# SOME ANGULAR CORRELATION FUNCTIONS FOR SUCCESSIVE NUCLEAR RADIATIONS

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## Abstract

Let $J'$, $J$, $J''$ represent the total angular momenta of the initial, intermediate, and final states of a nucleus respectively and $J_1$, $J_2$ the total angular momenta of the first and second emitted particles. Then, in terms of this notation, the following results can be found in this thesis.

$\alpha$-$\gamma$ and $\gamma$-$\gamma$ correlation functions have been calculated explicitly in terms of $\cos^2\theta$ for those transition schemes satisfying the following conditions:

(i) $J' = J+J_1$, $J = J''+J_2$ for arbitrary $J_1$, $J_2 = 1, 2$.

(ii) $J' = J-J_1$, $J = J''-J_2$ for arbitrary $J_1$, $J_2 = 1, 2$.

(iii) $J' = J_1-J$, $J = J''+J_2$ for arbitrary $J_1$, $J_2 = 1, 2$.

(iv) $J' = J-J_1$, $J = J_2-J''$ for $J_1 = 1, 2$, arbitrary $J_2$.

These are called the "special transitions" in the text.

$\alpha$-mixed$\gamma$ correlation functions have been tabulated explicitly in terms of $\cos^2\theta$ for an $\alpha$ particle with total angular momentum 1 or 2 and a photon corresponding to a mixture of electric quadrupole and magnetic dipole radiation. For an $\alpha$ particle with total angular momentum 3 the $\alpha$-mixed$\gamma$ correlation functions can be obtained from a table which lists the sums of products of angular momentum coefficients appearing in these correlation functions. These correlation functions are too clumsy to be expressed explicitly in terms of $\cos^2\theta$ in general, however they can be fairly easily evaluated once numerical values of the angular momenta of the nuclear states are prescribed.

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INTRODUCTION:-

If a nucleus emits two particles or photons in quick succession, there will be a certain angle $\theta$ between their directions of emission. The function, $W(\theta)$, representing the relative probability for an angle $\theta$ between the directions of emission of the particles or photons is called the directional or angular correlation function.

The general expression for $W(\theta)$ was first derived by Hamilton[1] for the successive emission of two photons ($\gamma$-$\gamma$ correlation). He calculated $W(\theta)$ explicitly for the cases in which the multipole orders of the emitted photons are either quadrupole or dipole. Goertzel[2] extended the theory of $\gamma$-$\gamma$ correlations by considering the effect on $W(\theta)$ due to the presence of an internal atomic field or an externally applied magnetic field. He showed that the effect of the extra-nuclear electrons on the angular correlation between the successive nuclear emissions can be neglected provided the radiation width of the intermediate state of the nucleus is much greater than the hyperfine splitting of that state. This was also shown by Hamilton. Goertzel also showed that an externally applied magnetic field may be used to reduce the effect of the extra-nuclear electrons on t he angular correlation. Falkoff and Uhlenbeck[3] calculated correlation functions in parametric form (the parameters depending on the types of particles emitted) for particles or photons with angular momentum 1 or 2. Ling and Falkoff[4] then extended the theory to include transitions in which mixtures of multipoles are emitted. They tabulated $\gamma$-mixed $\gamma$ correlation functions, where $\gamma$ refers to dipole or quadrupole radiation emitted in the first transition and mixed $\gamma$ to mixed electric quadrupole and magnetic dipole radiation emitted in the second transition. Finally, Spiers[5] has shown how the general angular correlation function for any successive emissions may be derived using the quantum mechanical addition of angular momenta. The same result was shown by Lloyd[6] using group theoretical methods. The above is a resume of some of the theoretical papers on angular correlation which have been used in the preparation of this thesis. For a more complete survey of such papers the reader is referred to reference 3.

This thesis consists of three main parts. In the first part, the general expression for the correlation function is derived by following Spier's method. The general expression is then written in a form useful for calculations. In the second part, a method of evaluating the summations which appear in the formula for $W(\theta)$ is presented. The method permits one to calculate angular correlation functions for any angular momentum for the first emitted particle or photon provided the transitions involved satisfy certain special conditions. Some $\alpha$-$\gamma$ and $\gamma$-$\gamma$ correlation functions are given explicitly for these special cases. In the third part, tables are given from which $\alpha$-mixed $\gamma$ correlation functions can be obtained for an $\alpha$ particle with angular momentum 1, 2, or 3, and a photon corresponding to mixed electric quadrupole and magnetic dipole radiation.

# I. GENERAL EXPRESSION FOR THE DIRECTIONAL CORRELATION FUNCTION.

The correlation function for the case in which two particles (not photons) are emitted in quick succession by a nucleus is derived below. Essentially, the derivation due to Spiers[5] is followed.

The following notation is used throughout this thesis:

$J'm', Jm, J''m'', J_1m_1, J_2m_2$ - represent the total angular momentum and its z component for the initial, intermediate, and final states of the nucleus and the total angular momentum (intrinsic plus orbital) and its z component for the first and second emitted particles respectively.

$\Psi_{Jm}$- is the normalized wave function for the nucleus in the state represented by the quantum numbers Jm.

$\phi_{Jm}$- is the normalized wave function for an emitted particle with quantized total angular momentum Jm.

Henceforth the types of correlation between particles and photons with given angular momenta will be denoted as follows:

$\alpha(J_1)-\gamma(J_2)$ means that an $\alpha$ particle with angular momentum $J_1$ is emitted in the first transition and a photon with angular momentum $J_2$ corresponding to a $2^{J_2}$ multipole is emitted in the second.

$\alpha(J_1)$-mixed $\gamma(J_2)$ means the same as above for the first transition, but indicates that a photon corresponding to a mixture of $2^{J_2}$ electric and $2^{J_2-1}$ magnetic multipole radiation is emitted in the second transition.

In the derivation, it is assumed that the effect of the extra-nuclear electrons on the correlation can be neglected. If this were not the case, the total angular momentum of the nucleus would process about the field due to the electrons and hence could change its value. However, this value is assumed constant throughout the derivation. This assumption is valid if the radiation width of the nuclear state is much larger than the hyperfine splitting of that state.

If $\psi^{i}_{J'm'}$ represents the state of the system consisting of the intermediate nucle us and the first emitted particle then, using the quantum mechanical addition of angular momenta,

$$\psi^{i}_{J'm'} = \sum_{J_1} a_{J_1} \sum_{m+m_1=m'} (J J_1\, m\, m_1 | J J_1 J' m')\, \psi_{Jm}\, \phi_{J_1 m_1}. \tag{1}$$

Here the $a_{J_1}$ are the probability amplitudes for the various possible values of $J_1$ ($|J'-J| \leq J_1 \leq J'+J$) which the first emitted particle can have; the bracketed expressions are the normalized angular momentum coefficients which may be considered defined by (1). Using (1), one may represent the two transitions of the nucleus by

$$\begin{aligned} \psi_{J'm'} &\rightarrow \sum_{J_1} a_{J_1} \sum_{m+m_1=m'} (J J_1\, m\, m_1 | J J_1 J' m')\, \psi_{Jm}\, \phi_{J_1 m_1} \\ \psi_{Jm} &\rightarrow \sum_{J_2} a_{J_2} \sum_{m''+m_2=m} (J'' J_2\, m''\, m_2 | J'' J_2 J m)\, \psi_{J''m''}\, \phi_{J_2 m_2}. \end{aligned} \tag{2}$$

Equation (2) may be written in the form

$$\begin{aligned} \psi_{J'm'} &\rightarrow \sum_{m} C^{(1)}_{m'm}\, \psi_{Jm} \\ \psi_{Jm} &\rightarrow \sum_{m''} C^{(2)}_{m m''}\, \psi_{J''m''}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} C^{(1)}_{m'm} &= \sum_{J_1} a_{J_1} (J J_1\, m\, m'-m | J J_1 J' m')\, \phi_{J_1\, m'-m} \\ C^{(2)}_{m m''} &= \sum_{J_2} a_{J_2} (J'' J_2\, m''\, m-m'' | J'' J_2 J m)\, \phi_{J_2\, m-m''}. \end{aligned} \tag{4}$$

From (3) it is seen that the final state, $\psi^f_{m'}$, of the system consisting of the final nucleus and the two emitted particles is given by

$$\psi^f_{m'} = \sum_{m m''} C^{(1)}_{m'm} C^{(2)}_{m m''} \psi_{J''m''} \quad . \tag{5}$$

Now, at any time, $|\psi^f_{m'}|^2 dV'' dV_1 dV_2$ is the probability that, with the initial nucleus in the state $J'm'$, particle 1 is in the volume $dV_1$ about the point $r_1\theta_1\phi_1$ with intrinsic angular momentum $S_1$ and its z component $\sigma_1$, particle 2 is in the volume $dV_2$ about the point $r_2\theta_2\phi_2$ with intrinsic angular momentum $S_2$ and its z component $\sigma_2$, and the final nucleus is in the volume $dV''$ about the point $r''\theta''\phi''$. $dV''$ and $r''\theta''\phi''$ represent symbolically the volume elements and coordinates of all the nucleons. In this thesis, only the directions of the two emitted particles will be of interest, i.e. the total probability that particle 1 is in $dV_1$ at $r_1\theta_1\phi_1$ and that particle 2 is in $dV_2$ at $r_2\theta_2\phi_2$ is desired. To get this it is necessary to sum the probabilities for the various spin orientations $\sigma_1, \sigma_2$ of the two emitted particles and to integrate the nuclear coordinates over all space. Normally, the initial nuclei will be randomly oriented i.e. the $2J'+1$ degenerate states $\psi_{J'm'}$ are equally populated. The average correlation for all nuclei is obtained by finding the weighted average over the $2J'+1$ initial states. In the normal case, this is done by summing over $m'$ and dividing by $2J'+1$. From the above statements, it is seen that the average correlation function $W(r_1\theta_1\phi_1, r_2\theta_2\phi_2)$ between the directions $\theta_1\phi_1$ and $\theta_2\phi_2$ may be defined by

$$W(r_1\theta_1\phi_1, r_2\theta_2\phi_2)\,dV_1\,dV_2 = \frac{dV_1\,dV_2}{2J'+1}\sum_{m'\sigma_1\sigma_2}\int_{V''}\left|\psi^f_{m'}\right|^2 dV'', \tag{6}$$

where $r_1$ and $r_2$ are taken to be large but constant. Only the relative variation of W with $\theta_1\phi_1$ and $\theta_2\phi_2$ is of interest experimentally and so, for this reason, factors independent of these angles will be omitted from W. From equations (5) and (6) one obtains (omitting $\frac{1}{2J'+1}$ )

$$W(r_1\theta_1\phi_1, r_2\theta_2\phi_2) = \sum_{m'm\,m''\sigma_1\sigma_2\,\alpha\beta} C^{(1)*}_{m'm}\,C^{(1)}_{m'\alpha}\,C^{(2)*}_{m\,m''}\,C^{(2)}_{\alpha\beta}\int_{V''}\psi^*_{J''m''}\,\psi_{J''\beta}\,dV''$$

$$= \sum_{m'm''\sigma_1\sigma_2}\left|\sum_m C^{(1)}_{m'm}\,C^{(2)}_{m\,m''}\right|^2 \tag{7}$$

using the orthogonality of the $\psi_{J''m''}$'s.

The summation over m can be brought outside the square modulus when $\theta_1=\phi_1=0$. For, the wave function $\phi_{Jm}$ of an emitted particle, which has total angular momentum J and z component m and intrinsic angular momentum S with z component $\sigma$, can be written, using the addition of angular momenta, as

$$\phi_{Jm}(r\theta\phi\sigma) = \sum_{L=|J-S|}^{J+S} b_L\,(L\,S\,m-\sigma\,\sigma|L\,S\,J\,m)\,f_L(r)\,Y_L^{m-\sigma}(\theta\phi)\,\chi_{S\sigma} \tag{8}$$

where $\chi_{S\sigma}$ is the wave function representing the intrinsic angular momentum and $f_L(r)Y_L^{m-\sigma}(\theta\phi)$ that representing the orbital angular momentum of the particle. The quantities $b_L$ are probability amplitudes for the various possible orbital angular momenta the particle can have. Since $Y_L^{m-\sigma}(00)=0$ unless $m=\sigma$, it is seen that $\phi_{J_1m_1}(r_1\,00\sigma_1)=0$ unless $m_1=\sigma_1$, and hence that $C^{(1)}_{m'm}(r_1\,00\sigma_1)=0$ unless $m'-m=\sigma_1$ from (4). Thus, if $\theta_1=\phi_1=0$ in (7), there is only one value for m which gives a non-vanishing term once m' and $\sigma_1$ are given and the summation over m can be discarded, i.e.

$$W(r_1\,00, r_2\theta_2\phi_2) = \sum_{m'm''}\sum_{\sigma_1}\left|C^{(1)}_{m',m-\sigma_1}(r_1\,00\sigma_1)\right|^2\sum_{\sigma_2}\left|C^{(2)}_{m'-\sigma_1,m''}(r_2\theta_2\phi_2\sigma_2)\right|^2 .$$

However, for later convenience, one can still sum over m', m, and m" knowing that the terms of the summation will vanish unless $m'-m=\sigma_1$, and so it is possible to write

$$W(r_1\,00, r_2\theta_2\phi_2) = \sum_{m'mm''}\sum_{\sigma_1}\left|C^{(1)}_{m'm}(r_1\,00\sigma_1)\right|^2\sum_{\sigma_2}\left|C^{(2)}_{mm''}(r_2\theta_2\phi_2\sigma_2)\right|^2. \qquad (9)$$

$W(r_1\,00, r_2\theta_2\phi_2)$ does not depend on $\phi_2$ because of the averaging processes used in obtaining it. It depends only on the angle $\theta_2=\theta$ between the directions of emission of the two particles.

From equations (3) it is seen that $P_{m'm}(0)$ and $P_{mm''}(\theta)$ defined by

$$\begin{aligned} P_{m'm}(0) &= \sum_{\sigma_1}\left|C^{(1)}_{m'm}(r_1\,00\sigma_1)\right|^2 \\ P_{mm''}(\theta) &= \sum_{\sigma_2}\left|C^{(2)}_{mm''}(r_2\theta_2\phi_2\sigma_2)\right|^2, \end{aligned} \qquad (10)$$

represent respectively the probability that the intermediate nucleus is in the state Jm with the emission of the first particle in the z direction (and at $r_1$) and the probability that the final nucleus is in the state J"m" with the emission of the second particle at an angle $\theta$ to the first (and at $r_2$). Using (9) and (10), one may write the directional correlation function in the form

$$W(\theta) = \sum_{m'mm''} P_{m'm}(0)\,P_{mm''}(\theta), \qquad (11)$$

where it is understood that $r_1$ and $r_2$ have large but constant values.

It should be noted that in equation (7) there exists an interference between the various ways in which a transition can occur from a given sublevel $J'm'$ to a final sublevel $J''m''$ via different intermediate sublevels $Jm$ because the probability amplitudes are summed over the intermediate sublevels before squaring rather than after and cross terms appear. As seen in equation (9), it is possible to remove this interference by taking the direction of emission of the first particle along the axis of quantization. This result was also obtained by Lloyd[6] and discussed by Lippmann[7].

If the direction of emission of the second particle instead of the first had been taken along the z axis of quantization, then $W(\theta)$ would have been written in the form

$$W(\theta)=\sum_{m'mm''} P_{m'm}(\theta)P_{mm''}(0). \tag{12}$$

The two expressions for $W(\theta)$ in (11) and (12) must obviously be equal.

By substituting (4) into $P_{mm''}(\theta)$ of (10) one obtains

$$P_{mm''}(\theta)=\sum_{\sigma_2}\left|\sum_{J_2} a_{J_2}(J''J_2\, m''\, m-m''|J''J_2\, Jm)\phi_{J_2\, m-m''}\right|^2. \tag{13}$$

The above formulae have been derived for the case that particles and not photons are emitted. Ling and Falkoff[4] have treated the case for the emission of a photon corresponding to mixed $2^{J_2}$ electric and $2^{J_2-1}$ magnetic multipole radiation. Their results are given by

$$P_{mm''}(\theta)=\frac{8\pi^2r^2}{c}\frac{ck^2}{8\pi}\vec{A}\cdot\vec{A}^*=\pi r^2k^2\sum_{\sigma_2=-1}^{+1}|A_{\sigma_2}|^2 \tag{14}$$

where $A_{-1}=\frac{A_x-iA_y}{\sqrt{2}}$, $A_0=A_z$, $A_{+1}=\frac{A_x+iA_y}{\sqrt{2}}$

and $A_{\sigma_2}=\alpha(J''J_2\, m''m_2|J''J_2\, Jm)A^E_{\sigma_2}(J_2,m_2)+\beta(J''J_2-1\,m''m_2|J''J_2-1\,Jm)A^M_{\sigma_2}(J_2-1,m_2)$.

Here $A^{E}_{\sigma_2}(J_2,m_2)$, $A^{M}_{\sigma_2}(J_2\text{-}1,\ m_2)$ are the components of the normalized vector potentials for a $2^{J_2}$ electric and a $2^{J_2-1}$ magnetic multipole respectively; $\alpha$ and $\beta$ represent the probability amplitudes for each multipole. Lloyd[8] has shown that $\alpha$ and $\beta$ can be made real by a prop er choice of the nuclear phases. However, in some calculations, it may be useful to have complex values, hence the formulae are left in the above form.

