FIELD THEORETICAL DESCRIPTION OF THE SUPERCONDUCTING
STATE

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ABSTRACT

The superconducting state is described in terms of the electronic plasma in interaction with the electromagnetic potentials. The interaction of the plasma with itself and with the lattice, is represented by a potential energy determined such that the equations of motion are self-consistent. The resulting non-linear field equations are solved in the linear approximation. There exist four independent modes of vibration of the field variables corresponding to the plasma oscillations, and one longitudinal and two transverse oscillations of the electromagnetic potentials. The contribution of these modes to the specific heat is discussed.

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I INTRODUCTION:

In this work an attempt is made to describe the phenomenon of superconductivity by representing the so-called "superconducting electrons" as a classical charged fluid. It will be shown that the hydrodynamic equations describing the fluid are consistent with London's phenomenological equations. Since as yet the behaviour of the specific heat of a superconducting metal below its transition temperature has not been satisfactorily explained, we shall try to describe this phenomenon in terms of the energies of the wave motions of the electric and magnetic field potentials. We shall begin our discussion by reviewing briefly some of the experimental facts associated with the phenomenon of superconductivity.

A The Experimental Facts.

In 1911, Kamerlingh Onnes discovered that at a certain very low temperature, T_c , the resistance of certain conductors dropped sharply to zero, indicating that no electric field could be maintained inside this 'superconductor'. Kamerlingh Onnes further discovered that this phenomenon of superconductivity was destroyed by the application of a sufficiently strong magnetic field, $H_c(T)$.

Considering the superconductor to be a perfect conductor (infinite conductivity), then it follows from Maxwell's equations that the magnetic field inside a superconductor is constant ($\dot{B} = 0$). One would then expect that if a superconducting metal were placed in a magnetic field and then cooled below its transition temperature, that the magnetic field would be 'frozen in'. It was found, however, by Meissner in 1933 that when the metal became superconducting the field in the neighborhood of the specimen increased, thus showing that the field inside the superconductor was pushed out, leaving no magnetic field inside. It was also shown that when a superconductor initially in zero field is placed in a magnetic field, the field inside the superconductor remains zero. Hence one concludes that this process is a reversible one and that the magnetic field may therefore be treated as a thermodynamic variable.

Treating H as a thermodynamic variable of the state of the superconductor, one can easily derive from the first and second laws of thermodynamics that the difference between the specific heat in the superconducting state and the specific heat in the normal state is

$$C_s - C_n = \frac{T}{8\pi} \frac{d^2 H_c^2}{dT^2} .$$

The specific heat at constant volume in the normal state at low temperatures can be represented by

$$C_n = \alpha T + 464.4 \left(\frac{T}{\Theta}\right)^3$$

The T^3 term, which is hereafter denoted by C_L , is attributed to the lattice vibrations according to the well known Debye theory of specific heats. The linear term has been shown (3) to result from the application of Fermi-Dirac statistics to a free electron gas, and is hence usually referred to as the normal electronic specific heat and is designated by C_n^{el} .

It is generally found that at the transition temperature T_c , where the change from normal to superconducting states takes place, the specific heat as a function of temperature is discontinuous (see Figure 1). It is seen from Figure 1 that the specific heat in the superconducting state, C_s , is greater than the lattice contribution, C_L , at all points, but not always greater than the normal electronic specific heat. Hence it is generally assumed that the lattice vibrations still contribute to the specific heat in the superconducting state, whereas the normally conducting electrons do not contribute. Since the difference $C_s - C_L$ cannot be accounted for by C_n^{el} , this difference is called the superconducting specific heat, and is designated by C_s^{el} . Experimental measurements of C_s^{el} generally indicate that C_s^{el} obeys a T^3 law. Recent work on the specific heat of superconducting Niobium (1) shows that in this case C_s^{el} rises much faster than any T^3 dependence. We shall later discuss the

contribution of various modes of vibration to the superconducting specific heat, and will give a possible reason for this deviation from a T^3 dependence.

B The Present State of the Theory of Superconductivity.

There exists at present no atomistic theory of superconductivity. One can, however, specify some of the conditions that a successful theory of superconductivity must fulfill. It must

- (i) agree with the phenomenological theory (London's equations),
- (ii) give the transition temperature and the critical magnetic field, and
- (iii) give the isotope effect ($M^{\frac{1}{2}} T_c = \text{const.}$).

