NUMERICAL ANALYSIS OF A UNITARY PARTICLE MODEL

by

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ABSTRACT

This thesis investigates a classical unitary field model consisting of a complex scalar source field coupled to a real scalar massless field, in an attempt to describe the family of heavy particles consisting of nucleons and hyperons.

The discussion is limited to spherically symmetric fields and approximate methods are used to solve the non-linear field equations. The solutions may be classified by the number of nodes in the source field. For each type of solution, the energy in the field is calculated as a function of the spatial extension of the particle described by that solution.

A striking feature of the theory is that for each type of solution the energy may be minimized with respect to the extension of the particle. The particular solution which yields the minimum energy is interpreted as representing a particle in its normal state. In this way, a discrete mass spectrum is obtained.

The mass ratios yielded by the theory compare reasonably well with the experimentally observed mass ratios for nucleons, Λ particles and Σ particles. A comparison of theoretical and experimental values determines the "bare" mass of the source field to correspond to 1185 electron masses, and yields a value for the ratio of the normalization constant of the source field to the coupling constant of the two fields, but does not specify these latter constants separately.

Limitations of this classical model are stated. A possible improvement of the model is investigated in an attempt to improve the theoretical mass ratios; however, the significant features of the model remain unchanged, and the change in the mass ratios, while in the right direction, is very slight.

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I. INTRODUCTION

1. Program

Ever since Mie (1912)⁽¹⁾ published his paper on the theory of matter, unitary theories of the so-called elementary particles have remained in what might be called a state of animated suspension. New attempts in the spirit of Mie keep appearing in the literature, but although they raise high hopes by establishing the feasibility of a unitary theory, none of them has yet succeeded in predicting quantitatively the outcome of a single experiment.

The aim of the present investigation is not to enlarge on the general feasibility of unitary particle theory, but rather to study the numerical feasibility of a known model which appears attractive from a general point of view, and, if possible, to develop a feeling for the order of magnitude of the parameters characterizing this model.

In particular, the family of heavy particles, consisting of nucleons and hyperons, will be taken as the reality to which the model under consideration should correspond. The reason for this particular choice is the fact that the model to be discussed below describes a "particle" in terms of self-maintained solutions of certain non-linear differential equations, which allow for the existence of a "ground state" together with "excited states", a situation resembling most closely the relation between the various members of the heavy particle family.

It will be found that one can obtain reasonable values for the mass spectrum. However, the description of the decays of the various particles and the prediction of their lifetimes is beyond the reach of present unitary theories.

2. Classification of Unitary Theories

Existing unitary theories may be separated naturally into three groups:

- (1) Purely classical field theories, in which the cohesion of particles is provided by either
 - (1a) coupling of the field to itself, or
 - (1b) coupling of two fields, each acting as the glue which holds the other together.

Models resulting from this approach have been studied by Rosen (1939)⁽²⁾, Finkelstein (1949)⁽³⁾, and others ⁽⁴⁾-⁽⁷⁾. The problems involved in the quantization of such theories have been studied by Finkelstein⁽³⁾.

- (2) Quantum field theories, with coupling of the field to itself. This approach has been followed by Heisenberg (1954)⁽⁸⁾ and others.
- (3) Field theories in which a classical "source" is coupled to a quantized "glue". The feasibility of this intermediate type of model was first pointed out by Heber (1955)⁽⁹⁾.

The interpretation of models resulting from the approach (2) meets with great difficulties, as has been shown by Kita (1956)⁽¹⁰⁾, and so far no model of type (3) is known which will correspond to a particle with positive mass. For these reasons, the present work will treat only models of type (1).

3. Formalism for Theories of Type (1)

We summarize the formalism used to describe theories of types (1a) and (1b).

The conditions that the theory be gauge invariant and have no singularities are imposed. A Lagrangian density $\mathcal{L} = \mathcal{L}(\psi_i, \frac{\partial \psi_i}{\partial x_\mu}, \dots)$ is defined, and the field equations are derived from it by requiring that the action integral

$$I.1) \quad I = \int \mathcal{L} d^4x$$

be an extremum. The energy-momentum density tensor, $T_{\mu\nu}$, and the charge-current density vector, s_μ , are introduced:

$$I.2) \quad T_{\mu\nu} = - \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i}{\partial x_\mu})} \frac{\partial \psi_i}{\partial x_\nu} + \mathcal{L} \delta_{\mu\nu}$$

$$I.3) \quad s_\mu = i\varepsilon \left\{ \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i}{\partial x_\mu})} \psi_i - \frac{\partial \mathcal{L}}{\partial(\frac{\partial \psi_i^*}{\partial x_\mu})} \psi_i^* \right\}.$$

The conservation laws for the momentum, energy, and electric charge follow from the invariance properties of the Lagrangian, and take the form

$$I.4) \quad \frac{\partial T_{\mu\nu}}{\partial x_\mu} = 0 \quad \nu = 1, 2, 3, 4.$$

$$\text{I.5)} \quad \frac{\partial S_{\mu}}{\partial x_{\mu}} = 0 .$$

Finally, any solution of the field equations which is free from singularities is interpreted as a particle of mass

$$\text{I.6)} \quad m = - \frac{1}{c^2} \int T_{44} d^3x$$

and charge

$$\text{I.7)} \quad e = - \frac{i}{c} \int S_4 d^3x .$$

It is evident from the definition of the charge-current density vector that a theory with real fields can only describe an electrically uncharged field and therefore neutral particles, and that charged particles may be obtained only if at least one of the field variables is complex and time-dependent.

II. IMPORTANCE OF NON-LINEAR TERMS IN THE FIELD EQUATIONS

1. Stability

The common feature of all three types of unitary theories is that the stability of the particle is provided by the non-linearity of the field equations.