The result (14) can be incorporated into formula (13) if $S_2$ is taken to be 1 ($\sigma_2$=1,0,-1) and $\phi_{J_2m_2}, \phi_{J_2-1\,m_2}$ are replaced by $\sqrt{\pi}rkA^{E}_{\sigma_2}(J_2,m_2)$ and $\sqrt{\pi}rkA^{M}_{\sigma_2}(J_2\text{-}1,m_2)$ respectively. Of course, $\alpha = a_{J_2}$ and $\beta = a_{J_2-1}$. Henceforth (13) will be considered valid for both particles and photons, the proper $\phi$'s being substituted in each case.

At this point one can see that the dependence of $P_{m'm}$ and $P_{mm''}$ and hence $W(\theta)$ on $r_1$ and $r_2$ can be factored out and thus omitted. For, in (13) the $\phi$'s (and A's) depend on r through the functions $f_L(r)$, which for large r are proportional to $(-i)^L \frac{e^{ikr}}{kr}$ (spherical wave), where k is the magnitude of the propagation vector.

If only one value of $J_2$ is possible for the particle or photon emitted in the transition $Jm \rightarrow J''m''$ then the equation (13) takes the form

$$\underset{m-m''=m_2}{P_{mm''}(\theta)} = (J''J_2\,m''m_2|\,J''J_2\,Jm)^2 \sum_{\sigma_2} |\phi_{J_2m_2}(r_2\theta_2\phi_2\sigma_2)|^2, \qquad (15)$$

the angular momentum coefficients being real (see equations (18) and (19)). Omitting the dependence on $r_2$ , which can be factored out, this has the form

$$\underset{m-m''=m_2}{P_{mm''}(\theta)} = (J'' J_2 m'' m_2 | J'' J_2 J m)^2 F^{m}_{J_2}(\theta), \qquad (16)$$

This formula can be found in Falkoff and Uhlenbeck's[3] paper. In this thesis only transitions in which an $\alpha$ particle or a photon is emitted will be considered. Since an $\alpha$ particle has no intrinsic angular momentum one can write $F^{m_2}_{J_2}(\theta) = |Y^{m_2}_{J_2}(\theta\,\phi_2)|^2$ for it, using (8) and (13). General expressions for the F's for a photon arising from pure and mixed multipole transitions are given by Ling and Falkoff[4] who substituted expressions in terms of spherical harmonics for the A's appearing in (13) and (14). A $2^{J_2}$ electric and a $2^{J_2-1}$ magnetic multipole have the same $F^{m_2}_{J_2}(\theta)$. For any parti cle or photon, $F^{m_2}_{J_2}(\theta) = F^{-m_2}_{J_2}(\theta)$ (see reference 3).

For a mixed $\gamma(J_2)$ transition, Ling and Falkoff[4] have given the following formula:

$$\begin{aligned} \underset{m-m''=m_2}{P_{mm''}(\theta)} = {} & |\alpha|^2 (J'' J_2 m'' m_2 | J'' J_2 J m)^2 F^{m_2}_{J_2}(\theta) + |\beta|^2 (J'' J_2{-}1\, m'' m_2 | J'' J_2{-}1\, J m)^2 F^{m_2}_{J_2-1}(\theta) \\ & + 2R(\alpha\beta^*)(J'' J_2 m'' m_2 | J'' J_2 J m)(J'' J_2{-}1\, m'' m_2 | J'' J_2{-}1\, J m) F^{m_2}_{J_2, J_2-1}(\theta). \end{aligned} \qquad (17)$$

Here $2R(\alpha\beta^*) = \alpha\beta^* + \alpha^*\beta$, and the $F^{m_2}_{J_2 J_2-1}(\theta)$ are the angular distribution functions for the interference contributions to $P_{mm''}(\theta)$ arising from the mixin g of the $2^{J_2}$ electric and $2^{J_2-1}$ magnetic multipole fields. The form of the result (17) follows from (13) and (14). Some $F^{m_2}_{J_2 J_2-1}(\theta)$ are listed in Table 1.

The angular momentum coefficients are given by (see Appendix A, equations (A4) and (A7))

$$(J'' J_2 m'' m_2 | J'' J_2 J m) = \left[\frac{(2J+1)\,(2J''-\lambda_2)!\,(2J_2-\lambda_2)!\,\lambda_2!}{(2J+\lambda_2+1)!}\right]^{\frac{1}{2}} C^{J m \lambda_2}_{J'' m'' J_2 m_2} \qquad (18)$$

where $J = J'' + J_2 - \lambda_2$; $0 \leq \lambda_2 \leq$ minimum of $2J''$, $2J_2$; $m = m'' + m_2$; and

$$C^{J\,m\,\lambda_2}_{J''m''J_2m_2} = \frac{1}{\lambda_2!}\left[\frac{(J+m)!\,(J-m)!}{(J''+m'')!\,(J''-m'')!\,(J_2+m_2)!\,(J_2-m_2)!}\right]^{\frac{1}{2}} \times$$
$$\sum_{\alpha=0}^{\lambda_2}(-1)^{\alpha}\frac{(J''+m'')!\,(J''-m'')!\,(J_2+m_2)!\,(J_2-m_2)!}{(J''+m''-\lambda_2+\alpha)!\,(J''-m''-\alpha)!\,(J_2+m_2-\alpha)!\,(J_2-m_2-\lambda_2+\alpha)!}\binom{\lambda_2}{\alpha}, \tag{19}$$

$\binom{\lambda_2}{\alpha} = \frac{\lambda_2!}{(\lambda_2-\alpha)!\,\alpha!}$. The summation over $\alpha$ is carried out with the understanding that each fraction $\frac{A!}{(A-\beta)!}$ $(A \geq 0, \beta \geq 0)$ appearing in the summand is to be written identically as $A(A-1)\ldots(A-\beta+1)$. Then, if $A-\beta<0$, the term containing the fraction $\frac{A!}{(A-\beta)!}$ will vanish. Using $J = J'' + J_2 - \lambda_2$ and $m = m'' + m_2$, it is easily shown that each axial quantum number appearing in $C^{J\,m\,\lambda_2}_{J''m''J_2m_2}$ must lie within the smallest range given by the following conditions:

$$-J \leq m \leq J, \quad -J'' \leq m'' \leq J'', \quad -J_2 \leq m_2 \leq J_2, \quad -(J''+J_2-\lambda_2) \leq m''+m_2 \leq J''+J_2-\lambda_2,$$
$$-(J-J_2+\lambda_2) \leq m-m_2 \leq J-J_2+\lambda_2, \quad -(J-J''+\lambda_2) \leq m-m'' \leq J-J''+\lambda_2. \tag{20}$$

The C's, defined by (19), have the following symmetry properties:

$$C^{J\,m\,\lambda_2}_{J''m''J_2m_2} = (-1)^{\lambda_2}\,C^{J\,-m\,\lambda_2}_{J''-m''J_2-m_2} \tag{21}$$

$$C^{J\,m\,\lambda_2}_{J_2m_2J''m''} = (-1)^{\lambda_2}\,C^{J\,m\,\lambda_2}_{J''m''J_2m_2} \tag{22}$$

$$C^{J\,m\,2J_2-\lambda_2}_{J''m''J_2m_2} = (-1)^{J_2+m_2+\lambda_2}\,C^{J''m''\lambda_2}_{J\,m\,J_2-m_2}. \tag{23}$$

The C's used in this thesis are listed in Table 2.

Using (11), (16), and (18) one can obtain the directional correlation function $W(\theta)$ for a particle $(J_1)$ or photon $(J_1)$ emitted in the first transition and a particle $(J_2)$ or photon $(J_2)$ emitted in the second transition. The result is

$$W(\theta) = \sum_{m_1m_2}\left[\sum_{m}\left(C^{J'\,m+m_1\,\lambda_1}_{J\,m\,J_1\,m_1}\right)^2\left(C^{J\,m\,\lambda_2}_{J''\,m-m_2\,J_2\,m_2}\right)^2\right]F^{m_1}_{J_1}(0)\,F^{m_2}_{J_2}(\theta). \tag{24}$$

Here, the normalization factors for the angular momentum coefficients have been factored out and omitted; the summation over $m'$ ($=m+m_1$), $m$, and $m''$ ($=m-m_2$) has been replaced by one over $m_1$, $m$, and $m_2$. The F's, as stated before, vary with the type of particle emitted. The angles 0 and $\theta$ can be interchanged (Cf. (11) and (12)).

The particle $(J_1)$-mixed $\gamma(J_2)$ or $\gamma(J_1)$-mixed $\gamma(J_2)$ correlation function is given by

$$
\begin{aligned}
W(\theta) = {} & |\alpha|^2(2J_2-\lambda_2)\lambda_2 \sum_{m_1,m_2}\Big[\sum_m \big(C^{J'\,m+m_1\,\lambda_1}_{J\,m\,J_1\,m_1}\big)^2\big(C^{J\,m\,\lambda_2}_{J''\,m-m_2\,J_2\,m_2}\big)^2\Big] F^{m_1}_{J_1}(0)\,F^{m_2}_{J_2}(\theta) \\
& + |\beta|^2(2J+\lambda_2+1)(2J''-\lambda_2+1)\sum_{m_1,m_2}\Big[\sum_m \big(C^{J'\,m+m_1\,\lambda_1}_{J\,m\,J_1\,m_1}\big)^2\big(C^{J\,m\,\lambda_2-1}_{J''\,m-m_2\,J_2-1\,m_2}\big)^2\Big] F^{m_1}_{J_1}(0)\,F^{m_2}_{J_2-1}(\theta) \qquad (25) \\
& + 2R(\alpha\beta^*)\big[(2J_2-\lambda_2)\lambda_2(2J+\lambda_2+1)(2J''-\lambda_2+1)\big]^{\frac{1}{2}}\sum_{m_1,m_2}\Big[\sum_m \big(C^{J'\,m+m_1\,\lambda_1}_{J\,m\,J_1\,m_1}\big)^2 C^{J\,m\,\lambda_2}_{J''\,m-m_2\,J_2\,m_2}\,C^{J\,m\,\lambda_2-1}_{J''\,m-m_2\,J_2-1\,m_2}\Big] F^{m_1}_{J_1}(0)\,F^{m_2}_{J_2,J_2-1}(\theta)
\end{aligned}
$$

where common factors have been omitted.

The formulae (24) and (25) are the ones which will be used in this thesis to calculate correlation functions. The formula (25) is valid except for $\lambda_2 = 0$ and $\lambda_2 = 2J_2$. For, from considerations of the vector addition of angular momenta, it can be seen that only a pure $2^{J_2}$ electric instead of a mixed $2^{J_2}$ electric and $2^{J_2-1}$ magnetic multipole transition will occur for these two cases. For $\lambda_2 = 0, 2J_2$ the formula (24) will be used.

A method of evaluating the summations appearing in the square brackets of (24) and (25) is given in Section II. of this thesis. once these summations are known, it is a fairly simple matter to obtain $W(\theta)$ if $J_1$ or $J_2$ is small.

$W(\theta)$, as given by (24), is also the correlation function for the reverse transition scheme $J'' \xrightarrow[J_2]{} J \xrightarrow[J_1]{} J'$ with the emission of the particles or photons occurring in the reverse order. For, by making use of the symmetry property (23) and the relation $F_J^{m}(\theta) = F_J^{-m}(\theta)$, and changing the summations over $m_1$ and $m_2$ to summations over $-m_1$ and $-m_2$, one can show that (24) is equal to

$$W(\theta) = \sum_{m_1 m_2} \left[ \sum_m \left( C^{J''\, m+m_2\, 2J_2-\lambda_2}_{J\, m\, J_2\, m_2} \right)^2 \left( C^{J\, m\, 2J_1-\lambda_1}_{J'\, m-m_1\, J_1\, m_1} \right)^2 \right] F_{J_1}^{m_1}(\theta)\, F_{J_2}^{m_2}(0), \quad (26)$$

which is the correlation function for the reversed process. If also $J_1 = J_2$, then $W(\theta)$ in (24) further represents the directional correlation function for the reverse transition scheme $J'' \xrightarrow[J_1]{} J \xrightarrow[J_2]{} J'$ but with the emission of the particles or photons occurring in the given order i.e. as in (24). This last result is obtained from (26) by interchanging $m_1$ and $m_2$, which interchange is possible since $m_1$ and $m_2$ run over the same range of values ($J_1 = J_2$). These results have been proved already, although in a different way, by Falkoff and Uhlenbeck[3]. These authors have also shown that the $F_J^m$'s are polynomials of degree at most $J$ in $\cos^2\theta$. Since the expressions (11) and (12) are equal, it is seen that $W(\theta)$ in (24) is a polynomial in $\cos^2\theta$ of degree at most the minimum of $J_1$ and $J_2$ (see also Yang[9]).

II. CALCULATION OF DIRECTIONAL CORRELATION FUNCTIONS.

In order to express $W(\theta)$ in (24) explicitly in terms of $\cos^2\theta$ and the angular momenta involved, it is necessary to evaluate the summations over m appearing in the square brackets, to substitute expressions for the F's in terms of $\cos^2\theta$, and then to carry out the summation over $m_1$ and $m_2$. It is the object of this section to present formulae which one can use to simplify and perform such calculations.

A. Symmetry of the Summations.

It is possible to reduce the amount of calculation required to obtain $W(\theta)$ explicitly by making use of the symmetry properties (21) and (22).

Applying (21) to the summation

$$\sum_{m} (C^{J'm+m_1\,\lambda_1}_{JmJ_1\,m_1})^2 (C^{Jm\;\lambda_2}_{J''m-m_2\,J_2\,m_2})^2, \tag{27}$$

appearing in (24), and changing the summation over m to one over -m, one can show that

$$(27) = \sum_{m} (C^{J'm-m_1\lambda_1}_{JmJ_1-m_1})^2 (C^{Jm\;\lambda_2}_{J''m+m_2\,J_2\,-m_2})^2. \tag{28}$$

As soon as the types of particles emitted are given, the formula (24) can be simplified since the $F^{m_1}_{J_1}(0)$ will vanish but for certain values of $m_1$. Because $\alpha$-$\gamma$ and $\gamma$-$\gamma$ correlations are the subject of this thesis, the formulas for their correlation functions will now be obtained.

If an $\alpha$ particle is emitted in the first transition along the axis of quantization then $F^{m_1}_{J_1}(0) = 0$ unless $m_1 = 0$.

Using this result and applying the relations (28) and $F^{m_2}_{J_2}(\theta) = F^{-m_2}_{J_2}(\theta)$ to (24) one obtains

$$W(\theta) = \left[\sum_m \left(C^{J'm\,\lambda_1}_{JmJ_1\,0}\right)^2 \left(C^{Jm\,\lambda_2}_{J''mJ_2\,0}\right)^2\right] F^0_{J_2}(\theta) + 2\sum_{m_2=1}^{J_2}\left[\sum_m \left(C^{J'm\lambda_1}_{JmJ_1\,0}\right)^2 \left(C^{Jm\,\lambda_2}_{J''m-m_2\,J_2\,m_2}\right)^2\right] F^{m_2}_{J_2}(\theta) \tag{29}$$

for the $\alpha(J_1)$-particle$(J_2)$ or $\alpha(J_1)-\gamma(J_2)$ correlation function. The interference summa tion appearing in the mixture term of (25) for $\alpha(J_1)$-mixed $\gamma(J_2)$ correlation functions can also be written in a similar form. The common factor $F^0_{J_1}(0)$ is omitted.

The emission of a photon in the first transition along the axis of quantization requires that $m_1 = \pm 1$ in (24) since $F^{m_1}_{J_1}(0) = 0$ unless $m_1 = \pm 1$ for a photon. From (24) and (28) then, one can write

$$W(\theta) = \sum_{m_2=-J_2}^{J_2}\left[\sum_m \left(C^{J'm+1\lambda_1}_{JmJ_1\,1}\right)^2 \left(C^{Jm\,\lambda_2}_{J''m-m_2\,J_2\,m_2}\right)^2\right] F^{m_2}_{J_2}(\theta) \tag{30}$$

for $\gamma(J_1)$-particle$(J_2)$ or $\gamma(J_1)-\gamma(J_2)$ correlation functions. Also, for $\gamma(J_1)$-mixed $\gamma(J_2)$ correlation functions, the summation in the interference term of (25) can be expressed in a similar form. The common factor $2F^1_{J_1}(0)$ is omitted.

B. Correlation Functions for Special Transitions.

The summations (27) can be evaluated quite easily for some special transitions determined by the following considerations. The expressions for $C^{Jm\,\lambda_2}_{J''m''J_2\,m_2}$ are the simplest when $\lambda_2 = 0$, $2J_2$, $2J''$ as one may check using Table 2 and the symmetry properties (22), (23). In each of the $(C)^2$'s in (27) the factor $(J+m)!\,(J-m)!$ will appe ar, either in the numerator or denominator, if the values of $\lambda_1$ and $\lambda_2$ are chosen from the set

$0$, $2J_1$, $2J$, and the set $0$, $2J_2$, $2J''$ respectively. The simplest summand in (27) is obtained when a combination of $\lambda_1$ and $\lambda_2$ is chosen from the above values in such a way that the factor $(J+m)!\,(J-m)!$ will be cancelled out. Of the combinations possible only four permit such a cancellation. These are:

(i). $\lambda_1=0$, $\lambda_2=0$.