Taking the usual view that superconductivity is caused by the conduction electrons, and is not a property of the lattice, one can say that a successful approach towards the problem has been made if one can explain the origin of a large gap in the energy levels of the conduction electrons near the surface of the Fermi sphere. In fact, if electrons were Bose particles, one could construct a fairly reasonable model of the superconductor, in which the existence of the critical magnetic field is explained in terms of the Zeeman effect field which would tend to close the gap between the ground state and the first excited state of the electrons in a superconductor of reasonable dimensions. This effect

comes out in the right order of magnitude in the approximation in which the interaction between the electrons themselves and between the electrons and the lattice is neglected. Such a model is unfortunately of little use in the case of a Fermi gas of electrons, since the level density near the surface of the Fermi sphere is so large that the magnetic field needed to make neighboring energy levels coincide by Zeeman effect is extremely small (10^{-3} oersteds). It can therefore be only by taking into account the interaction between electrons themselves, and between the electrons and lattice vibrations, that one can hope to obtain a successful start towards an atomistic theory of superconductivity. The very existence of the isotope effect indicates that the width of the energy gap at the surface of the Fermi sphere depends essentially on the interactions between the electrons and the lattice vibrations.

It is beyond the scope of the present work to try to describe the mechanism which sets up the superconducting state, which is presumably due to the interaction of electrons among themselves and with the lattice ions. It will be assumed that this interaction leads to a rigid ground state for the electrons, i.e., most of the electrons must be in their lowest states, and the probability that they can rise to states of higher energy must be small.

In the following, the electronic description of the superconductor will be characterized by a system of continuous

electric charges which interact with the electromagnetic field which they themselves produce. In other words, the electronic behaviour is characterized by a system of simultaneous differential equations consisting of Maxwell's equations and the classical hydrodynamic equations for a continuous fluid.

II THE LAGRANGIAN FORMULATION OF THE SUPERCONDUCTING STATE AND THE RESULTING EQUATIONS OF MOTION:

We shall try the following Lagrangian density:*

$$[1] \quad \mathcal{L} = -\rho \dot{\varphi} - \frac{1}{2} \rho v^2 + \frac{1}{8\pi} (\epsilon E^2 - H^2) - \frac{1}{8\pi} \left(\frac{\epsilon}{c} \dot{\Phi} + \nabla \cdot \vec{A} \right) - \frac{e}{m} \rho_i \bar{\Phi}_i$$

where $\underline{H}$ is the magnetic field, $\underline{E}$ the electric field, Φ and $\underline{A}$ the electric and magnetic vector potentials respectively, ϵ the dielectric constant, and $\underline{v}$ (defined by equation [2] below) is the velocity of the fluid. φ is the velocity potential, and ρ is the mass density of the fluid. The first two terms are the usual classical hydrodynamic terms for a free continuous fluid of density ρ ; by themselves, they give upon independent variation with respect to ρ and φ , the equation of continuity and the Bernouilli equation for a continuous fluid. The last three terms are the electro-magnetic terms which give, by themselves, upon variation with respect to $\underline{A}$ and Φ , Maxwell's equations. There is, however, one important difference: the usual term $\frac{e}{m} \rho \Phi$ has been replaced by $\frac{e}{m} \rho_i \bar{\Phi}_i$ where ρ_i and $\bar{\Phi}_i$ are the deviations from the average density ρ and the average potential Φ_0 respectively. This means that the background of positive lattice charge

* This Lagrangian is chosen for simplicity; one can try other Lagrangians which are derived from analogy with the classical Schroedinger field, but it turns out that this more complicated Lagrangian leads to the same solutions if one demands that London's first equation be satisfied. The treatment of this Lagrangian is outlined in Appendix A.

effectively cancels out the Coulomb interaction* of the fluid leaving only the interaction between the net electric charge density ρ_i , and the net electric field potential Φ_i . ρ and Φ are imagined to be written as $\rho = \rho_o + \rho_i$ and $\Phi = \Phi_o + \Phi_i$.

In addition we define the following:

$$[2] \quad \vec{V} = \nabla \varphi - \frac{e}{mc} \vec{A}$$

$$[3] \quad \vec{H} = \text{curl } \vec{A}$$

$$[4] \quad \vec{E} = -\nabla \Phi - \frac{1}{c} \dot{\vec{A}}$$

The choice [2] has been made to satisfy identically London's equation

$$[2a] \quad \text{curl } \vec{J}_E = -\frac{1}{c} \vec{H}, \quad \text{where} \quad \vec{J}_E = \frac{m}{e} \vec{V}$$

while [3] and [4] are the usual expressions for the electric and magnetic fields in terms of the potentials Φ and $\underline{A}$.

We will now assume that the density of the fluid is almost constant and that variations in density are small, i.e., we assume that in the expansion $\rho = \rho_o + \rho_i$, the deviation ρ_i is much smaller than ρ_o , and that ρ_o is constant. This assumption with the previous assumption that the positive lattice charge effectively cancels out the electric field due to the electrons, makes it reasonable to assume that φ , $\underline{A}$, and Φ_i , are small, so that second order terms in them can be neglected.

* This is shown to be true in Appendix A.