It is well known that in any physically reasonable linear field theory which yields plane wave solutions, $e^{i(\vec{k} \cdot \vec{x} - \omega t)}$, a wave packet spreads in the absence of external forces: the wave packet at time t is given in terms of the initial wave packet as

$$\begin{aligned} \text{II.1)} \quad \psi(\vec{x}, t) &= \frac{1}{(2\pi)^3} \int \left\{ \int \psi(\vec{x}', 0) e^{-i\vec{k} \cdot \vec{x}'} d^3\vec{x}' \right\} e^{i(\vec{k} \cdot \vec{x} - \omega t)} d^3\vec{k} \\ &= \frac{1}{(2\pi)^3} \int \psi(\vec{x}', 0) \left\{ \int e^{i[\vec{k} \cdot (\vec{x} - \vec{x}') - \omega t]} d^3\vec{k} \right\} d^3\vec{x}' \end{aligned}$$

where the form of $\omega = \omega(\vec{k})$ depends on the particular field theory under discussion, and determines the effect of $\int e^{i[\vec{k} \cdot (\vec{x} - \vec{x}') - \omega t]} d^3\vec{k}$ on $|\psi(\vec{x}, t)|^2$. Using any conveniently defined width of $|\psi(\vec{x}, t)|^2$ versus $\vec{x}$ as a measure of the spread of the wave packet, it can be shown that the physically reasonable relations

$$\text{II.2a)} \quad \omega \propto k^2 \quad (\text{non-relativistic Schrodinger equation})$$

$$\text{II.2b)} \quad \omega \propto \sqrt{\mu^2 + k^2} \quad (\text{relativistic Klein-Gordon and Dirac equations})$$

both lead to spreading of any initial wave packet. Thus we see that a free wave packet characterized by $|\psi(\vec{x}, t)|^2$ and interpreted as a free, spatially-confined piece of matter flows apart or is unstable.

We expect to obtain a stable localized solution if there is a source which leads to non-linear field equations. We illustrate this general feature by discussing in some detail a model of type (1a).

2. Qualitative Discussion of a Classical Model of Type (1a)

a) Equation to be solved: As a simple example of a theory of type (1a) we consider the real field characterized by the Lagrangian density

$$\text{II.3)} \quad \mathcal{L} = -\frac{1}{2} \left\{ \left(\frac{\partial \psi}{\partial x_\mu} \right)^2 + \kappa^2 \psi^2 \right\} + \frac{g}{n+2} \psi^{n+2}$$

where $n \geq 1$, and g determines the "strength" of the source term. The corresponding field equation is

$$\text{II.4)} \quad \square^2 \psi - \kappa^2 \psi = -g \psi^{n+1}.$$

If we investigate static, spherically symmetric solutions

$\psi = \psi(r)$, the field equation becomes

$$\text{II.5)} \quad \frac{d^2(r\psi)}{dr^2} - \kappa^2(r\psi) = -g r \psi^{n+1}.$$

The substitutions

$$\begin{aligned} \text{II.6)} \quad x &= \kappa r, \quad \kappa > 0 \\ y &= c r \psi \end{aligned}$$

reduce the field equation to

$$\text{II.7)} \quad \frac{d^2 y}{dx^2} = y \left(1 - \frac{g \kappa^{n-2}}{c^n} \frac{y^n}{x^n} \right).$$

We require that ψ be finite everywhere and that all observable integrals be finite; this imposes two boundary conditions on y :

$$\begin{aligned} \text{II.8)} \quad y &= 0 \quad \text{at} \quad x = 0 \\ y &\rightarrow 0 \quad \text{as} \quad x \rightarrow \infty. \end{aligned}$$

Graph 1: Solutions of equation II.10) for n odd.

Graph 2: Solutions of equation II.10) for n even.

The second condition implies that $g > 0$. Let

$$\text{II.9)} \quad c = (g \kappa^{n-2})^{\frac{1}{n}}$$

so that y now satisfies the non-linear equation

$$\text{II.10)} \quad y'' = y \left(1 - \left(\frac{y}{x} \right)^n \right).$$

b) Solutions of the equation: To discuss equation II.10) we distinguish between the cases n odd and n even. If n is odd, divide the half-plane $x \geq 0$ into three regions by the lines $y=x$ and $y=0$. If n is even, divide the half-plane into four regions by the lines $y=x$, $y=0$, and $y=-x$. The sign of y'' in each region is obtained by inspection of the differential equation, and is marked in graphs 1 and 2, which show qualitative sketches of the different types of solutions of equation II.10) which satisfy the first boundary condition; the solutions which satisfy in addition the second boundary condition are drawn more heavily. It is seen that for n odd there is one such solution, while for n even there are two.

c) Particles produced: The energy-momentum density tensor is

$$\text{II.11)} \quad T_{\mu\nu} = - \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi}{\partial x_\mu} \right)} \frac{\partial \psi}{\partial x_\nu} + \mathcal{L} \delta_{\mu\nu}.$$

Since $\psi = \psi(r)$,

$$\text{II.12)} \quad T_{rr} = - \frac{1}{2} \left\{ \left(\frac{d\psi}{dr} \right)^2 + \kappa^2 \psi^2 \right\} + \frac{g}{n+2} \psi^{n+2},$$

and the field equation is

$$\text{II.13)} \quad \frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr} - \kappa^2 \psi = - g \psi^{n+1}.$$

Integration by parts and application of the boundary conditions and the field equation yield

$$\text{II.14)} \quad m = -\frac{1}{c^2} \int T_{44} d^3x = \frac{q}{2(n+2)} \int \psi^{n+2} d^3x .$$

Thus if n is odd, the theory yields one particle of positive mass and no charge, while if n is even, it yields two particles of different, positive, mass and no charge.

