.9 were calculated over twice the time interval taken for the examples of section 3.4 (see figures 3.7 to 3.9). In spite of this increased time interval, which might be expected to produce greater error, the approximate solutions for both cases appear to be of comparable accuracy.

Again, it should be stressed that the error integral results of figures 4.1 to 4.6 lead to a pessimistic estimate of solution error. A comparison of the relative error of figures 3.1 to 3.6 and 4.1 to 4.6 is

complicated by the dependence of solution characteristics on the parameters p and β. If the envelope of the solution may be taken as a basis of comparison, then figures 3.4 to 3.6 and 4.4 to 4.6 are related and indicate that less error is incurred for the equation $\ddot{x} + x - px^3 + \beta f(x,\dot{x},t) = 0$; in general, however, it would be unwise to draw conclusions of this nature as solution error is dependent on a number of variables. The most interesting results are those shown in figures 4.2 and 4.6, as although the solution of the equation $\ddot{x} + x - px^3 - \beta x \cos \omega_0 t = 0$ does not decay in amplitude, for certain ranges of the parameters p and β the error (as defined by equation (29)) does not increase above a limiting value, at least for the time interval considered . The phases of the approximate and exact solutions are thus in accord over a long time interval, a feature which is rarely found even in approximations for quasi-linear equations, with the result that the error incurred per cycle of the solution remains approximately constant.

4.6 Conclusion

The analysis presented in this chapter has extended the application of elliptic functions to solution of the equation

$$\ddot{x} + x + rx^3 + \beta f(x,\dot{x},t) = 0$$

where $-1 < r < \infty$, and, together with chapter 3, enables approximate solutions of a wide class of non-autonomous grossly non-linear differential equations to be determined. The phase of solutions to such equations is strongly amplitude dependent, and the success of the approximation is, in part, due to the recognition of this property and the ability of elliptic functions to account for varying solution phase.

Although non-autonomous equations have been considered to demonstrate application of the method, it should be evident that autonomous equations are equally amenable to solution in the same manner. In the next chapter, two

important autonomous cases are investigated: a non-linear form of Van der Pol's equation, and an equation representing a non-linear system with heavy damping.

5. AUTONOMOUS NONLINEAR SYSTEMS

5.1 Introduction

The analysis of the preceding three chapters, although it has placed emphasis on the solution of non-autonomous equations, may equally be applied to a consideration of autonomous systems. Two such equations which have attracted considerable attention are Van der Pol's equation [24] - [31], and the unforced Duffing equation [32] - [34].

Much of the importance of Van der Pol's equation

$$\ddot{x} + x - \beta(1 - x^2)\dot{x} = 0 \tag{69}$$

derives from its description of oscillatory processes having a steady-state amplitude which is independent of initial conditions; its eventual solution is a stable sustained oscillation which, for the form of the equation given above, has a final amplitude of 2.0 for small β.

In the analysis of section 5.3 a modified version of Van der Pol's equation is considered, which applies to a stellar pulsation problem [12]:

$$\ddot{x} + x + px^3 - \beta(1 - \gamma x^2)\dot{x} = 0 \qquad (p > 0) \tag{70}$$

This particular example is also used to demonstrate application of the refined approximation of chapter 3 to systems where the initial amplitude is zero, but where $\dot{x}(0)$ is non-zero, and where the amplitude envelope is defined by an exponential function with a positive argument.

The main difficulty encountered by previous workers when investigating solution behaviour of the unforced Duffing equation

$$\ddot{x} + x + px^3 + \beta\dot{x} = 0 \tag{71}$$

was the prediction of solution phase (which, as might be anticipated from the discussion of section 4.6, is far from being a linear function of time), particularly when calculated from a quasi-linear approach. Considerable ingenuity has been applied to the construction of suitable mathematical models

using circular functions, and Ludecke and Wagner [34] in a recent paper adopted an approach which was conceptually similar to that used in chapter 3 (i.e., the variation of solution frequency with amplitude was taken into consideration), except that circular functions were still used in the development of the approximate solution.

In the following section it is shown that, for the damped Duffing equation of equation (71), the phase of the first order approximate solution can be obtained explicitly in terms of the amplitude envelope when elliptic function solutions are considered. This result makes possible a first order solution of equation (71) in a simple form, which is accurate even for comparatively large values of the parameter β.

5.2 Linear dissipation

In sections 3.3 and 4.2 it was shown that a partial cancellation of first order terms in the expressions for $\dot{\psi}$ (equation (29) of section 3.2 and equation (47) of section 4.2) was obtained for non-autonomous functions $f(x,\dot{x},t)$ in the equation $\ddot{x} + x + rx^3 + \beta f(x,\dot{x},t) = 0$ where $-1 < r < \infty$.

Now consider an autonomous function $f(x,\dot{x})$ in the equation

$$\ddot{x} + x + rx^3 + \beta f(x,\dot{x}) = 0 \tag{72}$$

If $r > 0$ (i.e. the case considered in chapter 3) the approximate solution is of the form $x(t) = a(t)\mathrm{Cn}\psi$, and $f(x,\dot{x})$ may be expressed as

$$f(x,\dot{x}) = -g_1\ a\ (1+pa^2)^{1/2}\ \mathrm{Sn}\psi\,\mathrm{Dn}\psi + g_2\ a\ \mathrm{Cn}\psi,$$

where for the moment no restriction is placed on the functions g_1 and g_2, except that they should not be functions of time. Comparing coefficients of $\mathrm{Sn}\psi\,\mathrm{Dn}\psi$ and $\mathrm{Cn}\psi$ when $f(x,\dot{x})$ is substituted in equation (33) gives (by the method of, for instance, example 1 of section 3.2):

$$A^{(1)}_{tt} + 4A^{(1)}\ (1+\varepsilon pa^2)(1+k^2) = -2g_1\ a(1+\varepsilon pa^2)$$

since $\frac{\partial g_1}{\partial t} = 0$, and the particular integral of this equation is:

$$A^{(1)} = \frac{-g_1 a}{2(1+k^2)}$$

where
$$k^2 = \frac{pa^2}{2(1+pa^2)} .$$

If $f(x,\dot{x})$ is a non-linear function then, although $f(x,\dot{x},)$ can be expressed in the linearized form

$$f(x,\dot{x}) = -g_1\ a(1+pa^2)^{1/2}\ Sn\psi\, Dn\psi + g_2\ aCn\psi$$

(see section 5.3), it is not strictly true that $g_1 \neq g_1(t)$, as g_1 will be the approximation of a term involving a mean value and periodic component. If, however, $f(x,\dot{x})$ is linear in x and $\dot{x}$, then g_1 and g_2 will be constant and $A^{(1)}_t$ will be zero.

Evaluating $B^{(1)}$ under the assumption $A^{(1)}_t = 0$, and substituting for $A^{(1)}$ and $B^{(1)}$ in equation (29) gives the first order approximation to $\dot{\psi}$ as:

$$\dot{\psi} = (1+pa^2)^{1/2} + \beta g_2 \frac{(1+pa^2)^{1/2}}{2(1+\varepsilon pa^2)} ,$$

i.e. if $f(x,\dot{x}) = g_1\dot{x}$, where g_1 is a constant, then first order terms in the expression for $\dot{\psi}$ cancel exactly.

A corresponding analysis for $-1 < r < 0$ (the case of chapter 4) where

$$f(x,\dot{x}) = g_1\ a\left[1-\frac{pa^2}{2}\right]^{1/2} Cn\psi Dn\psi + g_2\ a\ Sn\psi$$

gives, under the same assumptions of linearity,:

$$\dot{\psi} = \left[1-\frac{pa^2}{2}\right]^{1/2} + \frac{\beta g_2}{2(1-\delta pa^2)}\left[1-\frac{pa^2}{2}\right]^{1/2} + O(\beta^2)$$

(from equation (47)).

It therefore follows that, for equation (72) and $-1 < r < \infty$, first order terms in $\dot{\psi}$ cancel exactly; the significance of this result may be seen

from the following discussion.

Consider an approximate solution

$$x(t) = a(t)\ Cn\psi \qquad \text{(i.e. the case of } r>0\text{)}.$$

Then, if $\dot{\psi} = (1+pa^2)^{1/2}$:

$$\frac{d}{dt}(Cn\psi) = -Sn\psi\, Dn\psi\ \frac{\partial\psi}{\partial t}$$

and ψ may be obtained by integrating $\dot{\psi} = (1+pa^2)^{1/2}$ treating 'a' as constant. If the initial value of the phase is $\theta(0)$, then

$$x(t) = a(t)\ Cn\ [(1+pa^2)t + \theta(0)]$$

i.e. the phase-modifying term $\theta(t)$ is time-invariant (to first order). This further implies that solution frequency is only amplitude dependent, and the elliptic integral K, which defines the quarter-period T by

$$T = \frac{K}{(1+pa^2)^{1/2}},$$

may be regarded as invariant. This last result is particularly important when deriving the approximate solution. A similar argument may be applied to solutions of the form $x(t) = a(t)Sn\psi$, for which the same conclusions are reached.

The first order solution of the equation

$$\ddot{x} + x + rx^3 + \beta\dot{x} = 0$$

with $-1 < r < \infty$, and an initial amplitude defined by the initial conditions $x(0)$ and $\dot{x}(0)$ (see section 5.3), can then be obtained by calculating the amplitude envelope $a(t)$, and determining the local maxima, minima and zeroes of the solution from

$$\psi(t) = nK(0)$$

where $K(0)$ is the value of K corresponding to the initial amplitude $a(0)$ (derived from the value of k^2 for $a(0)$), and ψ is given by

$$\psi = (1+pa^2(t))^{1/2}t + \theta(0) \quad \text{for } r > 0 \quad .$$

and

$$\psi = \left[1 - \frac{pa^2(t)}{2}\right]^{1/2} t + \theta(0) \quad \text{for } -1 < r < 0.$$

Although this method relates directly to the equation considered in section 5.4, note that, because of the nonlinear nature of the function $(1 - \gamma x^2)\dot{x}$ in equation (70), it cannot be applied to solution of the modified Van der Pol equation.

5.3 The Modified Van der Pol equation

Consider the equation

$$\ddot{x} + x + px^3 - \beta(1 - \gamma x^2)\dot{x} = 0 \quad (p > 0) \tag{70}$$

where $x(0) = 0$. If the approximate solution of this equation is taken as

$$x(t) = a(t)\mathrm{Cn}\psi$$

where $\psi = \omega t - K + \theta(t)$, K is the complete elliptic integral of the first kind, $\omega = (1+pa^2)^{1/2}$ and $\theta(0) = 0$, then:

$$\dot{x} = \dot{a}\mathrm{Cn}\psi - a\dot{\psi}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi.$$

Now $\mathrm{Sn}(-K) = -1$, and since $\mathrm{Dn}^2u = 1 - k^2\mathrm{Sn}^2u$ [5] it therefore follows that $\mathrm{Dn}(-K) = (1 - k^2)^{1/2}$ where $k^2 = \dfrac{pa^2}{2(1+pa^2)}$.