By independent variation of the Lagrangian density with respect to φ , ρ , Φ , and $\underline{A}$, we get the following equations of motion:

$$[5] \quad \dot{\rho}_i + \rho_0 \nabla^2 \varphi - \frac{e}{mc} \rho_0 \nabla \cdot \vec{A} = 0$$

$$[6] \quad \dot{\varphi} + \frac{e}{m} \Phi_i = 0$$

$$[7] \quad \nabla^2 \vec{A} - \frac{\epsilon}{c^2} \ddot{\vec{A}} + \frac{4\pi e}{mc} \rho_0 (\nabla \varphi - \frac{e}{mc} \vec{A}) = 0$$

$$[8] \quad \nabla^2 \Phi - \frac{\epsilon}{c^2} \ddot{\Phi} + \frac{4\pi e}{m\epsilon} \rho = 0,$$

neglecting second order terms. Equation [5] is the usual equation of continuity expressing the law of conservation of matter. Equation [6] is Bernoulli's equation for the conservation of energy in the case that the kinetic energy of the fluid is negligible. Equations [7] and [8] are the usual electro^{MAGNETIC} field equations. London's first equation follows directly from equation [6]. We have

$$[6a] \quad \frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \frac{m}{2} \vec{v}^2}{\partial t} = \frac{m}{e} \nabla \dot{\varphi} - \frac{1}{c} \dot{\vec{A}} = -\nabla \Phi - \frac{1}{c} \dot{\vec{A}} = \vec{E}$$

which is London's first equation.

The equations [5] to [8] may be solved for ρ as follows: differentiate [5] with respect to time and substitute $\dot{\varphi}$ from [6]. Making use of the Lorentz condition $\nabla \cdot \vec{A} = -\frac{\epsilon}{c} \dot{\Phi}$, equation [5] becomes:

$$[9] \quad \ddot{\rho}_i + \frac{4\pi e^2}{m^2 \epsilon} \rho_0 \rho_i = 0$$

Hence $\rho = f(\vec{r}) e^{i\omega_0 t}$
where

$$[10] \quad \omega_0^2 = \frac{4\pi e^2}{m^2 \epsilon} \rho_0$$

and $f(\underline{r})$ is, as far as equation [9] is concerned, an arbitrary function of the co-ordinates. However, in order to solve the rest of the equations, we must know the explicit form of $f(\underline{r})$. It turns out that if $f(\underline{r})$ is taken to be a constant, the Lorentz condition cannot be satisfied. In Appendix A, where a more general Lagrangian is considered, $f(\underline{r})$ is shown to be of the form $\exp(i\vec{K}_0 \cdot \underline{r})$ where $\vec{K}_0$ is a vector whose magnitude depends on the coefficient of a term which does not appear in the Lagrangian used above. It is therefore convenient to take

$$[11] \quad f(\vec{r}) = \text{const.} e^{i\vec{K}_0 \cdot \vec{r}}$$

where for this discussion $\vec{K}_0$ can be taken to be any arbitrary non-zero vector. Hence we have:

$$[12] \quad \rho = \text{const.} e^{i\vec{K}_0 \cdot \vec{r} + i\omega_0 t}$$

We may now solve equation [8] for Φ_1 . We get:

$$[13] \quad \Phi_1 = \text{const.} e^{i\vec{K}_0 \cdot \vec{r} + i\omega t} + \frac{4\pi e}{m\epsilon(K_0^2 - \frac{\epsilon}{c^2}\omega_0^2)} \rho_1,$$

where

$$[14] \quad K^2 = \frac{\epsilon \omega^2}{c^2}$$

For brevity we put $\Phi' = \text{const.} e^{i\vec{k} \cdot \vec{r} + i\omega t}$ and $\alpha = \frac{4\pi e}{m\varepsilon(k_0^2 - \frac{\varepsilon}{c^2}\omega_0^2)}$.

Equations [5] and [6] yield the following solution for φ :

$$[15] \quad \varphi = \frac{e\alpha}{m\omega_0^2} \dot{\rho} + \frac{e}{m\omega_0^2} \dot{\Phi}' + \varphi'$$

where φ' is a general solution of $\dot{\varphi}' = 0$ and $\nabla^2 \varphi' = 0$. Finally equation [7] yields:

$$[16] \quad \vec{A} = \frac{mc}{e} \nabla \varphi' + \frac{c}{\omega_0^2} \nabla \dot{\Phi}' + \frac{\varepsilon}{c} \frac{\alpha}{K_0^2} \nabla \dot{\rho} + \vec{A}_0 e^{i\vec{k} \cdot \vec{r} + i\omega t}$$

where $\vec{A}_0$ is a constant vector, and

$$[17] \quad \omega_0^2 = \frac{c^2 K_1^2}{\varepsilon} + \omega_0^2$$

It is seen from [13] and [16] that the Lorentz condition is satisfied provided that $\vec{A}_0 \cdot \vec{K}_1 = 0$, i.e. provided that the vector potential

$$[18] \quad \vec{A}' = \vec{A}_0 e^{i\vec{K}_1 \cdot \vec{r} + i\omega t}$$

is a transverse wave.