III. NUMERICAL ANALYSIS OF A CLASSICAL MODEL OF TYPE (1b)

1. The Specific Model and Field Equations

As an example of a model of type (1b), consider a complex scalar source field, ψ , coupled to a real scalar massless field, φ , with Lagrangian density

$$\text{III.1)} \quad \mathcal{L} = - \frac{\partial \psi}{\partial x_\mu} \frac{\partial \psi^*}{\partial x_\mu} - \kappa^2 \psi \psi^* - \frac{\partial \varphi}{\partial x_\mu} \frac{\partial \varphi}{\partial x_\mu} + g \psi \psi^* \varphi^2; \\ g > 0.$$

The corresponding field equations are

$$\text{III.2)} \quad \square^2 \psi - \kappa^2 \psi = -g \psi \varphi^2 \\ \square^2 \varphi = -g \psi \psi^* \varphi.$$

By analogy with orthodox quantum field theory one may think of these as describing a "bare" nucleon field ψ of "bare" mass $m = \frac{\kappa \hbar}{c}$ coupled to a massless meson field φ . The main reason for the choice of this model is its simplicity and the fact that particles of positive mass result from it. It will be shown later that assignment of a "bare" mass μ to the meson field φ does not alter appreciably the features of this model, provided $\frac{\mu^2}{m^2} \ll 1$.

We investigate static, spherically symmetric solutions of the type

$$\text{III.3)} \quad \psi = \psi(r) \\ \varphi = \varphi(r)$$

where ψ is real. The field equations become

$$\text{III.4)} \quad \frac{1}{r} \frac{d^2(r\psi)}{dr^2} - \kappa^2 \psi = -g \psi \varphi^2 \\ \frac{1}{r} \frac{d^2(r\varphi)}{dr^2} = -g \psi^2 \varphi.$$

The substitutions

$$\begin{aligned} \text{III.5)} \quad x &= K r & u &= \sqrt{g} r \psi \\ v &= \sqrt{g} r \omega \end{aligned}$$

reduce the field equations to

$$\begin{aligned} \text{III.6)} \quad \frac{d^2 u}{dx^2} &= u \left(1 - \frac{v^2}{x^2} \right) \\ \frac{d^2 v}{dx^2} &= -v \left(\frac{u^2}{x^2} \right) . \end{aligned}$$

The boundary conditions on u and v for particle-like solutions are:

$$\begin{aligned} \text{III.7)} \quad u &= 0, \quad v = 0 \quad \text{at} \quad x = 0 \\ \frac{u}{x} &\rightarrow 0, \quad \frac{v}{x} \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty . \end{aligned}$$

In the region $x \gg 1$ the differential equations become

$$\begin{aligned} \text{III.8)} \quad u'' &= u \\ v'' &= 0 \end{aligned}$$

with solutions

$$\begin{aligned} \text{III.9)} \quad u &= C e^{-x} + D e^{x} \\ v &= A + B x . \end{aligned}$$

The boundary conditions require $B=D=0$. Thus for $x \gg 1$,

$$\begin{aligned} \text{III.10)} \quad u &= C e^{-x} \\ v &= A . \end{aligned}$$

By inspecting the differential equations, we see that

$$\begin{aligned} \text{III.11)} \quad u'' &< 0 & \text{if} & \quad \frac{v}{x} > 1 \\ u'' &> 0 & \text{if} & \quad \frac{v}{x} < 1 \\ v'' &< 0 & \text{for all} & \quad x > 0 \end{aligned}$$

The equations with the boundary conditions imposed will in general allow solutions with u having no nodes, one node, two nodes, etc. If each such solution is interpreted as a

**Graph 3: Nodeless solutions of equations III.6)
 u versus x .**

**Graph 4: Nodeless solutions of equations III.6)
 v versus x .**

particle, we have a model which allows, in principle, the calculation of a mass spectrum of various members of a family of electrically neutral particles described by these equations.

2. Approximate Solution of the Equations

Consider first the nodeless solution, of which a qualitative sketch is given in graphs 3 and 4. The exact solution of the equations appears quite hopeless, so we adopt the following approximation procedure: let

$$\text{III.12)} \quad v = \begin{cases} v_1 = ax & \text{for } x \leq x_0 \\ v_2 = ax_0 & \text{for } x \geq x_0 \end{cases}$$

Then the first of equations III.6) becomes

$$\text{III.13)} \quad u'' = \begin{cases} u(1-a^2) & \text{for } x \leq x_0 \\ u[1-(\frac{ax_0}{x})^2] & \text{for } x \geq x_0 \end{cases}$$

The a and x_0 which have been introduced are constant parameters which will be determined later. We may restrict a now, however, since for $a < 1$, we have from III.13)

$$\text{III.14)} \quad u'' > 0 \quad \text{for all } x,$$

while for $a=1$, we have

$$\text{III.15)} \quad \begin{aligned} u'' &= 0 & \text{for } x \leq x_0 \\ u'' &> 0 & \text{for } x > x_0 \end{aligned}$$

In either case, $\frac{u}{x}$ does not vanish as $x \rightarrow \infty$, thus violating the boundary condition. Therefore we require

$$\text{III.16)} \quad a > 1$$

Equation III.13) now yields

$$\text{III.17)} \quad u = \begin{cases} K \sin(\sqrt{a^2-1} x) & \text{for } x \leq x_0 \\ C e^{-x} & \text{asymptotically for } x \gg 1 \end{cases}$$

Graph 5:

Graph 6:

Approximate solutions of equations III.6): nodeless u .

Graph 7:

Graph 8:

Approximate solutions of equations III.4): nodeless ψ .

As a further rough approximation, we now let

$$\text{III.18)} \quad u = \begin{cases} u_1 = K \sin(\sqrt{a^2-1} x) & \text{for } x \leq x_0 \\ u_2 = C e^{-x} & \text{for } x \geq x_0, \end{cases}$$

and require that u and u' be continuous at $x=x_0$. These conditions yield

$$\text{III.19)} \quad K = \frac{C e^{-x_0}}{\sin(\sqrt{a^2-1} x_0)}$$

$$\text{III.20)} \quad x_0 = \frac{\tan^{-1}(-\sqrt{a^2-1})}{\sqrt{a^2-1}},$$

where we must have

$$\text{III.21)} \quad \frac{\pi}{2} < \sqrt{a^2-1} x_0 < \pi$$

for the solution with no nodes in u .