(ii). $\lambda_1=2J_1$, $\lambda_2=2J_2$.

(iii). $\lambda_1=2J$, $\lambda_2=0$.

(iv). $\lambda_1=2J_1$, $\lambda_2=2J''$.

Henceforth, the transi tions satisfying these conditions will be called the "special transitions". The summations (27) for these four cases are evaluated below. The results may be used to calculate any correlation functions for the special transitions. In particular, some $\alpha$-$\gamma$ and $\gamma$-$\gamma$ correlation functions are given.

(i). $\lambda_1=0$, $\lambda_2=0$ or $J'=J+J_1$, $J=J''+J_2$.

For this case the total angular momenta of the particles and nuclei resulting from each transition are parallel to one another. It is seen from condition (i) that $J_1$ and $J_2$ are the smallest angular momenta that can be emitted compatible with the angular momenta of the nuclear levels $J'$, $J$, $J''$. The summation (27) is evaluated as follows:

$$\sum_m (C^{J'm+m_1 0}_{JmJ_1 m_1})^2 (C^{Jm0}_{J''m-m_2 J_2 m_2})^2 =$$

$$=\sum_{m}\frac{(J'+m_1+m)!\,(J'-m_1-m)!}{(J_1+m_1)!\,(J_1-m_1)!\,(J''-m_2+m)!\,(J''+m_2-m)!\,(J_2+m_2)!\,(J_2-m_2)!}$$

(from Table 2)

$$=\binom{J_1+J_2+m_1+m_2}{J_1+m_1}\binom{J_1+J_2-m_1-m_2}{J_1-m_1}\sum_{m=-J+(J_2+m_2)}^{J-(J_2-m_2)}\binom{J'+m_1+m}{J_1+J_2+m_1+m_2}\binom{J'-m_1-m}{J_1+J_2-m_1-m_2}$$

(from (20))

$$=\binom{J_1+J_2+m_1+m_2}{J_1+m_1}\binom{J_1+J_2-m_1-m_2}{J_1-m_1}\sum_{v=0}^{2(J-J_2)}\binom{J_1+J_2+m_1+m_2+v}{J_1+J_2+m_1+m_2}\binom{2(J-J_2)+J_1+J_2-m_1-m_2-v}{J_1+J_2-m_1-m_2}$$

(using $m=v-J+J_2+m_2$)

$$=\binom{J_1+J_2+m_1+m_2}{J_1+m_1}\binom{J_1+J_2-m_1-m_2}{J_1-m_1}\binom{2J'+1}{2J''}, \tag{31}$$

using the summation formula (45) in Subsection C below.

One can substitute this result into (24) to obtain the correlation function between a particle($J_1$) or photon ($J_1$) and a particle ($J_2$) or photon ($J_2$), subject to the condition (i) on the angular momenta. In particular, if an $\alpha$ particle or photon is emitted a long the axis of quantization in the first transition, the correlation functions are obtained by substituting (31) into (29) and (30). The results are:

$\alpha(J_1)$-particle ($J_2$) or $\alpha(J_1)-\gamma(J_2)$:

$$W(\theta)=\binom{J_1+J_2}{J_1}2F^{0}_{J_2}(\theta)+2\sum_{m_2=1}^{J_2}\binom{J_1+J_2+m_2}{J_1}\binom{J_1+J_2-m_2}{J_1}F^{m_2}_{J_2}(\theta) \tag{32}$$

$\gamma(J_1)$-particle($J_2$) or $\gamma(J_1)-\gamma(J_2)$:

$$W(\theta)=\sum_{m_2=-J_2}^{J_2}\binom{J_1+J_2+1+m_2}{J_1+1}\binom{J_1+J_2-1-m_2}{J_1-1}F^{m_2}_{J_2}(\theta), \tag{33}$$

where the factor $\binom{2J'+1}{2J''}$ has been omitted. In the case that a photon is emitted in the second transition with angular momentum $J_2=1$ (dipole) or $J_2=2$ (quadrupole) one can obtain the correlation functions

$$\alpha(J_1)-\gamma(1): \quad W(\theta)= 1-\frac{J_1}{3J_1+4}\cos^2\theta$$

$$\alpha(J_1)-\gamma(2): \quad W(\theta)= 1+\frac{6J_1(J_1+1)}{(J_1+3)(5J_1+8)}\cos^2\theta-\frac{3J_1(J_1-1)}{(J_1+3)(5J_1+8)}\cos^4\theta$$

$$\gamma(J_1)-\gamma(1): \quad W(\theta) = 1-\frac{(J_1^2+J_1-3)}{(3J_1^2+7J_1+3)}\cos^2\theta \tag{34}$$

$$\gamma(J_1)-\gamma(2): \quad W(\theta) = 1+\frac{6J_1(J_1^2+2J_1-5)}{(J_1+1)(5J_1^2+23J_1+30)}\cos^2\theta-\frac{3(J_1-1)(J_1^2+J_1-10)}{(J_1+1)(5J_1^2+23J_1+30)}\cos^4\theta,$$

using the distribution functions listed in Table 1. Non-zero factors (i.e. those common factors which are polynomials in $J_1$ having no integral roots) have been omitted from these formulae.

(ii). $\lambda_1=2J_1$, $\lambda_2=2J_2$ .or $J'=J-J_1$, $J=J''-J_2$.

By using the third symmetry property (23) together with Table 2 one can sum (27) as in part (i) to get

$$\sum_m \left(C^{J'm+m_1\,2J_1}_{JmJ_1\,m_1}\right)^2\left(C^{Jm2J_2}_{J''m-m_2J_2\,m_2}\right)^2=\binom{J_1+J_2+m_1+m_2}{J_1+m_1}\binom{J_1+J_2-m_1-m_2}{J_1-m_1}\binom{2J''+1}{2J'}. \tag{35}$$

Since $\binom{2J''+1}{2J'}$ can be factored out of $W(\theta)$ just as $\binom{2J'+1}{2J''}$ was in part (i), then, comparing the results (35) and (31), one can see that the correlation functions for this transition scheme are the same as those given in part (i) i.e. the same as equations (32), (33), and (34).

(iii). $\lambda_1 = 2J$, $\lambda_2 = 0$ or $J' = J_1 - J$, $J = J'' + J_2$.

From this condition it is evident that $J_1 \geq J$ - in fact, $J_1$ is here the largest angular momentum which can be emitted compatible with the angular momenta $J', J$. The calculation of correlation functions for this transition requires the use of the summation formula (46). Thus, (27) becomes

$$\sum_m \left(C^{J'm+m_1\,2J}_{JmJ_1\,m_1}\right)^2 \left(C^{Jm0}_{J''m-m_2J_2m_2}\right)^2$$

$$= \sum_m \frac{(J_1+m_1)!\,(J_1-m_1)!\,(J+m)!\,(J-m)!}{(J'+m_1+m)!\,(J'-m_1-m)!\,(J+m)!\,(J-m)!\,(J''-m_2+m)!\,(J''+m_2-m)!\,(J_2+m_2)!\,(J_2-m_2)!}$$

(from Table~~s~~ 2 ~~and 5~~)

$$= \binom{J_1+m_1}{J_2-m_2}\binom{J_1-m_1}{J_2+m_1}\sum_m \binom{J'+J''+m_1+m_2}{J'+m_1+m}\binom{J'+J''-m_1-m_2}{J'-m_1-m}$$

$$= \binom{J_1+m_1}{J_2-m_2}\binom{J_1-m_1}{J_2+m_2}\sum_v \binom{J'+J''+m_1+m_2}{v}\binom{J'+J''-m_1-m_2}{2J'-v}$$

$$= \binom{J_1+m_1}{J_2-m_2}\binom{J_1-m_1}{J_2+m_2}\binom{2J'+2J''}{2J'}, \tag{36}$$

using the summation formula (46) in Subsection C below. This result with (24) gives the correlation function between particles or photons subject to condition (iii). If an $\alpha$ particle is emitted in the first transition, then the correlation function is obtained from (36) and (29) i.e.

$$W(\theta) = \binom{J_1}{J_2}^2 F^0_{J_2}(\theta) + 2\sum_{m_2=1}^{J_2}\binom{J_1}{J_2-m_2}\binom{J_1}{J_2+m_2} F^{m_2}_{J_2}(\theta). \tag{37}$$

If a photon is emitted in the first transition, then from (36) and (30),

$$W(\theta)=\sum_{m_2=-J_2}^{J_2}\binom{J_1+1}{J_2-m_2}\binom{J_1-1}{J_2+m_2}F_{J_2}^{m_2}(\theta). \qquad (38)$$

If a photon is also emitted in the second transition with angular momentum $J_2=1$, or 2 then, as in (i), one obtains the following correlation functions:

$$\alpha(J_1)-\gamma(1): \quad W(\theta) = J_1\left[1-\frac{(J_1-1)}{(3J_1-1)}\cos^2\theta\right]$$

$\alpha(J_1)-\gamma(2)$:

$$W(\theta) = J_1(J_1-1)\left[(J_1-2)+\frac{6J_1(J_1+1)}{(5J_1-3)}\cos^2\theta-\frac{3(J_1+1)(J_1+2)}{(5J_1-3)}\cos^4\theta\right] \qquad (39)$$

$$\gamma(J_1)-\gamma(1): \quad W(\theta) = 1-\frac{(J_1^2+J_1-3)}{(3J_1^2-J_1-1)}\cos^2\theta$$

$\gamma(J_1)-\gamma(2)$:

$$W(\theta) = (J_1-1)\left[J_1+\frac{6(J_1+1)(J_1^2-6)}{5J_1^2-13J_1+12}\cos^2\theta-\frac{3(J_1+2)(J_1^2+J_1-10)}{5J_1^2-13J_1+12}\cos^4\theta\right],$$

from (37), (38), and Table 1. Non-zero common factors have been omitted from the formulae (39). For those values of $J_1$ which make the factors appearing outside the square brackets vanish the transition can not take place. For, using the condition (iii), it is seen that $J_1=J'+J''+J_2$ i.e. the equation $J_1 \geq J_2$ must be satisfied if a transition occurs. Thus, for $0 \leq J_1 < J_2$ no transition producing an $\alpha(J_1)$ particle can occur and for $1 \leq J_1 < J_2$ no transition producing a photon $(J_1)$ can occur.

(iv). $\lambda_1=2J_1$, $\lambda_2=2J''$ or $J'=J-J_1$, $J=J_2-J''$.

Here $J_2=J''+J'+J_1$ and so it is larger than any of the other angular momenta. One cannot calculate correlation functions in terms of $\cos^2\theta$ for this case with arbitrary $J_1$ since by the last equation $J_2$ must also be arbitrary. However, correlation functions can be calculated with $J_2$ arbitrary and $J_1$ having some small value. Such correlation functions are given below.

To evaluate (27) for this case the summation formula (46) is used with Table 2 and the symmetry properties (22) and (23). Following the same method employed in (iii) one can show that

$$\sum_{m} (C^{J'm+m_1\, 2J_1}_{JmJ_1\, m_1})^2 (C^{Jm2J''}_{J''m-m_2 J_2\, m_2})^2 = \binom{J_2+m_2}{J_1-m_1}\binom{J_2-m_2}{J_1+m_1}\binom{2J'+2J''}{2J'}. \qquad (40)$$

If a photon is emitted along the axis of quantization in the second transition, then, omitting the factor $2P^1_{J_2}(0)\binom{2J'+2J''}{2J'}$, the correlation function will be

$$W(\theta) = \sum_{m_1=-J_1}^{J_1} \binom{J_2+1}{J_1-m_1}\binom{J_2-1}{J_1+m_1} F^{m_1}_{J_1}(\theta) \quad , \qquad (41)$$

from (21). Comparing (41) with (38), one observes that the $\gamma(1)-\gamma(J_2)$ and $\gamma(2)-\gamma(J_2)$ correlation functions for this case are the same as the $\gamma(J_1)-\gamma(1)$ and $\gamma(J_1)-\gamma(2)$ correlation functions respectively in (39) with $J_1$ replaced by $J_2$. The corresponding $\alpha-\gamma$ correlations are calculated from (41) using Table 1. The results are:

$\alpha(1)-\gamma(J_2)$: $\quad W(\theta) = 1 + \dfrac{J_2^2+J_2-3}{J_2^2-J_2+1}\cos^2\theta$

$\alpha(2)-\gamma(J_2)$:

$$W(\theta) = (J_2-1)\left[(J_2-2) + \frac{2(J_2^3-5J_2^2+36)}{3J_2^2-J_2+6}\cos^2\theta + \frac{3(J_2+2)(J_2^2+J_2-10)}{3J_2^2-J_2+6}\cos^4\theta\right]$$

$$(42)$$

$\gamma(1)-\gamma(J_2)$; $\quad W(\theta) = 1 - \dfrac{(J_2^2+J_2-3)}{3J_2^2-J_2-1}\cos^2\theta$

$\gamma(2)-\gamma(J_2)$:

$$W(\theta) = (J_2-1)\left[J_2 + \frac{6(J_2+1)(J_2^2-6)}{5J_2^2-13J_2+12}\cos^2\theta - \frac{3(J_2+2)(J_2^2+J_2-10)}{5J_2^2-13J_2+12}\cos^4\theta\right],$$

omitting non-zero factors. By the same reasoning as that in (iii), one can show that no transition occurs for $J_2=1$ in the $\alpha(2)-\gamma(J_2)$ and $\gamma(2)-\gamma(J_2)$ correlation functions of (42).

C. General Method.

In the preceding Section certain summation formulae ((45), (46)) have been used, which will be obtained now. To this purpose a general method of evaluating the summation (27) is presented here. The method can also be used to evaluate summations of the type appearing in the interference term of (25).

A very direct method of evaluating (27) is as follows. In Appendix B it is shown that the $(C\ )^2$ appearing in (27) are polynomials in m and hence the summand of (27) is a polynomial in m. Thus one can perform the indicated summation over m directly by using the known results for the sums $\sum_{m=-J}^{J} m^k$, where k is a positive integer. If the numerical values of $m_1$ and $m_2$ are also prescribed, then the final result is a polynomial in J. By finding the rational roots of this polynomial in J, one can express it in a partially factored form and some of the factors can be factored out of $W(\theta)$. However, for $J_1$ and $J_2 \geq 2$, the summand in (27) is a polynomial in m of degree $\geq 8$ and so the direct procedure of evaluating (27) and then calculating $W(\theta)$ becomes clumsy.

In the following method one does not have to expand the complete summand as a polynomial in m but only part of it. Furthermore, the final result is automatically factored in terms of some of the rational roots, in some cases all of them (e.g. the factor $\binom{2J'+1}{2J''}$ in (31) is factored out of $W(\theta)$). These advantages have permitted the calculation of correlation functions as given in part B which would be difficult to get using the direct method.

The method is based on the fact that (27) can be reduced to the problem of summing either

$$S^r_{pnq} \equiv \sum_{v=0}^{n} \binom{p+v}{p}\binom{n+q-v}{q}(A_r v^r + A_{r-1}v^{r-1} + \dots + A_0) \tag{43}$$

or

$$R^r_{PnQ} \equiv \sum_{v} \binom{P}{v}\binom{Q}{n-v}(A_r v^r + A_{r-1}v^{r-1} + \dots + A_0), \tag{44}$$

after substituting expressions for the C's and translating the summation over m to one over v. In (44), v takes on all those values for which $\binom{P}{v}, \binom{Q}{n-v}$ do not vanish, i.e. it has the range defined by $n-Q \leq v \leq n$ and $0 \leq v \leq P$. The summation (27) can always be placed in the form (43), since at the worst, $p=0$ and $q=0$, the binomial coefficients become 1, and (43) is then the direct summation of (27) whose summand is expanded into a polynomial in $v=m+J$. The summations (43) and (44) have been used to evaluate the summations (27) for the special transitions in B.

In Appendix C it is shown that

$$\sum_{v=0}^{n} \binom{p+v}{p}\binom{n+q-v}{q} v^k \begin{cases} = \binom{n+p+q+1}{n} & \text{if } k=0 \\ = \sum_{\alpha=1}^{k} a_{k\alpha} \frac{(p+\alpha)!}{p!} \binom{n+p+q+1}{n-\alpha} & \text{if } k \geq 1 \end{cases} \tag{45}$$

and that

$$\sum_{\nu}\binom{P}{\nu}\binom{Q}{n-\nu}\nu^{k}\begin{cases}=\binom{P+Q}{n} & \text{if } k=0\\[2mm] =\sum_{\alpha=1}^{k} a_{k\alpha}\dfrac{P!}{(P-\alpha)!}\dbinom{P+Q-\alpha}{n-\alpha} & \text{if } k\geq 1,\end{cases} \tag{46}$$

where the $a_{k\alpha}$'s are obtained from Table ~~4~~ 3.