It should be pointed out that the magnetic vector potential $\vec{A}$ given by [16] has a very unusual feature: it has a longitudinal component, $\frac{c}{\omega_0^2} \nabla \dot{\Phi}'$.

We see from the above, that the solutions are constructed from four independent modes of oscillation: ρ , Φ' , and the two components of $\vec{A}'$ which are orthogonal to its propagation vector $\vec{K}_1$. ρ has a fixed frequency, ω_0 , given by [10] and an arbitrary propagation vector, $\vec{K}_0$. Φ' has a frequency,

ω , and a propagation vector, $\underline{K}$, which are related by

$$\omega^2 = \frac{c^2 K^2}{\epsilon}$$

$\underline{A}'$ has a frequency ω_1 , and a propagation vector $\underline{K}_1$, related by

$$\omega_1^2 = \frac{c^2 K_1^2}{\epsilon} + \omega_0^2$$

It is shown in Section III, that, for thermodynamic considerations, ω_0^2 is negligible compared with $\frac{c^2 K_1^2}{\epsilon}$. The contribution of these four modes of oscillation to the specific heat of the system is considered below.

Equation [10] is the clue to a possible explanation of the high specific heats of superconductors. It states simply that the frequency of the classical charged fluid representing the superconducting electrons, i.e., the frequency of the so-called plasma oscillations, is a constant. We could then imagine the following model. In the normal state, the conducting electrons are essentially free, as shown by the existence of C_n^{el} . At some critical temperature the electrons, under the influence of the lattice ions, vibrate with some fixed frequency which is a sub-harmonic of the maximum lattice frequency, i.e. a sub-harmonic of the upper limit of the Debye spectrum of lattice frequencies. Let us assume that this plasma frequency is of order of magnitude $\frac{kT_c}{h} \sim 10^{10}$ to 10^{11} . This frequency results in an extremely large dielectric constant as was pointed out by Ginsburg (2). For example, if we take the number of superconducting

electrons per unit volume to be 5×10^{21} , then from equation [10] we have

$$\xi = \frac{4\pi e^2}{m\omega_o^2} \left(\frac{\rho_o}{m} \right) \sim 10^8 \text{ to } 10^{10}.$$

This large dielectric constant plays a very important role in the behaviour of the superconductor. Such a large ξ means a very small velocity of light ($\frac{c}{\sqrt{\xi}} \sim 10^5 \text{ cm/sec}$). Now the smaller the velocity of light, the larger is the momentum p associated with black body radiation of given energy E . Hence, since the number of photons which can have energy between E and $E+dE$ is proportional to $p^2 dp$, there are more photons with energy E when ξ is large than when ξ is small. In fact, the number of photons of given energy is proportional to $\xi^{3/2}$. Because of the greatly increased number of photons present, a much greater expenditure of energy will be required to raise the energy of the superconductor. This means a larger specific heat in the superconducting state than in the normal state. This subject is treated quantitatively in the next section.

III SPECIFIC HEAT

From the solutions for the equations of motion given in Section II, we see that there are four independent modes of vibration in the superconductor. They are:

- (1) the plasma oscillation,
- (2) the electric field potential,
- (3) and (4) the two independent modes of vibration due to $\underline{A}'$. There are two modes because $\underline{A}'$ is a transverse wave and hence there are two independent mutually orthogonal oscillations each perpendicular to the direction of propagation.

In thermal equilibrium, each of these modes will contribute separately to the total energy of the system. We will first show that the contribution of the plasma oscillations to the specific heat of the system is negligible and then proceed to calculate the contribution of the other three modes of vibration.

(1) Plasma Contribution

It was shown in Section II that the density of the continuous charged fluid representing the superconducting electrons was of the form

$$\rho = \rho_0 + \rho_1 = \rho_0 + \text{const.} e^{i\vec{k}_0 \cdot \vec{r} + i\omega_0 t}$$

We define the energy of such an oscillation as $E_0 = \hbar \omega_0$

and the momentum as $\vec{p}_0 = \hbar \vec{k}_0$. Then the total energy per unit

volume due to the plasma oscillation is

$$E_{\text{tot}} = \frac{4\pi}{(2\pi\hbar)^3} \int_0^{p_{\text{max}}} \frac{E_0 p_0^2 dp_0}{e^{\frac{E_0}{kT}} - 1}$$

where p_{max} is the maximum momentum associated with the plasma oscillations, k is Boltzmann's constant and T is the temperature. Because E_0 is independent of p_0 , the size of E_{tot} will be determined entirely by E_0 and p_{max} . Since we know nothing about the size of p_{max} , the best we can do is to assume some upper limit for it. We will assume that $p_{\text{max}} = \frac{\sqrt{E}}{c} E_0 = \frac{\hbar}{c} \left(\frac{4\pi e^2 \rho_0}{m^2} \right)^{1/2}$; presumably it would be much smaller than this. We therefore have:

$$E_{\text{tot}} = \frac{4\pi p_{\text{max}}^3 E_0}{3(2\pi\hbar)^3 (e^{\frac{E_0}{kT}} - 1)}$$