The approximate u and v and the corresponding ψ and q ,

$$\psi = \begin{cases} \psi_1 = \frac{C e^{-x_0}}{\sin(\sqrt{a^2-1} x_0)} \cdot \frac{\sin(\sqrt{a^2-1} \kappa r)}{\sqrt{g} \kappa} & \text{for } r \leq \frac{x_0}{\kappa} \\ \psi_2 = \frac{C e^{-\kappa r}}{\sqrt{g} \kappa} & \text{for } r \geq \frac{x_0}{\kappa} \end{cases}$$

$$\text{III.22)} \quad q = \begin{cases} q_1 = \frac{a \kappa}{\sqrt{g}} & \text{for } r \leq \frac{x_0}{\kappa} \\ q_2 = \frac{a x_0}{\sqrt{g} \kappa} & \text{for } r \geq \frac{x_0}{\kappa}, \end{cases}$$

are shown in graphs 5, 6, 7, and 8.

It should be pointed out at this stage that the preceding discussion also applies to the case of u having one node, or two nodes, etc., if we agree to keep the same approximation for v and if we understand that relation III.20) between a and x_0 becomes subject to

$$\text{III.23)} \quad \begin{aligned} \frac{3\pi}{2} < \sqrt{a^2-1} x_0 < 2\pi & \quad \text{for the one-node solution} \\ \frac{5\pi}{2} < \sqrt{a^2-1} x_0 < 3\pi & \quad \text{for the two-node solution etc.} \end{aligned}$$

instead of to III.21).

Graph 9:

Graph 10:

Approximate u and ψ for the one-node case .

Graph 11:

Graph 12:

Approximate u and ψ for the two-node case .

The approximate u and ψ for these next two cases are shown in graphs 9, 10, 11, and 12.

3. Normalization Procedure

The model is now characterized by the four parameters C, x_0, κ , and q . It is convenient, however, to express C in terms of a new constant, n , by a normalization procedure: now the dimensions of quantities appearing in $\mathcal{L}$ are specified, since $\mathcal{L}$ must be an energy density; a dimensional analysis reveals that $\int \psi^2 d^3x$ has the dimensions of $\frac{1}{\kappa q}$, and so we let

$$\text{III.24)} \quad \int \psi^2 d^3x = \frac{n}{\kappa q}$$

where n is a pure number (see Appendix 1). When we set

$$\text{III.25)} \quad 4\pi \int_0^{x_0/\kappa} \psi_1^2 r^2 dr + 4\pi \int_{x_0/\kappa}^{\infty} \psi_2^2 r^2 dr = \frac{n}{\kappa q},$$

we find

$$\text{III.26)} \quad C^2 = \frac{n}{2\pi} \frac{e^{2x_0}}{1 + \frac{1}{\sin^2(\sqrt{a^2-1} x_0)} [\sqrt{a^2-1} x_0 - \cos(\sqrt{a^2-1} x_0) \sin(\sqrt{a^2-1} x_0)]}.$$

The model is now characterized by n, x_0, κ , and q .

4. The Mass Equation

The energy-momentum density tensor is

$$\text{III.27)} \quad T_{\mu\nu} = \frac{\partial \psi}{\partial x_\mu} \frac{\partial \psi^*}{\partial x_\nu} + \frac{\partial \psi}{\partial x_\nu} \frac{\partial \psi^*}{\partial x_\mu} + 2 \frac{\partial \varphi}{\partial x_\mu} \frac{\partial \varphi}{\partial x_\nu} + \mathcal{L} \delta_{\mu\nu}.$$

For $\psi = \psi(r)$ we have

$$\text{III.28)} \quad T_{44} = - \{ (\kappa^2 - q\varphi^2) \psi^2 + (\nabla \psi)^2 + (\nabla \varphi)^2 \}.$$

TABLE I

VALUES OF $\frac{Hq}{nk}$ AND CORRESPONDING VALUES OF x_0
FOR THE NODELESS, ONE-NODE, AND TWO-NODE SOLUTIONS

NODELESS SOLUTION		ONE-NODE SOLUTION		TWO-NODE SOLUTION	
x_0	$\frac{Hq}{nk}$	x_0	$\frac{Hq}{nk}$	x_0	$\frac{Hq}{nk}$
0.70	3.377	1.50	4.249	2.50	4.409
1.20	2.336	2.75	2.546	4.00	2.708
2.00	1.950	4.50	2.033	6.00	2.153
2.40	1.9186	5.00	1.9942	8.00	2.001
2.45	1.9148	5.50	1.9777	8.50	1.9911
2.50	1.9185	5.75	1.9759	9.00	1.9884
2.60	1.921	6.00	1.9778	9.50	1.9921
3.00	1.950	6.50	1.9906	11.00	2.030
4.50	2.211	7.00	2.013	18.00	2.479
7.50	2.987	10.00	2.270		
10.00	3.713				

Integration by parts and application of the boundary conditions and the field equations yield

$$\begin{aligned}
 H &= mc^2 = - \int T_{44} d^3x \\
 &= \int g \psi^2 \psi^2 d^3x \\
 &= 4\pi g \int_0^{x_0} \psi_1^2 \psi_1^2 r^2 dr + 4\pi g \int_{x_0}^{\infty} \psi_2^2 \psi_2^2 r^2 dr \\
 \text{III.29)} \quad &= \frac{\pi \kappa}{g} \left(\frac{Y_1 + Y_2}{Y_3} \right)
 \end{aligned}$$

where

$$\begin{aligned}
 \text{III.30)} \quad Y_1 &= \frac{a^2}{\sqrt{a^2-1}} \cdot \frac{1}{\sin^2(\sqrt{a^2-1} x_0)} \left[\sqrt{a^2-1} x_0 - \cos(\sqrt{a^2-1} x_0) \sin(\sqrt{a^2-1} x_0) \right] \\
 Y_2 &= 2a^2 x_0 - 4(a x_0)^2 e^{2x_0} \int_{2x_0}^{\infty} \frac{e^{-u}}{u} du \\
 Y_3 &= 1 + \frac{1}{\sin^2(\sqrt{a^2-1} x_0)} \left[\sqrt{a^2-1} x_0 - \cos(\sqrt{a^2-1} x_0) \sin(\sqrt{a^2-1} x_0) \right].
 \end{aligned}$$