It is now possible to define $\dot{x}(0)$ in terms of the quantities p, a and k^2 as:

$$\dot{x}(0) = a(1+pa^2)^{1/2}\,(1-k^2)^{1/2} \tag{73}$$

i.e. a choice of the initial condition $\dot{x}(0)$ determines the initial solution envelope amplitude, the elliptic function solution of equation (70) then being defined by $x(0) = 0$ and $\dot{x}(0)$.

Note that this argument also applies to the analysis of chapter 3, and may obviously be extended to that of chapter 4 where $x(t) = a(t)\,\mathrm{Sn}\psi$.

These considerations of initial conditions are necessary to obtain a final solution, but do not affect the derivation of the equation defining

the functions $A^{(1)}$ and $B^{(1)}$ (i.e. equation (33)).

For equation (70):

$$f(x,\dot{x},t) = -(1 - \gamma x^2)\dot{x}$$

and therefore

$$f(x,\dot{x},t) = a\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi\,(1 - \gamma a^2\mathrm{Cn}^2\psi)(1+pa^2)^{1/2} + O(\beta) \qquad (74)$$

As before (see section 3.2), terms of $O(\beta)$ in $f(x,\dot{x},t)$ are equivalent to terms of $O(\beta^2)$ in $\beta f(x,\dot{x},t)$, and may be neglected in the first order analysis.

Equation (74) is of a more complex form than has been considered in previous chapters, as it contains a term $\mathrm{Cn}^2\psi\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi$ and may no longer be expressed in the simple form $f(x,\dot{x},t) = g(a,t)\mathrm{Sn}\psi\,\mathrm{Dn}\psi$ which had made possible a comparison of coefficients of $\mathrm{Cn}\psi$ and $\mathrm{Sn}\psi\,\mathrm{Dn}\psi$ in equation (33) (section 3.2).

Now $\mathrm{Cn}^2\psi = 1 - \mathrm{Sn}^2\psi$, i.e. $\mathrm{Cn}^2\psi\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi = \mathrm{Sn}\psi\,\mathrm{Dn}\psi - \mathrm{Sn}^3\psi\,\mathrm{Dn}\psi$

From Milne-Thompson [16]:

$$\mathrm{Sn}^3\psi\,\mathrm{Dn}\psi = \frac{8\pi^3}{K^3k^3}\left[\sum_{n=0}^{\infty} S_n \sin(2n+1)\frac{\pi\psi}{2K}\right]\left[\frac{\pi}{2K} + \frac{2\pi}{K}\sum_{n=1}^{\infty} d_n \cos n\frac{\pi\psi}{2K}\right]$$

where $S_n = \dfrac{q^{n+1/2}}{1 - q^{2n+1}}$, $d_n = \dfrac{q^n}{1 + q^{2n}}$ and $q = \exp\left[\dfrac{-\pi K'}{K}\right]$.

If k is small only s_0 and d_1 will be significant. Using Hermite's expansion for q in terms of k [7] and taking

$$K = \frac{\pi}{2}\left[1 + \frac{k^2}{4} + \frac{3k^4}{8} + \ldots\right]$$

(from Bowman [5]), the expression for $\mathrm{Sn}^3\psi\,\mathrm{Dn}\psi$ may be reduced to the form

$$\mathrm{Sn}^3\psi\,\mathrm{Dn}\psi = (1-\Delta)\sin\frac{\pi\psi}{2K} \qquad (75)$$

where

$$\Delta = 1 - \left[\frac{1 + 0.9375k^2}{1 + k^2}\right]\left[\frac{3}{4} - 0.0625k^2\right], \qquad (76)$$

terms of $O(k^4)$ have been neglected, and only the fundamental component of

the Fourier series expansion has been retained.

If the following approximation is made:

$$Sn^3\psi\, Dn\psi = (1-\Delta)Sn\psi\, Dn\psi \tag{77}$$

then $f(x,\dot{x},t)$ can be expressed in the form $f(x,\dot{x},t) = f(a)Sn\psi\, Dn\psi$, and the method of section 3.2 applied to determine $A^{(1)}$ and $B^{(1)}$.

From equations (74) and (77):

$$f(x,\dot{x},t) = \Gamma a Sn\psi\, Dn\psi\,(1+pa^2)^{1/2}$$

where $\Gamma = 1 - \gamma\Delta a^2$, (78)

and from equation (33), comparing coefficients of $Cn\psi$ and $Sn\psi\, Dn\psi$:

$$\Gamma a(1+pa^2)^{1/2} = aB^{(1)}_t + 2A^{(1)}(1+pa^2)^{1/2} + \left[\frac{2pa^2A^{(1)} + pa^2tA^{(1)}_t}{1+pa^2\ 1/2}\right] \tag{78}$$

$$0 = A^{(1)}_t - \frac{(1+\varepsilon pa^2)}{(1+pa^2)}\left[2aB^{(1)}(1+pa^2)^{1/2} + 2pa^2tA^{(1)}\right] \tag{80}$$

Taking the partial derivative of equation (80), substituting for $aB^{(1)}_t$ in equation (79) and setting $k^2 = \dfrac{pa^2}{2(1+pa^2)}$:

$$A^{(1)}_{tt} + 4A^{(1)}(1+\varepsilon pa^2)(1+k^2) = 2\Gamma a(1+\varepsilon pa^2).$$

The particular integral of this equation is:

$$A^{(1)} = \frac{\Gamma a}{2(1+k^2)}$$

and therefore, substituting for Γ from equation (78):

$$\dot{a} - \frac{\beta a}{2(1+k^2)} + \frac{\beta\gamma\Delta a^3}{2(1+k^2)} = 0 \tag{81}$$

Equation (81) is in the form of a Bernoulli equation [6], which may be integrated after the substitution

$$z = a^{-2}.$$

The final result for $a(t)$ is:

$$a(t) = \left[\gamma\Delta + \exp\left[\frac{-\beta t}{1+k^2}\right]\left[\frac{1}{a_0^{\,2}} - \gamma\Delta\right]\right]^{-1/2} \tag{82}$$

where a_0 is the initial value of the amplitude envelope (from equation (71)).

As Γ is not an explicit function of time $A_t^{(1)} = 0$ and hence, from equation (80):

$$B^{(1)} = \frac{-\Gamma pa^2 t}{2(1+k^2)(1+pa^2)^{1/2}}$$

or, substituting for Γ from equation (78):

$$B^{(1)} = \frac{-pa^2 t}{2(1+k^2)(1+pa^2)^{1/2}} + \frac{\gamma pa^4 t\Delta}{2(1+k^2)(1+pa^2)^{1/2}}$$

Integrating this expression, and treating 'a' as a constant [13]:

$$\theta(t) = \frac{\beta pa^2 t^2}{4(1+k^2)(1+pa^2)^{1/2}}\left[\gamma\Delta a^2 - 1\right] \tag{83}$$

Equations (82) and (83) now define the first order approximate solution. To reduce these results to apply to Van der Pol's original equation (see [30] and equation (69)), let $\gamma = 1$ and $p = 0$. Then, if $x(0) = 0$ and $\dot{x}(0) = a$, the approximate solution of equation (69) is given by:

$$x(t) = \left[\frac{1}{4} + \exp(-\beta t)\left[\frac{1}{a_0^{\,2}} - \frac{1}{4}\right]\right]^{-1/2} \sin t \tag{84}$$

since $\theta(t) = 0$.

The equation of the stable oscillation is obtained by letting t tend to infinity (i.e. $\exp(-\beta t) \to 0$), and hence in the limit:

$$x(t) = 2 \sin t.$$

The result $\theta(t) = 0$ was also found by Kryloff and Bogoliuboff [13], and produces a circular trajectory of the limit cycle in the phase plane $(x-\dot{x})$ for equation (69). In practice the trajectory is of a more complex nature [31], but it should be noted that the present analysis (and the K-B analysis) produces averaged results which do not take into consideration the second-order effect

of amplitude variation (e.g. the term $O(\beta)$ in equation (74)). An analysis of Van der Pol's equation by the quasi-linear method of Bogoliuboff-Mitropolsky can be found in Minorsky [17]. After some manipulation it can be shown that equation (84) is equivalent to the first order approximate solution in [17].

For the modified Van der Pol equation (equation (70)), the approximate solution is, from equations (82) and (83)

$$x(t) = \left[\gamma\Delta + \exp\left[\frac{-\beta t}{1+k^2}\right]\left[\frac{1}{a_0^{\,2}} - \gamma\Delta\right]\right]^{-1/2} \mathrm{Cn}\left[(1+pa^2)^{1/2}\, t - K + \frac{\beta p a^2 t^2}{4(1+k^2)(1+pa^2)^{1/2}}\left[\gamma\Delta a^2 - 1\right]\right] \tag{85}$$

where $x(0) = 0$, and $\dot{x}(0)$ is defined by equation (73) with $a = a_0$, the initial envelope amplitude.