We will now assume for the sake of the following order of magnitude calculation, that p_{max} and E_0 are temperature independent. In general, of course, they will not be independent of temperature, but taking this into account would not change the results obtained below by more than a small factor.

The contribution of the plasma oscillations to the specific heat is then

$$C_v = \frac{dE_{\text{tot}}}{dT} = \frac{4\pi k \rho_0^3}{3(2\pi\hbar)^3} \cdot \frac{x^2 e^x}{(e^x - 1)^2}$$

where $x = \frac{E_0}{kT}$. The expression $\frac{x^2 e^x}{(e^x - 1)^2}$ is a monotonously decreasing function, having its largest value (for positive x) at $x = 0$, at which point it is unity. It follows then that

C_V has its largest value when $E_0 \ll kT$; i.e.,

$$C_{V_{\max}} = \frac{4\pi k \rho_{\max}^3}{3(2\pi k)^3}$$

Taking the number of superconducting electrons per unit volume, $\frac{\rho_0}{m}$, to be about 5×10^{21} , we get:

$$C_{V_{\max}} = 5.48 \times 10^{-3} \text{ ergs/cc.}$$

Since measured values (1) of the specific heat in Niobium and other superconducting metals are about 10^4 ergs/cc, we conclude that the contribution from the plasma oscillations to the specific heat is negligible.

(2) Electric Field Potential Contribution.

We saw in Section II that the electric field potential Φ could be expressed as

$$\Phi = \Phi_0 + \Phi_1 = \Phi_0 + \text{const} e^{i\vec{k} \cdot \vec{r} + i\omega t} + \alpha \rho$$

with $K^2 = \frac{\epsilon \omega^2}{c^2}$. The contribution of the oscillation ρ to the energy of the system has been taken into account in III(1).

We will now consider the contribution of the mode of vibration represented by Φ_1 . We define the energy and momentum of this oscillation in the usual fashion by $E = \hbar\omega$ and $\underline{p} = \hbar\underline{K}$, respectively, and we define the total energy per unit volume of such oscillations by

$$[19] \quad E'_{\text{tot}} = \frac{4\pi}{(2\pi k)^3} \int_0^\infty \frac{E(\rho) \rho^2 d\rho}{e^{\frac{E}{kT}} - 1} = \frac{4\pi^5 k^4}{15(2\pi k c)^3} \xi^{3/2} T^4$$

(3) Contribution of the Magnetic Potential.

From equation [16] we see that the magnetic vector

potential $\underline{A}$ can be expressed as the sum of the waves $\text{grad } \dot{\phi}$, $\text{grad } \dot{\Phi}'$, and the complimentary function $\underline{A}' = \underline{A}_0 e^{i\vec{k} \cdot \vec{r} + i\omega t}$. (ϕ' , being independent of time, is not a wave motion.) The wave represented by $\underline{A}'$ can be treated in the same way as that represented by $\dot{\Phi}'$. We define the energy and momentum of the $\underline{A}'$ wave as $E_1 = \hbar \omega$, and $\vec{p}_1 = \hbar \vec{k}$ respectively. Equation [17] gives

$$E_1^2 = \frac{c^2 p_1^2}{\epsilon} + E_0^2$$

where $E_0 = \hbar \omega_0$. Hence the total energy of the $\underline{A}'$ wave is given by

$$[20] \quad E_{\text{tot}} = 2 \cdot \frac{4\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{E_1(p) p^2 dp}{e^{\frac{E_1}{\hbar T}} - 1} = \frac{8\pi \epsilon^{3/2}}{(2\pi\hbar c)^3} \int_{E_0}^\infty \frac{(E_1^2 - E_0^2)^{1/2} E_1^2 dE_1}{e^{\frac{E_1}{\hbar T}} - 1}$$

The factor of two, appearing in front of the integral in equation [20] comes from the fact that $\underline{A}'$ has two possible polarizations, i.e., two independent directions for $\underline{A}_0$ perpendicular to the propagation vector $\underline{k}_1$. Equation [20] is simplified a little by putting $u = \frac{E_1}{\hbar T}$ and $u_0 = \frac{E_0}{\hbar T}$, and by denoting by I the integral

$$I(u_0) = \int_{u_0}^\infty \frac{(1 - (\frac{u_0}{u})^2)^{1/2} u^3 du}{e^u - 1}$$