5. Numerical Results

a) Energy of the system: For each of the three cases investigated, namely no nodes, one node, and two nodes in u or ψ , values of the dimensionless quantity $\frac{Hg}{\pi \kappa}$ were calculated for a range of values of x_0 . A typical calculation is shown in Appendix 2, and the results are tabulated in Table I and plotted in graph 13 (opposite page 15).

Thus for each case, we have a family of approximate solutions of the differential equations, each yielding a characteristic value of $\frac{Hg}{\pi \kappa}$, corresponding to different values of x_0 . However, the $\frac{Hg}{\pi \kappa}$ versus x_0 curves show that there is one value of x_0 in each case for which $\frac{Hg}{\pi \kappa}$, and hence the energy, H , is a minimum.

Graph 13: The variation of $\frac{H_g}{n\kappa}$ with x_0 .

At the beginning of each curve is shown the number of nodes in the Ψ -field.

b) Mass and classical radius: The existence of a minimum for the energy is a striking feature of the theory, and enables us to obtain a discrete mass spectrum. We interpret the particular solutions which yield the minimum energy in the three cases as representing three neutral particles in their normal states, and we use the corresponding values of $\pi_0 = \frac{\chi_0}{\kappa}$ as a measure of the size of these particles.

The mass ratios are independent of n , κ , and q within the limits of the approximations used and are found to be

$$\text{III.31)} \quad m_2 : m_1 : m_0 = 1.038 : 1.032 : 1$$

The ratios of the radii are also independent of n , κ , and q in the same sense, and are

$$\text{III.32)} \quad r_2 : r_1 : r_0 = 3.67 : 2.34 : 1$$

The values of χ_0 used are those found by approximating $\frac{Hq}{n\kappa}(\chi_0)$ by a parabola in the vicinity of the minimum.

When we compare the theoretical mass spectrum with the mass spectrum of the known neutral members of the heavy particle family,

$$\text{III.33)} \quad m_\Sigma : m_\Lambda : m_N = 1.26 : 1.18 : 1,$$

the theoretical values appear reasonable in view of the simplicity of the model and the crudeness of the approximations used.

c) Parameters characterizing the model: We further relate this model to reality by specifying the classical radius and mass of the lightest particle produced by the theory to be those of a nucleon:

$$\begin{aligned} \text{III.34)} \quad r_0 &= 0.8 \times 10^{-13} \text{ cm.} \\ m_0 &= 1.67 \times 10^{-24} \text{ gm.} \end{aligned}$$

In this way we obtain

$$\begin{aligned} \text{III.35)} \quad \kappa &= 3.06 \times 10^{13} \text{ cm.} \\ \frac{n}{q} &= 2.56 \times 10^{-17} \text{ gm. cm.}^3 \text{ sec.}^{-2} \end{aligned}$$

This value of κ , which characterizes the ψ -field, corresponds to a "bare" nucleon field rest mass of

$$\text{III.36)} \quad m = 1185 \, m_{\text{electron}}.$$

This comparison of the theory with experimental values does not yield n and q separately.

d) "Quasicharge": Since we are at present discussing static solutions of the field equations, the particles described are necessarily electrically neutral. However, the concept of a "quasicharge" of the ψ -field particles, a property of the particles which is "seen" by the φ -field, may be introduced.

The massless φ -field must be regarded as a hidden variable of the theory which produces stability of the ψ -field particles. It has been introduced in analogy with an electrostatic field, but cannot be identified with the latter.

However, an interesting feature of the φ -field is brought to light when the analogy is continued and $-\int \nabla^2 \varphi \, d^3x$ is calculated for the solutions representing particles in their normal states. With Gauss' theorem in mind, we set

$$\text{III.37)} \quad -\int \nabla^2 \varphi \, d^3x = 4\pi q,$$

and call q the quasicharge of the particle. From one of the field equations III.4) with $\psi = \psi(r)$ we have

$$\begin{aligned} \text{III.38)} \quad -\int \nabla^2 \varphi \, d^3x &= 4\pi q \int_0^{\infty} \psi^2 \varphi_1 \, r^2 dr + 4\pi q \int_{r_0}^{\infty} \psi_2^2 \varphi_2 \, r^2 dr \\ &= \frac{n}{q} \cdot \frac{1}{Y_3} \left(\frac{Y_1}{a} + \frac{Y_4}{4} \right) \end{aligned}$$

where Y_1 and Y_3 are defined in III.30), and

$$\text{III.39)} \quad Y_4 = 2\alpha x_0 e^{2x_0} \int_{2x_0}^{\infty} \frac{e^{-u}}{u} du.$$

It is found that the quasicharges of the three particles represented by the nodeless, one-node, and two-node solutions are roughly equal:

$$\begin{aligned} \text{III.40)} \quad q_0 &= 1.401 \, n g^{-\frac{1}{2}} \, (\text{gm. cm.}^3 \text{ sec.}^{-2})^{\frac{1}{2}} \\ q_1 &= 1.432 \, n g^{-\frac{1}{2}} \, (\text{gm. cm.}^3 \text{ sec.}^{-2})^{\frac{1}{2}} \\ q_2 &= 1.401 \, n g^{-\frac{1}{2}} \, (\text{gm. cm.}^3 \text{ sec.}^{-2})^{\frac{1}{2}} \end{aligned}$$

e) Summary of results: This investigation of time-independent solutions has produced three particles with the following description:

NODES IN ψ	0	1	2	
REST MASS	2.130	2.198	2.215	$10^{-21} \frac{nk}{g} \text{ gm.}$
RADIUS	2.45	5.73	9.00	$k^{-1} \text{ cm.}$
ELECTRIC CHARGE	0	0	0	e.s.u.
QUASICHARGE	1.401	1.432	1.401	$n g^{-\frac{1}{2}} (\text{gm. cm.}^3 \text{ sec.}^{-2})^{\frac{1}{2}}$

However, there is no apparent limit to the number of particles which can be obtained by investigating static solutions and increasing the number of nodes in ψ , until the approximations used are no longer valid. The trend observed in the first three solutions indicates that the mass spectrum might tend to a limit.

6. Limitations of the Model

The preceding investigation was restricted to static spherically symmetric solutions. The methods used are easily extended to solutions of the type

$$\begin{aligned} \text{III.41)} \quad \psi &= \psi_0(r) e^{i\omega t} \\ Q &= Q(r) \end{aligned}$$

where ψ_0 and ω are real. The field equations are now

$$\begin{aligned} \text{III.42)} \quad \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi_0}{dr} \right) - \left(\kappa^2 - \frac{\omega^2}{c^2} \right) \psi_0 &= -g \psi_0 Q^2 \\ \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dQ}{dr} \right) &= -g \psi_0^2 Q \end{aligned}$$

The substitutions

$$\begin{aligned} \text{III.43)} \quad \alpha^2 &= \kappa^2 - \frac{\omega^2}{c^2} > 0 & u &= \sqrt{g} r \psi_0 \\ X &= \alpha r & v &= \sqrt{g} r Q \end{aligned}$$

reduce the field equations to the form III.6) again; the same approximate solutions are accepted, giving us ψ_0 and Q :

$$\begin{aligned} \text{III.44)} \quad \psi_0 &= \begin{cases} \psi_{01} = \frac{C e^{-\alpha r}}{\sin(\sqrt{\alpha^2 - 1} x_0)} \frac{\sin(\sqrt{\alpha^2 - 1} \alpha r)}{\sqrt{g} r} & \text{for } r \leq \frac{x_0}{\alpha} \\ \psi_{02} = \frac{C}{\sqrt{g}} \frac{e^{-\alpha r}}{r} & \text{for } r \geq \frac{x_0}{\alpha} \end{cases} \\ Q &= \begin{cases} Q_1 = \frac{\alpha \alpha'}{\sqrt{g}} & \text{for } r \leq \frac{x_0}{\alpha} \\ Q_2 = \frac{\alpha x_0}{\sqrt{g} r} & \text{for } r \geq \frac{x_0}{\alpha} \end{cases} \end{aligned}$$

where normalization

$$\text{III.45)} \quad \int \psi^* \psi d^3x = \frac{n}{\alpha g}$$

yields

$$\text{III.46)} \quad C^2 = \frac{n}{2\pi} e^{2x_0} \frac{1}{\chi_3}$$

as before. We now use these solutions to determine the mass and electrostatic charge of the particles described.

With $\psi = \psi_0(\kappa) e^{i\omega t}$, we have

$$\text{III.47)} \quad T_{44} = - \left[(\kappa^2 + \frac{\omega^2}{c^2} - g\varphi^2) \psi_0^2 - (\nabla \psi_0)^2 - (\nabla \varphi)^2 \right].$$

Integration by parts and application of the field equations and boundary conditions lead to

$$\begin{aligned} \text{III.48)} \quad H = m'c^2 &= \int (2 \frac{\omega^2}{c^2} + g\varphi^2) \psi_0^2 d^3x \\ &= 2 \frac{\omega^2}{c^2} \frac{n}{\alpha g} + \frac{n\alpha}{g} \left(\frac{\gamma_1 + \gamma_2}{\gamma_3} \right) \end{aligned}$$

with γ_1 , γ_2 , and γ_3 as defined in III.30). We see that for solutions with a given number of nodes and with a fixed value of ω , the curve $\frac{Hg}{n\alpha}$ versus x_0 has the same shape as the curve $\frac{Hg}{n\kappa}$ versus x_0 for the corresponding static case, and the minimum occurs for the same value of x_0 as before.

However, the whole curve is raised by the amount $2\frac{\omega^2}{c^2} \frac{1}{\alpha^2}$. The mass m' of a particle corresponding to an oscillatory solution with frequency ω is

$$\text{III.49)} \quad m' = 2 \frac{\omega^2}{c^4} \frac{n}{\alpha g} + m \frac{\alpha}{\kappa}$$

where m is the mass of the corresponding particle with a static solution. The classical radius is increased by the factor $\frac{\alpha}{\kappa}$.

The charge density for an oscillatory solution is

$$\text{III.50)} \quad S_4 = i\varepsilon \frac{\omega}{c} \psi_0^2.$$

This yields an electrostatic charge of

$$\begin{aligned} \text{III.51)} \quad e &= - \frac{i}{c} \int S_4 d^3x \\ &= \varepsilon \frac{\omega}{c^2} \int \psi_0^2 d^3x \\ &= \varepsilon \frac{\omega}{c^2} \frac{n}{\alpha g}, \end{aligned}$$

where ε is a scaling factor which cannot be determined by the theory. Clearly, since ω is not restricted to a discrete spectrum but to a continuous range of values, $\omega < \kappa c$, and since ε is unspecified, it is not possible to grasp with this classical model the quantization of charge.

It is also obvious that the instability and lifetimes of the heavy particles are beyond the capacity of this model.