A solution of the equation

$$\ddot{x} + x + 2x^3 - 0.1\,(1 - x^2)\,\dot{x} = 0\ , \tag{86}$$

using the first approach of section 3.4, is shown in figure 5.1. The initial amplitude a_0 was chosen as 1.0, giving (from equation (73)) $\dot{x}(0) = 1.414$. The parameters of equation (86) correspond to

$$\beta = 0.1,\quad \gamma = 1.0,\quad p = 2.0,\quad k^2 = 0.3333,\quad K = 1.734$$

and hence (from equation (76)):

$$\Delta = 0.283\ .$$

Substituting these values into equation (82), the amplitude envelope becomes:

$$a(t) = \left[\,0.283 + 0.717\,\exp\,(-0.075t)\right]^{-1/2}$$

and the amplitude of the stable oscillation a_s is, therefore,

$$a_s = (\gamma\Delta)^{-1/2} = (0.283)^{-1/2} = 1.88\ .$$

As $a_s^2 \quad \frac{1}{\gamma\Delta}$, the term $(\gamma\Delta a^2 - 1)$ in equation (83) vanishes when the stable oscillation is reached, and the solution of equation (86) then

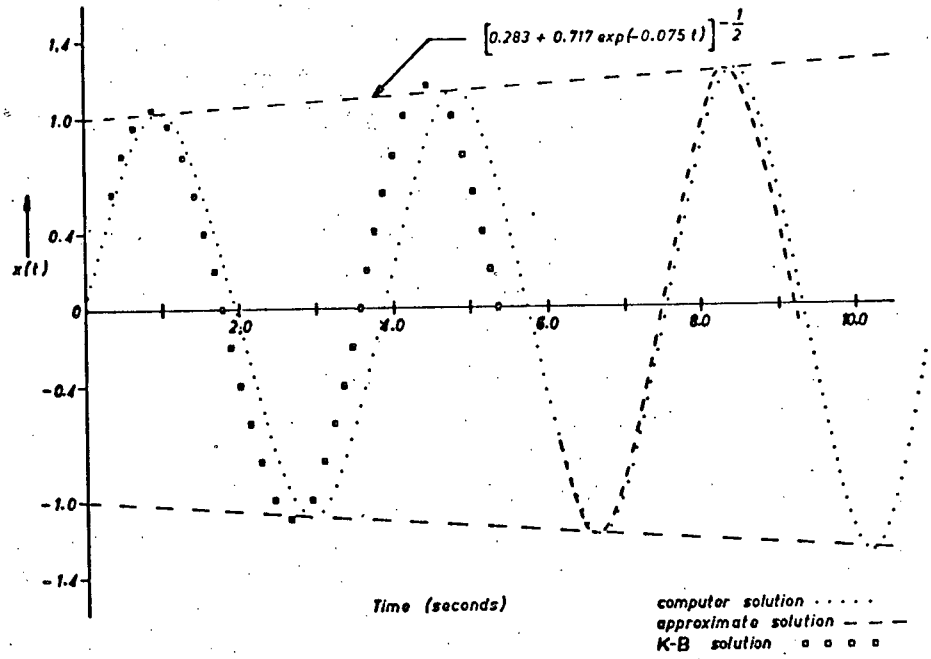

Fig. 5.1 Approximate and exact solutions of

$$\ddot{x} + x + 2x^3 - 0.1\ (1 - x^2)\ \dot{x} = 0$$

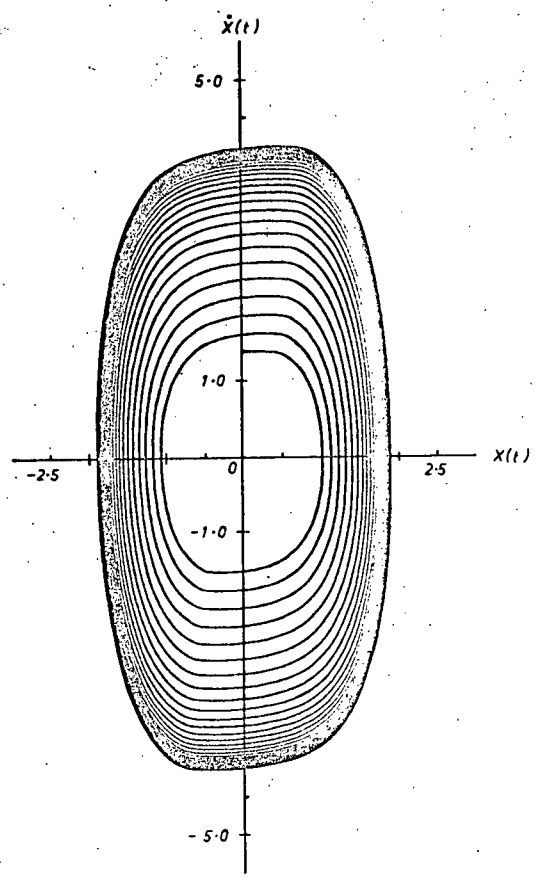

Fig. 5.2 Phase portrait of the solution to

$$\ddot{x} + x + 2x^3 - 0.1\ (1 - x^2)\ \dot{x} = 0$$

becomes

$$x(t) = 1.88\ \mathrm{Cn}\left[(8.06)^{1/2}\ t - 1.734\right]. \tag{87}$$

Differentiating equation (87) with respect to time, the maximum value of $\dot{x}(t)$ is obtained from the equation

$$\dot{x} = \dot{a}\ \mathrm{Cn}\psi - a\dot{\psi}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi .$$

When the stable oscillation is reached, $\dot{a} = 0$ and hence

$$\dot{x}_{max} = \left| a\dot{\psi}\,\mathrm{Sn}\psi\,\mathrm{Dn}\psi \right|_{max} .$$

For small values of k^2 (i.e. $k^2 \ll 1$) $|\mathrm{Sn}\psi\,\mathrm{Dn}\psi|$ will be close to its maximum value when $\psi = (2n+1)\ K$, and then

$$\dot{x}_{max} = a_s\ \dot{\psi}\ \left[1-k^2\right] \tag{88}$$

where

$$k^2 = \frac{pa_s^2}{2\left[1+pa_s^2\right]} \tag{89}$$

Finally, from equations (87), (88) and (89):

$$\dot{x}_{max} = 4.00 .$$

The phase portrait for equation (86), obtained from an analog computer simulation, is shown in figure 5.2, where the limit cycle behaviour may clearly be seen. Actual values of x_{max} and $\dot{x}_{max}$ (from this simulation) were 1.92 and 4.15 respectively, which compare favourably with the predicted value of 1.88 and 4.00.

5.4 The damped Duffing equation

For the Duffing equation

$$\ddot{x} + x + px^3 + \beta\dot{x} = 0 \qquad (p > 0) \tag{71}$$

with $x(0) = 1.0$ and $\dot{x}(0) = 0$, the function $\beta f(x,\dot{x},t)$ is, to a first approximation

$$\beta f(x,\dot{x},t) = \beta\dot{x} = -\beta a\ \mathrm{Sn}\psi\,\mathrm{Dn}\psi\ (1+pa^2)^{1/2} .$$

Following the method of section 3.2, and applying equation (33),

the expression defining $A^{(1)}$ may be obtained as:

$$A^{(1)}_{tt} + 4(1+\varepsilon pa^2)(1+k^2) = -2a\ (1+\varepsilon pa^2) \tag{83}$$

where

$$k^2 = \frac{pa^2}{2(1+pa^2)}$$

Hence, taking the particular integral of equation (83):

$$A^{(1)} = \frac{-a}{2(1+k^2)}$$

and

$$a(t) = \exp\left[\frac{-\beta t}{2(1+k^2)}\right] \tag{84}$$

From the analysis of section 5.2, the final solution for $x(t)$ is therefore

$$x(t) = \exp\left[\frac{-\beta t}{2(1+k^2)}\right] \mathrm{Cn}\left[\left[1 + p\exp\left[\frac{-\beta t}{1+k^2}\right]\right]^{1/2} t\right]$$

since $\theta(0)$, the initial phase, is zero (for the choice of initial conditions). The local maxima, minima and zeroes of the approximate solution can be obtained by a graphical solution of the equation

$$\psi(t) = \left[1 + p\exp\left[\frac{-\beta t}{1+k^2}\right]\right]^{1/2} t = n\,K\,(0) \qquad (n = 0,1,2,\ldots).$$

A solution of the equation

$$\ddot{x} + x + 2x^3 + \dot{x} = 0$$

with $x(0) = 1$, $\dot{x}(0) = 0$ is shown in figure 5.3, where although the parameter β in equation (71) is equal to 1, solution phase accuracy is seen to be maintained.

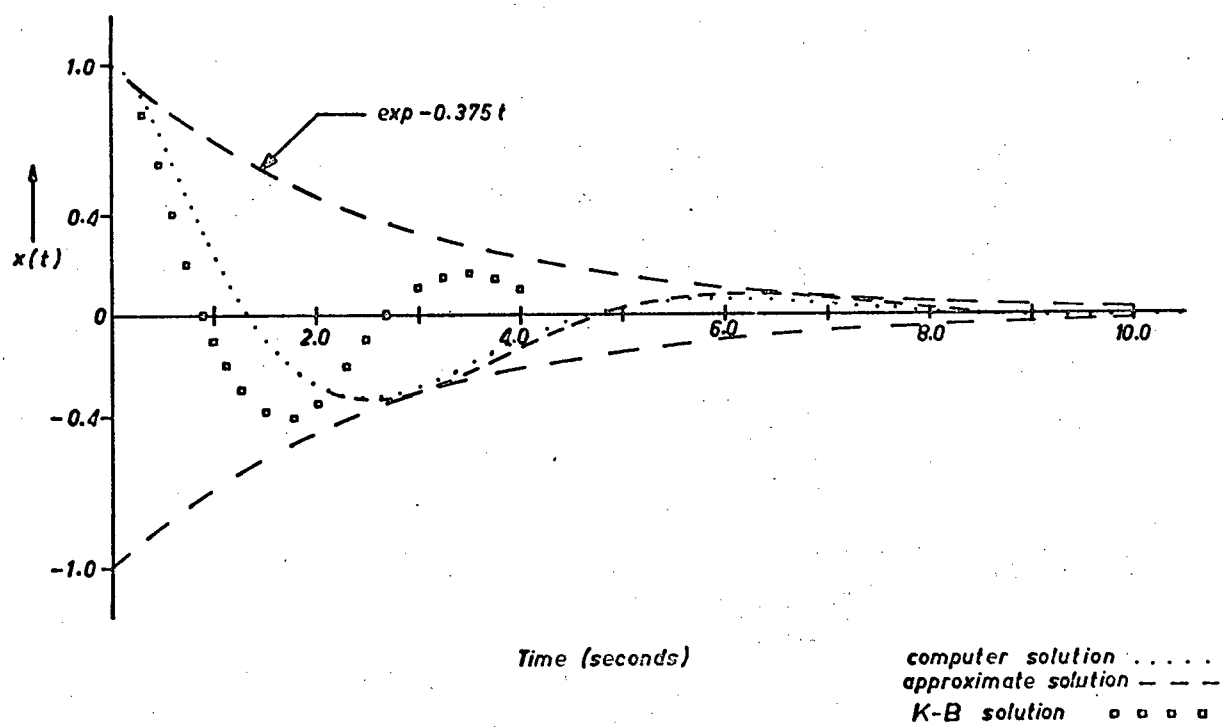

Fig. 5.3 Approximate and exact solutions of $\ddot{x} + x + 2x^3 + \dot{x} = 0$.

5.5 Comparison with the K-B Method

It has not previously been possible to make a comparison between the elliptic function approximation and the K-B approximation, as the latter cannot be applied to non-autonomous systems of the type considered in chapters 2-4. For the two nonlinear autonomous systems of section 5.3 and 5.4, however, such a comparison can be made.