Then

$$E_{\text{tot}} = \frac{8\pi \epsilon^{3/2} \hbar^4 T^4}{(2\pi\hbar c)^3} I$$

It will be shown below that the specific heat due to the energy of the $\dot{\Phi}'$ wave as given by equation [19] is of

FIG - 2 - VARIATION OF C_3^{gl} FROM AT^3 LAW

the same order of magnitude as experimentally measured values if the dielectric constant is taken to be about 10^{10} . With this ϵ and with the number of superconducting electrons taken to be $5 \times 10^{21}/\text{cc.}$, we get $u_0 \cong 0.05$. Hence a very good approximation is made to I by taking $u_0 = 0.00$, in which case $I = \frac{\pi^4}{15}$. We therefore have for the total energy of the Φ' and A' waves

$$[21] \quad E_{\text{TOT}} = E'_{\text{TOT}} + E_{\text{TOT}} = \frac{12 \pi^5 k^4 \epsilon^{3/2} T^4}{15 (2\pi \hbar c)^3}$$

Considering the dielectric constant to be a function of temperature, we get for the specific heat contribution from these waves:

$$[22] \quad C_v = \frac{dE_{\text{TOT}}}{dT} = a T^3 \left(\epsilon^{3/2} + \frac{1}{4} T \frac{d\epsilon^{3/2}}{dT} \right)$$

where
$$a = \frac{48 \pi^5 k^4}{15 (2\pi \hbar c)^3} = 4.38 \times 10^{-14} \text{ erg}/(\text{deg})^4 \text{ cc}$$

Fig. 2 shows a plot of C_s^{el}/aT^3 versus T calculated from the experimental data for Niobium (1).

Since we cannot expect to find the functional dependence of the dielectric constant on the temperature from this present theory, we must content ourselves with finding what function will give the closest fit between the experimental and theoretical specific heat curves. By a process of trial and error, a curve of ϵ versus T was found which could be used to construct the curve in Figure 2 by the relation

$$\frac{C_s^{el}}{aT^3} = \xi^{3/2} + \frac{1}{4} T \frac{d\xi^{3/2}}{dT}$$

It was found that this empirical curve of $\xi^{3/2}$ versus T could be described quite accurately by the function

$$[23] \quad \xi^{3/2} = \xi_0^{3/2} \cdot \frac{T_c}{T} \exp\left(-\frac{T_c}{T}\right)$$

where $\xi_0^{3/2} = 16.4 \times 10^{15}$ for Niobium.

Hence we can write

$$[24] \quad C_s^{el} = a \xi_0^{3/2} T_c T^2 e^{-\frac{T_c}{T}} \left(3 + \frac{T_c}{T}\right)$$

$$= 1.625 \times 10^3 T^2 e^{-\frac{8.7}{T}} \left(3 + \frac{8.7}{T}\right) \text{ for Niobium.}$$

Table I shows a comparison of C_s^{el} calculated from [24] with the experimental values (1) of C_s^{el} . It should be noted that the experimental values listed for $T < 2.5^\circ\text{K}$ are extrapolations and not measurements. Hence the large differences in the first four entries are not significant.

TABLE I

$T^{\circ}K$	$C_s^{el} \frac{\text{ergs}}{^{\circ}K \text{ cc.}}$ experimental	$C_s^{el} \frac{\text{ergs}}{^{\circ}K \text{ cc.}}$ Theoretical	% difference
.5	20	.00030	
1.0	70	3.23	
1.5	230	97.4	
2.0	700	616	
2.5	1900	2021	+5.9
3.0	4.56×10^3	4.74×10^3	+3.8
3.5	9.25×10^3	9.12×10^3	-1.4
4.0	1.601×10^4	1.528×10^4	-4.8
4.5	2.458×10^4	2.350×10^4	-4.4
5.0	3.48 " "	3.38 " "	-2.9
5.5	4.69 " "	4.64 " "	-1.1
6.0	6.12 " "	6.10 " "	-0.3
6.5	7.75 " "	7.78 " "	+0.4
7.0	9.59 " "	9.74 " "	+1.5
7.5	11.7 " "	11.9 " "	+1.7
8.0	14.2 " "	14.3 " "	+0.7
8.5	17.0 " "	17.0 " "	0.0
8.7	18.1 " "	18.1 " "	0.0

IV CONCLUSIONS

Demanding that London's phenomenological equations be consistent with the hydrodynamic description of a classical charged fluid has led us to find that solutions of the equations of motion contain four independent modes of oscillation. One of these, the plasma oscillation, does not contribute appreciably to the specific heat of the system. The other three oscillations, the one electric potential oscillation and the two magnetic vector potential oscillations, have been shown to contribute appreciably to the specific heat provided that the dielectric constant is, as was suggested by Ginsburg, of the order of 10^{10} .