Substituting these results into (43) and (44), it is easily shown that

$$S^{r}_{pnq}=\sum_{\alpha=1}^{r}\left[\sum_{\beta=\alpha}^{r}A_{\beta}a_{\beta\alpha}\right]\frac{(p+\alpha)!}{p!}\binom{n+p+q+1}{n-\alpha}+A_{0}\binom{n+p+q+1}{n} \tag{47}$$

$$R^{r}_{PnQ}=\sum_{\alpha=1}^{r}\left[\sum_{\beta=\alpha}^{r}A_{\beta}a_{\beta\alpha}\right]\frac{P!}{(P-\alpha)!}\binom{P+Q-\alpha}{n-\alpha}+A_{0}\binom{P+Q}{n}, \tag{48}$$

only the term containing $A_0$ appearing in each formula for the case $r=0$.

If p and q are set equal to zero in (45), the summation reduces to the sum of the $k^{th}$ power of the positive integers. As mentioned before, these sums are used in the direct method.

The coefficients $a_{k\alpha}$, listed in Table 3, are obtained from the induction formula

$$a_{k\alpha}\begin{cases}=1 & \text{for } k\geq 1, \alpha=1\\[2mm] =\sum_{\beta=1}^{k-\alpha+1}a_{k-\beta,\alpha-1}\dbinom{k-1}{\beta-1} & \text{for } k\geq 2, \alpha\geq 2,\end{cases} \tag{49}$$

proved in Appendix C.

For a transition scheme next to a special one, i.e. a transition scheme, characterized by $\lambda_1$, $\lambda_2$, for which the quantity $\lambda_1+\lambda_2$ differs from that for a special transition by

1, the evaluation of the summation is a little more difficult than for the special case. As an example, take the transition scheme $\lambda_1=0$, $\lambda_2=1$ or $J'=J+J_1$, $J=J''+J_2-1$ (next to the special transition (i)) and evaluate (27) as follows.

$$\sum_m (C^{J'm+m_1\,0}_{JmJ_1m_1})^2 (C^{Jm1}_{J''m-m_2J_2m_2})^2$$

$$=\sum_m \frac{(J'+m_1+m)!\,(J'-m_1-m)!\;4[mJ_2-m_2(J''+J_2)]^2}{(J_1+m_1)!\,(J_1-m_1)!\,(J''-m_2+m)!\,(J''+m_2-m)!\,(J_2+m_2)!\,(J_2-m_2)!} \quad \text{(Table 2)}$$

$$=K\sum_{m=-J-1+J_2+m_2}^{J+1-J_2+m_2} \binom{J'+m_1+m}{J_1+J_2-1+m_1+m_2}\binom{J'-m_1-m}{J_1+J_2-1-m_1-m_2}[mJ_2-m_2(J''+J_2)]^2$$

(limits from (20))

$$=K\sum_{v=0}^{2J''} \binom{J_1+J_2-1+m_1+m_2+v}{J_1+J_2-1+m_1+m_2}\binom{2J''+J_1+J_2-1-m_1-m_2-v}{J_1+J_2-1-m_1-m_2}\left[J_2v-J''(J_2+m_2)\right]^2$$

(using $m=v-J''+m_2$)

$$=K\Big[J_2^2(J_1+J_2+m_1+m_2+1)(J_1+J_2+m_1+m_2)\binom{2J'+1}{2J''-2}+$$

$$(J_2^2-2J_2J''(J_2+m_2))(J_1+J_2+m_1+m_2)\binom{2J'+1}{2J''-1}+J''^2(J_2+m_2)^2\binom{2J'+1}{2J''}\Big] \quad ,(50)$$

(using (47)). Here $K=\dfrac{4}{(J_1+J_2)^2-(m_1+m_2)^2}\dbinom{J_1+J_2+m_1+m_2}{J_1+m_1}\dbinom{J_1+J_2-m_1-m_2}{J_1-m_1}$

From (20), $-(J_1+J_2-1)\le m_1+m_2\le J_1+J_2-1$, hence the denominator of K does not vanish. The result (50) is awkward to use unless numerical values of the angular momenta are given. As the values of $\lambda_1$, $\lambda_2$ get farther away from those of the special transition the summation results will become more awkward.

## III. $\alpha$-MIXED $\gamma$ CORRELATION FUNCTIONS.

In this section, tables from which $\alpha$-mixed $\gamma(2)$ directional correlation functions can be calculated are given for an $\alpha$ particle having angular momentum 1, 2, or 3. In parts A and B it is shown how the $\alpha(1)$- and $\alpha(2)$-mixed (2) correlation functions are obtained from the Tables II. and III. of Ling and Falkoff[9] for $\gamma(1)$- and $\gamma(2)$-mixed $\gamma(2)$ correlation functions respectively. In part C, those summations appearing in the $\alpha(3)$-mixed $\gamma(2)$ correlation functions are expressed in the form (~~25~~ 43) from which the correlation functions can be fairly easily evaluated once the values of the angular momentum are prescribed.

Some $\alpha$-$\gamma$ correlation functions are listed in references 3 and 11. Some curves of $\alpha$-$\gamma$ correlation functions are given in reference 13. Devons[12] has listed some $\alpha$-mixed $\gamma$ correlation functions - however, some of his results do not agree with the results obtained in this thesis.

Since Hamilton's notation is used to tabulate the angular correlation functions listed in several papers, it is used to tabulate the correlation functions appearing here. The tables are listed in terms of $\Delta j$ and $\Delta J$ defined by $J' = J - \Delta j$ and $J'' = J + \Delta J$, i.e. $\Delta j = \lambda_1 - J_1$ and $\Delta J = \lambda_2 - J_2$.

### A. $\alpha(1)$-mixed $\gamma(2)$ correlation functions.

The $\alpha(J_1)$-mixed $\gamma(2)$ correlation functions can be calculated from

$$W(\theta) = |\alpha|^2 W_{J_1 2}(\theta) + |\beta|^2 W_{J_1 1}(\theta) + 2R(\alpha\beta^*) W_I(\theta), \qquad (51)$$

where

$$W_{J_1 2}(\theta) = \sum_m \left[\sum_{m_1} (JmJ_1m_1|JJ_1J'm+m_1)^2 F^{m_1}_{J_1}(\theta)\right]\left[\sum_{m_2} (J''m-m_2 2m_2|J''2Jm)^2 F^{m_2}_2(0)\right]$$

$$W_{J_1 1}(\theta) = \sum_m \left[\sum_{m_1} (JmJ_1m_1|JJ_1J'm+m_1)^2 F^{m_1}_{J_1}(\theta)\right]\left[\sum_{m_2} (J''m-m_2 1m_2|J''1Jm)^2 F^{m_2}_1(0)\right]$$

$$W_I(\theta) = \sum_m \left[\sum_{m_1} (JmJ_1m_1|JJ_1J'm+m_1)^2 F^{m_1}_{J_1}(\theta)\right] \times \qquad (52)$$

$$\left[\sum_{m_2} (J''m-m_2 2m_2|J''2Jm)(J''m-m_2 1m_2|J''1Jm) F^{m_2}_{21}(0)\right].$$

In the formulae (52), the angles $\phi$ and $\theta$ may be interchanged. Common, non-zero factors in $W(\theta)$ will be omitted. $\alpha$ and $\beta$ represent the probability amplitudes for the electric quadrupole and magnetic dipole radiation respectively. (This notation agrees with the text of Ling and Falkoff's paper, but in their tables II. and III., $\alpha$ and $\beta$ have been interchanged for some unexplained reason.)

In Appendix D it is shown that from the $\gamma(1)$-mixed $\gamma(2)$ correlation functions, $W(\theta) = Q + R\cos^2\theta$, listed in Table II. of Ling and Falkoff's paper, the $\alpha(1)$-mixed $\gamma(2)$ correlation functions, $W(\theta) = Q' + R'\cos^2\theta$, can be calculated using $Q' = \frac{Q+R}{2}$ and $R' = -R$. (53)

The $\alpha(1)$-mixed $\gamma(2)$ correlation functions thus obtained have been tabulated in Table 5. Common, non-zero factors have been omitted.

B. $\alpha(2)$-mixed $\gamma(2)$ correlation functions.

By the method used in A it can be shown that from the $\gamma(2)$-mixed $\gamma(2)$ correlation functions, $W(\theta) = Q + R\cos^2\theta + S\cos^4\theta$, listed in Table III. of Ling and Falkoff's paper the $\alpha(2)$-mixed $\gamma(2)$ correlation functions, $W(\theta) = Q' + R'\cos^2\theta + S'\cos^4\theta$ can be calculated using $Q' = \frac{1}{9}\left[6Q-(2R+3S)\right]$, $R' = \frac{2}{3}\left[2R+3S\right]$, and $S' = -S$. The $\alpha(2)$-mixed $\gamma(2)$ correlation functions thus obtained have been tabulated in Table 5. Common, non-zero factors are omitted from $W(\theta)$.

A misprint was noticed in Ling and Falkoff's Table III. In the $\gamma(2)-\gamma(1)$ correlation functions listed, Q and R are polynomials of the same degree in J for all transitions but $\Delta J = 1$ and $\Delta j = -1$. Calculating the $\gamma(2)-\gamma(1)$ correlation function for this case from Falkoff and Uhlenbeck's[3] paper one can obtain $W(\theta) = 1 + \frac{3(J+6)(2J-1)}{110J^2+269J+174}\cos^2\theta$. This shows that in Ling and Falkoff's Table $\frac{3}{2}(J+2)(110J^2+269J+174)$ in Q should be replaced by $\frac{3}{2}J(J+2)(110J^2+269J+174)$. This has been done to obtain the $\alpha(2)$-mixed $\gamma(2)$ correlation function for this transition in Table 6.

C. $\alpha(3)$-mixed $\gamma(2)$ correlation functions.

The (~~(2)xmixedx(2)xcorrel~~) summations $\sum_m (C^{J'm\lambda_1}_{Jm30})^2 (C^{Jm\,\lambda_2}_{J''m-m_2 2m_2})^2$, $\sum_m (C^{J'm\lambda_1}_{Jm30})^2 (C^{Jm\,\lambda_1-1}_{J''m-m_2 1m_2})^2$, $\sum_m (C^{J'm\lambda_1}_{Jm30})^2 C^{Jm\,\lambda_2}_{J''m-121} C^{Jm\,\lambda_2-1}_{J''m-111}$ are listed in Tables 7,8,9 respectively. They have been placed in the form (43) and tabulated for all values of $\Delta j$, $\Delta J$. They can then be directly evaluated using the formula (47). The coefficients, $A_1$, are too clumsy to tabulate for general values of the angular momenta $J'$, $J$, $J''$. However, since the $A_i$'s are obtained from products of binomials and monomials as in the equation

$$A_r v^r + A_{r-1} v^{r-1} + \dots + A_0 = \left[a_2 v^2 + a_1 v + a_0\right]\left[b_2 v^2 + b_1 v + b_0\right] \cdots \left[\ell_1 v + \ell_0\right],$$

the set of coefficients $a_2$, $a_1$, $a_0$; $b_2$, $b_1$, $b_0$; ...; $\ell_1$, $\ell_0$ can be tabulated instead. Once numerical values are assigned to $J'$, $J$, $J''$ the $A_i$'s are quite easily obtained and the summations can then be evaluated using (47).

The $\alpha(3)$-mixed $\gamma(2)$ correlation function is obtained in terms of $\cos^2\theta$ by substituting the evaluated sums together with the expressions for the F's given in Table 1 in the formula (25) (simplified by the result (29)). For $\alpha(3)-\gamma(2)$ or $\alpha(3)-\gamma(1)$ correlation functions the formula (29) is used.

An example showing how to ~~xxxxd~~ read the Tables 7, 8, and 9 is now given. For the transition in which $\Delta J = 1$, $\Delta j = 2$ or $\lambda_1 = 5$, $\lambda_2 = 3$ in Table 9 one obtains

$$\sum_m \left(C^{J'm5}_{Jm30}\right)^2 C^{Jm3}_{J''m-121} C^{Jm2}_{J''m-111} = 16\sqrt{3}S^3_{2,2J-4,4}$$ with $a_2 = 1$, $a_1 = -2(J-2)$, $a_0 = (J-2)^2$; $b_1 = 2$, $b_0 = -(J-4)$. Using (43), it is seen that this summation is equal to

$$16\sqrt{3}\sum_{v=0}^{2J-4}\binom{2+v}{2}\binom{2J-4+4-v}{4}\left[v^2-2(J-2)v+(J-2)^2\right]\left[2v-(J-4)\right],$$

which can now be evaluated using (47).

If $m_2$ has not been specified numerically in Tables 7 and 8, it means that the results are true for all possible positive and zero values of $m_2$. In Table 9, $m_2 = 1$ since $F^{m_2}_{2,1} = 0$ unless $m_2 = \pm 1$ and only $m_2 = 1$ is required from these two possibilities to calculate the $\alpha(3)$-mixed$\gamma(2)$ correlation function.

Table 1. Exhibition of the Angular Distribution Functions, $F_J^m(\theta)$.

$\alpha$ Particle:

J=1. $F_1^0 = 2\cos^2\theta$

$F_1^{\pm1} = 1-\cos^2\theta$

J=2 $F_2^0 = 1-6\cos^2\theta+9\cos^4\theta$

$F_2^{\pm1} = 6\cos^2\theta - 6\cos^4\theta$

$F_2^{\pm2} = \frac{3}{2}(1-2\cos^2\theta+\cos^4\theta)$

$F_J^m(0)=0$ unless m=0.

Photon:

(These distribution functions are properly weighted so that the correct relative effect of each multipole is represented when a mixed transition occurs.)

J=1. Electric or magnetic dipole.

$F_1^0 = \frac{3}{2}(2-2\cos^2\theta)$

$F_1^{\pm1} = \frac{3}{2}(1+\cos^2\theta)$

J=2. Electric or magnetic quadrupole.

$F_2^0 = \frac{5}{2}(6\cos^2\theta-6\cos^4\theta)$

$F_2^{\pm1} = \frac{5}{2}(1-3\cos^2\theta+4\cos^4\theta)$

$F_2^{\pm2} = \frac{5}{2}(1-\cos^4\theta)$

$F_J^m(0)=0$ unless m=±1.

Mixed electric quadrup ole and magnetic dipole distribution functions.

$F_{2,1}^0(\theta)=0$

$F_{2,1}^{\pm1}(\theta) = \pm\frac{\sqrt{15}}{2}(3\cos^2\theta-1)$

For any J, $F_{J,J-1}^m(\theta) = -F_{J,J-1}^{-m}(\theta)$.

For any particle or photon $F_J^m(\theta) = F_J^{-m}(\theta)$.

Table 2. Explicit expressions for some $C^{Jm\lambda_2}_{J''m''J_2m_2}$ (equation (19)).

$$C^{Jm0}_{J''m''J_2m_2} = \left[\frac{(J+m)!\,(J-m)!}{(J''+m'')!\,(J''-m'')!\,(J_2+m_2)!\,(J_2-m_2)!}\right]^{\frac{1}{2}}$$

$$C^{Jm1}_{J''m''J_2m_2} = 2\left[\frac{(J+m)!\,(J-m)!}{(J''+m'')!\,(J''-m'')!\,(J_2+m_2)!\,(J_2-m_2)!}\right]^{\frac{1}{2}}\left[m''J_2-m_2J''\right]$$

$$C^{JJ\lambda_2}_{J''m''J_2\,J-m''} = (-1)^{J''-m''}\left[\frac{(2J)!}{(2J''-\lambda_2)!\,(2J_2-\lambda_2)!}\binom{J''+m''}{2J''-\lambda_2}\binom{J''+2J_2-\lambda_2-m''}{2J_2-\lambda_2}\right]^{\frac{1}{2}}$$

$J_2=1$:

$$C^{Jm0}_{J-1\,m-m_2\,1m_2} = \left[\frac{(J+m)!\,(J-m)!}{(J-1+m_2-m)!\,(J-1-m_2+m)!\,(1+m_2)!\,(1-m_2)!}\right]^{\frac{1}{2}}$$

$$C^{Jm1}_{Jm-m_2\,1m_2} = 2\left[\frac{(J+m)!\,(J-m)!}{(J-m_2+m)!\,(J+m_2-m)!\,(1+m_2)!\,(1-m_2)!}\right]^{\frac{1}{2}}\left[m-m_2(J+1)\right]$$

$$C^{Jm2}_{J+1\,m-m_2\,1m_2} = (-1)^{1+m_2}\left[\frac{(J+1-m_2+m)!\,(J+1+m_2-m)!}{(J+m)!\,(J-m)!\,(1-m_2)!\,(1+m_2)!}\right]^{\frac{1}{2}}$$