Consider first the modified Van der Pol equation

$$\ddot{x} + x + px^3 - \beta(1 - x^2)\dot{x} = 0$$

with $x(0) = 0$, and assume (following the K-B method [13]) a solution of the form

$$x(t) = a(t) \sin(\omega t + \theta(t)) , \quad \text{where } \omega = 1.$$

Now the function $f(x,\dot{x})$ in the equation

$$\ddot{x} + \omega^2 x + \varepsilon f(x,\dot{x}) = 0$$

is given by $f(x,\dot{x}) = px^3 - \beta(1 - x^2)\dot{x}$,

assuming $\varepsilon = 1$.

Substituting for x and $\dot{x}$:

$$f(a \sin\psi, a\dot{\psi}\cos\psi) = \frac{pa^3}{4}\left[3\sin\psi - \sin 3\psi\right] + \beta\left[a\left[\frac{a^2}{4} - 1\right]\cos\psi - \frac{a^3}{4}\cos 3\psi\right]$$

where $\psi = \omega t + \theta$. The first order K-B approximation equations for $\dot{a}$ and $\dot{\theta}$ may then be obtained as [13]:

$$\dot{a} = \frac{\beta a}{2}\left[1 - \frac{a^2}{4}\right]$$

$$\theta = \frac{3}{8} pa^2$$

For the equation chosen in section 5.3, where $p = 2$, $\beta = 0.1$, and assuming $a(0) = 1.0$, the resulting K-B approximation solution is:

$$x(t) = \left[0.25 + 0.75 \exp(-0.1t)\right]^{-1/2} \sin(1.75t) .$$

The first one and a half periods of this solution are plotted in figure 5.1, where the superior accuracy of the elliptic function approximation is evident.

Now consider the heavily damped Duffing equation considered in section 5.4:

$$\ddot{x} + x + px^3 + \beta\dot{x} = 0 ,$$

with $x(0) = 1$ and $\dot{x}(0) = 0$. Assuming an approximate solution of the form

$$x(t) = a(t) \cos(\omega t + \theta(t)),$$

where again $\omega = 1$, the K-B approximations for $\dot{a}$ and $\dot{\theta}$ are:

$$\dot{a} = -\beta\frac{a}{2}$$

and

$$\dot{\theta} = \frac{3}{8}pa^2 .$$

For the equation

$$\ddot{x} + x + 2x^3 + \dot{x} = 0$$

with $x(0) = 1.0$ and $\dot{x}(0) = 0$, the K-B approximate solution is then

$$x(t) = \exp(-0.5t) \cos(1.75t) .$$

The first period of this solution is shown in figure 5.3, and again the elliptic function approximation is seen to be considerably more accurate.

In fairness to the K-B method it should be noted that both of the equations considered here are grossly nonlinear and, as such, are not strictly amenable to analysis by a quasi-linear approach. This comparison does, however, serve to accentuate the misrepresentation of nonlinear system performance when linearity, or even quasilinearity, is assumed.

5.6 Conclusion

The application of the refined approximation of chapters 3 and 4 to non-linear autonomous equations of the normalized form

$$\ddot{x} + x + rx^3 + \beta f(x,\dot{x}) = 0 \qquad (-1 < r < \infty)$$

is demonstrated in this chapter, by considering a modified Van der Pol equation [17] and a damped, forced Duffing equation [34]. It is shown, in section 5.2, that an explicit form of the solution to the equation

$$\ddot{x} + x + rx^3 + \beta\dot{x} = 0 \qquad (-1 < r < \infty)$$

can be obtained, making use of a complete cancellation of first-order terms in the expression for $\dot{\psi}$, where ψ is the argument of the elliptic function used in deriving the approximate solution. This is a significant result, and represents an improvement both in accuracy and simplicity over existing solution methods.

It is also shown in section 5.3, as the defining equations of the refined approximation (equations (33) and (54)) are independent of initial conditions, approximate solutions can readily be obtained for $x(0) = 0$ when $\dot{x}(0)$ is defined.

A comparison of solutions obtained from the elliptic function approach and from the K-B method, demonstrates the superior accuracy of the elliptic function approximation.

The analysis of this chapter completes the development and application of the first order elliptic function approximation to solutions of the normalized equation

$$\ddot{x} + x + rx^3 + \beta f(x,\dot{x},t) = 0 \qquad (-1 < r < \infty).$$

In chapter 6 some physical systems described by this equation are considered to demonstrate possible areas of application of the approximation method.

6. SOME APPLICATIONS

6.1 Introduction

The theory developed in the preceding chapters relates to second order differential equations exhibiting a cubic nonlinearity and small perturbing functions. It is the purpose of this chapter to show how such equations arise in practice by considering examples chosen from the fields of mechanics, astrophysics, circuit analysis and control systems. Detailed derivations have been avoided, the intention being to obtain differential equations corresponding to those chosen as examples in previous chapters.

6.2 Environmental studies of mechanical systems

As a familiar example of an oscillatory mechanical system, consider a particle, of mass m, suspended on a non-linear spring of the hardening type [6], which exerts a restoring force $F = c\,(x + px^3)$ $(p > 0)$. Suppose that the system is operating in an environment subject to rapid temperature change, and that the effect of linearly decreasing temperature on system performance is to be investigated.

Contraction of the spring resulting from the temperature decrease [19] will produce an increase in spring stiffness with time. If x is the displacement of the particle from its equilibrium position, then this increase in stiffness can be represented by a term βtx in the expression for the restoring force F of the spring:

$$F = c(x + px^3 + \beta tx)$$

where c, p and β are constant positive parameters, and β is small.

The equation of motion of the particle can then be written as

$$\ddot{x} + \frac{c}{m}\,(x + px^3 + \beta tx) = 0$$

or, taking for simplicity $\frac{c}{m} = 1$:

$$\ddot{x} + x + px^3 + \beta tx = 0 ,$$

which is the equation of the first example of chapters 2 and 3. The first example of chapter 4 applies to a spring with a softening characteristic [6], but subject to the same increase in stiffness as a result of decreasing temperature, and for which the restoring force F is of the form

$$F = c\,(x - px^3 + \beta tx)$$

Then, if $\frac{c}{m} = 1$, the equation of motion of the particle is

$$\ddot{x} + x - px^3 + \beta tx = 0 . \qquad (0 < p < 1)$$

Now consider the same mass-nonlinear spring system which is operating in an environment of varying viscosity, such as a liquid which is rapidly cooling [8]. If initially the viscosity is negligible, and increases linearly with time and decreasing temperature, then the damping force may be taken as $\lambda t\dot{x}$ $(\lambda > 0)$. The equation of motion of the particle is then

$$\ddot{x} + \frac{c}{m}(x + rx^3) + \frac{\lambda}{m}\, t\dot{x} = 0 .$$

Taking $\frac{c}{m} = 1$ and $\frac{\lambda}{m} = \beta$ gives, for $r > 0$, the equation considered in the second example of chapters 2 and 3, and for $-1 < r < 0$, the equation of the second example of chapter 4.

6.3 Stellar pulsation

Stars of the Cepheid type exhibit variations in luminosity which are of a periodic nature, and a possible explanation of this phenomenon is that the surface area of such stars is varying. If the star is assumed to be radially symmetric, quantitative results for the periodic variation in radius can be obtained by observing frequency changes in the emission spectrum of the Cepheid. As the velocity of the light-radiating source is readily determined from such observations, it is usual to plot radial velocity as a function of time. Such a characteristic is called a Cepheid velocity curve.

The phenomenon of stellar pulsation has attracted the attention of many astrophysicists, and W.S. Krogdahl [12] proposed a particularly interesting mathematical model. Krogdahl based his analysis on the existing idealized models of Cepheids which, after certain approximations, he derived in the form

$$\frac{d^2q}{d\tau^2} = -q + \frac{2}{3}q^2 + \frac{14}{27}q^3 + \dots + \frac{2}{3}(1-q)\left[\frac{dq}{d\tau}\right]^2 + \dots \quad .$$

Here q is given by $q = \frac{g(a^3, \tau)}{a^3}$ where 'a' is the mean or equilibrium radius of the star, $g(a^3, \tau)$ being a function which is assumed small in comparison with a^3.

From the assumptions that

a) Cepheids are statically unstable stars

b) Real thermodynamic systems are dissipative

and a somewhat heuristic argument, he then proposed that this equation be modified by the addition of a term

$$\mu\left[1 - \frac{q^2}{\lambda^2}\right]\frac{dq}{d\tau} ,$$

where μ and λ are small empirical constants.

The resulting equation is self-excited [6], and after a change of variable can be obtained in the final form

$$Q'' + Q - \frac{2}{3}\lambda Q^2 + \frac{14}{27}\lambda^2 Q^3 - \mu(1 - Q^2)Q' - \frac{2\lambda}{3}(1 - \lambda Q)(Q')^2 = 0 .$$

When $\lambda = 0$ this equation reduces to Van der Pol's equation (equation (69)) and therefore should, for small λ, exhibit limit-cycle behaviour in the $Q - Q'$ phase-plane. Krogdahl used a graphical technique to determine limit-cycle trajectories which exhibited behaviour similar to that of Cepheid velocity curves. To quote H.T. Davis [7] (p. 371), "the general argument of

the author seems to be confirmed".

In section 5.3 it was shown that the amplitude of the stable oscillation of the modified Van der Pol equation was $a_s = \left[\gamma\Delta\right]^{-1/2}$. Now for Krogdahl's equation, and comparing the terms

$$\beta(1 - \gamma x^2)\dot{x} \qquad \text{(from equation 70)} \qquad \text{and} \qquad \mu(1 - Q^2)\ Q',$$

it can be seen that $\gamma = 1$. Taking, for purposes of approximation, $\Delta = 0.25$, the final amplitude of the solution Q will be approximately 2. This implies that the term $\frac{14}{27}\lambda^2 Q^3$ is not negligible, and that Krogdahl's equation may be written in the form

$$Q'' + Q + \frac{14}{27}\lambda^2 Q^3 - \lambda\left[\frac{2}{3}Q^2 + \frac{\mu}{\lambda}(1 - Q^2)\ Q' + \frac{2}{3}(1 - \lambda Q)(Q')^2\right] = 0 .$$

This equation is comparable to that of the modified Van der Pol equation considered in section 5.3, and may be analysed in a similar manner.

6.4 Nonlinear frequency modulation

Many applications of non-linear and time-varying differential equations are to be found in the field of circuit theory, but the analyses of recent papers [35] - [43] are limited by existing techniques to a quasi-linear approach. The modified Mathieu equation [15], [17], considered in chapters 2, 3 and 4 is not restricted by considerations of quasi-linearity, and a circuit described by such an equation is discussed below.