Whether or not this is the correct interpretation of the high specific heat in superconducting metals can only be ascertained by an experiment* determining the dielectric constant in superconductors.

* Ginsburg (2) mentions that an experiment has been performed by Galkin and Besugly which indicates that the dielectric constant of tin is about 5×10^9 . The author, however, has been unable to find the paper referred to by Ginsburg.

APPENDIX A

In analogy with the classical Schroedinger field we could start with the Lagrangian density:

$$[1] \quad \mathcal{L} = -\rho \dot{\varphi} - \frac{\hbar^2}{8m^2} \rho \left(\frac{\nabla \rho}{\rho} \right)^2 - \frac{1}{2} \rho v^2 + \frac{1}{8\pi} (\varepsilon E^2 - H^2) - \frac{1}{8\pi} \left(\frac{\varepsilon}{c} \dot{\Phi} + \nabla \cdot \vec{A} \right)^2 + \mathcal{U}(\rho, \Phi)$$

where $\mathcal{U}(\rho, \Phi)$ is some energy density which is to be determined. By independent variation with respect to φ , ρ , $\vec{A}$, and Φ , we get the equations of motion:

$$[5] \quad \dot{\rho} + \nabla \cdot (\rho \vec{v}) = 0$$

$$[6] \quad \dot{\varphi} + \frac{1}{2} v^2 + \frac{\hbar^2}{4m^2} \left(\frac{1}{2} \left(\frac{\nabla \rho}{\rho} \right)^2 - \frac{1}{\rho} \nabla^2 \rho \right) = \frac{\partial \mathcal{U}}{\partial \rho}$$

$$[7] \quad \nabla^2 \vec{A} - \frac{\varepsilon}{c^2} \ddot{\vec{A}} + \frac{4\pi e}{m c} \rho \vec{v} = 0$$

$$[8] \quad \nabla^2 \Phi - \frac{\varepsilon}{c^2} \ddot{\Phi} = \frac{4\pi}{\varepsilon} \frac{\partial \mathcal{U}}{\partial \Phi}$$

Making the expansions:

$$\rho = \rho_0 + \rho_1$$

$$\Phi = \Phi_0 + \Phi_1$$

$$\mathcal{U}(\rho, \Phi) = \mathcal{U}(\rho_0, \Phi_0) + A_1 \Phi_1 + B_1 \rho_1 + C_1 \rho_1 \Phi_1 + \frac{A_2}{2!} \Phi_1^2 + \frac{B_2}{2!} \rho_1^2 + \dots$$

where

$$A_1 = \left(\frac{\partial \mathcal{U}}{\partial \Phi} \right)_{\rho_0, \Phi_0}, \quad A_2 = \left(\frac{\partial^2 \mathcal{U}}{\partial \Phi^2} \right)_{\rho_0, \Phi_0},$$

$$B_1 = \left(\frac{\partial \mathcal{U}}{\partial \rho} \right)_{\rho_0, \Phi_0}, \quad B_2 = \left(\frac{\partial^2 \mathcal{U}}{\partial \rho^2} \right)_{\rho_0, \Phi_0},$$

and

$$C_1 = \left(\frac{\partial^2 \mathcal{U}}{\partial \rho \partial \Phi} \right)_{\rho_0, \Phi_0}$$

$$\frac{\partial \mathcal{U}}{\partial \Phi} = A_1 + A_2 \Phi_1 + C_1 \rho_1 + \dots$$

$$\frac{\partial \mathcal{U}}{\partial \rho} = B_1 + B_2 \rho_1 + C_1 \Phi_1 + \dots$$

and assuming that $\underline{A}$ and φ are small such that second order terms in them may be neglected, we find that in order to satisfy the zero order equations we must have $A_1 = B_1 = 0$. Hence the first order equations of motion become:

$$[5''] \quad \dot{\rho}_1 + \rho_0 \nabla^2 \varphi - \frac{e}{m c} \rho_0 \nabla \cdot \vec{A} = 0$$

$$[6''] \quad \dot{\Phi} - \frac{\hbar^2}{4m^2 \rho_0} \nabla^2 \rho_1 = B_2 \rho_1 + C_1 \Phi_1$$

$$[7''] \quad \nabla^2 \vec{A} - \frac{\epsilon}{c^2} \ddot{\vec{A}} + \frac{4\pi e}{m c} \rho_0 \nabla \varphi - \frac{4\pi e^2}{m^2 c^2} \rho_0 \vec{A} = 0$$

$$[8''] \quad \nabla^2 \Phi_1 - \frac{\epsilon}{c^2} \ddot{\Phi}_1 = \frac{4\pi}{\epsilon} (A_2 \Phi_1 + C_1 \rho_1)$$

We can determine A_2 and C_1 as follows: take the divergence of [7''] and demand that the Lorentz condition, $\nabla \cdot \vec{A} = -\frac{\epsilon}{c} \dot{\Phi}$, holds.