IV. POSSIBLE IMPROVEMENT OF THE MODEL

1. Specific Model and Field Equations

In an attempt to improve the theoretical mass spectrum we assign a "bare" mass to the φ -field as well as to the ψ -field. Thus we consider two coupled fields with Lagrangian density

$$\text{IV.1)} \quad \mathcal{L} = -\frac{\partial\psi}{\partial x_\mu} \frac{\partial\psi^*}{\partial x_\mu} - \kappa^2 \psi \psi^* - \frac{\partial\varphi}{\partial x_\mu} \frac{\partial\varphi}{\partial x_\mu} - \lambda^2 \varphi^2 + g \psi \psi^* \varphi^2$$

$$g > 0$$

and we require that

$$\text{IV.2)} \quad \frac{\lambda^2}{\kappa^2} \ll 1.$$

The field equations

$$\text{IV.3)} \quad \begin{aligned} \square^2 \psi - \kappa^2 \psi &= -g \psi \varphi^2 \\ \square^2 \varphi - \lambda^2 \varphi &= -g \psi \psi^* \varphi \end{aligned}$$

become

$$\text{IV.4)} \quad \begin{aligned} \frac{1}{r^2} \frac{d^2}{dr^2} (r\psi) - \kappa^2 \psi &= -g \psi \varphi^2 \\ \frac{1}{r^2} \frac{d^2}{dr^2} (r\varphi) - \lambda^2 \varphi &= -g \psi^2 \varphi \end{aligned}$$

for static, spherically symmetric solutions

$$\text{IV.5)} \quad \begin{aligned} \psi &= \psi(r) \\ \varphi &= \varphi(r) \end{aligned}.$$

The substitutions

$$\text{IV.6)} \quad \begin{aligned} x &= \kappa r & u &= \sqrt{g} \, r \psi \\ v &= \sqrt{g} \, r \varphi \end{aligned}$$

lead to

$$\text{IV.7)} \quad \begin{aligned} u'' &= u \left(1 - \frac{v^2}{x^2} \right) \\ v'' &= v \left(\frac{\lambda^2}{\kappa^2} - \frac{u^2}{x^2} \right). \end{aligned}$$

Graph 14: Nodeless solutions of equations IV.7)
 u versus x .

Graph 15: Nodeless solutions of equations IV.7)
 v versus x .

The boundary conditions on u and v are

$$\begin{aligned} &u=0, \quad v=0 \quad \text{at } x=0 \\ \text{IV.8)} \quad &\frac{u}{x} \rightarrow 0, \quad \frac{v}{x} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \end{aligned}$$

In the region $x \gg 1$ we have

$$\begin{aligned} &u'' = u \\ \text{IV.9)} \quad &v'' = \frac{\lambda^2}{k^2} v \end{aligned}$$

with acceptable solutions

$$\begin{aligned} &u = C e^{-x} \\ \text{IV.10)} \quad &v = B e^{-\frac{\lambda}{k} x}. \end{aligned}$$

Qualitative solutions of equations IV.7) with boundary conditions IV.8) are drawn in graphs 14 and 15.

2. Approximate Solution of the Equations

We must again adopt approximate methods of solution because of the non-linearity of the equations. We put

$$\begin{aligned} &v = ax \\ \text{IV.11)} \quad &u = A \sin(\sqrt{a^2-1} x) \end{aligned}$$

in the region of small x , and join these to the asymptotic expressions IV.10) at x_0 . The approximate solutions are shown in graphs 16 and 17.

Continuity of u , v , and u' at x_0 require

$$\text{IV.12)} \quad A = \frac{C e^{-x_0}}{\sin(\sqrt{a^2-1} x_0)}, \quad B = a x_0 e^{\frac{\lambda}{k} x_0}$$

$$\text{IV.13)} \quad x_0 = \frac{\tan^{-1}(-\sqrt{a^2-1})}{\sqrt{a^2-1}},$$

which lead to:

Graph 16: Approximate solution of equations IV.7)
for the nodeless case: u versus x .

Graph 17: Approximate solution of equations IV.7)
for the nodeless case: v versus x .

$$\text{IV.14)} \quad \psi = \begin{cases} \psi_1 = \frac{C e^{-x_0}}{\sin(\sqrt{a^2-1} x_0)} \frac{\sin(\sqrt{a^2-1} K r)}{\sqrt{g} r} & \text{for } r_0 \leq \frac{x_0}{K} \\ \psi_2 = \frac{C e^{-K r}}{\sqrt{g} r} & \text{for } r_0 \geq \frac{x_0}{K} \end{cases}$$

$$\text{IV.15)} \quad \varphi = \begin{cases} \varphi_1 = \frac{a K}{\sqrt{g}} & \text{for } r_0 \leq \frac{x_0}{K} \\ \varphi_2 = a x_0 e^{\frac{2}{K} x_0} \frac{e^{-K r}}{\sqrt{g} r} & \text{for } r_0 \geq \frac{x_0}{K} \end{cases}$$

Since the ψ -field is the same as III.21), normalization gives expression III.26) again for C^2 .

To calculate the energy of the system we form

$$\text{IV.16)} \quad T_{44} = -[(\kappa^2 - g\varphi^2)\psi^2 + \lambda^2\varphi^2 + (\nabla\psi)^2 + (\nabla\varphi)^2]$$

and reduce

$$\text{IV.17)} \quad H = mc^2 = - \int T_{44} d^3x$$

by integration by parts and use of the boundary conditions and field equations, to find:

$$\begin{aligned} \text{IV.18)} \quad H &= 4\pi \int_0^\infty g \varphi^2 \psi^2 r^2 dr \\ &= \frac{\pi K}{g} \left(\frac{Y_1 + Y_5}{Y_3} \right) \end{aligned}$$

where Y_1 and Y_3 are defined in III.30) and

$$\text{IV.19)} \quad Y_5 = 2a^2 x_0 - 4(a x_0)^2 \left(1 + \frac{\lambda}{K}\right) e^{2x_0 \left(1 + \frac{\lambda}{K}\right)} \int_{\frac{2x_0 \left(1 + \frac{\lambda}{K}\right)}{u}}^\infty \frac{e^{-u}}{u} du$$