Consider the circuit shown in figure 6.1.

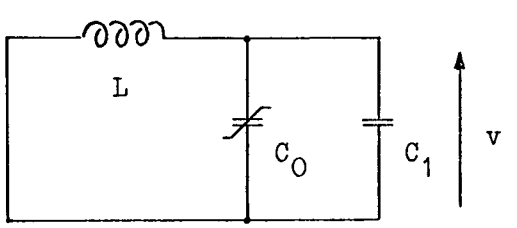

Fig. 6.1 L-C circuit with nonlinear and time-varying capacitors.

where L is a linear inductor, C_0 represents nonlinear capacitance effects in the circuit and C_1 is a time-varying capacitance device such as a varactor diode [39], [43]. Such a resonant circuit is found in frequency-modulated oscillators [6].

Suppose C_0 is a saturating capacitance such that

$$C_0 = a - bq^2$$

where 'a' and 'b' are positive constants, b is small and q is the total charge stored in the circuit.

If the varactor diode is modulated by a simple harmonic signal, then the capacitance C_1 may be written as [6]

$$C_1 = c\ (1 + d \cos \omega_0 t)$$

where c is the mean capacitance of the varactor diode, d is a small constant and ω_0 is the frequency of the modulating signal. Neglecting terms of order b^2 and d^2, the differential equation describing the system may be obtained in terms of the total stored charge q as:

$$\ddot{q} + \frac{q}{acL}\left[a + c + \frac{bc}{a}\ q^2 - ad \cos \omega_0 t \right] = 0 .$$

Taking, for simplicity, $p = \frac{bc}{a}$, $acL = 1$, $a + c = 1$ and $ad = \beta$, this equation becomes

$$\ddot{q} + q + pq^3 - \beta q \cos \omega_0 t = 0$$

which is the form considered by Minorsky [17] and in the final examples of chapters 2 and 3.

6.5 Nonlinear control systems

As a final example, a feedback control system containing a nonlinear element is considered. Such systems have attracted considerable attention [46] - [54], and control systems containing both linear and nonlinear com-

ponents of the form shown in figure 6.2 are generically designated as "problems of Lur'e" [51], provided $ef(e) > 0$ and $f(0) = 0$. An acceptable nonlinearity would be an odd function $f(e) = e + re^3$.

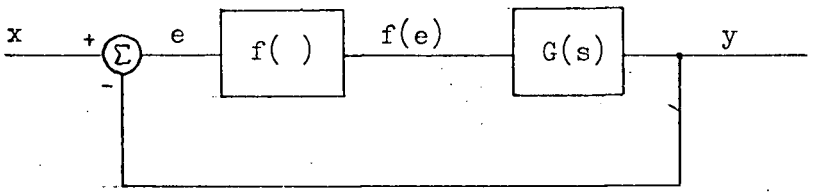

Fig. 6.2 The Lur'e type control system.

G(s) represents a linear element in operator form, and f( ) represents a nonlinear element.

To investigate control systems which can give rise to nonlinear differential equations of the type analysed in chapters 2-5, consider a servomechanism preceded by a saturating amplifier. The servomechanism may be represented by the transfer function

$$G(s) = \frac{c}{s^2 + as + b}$$

where a, b and c are constant parameters, and the saturating amplifier represented by the function $f(e) = e - pe^3$ $(0 < p < 1)$.

Then, if $x(0^-) = y(0^-)$ and $x(t) = 0$ $\forall\, t > 0$, and since $f(-e) = -f(e)$, the differential equation of the control system shown in figure 6.2 will be

$$\ddot{y} + a\dot{y} + y(b+c) - pcy^3 = 0 .$$

This is of the form considered in section 5.4 for the damped Duffing equation, and hence certain nonlinear control systems are amenable to a transient analysis by the methods developed in chapters 2-5.

6.6 Conclusion

In this chapter some possible areas of application of the approximation method have been indicated by considering four systems which result in equations of the form

$$\ddot{x} + x + rx^3 + \beta f(x,\dot{x},t) = 0 \qquad (-1 < r < \infty).$$

An exhaustive survey of practical applications was not intended, but the examples chosen serve to show some areas to which the general analysis pertains.

7. CONCLUSION

In this thesis a method has been presented for determining approximate solutions to a class of grossly nonlinear, non-autonomous second order differential equations characterized by

$$\frac{d^2x}{d\tau^2} + m^2 (x + rx^3) + \mu f (x, \frac{dx}{d\tau}, \tau) = 0 \qquad (-1 < r < \infty)$$

with the restriction that resonance effects be negligible. A solution is assumed in the form of a Jacobian elliptic function with variable amplitude, 'a', and a phase modification term, θ. The analysis takes, as its starting point, perturbation series representations of $\dot{a}$ and $\dot{\theta}$ similar to that chosen by Bogoliuboff and Mitropolsky. The approximate solution is obtained by integrating expressions for $\dot{a}$ and $\dot{\theta}$ but, in contrast to the K-B and B-M methods, no assumptions about solution frequency need be made. This is an important advance over existing techniques, as substantially less error is incurred during this integration.

The refined approximation takes into consideration the variation of solution frequency with amplitude, a characteristic of nonlinear systems. Factors involving k, the modulus of the elliptic function, are introduced into the expressions for $\dot{a}$ and $\dot{\theta}$ by this approach and give solutions which are directly related to the degree of nonlinearity. This feature is unique to the elliptic function approximation, and is particularly important in the analysis of grossly nonlinear oscillatory systems.

It is shown that a differential equation which will generate the approximate solution can be formulated for any particular example, where the coefficients are derivatives of the amplitude envelope and the argument of the elliptic function. The error of the approximate solution can then be determined, and an integral-error definition is used in deriving error results for the three general examples considered in the analysis. In practice

approximate solutions show much greater accuracy than would be expected from the error analysis, and use of the integral error diagram leads to a pessimistic estimate of solution accuracy.

In the application of the analysis to autonomous systems, a comparison is possible between results obtained from the elliptic function approach and those obtained from the K-B method. Such a comparison cannot be made for non-autonomous systems of the type considered here because of limitations of the K-B method. It may be seen that such a task exceeds the capabilities of the K-B method, but first-order approximate solutions of high accuracy are obtained from the elliptic function approach. Because of an exact cancellation of certain first order terms, a solution of the heavily damped Duffing equation

$$\ddot{x} + x + rx^3 + \beta\dot{x} = 0 \qquad (-1 < r < \infty)$$

can be obtained in a simple but accurate form which represents a substantial improvement in accuracy over existing approximate solutions.

A useful extension of this work would be to include systems involving time-delay, and also to analyse the effect of retaining more terms in the perturbation series. The analysis can, in principle, be extended to any order of accuracy at the expense of tedium. Polynomial truncation to a cubic would enable approximate solutions of the equation

$$\ddot{x} + m^2 (x + rx^3) + \sum_{n=2}^{\infty} a_{(2n+1)} x^{(2n+1)} + \beta f(x,\dot{x},t) = 0$$

to be determined. This problem also merits attention.

In conclusion, a new analytical method has been presented for investigating the transient response of non-resonant, non-autonomous, grossly nonlinear second order systems. The approximation technique is shown to have application to the fields of astrophysics, mechanical systems, electronic

oscillators, frequency modulation processes and nonlinear control systems, but can in general be applied to most oscillatory nonlinear problems where the K-B method has been used in the past.

8. REFERENCES

The references given below are listed alphabetically, and are followed by four sub-sections containing references to recent work on Van der Pol's equation, the Duffing equation, circuit theory and control systems. Some books, in particular Cunningham [6], Minorsky [17] and Bowman [5], are referenced repeatedly in the text. For information on a specific topic the reader is referred to the index of the book in question.

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9. APPENDIX

9.1 Elliptic function relationships

$$Sn^2 u + Cn^2 u = 1$$

$$Dn^2 u + k^2 \, Sn^2 u = 1$$

$$k' = \left[1 - k^2\right]^{1/2}$$

$$\frac{d}{du}(Snu) = Cnu \; Dnu$$

$$\frac{d}{du}(Cnu) = - Snu \; Dnu$$

$$\frac{d}{du}(Dnu) = - k^2 \; Snu \; Cnu$$

$$K = \int_0^{\pi/2} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}}$$

$$K' = \int_0^{\pi/2} \frac{d\phi}{(1 - (k')^2 \sin^2 \phi)^{1/2}}$$

These formulae summarize the relationships of elliptic functions used, or implied, in the text. For additional information the reader is referred to Bowman [5], which also contains a selected bibliography on elliptic functions.

9.2 Tabulation of ε and δ

The constants ε and δ, defined by equations (10) and (53) respectively, were evaluated using the UBC IBM 360/67 digital computer. The constants p, k^2, ε and K appear in table 1, for increments in p of 0.01. The constants p, k^2, δ and K are tabulated in Table 2, for increments in p of 0.005.