We get

$$[7''a] \quad \nabla^2 \dot{\Phi}_1 - \frac{\epsilon}{c^2} \ddot{\Phi}_1 - \frac{4\pi e \rho_0}{m \epsilon} \nabla^2 \varphi - \frac{4\pi e^2 \rho_0}{m^2 c^2} \dot{\Phi}_1 = 0$$

Substituting $\nabla^2 \varphi$ from [5''] we get

$$[7''b] \quad \nabla^2 \dot{\Phi}_1 - \frac{\epsilon}{c^2} \ddot{\Phi}_1 = \frac{4\pi}{\epsilon} (A_2 \dot{\Phi}_1 + C_1 \dot{\rho}_1) = -\frac{4\pi e}{m \epsilon} \dot{\rho}_1$$

Differentiating equation [8''] with respect to t, we get

$$[8''b] \quad \nabla^2 \ddot{\Phi}_1 - \frac{\epsilon}{c^2} \dddot{\Phi}_1 = \frac{4\pi}{\epsilon} (A_2 \ddot{\Phi}_1 + C_1 \dot{\rho}_1)$$

Comparison of equation [7''b] and [8''b] shows that $A_2 = 0$ and $C_1 = -\frac{e}{m}$.

Therefore $\frac{\partial \mathcal{U}}{\partial \Phi} = -\frac{e}{m} \rho_1 + \dots$ higher order terms

and $\frac{\partial \mathcal{U}}{\partial \rho} = -\frac{e}{m} \Phi_1 + B_2 \rho_1 + \dots$ higher order terms

Hence $\mathcal{U} = \mathcal{U}_0 - \frac{e}{m} \rho_1 \Phi_1 + \frac{B_2}{2} \rho_1^2 + \dots$ higher order terms

The absence of any terms in U like $\frac{e}{m} \rho_1 \Phi_0$ or $\frac{e}{m} \rho_0 \Phi_1$ means that there is no Coulomb interaction between the electrons on the average. This means essentially that the interaction with the positive lattice charges cancels the interaction with the average negative electron charges represented by ρ_0 and Φ_0 . Using these coefficients, the equations of motion become:

$$[5b] \quad \dot{\rho}_1 + \rho_0 \nabla^2 \varphi - \frac{e}{mc} \rho_0 \nabla \cdot \vec{A} = 0$$

$$[6b] \quad \dot{\varphi} - \frac{\hbar^2}{4m^2 \rho_0} \nabla^2 \rho_1 + \frac{e}{m} \Phi_1 = B_2 \rho_1$$

$$[7b] \quad \nabla^2 \vec{A} - \frac{\varepsilon}{c^2} \ddot{\vec{A}} + \frac{4\pi e}{mc} \rho_0 \nabla \varphi - \frac{4\pi e^2}{m^2 c^2} \rho_0 \vec{A} = 0$$

$$[8b] \quad \nabla^2 \Phi_1 - \frac{\varepsilon}{c^2} \ddot{\Phi}_1 = -\frac{4\pi e}{m \varepsilon} \rho_1$$

We will now demand that London's first equation is satisfied.

Taking the gradient of all terms in equation [6b] we have:

$$\frac{m}{e} \nabla \dot{\varphi} + \nabla \Phi_1 = \frac{m}{e} \nabla \left(B_2 \rho_1 + \frac{\hbar^2}{4m^2 \rho_0} \nabla^2 \rho_1 \right)$$

$$\text{or} \quad \frac{\partial \vec{A}}{\partial t} - \vec{E} = \frac{m}{e} \nabla \left(B_2 \rho_1 + \frac{\hbar^2}{4m^2 \rho_0} \nabla^2 \rho_1 \right)$$

$$\text{Hence} \quad \nabla \left(B_2 \rho_1 + \frac{\hbar^2}{4m^2 \rho_0} \nabla^2 \rho_1 \right) = 0$$

$$\text{or} \quad \nabla^2 \rho_1 + \frac{4m^2 \rho_0 B_2}{\hbar^2} \rho_1 = \text{CONST} = C_1$$

The solution of the above equation is:

$$[12a] \quad \rho_1 = g(t) e^{i \vec{k}_0 \cdot \vec{r}} + \frac{\hbar^2 C_1}{4m^2 \rho_0 B_2} \quad \text{where } K_0^2 = \frac{4m^2 \rho_0 B_2}{\hbar^2}$$

If we demand that the spatial average of ρ_1 is zero, then it

follows immediately from [12a] that the constant C_1 must be zero. Hence we have:

$$[12b] \quad \rho = g(t) e^{i\vec{k}_0 \cdot \vec{r}}$$

Proceeding as before we can now construct the solutions to the equations of motion.

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