$J_2=2$:

$$C^{Jm0}_{J-2\,m-m_2\,2m_2} = \left[\frac{(J+m)!\,(J-m)!}{(J-2-m_2+m)!\,(J-2+m_2-m)!\,(2+m_2)!\,(2-m_2)!}\right]^{\frac{1}{2}}$$

$$C^{Jm1}_{J-1\,m-m_2\,2m_2} = 2\left[\frac{(J+m)!\,(J-m)!}{(J-1-m_2+m)!\,(J-1+m_2-m)!\,(2+m_2)!\,(2-m_2)!}\right]^{\frac{1}{2}}\left[2m-m_2(J+1)\right]$$

$$C^{Jm2}_{Jm20} = 3m^2-J(J+1)$$

$$C^{Jm2}_{Jm-121} = -\left[\frac{3}{2}(J+m)(J+1-m)\right]^{\frac{1}{2}}\left[2m-1\right]$$

$$C^{Jm2}_{Jm-222} = \left[\frac{3}{2}\,\frac{(J+m)!\,(J+2-m)!}{(J+m-2)!\,(J-m)!}\right]^{\frac{1}{2}}$$

$$C^{Jm3}_{J+1\,m-m_2\,2m_2} = (-1)^{1+m_2}\,2\left[\frac{(J+1-m_2+m)!\,(J+1+m_2-m)!}{(J+m)!\,(J-m)!\,(2+m_2)!\,(2-m_2)!}\right]^{\frac{1}{2}}\left[2m+m_2J\right]$$

$$C^{Jm4}_{J+2\,m-m_2\,2m_2} = (-1)^{m_2}\left[\frac{(J+2-m_2+m)!\,(J+2+m_2-m)!}{(J+m)!\,(J-m)!\,(2+m_2)!\,(2-m_2)!}\right]^{\frac{1}{2}}$$

$J_1=3$, $m_1=0$:

$$C^{J+3m0}_{Jm30} = \frac{1}{3!}\left[\frac{(J+3+m)!\,(J+3-m)!}{(J+m)!\,(J-m)!}\right]^{\frac{1}{2}}$$

$$C^{J+2m1}_{Jm30} = \left[\frac{(J+2+m)!\,(J+2-m)!}{(J+m)!\,(J-m)!}\right]^{\frac{1}{2}}m$$

$$C^{J+1m2}_{Jm30} = \frac{1}{2}\left[(J+1+m)(J+1-m)\right]^{\frac{1}{2}}\left[5m^2-J(J+2)\right]$$

$$C^{Jm3}_{Jm30} = \frac{2}{3}m\left[5m^2-(3J^2+3J-1)\right]$$

$$C^{J-1m4}_{Jm30} = -\frac{1}{2}\left[(J+m)(J-m)\right]^{\frac{1}{2}}\left[5m^2-(J^2-1)\right]$$

$$C^{J-2m5}_{Jm30} = \left[\frac{(J+m)!\,(J-m)!}{(J+m-2)!\,(J-m-2)!}\right]^{\frac{1}{2}}m$$

$$C^{J-3m6}_{Jm30} = -\frac{1}{3!}\left[\frac{(J+m)!\,(J-m)!}{(J+m-3)!\,(J-m-3)!}\right]^{\frac{1}{2}}$$

Table 3. A tabulation of the coefficients $a_{k\alpha}$ defined by (49).

| $k \backslash \alpha$ | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | . | . | . | . | . | . | . | . | . | 1 |
| 2 | . | . | . | . | . | . | . | . | 1 | 1 |
| 3 | . | . | . | . | . | . | . | 1 | 3 | 1 |
| 4 | . | . | . | . | . | . | 1 | 6 | 7 | 1 |
| 5 | . | . | . | . | . | 1 | 10 | 25 | 15 | 1 |
| 6 | . | . | . | . | 1 | 15 | 65 | 90 | 31 | 1 |
| 7 | . | . | . | 1 | 21 | 140 | 350 | 301 | 63 | 1 |
| 8 | . | . | 1 | 28 | 266 | 1050 | 1701 | 966 | 127 | 1 |
| 9 | . | 1 | 36 | 462 | 2646 | 6951 | 7770 | 3025 | 255 | 1 |
| 10 | 1 | 45 | 750 | 5880 | 22827 | ~~(42[illegible]25)~~ 42525 | 34105 | 9330 | 511 | 1 |

Table 4. The following abbreviations are used in Tables 5 and 6.

$$d_1 = [15J(J+2)]^{\frac{1}{2}}$$
$$d_2 = (2J-1)[15J(J+2)]^{\frac{1}{2}}$$
$$d_3 = [5(2J-1)(2J+3)]^{\frac{1}{2}}$$
$$d_4 = (2J+3)[15(J^2-1)]^{\frac{1}{2}}$$
$$d_5 = [15(J^2-1)]^{\frac{1}{2}}$$
$$d_6 = (J-5)[15J(J+2)]^{\frac{1}{2}}$$
$$d_7 = (2J-3)(2J+5)[15J(J+2)]^{\frac{1}{2}}$$
$$d_8 = (J+2)(2J-1)(J+6)[15J(J+2)]^{\frac{1}{2}}$$
$$d_9 = (J+2)(2J-1)[15J(J+2)]^{\frac{1}{2}}$$
$$d_{10} = 3[5(2J-1)(2J+3)]^{\frac{1}{2}}$$
$$d_{11} = 3(J-5)[5(2J-1)(2J+3)]^{\frac{1}{2}}$$
$$d_{12} = (2J-3)(2J+5)[5(2J-1)(2J+3)]^{\frac{1}{2}}$$
$$d_{13} = 3(J+6)[5(2J-1)(2J+3)]^{\frac{1}{2}}$$
$$d_{14} = 3[5(2J-1)(2J+3)]^{\frac{1}{2}}$$
$$d_{15} = (J-1)(2J+3)[15(J^2-1)]^{\frac{1}{2}}$$
$$d_{16} = (J-1)(2J+3)(J-5)[15(J^2-1)]^{\frac{1}{2}}$$
$$d_{17} = \frac{1}{3}(2J-3)(2J+5)[15(J^2-1)]^{\frac{1}{2}}$$
$$d_{18} = (J+6)[15(J^2-1)]^{\frac{1}{2}}$$

Table 5. $\alpha(1)$-mixed$\delta(2)$ correlation functions.

$$W(\theta) = Q' + R'\cos^2\theta.*$$

| $\Delta J$ | $\Delta j$ | | $\lvert\alpha\rvert^2$ | $\lvert\beta\rvert^2$ | $2\,R(\alpha\beta^*)$ |
|---|---|---|---|---|---|
| 2 ($\beta=0$) | 1 | Q' | 13 | | |
| | | R' | 3 | | |
| | 0 | Q' | $16J-7$ | | |
| | | R' | $-3(2J-7)$ | | |
| | -1 | Q' | $26J^2+71J+42$ | | |
| | | R' | $3J(2J-1)$ | | |
| 1 | 1 | Q' | $\frac{5}{21}(29J+6)$ | $7J$ | $-\frac{1}{3}d_1$ |
| | | R' | $-\frac{5}{7}(J+6)$ | $-J$ | $d_1$ |
| | 0 | Q' | $\frac{5}{21}(26J^2+17J+6)$ | $J(6J+7)$ | $\frac{1}{3}d_2$ |
| | | R' | $\frac{5}{7}(J+6)(2J-1)$ | $J(2J-1)$ | $-d_2$ |
| | -1 | Q' | $\frac{5}{21}(58J^2+151J+78)$ | $14J^2+33J+20$ | $-\frac{1}{3}d_2$ |
| | | R' | $-\frac{5}{7}(J+6)(2J-1)$ | $-J(2J-1)$ | $d_2$ |
| 0 | 1 | Q' | $\frac{5}{7}(20J^2-8J-5)$ | $(2J-1)(6J-1)$ | $d_3$ |
| | | R' | $-\frac{5}{7}(2J+5)(2J-3)$ | $(2J-1)(2J+3)$ | $-3d_3$ |
| | 0 | Q' | $\frac{5}{7}(8J^2+8J+5)$ | $8J^2+8J-1$ | $-d_3$ |
| | | R' | $\frac{5}{7}(2J-3)(2J+5)$ | $-(2J-1)(2J+3)$ | $3d_3$ |
| | -1 | Q' | $\frac{5}{7}(20J^2+48J+23)$ | $(2J+3)(6J+7)$ | $d_3$ |
| | | R' | $-\frac{5}{7}(2J-3)(2J+5)$ | $(2J-1)(2J+3)$ | $-3d_3$ |

*Correlation functions for those transitions in which only a pure electric multipole ($\beta=0$) is emitted, have been included in the Tables 5 and 6 for the sake of completeness. These were obtained from reference 3.

Table 5. (continued).

| ΔJ | Δj | | $\lvert\alpha\rvert^2$ | $\lvert\beta\rvert^2$ | $2R(\alpha\beta^*)$ |
|---|---|---|---|---|---|
| -1 | 1 | Q' | $\frac{5}{21}(58J^2-35J-15)$ | $14J^2-5J+1$ | $\frac{1}{3}d_4$ |
| | | R' | $-\frac{5}{7}(2J+3)(J-5)$ | $-(J+1)(2J+3)$ | $-d_4$ |
| | 0 | Q' | $\frac{5}{21}(26J^2+35J+15)$ | $(J+1)(6J-1)$ | $-\frac{1}{3}d_4$ |
| | | R' | $\frac{5}{7}(2J+3)(J-5)$ | $(J+1)(2J+3)$ | $d_4$ |
| | -1 | Q' | $\frac{5}{21}(29J+23)$ | $7(J+1)$ | $\frac{1}{3}d_5$ |
| | | R' | $-\frac{5}{7}(J-5)$ | $-(J+1)$ | $-d_5$ |
| -2 ($\beta=0$) | 1 | Q' | $26J^2-19J-3$ | | |
| | | R' | $3(J+1)(2J+3)$ | | |
| | 0 | Q' | $16J+3$ | | |
| | | R' | $-3(2J+3)$ | | |
| | -1 | Q' | $13$ | | |
| | | R' | $3$ | | |

Example: For $\Delta J = -1$, $\Delta j = 1$, one obtains

$$Q' = \frac{5}{21}(58J^2-35J-15)\lvert\alpha\rvert^2 + (14J^2-5J+1)\lvert\beta\rvert^2 + \frac{1}{3}d_4\,2R(\alpha\beta^*)$$

$$R' = -\frac{5}{7}(2J+3)(J-5)\lvert\alpha\rvert^2 - (J+1)(2J+3)\lvert\beta\rvert^2 - d_4\,2R(\alpha\beta^*),$$

from which $W(\theta) = Q'+R'\cos^2\theta$ is easily obtained.

Table 6. $\alpha(2)$-mixed $\gamma(2)$ correlation functions.

$$W(\theta) = Q' + R'\cos^2\theta + S'\cos^4\theta.$$

| $\Delta J$ | $\Delta j$ | | $\lvert\alpha\rvert^2$ | $\lvert\beta\rvert^2$ | $2R(\alpha\beta^*)$ |
|---|---|---|---|---|---|
| 2 ($\beta = 0$) | 2 | Q' | 15 | | |
| | | R' | 6 | | |
| | | S' | -1 | | |
| | 1 | Q' | -2(3J+2) | | |
| | | R' | 2(J-3) | | |
| | | S' | $-\frac{2}{3}(2J-3)$ | | |
| | 0 | Q' | -4(2J+1)(J+2) | | |
| | | R' | 4(2J-3) | | |
| | | S' | $\frac{4}{3}(J-1)(2J-3)$ | | |
| | -1 | Q' | $-(12J^3+54J^2+78J+30)$ | | |
| | | R' | $2(2J-1)(J^2+2J+6)$ | | |
| | | S' | $-\frac{2}{3}(J-1)(2J-1)(2J-3)$ | | |
| | -2 | Q' | $-(20J^4+148J^3+391J^2+437J+168)$ | | |
| | | R' | $-2J(2J-1)(2J^2+7J+9)$ | | |
| | | S' | $\frac{1}{3}J(J-1)(2J-1)(2J-3)$ | | |
| 1 | 2 | Q' | 5(3J+1) | 15J | $-d_1$ |
| | | R' | -5(J+4) | -3J | $3d_1$ |
| | | S' | $\frac{5}{3}(2J+5)$ | | |
| | 1 | Q' | $5(5J^2+5J+6)$ | 3J(9J+1) | $d_6$ |
| | | R' | 25(J+3)(J-2) | 3J(J-5) | $-3d_6$ |
| | | S' | $-\frac{20}{3}(2J-3)(2J+5)$ | | |
| | 0 | Q' | $5(4J^3+8J^2+3J+6)$ | $\frac{1}{3}(52J^3+136J+99)$ | $\frac{1}{3}d_7$ |
| | | R' | -5(2J-3)(2J+5)(J-2) | J(2J-3)(2J+5) | $-d_7$ |

Table 6 (continued).

| $\Delta J$ | $\Delta j$ | | $\lvert\alpha\rvert^2$ | $\lvert\beta\rvert^2$ | $2R(\alpha\beta^*)$ |
|---|---|---|---|---|---|
| | | S' | $\frac{20}{3}(2J-3)(2J+5)(J-1)$ | | |
| | -1 | Q' | $5(10J^4+47J^3+62J^2+16J+12)$ | $3J(J+2)(18J^2+43J+30)$ | $d_8$ |
| | | R' | $5(2J-1)(5J^3+6J^2+4J+48)$ | $3J(J+2)(J+6)(2J-1)$ | $-3d_8$ |
| | | S' | $-\frac{20}{3}(J-1)(2J-3)(2J+5)(2J-1)$ | | |
| | -2 | Q' | $15(2J^3+9J^2+13J+5)$ | $3(J+2)(10J^2+23J+14)$ | $-d_9$ |
| | | R' | $-5(2J-1)(J^2+2J+6)$ | $-3J(J+2)(2J-1)$ | $3d_9$ |
| | | S' | $\frac{5}{3}(2J-1)(J-1)(2J-3)$ | | |
| 0 | 2 | Q' | $15(2J+1)(J-1)$ | $3(2J-1)(4J-1)$ | $d_{10}$ |
| | | R' | $15(2J+5)$ | $3(2J-1)(2J+3)$ | $-3d_{10}$ |
| | | S' | $-5(J+2)(2J+5)$ | | |
| | 1 | Q' | $15(4J^3+4J^2-J-7)$ | $3(2J-1)(10J^2+7J-5)$ | $-d_{11}$ |
| | | R' | $-15(2J-3)(2J+5)(J+3)$ | $-3(2J-1)(2J+3)(J-5)$ | $3d_{11}$ |
| | | S' | $20(2J-3)(2J+5)(J+2)$ | | |
| | 0 | Q' | $\frac{5}{3}(16J^4+32J^3+40J^2+24J-63)$ | $\frac{1}{3}(2J-1)(2J+3)(32J^2+32J-15)$ | $-d_{12}$ |
| | | R' | $5(2J-3)(2J+5)(4J^2+4J-9)$ | $-(2J-1)(2J+3)(2J-3)(2J+5)$ | $3d_{12}$ |
| | | S' | $-20(2J-3)(2J+5)(J-1)(J+2)$ | | |
| | -1 | Q' | $15(4J^3+8J^2+3J+6)$ | $3(2J+3)(10J^2+13J-2)$ | $-d_{13}$ |
| | | R' | $-15(2J-3)(2J+5)(J-2)$ | $-3(2J+3)(2J-1)(J+6)$ | $3d_{13}$ |
| | | S' | $20(2J-3)(2J+5)(J-1)$ | | |
| | -2 | Q' | $15(2J+1)(J+2)$ | $3(2J+3)(4J+5)$ | $d_{14}$ |
| | | R' | $-15(2J-3)$ | $3(2J+3)(2J-1)$ | $-3d_{14}$ |
| | | S' | $-5(J-1)(2J-3)$ | | |
| -1 | 2 | Q' | $15(2J^3-3J^2+J+1)$ | $3(J-1)(10J^2-3J+1)$ | $d_{15}$ |
| | | R' | $-5(2J+3)(J^2+5)$ | $-3(J-1)(2J+3)(J+1)$ | $-3d_{15}$ |
| | | S' | $\frac{5}{3}(2J+3)(J+2)(2J+5)$ | | |

Table 6 (continued).

| $\Delta J$ | $\Delta j$ | | $\|\alpha\|^2$ | $\|\beta\|^2$ | $2R(\alpha\beta^*)$ |
|---|---|---|---|---|---|
| | 1 | Q' | $5(10J^4-7J^3-19J^2+7J+21)$ | $3(J+1)(J-1)(18J^2-7J+5)$ | $-d_{16}$ |
| | | R' | $5(2J+3)(5J^3+9J^2+7J-45)$ | $3(J+1)(J-1)(2J+3)(J-5)$ | $3d_{16}$ |
| | | S' | $-\frac{20}{3}(2J+3)(2J-3)(2J+5)(J+2)$ | | |
| | 0 | Q' | $5(4J^3+4J^2-J-7)$ | $\frac{1}{3}(J+1)(52J^2-32J+15)$ | $-d_{17}$ |
| | | R' | $-5(2J-3)(2J+5)(J+3)$ | $(J+1)(2J-3)(2J+5)$ | $3d_{17}$ |
| | | S' | $\frac{20}{3}(J+2)(2J-3)(2J+5)$ | | |
| | -1 | Q' | $5(5J^2+5J+6)$ | $3(J+1)(9J-2)$ | $-d_{18}$ |
| | | R' | $5(5J-10)(J+3)$ | $3(J+1)(J+6)$ | $3d_{18}$ |
| | | S' | $-\frac{20}{3}(2J-3)(2J+5)$ | | |
| | -2 | Q' | $5(3J+2)$ | $15(J+1)$ | $d_5$ |
| | | R' | $-5(J-3)$ | $-3(J+1)$ | $-3d_5$ |
| | | S' | $\frac{5}{3}(2J-3)$ | | |
| -2 ($\beta=0$) | 2 | Q' | $-(20J^4-68J^3+67J^2-19J-6)$ | | |
| | | R' | $-2(J+1)(2J+3)(2J^2-3J+4)$ | | |
| | | S' | $\frac{1}{3}(J+1)(2J+3)(J+2)(2J+5)$ | | |
| | 1 | Q' | $-6(2J^3-3J^2+J+1)$ | | |
| | | R' | $2(2J+3)(J^2+5)$ | | |
| | | S' | $-\frac{2}{3}(2J+3)(J+2)(2J+5)$ | | |
| | 0 | Q' | $-4(2J^2-J-1)$ | | |
| | | R' | $-4(2J+5)$ | | |
| | | S' | $\frac{4}{3}(J+2)(2J+5)$ | | |
| | -1 | Q' | $-2(3J+1)$ | | |
| | | R' | $-(J+10)$ | | |
| | | S' | $-\frac{2}{3}(2J+5)$ | | |
| | -2 | Q' | 15 | | |
| | | R' | 6 | | |
| | | S' | -1 | | |