p	k^2	ε	K	p	k^2	ε	K
0.010	0.0050	0.6664	1.5727	0.510	0.1689	0.6585	1.6443
0.020	0.0098	0.6662	1.5747	0.520	0.1711	0.6584	1.6453
0.030	0.0146	0.6660	1.5766	0.530	0.1732	0.6583	1.6464
0.040	0.0192	0.6658	1.5784	0.540	0.1753	0.6581	1.6474
0.050	0.0238	0.6656	1.5803	0.550	0.1774	0.6580	1.6484
0.060	0.0283	0.6654	1.5821	0.560	0.1795	0.6579	1.6494
0.070	0.0327	0.6652	1.5839	0.570	0.1815	0.6578	1.6504
0.080	0.0370	0.6650	1.5856	0.580	0.1835	0.6577	1.6514
0.090	0.0413	0.6648	1.5874	0.590	0.1855	0.6576	1.6524
0.100	0.0455	0.6646	1.5891	0.600	0.1875	0.6575	1.6534
0.110	0.0495	0.6644	1.5908	0.610	0.1894	0.6574	1.6543
0.120	0.0536	0.6642	1.5925	0.620	0.1914	0.6573	1.6553
0.130	0.0575	0.6640	1.5941	0.630	0.1933	0.6572	1.6562
0.140	0.0614	0.6639	1.5958	0.640	0.1951	0.6571	1.6572
0.150	0.0652	0.6637	1.5974	0.650	0.1970	0.6570	1.6581
0.160	0.0690	0.6635	1.5990	0.660	0.1988	0.6569	1.6590
0.170	0.0726	0.6633	1.6005	0.670	0.2006	0.6568	1.6599
0.180	0.0763	0.6632	1.6021	0.680	0.2024	0.6567	1.6608
0.190	0.0798	0.6630	1.6036	0.690	0.2041	0.6566	1.6617
0.200	0.0833	0.6628	1.6051	0.700	0.2059	0.6565	1.6626
0.210	0.0868	0.6626	1.6066	0.710	0.2076	0.6564	1.6635
0.220	0.0902	0.6625	1.6081	0.720	0.2093	0.6563	1.6643
0.230	0.0935	0.6623	1.6096	0.730	0.2110	0.6562	1.6652
0.240	0.0968	0.6622	1.6110	0.740	0.2126	0.6561	1.6661
0.250	0.1000	0.6620	1.6124	0.750	0.2143	0.6560	1.6669
0.260	0.1032	0.6618	1.6138	0.760	0.2159	0.6559	1.6677
0.270	0.1063	0.6617	1.6152	0.770	0.2175	0.6558	1.6686
0.280	0.1094	0.6615	1.6166	0.780	0.2191	0.6558	1.6694
0.290	0.1124	0.6614	1.6180	0.790	0.2207	0.6557	1.6702
0.300	0.1154	0.6612	1.6193	0.800	0.2222	0.6556	1.6710
0.310	0.1183	0.6611	1.6206	0.810	0.2238	0.6555	1.6718
0.320	0.1212	0.6609	1.6219	0.820	0.2253	0.6554	1.6726
0.330	0.1241	0.6608	1.6232	0.330	0.2268	0.6553	1.6734
0.340	0.1269	0.6607	1.6245	0.840	0.2283	0.6552	1.6742
0.350	0.1296	0.6605	1.6258	0.850	0.2297	0.6552	1.6749
0.360	0.1324	0.6604	1.6270	0.860	0.2312	0.6551	1.6757
0.370	0.1350	0.6602	1.6283	0.870	0.2326	0.6550	1.6765
0.380	0.1377	0.6601	1.6295	0.880	0.2340	0.6549	1.6772
0.390	0.1403	0.6600	1.6307	0.890	0.2354	0.6548	1.6779
0.400	0.1429	0.6598	1.6319	0.900	0.2368	0.6547	1.6787
0.410	0.1454	0.6597	1.6331	0.910	0.2382	0.6547	1.6794
0.420	0.1479	0.6596	1.6343	0.920	0.2396	0.6546	1.6801
0.430	0.1503	0.6595	1.6354	0.930	0.2409	0.6545	1.6809
0.440	0.1528	0.6593	1.6366	0.940	0.2423	0.6544	1.6816
0.450	0.1552	0.6592	1.6377	0.950	0.2436	0.6544	1.6823
0.460	0.1575	0.6591	1.6388	0.960	0.2449	0.6543	1.6830
0.470	0.1599	0.6590	1.6399	0.970	0.2462	0.6542	1.6837
0.480	0.1622	0.6588	1.6410	0.980	0.2475	0.6541	1.6844
0.490	0.1644	0.6587	1.6421	0.990	0.2487	0.6541	1.6851
0.500	0.1667	0.6586	1.6432	1.000	0.2500	0.6540	1.6857

Table 1

p	k^2	ε	K	p	k^2	ε	K
1.010	0.2512	0.6539	1.6864	1.510	0.3008	0.6509	1.7144
1.020	0.2525	0.6538	1.6871	1.520	0.3016	0.6509	1.7148
1.030	0.2537	0.6538	1.6878	1.530	0.3024	0.6508	1.7153
1.040	0.2549	0.6537	1.6884	1.540	0.3031	0.6508	1.7157
1.050	0.2561	0.6536	1.6891	1.550	0.3039	0.6507	1.7162
1.060	0.2573	0.6536	1.6897	1.560	0.3047	0.6507	1.7166
1.070	0.2585	0.6535	1.6904	1.570	0.3054	0.6506	1.7171
1.080	0.2596	0.6534	1.6910	1.580	0.3062	0.6506	1.7175
1.090	0.2608	0.6534	1.6916	1.590	0.3069	0.6506	1.7180
1.100	0.2619	0.6533	1.6923	1.600	0.3077	0.6505	1.7184
1.110	0.2630	0.6532	1.6929	1.610	0.3084	0.6505	1.7188
1.120	0.2641	0.6532	1.6935	1.620	0.3092	0.6504	1.7193
1.130	0.2653	0.6531	1.6941	1.630	0.3099	0.6504	1.7197
1.140	0.2664	0.6530	1.6947	1.640	0.3106	0.6503	1.7201
1.150	0.2674	0.6530	1.6953	1.650	0.3113	0.6503	1.7206
1.160	0.2685	0.6529	1.6959	1.660	0.3120	0.6502	1.7210
1.170	0.2696	0.6528	1.6965	1.670	0.3127	0.6502	1.7214
1.180	0.2706	0.6528	1.6971	1.680	0.3134	0.6501	1.7218
1.190	0.2717	0.6527	1.6977	1.690	0.3141	0.6501	1.7222
1.200	0.2727	0.6526	1.6983	1.700	0.3148	0.6501	1.7227
1.210	0.2738	0.6526	1.6988	1.710	0.3155	0.6500	1.7231
1.220	0.2748	0.6525	1.6994	1.720	0.3162	0.6500	1.7235
1.230	0.2758	0.6525	1.7000	1.730	0.3168	0.6499	1.7239
1.240	0.2768	0.6524	1.7006	1.740	0.3175	0.6499	1.7243
1.250	0.2778	0.6523	1.7011	1.750	0.3182	0.6498	1.7247
1.260	0.2788	0.6523	1.7017	1.760	0.3188	0.6498	1.7251
1.270	0.2797	0.6522	1.7022	1.770	0.3195	0.6498	1.7255
1.280	0.2807	0.6522	1.7028	1.780	0.3201	0.6497	1.7259
1.290	0.2817	0.6521	1.7033	1.790	0.3208	0.6497	1.7262
1.300	0.2826	0.6521	1.7039	1.800	0.3214	0.6496	1.7266
1.310	0.2835	0.6520	1.7044	1.810	0.3221	0.6496	1.7270
1.320	0.2845	0.6519	1.7049	1.820	0.3227	0.6496	1.7274
1.330	0.2854	0.6519	1.7054	1.830	0.3233	0.6495	1.7278
1.340	0.2863	0.6518	1.7060	1.840	0.3239	0.6495	1.7282
1.350	0.2872	0.6518	1.7065	1.850	0.3246	0.6494	1.7285
1.360	0.2881	0.6517	1.7070	1.860	0.3252	0.6494	1.7289
1.370	0.2890	0.6517	1.7075	1.870	0.3258	0.6494	1.7293
1.380	0.2899	0.6516	1.7080	1.880	0.3264	0.6493	1.7296
1.390	0.2908	0.6516	1.7085	1.890	0.3270	0.6493	1.7300
1.400	0.2917	0.6515	1.7090	1.900	0.3276	0.6492	1.7304
1.410	0.2925	0.6514	1.7095	1.910	0.3282	0.6492	1.7307
1.420	0.2934	0.6514	1.7100	1.920	0.3288	0.6492	1.7311
1.430	0.2942	0.6513	1.7105	1.930	0.3293	0.6491	1.7315
1.440	0.2951	0.6513	1.7110	1.940	0.3299	0.6491	1.7318
1.450	0.2959	0.6512	1.7115	1.950	0.3305	0.6491	1.7322
1.460	0.2967	0.6512	1.7120	1.960	0.3311	0.6490	1.7325
1.470	0.2976	0.6511	1.7125	1.970	0.3316	0.6490	1.7329
1.480	0.2984	0.6511	1.7129	1.980	0.3322	0.6489	1.7332
1.490	0.2992	0.6510	1.7134	1.990	0.3328	0.6489	1.7336
1.500	0.3000	0.6510	1.7139	2.000	0.3333	0.6489	1.7339

Table 1

p	k^2	ε	K	p	k^2	ε	K
2.010	0.3339	0.6488	1.7343	2.510	0.3575	0.6473	1.7492
2.020	0.3344	0.6488	1.7346	2.520	0.3579	0.6472	1.7494
2.030	0.3350	0.6488	1.7349	2.530	0.3584	0.6472	1.7497
2.040	0.3355	0.6487	1.7353	2.540	0.3588	0.6472	1.7499
2.050	0.3361	0.6487	1.7356	2.550	0.3592	0.6472	1.7502
2.060	0.3366	0.6487	1.7359	2.560	0.3595	0.6471	1.7505
2.070	0.3371	0.6486	1.7363	2.570	0.3599	0.6471	1.7507
2.080	0.3377	0.6486	1.7366	2.580	0.3603	0.6471	1.7510
2.090	0.3382	0.6486	1.7369	2.590	0.3607	0.6471	1.7512
2.100	0.3387	0.6485	1.7372	2.600	0.3611	0.6470	1.7515
2.110	0.3392	0.6485	1.7376	2.610	0.3615	0.6470	1.7517
2.120	0.3397	0.6485	1.7379	2.620	0.3619	0.6470	1.7520
2.130	0.3403	0.6484	1.7382	2.630	0.3623	0.6470	1.7522
2.140	0.3408	0.6484	1.7385	2.640	0.3626	0.6469	1.7525
2.150	0.3413	0.6484	1.7388	2.650	0.3630	0.6469	1.7527
2.160	0.3418	0.6483	1.7392	2.660	0.3634	0.6469	1.7529
2.170	0.3423	0.6483	1.7395	2.670	0.3638	0.6469	1.7532
2.180	0.3428	0.6483	1.7398	2.680	0.3641	0.6468	1.7534
2.190	0.3433	0.6482	1.7401	2.690	0.3645	0.6468	1.7537
2.200	0.3437	0.6482	1.7404	2.700	0.3649	0.6468	1.7539
2.210	0.3442	0.6482	1.7407	2.710	0.3652	0.6468	1.7541
2.220	0.3447	0.6481	1.7410	2.720	0.3656	0.6467	1.7544
2.230	0.3452	0.6481	1.7413	2.730	0.3659	0.6467	1.7546
2.240	0.3457	0.6481	1.7416	2.740	0.3663	0.6467	1.7548
2.250	0.3461	0.6480	1.7419	2.750	0.3667	0.6467	1.7551
2.260	0.3466	0.6480	1.7422	2.760	0.3670	0.6466	1.7553
2.270	0.3471	0.6480	1.7425	2.770	0.3674	0.6466	1.7555
2.280	0.3476	0.6479	1.7428	2.780	0.3677	0.6466	1.7558
2.290	0.3480	0.6479	1.7431	2.790	0.3681	0.6466	1.7560
2.300	0.3485	0.6479	1.7434	2.800	0.3684	0.6465	1.7562
2.310	0.3489	0.6478	1.7437	2.810	0.3688	0.6465	1.7565
2.320	0.3494	0.6478	1.7440	2.820	0.3691	0.6465	1.7567
2.330	0.3498	0.6478	1.7442	2.830	0.3694	0.6465	1.7569
2.340	0.3503	0.6478	1.7445	2.840	0.3698	0.6464	1.7571
2.350	0.3507	0.6477	1.7448	2.850	0.3701	0.6464	1.7573
2.360	0.3512	0.6477	1.7451	2.860	0.3705	0.6464	1.7576
2.370	0.3516	0.6477	1.7454	2.870	0.3708	0.6464	1.7578
2.380	0.3521	0.6476	1.7457	2.880	0.3711	0.6464	1.7580
2.390	0.3525	0.6476	1.7459	2.890	0.3715	0.6463	1.7582
2.400	0.3529	0.6476	1.7462	2.900	0.3718	0.6463	1.7584
2.410	0.3534	0.6476	1.7465	2.910	0.3721	0.6463	1.7587
2.420	0.3538	0.6475	1.7468	2.920	0.3724	0.6463	1.7589
2.430	0.3542	0.6475	1.7470	2.930	0.3728	0.6462	1.7591
2.440	0.3546	0.6475	1.7473	2.940	0.3731	0.6462	1.7593
2.450	0.3551	0.6474	1.7476	2.950	0.3734	0.6462	1.7595
2.460	0.3555	0.6474	1.7478	2.960	0.3737	0.6462	1.7597
2.470	0.3559	0.6474	1.7481	2.970	0.3741	0.6462	1.7599
2.480	0.3563	0.6474	1.7484	2.980	0.3744	0.6461	1.7601
2.490	0.3567	0.6473	1.7486	2.990	0.3747	0.6461	1.7604
2.500	0.3571	0.6473	1.7489	3.000	0.3750	0.6461	1.7606