3. Numerical Results

We now tie this model down to reality by considering the ψ -field as the description of a bare nucleon, and the φ -field as the description of a bare meson field acting as glue. In particular, we assume that the bare mass ratio reflects approximately the observable mass ratio of the corresponding dressed particles, and we substitute

$$\text{IV.20)} \quad \frac{\lambda}{K} = \frac{\mu}{m_N} = \frac{1}{6}$$

TABLE II

VALUES OF $\frac{Hq}{nk}$ AND CORRESPONDING VALUES OF x_0
FOR THE NODELESS AND ONE-NODE SOLUTIONS

NODELESS SOLUTION		ONE-NODE SOLUTION	
x_0	$\frac{Hq}{nk}$	x_0	$\frac{Hq}{nk}$
0.4285	5.092		
0.8573	2.813	1.800	3.479
1.2860	2.206	3.001	2.368
1.7140	1.9825	4.028	2.072
2.1000	1.9085	4.716	1.993
2.3580	1.8950	5.143	1.9695
2.4000	1.8948	5.400	1.963
2.4856	1.8957	5.571	1.960
2.7000	1.9050	5.700	1.960
2.8716	1.9188	6.000	1.963
3.4286	1.991	6.427	1.976
5.3560	2.404		

Calculations of $\frac{Hq}{nk}$ were carried out for different x_0 and for the case of ψ having no nodes and one node.

The results are tabulated in Table II.

We see that $\frac{Hq}{nk}(x_0)$ again has a minimum in each case; the minimum values and the corresponding values of x_0 are

$$\begin{aligned} \text{IV.21)} \quad \frac{Hq}{nk} &= 1.895 \quad \text{for } x_0 \doteq 2.4 \text{ (nodeless case)} \\ \frac{Hq}{nk} &= 1.960 \quad \text{for } x_0 \doteq 5.6 \text{ (one-node case)} \end{aligned}$$

as compared with

$$\begin{aligned} \text{IV.22)} \quad \frac{Hq}{nk} &= 1.915 \quad \text{for } x_0 \doteq 2.45 \text{ (nodeless case)} \\ \frac{Hq}{nk} &= 1.976 \quad \text{for } x_0 \doteq 5.75 \text{ (one-node case)} \end{aligned}$$

in the massless Q -field model. The mass ratio of the two lightest particles has increased only very slightly, from 1.032 : 1 to 1.034 : 1.

Since no significant changes in the features of the model have been produced, this model was not investigated any further. The assumption that bare mass ratios reflect dressed mass ratios is very dubious in any case.

APPENDIX I

Dimensional Analysis of Quantities Occurring in the Type (1b) Model Discussed

$$\text{III.1)} \quad \mathcal{L} = - \frac{\partial \psi}{\partial x_\mu} \frac{\partial \psi^*}{\partial x_\mu} - \kappa^2 \psi \psi^* - \frac{\partial \varphi}{\partial x_\mu} \frac{\partial \varphi}{\partial x_\mu} + g \psi \psi^* \varphi^2 .$$

Also, since $\mathcal{L}$ must be an energy density, we have

$$[\mathcal{L}] = M L^{-1} T^{-2} .$$

From this we conclude that

$$[\psi] = M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}$$

$$[\varphi] = M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}$$

$$[\kappa] = L^{-1}$$

$$[g] = M^{-1} L^{-3} T^2$$

$$\begin{aligned} [\int \psi^* \psi d^3x] &= M L^4 T^{-2} \\ &= L (M L^3 T^{-2}) \end{aligned}$$

$$= \left[\frac{1}{\kappa g} \right]$$

$$[\int \nabla^2 \varphi d^3x] = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} .$$

APPENDIX II

A Sample of the Calculations Made for the Model

in which ϕ is a Massless Field:

Calculation of $\frac{Hg}{n\kappa}$ corresponding to $x_0 = 5.75$

for the case of ψ having one node.

$\tan^{-1}(-\sqrt{a^2-1})$	5.5183	$\cos(\sqrt{a^2-1} x_0)$	0.72149
$\sqrt{a^2-1}$	0.9597	$\sin(\sqrt{a^2-1} x_0)\cos(\sqrt{a^2-1} x_0)$	0.4996
$x_0 = \frac{\tan^{-1}(\sqrt{a^2-1})}{\sqrt{a^2-1}}$	5.7500	$\frac{1}{\sin^2(\sqrt{a^2-1} x_0)}$	2.0857
$2x_0$	11.500	$\frac{a^2}{\sqrt{a^2-1}}$	2.0017
$\int_{2x_0}^{\infty} \frac{e^{-u}}{u} du$	8.1498×10^{-7}	$2a^2x_0$	22.0915
e^{2x_0}	98716.	$4(ax_0)^2$	254.05
$e^{2x_0} \int_{2x_0}^{\infty} \frac{e^{-u}}{u} du$	0.080452	$4(ax_0)^2 e^{2x_0} \int_{2x_0}^{\infty} \frac{e^{-u}}{u} du$	20.439
a^2-1	0.9210	$(\sqrt{a^2-1} x_0) - \cos(\sqrt{a^2-1} x_0)\sin(\sqrt{a^2-1} x_0)$	6.0179
a^2	1.9210	Y_1	25.124
$\sqrt{a^2-1} x_0$	5.5183	Y_2	1.653
$\sin(\sqrt{a^2-1} x_0)$	-0.69242	Y_3	13.552

$$\frac{Hg}{n\kappa} = \frac{Y_1 + Y_2}{Y_3} = 1.9759 .$$

Tables which were found useful were:

- 1) British Association for the Advancement of Science
Mathematical Tables: Volume I (for the exponential
integral);
- 2) U. S. Department of Commerce, National Bureau of
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