Table 7. The summations $\sum_{m}(C^{J'm\lambda_1}_{Jm30})^2(C^{Jm\ \lambda_2}_{J''m-m_2 2m_2})^2$. $J'=J-\Delta j$, $J=J''-\Delta J$ or $\Delta J=\lambda_2-2$, $\Delta j=\lambda_1-3$.

| $\Delta J$ | $\Delta j$ | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| 2 | 3 | $m_2$ | $\binom{5+m_2}{3}\binom{5-m_2}{3} S^0_{5-m_2,2J-6,5+m_2}$ | | | 1 |
| | 2 | $m_2$ | $\binom{4+m_2}{2}\binom{4-m_2}{2} 4S^2_{4-m_2,2J-4,4+m_2}$ | | | |
| | | | | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | 1 | $m_2$ | $\frac{9-m_2^2}{4}S^4_{3-m_2,2J-2,3+m_2}$ | | | |
| | | | | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | | | | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | 0 | $m_2$ | $\frac{4}{9}S^6_{2-m_2,2J,2+m_2}$ | 5 | $-10J$ | $(J-1)(2J-1)$ |
| | | | | 5 | $-10J$ | $(J-1)(2J-1)$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | -1 | $m_2$ | $-\frac{1}{4}S^6_{2-m_2,2J,2+m_2}$ | 5 | $-10J$ | $2J(2J-1)$ |
| | | | | 5 | $-10J$ | $2J(2J-1)$ |
| | | | | 1 | $-2J$ | $-2J-1$ |
| | -2 | $m_2$ | $S^6_{2-m_2,2J,2+m_2}$ | 1 | $-2J$ | $-4J-4$ |
| | | | | 1 | $-2J$ | $-2J-1$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | -3 | 0 | $\frac{1}{4}S^4_{3,2J,3}$ | 1 | 3 | 2 |
| | | | | 1 | $-4J-3$ | $(2J+1)(2J+2)$ |
| | | 1 | $-\frac{1}{6}S^4_{3,2J,3}$ | 1 | $-2J-2$ | $-2J-3$ |
| | | | | 1 | $-4J-3$ | $(2J+1)(2J+2)$ |
| | | 2 | $-\frac{1}{6}S^3_{3,2J,4}$ | 1 | $-4J-5$ | $(2J+2)(2J+3)$ |
| | | | | | 1 | $-2J-1$ |
| 1 | 3 | 0,1 | $16\binom{4+m_2}{2}\binom{4-m_2}{2} S^2_{4-m_2,2J-6,4+m_2}$ | | | |
| | | | | 4 | $-4(2J-m_2J-6)$ | $(2J-m_2J-6)^2$ |

Table 7 (continued).

| $\Delta J$ | $\Delta j$ | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| 1 | 3 | 2 | $80S^1_{3,2J-6,6}$ | | 1 | 3 |
| | 2 | 0,1 | $4(9-m_2^2)S^4_{3-m_2,2J-4,3+m_2}$ | | | |
| | | | | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 4 | $-4(2J-m_2J-4)$ | $(2J-m_2J-4)^2$ |
| | | 2 | $160S^3_{2,2J-4,5}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | | 1 | 2 |
| | 1 | $m_2$ | $S^6_{2-m_2,2J-2,2+m_2}$ | | | |
| | | | | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | | | | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | | | | 4 | $4(2-2J+m_2J)$ | $(2-2J+m_2J)^2$ |
| | 0 | 0,1 | $\frac{16}{9(4-m_2^2)}S^8_{1-m_2,2J,1+m_2}$ | | | |
| | | | | 5 | $-10J$ | $(2J-1)(J-1)$ |
| | | | | 5 | $-10J$ | $(2J-1)(J-1)$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | | | | 4 | $-4J(2-m_2)$ | $J^2(2-m_2)^2$ |
| | | 2 | $\frac{16}{9}S^6_{1,2J-1,3}$ | 5 | $-10(J-1)$ | $(2J-1)(J-6)$ |
| | | | | 5 | $-10(J-1)$ | $(2J-1)(J-6)$ |
| | | | | 1 | $-2(J-1)$ | $(J-1)^2$ |
| | -1 | 0 | $-S^8_{1,2J,1}$ | 5 | $-10J$ | $J(4J-2)$ |
| | | | | 5 | $-10J$ | $J(4J-2)$ |
| | | | | 1 | $-2J$ | $-2J-1$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | | 1 | $-\frac{1}{3}S^7_{1,2J,2}$ | 5 | $-10J$ | $J(4J-2)$ |
| | | | | 5 | $-10J$ | $J(4J-2)$ |
| | | | | 4 | $-4J$ | $J^2$ |
| | | | | | 1 | $-2J-1$ |

Table 7 (continued).

| $\Delta J$ | $\Delta j$ | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| 1 | -1 | 2 | $-2S^5_{2,2J-1,3}$ | 5 | $-10(J-1)$ | $(2J-1)(2J-5)$ |
| | | | | 5 | $-10(J-1)$ | $(2J-1)(2J-5)$ |
| | | | | | 1 | $-2J$ |
| | -2 | 0 | $-16S^6_{2,2J,2}$ | 1 | $-2J$ | $J^2$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | | | | 1 | $-2J$ | $-2J-1$ |
| | | 1 | $-\frac{8}{3}S^6_{2,2J,2}$ | 1 | $-2J$ | $J^2$ |
| | | | | 4 | $-4J$ | $J^2$ |
| | | | | 1 | $-4J-3$ | $(2J+1)(2J+2)$ |
| | | 2 | $24S^4_{3,2J-1,3}$ | 1 | $-2(J-1)$ | $(J-1)^2$ |
| | | | | 1 | $-4J-1$ | $2J(2J+1)$ |
| | -3 | 0 | $-4S^4_{3,2J,3}$ | 1 | $-2J$ | $-2J-1$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | | 1 | $\frac{2}{3}S^4_{3,2J,3}$ | 1 | $-4J-3$ | $(2J+2)(2J+1)$ |
| | | | | 4 | $-4J$ | $J^2$ |
| | | 2 | $-\frac{8}{3}S^3_{4,2J-1,3}$ | 1 | $-4J-3$ | $(2J+1)(2J+2)$ |
| | | | | | 1 | $-2J$ |
| 0 | 3 | 0 | $S^4_{3,2J-6,3}$ | 3 | $-6(J-3)$ | $2J^2-19J+27$ $\cancel{(2J-1)(J-9)}$ |
| | | | | 3 | $-6(J-3)$ | $2J^2-19J+27$ $\cancel{(2J-1)(J-9)}$ |
| | | 1 | $6S^3_{3,2J-6,4}$ | 4 | $-4(2J-5)$ | $(2J-5)^2$ |
| | | | | | 1 | 3 |
| | | 2 | $30S^2_{3,2J-6,5}$ | 1 | 5 | 6 |
| | 2 | 0 | $4S^6_{2,2J-4,2}$ | 3 | $-6(J-2)$ | $2J^2-13J+12$ $\cancel{(2J-1)(J-8)}$ |
| | | | | 3 | $-6(J-2)$ | $2J^2-13J+12$ |
| | | | | 1 | $-2(J-2)$ | $(J-2)^2$ |

Table 7 (continued).

| J | j | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| 0 | 2 | 1 | $18S^5_{2,2J-4,3}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 4 | $-4(2J-3)$ | $(2J-3)^2$ |
| | | | | | 1 | 2 |
| | | 2 | $72S^4_{2,2J-4,4}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 1 | 3 | 2 |
| | 1 | 0 | $\frac{1}{4}S^8_{1,2J-2,1}$ | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | | | | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | | | | 3 | $-6(J-1)$ | $2J^2-7J+3$ |
| | | | | 3 | $-6(J-1)$ | $2J^2-7J+3$ |
| | | 1 | $\frac{3}{4}S^7_{1,2J-2,2}$ | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | | | | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | | | | 4 | $-4(2J-1)$ | $(2J-1)^2$ |
| | | | | | 1 | 1 |
| | | 2 | $\frac{9}{2}S^5_{2,2J-3,3}$ | 5 | $-10(J-2)$ | $4J^2-20J+21$ |
| | | | | 5 | $-10(J-2)$ | $4J^2-20J+21$ |
| | | | | | 1 | 2 |
| | 0 | 0 | $\frac{4}{9}S^{10}_{0,2J,0}$ | 5 | $-10J$ | $(2J-1)(J-1)$ |
| | | | | 5 | $-10J$ | $(2J-1)(J-1)$ |
| | | | | 3 | $-6J$ | $J(2J-1)$ |
| | | | | 3 | $-6J$ | $J(2J-1)$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | | 1 | $\frac{2}{3}S^8_{1,2J-1,1}$ | 5 | $-10(J-1)$ | $(2J-1)(J-6)$ |
| | | | | 5 | $-10(J-1)$ | $(2J-1)(J-6)$ |
| | | | | 1 | $-2(J-1)$ | $(J-1)^2$ |
| | | | | 4 | $-4(2J-1)$ | $(2J-1)^2$ |

Table 7 (continued).

| $\Delta J$ | $\Delta j$ | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | $\frac{8}{3}S^6_{2,2J-2,2}$ | 5 | $-10(J-2)$ | $(2J-1)(J-21)$ |
| | | | | 5 | $-10(J-2)$ | $(2J-1)(J-21)$ |
| | | | | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | -1 | 0 | $\frac{1}{4}S^8_{1,2J,1}$ | 5 | $-10J$ | $J(3J-2)$ |
| | | | | 5 | $-10J$ | $J(3J-2)$ |
| | | | | 3 | $-6J$ | $J(2J-1)$ |
| | | | | 3 | $-6J$ | $J(2J-1)$ |
| | | 1 | $-\frac{3}{4}S^7_{2,2J-1,1}$ | 5 | $-10(J-1)$ | $(2J-1)(2J-5)$ |
| | | | | 5 | $-10(J-1)$ | $(2J-1)(2J-5)$ |
| | | | | 4 | $-4(2J-1)$ | $(2J-1)^2$ |
| | | | | | 1 | $-2J$ |
| | | 2 | $-\frac{9}{2}S^5_{3,2J-2,2}$ | 5 | $-10(J-2)$ | $4J^2-22J+20$ |
| | | | | 5 | $-10(J-2)$ | $4J^2-22J+20$ |
| | | | | | 1 | $-(2J-1)$ |
| | -2 | 0 | $4S^6_{2,2J,2}$ | 3 | $-6J$ | $J(2J-1)$ |
| | | | | 3 | $-6J$ | $J(2J-1)$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | | 1 | $-18S^3_{3,2J-1,2}$ | 4 | $-4(2J-1)$ | $(2J-1)^2$ |
| | | | | | 1 | $-2J$ |
| | | 2 | $72S^4_{4,2J-2,2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 1 | $-(4J-1)$ | $2J(2J-1)$ |
| | -3 | 0 | $S^4_{3,2J,3}$ | 3 | $-6J$ | $J(2J-1)$ |
| | | | | 3 | $-6J$ | $J(2J-1)$ |
| | | 1 | $-6S^3_{4,2J-1,3}$ | 4 | $-4(2J-1)$ | $(2J-1)^2$ |
| | | | | | 1 | $-2J$ |

Table 7 (continued).

| $\Delta J$ | $\Delta j$ | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| 0 | -3 | 2 | $30S^2_{5,2J-2,3}$ | 1 | $-(4J-1)$ | $2J(2J-1)$ |
| -1 | 3 | 0 | $-4S^4_{3,2J-6,3}$ | 1 | $-2(J-3)$ | $-3(2J-3)$ |
| | | | | 1 | $-2(J-3)$ | $(J-3)^2$ |
| | | 1 | $\frac{2}{3}S^4_{3,2J-6,3}$ | 4 | $-4(3J-5)$ | $(3J-5)^2$ |
| | | | | 1 | 5 | 6 |
| | | 2 | $\frac{8}{3}S^3_{3,2J-6,4}$ | 1 | 3 | 2 |
| | | | | | 1 | 3 |
| | 2 | 0 | $-16S^6_{2,2J-4,2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 1 | $-2(J-2)$ | $-4(J-1)$ |
| | | 1 | $\frac{8}{3}S^6_{2,2J-4,2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 1 | 3 | 2 |
| | | | | 4 | $-12(J-1)$ | $9(J-1)^2$ |
| | | 2 | $24S^4_{3,2J-5,3}$ | 1 | $-2(J-3)$ | $(J-3)^2$ |
| | | | | 1 | 5 | 6 |
| | 1 | 0 | $-S^8_{1,2J-2,1}$ | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | | | | 5 | $-10(J-1)$ | $(J-1)(4J-6)$ |
| | | | | 1 | $-2(J-1)$ | $(J-1)^2$ |
| | | | | 1 | $-2(J-1)$ | $-(2J-1)$ |
| | | 1 | $\frac{1}{3}S^7_{2,2J-3,1}$ | 5 | $-10(J-2)$ | $4J^2-20J+21$ |
| | | | | 5 | $-10(J-2)$ | $4J^2-20J+21$ |
| | | | | 4 | $-12(J-1)$ | $9(J-1)^2$ |
| | | | | | 1 | 2 |
| | | 2 | $2S^5_{3,2J-4,2}$ | 5 | $-10(J-3)$ | $4J^2-30J+46$ |
| | | | | 5 | $-10(J-3)$ | $4J^2-30J+46$ |
| | | | | | 1 | 3 |

Table 1 (continued).

| $\Delta J$ | $\Delta j$ | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| -1 | 0 | 0,1 | $\frac{16}{9(4-m_2^2)}S^8_{1+m_2,2J-2,1-m_2}$ | | | |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-(3J^2+3J-1)$ |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-(3J^2+3J-1)$ |
| | | | | 1 | $-2(J-1-m_2)$ | $(J-1-m_2)^2$ |
| | | | | 4 | $-4(J-1)(2+m_2)$ | $(J-1)^2(2+m_2)^2$ |
| | | 2 | $\frac{16}{9}S^6_{3,2J-3,1}$ | 5 | $-10(J-3)$ | $2J^2-33J+46$ |
| | | | | 5 | $-10(J-3)$ | $2J^2-33J+46$ |
| | | | | 1 | $-2(J-3)$ | $(J-3)^2$ |
| | -1 | $m_2$ | $S^6_{2+m_2,2J-2,2-m_2}$ | | | |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-J(J+2)$ |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-J(J+2)$ |
| | | | | 4 | $-4(J-1)(2+m_2)$ | $(J-1)^2(2+m_2)^2$ |
| | -2 | $m_2$ | $4(9-m_2^2)S^4_{3+m_2,2(J-1),3-m_2}$ | | | |
| | | | | 1 | $-2(J-1-m_2)$ | $(J-1-m_2)^2$ |
| | | | | 4 | $-4(J-1)(2+m_2)$ | $(J-1)^2(2+m_2)^2$ |
| | -3 | $m_2$ | $\frac{4}{9}\binom{4+m_2}{2}\binom{4-m_2}{2}S^2_{4+m_2,2J-2,4-m_2}$ | | | |
| | | | | 4 | $-4(J-1)(2+m_2)$ | $(J-1)^2(2+m_2)^2$ |
| -2 | 3 | 0 | $\frac{1}{4}S^4_{3,2J-6,3}$ | 1 | $-2(J-3)$ | $-3(2J-3)$ |
| | | | | 1 | $-2(J-3)$ | $-4(J-2)$ |
| | | 1 | $-\frac{1}{6}S^4_{3,2J-6,3}$ | 1 | $-2(J-3)$ | $-3(2J-3)$ |
| | | | | 1 | 3 | 2 |
| | | 2 | $\frac{1}{6}S^3_{4,2J-7,3}$ | 1 | 5 | 6 |
| | | | | | 1 | 4 |
| | 2 | 0 | $S^6_{2,2J-4,2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 1 | $-2(J-2)$ | $-4(J-1)$ |
| | | | | 1 | $-2(J-2)$ | $-2J+3$ |

Table 7 (continued).