Table 1

p	k^2	ε	K	p	k^2	ε	K
3.010	0.3753	0.6461	1.7608	3.510	0.3891	0.6451	1.7701
3.020	0.3756	0.6460	1.7610	3.520	0.3894	0.6451	1.7702
3.030	0.3759	0.6460	1.7612	3.530	0.3896	0.6451	1.7704
3.040	0.3762	0.6460	1.7614	3.540	0.3899	0.6451	1.7706
3.050	0.3765	0.6460	1.7616	3.550	0.3901	0.6450	1.7707
3.060	0.3768	0.6460	1.7618	3.560	0.3903	0.6450	1.7709
3.070	0.3771	0.6459	1.7620	3.570	0.3906	0.6450	1.7710
3.080	0.3774	0.6459	1.7622	3.580	0.3908	0.6450	1.7712
3.090	0.3777	0.6459	1.7624	3.590	0.3911	0.6450	1.7714
3.100	0.3780	0.6459	1.7626	3.600	0.3913	0.6450	1.7715
3.110	0.3783	0.6459	1.7628	3.610	0.3915	0.6449	1.7717
3.120	0.3786	0.6458	1.7630	3.620	0.3918	0.6449	1.7719
3.130	0.3789	0.6458	1.7632	3.630	0.3920	0.6449	1.7720
3.140	0.3792	0.6458	1.7634	3.640	0.3922	0.6449	1.7722
3.150	0.3795	0.6458	1.7636	3.650	0.3925	0.6449	1.7723
3.160	0.3798	0.6458	1.7638	3.660	0.3927	0.6449	1.7725
3.170	0.3801	0.6457	1.7640	3.670	0.3929	0.6448	1.7726
3.180	0.3804	0.6457	1.7641	3.680	0.3932	0.6448	1.7728
3.190	0.3807	0.6457	1.7643	3.690	0.3934	0.6448	1.7730
3.200	0.3809	0.6457	1.7645	3.700	0.3936	0.6448	1.7731
3.210	0.3812	0.6457	1.7647	3.710	0.3938	0.6448	1.7733
3.220	0.3815	0.6456	1.7649	3.720	0.3941	0.6448	1.7734
3.230	0.3818	0.6456	1.7651	3.730	0.3943	0.6447	1.7736
3.240	0.3821	0.6456	1.7653	3.740	0.3945	0.6447	1.7737
3.250	0.3823	0.6456	1.7655	3.750	0.3947	0.6447	1.7739
3.260	0.3826	0.6456	1.7657	3.760	0.3950	0.6447	1.7740
3.270	0.3829	0.6455	1.7658	3.770	0.3952	0.6447	1.7742
3.280	0.3832	0.6455	1.7660	3.780	0.3954	0.6447	1.7743
3.290	0.3834	0.6455	1.7662	3.790	0.3956	0.6447	1.7745
3.300	0.3837	0.6455	1.7664	3.800	0.3958	0.6446	1.7746
3.310	0.3840	0.6455	1.7666	3.810	0.3960	0.6446	1.7748
3.320	0.3843	0.6455	1.7668	3.820	0.3963	0.6446	1.7749
3.330	0.3845	0.6454	1.7669	3.830	0.3965	0.6446	1.7751
3.340	0.3848	0.6454	1.7671	3.840	0.3967	0.6446	1.7752
3.350	0.3851	0.6454	1.7673	3.850	0.3969	0.6446	1.7754
3.360	0.3853	0.6454	1.7675	3.860	0.3971	0.6445	1.7755
3.370	0.3856	0.6454	1.7676	3.870	0.3973	0.6445	1.7757
3.380	0.3858	0.6453	1.7678	3.880	0.3975	0.6445	1.7758
3.390	0.3861	0.6453	1.7680	3.890	0.3977	0.6445	1.7760
3.400	0.3864	0.6453	1.7682	3.900	0.3980	0.6445	1.7761
3.410	0.3866	0.6453	1.7683	3.910	0.3982	0.6445	1.7762
3.420	0.3869	0.6453	1.7685	3.920	0.3984	0.6445	1.7764
3.430	0.3871	0.6453	1.7687	3.930	0.3986	0.6444	1.7765
3.440	0.3874	0.6452	1.7689	3.940	0.3988	0.6444	1.7767
3.450	0.3876	0.6452	1.7690	3.950	0.3990	0.6444	1.7768
3.460	0.3879	0.6452	1.7692	3.960	0.3992	0.6444	1.7770
3.470	0.3881	0.6452	1.7694	3.970	0.3994	0.6444	1.7771
3.480	0.3884	0.6452	1.7695	3.980	0.3996	0.6444	1.7772
3.490	0.3886	0.6451	1.7697	3.990	0.3998	0.6444	1.7774
3.500	0.3889	0.6451	1.7699	4.000	0.4000	0.6443	1.7775

Table 1

p	k^2	ε	K	p	k^2	ε	K
4.010	0.4002	0.6443	1.7777	4.510	0.4093	0.6437	1.7840
4.020	0.4004	0.6443	1.7778	4.520	0.4094	0.6437	1.7841
4.030	0.4006	0.6443	1.7779	4.530	0.4096	0.6437	1.7842
4.040	0.4008	0.6443	1.7781	4.540	0.4097	0.6436	1.7843
4.050	0.4010	0.6443	1.7782	4.550	0.4099	0.6436	1.7844
4.060	0.4012	0.6443	1.7783	4.560	0.4101	0.6436	1.7846
4.070	0.4014	0.6442	1.7785	4.570	0.4102	0.6436	1.7847
4.080	0.4016	0.6442	1.7786	4.580	0.4104	0.6436	1.7848
4.090	0.4018	0.6442	1.7787	4.590	0.4106	0.6436	1.7849
4.100	0.4020	0.6442	1.7789	4.600	0.4107	0.6436	1.7850
4.110	0.4021	0.6442	1.7790	4.610	0.4109	0.6436	1.7851
4.120	0.4023	0.6442	1.7791	4.620	0.4110	0.6435	1.7852
4.130	0.4025	0.6442	1.7793	4.630	0.4112	0.6435	1.7854
4.140	0.4027	0.6441	1.7794	4.640	0.4113	0.6435	1.7855
4.150	0.4029	0.6441	1.7795	4.650	0.4115	0.6435	1.7856
4.160	0.4031	0.6441	1.7797	4.660	0.4117	0.6435	1.7857
4.170	0.4033	0.6441	1.7798	4.670	0.4118	0.6435	1.7858
4.180	0.4035	0.6441	1.7799	4.680	0.4120	0.6435	1.7859
4.190	0.4037	0.6441	1.7801	4.690	0.4121	0.6435	1.7860
4.200	0.4038	0.6441	1.7802	4.700	0.4123	0.6435	1.7861
4.210	0.4040	0.6441	1.7803	4.710	0.4124	0.6434	1.7862
4.220	0.4042	0.6440	1.7804	4.720	0.4126	0.6434	1.7863
4.230	0.4044	0.6440	1.7806	4.730	0.4127	0.6434	1.7865
4.240	0.4046	0.6440	1.7807	4.740	0.4129	0.6434	1.7866
4.250	0.4048	0.6440	1.7808	4.750	0.4130	0.6434	1.7867
4.260	0.4049	0.6440	1.7810	4.760	0.4132	0.6434	1.7868
4.270	0.4051	0.6440	1.7811	4.770	0.4133	0.6434	1.7869
4.280	0.4053	0.6440	1.7812	4.780	0.4135	0.6434	1.7870
4.290	0.4055	0.6439	1.7813	4.790	0.4136	0.6434	1.7871
4.300	0.4057	0.6439	1.7815	4.800	0.4138	0.6433	1.7872
4.310	0.4058	0.6439	1.7816	4.810	0.4139	0.6433	1.7873
4.320	0.4060	0.6439	1.7817	4.820	0.4141	0.6433	1.7874
4.330	0.4062	0.6439	1.7818	4.830	0.4142	0.6433	1.7875
4.340	0.4064	0.6439	1.7820	4.840	0.4144	0.6433	1.7876
4.350	0.4065	0.6439	1.7821	4.850	0.4145	0.6433	1.7877
4.360	0.4067	0.6439	1.7822	4.860	0.4147	0.6433	1.7878
4.370	0.4069	0.6438	1.7823	4.870	0.4148	0.6433	1.7879
4.380	0.4071	0.6438	1.7824	4.880	0.4150	0.6433	1.7880
4.390	0.4072	0.6438	1.7826	4.890	0.4151	0.6433	1.7881
4.400	0.4074	0.6438	1.7827	4.900	0.4153	0.6432	1.7882
4.410	0.4076	0.6438	1.7828	4.910	0.4154	0.6432	1.7883
4.420	0.4077	0.6438	1.7829	4.920	0.4155	0.6432	1.7884
4.430	0.4079	0.6438	1.7830	4.930	0.4157	0.6432	1.7885
4.440	0.4081	0.6438	1.7832	4.940	0.4158	0.6432	1.7886
4.450	0.4083	0.6438	1.7833	4.950	0.4160	0.6432	1.7888
4.460	0.4084	0.6437	1.7834	4.960	0.4161	0.6432	1.7889
4.470	0.4086	0.6437	1.7835	4.970	0.4162	0.6432	1.7890
4.480	0.4088	0.6437	1.7836	4.980	0.4164	0.6432	1.7891
4.490	0.4089	0.6437	1.7838	4.990	0.4165	0.6431	1.7892
4.500	0.4091	0.6437	1.7839	5.000	0.4167	0.6431	1.7893

Table 1

p	k^2	δ	K	p	k^2	δ	K
0.005	0.0025	0.6668	1.5718	0.255	0.1461	0.6737	1.6334
0.010	0.0050	0.6669	1.5728	0.260	0.1494	0.6739	1.6350
0.015	0.0076	0.6670	1.5738	0.265	0.1527	0.6741	1.6365
0.020	0.0101	0.6671	1.5748	0.270	0.1561	0.6742	1.6381
0.025	0.0127	0.6672	1.5758	0.275	0.1594	0.6744	1.6397
0.030	0.0152	0.6673	1.5768	0.280	0.1628	0.6746	1.6413
0.035	0.0178	0.6675	1.5779	0.285	0.1662	0.6748	1.6430
0.040	0.0204	0.6676	1.5789	0.290	0.1696	0.6750	1.6446
0.045	0.0230	0.6677	1.5800	0.295	0.1730	0.6751	1.6463
0.050	0.0256	0.6678	1.5810	0.300	0.1765	0.6753	1.6479
0.055	0.0283	0.6679	1.5821	0.305	0.1799	0.6755	1.6496
0.060	0.0309	0.6681	1.5832	0.310	0.1834	0.6757	1.6514
0.065	0.0336	0.6682	1.5842	0.315	0.1869	0.6759	1.6531
0.070	0.0363	0.6683	1.5853	0.320	0.1905	0.6761	1.6549
0.075	0.0390	0.6684	1.5864	0.325	0.1940	0.6763	1.6566
0.080	0.0417	0.6686	1.5876	0.330	0.1976	0.6765	1.6584
0.085	0.0444	0.6687	1.5887	0.335	0.2012	0.6767	1.6602
0.090	0.0471	0.6688	1.5898	0.340	0.2048	0.6769	1.6621
0.095	0.0499	0.6689	1.5909	0.345	0.2085	0.6771	1.6639
0.100	0.0526	0.6691	1.5921	0.350	0.2121	0.6773	1.6658
0.105	0.0554	0.6692	1.5933	0.355	0.2158	0.6775	1.6677
0.110	0.0582	0.6693	1.5944	0.360	0.2195	0.6777	1.6696
0.115	0.0610	0.6695	1.5956	0.365	0.2232	0.6779	1.6715
0.120	0.0638	0.6696	1.5968	0.370	0.2270	0.6782	1.6735
0.125	0.0667	0.6697	1.5980	0.375	0.2308	0.6784	1.6755
0.130	0.0695	0.6699	1.5992	0.380	0.2346	0.6786	1.6775
0.135	0.0724	0.6700	1.6004	0.385	0.2384	0.6788	1.6795
0.140	0.0753	0.6702	1.6017	0.390	0.2422	0.6791	1.6816
0.145	0.0782	0.6703	1.6029	0.395	0.2461	0.6793	1.6836
0.150	0.0811	0.6704	1.6042	0.400	0.2500	0.6795	1.6857
0.155	0.0840	0.6706	1.6054	0.405	0.2539	0.6798	1.6879
0.160	0.0870	0.6707	1.6067	0.410	0.2579	0.6800	1.6900
0.165	0.0899	0.6709	1.6080	0.415	0.2618	0.6802	1.6922
0.170	0.0929	0.6710	1.6093	0.420	0.2658	0.6805	1.6944
0.175	0.0959	0.6712	1.6106	0.425	0.2698	0.6807	1.6967
0.180	0.0989	0.6713	1.6120	0.430	0.2739	0.6810	1.6989
0.185	0.1019	0.6715	1.6133	0.435	0.2780	0.6812	1.7012
0.190	0.1050	0.6716	1.6146	0.440	0.2821	0.6815	1.7035
0.195	0.1080	0.6718	1.6160	0.445	0.2862	0.6817	1.7059
0.200	0.1111	0.6719	1.6174	0.450	0.2903	0.6820	1.7083
0.205	0.1142	0.6721	1.6188	0.455	0.2945	0.6823	1.7107
0.210	0.1173	0.6722	1.6202	0.460	0.2987	0.6825	1.7131
0.215	0.1204	0.6724	1.6216	0.465	0.3029	0.6828	1.7156
0.220	0.1236	0.6725	1.6230	0.470	0.3072	0.6831	1.7181
0.225	0.1268	0.6727	1.6245	0.475	0.3115	0.6834	1.7207
0.230	0.1299	0.6729	1.6259	0.480	0.3158	0.6836	1.7232
0.235	0.1331	0.6730	1.6274	0.485	0.3201	0.6839	1.7259
0.240	0.1364	0.6732	1.6289	0.490	0.3245	0.6842	1.7285
0.245	0.1396	0.6734	1.6304	0.495	0.3289	0.6845	1.7312
0.250	0.1429	0.6735	1.6319	0.500	0.3333	0.6848	1.7339

Table 2

p	k^2	δ	K	p	k^2	δ	K
0.505	0.3378	0.6851	1.7367	0.755	0.6064	0.7084	1.9566
0.510	0.3423	0.6854	1.7395	0.760	0.6129	0.7091	1.9638
0.515	0.3468	0.6857	1.7423	0.765	0.6194	0.7099	1.9711
0.520	0.3514	0.6860	1.7452	0.770	0.6260	0.7107	1.9787
0.525	0.3559	0.6864	1.7481	0.775	0.6327	0.7115	1.9865
0.530	0.3605	0.6867	1.7511	0.780	0.6393	0.7123	1.9945
0.535	0.3652	0.6870	1.7541	0.785	0.6461	0.7131	2.0027
0.540	0.3699	0.6873	1.7572	0.790	0.6529	0.7140	2.0112
0.545	0.3746	0.6877	1.7603	0.795	0.6597	0.7148	2.0200
0.550	0.3793	0.6880	1.7634	0.800	0.6667	0.7157	2.0290
0.555	0.3841	0.6884	1.7666	0.805	0.6736	0.7167	2.0382
0.560	0.3889	0.6887	1.7699	0.810	0.6807	0.7176	2.0478
0.565	0.3937	0.6891	1.7732	0.815	0.6878	0.7186	2.0577
0.570	0.3986	0.6894	1.7765	0.820	0.6949	0.7196	2.0679
0.575	0.4035	0.6898	1.7800	0.825	0.7021	0.7207	2.0785
0.580	0.4085	0.6902	1.7834	0.830	0.7094	0.7218	2.0895
0.585	0.4134	0.6906	1.7869	0.835	0.7167	0.7229	2.1008
0.590	0.4184	0.6909	1.7905	0.840	0.7241	0.7240	2.1126
0.595	0.4235	0.6913	1.7942	0.845	0.7316	0.7252	2.1248
0.600	0.4286	0.6917	1.7979	0.850	0.7391	0.7264	2.1375
0.605	0.4337	0.6921	1.8016	0.855	0.7467	0.7277	2.1507
0.610	0.4388	0.6925	1.8055	0.860	0.7544	0.7290	2.1644
0.615	0.4440	0.6930	1.8094	0.865	0.7621	0.7304	2.1788
0.620	0.4493	0.6934	1.8133	0.870	0.7699	0.7318	2.1938
0.625	0.4545	0.6938	1.8174	0.875	0.7778	0.7333	2.2095
0.630	0.4599	0.6943	1.8215	0.880	0.7857	0.7349	2.2259
0.635	0.4652	0.6947	1.8257	0.885	0.7937	0.7365	2.2432
0.640	0.4706	0.6952	1.8299	0.890	0.8018	0.7382	2.2613
0.645	0.4760	0.6956	1.8343	0.895	0.8100	0.7399	2.2804
0.650	0.4815	0.6961	1.8387	0.900	0.8182	0.7418	2.3006
0.655	0.4870	0.6966	1.8432	0.905	0.8265	0.7437	2.3220
0.660	0.4925	0.6971	1.8478	0.910	0.8349	0.7457	2.3447
0.665	0.4981	0.6976	1.8525	0.915	0.8433	0.7479	2.3689
0.670	0.5038	0.6981	1.8573	0.920	0.8519	0.7502	2.3947
0.675	0.5094	0.6986	1.8621	0.925	0.8605	0.7526	2.4225
0.680	0.5152	0.6991	1.8671	0.930	0.8692	0.7552	2.4523
0.685	0.5209	0.6996	1.8722	0.935	0.8779	0.7580	2.4847
0.690	0.5267	0.7002	1.8774	0.940	0.8868	0.7609	2.5199
0.695	0.5326	0.7007	1.8827	0.945	0.8957	0.7641	2.5584
0.700	0.5385	0.7013	1.8881	0.950	0.9048	0.7676	2.6011
0.705	0.5444	0.7019	1.8936	0.955	0.9139	0.7714	2.6486
0.710	0.5504	0.7025	1.8993	0.960	0.9231	0.7756	2.7022
0.715	0.5564	0.7031	1.9051	0.965	0.9324	0.7803	2.7635
0.720	0.5625	0.7037	1.9110	0.970	0.9417	0.7855	2.8349
0.725	0.5686	0.7043	1.9170	0.975	0.9512	0.7916	2.9202
0.730	0.5748	0.7050	1.9232	0.980	0.9608	0.7989	3.0257
0.735	0.5810	0.7056	1.9296	0.985	0.9704	0.8078	3.1631
0.740	0.5873	0.7063	1.9361	0.990	0.9802	0.8195	3.3590
0.745	0.5936	0.7070	1.9427	0.995	0.9900	0.8373	3.6980
0.750	0.6000	0.7077	1.9496	1.000	1.0000	0.9313	7.6708

Table 2