| $\Delta J$ | $\Delta j$ | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| -2 | 2 | 1 | $-2S^5_{3,2J-5,2}$ | 1 | $-2(J-3)$ | $-3(2J-3)$ |
| | | | | 1 | $-2(J-3)$ | $(J-3)^2$ |
| | | | | | 1 | 2 |
| | | 2 | $2S^4_{4,2J-6,2}$ | 1 | $-2(J-4)$ | $(J-4)^2$ |
| | | | | 1 | 7 | 12 |
| | 1 | 0,1 | $-\frac{1}{4}S^6_{2+m_2,2J-4,2-m_2}$ | | | |
| | | | | 5 | $-10(J-2-m_2)$ | $5(J-2-m_2)^2-(J^2-1)$ |
| | | | | 5 | $-10(J-2-m_2)$ | $5(J-2-m_2)^2-(J^2-1)$ |
| | | | | 1 | $-2(J-2-m_2)$ | $(J-2-m_2)^2-J^2$ |
| | | 2 | $\frac{1}{4}S^5_{4,2J-5,1}$ | 5 | $-10(J-4)$ | $4J^2-40J+81$ |
| | | | | 5 | $-10(J-4)$ | $4J^2-40J+81$ |
| | | | | | 1 | 4 |
| | 0 | $m_2$ | $\frac{4}{9}S^6_{2+m_2,2J-4,2-m_2}$ | | | |
| | | | | 5 | $-10(J-2-m_2)$ | $5(J-2-m_2)^2-(3J^2+3J-1)$ |
| | | | | 5 | $-10(J-2-m_2)$ | $5(J-2-m_2)^2-(3J^2+3J-1)$ |
| | | | | 1 | $-2(J-2-m_2)$ | $(J-2-m_2)^2$ |
| | -1 | $m_2$ | $\frac{(9-m_2^2)}{4}S^4_{3+m_2,2J-4,3-m_2}$ | | | |
| | | | | 5 | $-10(J-2-m_2)$ | $5(J-2-m_2)^2-J(J+2)$ |
| | | | | 5 | $-10(J-2-m_2)$ | $5(J-2-m_2)^2-J(J+2)$ |
| | -2 | $m_2$ | $4\binom{4+m_2}{2}\binom{4-m_2}{2}S^2_{4+m_2,2J-4,4-m_2}$ | | | |
| | | | | 1 | $-2(J-2-m_2)$ | $(J-2-m_2)^2$ |
| | -3 | $m_2$ | $\binom{5+m_2}{3}\binom{5-m_2}{3}S^0_{5+m_2,2J-4,5-m_2}$ | | | 1 |

Table 8. The summations $\sum_{m}(C^{J'm\lambda_1}_{Jm30})^2(C^{Jm\lambda_2-1}_{J''m-m_2 1m_2})^2$. $J'=J-\Delta j$, $J=J''-\Delta J$ or $\Delta J=\lambda_2-2$, $\Delta j=\lambda_1-3$.

| $\Delta J$ | $\Delta j$ | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| 1 | 3 | $m_2$ | $\binom{4+m_2}{3}\binom{4-m_2}{3}S^0_{4-m_2,2J-6,4+m_2}$ | | | 1 |
| | 2 | $m_2$ | $(9-m_2^2)(4-m_2^2)S^2_{3-m_2,2J-4,3+m_2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | 1 | $m_2$ | $\frac{4-m_2^2}{4}S^4_{2-m_2,2J-2,2+m_2}$ | 5 | $-10(J-1)$ | $4J^2-10J+6$ |
| | | | | 5 | $-10(J-1)$ | $4J^2-10J+6$ |
| | 0 | $m_2$ | $\frac{4}{9}S^6_{1-m_2,2J,1+m_2}$ | 1 | $-2J$ | $J^2$ |
| | | | | 5 | $-10J$ | $2J^2-3J+1$ |
| | | | | 5 | $-10J$ | $2J^2-3J+1$ |
| | -1 | $m_2$ | $-\frac{1}{4}S^6_{1-m_2,2J,1+m_2}$ | 5 | $-10J$ | $2J(2J-1)$ |
| | | | | 5 | $-10J$ | $2J(2J-1)$ |
| | | | | 1 | $-2J$ | $-2J-1$ |
| | -2 | 0 | $-4S^4_{2,2J,2}$ | 1 | $-2J$ | $J^2$ |
| | | | | 1 | $-2J$ | $-2J-1$ |
| | | 1 | $2S^4_{2,2J,2}$ | 1 | $-2J$ | $J^2$ |
| | | | | 1 | $-4J-3$ | $(2J+1)(2J+2)$ |
| | -3 | 0 | $-S^2_{3,2J,3}$ | 1 | $-2J$ | $-2J-1$ |
| | | 1 | $\frac{1}{2}S^2_{3,2J,3}$ | 1 | $-4J-3$ | $(2J+1)(2J+2)$ |
| 0 | 3 | 0 | $4S^2_{3,2J-6,3}$ | 1 | $-2(J-3)$ | $(J-3)^2$ |
| | | 1 | $8S^1_{3,2J-6,4}$ | | 1 | 3 |

Table 8 (continued).

| J | j | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0$ 1 |
|---|---|---|---|---|---|---|
| 0 | 2 | 0 | $16S^4_{2,2J-4,2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | 1 | $24S^3_{2,2J-4,3}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | | 1 | 2 |
| | 0 | 0 | $\frac{16}{9}S^8_{0,2J,0}$ | 1 | $-2J$ | $J^2$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | | | | 5 | $-10J$ | $2J^2-3J+1$ |
| | | | | 5 | $-10J$ | $2J^2-3J+1$ |
| | | 1 | $\frac{8}{9}S^6_{1,2J-1,1}$ | 1 | $-2(J-1)$ | $(J-1)^2$ |
| | | | | 5 | $-10(J-1)$ | $2J^2-13J+6$ |
| | | | | 5 | $-10(J-1)$ | $2J^2-13J+6$ |
| | 1 | 0 | $S^6_{1,2J-2,1}$ | 5 | $-10(J-1)$ | $2(J-1)(2J-3)$ |
| | | | | 5 | $-10(J-1)$ | $2(J-1)(2J-3)$ |
| | | | | 1 | $-2(J-1)$ | $(J-1)^2$ |
| | | 1 | $S^5_{1,2J-2,2}$ | 5 | $-10(J-1)$ | $2(J-1)(2J-3)$ |
| | | | | 5 | $-10(J-1)$ | $2(J-1)(2J-3)$ |
| | | | | | 1 | 1 |
| | -1 | $m_2$ | $S^6_{1+m_2,2J,1-m_2}$ | 5 | $-10(J-m_2)$ | $5(J-m_2)^2-J(J+2)$ |
| | | | | 5 | $-10(J-m_2)$ | $5(J-m_2)\ -J(J+2)$ |
| | | | | 1 | $-2J(1+m_2)$ | $J^2(1+m_2)^2$ |
| | -2 | 0 | $16S^4_{2,2J,2}$ | 1 | $-2J$ | $J^2$ |
| | | | | 1 | $-2J$ | $J^2$ |
| | | 1 | $12S^4_{3,2J,1}$ | 1 | $-2(J-1)$ | $(J-1)^2$ |
| | | | | 1 | $-4J$ | $4J^2$ |
| | -3 | $m_2$ | $\frac{4}{9}\binom{3+m_2}{2}\binom{3-m_2}{2}S^2_{3+m_2,2J,3-m_2}$ | 1 | $-2J(1+m_2)$ | $J^2(1+m_2)^2$ |

Table 8 (continued).

| $\Delta J$ | $\Delta j$ | $m_2$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|---|
| -1 | 3 | 0 | $-S^2_{3,2J-6,3}$ | 1 | $-2(J-3)$ | $-3(2J-3)$ |
| | | 1 | $\frac{1}{2}S^2_{3,2J-6,3}$ | 1 | 5 | 6 |
| | 2 | 0 | $-4S^4_{2,2J-4,2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 1 | $-2(J-2)$ | $-4(J-1)$ |
| | | 1 | $2S^4_{2,2J-4,2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 1 | $-(4J-5)$ | $(2J-2)(2J-3)$ |
| | 1 | $m_2$ | $-\frac{1}{4}S^6_{1+m_2,2J-2,1-m_2}$ | | | |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-(J^2-1)$ |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-(J^2-1)$ |
| | | | | 1 | $-2(J-1-m_2)$ | $(J-1-m_2)^2-J^2$ |
| | 0 | $m_2$ | $\frac{4}{9}S^6_{1+m_2,2J-2,1-m_2}$ | | | |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-(3J^2+3J-1)$ |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-(3J^2+3J-1)$ |
| | | | | 1 | $-2(J-1-m_2)$ | $(J-1-m_2)^2$ |
| | -1 | $m_2$ | $\frac{(4-m_2^2)}{4}S^4_{2+m_2,2J-2,2-m_2}$ | | | |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-J(J+2)$ |
| | | | | 5 | $-10(J-1-m_2)$ | $5(J-1-m_2)^2-J(J+2)$ |
| | -2 | $m_2$ | $4\binom{3+m_2}{2}\binom{3-m_2}{2}S^2_{3+m_2,2J-2,3-m_2}$ | | | |
| | | | | 1 | $-2(J-1-m_2)$ | $(J-1-m_2)^2$ |
| | -3 | $m_2$ | $\binom{4+m_2}{3}\binom{4-m_2}{3}S^0_{4+m_2,2J-2,4-m_2}$ | | | 1 |

Table 9. The summations $\sum_m (C^{J'm\lambda_1}_{Jm30})^2 C^{Jm\,\lambda_2}_{J''m-121} C^{Jm\lambda_2-1}_{J''m-111}$. $J'=J-\Delta j$, $J=J''-\Delta J$ or $\Delta J=\lambda_2-2$, $\Delta j=\lambda_1-3$.

| $\Delta J$ | $\Delta j$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|
| 1 | 3 | $\frac{20}{\sqrt{3}}S^1_{3,2J-6,5}$ | | 2 | $-(J-6)$ |
| | 2 | $16\sqrt{3}S^3_{2,2J-4,4}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 2 | $-(J-4)$ |
| | 1 | $\frac{\sqrt{3}}{2}S^5_{1,2J-2,3}$ | 5 | $-10(J-1)$ | $2(J-1)(2J-3)$ |
| | | | 5 | $-10(J-1)$ | $2(J-1)(2J-3)$ |
| | | | | 2 | $-(J-2)$ |
| | 0 | $\frac{8}{9\sqrt{3}}S^7_{0,2J,2}$ | 5 | $-10J$ | $2J^2-3J+1$ |
| | | | 5 | $-10J$ | $2J^2-3J+1$ |
| | | | 1 | $-2J$ | $J^2$ |
| | | | | 2 | $-J$ |
| | -1 | $\frac{-1}{2\sqrt{3}}S^6_{1,2J,2}$ | 5 | $-10J$ | $2J(2J-1)$ |
| | | | 5 | $-10J$ | $2J(2J-1)$ |
| | | | 2 | $-5J-2$ | $J(2J+1)$ |
| | -2 | $\frac{4}{\sqrt{3}}S^5_{2,2J,2}$ | 1 | $-2J$ | $J^2$ |
| | | | 1 | $-4J-3$ | $(2J+1)(2J+2)$ |
| | | | | 2 | $-J$ |
| | -3 | $\frac{1}{\sqrt{3}}S^2_{3,2J,3}$ | 1 | $-4J-3$ | $(2J+1)(2J+2)$ |
| 0 | 3 | $4\sqrt{3}S^2_{3,2J-\underline{6},4}$ | 2 | $-(2J-11)$ | $-3(2J-5)$ |
| | 2 | $12\sqrt{3}S^4_{2,2J-4,3}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | 2 | $-(2J-7)$ | $-2(2J-3)$ |
| | 1 | $\frac{\sqrt{3}}{2}S^6_{1,2J-2,2}$ | 5 | $-10(J-1)$ | $2(J-1)(2J-3)$ |
| | | | 5 | $-10(J-1)$ | $2(J-1)(2J-3)$ |
| | | | 2 | $-(2J-3)$ | $-(2J-1)$ |

Table 9.(continued).

| $\Delta J$ | $\Delta j$ | $KS^r_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|
| 0 | 0 | $\frac{4}{3\sqrt{3}}S^7_{1,2J-1,1}$ | 5 | $-10(J-1)$ | $(2J-1)(J-6)$ |
| | | | 5 | $-10(J-1)$ | $(2J-1)(J-6)$ |
| | | | 1 | $-2(J-1)$ | $(J-1)^2$ |
| | | | | 2 | $-(2J-1)$ |
| | -1 | $-\frac{\sqrt{3}}{2}S^6_{2,2J-1,1}$ | 5 | $-10(J-1)$ | $(2J-1)(2J-5)$ |
| | | | 5 | $-10(J-1)$ | $(2J-1)(2J-5)$ |
| | | | 2 | $-(6J-1)$ | $2J(2J-1)$ |
| | -2 | $-12\sqrt{3}S^4_{3,2J-1,2}$ | 1 | $-2(J-1)$ | $(J-1)^2$ |
| | | | 2 | $-(6J-1)$ | $2J(2J-1)$ |
| | -3 | $-4\sqrt{3}S^2_{4,2J-1,3}$ | 2 | $-(6J-1)$ | $2J(2J-1)$ |
| -1 | 3 | $\frac{1}{\sqrt{3}}S^3_{3,2J-6,3}$ | 1 | 5 | 6 |
| | | | | 2 | $-(3J-5)$ |
| | 2 | $\frac{4}{\sqrt{3}}S^5_{2,2J-4,2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | 1 | 3 | 2 |
| | | | | 2 | $-3(J-1)$ |
| | 1 | $\frac{1}{2\sqrt{3}}S^6_{2,2J-3,1}$ | 5 | $-10(J-2)$ | $(2J-3)(2J-7)$ |
| | | | 5 | $-10(J-2)$ | $(2J-3)(2J-7)$ |
| | | | 2 | $-(3J-7)$ | $-6(J-1)$ |
| | 0 | $\frac{8}{9\sqrt{3}}S^7_{2,2J-2,0}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | 5 | $-10(J-2)$ | $(J-1)(2J-21)$ |
| | | | 5 | $-10(J-2)$ | $(J-1)(2J-21)$ |
| | | | | 2 | $-3(J-1)$ |
| | -1 | $\frac{\sqrt{3}}{2}S^5_{3,2J-2,1}$ | 5 | $-10(J-2)$ | $2(2J^2-11J+10)$ |
| | | | 5 | $-10(J-2)$ | $2(2J^2-11J+10)$ |
| | | | | 2 | $-3(J-1)$ |

Table 9 (continued).

| $\Delta J$ | $\Delta j$ | $KS^{r}_{pnq}$ | $v^2$ | $v^1$ | $v^0=1$ |
|---|---|---|---|---|---|
| -1 | -2 | $16\sqrt{3}S^{3}_{4,2J-2,2}$ | 1 | $-2(J-2)$ | $(J-2)^2$ |
| | | | | 2 | $-3(J-1)$ |
| | -3 | $\frac{20}{\sqrt{3}}S^{1}_{5,2J-2,3}$ | | 2 | $-3(J-1)$ |

Appendix A. The Normalized Angular Momentum Coefficients.

In equations (A3) and (A4) below, two expressions are given for the normalized angular momentum coefficients. Since the summation in (A4) is easier to evaluate than that in (A3), an expression for the factor $f(JJ''J_2)$ is desired so that (A4) can be used in the calculations. It is the object of this appendix to show how the normalization factor $f(JJ''J_2)$ can be obtained and then to show that the two formulae (A3) and (A4) are the same by proving the summations in each formula are equivalent. From either of these two results the equations (18) and (19) can be obtained.

The normalized angular momentum coefficients $(J''J_2m''m_2|J''J_2Jm)$ are defined by

$$\psi^1_{Jm} = \sum_{m''+m_2=m} (J''J_2m''m_2|J''J_2Jm)\psi_{J''m''}\phi_{J_2m_2}, \qquad \text{(A1)}$$

where $\psi^1_{Jm}$ is the normalized wave function describing the state of the system consisti ng of a nucleus (angular momentum $J''$) and an emitted particle (angular momentum $J_2$) which are in the states represented by the normalized wave functions $\psi_{J''m''}$, $\phi_{J_2m_2}$ respectively. Jm refers to the total angular momentum and its z component, r espectively, of the system. If $\int|\psi^1_{Jm}|^2dV=1$ is formed it can be seen that the normalized coefficients satisfy

$$\sum_{m''+m_2=m} (J''J_2m''m_2|J''J_2Jm)^2=1. \qquad \text{(A2)}$$

This follows from the orthogonality of the wave functions and the fact that the coefficients are real numbers (Cf A3).

The expression given by Wigner[14] for the normalized coefficients is (using different notation)

$$(J''J_2m''m_2|J''J_2Jm) = \left[\frac{(2J+1)(2J''-\lambda_2)!(2J-\lambda_2)!\lambda_2!}{(2J+\lambda_2+1)!}\right]^{\frac{1}{2}} \times$$

$$\left[\frac{(J+m)!\quad(J-m)!}{(J''+m'')!(J''-m'')!(J_2+m_2)!(J_2-m_2)!}\right]^{\frac{1}{2}} \sum_{\alpha} \frac{(-1)^{\alpha+J_2+m_2}(2J_2+J''+m''-\lambda_2-\alpha)!(J''-m''+\alpha)!}{(2J_2-\lambda_2-\alpha)!(J+m-\alpha)!\alpha!(J''-J_2-m''-m_2+\alpha)!} \tag{A3}$$

where $\lambda_2$ is defined by $J=J''+J_2-\lambda_2$ which means its values must lie in the range $0\leq\lambda_2\leq$ minimum of $2J''$, $2J_2$ since $|J''-J_2|\leq J\leq J''+J_2$; $m=m''+m_2$; $\alpha$ takes all those values for which negative arguments do not appear in the factorials $(0!=1)$ (All summation indices which do not have their range given will be summed in this manner.). This formula is not easy to use because it is not very symmetrical in the J's and m's. Van der Waerden[15] gives a symmetrical formula for the unnormalized coefficients. Using this formula the normalized angular momentum coefficients may be written in the form

$$(J''J_2m''m_2|J''J_2Jm) = f(J\,J''J_2)\times$$

$$\left[\frac{(J+m)!\quad(J-m)!}{(J''+m'')!(J''-m'')!(J_2+m_2)!(J_2-m_2)!}\right]^{\frac{1}{2}} \sum_{\alpha} \frac{(-1)^{\alpha}(J''+m'')!\quad(J''-m'')!\quad(J_2+m_2)!\quad(J_2-m_2)!}{(J''+m''-\lambda_2+\alpha)!(J''-m''-\alpha)!(J_2+m_2-\alpha)!(J_2-m_2-\lambda_2+\alpha)!\alpha!(\lambda_2-\alpha)!}$$

$$= f(JJ''J_2)\,C^{Jm\,\lambda_2}_{J''m''J_2m_2}, \tag{A5}$$

using (19). Here $f(JJ''J_2)$ is the normalization factor necessary for (A4) to satisfy (A2).

The normalization factor f is now calculated following the method used by Keller[16]. By substituting (A5) into (A2) with $m=J$ one can obtain

$$f^2\sum_{m''=J''-\lambda_2}^{J''}\left(C^{JJ\,\lambda_2}_{J''m''J_2\,J-m''}\right)^2 = 1, \tag{A6}$$

the range for $m''$ being determined from the relations in (20). The summation in (A6) is evaluated as follows: first, express the summand explicitly in terms of $m''$ using Table 2; then change the summation over $m''$ to one over $v = m'' - J'' + \lambda_2$; finally, use the summation formula (45) with $k=0$ to get the sum. Combining the result with (A6) one obtains

$$f(JJ''J_2) = \left[\frac{(2J+1)(2J''-\lambda_2)!\,(2J_2-\lambda_2)!\,\lambda_2!}{(2J+\lambda_2+1)!}\right]^{\frac{1}{2}} \tag{A7}$$

The equivalence of (A3) and (A4) is now shown by proving that

$$\sum_\alpha \frac{(-1)^{\alpha+J_2+m_2}(J''+2J_2+m''-\lambda_2-\alpha)!\,(J''-m''+\alpha)!}{(2J_2-\lambda_2-\alpha)!\,(J''+J_2+m''+m_2-\lambda_2-\alpha)!\,(J''-J_2-m''-m_2+\alpha)!\,\alpha!} \tag{A8a}$$

$$=\sum_\alpha \frac{(-1)^\alpha (J''+m'')!\,(J''-m'')!\,(J_2+m_2)!\,(J_2-m_2)!}{(J''+m''-\lambda_2+\alpha)!\,(J''-m''-\alpha)!\,(J_2+m_2-\alpha)!\,(J_2-m_2-\lambda_2+\alpha)!\,\alpha!\,(\lambda_2-\alpha)!} \tag{A8b}$$

$$= \frac{(-1)^{J_2+m_2}}{(2J_2-\lambda_2)!}\left(\frac{\partial}{\partial Y}\right)^{J_2-m_2}_{Y=1}\left(\frac{\partial}{\partial X}\right)^{J_2+m_2}_{X=1}\left[X^{J''-m''}Y^{J''+m''}(Y-X)^{2J_2-\lambda_2}\right] \tag{A8c}$$

Obviously, (b) cannot be obtained from (a) by a linear substitution for $\alpha$. It is necessary to prove (a) = (b) by showing that both can be obtained from (c).

To get (a) from (c):

Use $X^{J''-m''}Y^{J''+m''}(Y-X)^{2J_2-\lambda_2} = \sum_\alpha (-1)^\alpha \binom{2J_2-\lambda_2}{\alpha} Y^{J''+2J_2+m''-\lambda_2-\alpha} X^{J''-m''+\alpha}$

and perform the indicated differentiation using

$$\left(\frac{d}{dX}\right)^S X^K = \frac{K!}{(K-S)!}\,X^{K-S} \tag{A9}$$

To get (b) from (c):

First form $\left(\frac{\partial}{\partial X}\right)^{J_2+m_2}_{X=1}\left[X^{J''-m''}(Y-X)^{2J_2-\lambda_2}\right]$

$$= (-1)^{2J_2-\lambda_2}\sum_\alpha \binom{J_2+m_2}{\alpha}\left[\left(\frac{\partial}{\partial X}\right)^{\alpha}_{X=1} X^{J''-m''}\right]\left[\left(\frac{\partial}{\partial X}\right)^{J_2+m_2-\alpha}_{X=1}(X-Y)^{2J_2-\lambda_2}\right]$$

(using Leibnitz' theorem for the differentiation of a product of two functions)

$$= (-1)^{2J_2-\lambda_2}\sum_{\alpha}\frac{(J_2+m_2)!\;\;(J''-m'')!\;\;(2J_2-\lambda_2)!}{(J_2+m_2-\alpha)!\;\alpha!\,(J''-m''-\alpha)!\;(J_2-m_2-\lambda_2+\alpha)!}(1-Y)^{J_2-m_2-\lambda_2+\alpha}$$

(using (A9)). Then form $\left(\frac{\partial}{\partial Y}\right)^{J_2-m_2}\left[Y^{J''-m''}(1-Y)^{J_2-m_2-\lambda_2+\alpha}\right]$

$$=(-1)^{J_2-m_2-\lambda_2+\alpha}\sum_{\beta}\binom{J_2-m_2}{\beta}\left[\left(\frac{\partial}{\partial Y}\right)^{\beta}Y^{J''+m''}\right]\left[\left(\frac{\partial}{\partial Y}\right)^{J_2+m_2-\beta}(Y-1)^{J_2-m_2-\lambda_2+\alpha}\right]$$

(using Leibnitz' theorem)

$$=(-1)^{J_2-m_2-\lambda_2+\alpha}\sum_{\beta}\frac{(J_2-m_2)!\;\;(J''\;m'')!\;(J_2-m_2-\lambda_2+\alpha)!}{(J_2-m_2-\beta)!\,\beta!\,(J''+m''-\beta)!\;(\beta-\lambda_2+\alpha)!}Y^{J''+m''-\beta}(Y-1)^{\beta-\lambda_2+\alpha}$$

(using (A9)). Setting $Y=1$ in the latter summation leaves only the term for which $\beta=\lambda_2-\alpha$. Substituting these formulae in (c) one obtains (a).

Thus (a) = (b). This equality has also been shown by Racah[17] by a different method.

The expression for $f(JJ''J_2)$ given in (A7) could have been obtained by using the result (A8) with (A3) and (A4).

Appendix B. To show that $(C^{J'm+m_1\ \lambda_1}_{JmJ_1m_1})^2$ and $(C^{Jm\ \lambda_2}_{J''m-m_2J_2m_2})^2$ are polynomials in m.

If the latter quantity is a polynomial in m then so is the former. Hence all that is necessary to show is the proof that $(C^{Jm\ \lambda_2}_{J''m-m_2J_2m_2})^2$ is a polynomial in m and this is given below.

From the symmetry properties (21), (23) one can see that if $(C^{Jm\ \lambda_2}_{J''m-m_2J_2m_2})^2$ is a polynomial in m for $0\leq\lambda_2\leq J_2$ and $0\leq m_2\leq J_2$ then it is a polynomial in m for any $\lambda_2$ and $m_2$ in the ranges $0\leq\lambda_2\leq 2J_2$, $-J_2\leq m_2\leq J_2$. In the definition (19), it is seen that the summand is a polynomial in m. It must be shown that the factors $(J''+m'')!\,(J''-m'')!$ appearing in the denominator of $(C\ )^2$ are cancelled out for $0\leq\lambda_2\leq J_2$, $0\leq m_2\leq J_2$ to complete the proof.

Now, $J''+m'' = J+m-(J_2+m_2-\lambda_2)$ and is $\leq J+m$ for $0\leq\lambda_2\leq J_2$ and $m_2\geq 0$. Hence $(J''+m'')!$ divides $(J+m)!$ for $0\leq\lambda_2\leq J_2$ and $0\leq m_2\leq J_2$. $J''-m'' = J-m-(J_2-m_2-\lambda_2)$ and is $\leq J-m$ if $J_2-m_2-\lambda_2\geq 0$. For the case that $J_2-m_2-\lambda_2<0$, the factor $\frac{(J-m)!}{(J''-m'')!} = \frac{(J-m)!}{(J-m-(J_2-m_2-\lambda_2))!}$ is cancelled by the common factor $\frac{(J''-m'')!}{(J-m)!} = \frac{(J''-m'')!}{(J''-m''+J_2-m_2-\lambda_2)}$ which appears in all the terms, $\frac{(J''-m'')!}{(J''-m''-\alpha)!}$, of the summand in (19) since $J_2-m_2-\lambda_2+\alpha\geq 0$ for a non-zero term.

Appendix C. The Summation Formulae.

To derive the formulae (45) and (49):

It will now be shown that

$$\sum_{v=0}^{n}\binom{p+v}{p}\binom{n+q-v}{q}v^k \begin{cases} = \binom{n+p+q+1}{n} \quad \text{if } k=0 & \text{(B1)} \\ = \sum_{\alpha=1}^{k} a_{k\alpha}\frac{(p+\alpha)!}{p!}\binom{n+p+q+1}{n-\alpha} \quad \text{if } k\geq 1 & \text{(B2)} \end{cases}$$

$$\text{where } a_{k\alpha} \begin{cases} = 1 \text{ if } k\geq 1, \alpha=1 \\ = \sum_{\beta=1}^{k-\alpha+1} a_{k-\beta,\alpha-1}\binom{k-1}{\beta-1} \text{ if } k\geq 2,\ \alpha\geq 2. \end{cases} \quad \text{(B3)}$$

These formulae are the same as (45) and (49) of the text.

The result (B1) will first be proven, (B2) and (B3) then follow from it by induction.

To prove (B1), one forms $(1-X)^{-p-1}(1-X)^{-q-1} = (1-X)^{-p-q-2}$ and expands each binomial to get

$$\sum_{v=0}^{\infty}\binom{p+v}{p}X^v\sum_{w=0}^{\infty}\binom{q+w}{q}X^w = \sum_{n=0}^{\infty}\binom{n+p+q+1}{n}X^n.$$

Collecting the coefficient of $X^n$ on both sides of this equation one arrives at the result (B1).

The formula (B2) is now proven for the case $k=1$.

$$\sum_{v=0}^{n}\binom{p+v}{p}\binom{n+q-v}{q}v = \sum_{v=1}^{n}\frac{(p+v)!}{(v-1)!\,p!}\binom{n+q-v}{q} = (p+1)\sum_{w=0}^{n-1}\binom{p+1+w}{p+1}\binom{n-1+q-w}{q}$$

$$= (p+1)\binom{n+p+q+1}{n-1} \text{ using (B1).}$$

Now, assume (B2) is true for all values of k up to $k=r-1$ $(r\geq 2)$. Then $\sum_{v=0}^{n}\binom{p+v}{p}\binom{n+q-v}{q}v^r = \sum_{v=1}^{n}\frac{(p+v)!}{(v-1)!\,p!}\binom{n+q-v}{q}v^{r-1}$

$$= (p+1)\sum_{w=0}^{n-1}\binom{p+1+w}{p+1}\binom{n-1+q-w}{q}(w+1)^{r-1}$$

$$= (p+1)\sum_{w=0}^{n-1}\binom{p+1+w}{p+1}\binom{n-1+q-w}{q}\left[\sum_{\gamma=1}^{r-1}\binom{r-1}{\gamma}w^{\gamma}+1\right]$$

$$= (p+1)\sum_{\gamma=1}^{r-1}\binom{r-1}{\gamma}\sum_{\lambda=1}^{\gamma}a_{\gamma\lambda}\frac{(p+1+\lambda)!}{(p+1)!}\binom{n+p+q+1}{n-(\lambda+1)} + (p+1)\binom{n+p+q+1}{n-1}$$

(using the induction assumption)

$$=(p+1)\sum_{\lambda=1}^{r-1}\left[\sum_{\gamma=\lambda}^{r-1}a_{\gamma\lambda}\binom{r-1}{\gamma}\right]\frac{(p+1+\lambda)!}{(p+1)!}\binom{n+p+q+1}{n-(\lambda+1)} + (p+1)\binom{n+p+q+1}{n-1}$$

$$=\sum_{\alpha=2}^{r}\left[\sum_{\gamma=\alpha-1}^{r-1}a_{\gamma,\alpha-1}\binom{r-1}{\gamma}\right]\frac{(p+\alpha)!}{p!}\binom{n+p+q+1}{n-\alpha} + (p+1)\binom{n+p+q+1}{n-1}$$

$$=\sum_{\alpha=2}^{r}\left[\sum_{\beta=1}^{r-\alpha+1}a_{r-\beta,\alpha-1}\binom{r-1}{r-\beta}\right]\frac{(p+\alpha)!}{p!}\binom{n+p+q+1}{n-\alpha} + (p+1)\binom{n+p+q+1}{n-1}$$

$$=\sum_{\alpha=1}^{r}a_{r\alpha}\frac{(p+\alpha)!}{p!}\binom{n+p+q+1}{n-\alpha}$$

where $a_{r\alpha}=\sum_{\beta=1}^{r-\alpha+1}a_{r-\beta,\alpha-1}\binom{r-1}{\beta-1}$ if $r\geq 2, \alpha\geq 2$

$a_{r1}=1$ if $r\geq 1$.

This completes the proof of (B2) and (B3).

To derive formula (46):

The formula (46) may be derived by using the same method of proof as that already given for the formula (45) starting with the equation $(1-X)^P(1-X)^Q=(1-X)^{P+Q}$ instead of $(1-X)^{-p-1}(1-X)^{-q-1}=(1-X)^{-p-q-2}$ and expanding the binomials using the formula $(1-X)^P=\sum_{v=0}^{P}\binom{P}{v}(-X)^v$ instead of $(1-X)^{-P}=\left[\sum_{v=0}^{\infty}\binom{-P}{v}(-X)^v\right]=\sum_{v=0}^{\infty}(-1)^v\binom{P-1+v}{v}(-X)^v$. However, the relation $\binom{-P}{v}=(-1)^v\binom{P-1+v}{v}$ is all that is needed to get (46) from (45). If one sets $-p-1=P$ and $-q-1=Q$ in (45) then (46) can be obtained using this relation. The limits for the summations are determined from the range for which the binomial coefficients in the summand do not vanish.

Appendix D. The proof of formula (53).

The $\alpha(1)$- and $\gamma(1)$-mixed $\gamma(2)$ correlation functions can be calculated from (51) and (52). The second bracketed expression in each $W(\theta)$ in (52) is independent of $\theta$. $W_{12}(\theta)$ can thus be written symbolically as (see reference 3)

$$W_{12}(\theta) = gGF_1^0(\theta) + dGF_1^1(\theta), \tag{D1}$$

where gG and dG are the coefficients of $F_1^0(\theta)$ and $F_1^1(\theta)$ obtained from (52).

The $\alpha(1)$-$\gamma(2)$ correlation ~~wi~~ function $W_{12}(\theta)$ will first be obtained from the $\gamma(1)$-$\gamma(2)$ correlation function $W_{12}(\theta)$ which are tabulated in Ling and Falkoff's paper. Substituting in (D1) expressions for the F's from Table 1 for an $\alpha(1)$ particle and a $\gamma(1)$ ray one obtains

$$\alpha(1)\text{-}\gamma(2): \quad W_{12}(\theta) = dG + (2gG - dG)\cos^2\theta \tag{D2}$$

$$\gamma(1)\text{-}\gamma(2): \quad W_{12}(\theta) = 2gG + dG + (dG - 2gG)\cos^2\theta. \tag{D3}$$

The expression (D3) is given in Table II. of Ling and Falkoff's paper in the form

$$\gamma(1)\text{-}\gamma(2): \quad W_{12}(\theta) = Q + R\cos^2\theta, \tag{D4}$$

with $\alpha = 0$.

Hence, $$2gG + dG + (dG - 2gG)\cos^2\theta = K(Q + R\cos^2\theta), \tag{D5}$$

where K is a possible common factor that has been omitted. Equating the coefficients of $\cos^2\theta$, one obtains two equations from which one can solve for gG and dG in terms of K, Q, and R. Substituting the result in (D2) gives

$$\alpha(1)\text{-}\gamma(2): \quad W_{12}(\theta) = K\left(\frac{Q+R}{2} - R\cos^2\theta\right),$$

from which K can be omitted. This result is exactly the same for $W_{22}(\theta)$ and $W_L(\theta)$, a different G being used in each case. The common factor K is the same in each case. Hence the $\alpha(1)$-mixed $\gamma(2)$ correlation functions can be obtained from the $\gamma(1)$-mixed $\gamma(2)$ correlation functions listed in Table II. of Ling and Falkoff's paper by the relation (53).

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